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Tangential Calculus and Variational Integrals with Respect to a Measure

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Contents

I. Introduction

Several theories have been developed in the last decades in order to give a weak description of sets: this is the main aim of the mathematical branch nowadays addressed as "Geometric Measure Theory". Actually, many mathematical problems, even very classical such as the Euler's elastica [49], involve manifolds as unknown; in particular, from the Calculus of Variations' point of view, it is interesting to establish existence and regularity results for minimum problems concerning sets of any dimension (think for instance of minimal surfaces [45]). The modern approach to this fascinating area is based on the theories of rectifiable or general varifolds [4, 5, 6, 64], and of cartesian currents [44, 54], which both provide useful compactness theorems, and have by now found a systematic treatment. Nevertheless, there are still many unsolved problems. To begin with, the above mentioned theories deal with generalized manifolds, embedded in $\mathbb{R}^n$, which have a fixed (integer) dimension, while it is not difficult to be convinced that often in shape optimization problems the dimension of the optimal structure is *a priori* unknown, and even possibly non-constant. Thus, it is not clear in which class the solution is to be searched. Furthermore, very little progress has been made in the study of functional spaces over these weak objects, so that no technique allows at present to handle even simple problems involving a state equation on the unknown manifold.

The aim of this Thesis is to propose a new approach that can be synthetically defined as the Calculus of Variations with respect to a measure. It consists in adopting a quite large class of competitors for the above described kind of variational problems, namely, the class $\mathcal{M}^+$ of all positive and finite Borel regular measures on $\mathbb{R}^n$, or, more precisely, some suitable subclasses of $\mathcal{M}^+$. In particular, the integral functionals under consideration will be all of the form

$$F(u) = \int_{\mathbb{R}^n} f(x, u, \nabla u)\, d\mu \ ,$$

where the measure μ will no longer be necessarily the Lebesgue measure $\mathcal{L}^n$, but more in general a positive measure μ on $\mathbb{R}^n$: it can be chosen for instance to be the Hausdorff measure $\mathcal{H}^k$ concentrated over a set of dimension $k < n$, possibly multiplied by a positive density. By this way, we are in a position to treat for instance problems suggested by Mechanics, where one needs to model complex elastic structures, such as fibers, beams, membranes, plates, with possible joints between parts of different dimensions.

In order to exploit the good compactness properties offered by the weak setting of measures, one has to develop a sort of tangential calculus with respect to a fixed element μ of $\mathcal{M}^+$. Indeed, the starting point of our theory is a new notion of tangent space T_μ to a positive measure μ, which has been first introduced by Bouchitté, Buttazzo, and Seppecher in [18]; when μ is the

measure $\mathcal{H}^k$ over a Lipschitz k-manifold M in $\mathbb{R}^n$, then T_μ reduces to the classical definition of tangent bundle to M. Further, this notion seems to have an autonomous interest from the Geometric Measure Theory point of view: in [42 (Fragalà and Mantegazza)], we have studied the relations with some other definitions of tangent space to a measure, already used in the literature and based on the idea of blow-up; we have also proved a rectifiability theorem for the restriction of a measure μ to the set of points where its tangent space has a dimension k (for $1 \leq k \leq n$).
Starting from the notion of tangent bundle to μ, other classic concepts of Differential Geometry can be reformulated in the same weak setting. For instance, in [16 (Bouchitté, Buttazzo, and Fragalà)], we have given the notion of vector mean curvature H of a measure, useful to consider elastic structures submitted to a bending force. Our definition identifies H with a vector-valued measure, and it is comprehensive of a boundary term; in fact, when μ is the weight measure of a rectifiable varifold V, $H(\mu)$ reduces to the well-known notion of first variation δV. In the same paper, we have studied the lower semicontinuity of functionals depending on $H(\mu)$ with respect to the weak convergence of measures: it turns out to be strictly related with the behaviour of the dimension of the tangent spaces; indeed, we obtain the weak convergence of the curvatures as soon as $\mu_h \rightharpoonup \mu$ implies $(\dim T_{\mu_h})\mu_h \rightharpoonup (\dim T_\mu)\mu$. Unfortunately, the conditions on μ_h that we have been able to give for the validity of such implication include the constraint $\dim T_{\mu_h} = k$, where k is a fixed integer (not depending on h). This considerably limits the present power of our method, and the delicate removing of this assumption would be worth, in view of its many applications, of further investigation. Nevertheless, via our lower semicontinuity results, we can prove the existence of a solution to minimum problems involving curvatures and posed on a class of measures of fixed dimension, such as Willmore-type problems for surfaces with a prescribed boundary.
Passing to the functional aspect of our method, the first definition of functional space with respect to a measure μ was given in [18], where the authors introduced the μ-Sobolev spaces "of type H". Indeed, for any smooth function u belonging to $\mathcal{D} := \mathcal{C}_0^\infty(\mathbb{R}^n)$, its μ-tangential gradient can be easily obtained as the orthogonal projection onto T_μ of the usual gradient of u; hence the spaces $H^{1,p}_\mu = H^{1,p}_\mu(\mathbb{R}^n)$ can be defined in a natural way by completion. Then, a relaxation result holds, when $p > 1$, for p-growth energies given on $\mathcal{D}$ by an integral functional of the kind

$$F(u) = \int_{\mathbb{R}^n} f(x, \nabla u)\, d\mu \ .$$

The domain of the relaxed functional $\overline{F}$ (with respect to the L^p_μ topology) turns out to coincide with $H^{1,p}_\mu$, and the explicit expression of $\overline{F}(u)$ can be computed for any $u \in H^{1,p}_\mu$.
The extension of this approach to the case $p = 1$ led us to consider the space of functions with bounded variation with respect to a measure, and some aspects of the related perimeter theory: in [13 (Bellettini, Bouchitté, and Fragalà)], we have introduced a new distributional definition of the space $BV_\mu = BV_\mu(\mathbb{R}^n)$, which properly contains $H^{1,1}_\mu$ and reduces to the classical space

$BV(\Omega)$ by taking $\mu = \mathcal{L}^n \llcorner \Omega$. Such space turns out to be the natural relaxation domain of a metric integral functional in $d\mu$, namely, of an energy $F(u)$ of the same kind as above, but where the integrand f has now a linear growth (more precisely, f is a so-called Finsler metric [12] on the tangent bundle to μ). We have studied in detail the problem of giving an integral representation for $\overline{F}(u)$, which is considerably more delicate than in the super-linear case: at least when u does not belong to $H^{1,1}_\mu$, suitable regularity assumptions are needed on both μ and f. On the other hand, a rather short and elegant proof, enforcing the distributional nature of our μ-total variation and a commutation argument between supremum and integral, yields the validity of a quite general coarea-type formula on $L^1_\mu = L^1_\mu(\mathbb{R}^n)$.
More recently, having in mind applications to shape optimization problems where the control variable is represented by a manifold, we have investigated in [17 (Bouchitté, Buttazzo, and Fragalà)] the convergence of state equations corresponding to a variational problem posed on the Sobolev spaces $H^{1,p}_{\mu_h}(\Omega)$, being μ_h an assigned sequence of controls, with $\mu_h \rightharpoonup \mu$, and Ω an open subset of $\mathbb{R}^n$. We have discovered that, when dealing with the Γ-convergence of the spaces $H^{1,p}_{\mu_h}(\Omega)$, one may fall upon the space $H^{1,p}_\mu(\Omega)$ associated with the limit measure μ, as well as upon a larger μ-Sobolev space $W^{1,p}_\mu(\Omega)$. It consists in a space "of type W", *i.e.* of distributional nature, whose definition makes sense only under suitable assumptions on the mean curvature of μ: this topic is treated in detail in Chapter V. In particular, when μ is the Lebesgue measure, $H^{1,p}_\mu(\Omega)$ and $W^{1,p}_\mu(\Omega)$ respectively reduce to the classical spaces $H^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$. Therefore, the possible well-known phenomenon "$H \neq W$" may occur, and in addition, when μ is supported by a lower dimensional set, such inequality may hold independently from the regularity of the boundary of Ω (even with $\Omega = \mathbb{R}^n$). This appears to be a serious obstacle in the way to derive general existence results: it seems reasonable to overcome this difficulty by posing the state equation directly on the weak spaces, but this perspective is concerned with a delicate approximation problem at present unsolved.
A more straightforward application of our method is the homogenization of thin reticulated elastic structures: in a forthcoming paper by Bouchitté and Fragalà, we identify each of them with a periodic positive measure μ, and we carry out the tangential calculus with respect to such a kind of measure by studying the subspace of $H^{1,p}_\mu(\mathbb{R}^n)$ given by the periodic functions. This allows to formulate a unit-cell problem on the support of μ, whose solution is the candidate homogenized density for a sequence of energies of the form $\int_{\mathbb{R}^n} f(\frac{x}{\varepsilon}, \nabla u) d\mu_\varepsilon$ (here f is a convex integrand with a p-growth condition, and μ_ε are the ε-scalings of μ, defined by $\mu_\varepsilon(B) = \varepsilon^n \mu(\frac{B}{\varepsilon})$). Then, the technique adopted to prove the conjectured homogenization result, makes use of a new concept of two-scale convergence with respect to measures, and of some crucial assumptions of "connectedness" on the measure μ. This seems to be an unifying approach, valid for structures of any dimension, independent of the shape of the domain under consideration (which is not the case for the more classical technique of the operators' extension), and with possible generalizations to the case of non-local limit energies.

What has been exposed in a descriptive way above, will be now presented in a more rigorous framework in the following five compendia, throughout which all efforts have been done in order to keep the notation consistent. Each compendium summarizes the contents of a corresponding paper, annexed as a self-contained chapter of the Thesis, in which the proofs of all the results can be found. So, equations and theorems are numbered independently in every chapter, while a common bibliography has been postponed at the end of the Thesis. Furthermore, as it has been already pointed out in this Introduction, the subject presents several still open problems. Some selected ones, of both technical and substantial character, which have more drawn our attention, are mentioned throughout the compendia. We address them with the hope of solving some in the years to come.

Compendium of Chapter II

In Chapter II, we study different notions of tangent space to a positive Borel regular measure $\mu \in \mathcal{M}^+$, with special attention to the one given by Bouchitté, Buttazzo, and Seppecher in [18]. We slightly modify their definition (*cf.* the compendium of Chapter V), by considering, as a class of tangent fields to μ, the vector space

$$X_\mu := \{\Phi \in (L^1_\mu)^n \ : \ \mathrm{div}(\Phi\mu) \in \mathcal{M}\} ,$$

where $(L^1_\mu)^n := L^1_\mu(\mathbb{R}^n;\mathbb{R}^n)$, the divergence operator is intended in the distributional sense, and $\mathcal{M}$ denotes the set of signed Borel measures with finite total variation on $\mathbb{R}^n$.

Definition 1.1. *For μ-a.e. $x \in \mathbb{R}^n$, we define the tangent space to μ by*

$$T_\mu(x) := \mu - \mathrm{ess}\bigcup\{\Phi(x) : \Phi \in X_\mu\} \ . \tag{1.1}$$

The above definition means that T_μ is the unique μ-measurable, closed valued multifunction on $\mathbb{R}^n$ characterized by:

(i) $\Phi \in X_\mu \Rightarrow \Phi(x) \in T_\mu(x)$ for μ-a.e. $x \in \mathbb{R}^n$;

(ii) T_μ is the minimal μ-measurable multifunction satisfying (i), that is, $T_\mu(x) \subseteq \Sigma(x)$ for μ-a.e. $x \in \mathbb{R}^n$, whenever Σ is a μ-measurable, closed valued multifunction on $\mathbb{R}^n$ such that $\Phi \in X_\mu \Rightarrow \Phi(x) \in \Sigma$ for μ-a.e. $x \in \mathbb{R}^n$.

It is easy to check that, for any $\mu \in \mathcal{M}^+$ and μ-almost every $x \in \mathbb{R}^n$, $T_\mu(x)$ individuates a linear subspace of $\mathbb{R}^n$, whose dimension may depend on x. For further details on the properties of the μ-essential union, we refer to [68, 18]. Here we rather focus attention on the comparison between Definition 1.1 and other different notions of tangent space to a measure given by Simon [64] and by Preiss [59], both based on the crucial idea of blow-up. For any fixed $x \in \mathbb{R}^n$ and any positive real number ρ, let $\mu_{x,\rho}$ be the measure defined on Borel subsets of $\mathbb{R}^n$ by $\mu_{x,\rho}(B) = \mu(x+\rho B)$. Then the above mentioned definitions read respectively as follows.

Definition 1.2. *Let $P_\mu = P_\mu(x)$ be a k-dimensional subspace of $\mathbb{R}^n$, with $k \leq n$. Then P_μ is said to be the approximate tangent space for μ at x if there exists a positive constant $\theta = \theta(x)$, which is called the multiplicity at x, such that, when $\rho \to 0$,*

$$\frac{1}{\rho^k}\mu_{x,\rho} \rightharpoonup \theta(x)\mathcal{H}^k \llcorner P_\mu(x) \tag{1.2}$$

weakly * in the sense of measures.

Definition 1.3. *Let ν be a measure on the unit ball B of $\mathbb{R}^n$. We say that ν belongs to the set $\mathrm{Tan}(\mu, x)$ of tangent measures for μ at the point $x \in \mathbb{R}^n$, if there exists a sequence of positive real numbers ρ_i converging to zero such that, when $i \to \infty$,*

$$\frac{\mu_{x,\rho_i}}{\mu(B_{\rho_i}(x))} \rightharpoonup \nu$$

weakly * in the sense of measures.

The μ-almost everywhere existence of a k-dimensional space satisfying (1.2) is fulfilled if and only if μ is, up to a multiplicative density, the measure $\mathcal{H}^k$ over a k-rectifiable set [64, Theorem 11.8], while a straightforward compactness argument ensures that the class of tangent measures $\mathrm{Tan}(\mu, x)$ is non-empty for any measure $\mu \in \mathcal{M}^+$ and for every point $x \in \mathbb{R}^n$. The relations with Definition 1.1 are expressed by the next two lemmas.

Lemma 1.4. *Suppose that P_μ associates to μ-a.e. $x \in \mathbb{R}^n$ the k-dimensional approximate tangent space $P_\mu(x)$ to μ at x. Then*

$$T_\mu(x) \subseteq P_\mu(x) \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n , \tag{1.3}$$

with a possibly strict inclusion.

Lemma 1.5. *For μ-a.e. $x \in \mathbb{R}^n$, any measure $\nu \in \mathrm{Tan}(\mu, x)$ is of the form*

$$\left(\mathcal{H}^{k(x)}_{T_\mu(x)} \times \sigma\right) \llcorner B ,$$

where $k(x)$ is the dimension of $T_\mu(x)$, $\mathcal{H}^{k(x)}_{T_\mu(x)}$ is the $k(x)$-dimensional Hausdorff measure over $T_\mu(x)$, and σ is a measure of locally finite mass on $T_\mu(x)^\perp$.

Enforcing the above lemmas, we are enabled to explicitly identify the tangent space T_μ for some relevant class of measures. We give here a short list of examples, whose detailed discussion can be found in section II.3.

Example 1.6. (*Lipschitz manifolds*) If μ is the measure $\mathcal{H}^k$ over a k-Lipschitz manifold M, the tangent space T_μ coincides μ-a.e. with the usual tangent space to M.

Example 1.7. (*Normal currents*) There is a one-to one correspondence between the normal 1-currents T in $\mathbb{R}^n$, endowed with an orientating field Φ, and the (Radon) measures μ, coupled with a unitary tangent field in the class X_μ. This is obtained through the maps $(T, \Phi) \mapsto (\mu(T), \Phi)$, being $\mu(T)$ the total variation measure of T, and conversely $(\mu, \Phi) \mapsto (T(\mu), \Phi)$, being $T(\mu)$ the current defined on smooth 1-forms ω by $T(\mu)[\omega] = \int_{\mathbb{R}^n} \langle \omega(x), \Phi(x) \rangle \, d\mu(x)$.

Example 1.8. (*Dirac masses*) It is immediate to check that, when μ is the sum of a finite number of Dirac masses, T_μ is reduced to zero μ-a.e. Indeed, any tangent field Φ belonging to the space X_μ must vanish μ-a.e.

Example 1.9. (*Cantor-like sets*) Let us take, for $n = 1$, $\mu = \mathcal{H}^\alpha \llcorner C$, where $0 < \alpha \leq 1$ and C is an α-dimensional Cantor set, that is, a closed subset of the interval $[0, 1]$ with a dense complement and $\mathcal{H}^\alpha(C) \in (0, +\infty)$. Then one can check that T_μ is reduced to zero μ-a.e. on $\mathbb{R}$. As immediate consequences of this, the tangent space T_μ is not local on Borel subsets of $\mathbb{R}^n$ (actually it is local only on open sets), and the inclusion (1.3) may be strict (indeed it is easy to verify that P_μ coincides in this case with the whole $\mathbb{R}$).

Example 1.10. (*The Koch curve*) Let μ be the measure $\mathcal{H}^\alpha$ ($\alpha = \log 4 / \log 3$) concentrated on the fractal set K in $\mathbb{R}^2$ usually called the Koch curve, or snow-flake (its construction is explicitly recalled in Section II.3). Then, due to the self-similarity of K, a blow-up argument based on Lemma 1.5 allows to prove that T_μ is again reduced to zero μ-a.e. on $\mathbb{R}^2$.

Example 1.11. (*Rectifiable varifolds*) For an appropriate class of measures concentrated on rectifiable sets, the equality holds in (1.3). More precisely, this can be proved whenever $\mu_V = \theta \mathcal{H}^k \llcorner M$ is the weight measure of a rectifiable k-varifold $V = V(M, \theta)$ [64], with bounded first variation δV.

We now present a result about the dimension of T_μ as a linear subspace of $\mathbb{R}^n$, which will be of crucial importance for the study of the weak convergence of the curvatures in Chapter III. The main estimate is based on the non-finiteness of the α-dimensional upper density $\theta^*_\alpha(\mu, x)$, which is defined as

$$\theta^*_\alpha(\mu, x) := \limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\omega_\alpha \rho^\alpha} \,, \tag{1.4}$$

where ω_α is a normalization constant coinciding for α integer with the Lebesgue measure of the unit ball in $\mathbb{R}^\alpha$ [64, p.10]. The α-lower density $\theta_{*\alpha}(\mu, x)$ is analogously defined, replacing the limsup in (1.4) by a liminf ; whenever $\theta^*_\alpha(\mu, x)$ and $\theta_{*\alpha}(\mu, x)$ have a common value, it is called the *density* of μ at x, and it is denoted by $\theta_\alpha(\mu, x)$.

Theorem 1.12. *Let α be a real number, $0 \leq \alpha \leq n$, and let $E \subseteq \mathbb{R}^n$ be a set where $\theta^*_\alpha(\mu, x) = +\infty$. Then, for μ-a.e. $x \in E$, the dimension of $T_\mu(x)$ cannot be larger than the integer part of α.*

Easy corollaries of such result are:

- for any subset E of $\mathbb{R}^n$, the dimension of T_μ on E cannot be larger than the Hausdorff dimension of E;
- if μ has a k-dimensional (strictly) positive lower density for some integer k, then the dimension of T_μ is less or equal than k.

The concluding section of the chapter is devoted to the relation between the existence of a k-dimensional tangent space T_μ and the rectifiability properties of μ. This is interesting in order to know what kind of measures one actually deals with when studying minimum problems on the class of measures μ with $\dim T_\mu = k$. Again, the corresponding densities play a central rôle; in fact, the following result holds, as a consequence of a rather technical rectifiability criterion by Preiss [Chapter II, Theorem 4.6].

Theorem 1.13. *Let k be an integer less than or equal to n, and let E_k be the set of the points $x \in \mathbb{R}^n$ such that*

(i) the dimension of $T_\mu(x)$ is equal to k;

(ii) $\theta_{*k}(\mu, x)$ is positive and finite;

(iii) the "doubling condition" holds at x, i.e.

$$\limsup_{\rho \to 0^+} \frac{\mu(B_{2\rho}(x))}{\mu(B_\rho(x))} < +\infty \ .$$

Then the measure $\mu \llcorner E_k$ is of the kind $\theta \mathcal{H}^k \llcorner E$, with k an integer, E a countably k-rectifiable subset of $\mathbb{R}^n$, and θ a positive function $\mathcal{H}^k$-integrable on E.

We stress that the doubling condition is implied by the finiteness of the upper k-dimensional density, so that both (ii) and (iii) are satisfied when μ admits a positive and finite k-density. Notice also that Theorem 1.13 includes, differently from the above quoted rectifiability criterion based on the approximated tangent space P_μ [64, Theorem 11.8], the case of measures with variable dimension.

Compendium of Chapter III

In Chapter III, we introduce the notion of generalized mean curvature H of a measure μ, and then we investigate the weak lower semicontinuity of integral functionals depending on $H(\mu)$.

As it often occurs with the definition of "weak objects", our notion of mean curvature is inspired by an integration by parts formula: the divergence theorem on a smooth k-manifold M. For any vector field X of class $\mathcal{C}^1$ on M, it reads as

$$\int_M \mathrm{div}_M X \, d\mathcal{H}^k = -\int_M X \cdot h(M) \, d\mathcal{H}^k - \int_{\partial M} X \cdot \eta(M) \, d\mathcal{H}^{k-1} , \tag{1.5}$$

where $h(M)$ is the mean curvature vector of M, and $\eta(M)$ is the inward conormal unit vector on ∂M. By this way, replacing in (1.5) the measure $\mathcal{H}^k \llcorner M$ by a generic element μ of $\mathcal{M}^+$, we arrive at the following definition.

Definition 1.14. *For any $\mu \in \mathcal{M}^+$, its vector mean curvature is given by*

$$H(\mu) := \mathrm{div}(P_\mu \mu) \in (\mathcal{D}')^n \ .$$

In other words, we have by definition

$$\langle H(\mu), X \rangle = -\int_{\mathbb{R}^n} \mathrm{div}_\mu X \, d\mu \qquad \forall X \in \mathcal{D}^n \ ,$$

where the bracket at the left hand side is intended between the space $(\mathcal{D}')^n$ of vector-valued distributions on $\mathbb{R}^n$ and the space $\mathcal{D}^n$ of smooth compactly supported vector fields, while the μ-tangential divergence at the right hand side is defined as $\mathrm{div}_\mu X := \sum_{i=1}^n (\nabla_\mu X^i)_i$.
Notice that, as well as the tangent space T_μ, $H(\mu)$ is well-defined for any μ in $\mathcal{M}^+$. Nevertheless, we are interested in considering measures μ such that $H(\mu)$ is not so wild.

Definition 1.15. *The subclass of measures $\mu \in \mathcal{M}^+$ such that $H(\mu)$ belongs to the space $\mathcal{M}^n$ of vector-valued Borel measures with finite total variation on $\mathbb{R}^n$, will be denoted by $\mathcal{M}_{\mathrm{BC}}$ (which stays for "measures with bounded curvature").*

For any $\mu \in \mathcal{M}_{\mathrm{BC}}$, we shall use the decomposition of $H(\mu)$, given by the Radon-Nykodim Theorem, into

$$H(\mu) = h(\mu)\mu + \partial\mu \ ,$$

where $h(\mu)$ is in $(L^1_\mu)^n$, and $\partial\mu$, belonging to $\mathcal{M}^n$, is singular with respect to μ. The function $h(\mu)$ and the measure $\partial\mu$ will be called respectively the pointwise mean curvature and the boundary of μ.
A short list of examples of measures in $\mathcal{M}_{\mathrm{BC}}$ is given below. In particular, the first example justifies the terminology "boundary of μ", that we shall use for the measure $\partial\mu$.

Example 1.16. (*Smooth manifolds*) By (1.5), the measure $\mathcal{H}^k$ over a smooth k-manifold M belongs to $\mathcal{M}_{\mathrm{BC}}$: the density $h(\mu)$ of the absolutely continuous part of $H(\mu)$ coincides with the pointwise mean curvature vector $h(M)$, while the singular part of $H(\mu)$ is concentrated on the boundary of M, and it is given by $\partial\mu = \eta(M)\mathcal{H}^{k-1} \llcorner \partial M$.

Example 1.17. (*Rectifiable varifolds*) Let μ_V be the measure associated with a rectifiable k-varifold $V = V(M,\theta)$. If V has a bounded first variation δV, by Example 1.11 $H(\mu)$ coincides with δV, and in particular μ belongs to $\mathcal{M}_{\mathrm{BC}}$.

Example 1.18. (*Dirac masses - Fractal sets*) In all the cases when T_μ is reduced to zero (see Examples 1.8, 1.9, and 1.10), μ trivially belongs to $\mathcal{M}_{\mathrm{BC}}$ and it has zero mean curvature.

Example 1.19. (*Singular or end-points*) The presence of isolated corner or end-points in the support of μ does not prevent μ to belong to $\mathcal{M}_{\mathrm{BC}}$, but simply adds some vector Dirac masses to the boundary of μ. This happens for instance when μ is the measure $\mathcal{H}^1$ over one of the structures represented in Figures 1 and 2 of Chapter III: integrating by parts, it is easy to check that in both cases $H(\mu)$ coincides with $\partial\mu$, and it is given by the sum of three oriented Dirac masses concentrated at the extremal points. Notice that, in the case of Figure 1, the oscillations of the tangent space at the central point are balanced by virtue of the symmetry of the structure, whose branches meet at 120 degrees.

The concept of mean curvature of a measure allows to give a weak formulation of some geometric problems, avoiding the heaviness of parametrizations and adopting a framework quite suitable to prove existence results. Consider for instance the problem of minimizing the Willmore functional

$$\int_M \left(1 + |h(M)|^2\right) d\mathcal{H}^k$$

over the class of the smooth k-manifolds M having a prescribed $(k-1)$-manifold Γ as a boundary, and a prescribed field γ as a co-normal vector on Γ. When $k=1$, if Γ, γ are given respectively by two points P_1 and P_2 in $\mathbb{R}^n$, and two vectors v_1 at P_1 and v_2 at P_2, we recover the classical Euler's elastica problem; when $k=2$, we can assign a smooth closed curve Γ with a vector field γ on it, and then search for the best surface which leans on Γ with γ as a co-normal direction. In our weak setting, the above problem can be reformulated as

$$\min\left\{\int_{\mathbb{R}^n} \left(1 + |h(\mu)|^2\right) d\mu \ : \ \mu \in \mathcal{M}_{\mathrm{BC}}\ ,\ \partial\mu = \lambda\right\}\ , \tag{1.6}$$

where the vector measure λ is given by $\lambda = \gamma\mathcal{H}^{k-1} \llcorner \Gamma$.
Notice that both the absolutely continuous and the singular parts of $H(\mu)$ enter in (1.6), and in particular the notion of $\partial\mu$ allows us to unify in only one condition the Dirichlet boundary data of the problem. Notice also that λ could be *a priori* assigned as a measure of variable dimension, and that no connectedness assumption is made in the support of the admissible competitors; we shall see in Chapter V that such topological constraint can be added when dealing with one-dimensional structures.

To the aim of proving the existence of a solution for a model problem such as (1.6), let us consider the following more general situation. We want to minimize a functional F defined on $\mathcal{M}^+$ through

$$F(\mu) := \begin{cases} J(\mu, H(\mu)) & \text{if } \mu \in \mathcal{A} \ , \\ +\infty & \text{otherwise} \ , \end{cases}$$

where the domain $\mathcal{A}$ of F is supposed to be a suitable subclass of $\mathcal{M}_{\rm BC}$. We assume that the cost functional J is both coercive and lower semicontinuous with respect to the weak convergence of measures on the product space $\mathcal{M}_{\rm BC} \times \mathcal{M}^n$. Then, a straightforward application of the direct method of the Calculus of Variations yields that the minimum problem

$$\min\left\{F(\mu) \ : \ \mu \in \mathcal{M}^+\right\} \tag{1.7}$$

admits a solution as soon as the class of couples $\{(\mu, H(\mu)) \ : \ \mu \in \mathcal{A}\}$ is weakly closed in $\mathcal{M}_{\rm BC} \times \mathcal{M}^n$. This of course depends on the choice of the class $\mathcal{A}$, which must satisfy the following condition:

$$\mathcal{A} \supset \mu_h \rightharpoonup \mu \ , \quad H(\mu_h) \rightharpoonup \nu \quad \Longrightarrow \quad \mu \in \mathcal{A} \ , \quad \nu = H(\mu) \ . \tag{1.8}$$

Thus, we are looking for closed subclasses of $\mathcal{M}_{\rm BC}$ on which the weak convergence of the curvatures holds.

Remark 1.20. The class $\mathcal{A}$ considered in (1.6), *i.e.* defined by a condition of the kind $\partial\mu = \lambda$, does not satisfy (1.8), as it can be easily seen by taking, for $n = 1$, $\lambda = 0$, and considering the sequence $\mu_h = \frac{1}{h}\sum_{i=1}^h \delta_{i/h}$, where $\delta_{i/h}$ is the Dirac mass at i/h. Then $\{\mu_h\}$ weakly converges to $\mu = \mathcal{L}^1 \llcorner (0,1)$, and the measure $H(\mu_h)$ turns out to be zero for any h. Therefore, neither μ belongs to $\mathcal{A}$, nor the weak limit of the curvatures (which is obviously zero) coincides with $H(\mu)$. We stress that, while the tangent space to each μ_h is reduced to zero, the limit measure is one-dimensional.

The above observation about the behaviour of the dimension of the tangent space when passing to the limit, is actually the signal of a more general phenomenon. Indeed, by applying the following lemma (with $p = +\infty$) to the projecting operators, we can prove that the dimension of the tangent space is in general weakly upper semicontinuous, and that the convergence of the curvatures holds provided the dimension does pass to the limit (see Theorem 3.4 below).

Lemma 1.21. *Let $\{\mu_h\} \subset \mathcal{M}^+$, and let $f_h \in (L^p_{\mu_h})^d$ for some $p > 1$. If $\mu_h \rightharpoonup \mu$, and $\|f_h\|_{p,\mu_h} \leq M$, then the sequence $\{f_h \mu_h\}$ is bounded in $\mathcal{M}^d$, and any of its limit points is absolutely continuous with respect to μ, with a density $f \in (L^p_\mu)^d$. Furthermore, for every convex and lower semicontinuous function $\Phi : \mathbb{R}^d \to [0, +\infty)$,*

$$\liminf_{h \to +\infty} \int_{\mathbb{R}^n} \Phi(f_h)\, d\mu_h \geq \int_{\mathbb{R}^n} \Phi(f)\, d\mu \ .$$

Theorem 1.22. *Let $\{\mu_h\} \subset \mathcal{M}_{\rm BC}$, with $\mu_h \rightharpoonup \mu$, $|H(\mu_h)| \leq C$, and $(\dim T_{\mu_h})\mu_h \rightharpoonup f\mu$. Then*

(i) $\dim T_\mu \geq f$ μ-a.e.*;*

(ii) if $\dim T_\mu = f$ μ-a.e.*, then $P_{\mu_h}\mu_h \rightharpoonup P_\mu\mu$, and in particular $H(\mu_h) \rightharpoonup H(\mu)$.*

Thus, in order to find suitable classes $\mathcal{A}$ satisfying (1.8), we have to determine what conditions ensure that the equality $\dim T_\mu = f$ of Theorem 1.22 (ii) holds. Unfortunately, until now we have been able to give an answer only restricting the problem to the class $\mathcal{M}_k$ of all the measures μ in $\mathcal{M}_{\rm BC}$ which have μ-a.e. a k-dimensional tangent space. Indeed in this case a crucial role is played by the k-dimensional density $\theta_k(\mu,x)$: by a monotonicity argument, it can be proved to exist and to be finite, see Lemma 1.23; moreover, as a consequence of Theorem 1.12, an upper bound for the dimension of T_μ follows from the assumption that $\theta_k(\mu,x)$ does not vanish, see Lemma 1.24.

Lemma 1.23. *Let k be a fixed integer, $0 \leq k \leq n$. For any $\mu \in \mathcal{M}_k$ and for μ-a.e. point $x \in \mathbb{R}^n$, the k-dimensional density $\theta_k(\mu,x)$ of μ at x exists and it is finite.*

Lemma 1.24. *Let $\mu \in \mathcal{M}_k$. If $\theta_k(\mu,x)$ is strictly positive for μ-a.e. $x \in \mathbb{R}^n$, then it holds* $\dim T_\mu \leq k$ μ-a.e.

Enforcing Lemma 1.24, we arrive at the following result.

Theorem 1.25. *Let $\{\mu_h\} \subset \mathcal{M}_k$ with $\mu_h \rightharpoonup \mu$. Then μ belongs to $\mathcal{M}_k$ provided one of the following assumptions is satisfied:*

$$\theta_k(\mu_h,x) \geq c > 0 \quad \mu_h\text{-a.e.}\,, \qquad |H(\mu_h)| \leq C < +\infty\,; \tag{1.9}$$

$$\int_{\mathbb{R}^n} (\log\theta_k(\mu_h,x))^- \, d\mu_h \leq C < +\infty\,, \qquad \partial\mu_h = 0\,, \qquad \|h(\mu_h)\|_{p,\mu_h} \leq C < +\infty\,. \tag{1.10}$$

The counterpart of this theorem in the theory of varifolds exists to our knowledge only in the case of hypotheses (1.9) [64]. Coupling now Theorems 1.22 and 1.25, we can deduce sufficient conditions for the weak convergence of the curvatures: for instance, we shall have $H(\mu_h) \rightharpoonup H(\mu)$ if $\mu_h = \mathcal{H}^k \llcorner M_h$, being M_h smooth manifolds such that $\mu_h \rightharpoonup \mu$ and $|H(\mu_h)| \leq C$. Furthermore, applying the direct method as illustrated above, we can obtain existence results for minimum problems like (1.7). For instance, we get

Corollary 1.26. *The minimum problem (1.6) admits a solution if we take as admissible measures the class $\mathcal{A}$ of all the elements $\mu \in \mathcal{M}_k$ such that $\theta_k(\mu,x)$ is larger than or equal to a fixed positive constant c (where we implicitly assume that $\{\mu \in \mathcal{A} : \partial\mu = \lambda\} \neq \emptyset$).*

A lower semicontinuity result for the dimension of the tangent space more general than Theorem 1.25, *i.e.* holding on the whole $\mathcal{M}_{\rm BC}$ instead of $\mathcal{M}_k$, would allow to treat minimum problems for structures of *a priori* unknown dimension, which besides cannot be attacked by means of the theories of varifolds or currents. This perspective is however related with some open questions which are rather delicate from the Geometric Measure Theory point of view, such as the validity of the following implications:

- $\mu \in \mathcal{M}_k \quad \Longrightarrow \quad \partial\mu \ll \mathcal{H}^{k-1}$;
- $\mu \in \mathcal{M}_{\rm BC}\ , \quad \mu = \sum_{k=1}^n \mu_k\ , \quad \dim T_{\mu_k} = k \quad \Longrightarrow \quad \mu_k \in \mathcal{M}_k\,.$

Compendium of Chapter IV

Before entering into the subject of this chapter, we make a short prologue, in order to recall the notion of μ-Sobolev spaces and a related relaxation result, which were given in [18]. To this aim, a definition of tangent space to μ slightly different from (1.1) is needed. For a fixed exponent $p \in [1, +\infty]$, let

$$X_\mu^{p'} := \left\{ \Phi \in (L_\mu^{p'})^n \ :\ \mathrm{div}(\Phi\mu) \in L_\mu^{p'} \right\}\ ,$$

where $p' := \frac{p}{p-1}$ is the conjugate exponent of p, and

$$T_\mu^p(x) := \mu - \mathrm{ess} \bigcup \left\{ \Phi(x)\ :\ \Phi \in X_\mu^{p'} \right\}\ , \qquad x \in \mathbb{R}^n\ .$$

The obvious inclusion $T_\mu^p(x) \subseteq T_\mu(x)$ is actually an equality for all the measures μ we shall deal with; however, it is still an open question whether any particular μ exists such that the notion of T_μ^p does depend on p.

The Sobolev space $H_\mu^{1,p} = H_\mu^{1,p}(\mathbb{R}^n)$ is defined as the completion of $\mathcal{D}$ with respect to the norm $\|\psi\|_{1,p,\mu} := \|\psi\|_{p,\mu} + \|\nabla_\mu \psi\|_{p,\mu}$, where $\nabla_\mu\psi(x)$ is the orthogonal projection $P_\mu(x)$ of $\nabla\psi(x)$ onto $T_\mu^p(x)$, and $\|\cdot\|_{p,\mu}$ is the usual L_μ^p-norm. In other words, it turns out that the linear operator $A : D(A) \subset L_\mu^p \to (L_\mu^p)^n$ defined by $D(A) = \mathcal{D}$ and $A\psi = \nabla_\mu\psi$ is closable ([18, Proposition 2.1]), and the domain of its unique closed extension $\overline{A}$ is called $H_\mu^{1,p}$. We adopt the notation $H_\mu^{1,p}$ since this is in fact a definition "of type H": for instance, in the case $\mu = \mathcal{L}^n \llcorner \Omega$, we obtain the usual space $H^{1,p}(\Omega)$. A distributional notion of μ-Sobolev space "of type W", corresponding to $W_\mu^{1,p}(\Omega)$ when $\mu = \mathcal{L}^n \llcorner \Omega$, will be studied in Chapter V.

Coming back to $H_\mu^{1,p}$, it is by definition a Banach space, and it is reflexive when $p \in (1, +\infty)$. For any $u \in H_\mu^{1,p}$, we set $\nabla_\mu u := \overline{A}u$. Then a density argument gives straightforward the following useful integration by parts:

$$\text{(1.11)} \qquad \int_{\mathbb{R}^n} \nabla_\mu u \cdot \Phi \, d\mu = -\langle u\,,\, \mathrm{div}(\Phi\mu)\rangle_{(L_\mu^p, L_\mu^{p'})} \qquad \forall u \in H_\mu^{1,p},\ \forall \Phi \in X_\mu^{p'}\ .$$

Notice that the adjoint operator $A^* : (L^{p'}_\mu)^n \to L^{p'}_\mu$ equals $-\operatorname{div}(P_\mu\sigma\mu)$ when applied to any vector field σ belonging to its domain $D(A^*)$, given by

$$D(A^*) = \left\{\sigma \in (L^{p'}_\mu)^n \ : \ \operatorname{div}(P_\mu\sigma\mu) \in L^{p'}_\mu\right\} = \left\{\sigma \in (L^{p'}_\mu)^n \ : \ P_\mu\sigma \in X^{p'}_\mu\right\} =: Y^{p'}_\mu \ .$$

In particular, when $p > 1$, the following characterization by duality of $H^{1,p}_\mu$ can be deduced from Lemma 1.27 below, whose proof is an easy consequence of the equality $\overline{T} = T^{**}$ [22], holding for any closable and densily defined linear operator:

$$H^{1,p}_\mu = \left\{u \in L^p_\mu \ \text{s.t.}\ \exists\, C > 0 \ : \ |\langle u, \operatorname{div}(P_\mu\sigma\mu)\rangle| \le C\|\sigma\|_{p',\mu} \quad \forall\sigma \in Y^{p'}_\mu\right\} . \tag{1.12}$$

Lemma 1.27. *Let V, W be two Banach spaces, with W reflexive. Assume that $T : D(T) \subset V \to W$ is a linear and closable operator with dense domain. Then*

$$D(\overline{T}) = \left\{v \in V \ s.t.\ \exists\, C > 0 \ : \ |\langle v, T^*w'\rangle_{(V,V')}| \le C\|w'\|_{W'} \ \forall w' \in D(T^*)\right\} .$$

Consider now an integrand $f : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ satisfying the following assumptions:

$$\begin{cases} \text{for any } z \in \mathbb{R}^n \ , \text{ the map } x \mapsto f(x,z) \text{ is } \mu\text{-measurable on } \mathbb{R}^n \ ; \\ \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n \ , \text{ the map } z \mapsto f(x,z) \text{ is convex on } \mathbb{R}^n \ ; \\ \text{it exists } C > 0 \text{ such that } C^{-1}|z|^p \le f(x,z) \le C(1+|z|^p) \text{ for } (x,z) \in \mathbb{R}^n \times \mathbb{R}^n \ . \end{cases} \tag{1.13}$$

Then, let $F : L^p_\mu \to [0,+\infty]$ be defined by

$$F(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^n} f(x,\nabla u)\, d\mu & \text{if } u \in \mathcal{D} \ , \\ +\infty & \text{if } u \in L^p_\mu \setminus \mathcal{D} \ . \end{cases} \tag{1.14}$$

The relaxed functional $\overline{F}$ of F with respect to the L^p_μ-topology, which is obtained by definition as

$$\overline{F}(u) = \inf\left\{\liminf_{h\to+\infty} F(u_h) \ : \ u_h \to u \text{ in } L^p_\mu\right\} ,$$

was computed, for $p > 1$, in [18]: it turns out that its finiteness domain is the space $H^{1,p}_\mu$, and its explicit expression is

$$\overline{F}(u) = \int_{\mathbb{R}^n} f_\mu(x, \nabla_\mu u)\, d\mu \ , \qquad u \in H^{1,p}_\mu \ , \tag{1.15}$$

where

$$f_\mu(x,z) := \inf\left\{f(x,z+\xi) \ : \ \xi \in [T^p_\mu(x)]^\perp\right\} . \tag{1.16}$$

The proof of this result is obtained by duality: indeed, since F is convex, $\overline{F}$ coincides with the bipolar functional F^{**} (in the duality $(L^p_\mu, L^{p'}_\mu)$). The key assumption $p>1$ is related to the validity of (1.12), which motivated the equality $H^{1,p}_\mu = D(\overline{F})$; on the other hand, the main tools for the integral representation (1.15) are the closability of the operator A and the consequent integration by parts formula (1.11).
The interest in studying the same problem in the case $p=1$, *i.e.* the relaxation in L^1_μ of integral functionals in $d\mu$ with a linear growth, was the initial motivation which pushed us towards the definition of the space BV_μ. Furthermore, following the geometric approach proposed in [7], one can look at the integrand f as a Finsler metric, think of $\mathbb{R}^n$ as endowed with a couple (μ, f), and try to develop a related perimeters' theory. It turns out that both the regularity of μ and f are crucial for the extension of (1.15), while it is surprising how other results, such as a coarea formula on L^1_μ and a chain rule on $H^{1,1}_\mu$, can be proved in a quite general context.
Let us now pass to expose synthetically the contents of Chapter IV. We intentionally drop the many details and technicalities as possible, recalling only the main results. In the spirit of the geometric approach above mentioned, the integrand f will be denoted by φ, which is the usual notation for Finsler metrics, while, as it usually happens for metrics on Riemannian manifolds, the definition domain of φ will be restricted to be the tangent bundle $T\mu := \{(x,z) \ : \ x \in \mathbb{R}^n\,,\ z \in T_\mu(x)\}$. The hypotheses made on φ are the μ-measurability in x, the convexity in z, the linear growth, and the homogeneity condition

$$\varphi(x,tz) = t\,\varphi(x,z) \qquad \forall\,(x,z)\in T\mu\ ,\ \forall\, t>0\ .$$

The coercivity assumption $\varphi(x,z)\geq C^{-1}|z|$ is dropped when not otherwise specified, in order to allow *a priori* some degeneracy directions. We recall that the dual metric φ^o is defined on the dual bundle $T^*\mu$ by

$$\varphi^o(x,z^*) := \sup\{z\cdot z^* \ : \ z\in T_\mu(x)\,,\ \varphi(x,z)\leq 1\}\ .$$

We are now in a position to introduce the space $BV_{\mu,\varphi}$.

Definition 1.28. *For any $u \in L^1_\mu$, we define the (μ,φ)-total variation $|D_\mu u|_\varphi$ as*

$$|D_\mu u|_\varphi := \sup\Big\{-\langle \mathrm{div}(\Phi\mu), u\rangle_{(L^\infty_\mu, L^1_\mu)} \ : \ \Phi\in X^\infty_\mu\ ,\ \varphi^o(x,\sigma(x))\leq 1 \text{ for } \mu\text{-a.e. } x\Big\} \tag{1.17}$$

and we set

$$BV_{\mu,\varphi} := \{u\in L^1_\mu \ : \ |D_\mu u|_\varphi < +\infty\}\ .$$

In particular, if E is a μ-measurable subset of $\mathbb{R}^n$ such that the characteristic function χ_E belongs to L^1_μ, we define the (μ,φ)-perimeter $P_{\mu,\varphi}(E)$ of E as $P_{\mu,\varphi}(E) := |D_\mu\chi_E|_\varphi$.

Remark 1.29. (i) When φ is controlled from below by the euclidean norm, $BV_{\mu,\varphi}$ depends only on μ; in particular, when $\varphi(x,z)=|z|$, $BV_{\mu,\varphi}$ will be denoted by BV_μ.
(ii) When μ is the Lebesgue measure over an open set Ω with a Lipschitz boundary, and $\varphi(x,z)=|z|$, it is possible to show that $|D_\mu u|_\varphi$ and $BV_{\mu,\varphi}$ reduce respectively to the classical notions of $|Du|(\Omega)$ and $BV(\Omega)$ (see Proposition 3.1 of Chapter IV).

The distributional nature of Definition 1.17 allows us to prove a quite general coarea formula on L^1_μ, which extends previous results when applied to suitable measures μ [41, 9, 43, 63]. We stress that the technique of our proof is different from the classical one: it is based on a commutation argument between supremum and integral [21, Theorem 1], rather than on the approximation with smooth functions [39].

Theorem 1.30. *For any $u\in L^1_\mu$, the map $t\mapsto P_{\mu,\varphi}(\{u>t\})$ is Lebesgue-measurable and it holds*

$$|D_\mu u|_\varphi=\int_{\mathbb{R}} P_{\mu,\varphi}(\{u>t\})\,dt\ . \tag{1.18}$$

As a corollary of (1.18), we get the stability of $BV_{\mu,\varphi}$ by composition with Lipschitz functions, *i.e.* $A\circ u\in BV_{\mu,\varphi}$ for any $u\in BV_{\mu,\varphi}$ and any $A\in \mathrm{Lip}(\mathbb{R})$; if in addition A is monotone non-decreasing, the (μ,φ)-total variation of $A\circ u$ can be written as

$$|D(A\circ u)|_{\mu,\varphi}=\int_{\mathbb{R}} A'(t)\,P_{\mu,\varphi}(\{u>t\})\,dt\ .$$

Another interesting consequence of (1.18) is the following chain rule on $H^{1,1}_\mu$, which is obtained using also Theorem 1.33 below.

Theorem 1.31. *For any $u\in H^{1,1}_\mu$ and any $a\in L^\infty(\mathbb{R})$, setting $A(t):=\int_0^t a(s)\,ds$, $A\circ u$ belongs to $H^{1,1}_\mu$, and it holds*

$$\nabla_\mu(A\circ u)=a(u)\nabla_\mu u \qquad \mu\text{-a.e.}\ .$$

Let us now focus attention on the relaxation on L^1_μ of an integral functional of the kind

$$J(u):=\begin{cases}\displaystyle\int\varphi(x,\nabla_\mu u)\,d\mu & \text{if } u\in\mathcal{D}\ ,\\ +\infty & \text{if } u\in L^1_\mu\setminus\mathcal{D};\end{cases}$$

notice the difference with (1.14), where f depends also on the orthogonal part of ∇u. Taking this observation into account, and comparing the definition of $BV_{\mu,\varphi}$ with (1.12), the statement of the following theorem appears natural.

Theorem 1.32. *The equality $\overline{J}(u) = |D_\mu u|_\varphi$ holds in L^1_μ. In particular, $BV_{\mu,\varphi}$ is the finiteness domain of $\overline{J}$.*

The problem of finding an integral representation for $\overline{J}$ is now considerably more delicate than in the case $p > 1$; we are going to treat it first for u belonging to the (proper and closed) subspace of $BV_{\mu,\varphi}$ given by $H^{1,1}_\mu$. In this case, enforcing the same commutation argument used in the proof of Theorem 1.30, we get the following result.

Theorem 1.33. *For any $u \in H^{1,1}_\mu$, we have*

$$|D_\mu u|_\varphi = \int_{\mathbb{R}^n} h(x, \nabla_\mu u)\, d\mu \ , \quad u \in H^{1,1}_\mu \ ,$$

where

$$h(x,z) := \mu - \operatorname*{ess\,sup}_{\Phi \in \mathcal{K}} \{\sigma(x) \cdot z\} \ , \qquad \mathcal{K} := \{\Phi \in X^\infty_\mu \ : \ \varphi^o(x, \Phi(x)) \leq 1\} \ .$$

As the next examples show, the equality $h = \varphi$ may not hold, due to the lack of regularity of φ or μ.

Example 1.34. Let $n = 1$, and let $F \subset (0,1)$ be a closed set with empty interior and $\mathcal{L}^1(F) > 0$. Choosing $\mu = \mathcal{L}^1 \llcorner (0,1)$ and $\varphi(x,z) = \chi_F(x)|z|$, one gets $h = 0$; choosing $\varphi(x,z) = |z|$ and $\mu(x) = a(x)\mathcal{L}^1 \llcorner (0,1)$, with $a(x) = 2\chi_F(x) + \chi_{(0,1)\setminus F}(x)$, one gets $h(x,z) = a(x)^{-1}|z|$ (see Remark 5.6 of Chapter IV for more details). In both cases it is evident that h is different from φ.

A comparison result between h and φ, and a sufficient condition to make them coincide, are given in the next theorem. We adopt for the sake of brevity the notation

$$h_1 \preceq h_2 \iff \quad h_1(x,z) \leq h_2(x,z) \text{ for all } x \in \mathbb{R}^n \setminus E, \text{ with } \mu(E) = 0 \ , \ z \in T_\mu(x) \ ,$$

and we set

$$\begin{aligned} &\overline{\varphi}_\mu(x,z) := \sup \Big\{ \sigma(x) \cdot z \ : \ \sigma \in \mathcal{C}^n \ , \ \sigma(x) \cdot z \preceq \varphi(x,z) \ , \ P_\mu \circ \sigma \in X_\mu \Big\} \ , \\ &\mathcal{V}_\mu := \{\sigma \in \mathcal{C}^n_0 \ : \ P_\mu \circ \sigma \in X_\mu \ \} \ , \end{aligned}$$

where $\mathcal{C}^n$ and $\mathcal{C}^n_0$ are respectively the class of all the continuous vector fields on $\mathbb{R}^n$, and the subclass of those which vanish at infinity.

Theorem 1.35. *With the notation above, we have* $\overline{\varphi}_\mu \preceq h \preceq \varphi$. *Conversely, in order to have* $\varphi \preceq \overline{\varphi}_\mu$, *it is sufficient that the following conditions hold:*

$$(1.19) \qquad \begin{cases} (i) & \varphi \text{ is coercive and lower semicontinuous on } T\mu \ ; \\ (ii) & \mathcal{V}_\mu \text{ is dense in } \mathcal{C}_0^n \text{ with respect to the uniform norm .} \end{cases}$$

Notice that the validity of condition (1.19) (ii) above is related to the regularity of the mean curvature $H(\mu)$: for instance, if $H(\mu) \in (L^\infty_\mu)^n$, one can check that $\mathcal{V}_\mu$ contains $\mathcal{D}^n$, so that (1.19) (ii) holds.
Moreover, assumption (1.19) (ii) turns out to be crucial also for the existence of an integral representation of $|D_\mu u|_\varphi$ when $u \in BV_{\mu,\varphi} \setminus H^{1,1}_\mu$. Consider indeed the following example, which is rather surprising and is discussed in detail in Section 6 of Chapter IV.

Example 1.36. Let μ be the one-dimensional measure over the curve in $\mathbb{R}^2$ illustrated in Figure 1 below. In this case it can be proved that the closure $\overline{\mathcal{V}_\mu}$ of $\mathcal{V}_\mu$ in $\mathcal{C}_0^2$ is given by the fields which are at P parallel to the x axis, so that $\overline{\mathcal{V}_\mu}$ is strictly contained into $\mathcal{C}_0^2$. Assume that φ does not depend explicitly on x, and take u equal to the characteristic function χ_E of the right branch E; explicit calculations yield the equality $P_{\mu,\varphi}(E) = \min\{\varphi(\nu^+), \varphi(\nu^-)\}$ (see Figure 1 for the definition of the unit vectors $\nu^\pm$). In particular, an integral representation of the type $|D_\mu u|_\varphi = \int_{\mathbb{R}^n} \varphi(\alpha_\mu(u))$ cannot hold for any vector measure $\alpha_\mu(u)$, as the map $\varphi \mapsto |D_\mu u|_\varphi$ (with $u = \chi_E$) fails to be linear.

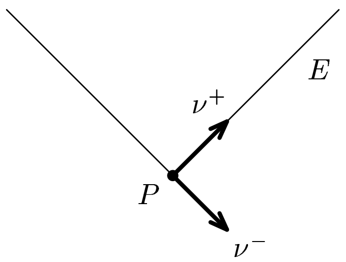

Figure 1

Actually, condition (1.19) (ii) can be proved to be necessary and sufficient for the closability of the operator $T : D(T) \subset L^1_\mu \to \mathcal{M}^n$ defined by $D(T) = H^{1,1}_\mu$, $Tu = (\nabla_\mu u)\mu$ (being $\mathcal{M}^n$ endowed with the weak $*$ topology). In this case, $\overline{T}u \in \mathcal{M}^n$ is defined by

$$(1.20) \qquad \langle \overline{T}u, \sigma \rangle_{(\mathcal{M}^n, \mathcal{C}_0^n)} := -\langle u, \operatorname{div}[(P_\mu \sigma)\mu] \rangle \ , \qquad \sigma \in \mathcal{V}_\mu \ ;$$

in particular, $D(\overline{T})$ turns out to contain BV_μ, and we can set $D_\mu u := \overline{T}u$ for any $u \in BV_\mu$. Thus, (1.20) plays now the role of the integration by parts formula (1.11), and we can arrive at the following representation result.

Theorem 1.37. *Let $\mu \in \mathcal{M}^+$ satisfy the regularity assumption (1.19) (ii) and the additional condition*

$$\sup\Big\{\langle u, \mathrm{div}\,[(P_\mu\sigma)\mu]\rangle \ : \ \sigma \in \mathcal{V}_\mu \ , \ |\sigma| \leq 1\Big\} = \sup\Big\{\langle u, \mathrm{div}(\Phi\mu)\rangle \ : \ \Phi \in X_\mu \ , \ |\Phi| \leq 1 \ \mu\text{-a.e.}\Big\} \ .$$

Assume that φ is coercive and admits a continuous extension on $\mathbb{R}^n \times \mathbb{R}^n$. Then it holds

$$|D_\mu u|_\varphi = \int_{\mathbb{R}^n} \varphi(x, D_\mu u) \ , \qquad u \in BV_{\mu,\varphi} \ . \tag{1.21}$$

where the right hand side is intended in the usual sense of integration theory, namely $\int_{\mathbb{R}^n} f(x, \frac{dD_\mu u}{d|D_\mu u|})\, d|D_\mu u|$.

The hypotheses made on μ in the theorem above are satisfied for instance if $\mu = a\mathcal{H}^k \llcorner M$, where the manifold M and the density a satisfy suitable smoothness assumptions (*cf.* Example 3.2 of Chapter IV). On the other hand, it is an open problem to understand which kind of general representation formula may replace (1.21) in the case of even simple Lipschitz manifolds, such as the one considered in Example 1.36.

Compendium of Chapter V

In Chapter V we treat a problem which can be seen as an extension of the relaxation result recalled in the previous compendium (*cf.* formula (1.15)); namely, we study the convergence of energies with respect to varying measures. More precisely, for a given sequence $\{\mu_h\} \subset \mathcal{M}^+$, weakly converging to μ, we consider the functionals F_h defined on $L^p_{\mu_h}(\Omega)$ by

$$F_h(u) := \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\, d\mu_h & \text{if } u \in \mathcal{D} \qquad (\ \textit{variant}\text{: if } u \in \mathcal{D}(\Omega)\ ) \ , \\ +\infty & \text{otherwise} \ . \end{cases}$$

Here the real exponent p is supposed to be strictly larger than 1, and the integrand f is assumed to satisfy conditions (1.13). Notice that, with respect to (1.14), the integration domain $\mathbb{R}^n$ has been changed for more generality into an open set Ω: this does not affect the relaxation of each F_h, since the procedure followed to define $H^{1,p}_\mu$ can be localized on Ω. In other words, the space $H^{1,p}_\mu(\Omega)$ can be defined as the completion of $\mathcal{D}$ with respect to the norm $\|\psi\|_{1,p,\mu,\Omega} := \|\psi\|_{p,\mu,\Omega} + \|\nabla_\mu\psi\|_{p,\mu,\Omega}$, where $\|\cdot\|_{p,\mu,\Omega}$ is the usual $L^p_\mu(\Omega)$-norm; in the variant when $\mathcal{D}$ is replaced by $\mathcal{D}(\Omega)$, one obtains in the same way the closed subspace $H^{1,p}_{\mu,0}(\Omega)$ of $H^{1,p}_\mu(\Omega)$ given by the Sobolev functions which vanish outside Ω.

The study of the asymptotic behaviour of the energies F_h is intended in the sense of Γ-convergence. We recall that $\{F_h\}$ Γ-converges to $\mathcal{F}$ if both the Γ-liminf and the Γ-limsup of $\{F_h\}$ agree with $\mathcal{F}$ on $L^p_\mu(\Omega)$; they are defined respectively by [34]

$$\Gamma\text{-}\liminf_{h\to+\infty} F_h(u) := \inf\left\{ \liminf_{h\to+\infty} F_h(u_h) \ : \ u_h\mu_h \rightharpoonup u\mu \right\} , \qquad \forall u \in L^p_\mu(\Omega) \ ;$$
$$\Gamma\text{-}\limsup_{h\to+\infty} F_h(u) := \inf\left\{ \limsup_{h\to+\infty} F_h(u_h) \ : \ u_h\mu_h \rightharpoonup u\mu \right\} , \qquad \forall u \in L^p_\mu(\Omega) \ .$$

Notice that the adopted convergence $u_h\mu_h \rightharpoonup u\mu$ is the most natural one, since the spaces $L^p_{\mu_h}(\Omega)$ vary with μ_h.
The same problem with μ_h equal to the Lebesgue measure on varying open sets Ω_h was recently studied in [24, 25]. The idea of considering varying manifolds instead of varying open subsets of $\mathbb{R}^n$ was suggested by the possible application of this topic to optimization problems with manifolds as controls. Such problems arise in a natural way in Mechanics, when one has to find the best low-dimensional structure with respect to some criterion. We shall come back to this aspect at the end of the compendium, giving some new perspectives for existence results: in spite they are a weaker conclusion than the one we expected, to the best of our knowledge no general method exists in the literature to deal with such a kind of problems.
We now pass to the survey of the results. As a first fact, using the relaxation already known for each fixed functional F_h, it is not difficult to prove the following upper bound for the Γ-limsup of the sequence $\{F_h\}$.

Theorem 1.38. *If $\mu_h \rightharpoonup \mu$, it holds*

$$\Gamma\text{-}\limsup_{h\to+\infty} F_h(u) \le F(u) \qquad \forall u \in L^p_\mu(\Omega) \ ,$$

where

$$F(u) := \begin{cases} \displaystyle\int_\Omega f_\mu(x, \nabla_\mu u)\, d\mu & \text{if } u \in H^{1,p}_\mu(\Omega) \ , \\ +\infty & \text{otherwise} \ , \end{cases}$$

being f_μ defined by (1.16).

The identification of a lower bound for the Γ-liminf of F_h is not so immediate: more powerful convergence and closedness results than those of Chapter III, as well as a deeper understanding of the μ-Sobolev spaces, are required. Let us begin by the following convergence result for the projecting operators onto the tangent space.

Theorem 1.39. *Let $\mu_h \rightharpoonup \mu$, and assume that $P_{\mu_h}\mu_h \rightharpoonup P_\mu\mu$. Then $\{P_{\mu_h}\}$ actually converges to P_μ strongly, in the sense that*

$$\text{(1.22)} \qquad \lim_{h\to+\infty} \int_{\mathbb{R}^n} \varphi(x, P_{\mu_h}(x))\, d\mu_h = \int_{\mathbb{R}^n} \varphi(x, P_\mu(x))\, d\mu(x) \qquad \forall \varphi \in \mathcal{C}_0(\mathbb{R}^n \times \mathbb{R}^{n^2}) \ .$$

Let us make some remarks on this key theorem. First, it can be interpreted as a non-oscillation result for the tangent spaces. Indeed, when $P_{\mu_h}\mu_h \rightharpoonup P_\mu\mu$, the sequence $\{\nu_h\}$ of the measures defined on $\mathcal{C}_0(\mathbb{R}^n \times \mathbb{R}^{n^2})$ by the integral at the left hand side of (1.22), turns out to be bounded. Then, by the theory of Young measures [69], any of the limit points of $\{\nu_h\}$ can be disintegrated as $\mu \otimes (\nu_x)$, being (ν_x) a family of probabilities on $\mathbb{R}^{n^2}$. So (1.22) means that the family (ν_x) does not depend on the choice of the subsequence, and moreover it is a family of Dirac masses at $P_\mu(x)$.

We also stress that the class of the test functions φ admissible in (1.22) can be enlarged to all the continuous functions φ on $\mathbb{R}^n \times \mathbb{R}^{n^2}$ satisfying a growth condition of the kind $|\varphi(x,z)| \leq C(1+|z|^p)$ for some real exponent p (see Proposition 3.4 of Chapter V).

The proof of (1.22) is based on an extremality argument from Convex Analysis: the support of each measure ν_x turns out to be contained in the closed convex subset K of $\mathbb{R}^{n^2}$ given by the matrices whose spectrum is contained in the interval $[0,1]$, while the barycenter of ν_x coincides with $P_\mu(x)$, which is an extremal point of K (because its spectrum is contained in $\{0,1\}$). Consequently, ν_x can be nothing else then a Dirac mass concentrated at $P_\mu(x)$.

To use the power of Theorem 1.39, we need to combine it with the following weak-strong convergence result. Given $\mu_h \rightharpoonup \mu$, $f_h \in (L^p_{\mu_h})^d$, $f \in (L^p_\mu)^d$, we shall say that $\{f_h\}$ weakly or strongly converges to f, and we shall write $f_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} f$ or $f_h \stackrel{s(\mu_h,\mu)}{\longrightarrow} f$, respectively if $f_h\mu_h \rightharpoonup f\mu$, or if (1.22) holds (with $P_{\mu_h}, P_\mu, \mathbb{R}^{n^2}$ replaced by $f_h, f, \mathbb{R}^d$).

Theorem 1.40. *Let $f_h \in L^p_{\mu_h}$, $g_h \in L^q_{\mu_h}$, with $\frac{1}{p}+\frac{1}{q}<1$. If $f_h \stackrel{s(\mu_h,\mu)}{\longrightarrow} f$ and $g_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} g$, then $f_h g_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} f\,g$, provided $\max\{\|f_h\|_{p,\mu_h}, \|g_h\|_{q,\mu_h}\} \leq C$ for some positive constant C independent of h. The same holds if (f_h,g_h) takes values in $\mathbb{R}^{n^2}\times\mathbb{R}^n$ or in $\mathbb{R}^{n^2}\times\mathbb{R}$ (intending the product f_hg_h as the matrix f_h applied to the vector g_h or multiplied by the scalar g_h).*

We now claim that Theorems 1.39 and 1.40 can be used for studying the Γ-liminf of F_h. Indeed, assume without loss of generality that $u_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} u$ with $F_h(u_h)$ bounded. In order to have (1.22), suppose that μ_h belong to $\mathcal{M}_k$ and satisfy the assumptions (1.9) of Theorem 1.25, so that Theorem 1.39 applies. Thus, by virtue of Theorem 1.40 (taking $(f_h,g_h)=(P_{\mu_h},u_h)$), it holds $u_hP_{\mu_h} \stackrel{w(\mu_h,\mu)}{\longrightarrow} uP_\mu$, which yields

$$\operatorname{div}(u_hP_{\mu_h}\mu_h) \rightharpoonup \operatorname{div}(uP_\mu\mu) \qquad \text{in} \ \ (\mathcal{D}'(\Omega))^n \ . \tag{1.23}$$

Let us focus attention on the vector distribution $\operatorname{div}(uP_\mu\mu)$. If u is smooth, it is easy to check, integrating by parts, that

$$\operatorname{div}(uP_\mu\mu) = [\nabla_\mu u + u\,h(\mu)]\,\mu + u\,\partial\mu \ . \tag{1.24}$$

This identity inspires the following distributional definition of μ-tangential gradient for a function $u \in L^p_\mu(\Omega)$. It makes sense whenever $\mu \in \mathcal{M}_{\mathrm{BC}}$ with $h(\mu) \in (L^\infty_\mu)^n$. Letting $\Omega_\mu := \Omega \setminus \operatorname{spt}(\partial\mu)$, we set

$$D_\mu u := \operatorname{div}(uP_\mu\mu) \llcorner \Omega_\mu - u\,h(\mu) \ .$$

Consequently, under the regularity assumption that $\mathrm{spt}(\mu)$ is μ-negligible, we define the Sobolev space $W^{1,p}_\mu(\Omega)$ as the class of all functions u in $L^p_\mu(\Omega)$ such that $D_\mu u$ belongs to $(L^p_\mu(\Omega))^n$. Endowed with the natural norm, $W^{1,p}_\mu(\Omega)$ turns out to be a Banach space, of which $H^{1,p}_\mu(\Omega)$ is a closed subspace: indeed, since (1.24) is satisfied by density on $H^{1,p}_\mu(\Omega)$, it holds $D_\mu u = \nabla_\mu u$ for any $u \in H^{1,p}_\mu(\Omega)$. The Dirichlet space $W^{1,p}_{\mu,0}(\Omega)$ can be obtained by considering the functions on Ω such that their extension to zero outside Ω belongs to $W^{1,p}_\mu(\mathbb{R}^n)$. Notice also that, when $\mu = \mathcal{L}^n$, $W^{1,p}_\mu(\Omega)$ and $H^{1,p}_\mu(\Omega)$ reduce respectively to the classical spaces $W^{1,p}(\Omega)$ and $H^{1,p}(\Omega)$. In view of the convergence result (1.23), and of the above notion of μ-Sobolev functions, we are interested in working on the subclass $\mathcal{A}_\lambda$ of $\mathcal{M}_{\mathrm{BC}}$ described by the following conditions, where λ is a prescribed element of $\mathcal{M}^n$ and c, C are fixed positive constants such that $\mathcal{A}_\lambda \neq \emptyset$:

$$\begin{aligned} &\dim T_\mu(x) = k \ \text{ for } \mu\text{-a.e. } x\ , \qquad \theta_k(\mu, x) \geq c \ \text{ for } \mu\text{-a.e. } x\ , \\ &\|h_\mu\|_{\infty,\mu} \leq C\ , \qquad \partial\mu = \lambda\ , \qquad P_\mu(x) h_\mu(x) = 0 \ \text{ for } \mu\text{-a.e. } x\ . \end{aligned}$$

Some comments about the measures in $\mathcal{A}_\lambda$ are collected in Remark 3.14 of Chapter V. What is important is that the class $\mathcal{A}_\lambda$ can be shown to be weakly closed in $\mathcal{M}^+$. Moreover, it is interesting to notice that some topological properties of the support of measures in $\mathcal{A}_\lambda$ are preserved under weak convergence. We make a short diversion recalling the following two results proved in Chapter V: the former is about the stability of connectedness, the latter is a stronger statement holding for one-dimensional structures.

Theorem 1.41. *Let $\{\mu_h\} \subset \mathcal{A}_\lambda$, $\mu_h \rightharpoonup \mu$, and assume that for every h the set $S_h := \mathrm{spt}(\mu_h)$ is contained into a fixed ball of $\mathbb{R}^n$. Then $\{S_h\}$ converges in the Hausdorff metric to the support of μ. In particular, if every S_h has a number of connected components smaller than a fixed integer N, the same holds for the support of μ.*

Theorem 1.42. *Let $\mu_h = \theta_h \mathcal{H}^1 \llcorner S_h$ be a sequence in $\mathcal{M}_{BC}$, with S_h compact and connected subsets of a fixed ball of $\mathbb{R}^n$, $\theta_h \geq c > 0$, and $|H(\mu_h)| \leq C < +\infty$. If $\mu_h \rightharpoonup \mu$, then μ is of the form $\mu = \theta \mathcal{H}^1 \llcorner S$, where S is compact and connected, and $\theta \geq c$.*

Turning back to the Γ- liminf of F_h, assume now that the measures μ_h belong to the class $\mathcal{A}_\lambda$. Starting from (1.23), one can prove that $u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u$ with $\{F_h(u_h)\}$ bounded entails

$$u \in W^{1,p}_\mu(\Omega)\,; \qquad \nabla u_h \overset{w(\mu_h,\mu)}{\longrightarrow} D_\mu u + \xi\ , \quad \text{with } \ \xi(x) \in T^p_\mu(x)^\perp \ \text{ for } \mu\text{-a.e. } x\ .$$

We arrive thus at the following result.

Theorem 1.43. *If $\{\mu_h\} \subset \mathcal{A}_\lambda$, $\mu_h \rightharpoonup \mu$, it holds*

$$\Gamma\text{-}\liminf_{h\to+\infty} F_h(u) \geq \widetilde{F}(u) \qquad \forall u \in L^p_\mu(\Omega)\ ,$$

where

$$\widetilde{F}(u) := \begin{cases} \displaystyle\int_\Omega f_\mu(x, D_\mu u)\, d\mu & \textit{if } u \in W^{1,p}_\mu(\Omega)\ , \\ +\infty & \textit{otherwise}\ , \end{cases}$$

being f_μ *defined by (1.16).*

In the variant when the domain of F_h is $\mathcal{D}(\Omega)$, Theorems 1.38 and 1.43 still hold replacing $H^{1,p}_\mu(\Omega)$ and $W^{1,p}_\mu(\Omega)$ respectively by $H^{1,p}_{\mu,0}(\Omega)$ and $W^{1,p}_{\mu,0}(\Omega)$.

The main point is now to compare the upper and lower bounds found in Theorems 1.38 and 1.43. It is clear that a gap phenomenon may occur between F and $\widetilde{F}$ when the equality $H^{1,p}_\mu(\Omega) = W^{1,p}_\mu(\Omega)$ is not satisfied. So, which is between F and $\widetilde{F}$ the best candidate to be the Γ-limit of $\{F_h\}$? Of course, if the sequence $\{\mu_h\}$ is identically equal to μ, one gets F as a Γ-limit. On the other hand, we can exhibit some examples of measures μ such that "$H \neq W$", and, suitably choosing the sequence $\{\mu_h\}$, the energies F_h Γ-converge to $\widetilde{F}$. For instance, if we take (in $\mathbb{R}^2$) $\mu = \mathcal{L}^2$ and Ω equal to the unit ball minus a radius (*cf.* Figure 4.1a of Chapter V), or $\mu = \mathcal{L}^2 \llcorner E$ with E as in Figure 4.2a of Chapter V and $\Omega = \mathbb{R}^2$, we recover two well-known cases where H and W are different form each other. Choosing the approximating sequences described respectively in Figures 4.1b and 4.2b, we get $F_h \xrightarrow{\Gamma} \widetilde{F}$. Nevertheless, one may think that this is due to the fact that the boundary of the approximating measures varies, but this is not the reason. Indeed, consider the case of a one-dimensional measure concentrated over a curve S having a double point, or weighted by a double density on a portion of its support (*cf.* respectively Figures 4.3a and 4.4a of Chapter V). In both cases, one has $H \neq W$, and, choosing the approximating measures drawn in Figures 4.3b and 4.4b (in codimension strictly higher than 1 in the former case), one finds the energy $\widetilde{F}$ as a Γ-limit. The upshot is that a general Γ-convergence result can be stated if and only if the limit measure μ satisfies the equality $H = W$.

We conclude by reconsidering the initial perspective of applying our approach in order to obtain the existence of a solution for optimization problems with manifolds as controls. The weak formulation provided by the use of measures consists in taking, as a class of admissible controls, all the measures μ in $\mathcal{A}_\lambda$, where λ is some prescribed element of $\mathcal{M}^n$. By Theorem 1.41, we may impose in addition that the support of any competitor μ is a connected portion of a fixed compact set K in $\mathbb{R}^n$. As a cost, we take for instance a functional of the kind

$$J(\mu) := \int_{\mathbb{R}^n} j(x, u_\mu)\, d\mu\ ,$$

where j is continuous in x, convex in z, with a quadratic growth, and such that $j(x,z) \geq 1$ on $\mathbb{R}^n \times \mathbb{R}^n$. The function u_μ is supposed to satisfy a state equation on the support of μ, which can be formulated in a distributional sense either as in (1.25) or as in (1.26):

$$-\operatorname{div}\big((\nabla_\mu u_\mu)\mu\big) + u_\mu = g\mu \quad \text{in } \mathcal{D}'\ , \qquad u \in H^{1,2}_\mu\ , \tag{1.25}$$

$$-\operatorname{div}\big((D_\mu u_\mu)\mu\big) + u_\mu = g\mu \quad \text{in } \mathcal{D}'\ , \qquad u \in W^{1,2}_\mu\ , \tag{1.26}$$

where the source term g is a prescribed function in $\mathcal{C}_0(\mathbb{R}^n)$.
By the closedness of the class $\mathcal{A}_\lambda$ and the coercivity condition on the integrand j, the key step to obtain the existence of a minimizer for J via the direct method is to establish the lower semicontinuity of J on $\mathcal{A}_\lambda$; as J is of course lower semicontinuous with respect to the convergence $u_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} u$, this is equivalent to study the weak continuity of the map $\mu \mapsto u_\mu \mu$. To this aim, notice that the state equations (1.25) and (1.26) can be rewritten in the form $u \in \operatorname{argmin} G(\mu, \cdot)$ (they are in fact the Euler-Lagrange equations of an integral functional G in $d\mu$ whose principal part is the energy on $H^{1,2}_\mu$ and on $W^{1,2}_\mu$, respectively). Hence, the weak continuity of the map $\mu \mapsto u_\mu \mu$ is directly concerned with the convergence of the minima of $G(\mu_h, \cdot)$ to the minima of $G(\mu, \cdot)$. By the main property of Γ-convergence, this is achieved if $G(\mu_h, \cdot) \xrightarrow{\Gamma} G(\mu, \cdot)$. Here is the link with our results: the choice of adopting (1.25) or (1.26) depends now on the problem we deal with, as it implies a different strategy.
There are in fact some interesting problems for one dimensional structures where we think possible to prove *a priori* that the optimal measure μ must satisfy the equality $H = W$. This is for instance the case where λ is a finite sum of oriented Dirac masses (see Figure 4.6 of Chapter V): indeed, the cost seems to decrease if all the singularities of μ are of the kind "triple junction" (see Figure 4.7 of Chapter V). Thus, our Γ-convergence result could be useful to prove the existence of optimal truss-like structures.
On the other hand, this approach does not suit problems involving higher-dimensional measures, for instance when $k = 2$ and λ is equal to $\gamma \mathcal{H}^1 \llcorner \Gamma$, being Γ a smooth closed curve, and γ a prescribed vector field on Γ (see Figure 2 below). We suggest in this case to adopt (1.26), which corresponds to consider the energies $\widetilde{F}_h$ defined by enlarging the domain of F_h to the weak spaces $W^{1,p}_{\mu_h}(\Omega)$. Indeed we conjecture that, when $\{\mu_h\} \subset \mathcal{A}_\lambda$, $\{\widetilde{F}_h\}$ Γ-converges to $\widetilde{F}$. In spite of the fact that we are by now able to prove only the Γ- liminf inequality, we stress that the conjecture is fulfilled in all the pathological examples of Chapter V, Section 4, in which the boundary is fixed.

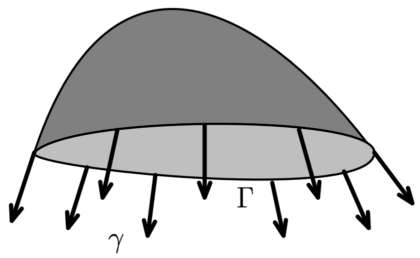

Figure 2

Compendium of Chapter VI

In Chapter VI, we are again concerned with the study of the asymptotic behaviour of a sequence of variational integrals. This time they are of the form

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right)\, d\mu_\varepsilon\ , \qquad u \in \mathcal{C}^1_0(\Omega)\ .$$

Here f, in addition to the usual assumptions (1.13), is supposed to be periodic in the first variable, while μ_ε are the ε-scalings of a fixed periodic measure $\mu \in \mathcal{M}^+$, defined by $\mu_\varepsilon(B) := \varepsilon^n \mu\big(\frac{B}{\varepsilon}\big)$. When the small parameter ε tends to zero, we recover the classical setting of homogenization: according to the Hausdorff dimension of the set where μ is concentrated, this unifying approach encloses both the widely studied cases of perforated domains [28, 55], and of thin reticulated elastic structures [10, 32].

To individuate the homogenized density of F_ε we do not use the classical technique of the operators' extension, which strongly depends on the shape of the domain under consideration, but we rather apply a generalization of the two-scale method proposed in [56, 2, 3]. The key argument is a structure result for all the possible two-scale limits reached by the sequences $(u_\varepsilon, \nabla u_\varepsilon)$ with $\sup_\varepsilon \int_\Omega |u_\varepsilon|^p + |\nabla u_\varepsilon|^p\, d\mu_\varepsilon < +\infty$. To prove such theorem, we need to adapt the μ-tangential calculus developed in the previous chapters to the periodic case, and to postulate suitable connectedness assumptions on μ.

Let us explain in the main lines the new concept of two-scale convergence with respect to measures. We fix the periodicity cell of μ to be the unit cube Y of $\mathbb{R}^n$, and we make the (not restrictive) simplifications $\mu(\partial Y) = 0$ and $\mu(Y) = 1$; in particular, $\{\mu_\varepsilon\}$ weakly converges to the Lebesgue measure $\mathcal{L}^n$ as $\varepsilon \to 0$. For brevity, we also adopt the notation $\mathbf{T} := \mathbb{R}^n/\mathbf{Z}^n$ and $m := (\mathcal{L}^n \llcorner \Omega) \otimes (\mu \llcorner Y)$. Finally, we assume from now on the condition $p > 1$.

Definition 1.44. *Let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$ and $u_0 \in L^p_m(\Omega \times \mathbf{T})$. We say that $\{u_\varepsilon\}$ two-scale converges to u_0 (with respect to μ and as $\varepsilon \to 0$), and we write $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0$, if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi\big(x, \frac{x}{\varepsilon}\big)\, d\mu_\varepsilon(x) = \int_{\Omega\times Y} u_0(x,y)\varphi(x,y)\, dm(x,y) \qquad \forall \varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))\ .$$

If in addition the $L^p_{\mu_\varepsilon}(\Omega)$-norms of u_ε converge to the $L^p_m(\Omega\times Y)$-norm of u_0, we say that $\{u_\varepsilon\}$ strongly two-scale converges to u_0, and we write $u_\varepsilon \twoheadrightarrow\!\!\!\!\twoheadrightarrow u_0$.

Similarly as in the well-known case when $\mu_\varepsilon = \mathcal{L}^n$ for any ε, the two-scale convergence gives more information on the sequence $\{u_\varepsilon \mu_\varepsilon\}$ than the pure weak convergence. Indeed, when $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0$, $\{u_\varepsilon\mu_\varepsilon\}$ weakly converges to $u\mathcal{L}^n$, where $u(x) := \int_Y u_0(x,y)\, d\mu(y)$, but in addition the oscillations of $u_\varepsilon\mu_\varepsilon$ at the scale ε are captured. When this completely describes the behaviour of $u_\varepsilon\mu_\varepsilon$, *i.e.* when all the oscillations of $u_\varepsilon\mu_\varepsilon$ are in resonance with those of the test functions $\varphi(x, \frac{x}{\varepsilon})$, we

say that the two-scale convergence is strong. This happens for instance if $u_\varepsilon = u_0(x, \frac{x}{\varepsilon})$, with $u_0 \in \mathcal{C}(\overline{\Omega}; L^p_\mu(\mathbf{T}))$. We also stress that, when $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0$, the only condition which must be checked in order to verify whether $u_\varepsilon \twoheadrightarrow u_0$, is $\limsup_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon \le \int_{\Omega\times Y} |u_0|^p \, dm$; indeed, the converse inequality for the liminf is always satisfied by a lower-semicontinuity property of the two-scale convergence (see Proposition 2.5 of Chapter VI).
In the following, we shall use the lemmas below about two-scale convergence: the former is a compactness lemma, the latter will be used to prove a corrector-type result.

Lemma 1.45. *Let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$ such that $\sup_\varepsilon \int_\Omega |u_\varepsilon|^p < +\infty$. Then we can extract a subsequence two-scale converging to some element $u_0 \in L^p_m(\Omega \times Y)$.*

Lemma 1.46. *Let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$, $v_\varepsilon \in L^{p'}_{\mu_\varepsilon}(\Omega)$ such that $u_\varepsilon \twoheadrightarrow u_0$, $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v_0$, and $\int_\Omega |v_\varepsilon|^{p'} \, d\mu_\varepsilon \le M$, with $u_0 \in L^p_m(\Omega\times Y)$, $v_0 \in L^{p'}_m(\Omega \times Y)$, and $M > 0$ independent of ε. Then*

(i) $u_\varepsilon v_\varepsilon \mu_\varepsilon \rightharpoonup u_0 v_0 m$;

(ii) $\displaystyle\lim_{\varepsilon\to 0} \int_\Omega \left|u_\varepsilon - u_0\left(x, \frac{x}{\varepsilon}\right)\right|^p d\mu_\varepsilon = 0$ *if* $u_0 \in \mathcal{C}(\Omega \times \mathbf{T})$.

We now summarize the essential features about the periodic μ-Sobolev spaces $H^{1,p}_\mu(\mathbf{T})$. They are defined in the following natural way. Consider the linear operator $A_\sharp : D(A_\sharp) \subset L^p_\mu(\mathbf{T}) \to (L^p_\mu(\mathbf{T}))^n$ given by $D(A_\sharp) = \mathcal{C}^\infty(\mathbf{T})$, $A_\sharp\psi = \nabla_\mu\psi$. The space $H^{1,p}_\mu(\mathbf{T})$ is by definition the domain of the unique closed extension $\overline{A}_\sharp$ of $A_\sharp$, and we set as usual $\nabla_\mu u := \overline{A}_\sharp u$ for any $u \in H^{1,p}_\mu(\mathbf{T})$. However we stress that, due to the role of periodicity in the integration by parts, the proof of the closability of $A_\sharp$ is in this case quite delicate, and requires some preliminary lemmas here omitted (see Lemmas 3.2, 3.3, and 3.4 of Chapter VI).
On the other hand, it is rather easy to adapt the relaxation result of [18] to the periodic case. More precisely, let F be defined by (1.14), replacing $\mathcal{D}$ by $\mathcal{C}^\infty(\mathbf{T})$; then (1.15) holds, replacing $H^{1,p}_\mu$ by $H^{1,p}_\mu(\mathbf{T})$.
As a last preliminary, we give below a list of some different notions of p-connectedness for μ, in a weak or in a strong sense, on the torus or on the whole $\mathbb{R}^n$. We set $H^{1,p}_{\mu,\mathrm{loc}} := \{u \in L^p_{\mu,\mathrm{loc}} : u\psi \in H^{1,p}_\mu \ \forall \psi \in \mathcal{D}\}$, and we refer to Section 4 of Chapter VI for a brief discussion about the following four conditions $(H1)$-$(H4)$.

- μ is *weakly p-connected on* $\mathbf{T}$ if:

$$(H1) \qquad u \in H^{1,p}_\mu(\mathbf{T}), \quad \nabla_\mu u = 0 \ \mu\text{-a.e.} \quad \Rightarrow \quad \exists\, c \in \mathbb{R} \ : \ u = c \ \mu\text{-a.e.};$$

- μ is *weakly p-connected on* $\mathbb{R}^n$ if:

$$(H2) \qquad u \in H^{1,p}_{\mu,loc}, \quad \nabla_\mu u = 0 \ \mu\text{-a.e.} \quad \Rightarrow \quad \exists\, c \in \mathbb{R} \ : \ u = c \ \mu\text{-a.e.};$$

- μ is *strongly p-connected on* $\mathbf{T}$ if:

$$(H3) \qquad \exists\, C > 0 \ : \ \int_Y |u|^p \, d\mu \le C \int_Y |\nabla_\mu u|^p \, d\mu \quad \forall u \in H^{1,p}_\mu(\mathbf{T}) \text{ with } \int_Y u \, d\mu = 0 \, ;$$

- μ is *strongly p-connected on* $\mathbb{R}^n$ if:

$$(H4) \qquad \exists\, C > 0 : \int_{kY} |u|^p \, d\mu \le kC \int_{kY} |\nabla_\mu u|^p \, d\mu \quad \forall k \in \mathbb{N}\, ,\ \forall u \in H^{1,p}_{\mu,loc} \text{ with } \int_{kY} u \, d\mu = 0 \, .$$

All the assumptions $(H1)$-$(H4)$ are satisfied for instance if $\mu \llcorner Y$ is the measure $\mathcal{H}^1$ over one of the structures represented in Figures 4.2a, 4.2b, and 4.2c of Chapter VI. Notice that the condition $(H4)$ implies all the others, as $H^{1,p}_\mu(\mathbf{T})$ is by its definition a closed subspace of $H^{1,p}_{\mu,\mathrm{loc}}$. In particular, $(H4)$ entails that a Poincaré constant independent of ε rules as well for all the measures μ_ε, and this can be used to derive the following compactness lemma.

Lemma 1.47. *If μ satisfies $(H4)$, and $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega)$ is such that $F_\varepsilon(u_\varepsilon) \le M$, then $\sup_\varepsilon \int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon < +\infty$; moreover, $\{u_\varepsilon \mu_\varepsilon\}$ is relatively compact in $\mathcal{M}$ and any of its limit points is absolutely continuous with respect to μ, with a density $u \in W^{1,p}_0(\Omega)$.*

On the other hand, the less restrictive conditions $(H1)$-$(H3)$ are the exact assumptions needed in the next crucial result for the possible two-scale limits of a sequence of gradients.

Theorem 1.48. *Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $\mathcal{C}^1$, and let $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega)$ satisfy $\int_\Omega \big(|u_\varepsilon|^p + |\nabla u_\varepsilon|^p\big) \, d\mu_\varepsilon \le M$; possibly passing to a subsequence, assume that $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0 \in L^p_m(\Omega \times Y)$ and $\nabla u_\varepsilon \rightharpoonup\!\!\rightharpoonup \chi \in \big(L^p_m(\Omega \times Y)\big)^n$. Then:*

(i) if μ satisfies $(H1)$, $u_0(x,y) = u(x)$ (i.e. u_0 is independent of y), where the function u belongs to $W^{1,p}_0(\Omega)$ provided μ satisfies also $(H2)$ and $(H3)$;

(ii) under assumptions $(H2)$ and $(H3)$ on μ, there exists $u_1 \in L^p\big(\Omega, W^{1,p}_\mu(\mathbf{T})\big)$ such that $\chi(x,y) = \nabla u(x) + \nabla_{\mu,y} u_1(x,y) + \xi(y)$, with $\xi(y) \in \big[T_\mu(y)\big]^\perp$ μ-a.e. on Y; in addition, $\nabla_{\mu_\varepsilon} u_\varepsilon \rightharpoonup\!\!\rightharpoonup \nabla u(x) + \nabla_{\mu,y} u_1(x,y)$.

We briefly explain which are the intermediate results that allow to prove Theorem 1.48.
The independence on y of the two-scale limit u_0 under the assumption $(H1)$ on μ comes from the fact that the constant functions are the orthogonal space of $V = \{\mathrm{div}(\Phi\mu) \ : \ \Phi \in X^{p'}_\mu(\mathbf{T})\}$ in $L^{p'}_\mu(\mathbf{T})$, being $X^{p'}_\mu(\mathbf{T}) := \{\Phi \in (L^{p'}_\mu(\mathbf{T}))^n \ : \ \mathrm{div}(\Phi\mu) \in L^{p'}_{\mu,\mathrm{loc}}\}$ (see Lemma 5.2 of Chapter VI).
The belonging of u to $W^{1,p}_0(\Omega)$ under $(H2)$ and $(H3)$ is rather delicate. It is based on the coercivity of the integrand j^{hom}, which will be the density of the homogenized energy of F_ε. It is defined, for any $z \in \mathbb{R}^n$, by

$$(1.27) \qquad \begin{aligned} f^{\mathrm{hom}}(z) :&= \inf \Big\{ \int_Y f(y, z + \nabla u\,(y)) \, d\mu \ : \ u \in \mathcal{C}^\infty(\mathbf{T}) \Big\} \\ &= \inf \Big\{ \int_Y f_\mu(y, P_\mu(y)[z] + \nabla_\mu u\,(y)) \, d\mu \ : \ u \in H^{1,p}_\mu(\mathbf{T}) \Big\} \, , \end{aligned}$$

where the last equality follows from the above mentioned relaxation result on $H^{1,p}_\mu(\mathbf{T})$.
It turns out that $f^{\hom}$ is convex and has a growth of order p from above, but only imposing $(H2)$ and $(H3)$ the coercivity condition $f^{\hom}(z) \geq c|z|^p$ is gained (see Lemma 5.4 of Chapter VI). This implies in particular that, under $(H2)$ and $(H3)$, the Fenchel conjugate $(j^{\hom})^*$ is continuous on the whole $\mathbb{R}^n$, which entails $u \in W^{1,p}(\Omega)$. Then, the zero condition on the boundary, *i.e.* $u \in W^{1,p}_0(\Omega)$, follows from the regularity assumption on $\partial\Omega$.
Finally, part (ii) in the statement of Theorem 1.48 is obtained by observing that the orthogonal space in $L^{p'}_\mu(\mathbf{T})$ of $M := \{\sigma \in (L^{p'}_\mu(\mathbf{T}))^n \ : \ P_\mu\sigma \in X^{p'}_\mu(\mathbf{T}), \operatorname{div}(P_\mu\sigma\mu) = 0\}$ is given by $M^\perp = \{\nabla_\mu u \, : \, u \in H^{1,p}_\mu(\mathbf{T})\}$ (see Lemma 5.5 of Chapter VI).
Using Lemma 1.47 and Theorem 1.48, we can prove the following Γ-convergence result. We think of F_ε as extended on $\mathcal{M}$ by setting $F_\varepsilon(\nu) = \int_\Omega f(\frac{x}{\varepsilon}, \nabla u)\, d\mu_\varepsilon$ if $\nu = u\mu_\varepsilon$, and $+\infty$ otherwise.

Theorem 1.49. *Under the assumption $(H4)$ on μ, the sequence $\{F_\varepsilon\}$ Γ-converges on $\mathcal{M}$ (as ε tends to 0 and with respect to the weak star convergence), to the homogenized functional $F^{\hom}$ given by*

$$F^{\hom}(\nu) = \begin{cases} \displaystyle\int_\Omega f^{\hom}(\nabla u\,(x))\, dx & \textit{if } \nu = u\,\mathcal{L}^n \ , \ u \in W^{1,p}_0(\Omega) \\ +\infty & \textit{otherwise} \ , \end{cases}$$

where the integrand $f^{\hom}$ is defined via the unit-cell problem (1.27).

As a consequence of Theorem 1.49, we get information on the asymptotic behaviour as $\varepsilon \to 0$ of the minimizing sequences u_ε for

$$\inf \left\{ F_\varepsilon(u) - \int_\Omega f u \, d\mu_\varepsilon \ , \quad u \in \mathcal{C}^1_0(\Omega) \right\} \ ,$$

where f is a prescribed function continuous on $\overline{\Omega}$. Indeed we can say that, up to subsequences, $u_\varepsilon\mu_\varepsilon \rightharpoonup u\mathcal{L}^n$, where u solves the limit problem

$$\min \left\{ F^{\hom}(u) - \int_\Omega f u \, d\mu_\varepsilon \ , \quad u \in W^{1,p}_0(\Omega) \right\} \ .$$

In the special case when f is a quadratic form $f(y,z) = A(y)z \cdot z$, induced by a symmetric matrix $A(y)$ with coefficients in $L^\infty(\mathbf{T})$, we obtain

$$(u_\varepsilon, \nabla_{\mu_\varepsilon} u_\varepsilon) \ \rightharpoonup\rightharpoonup \ \Big(u(x) \, , \, \sum_i (e_i + \nabla_{\mu,y}\, \chi_i(y)) \frac{\partial u}{\partial x_i}(x) \Big) \ , \tag{1.28}$$

where $u \in W^{1,2}_0(\Omega)$ and $\chi_i \in H^{1,2}_\mu(\mathbf{T})$ are respectively the unique solutions to the homogenized problem (1.29) and to the n linear problems (1.30) on the unit cell (where $e_{i,\mu}(y) := P_\mu(y)e_i$ and $A^{\hom}_{i,j} := \int_Y \big(e_{i,\mu} + \nabla_{\mu,y}\,\chi_i\big) \cdot \big(e_{j,\mu} + \nabla_{\mu,y}\,\chi_j\big)\, d\mu(y)$):

$$- \operatorname{div}_x \big(A^{\hom}\nabla u(x)\big) = f(x) \ , \qquad u \in W^{1,2}_0(\Omega) \ , \tag{1.29}$$

$$- \operatorname{div}_y \Big[A(y) \Big(e_{i,\mu}(y) + \nabla_\mu\, \chi_i(y) \Big) \Big] = 0 \ , \qquad \chi_i \in H^{1,2}_\mu(\mathbf{T}) \ , \qquad i = 1, 2, \cdots, n \ . \tag{1.30}$$

Moreover, we can also prove a corrector-type result, more precisely, setting $u_1(x,y) := \sum_i \dfrac{\partial u}{\partial x_i}(x)\,\chi_i(y)$, it holds

$$\lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon(x) - u(x)|^2 \, d\mu_\varepsilon = 0 \tag{1.31}$$

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u - u_1(x, \frac{x}{\varepsilon})\|_{H^{1,2}_{\mu_\varepsilon}(\Omega)} = 0 \ . \tag{1.32}$$

Example 1.50. Theorem 1.49 applies for instance if μ is given by $(\mathcal{H}^1(G))^{-1}\mathcal{H}^1 \llcorner G$, being G one of the gridworks in $\mathbb{R}^2$ whose unit cell is represented by Figures 4.2a, 4.2b, and 4.2c of Chapter VI. If in addition we let f be a positive-definite, quadratic energy density of the type

$$f(z) := \frac{1}{2}\left(a_{11}z_1^2 + 2a_{12}z_1z_2 + a_{22}z_2^2\right) \ , \quad \text{with } \det(a_{ij}) > 0 \text{ and } a_{11}, a_{22} > 0 \ ,$$

then also (1.31)-(1.32) hold and further the calculation of f^{hom} can be done explicitly. For instance, in the case of the network in Figure 4.2b, we obtain

$$f^{\text{hom}}(z) = \frac{\det(a_{ij})}{4}\left(\frac{z_1^2}{a_{22}} + \frac{z_2^2}{a_{11}}\right) \ .$$

We conclude by pointing out the interest in treating also the case of non-connected measures μ. This comes out from the aim of studying a phenomenon which appears when dealing with the passage to the limit in different parameters. Actually, our framework includes in a natural way the case of thin reinforced structures: they are made of identical cells, periodically repeated, in which the material is distributes along bars of cross-section $\delta \ll \varepsilon$. Thus, one can try to apply Theorem 1.49 either to each of the measures μ_δ which describe the periodic structure of thickness δ (and then let δ tend to zero), or to the measure μ obtained as the weak limit of μ_δ as $\delta \to 0$.
An example of fibered structure in $\mathbb{R}^3$, corresponding to a disconnected measure μ, can be used to convince oneself that the order chosen to pass to the limit in the parameters ε and δ is crucial: indeed, it may lead to different homogenized energies, possibly containing a non-local term (*cf.* Section 7 of Chapter VI). We think that this is a general phenomenon due to the lack of connectedness, which makes fail Theorem 1.49, and requires a more general Γ-convergence result.

II. On some notions of tangent space to a measure*

1. Introduction

We study some different notions of tangent space to a measure μ on $\mathbb{R}^n$, with special attention to the one given by Bouchitté, Buttazzo and Seppecher in [18].
We compare this definition with others usually employed in Geometric Measure Theory, based on the fundamental idea of blow-up of the measure μ. One is the definition of approximate tangent space to a measure [64] which generalizes the notion of approximate tangent space to a rectifiable subset of $\mathbb{R}^n$; another identifies the tangent space to a measure with a set of "tangent measures" (see Preiss [59]).
We will denote respectively by T_μ, P_μ and $\mathrm{Tan}(\mu)$ the tangent spaces to a measure defined in [18], [64], [59]. While the last two are very useful to study geometric properties of a measure (in particular they allow rectifiability properties of μ to be deduced from the behaviour of μ on the balls of $\mathbb{R}^n$), the main interest of the definition proposed in [18] is in the applications to variational problems. Actually it suits very well to the study of functionals depending on measures, and therefore it can be quite useful in problems of calculus of variations and shape optimization.
An example of variational problem in which measures arise in a quite natural way can be found in [18], where the authors give a model for the elastic energy of low-dimensional structures. The traditional method to study these objects relies on Γ-convergence; more precisely, if one considers for instance a membrane S, the idea is to fatten S in order to get a sequence of three-dimensional sets S_ε approximating S as ε tends to zero. Each S_ε is seen as the region occupied by an elastic body, hence, for every ε, one has an energy functional

$$F_\varepsilon(u) = \int_{S_\varepsilon} f(x, \nabla u(x))\, dx \ ,$$

where $u : S_\varepsilon \to \mathbb{R}^3$ represents a deformation of S_ε and f is a bulk energy density. Then, taking a suitable Γ-limit F_0, one obtains the required energy (see for instance [58]).
The same result can be obtained by the new method proposed in [18], which is more intrinsic to the elastic structure. It consists in taking a suitable measure μ (for example, in the case of the membrane S, one evidently chooses $\mu = \mathcal{H}^2 \llcorner S$), and performing the relaxation of an integral functional of the type

$$F(u) = \int_{\mathbb{R}^n} f(x, \nabla u(x))\, d\mu(x) \ , \qquad u \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \ .$$

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* Extended version of the paper by I. Fragalà and C. Mantegazza appeared on the Proceedings of the Royal Society of Edinburgh (1999) [42].

In this way one obtains, as elastic energy of the structure associated to μ, another integral functional:

$$\overline{F}(u) = \int_{\mathbb{R}^n} f_\mu(x, D_\mu u(x))\, d\mu(x) \quad , \quad u \in W^{1,p}_\mu(\mathbb{R}^n, \mathbb{R}^n) \ .$$

Here the relaxation is made with respect to the weak topology of a Sobolev space $W^{1,p}_\mu$ associated to the measure μ, f_μ is an integrand related to f and $D_\mu u$ is the "tangential gradient of u with respect to μ", which can be defined just starting from the notion of tangent space to μ.
One of the advantages of this method is that it can be applied also to very irregular structures, as well as to objects of variable dimension.
Another field where the use of measures comes out to be even more natural is shape optimization. Indeed, it is well known that often, and in the most interesting cases, the solutions to shape optimization problems are not represented by sets. Then, it is quite natural to give a general formulation of such problems in which the unknown is represented by a measure. Recently, this approach has been developed in [19], where the following optimization problem was considered:

$$(\mathcal{P}) \qquad \max \left\{ \mathcal{E}(\mu) \ : \ \mu \text{ a non-negative measure with } \int_{\mathbb{R}^n} d\mu = m \right\} \ .$$

The energy functional $\mathcal{E}(\mu)$ is given by

$$\mathcal{E}(\mu) = \inf \left\{ E(u, \mu) \ : \ u \in \mathcal{D}(\mathbb{R}^n) \right\} \ ,$$

where the energy $E(u,\mu)$ is expressed, by means of the gradient of u and of a given signed measure f on $\mathbb{R}^n$ having zero average, as

$$E(u,\mu) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu - \langle f, u \rangle \ .$$

This problem can be interpreted as follows: f represents a given density of heat sources, u is a smooth distribution of temperature and one looks for a measure μ representing the optimal conductivity density. An existence result for $(\mathcal{P})$ has been proved in [19], together with a characterization of the solutions in terms of a suitable relaxed version of the Monge-Kantorovich equation.

An outline of the paper is as follows.
In Section 2 we recall the different definitions and state some comparison results between them. In particular we prove the inclusion $T_\mu \subseteq P_\mu$ (see Lemma 2.4), which in some cases may be strict, and a useful decomposition lemma for the measures of $\mathrm{Tan}(\mu)$ showing that all such measures are the product of a Hausdorff measure on T_μ with an arbitrary measure on the orthogonal complement of T_μ (see Lemma 2.6). Both these results are proved by means of blow-up techniques.
In Section 3 we consider some interesting classes of examples, some of them "regular", in which T_μ coincides with P_μ (see Examples 1-3), others "pathological", like self-similar sets for which

T_μ reduces to zero (see Examples 4 and 5). In this contest some relations with normal currents and rectifiable varifolds are proved.
In Section 4 we first prove an estimate from above for the dimension of T_μ, then we establish a rectifiability theorem based on the behaviour of tangent spaces T_μ. Our estimate for the dimension of T_μ says, roughly speaking, that it cannot be greater than the Hausdorff dimension of the support of μ, while the rectifiability theorem, which is deduced from a criterion of Preiss, also includes the case of a sum of Hausdorff measures of variable dimension, each one concentrated on a different rectifiable set. This is possible by a nice property of the tangent space T_μ which we want finally to point out: its dimension may depend on the point of the support of μ, differently from what happens for the approximate tangent space P_μ.

Throughout the paper, all the measures considered will be positive Radon measures on $\mathbb{R}^n$, unless otherwise stated.

2. Comparison results

Different notions of tangent space to a Radon measure have been proposed in the literature; we focus our attention mainly on the one which has been recently introduced by Bouchitté, Buttazzo and Seppecher in [18]. We slightly modify their definition by considering a larger set of admissible tangent fields: we consider the space of vector fields on $\mathbb{R}^n$ given by

$$X_\mu = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ |\Phi| \in L^1_{\mathrm{loc}}(\mu) \ , \mathrm{div}(\Phi\mu) \in \mathcal{M}\} \ ,$$

where the divergence operator is in the distributional sense and $\mathcal{M}$ denotes the set of Radon measures on $\mathbb{R}^n$. We meke the following

Definition 2.1. *For μ-a.e. $x \in \mathbb{R}^n$, the tangent space to μ is given by*

$$T_\mu(x) = \mu - \mathrm{ess}\bigcup\{\Phi(x) : \Phi \in X_\mu\} \ .$$

For the existence and properties of the μ-essential union, we refer to [68]. We only recall that, by definition, T_μ is the only μ-measurable, closed valued multifunction on $\mathbb{R}^n$ such that

(i) $\Phi \in X_\mu \Longrightarrow \Phi(x) \in T_\mu(x)$ for μ-a.e. $x \in \mathbb{R}^n$;

(ii) for any other μ-measurable, closed valued multifunction Σ on $\mathbb{R}^n$ satisfying (i), *i.e.* $\Phi \in \Sigma \Longrightarrow \Phi(x) \in \Sigma$ for μ-a.e. $x \in \mathbb{R}^n$, we have $T_\mu(x) \subseteq \Sigma(x)$ for μ-a.e. $x \in \mathbb{R}^n$.

We also point out that, for every measure μ, the vector space $T_\mu(x)$ is well-defined up to a μ-negligible set, and its dimension as a linear subspace of $\mathbb{R}^n$ may depend on the point x.

We now aim to relate Definition 2.1 to the notions of tangent space to a measure given by Simon [64] and by Preiss [59], both based on the crucial idea of blow-up.
For any fixed $x^0 \in \mathbb{R}^n$ and any positive real number ρ, let us denote by $\mu_{x^0,\rho}$ the Borel measure on $\mathbb{R}^n$ defined by

$$\mu_{x^0,\rho}(B) = \mu(x^0 + \rho B) \qquad \forall B \in \mathcal{B}(\mathbb{R}^n) \ ,$$

where $\mathcal{B}(\mathbb{R}^n)$ is the class of Borel subsets of $\mathbb{R}^n$.

Definition 2.2. *Let $P_\mu = P_\mu(x^0)$ be a k-dimensional subspace of $\mathbb{R}^n$, with $k \leq n$. Then P_μ is said to be the approximate tangent space for μ at x if there exists a positive constant $\theta = \theta(x^0)$, which is called the multiplicity at x^0, such that, when ρ converges to zero*

$$\frac{1}{\rho^k}\mu_{x^0,\rho} \ \rightharpoonup \ \theta(x^0)\mathcal{H}^k \llcorner P_\mu(x^0) \tag{2.1}$$

in the vague topology on measures, or equivalently

$$\frac{1}{\rho^k}\int_{\mathbb{R}^n} \varphi\left(\frac{x-x^0}{\rho}\right)\, d\mu(x) \ \to \ \theta(x^0)\int_{P_\mu(x^0)} \varphi(x)\, d\mathcal{H}^k(x) \qquad \forall \varphi \in C^0_c(\mathbb{R}^n) \ .$$

For a general Radon measure μ, the existence μ-almost everywhere of a k-dimensional linear space $P_\mu = P_\mu(x^0)$ satisfying (2.1) is not guaranteed. Actually, if we assume that μ has a tangent space of constant dimension k for μ-a.e. $x \in \mathbb{R}^n$, it is reasonable to expect that μ must have in some sense a "k-dimensional behaviour". We have the following fundamental characterization of the measures admitting a k-dimensional approximate tangent space μ-almost everywhere (see [64], Theorem 11.8).

Theorem 2.3. *The measure μ has a k-dimensional approximate tangent space $P_\mu(x)$ at μ-a.e. $x \in \mathbb{R}^n$ if and only if $\mu = \theta\mathcal{H}^k \llcorner M$, where M is a countably k-rectifiable subset of $\mathbb{R}^n$ and θ is a non-negative $\mathcal{H}^k$-measurable function on $\mathbb{R}^n$.*

A more general rectifiability criterion based on tangent spaces T_μ will be proved in Section 4. For the moment we state a comparison result between definitions 2.1 and 2.2, which is proved analogously to Lemma 5.2 of [18].

Lemma 2.4. *Suppose that $P_\mu = P_\mu(x)$ associates to μ-a.e. point x the k-dimensional approximate tangent space $P_\mu(x)$ to μ at x. Then*

$$T_\mu(x) \subseteq P_\mu(x) \qquad \textit{for } \mu\textit{-a.e. } x \in \mathbb{R}^n \ . \tag{2.2}$$

Proof. For the minimality property ii) of T_μ, it is sufficient to prove that, for every tangent field Φ, we have $\Phi(x) \in P_\mu(x)$ for μ-a.e. $x \in \mathbb{R}^n$.
Let $\Phi \in X_\mu$, we observe that $P_\mu(x)^\perp$ is spanned by

$$\left\{ \int_{P_\mu(x)} \nabla\psi(y)\ d\mathcal{H}^k(y) \quad : \quad \psi \in \mathcal{D}(B) \right\} ,$$

where B is the unit ball of $\mathbb{R}^n$; indeed, the integral on $P_\mu(x)$ of the gradient of any test function ψ has no component on $P_\mu(x)$, since the orthogonal projection of $\nabla\psi$ on $P_\mu(x)$ can be integrated by parts.
Thus, we only need to prove that, for μ-a.e. $x \in \mathbb{R}^n$,

$$\Phi(x) \cdot \int_{P_\mu(x)} \nabla\psi(y)\ d\mathcal{H}^k(y) = 0 \qquad \forall \psi \in \mathcal{D}(B) .$$

We can suppose that x is a Lebesgue point for Φ with respect to μ and that both μ and the measure $m = |\operatorname{div}(\Phi\mu)|$ have finite k-dimensional density at x, these conditions being satisfied μ-almost everywhere (by Theorem 2.3, $\mu = \theta\mathcal{H}^k \llcorner M$). Let us set $\psi_\rho(y) = \psi\left(\frac{y-x}{\rho}\right)$ and $M = \max\limits_B |\psi|$. Then

$$\begin{aligned}
\theta(x)\left|\Phi(x) \cdot \int_{P_\mu(x)} \nabla\psi(y)\ d\mathcal{H}^k(y)\right| &= \lim_{\rho\to 0} \frac{1}{\rho^k}\left|\Phi(x) \cdot \int_{B_\rho(x)} \nabla\psi\left(\frac{y-x}{\rho}\right)\ d\mu(y)\right| \\
&= \lim_{\rho\to 0} \frac{1}{\rho^k}\left|\int_{B_\rho(x)} \Phi(y) \cdot \nabla\psi\left(\frac{y-x}{\rho}\right)\ d\mu(y)\right| \\
&= \lim_{\rho\to 0} \frac{\rho}{\rho^k}\left|\int_{B_\rho(x)} \Phi(y) \cdot \nabla\psi_\rho(y)\ d\mu(y)\right| \\
&= \lim_{\rho\to 0} \frac{\rho}{\rho^k}\left|\int_{B_\rho(x)} \psi_\rho(y)\ d(\operatorname{div}(\Phi\mu))(y)\right| \\
&\le \lim_{\rho\to 0} \frac{\rho}{\rho^k}\int_{B_\rho(x)} |\psi_\rho(y)|\ dm(y) \\
&\le \lim_{\rho\to 0} \rho M \frac{m(B_\rho(x))}{\rho^k}
\end{aligned}$$

and the last limit is zero because the k-dimensional density of m at x is finite. ∎

Next we want to show the strict relation between the linear space T_μ and the quite general concept of tangent measure for μ at a point x^0, studied by Preiss in [59].
He considers all the possible limits of sequences given by blowing up μ around x^0. More precisely, he calls a measure ν on $\mathbb{R}^n$ a tangent measure for μ at x^0 if ν is non-zero and if there exist two sequences of positive real numbers λ_i and ρ_i (with ρ_i converging to zero) such that

$$\lim_{i\to+\infty} \frac{\mu_{x^0,\rho_i}}{\lambda_i} = \nu$$

in the vague topology on measures.
Adopting this definition, the set of tangent measures for μ at x is shown to be non-empty only for μ-a.e. x (see [59], Theorem 2.5). Therefore, we prefer to take as normalization constants $\lambda_i = \mu(B_{\rho_i}(x^0))$, instead of arbitrary positive numbers. In this way the mass on the unit ball B of the measure $\frac{\mu_{x^0,\rho_i}}{\mu(B_{\rho_i}(x^0))}$ is equal to one for every i, assuring the existence of a weakly convergent subsequence.

Definition 2.5. *We say that a measure ν on the unit ball B of $\mathbb{R}^n$ belongs to the set $\mathrm{Tan}(\mu, x)$ of tangent measures for μ at the point x^0, if there exists a sequence of positive real numbers ρ_i converging to zero such that*

$$\lim_{i\to+\infty} \frac{\mu_{x^0,\rho_i}}{\mu(B_{\rho_i}(x^0))} = \nu \tag{2.3}$$

as measures on the unit ball B, or equivalently

$$\frac{1}{\mu(B_{\rho_i}(x^0))} \int_B \varphi\left(\frac{x-x^0}{\rho_i}\right) d\mu(x) \to \int_B \varphi(x)\, d\nu(x) \qquad \forall \varphi \in C^0_c(B) .$$

We can prove a structure result for measures belonging to $\mathrm{Tan}(\mu, x)$, showing that all such measures are of product type.

Lemma 2.6. *For μ-a.e. $x \in \mathbb{R}^n$, any measure $\nu \in \mathrm{Tan}(\mu, x)$ is of the form*

$$\left(\mathcal{H}^{k(x)}_{T_\mu(x)} \times \sigma\right) \llcorner B , \tag{2.4}$$

where $k(x)$ is the dimension of $T_\mu(x)$, $\mathcal{H}^{k(x)}_{T_\mu(x)}$ is the $k(x)$-dimensional Hausdorff measure on $T_\mu(x)$, and σ is a measure of locally finite mass on $T_\mu(x)^\perp$.

Proof. Let $L_\mu(x)$ be the set of vectors $v \in \mathbb{R}^n$ such that every measure $\nu \in \mathrm{Tan}(\mu, x)$ can be decomposed as a product $\left(\mathcal{H}^1_{\langle v\rangle} \times \sigma\right) \llcorner B$, where $\mathcal{H}^1_{\langle v\rangle}$ is the one-dimensional Hausdorff measure on the line $\langle v\rangle$ and σ is a measure on $v^\perp$.
The conclusion of the lemma is equivalent to $T_\mu(x) \subseteq L_\mu(x)$ for μ-a.e. x, since this inclusion implies that

$$\nu = \left(\mathcal{H}^1_{\langle v_1\rangle} \times \ldots \times \mathcal{H}^1_{\langle v_{k(x)}\rangle} \times \tau\right) \llcorner B ,$$

being v_i a basis of $T_\mu(x)$ and τ a Radon measure on $T_\mu(x)^\perp$.
So, by using the minimality property (ii) of T_μ, it is enough to show that every tangent field Φ satisfies μ-almost everywhere $\Phi(x) \in L_\mu(x)$.
Let $\Phi \in X_\mu$, and let us suppose that $D_{\Phi(x)}\nu = 0$ for every measure $\nu \in \mathrm{Tan}(\mu, x)$, where $D_{\Phi(x)}\nu$ is intended in the distributional sense, *i.e.*

$$\langle D_{\Phi(x)}\nu, \psi\rangle = \int_B \frac{\partial\psi(y)}{\partial\Phi(x)}\, d\nu(y) \qquad \forall \psi \in \mathcal{D}(B) .$$

Then we can prove by regularization that $\Phi(x) \in L_\mu(x)$. Indeed, if we take a sequence of smooth convolution kernels ρ_ε, the functions ν_ε defined by

$$\nu_\varepsilon(y) = \int_B \rho_\varepsilon(y-z)d\nu(z)$$

turn out to be of class C^∞ on $B_{1-\varepsilon}$, with zero derivative in the direction $\Phi(x)$. If we decompose $y \in \mathbb{R}^n$ as $y = (y', y'') \in \langle \Phi(x) \rangle \times \Phi(x)^\perp$, we have $\nu_\varepsilon = \nu_\varepsilon(y'')$, that is, for every $\varepsilon > 0$, ν_ε does not depend on y'. By using the inequality

$$\|\nu_\varepsilon\|_{L^1(B_{1-\varepsilon})} \leq 1 \qquad \forall \varepsilon > 0 ,$$

we get that $\nu_\varepsilon(y'')$ is bounded in $L^1_{\text{loc}}(B_{1-\varepsilon} \cap \Phi(x)^\perp)$ with respect to the measure $\mathcal{H}^{n-1}_{\Phi(x)^\perp}$; hence, possibly choosing a subsequence, we can suppose that $\nu_\varepsilon(y'')\mathcal{H}^{n-1}$ weakly converges to a measure σ of locally finite mass on $B \cap \Phi(x)^\perp$.
Therefore, we have

$$\nu_\varepsilon \mathcal{H}^n \rightharpoonup \left(\mathcal{H}^1_{\langle \Phi(x) \rangle} \times \sigma\right) \llcorner B ,$$

which yields $\nu = \left(\mathcal{H}^1_{\langle \Phi(x) \rangle} \times \sigma\right) \llcorner B$, that is $\Phi(x) \in L_\mu(x)$.
It only remains to prove that for $\Phi \in X_\mu$ and $\nu \in \mathrm{Tan}(\mu, x)$, at μ-a.e. x, we have $D_{\Phi(x)}\nu = 0$, that is,

$$\Phi(x) \cdot \int_B \nabla\psi(y)\ d\nu(y) = 0 \qquad \forall \psi \in \mathcal{D}(B) . \tag{2.5}$$

Repeating the argument used in the proof of Lemma 2.4 , we set $M = \max_B |\psi|$, $m = |\operatorname{div}(\Phi\mu)|$, and we suppose that x is a Lebesgue point for Φ with respect to μ as well as a finite density point for m with respect to μ. In this way we get the inequality

$$\frac{1}{\mu(B_\rho(x))} \left| \Phi(x) \cdot \int_{B_\rho(x)} \nabla\psi\left(\frac{y-x}{\rho}\right)\ d\mu(y) \right| \leq \rho M \frac{m(B_\rho(x))}{\mu(B_\rho(x))} .$$

Finally, when ρ converges to zero, we take into account (2.3) and we get (2.5). ■

3. Examples

Example 1 (*Lipschitz manifolds*).

We improve a result of [18] by showing that if μ is equal to $\mathcal{H}^k$ on a k-dimensional manifold M, then T_μ coincides with the classical tangent space, even if M is Lipschitz rather than of class C^2. We denote by $T_M(x)$ the classical tangent space to M at a point $x \in M$, which exists $\mathcal{H}^k$-almost everywhere on M by Rademacher's theorem.

Theorem 3.1. *Let $\mu = \mathcal{H}^k \llcorner M$, where $k \leq n$ and M is a k-dimensional Lipschitz manifold in $\mathbb{R}^n$. Then*

$$T_\mu(x) = T_M(x) \qquad \text{for } \mu\text{-a.e. } x \in M \ .$$

Proof. First let us recall that, for a measure μ as in the hypotheses, the approximate tangent space to μ given by Definition 2.2 exists μ-almost everywhere and coincides with T_M. So, by Lemma 2.4, we have immediately

$$T_\mu(x) \subseteq T_M(x) \qquad \text{for } \mu\text{-a.e. } x \in M \ .$$

In order to prove the opposite inclusion we observe that, in general, T_μ is local on open sets of $\mathbb{R}^n$, in other words, if $A \subseteq \mathbb{R}^n$ is open and we have measures μ_1 and μ_2 with $\mu_1 \llcorner A = \mu_2 \llcorner A$, then $T_{\mu_1}(x) = T_{\mu_2}(x)$ μ-a.e. in A. Therefore we can suppose without loss of generality that M is the graph of a Lipschitz function u from a regular open set $\Omega \subseteq \mathbb{R}^k$ to $\mathbb{R}^{n-k}$. The Jacobian of the map $x \mapsto (x, u(x))$ will be indicated by $J(x)$, we notice that it is bounded from below by definition and from above by the Lipschitz condition satisfied by u.
Then, we will get the inclusion

$$T_M(x) \subseteq T_\mu(x) \qquad \text{for } \mu\text{-a.e. } x \in M$$

if we prove that, for any fixed $v \in \mathbb{R}^k$, the vector field

$$\Phi(x) = \frac{(v, \partial_v u(x))}{J(x)}$$

belongs to X_μ. We show that $\operatorname{div}(\Phi\mu)$ is a Radon measure. For any test function $\psi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{aligned} \int \Phi \cdot \nabla\psi \; d\mu &= \int_M \Phi \cdot \nabla\psi \; d\mathcal{H}^k \\ &= \int_\Omega (v, \partial_v u(x)) \cdot \nabla\psi(x, u(x)) \; dx \\ &= \int_\Omega \partial_v \left[\psi(x, u(x))\right] \; dx \\ &= \int_{\partial\Omega} \psi(x, u(x))\, v \cdot n \; d\mathcal{H}^{k-1} \end{aligned}$$

where in the last equality we have applied the divergence theorem (as usual n is the external unit normal vector to $\partial\Omega$).

Example 2 (*relations with currents*).

Here we want to discuss some relations between tangent spaces to a measure and normal currents in $\mathbb{R}^n$ [54]. Let us briefly introduce some standard notations.
By $\bigwedge_k \mathbb{R}^n$ and $\bigwedge^k \mathbb{R}^n$ we denote respectively the spaces of k-vectors and k-covectors in $\mathbb{R}^n$, and by $\langle\ ,\ \rangle$ the corresponding dual pairing; on these vector spaces is canonically defined a scalar product inherited by the usual one of $\mathbb{R}^n$. We indicate by $\mathcal{D}_k(\mathbb{R}^n)$ the space of smooth k-forms with compact support

$$\omega = \sum_{1\leq i_1<\ldots<i_k\leq n} \omega_{i_1\ldots i_k}\ dx^{i_1}\wedge\ldots\wedge dx^{i_k}\ ,$$

endowed with the usual locally convex topology of distributions. A k-dimensional current in $\mathbb{R}^n$ is an element of the dual space of $\mathcal{D}_k(\mathbb{R}^n)$, which will be denoted by $\mathcal{D}^k(\mathbb{R}^n)$.
We recall that a current T is said to be normal if both T and its boundary ∂T have locally finite mass (the boundary of a k-current T is the $(k-1)$-current ∂T determined by the relation $\partial T(\omega) = T(d\omega)$, see [64]). In particular, by the Riesz Representation Theorem, for every normal current $T \in \mathcal{D}^k(\mathbb{R}^n)$ there exist a Radon measure μ_T on $\mathbb{R}^n$ and a μ_T-measurable function $\tau : \mathbb{R}^n \to \bigwedge_k(\mathbb{R}^n)$, with $|\tau| = 1$ μ-a.e., such that

$$T(\omega) = \int_{\mathbb{R}^n} \langle\omega(x),\tau(x)\rangle\ d\mu_T(x) \qquad \forall\omega\in\mathcal{D}_k(\mathbb{R}^n)\ .$$

The measure μ_T is called the total variation of T, while the function τ is usually referred as the orientation of the current T.
We first show that there exists a natural one-to-one correspondence between the set of normal 1-currents in $\mathbb{R}^n$ and the set of pairs (μ,Φ), where μ is a Radon measure on $\mathbb{R}^n$ and Φ is a unit tangent vector field to μ.

Theorem 3.2. *If T is a normal 1-current with total variation μ_T and orientation τ, the vector field τ is tangent to μ_T, that is*

$$\tau \in X_{\mu_T}\ .$$

Conversely, for any given Radon measure μ on $\mathbb{R}^n$ which admits a tangent field $\Phi\in X_\mu$ with $|\Phi| = 1$ μ-a.e., we obtain a normal 1-current by setting

$$T(\omega) = \int_{\mathbb{R}^n} \langle\omega(x),\Phi(x)\rangle\ d\mu_T(x) \qquad \forall\omega\in\mathcal{D}_1(\mathbb{R}^n)\ . \tag{3.1}$$

Proof. In order to prove the first part of the statement, we have to show that for every compact subset $K\subset\mathbb{R}^n$ there exists a positive constant c_K such that, for every smooth test function ψ, with support contained in K,

$$\left|\int_{\mathbb{R}^n} \tau\cdot\nabla\psi\ d\mu_T\right| \leq c_K\|\psi\|_\infty\ .$$

This is a straightforward consequence of the assumption that ∂T has locally finite mass since we have

$$\int_{\mathbb{R}^n} \tau \cdot \nabla\psi \; d\mu_T = \int_{\mathbb{R}^n} \langle d\psi, \tau\rangle \; d\mu_T = T(d\psi) = \partial T(\psi) \; .$$

Conversely, the 1-current T defined by (3.1) is immediately seen to be normal. Indeed T has locally finite mass because μ is a Radon measure, and the same property holds also for the boundary of T since we have, as above,

$$\partial T(\psi) = \int_{\mathbb{R}^n} \Phi \cdot \nabla\psi \; d\mu \; ,$$

where now Φ belongs to X_μ by hypothesis. ∎

We now consider the case of k-dimensional currents. We say that a normal k-current is simple if its orientation τ takes values in the simple k-vectors of $\mathbb{R}^n$, that is, τ is of the form $v_1 \wedge \ldots \wedge v_k$. Then there exists a natural way to produce, starting from a normal and simple k-current T, a pair $(\mu, \{\Phi_1, \ldots, \Phi_{\binom{n}{k-1}}\})$, where μ is a Radon measure (which, as one expects, will be the total variation of T) and $\{\Phi_1, \ldots, \Phi_{\binom{n}{k-1}}\}$ is a set of tangent fields to μ.

We indicate by c the contraction operator from $\bigwedge^{k-1}(\mathbb{R}^n) \times \bigwedge_k(\mathbb{R}^n)$ into $\mathbb{R}^n$ defined by

$$c(\omega, v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \omega, v_1 \wedge \ldots \wedge \widehat{v}_i \wedge \ldots \wedge v_k \rangle v_i \; .$$

Thus if we take a smooth $(k-1)$-form ω and k vector fields v_i on $\mathbb{R}^n$, $c(\omega, v_1 \wedge \ldots \wedge v_k)$ gives a vector field on $\mathbb{R}^n$. We denote by $dx^{i_1 \cdots i_{k-1}}$ the $(k-1)$-form $dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$.

Theorem 3.3. *If T is a normal and simple k-current with total variation μ_T and orientation $\tau_1 \wedge \ldots \wedge \tau_k$, we have*

$$c(dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k) \in X_{\mu_T}$$

for every multi-index $(i_1, \ldots, i_{k-1})$ with $1 \le i_1 < \ldots < i_{k-1} \le n$.

Proof. Let ψ be a test function in $\mathcal{D}(\mathbb{R}^n)$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} c(dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k) \cdot \nabla\psi \; d\mu_T &= \int_{\mathbb{R}^n} \langle d\psi, c(dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k)\rangle \; d\mu_T \\
&= \int_{\mathbb{R}^n} \langle d\psi \wedge dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k\rangle \; d\mu_T \\
&= \int_{\mathbb{R}^n} \langle d(\psi dx^{i_1 \cdots i_{k-1}}), \tau_1 \wedge \ldots \wedge \tau_k\rangle \; d\mu_T \\
&= T(d(\psi dx^{i_1 \cdots i_{k-1}})) \\
&= \partial T(\psi dx^{i_1 \cdots i_{k-1}}) \; .
\end{aligned}$$

Then, since T has locally finite mass, for every compact subset K of $\mathbb{R}^n$ there exists a constant c_K such that, if the support of ψ is contained in K, we have

$$\left|\int_{\mathbb{R}^n} c(dx^{i_1\dots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k) \cdot \nabla\psi \; d\mu_T\right| \leq c_K \|\psi\|_\infty \;,$$

which means that the field $c(dx^{i_1\dots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k)$ belongs to X_{μ_T}. ■

Remark 3.4. For $k > 1$ it does not seem possible to repeat the same argument as the case $k = 1$ to produce an orientation for a normal k-current, starting from a set of k linearly independent vector fields $\Phi_1, \ldots, \Phi_k$, tangent to a measure μ. Indeed, in general the boundary of the k-current

$$T(\omega) = \int_{\mathbb{R}^n} \langle \omega, \Phi_1 \wedge \ldots \wedge \Phi_k \rangle \; d\mu_T \qquad \forall \omega \in \mathcal{D}_k(\mathbb{R}^n)$$

may fail to have locally finite mass. Actually, if we take a $(k-1)$-test form of the type $\omega = \psi dx^{i_1\dots i_{k-1}}$, with $\psi \in \mathcal{D}(\mathbb{R}^n)$, the same computations made in the proof of Theorem 3.3 lead to

$$\partial T(\omega) = \int_{\mathbb{R}^n} \nabla\psi \cdot \sum_{i=1}^{k} (-1)^{i-1} \langle dx^{i_1,\dots,i_{k-1}}, \Phi_1 \wedge \ldots \wedge \widehat{\Phi}_i \wedge \ldots \wedge \Phi_k \rangle \Phi_i \; d\mu \;.$$

Therefore, only if we suppose that each function

$$\varphi_i = (-1)^{i-1} \langle dx^{i_1,\dots,i_{k-1}}, \Phi_1 \wedge \ldots \wedge \widehat{\Phi}_i \wedge \ldots \wedge \Phi_k \rangle$$

is smooth (at least of class C^1), then we can write

$$\partial T(\omega) = \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left[\nabla(\psi\varphi_i) \cdot \Phi - \psi \nabla\varphi_i \cdot \Phi_i\right] \; d\mu$$

and deduce, from the choice of the fields Φ_i, that ∂T has locally finite mass.

Example 3 (*rectifiable k-varifolds*).

For an appropriate class of rectifiable k-varifolds V (see [64]) the approximate tangent space to the weight measure μ_V coincides with T_{μ_V}.

Theorem 3.5. *Let $\mu_V = \theta \mathcal{H}^k \llcorner M$ be the weight measure of a rectifiable k-varifold $V = V(M, \theta)$, and suppose that the first variation δV is a Radon measure. Then*

$$T_{\mu_V} = P_{\mu_V} \qquad \mu_V\text{-a.e.}$$

Proof. By (2.2) we only need to prove the inclusion $P_{\mu_V} \subseteq T_{\mu_V}$ for μ_V-a.e. $x \in M$. If e_i are the coordinate fields, it is sufficient to show that $e_i^M \in X_{\mu_V}$ for $i = 1, \ldots, n$, where e_i^M denotes the projection of e_i on P_{μ_V}. For every test function $\psi \in \mathcal{D}(\mathbb{R}^n)$, if $\nabla^M \psi$ is the projection of $\nabla\psi$ on P_{μ_V}, we have

$$\int_{\mathbb{R}^n} e_i^M \cdot \nabla\psi \, d\mu_V = \int_{\mathbb{R}^n} \nabla_i^M \psi \, d\mu_V = \langle (\delta V)_i, \psi \rangle \ .$$

Since, by hypothesis, the measure δV is locally finite, for every compact subset K of $\mathbb{R}^n$ there exists a constant c_K such that, if the support of ψ is included in K, we have

$$\left| \int_{\mathbb{R}^n} e_i^M \cdot \nabla\psi \, d\mu_V \right| \leq c_K \|\psi\|_\infty \ .$$

Therefore $e_i^M \in X_{\mu_V}$ $\forall i = 1, \ldots, n$. ∎

Example 4 (*Cantor-like sets*).

We consider now the case of measures μ concentrated on Cantor-like sets. The interest in such examples arises from the fact that the study of the corresponding T_μ throws some light on possible pathological behaviours of tangent spaces.

(i) For $n = 1$, we take $\mu = \mathcal{H}^\alpha \llcorner C$, where $0 < \alpha \leq 1$ and C is an α-dimensional Cantor set, that is, a closed subset of $[0,1]$ with a dense complement and $\mathcal{H}^\alpha(C) \in (0, +\infty)$. Then

$$T_\mu(x) = \{0\} \qquad \text{for } \mu\text{-a.e. } x \ . \tag{3.2}$$

Indeed, one can see that every vector field $\Phi \in X_\mu$ must be identically zero. In fact, since by definition the distributional derivative of $\Phi\mu$ is a Radon measure, we have $\Phi\mu = f\mathcal{H}^1$, where f is a function with bounded variation. Thus, in the case $\alpha < 1$ we infer that $\Phi = 0$ μ-a.e. from the fact that μ is singular with respect to $\mathcal{H}^1$. In the case $\alpha = 1$ we get the same conclusion by observing that f must be absolutely continuous and it vanishes on the dense set $[0,1] \setminus C$, being equal to $\Phi\chi_C$.
As an immediate consequence of (3.2), we get the following corollaries.

Corollary 3.6. *The tangent space T_μ is not local on Borel subsets of $\mathbb{R}^n$.*

Corollary 3.7. *The inclusion (2.2) may be strict.*

In fact, the measures $\mathcal{H}^1$ and $\mathcal{H}^1 \llcorner C$ coincide on the Borel set C but their tangent spaces are different almost everywhere on C.

(ii) For $n = 2$, let us consider a product measure $\mu = \mu_1 \times \mu_2$ on $\mathbb{R}^2$, where μ_1 is the Lebesgue measure on $(0,1)$ and μ_2 is a measure as in the previous example, *i.e.* μ_2 is given by $\mathcal{H}^1 \llcorner C$, with C a 1-dimensional Cantor set. One can easily check that $T_\mu(x) = T_{\mu_1}(x) \times T_{\mu_2}(x)$ whenever $\mu = \mu_1 \times \mu_2$. Therefore in this case, taking into account (3.2),

$$T_\mu(x) = \mathbb{R} \qquad \text{for } \mu\text{-a.e. } x \ .$$

Corollary 3.8. *If μ is concentrated on a set S and if the dimension of $T_\mu(x)$ is a positive constant k for μ-a.e. $x \in S$, the Hausdorff dimension of S may be strictly larger than k, so that in general we have no relation between the measures μ and $\mathcal{H}^k \llcorner S$.*

Example 5 (*the Koch curve*).

For $i = 1, \ldots, 4$, let us consider in $\mathbb{R}^2$ the similitudes

$$S_i(x) = s_i + \frac{1}{3} R_i(x) ,$$

where the points s_i and the rotations R_i are respectively given by

$$s_1 = (0,0) \quad , \quad s_2 = \left(\frac{1}{3}, 0\right) \quad , \quad s_3 = \left(\frac{2}{3}, 0\right) \quad , \quad s_4 = (0,1)$$

and

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ R_2 = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \ , \ R_3 = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} \ , \ R_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The Koch curve K is by definition the only compact subset of $\mathbb{R}^2$ such that

$$\bigcup_{i=1}^{4} S_i(K) = K . \tag{3.3}$$

The existence of a unique compact subset of $\mathbb{R}^2$ which is invariant for the family $\mathcal{S} = \{S_1, \ldots, S_4\}$ in the sense of (3.3) is proved in [47]. Therefore K is well defined and it turns out to be a self-similar fractal of Hausdorff dimension $\alpha = \log 4 / \log 3$.

Theorem 3.9. *If $\mu = \mathcal{H}^\alpha \llcorner K$, then*

$$T_\mu(x) = \{0\} \qquad \text{for } \mu\text{-a.e. } x . \tag{3.4}$$

Proof. For a fixed $x^0 \in K$, consider the sets $C_n = 3^n(K - x^0)$. It is easy to see that, because of the self-similarity of K, each set C_n is the union of 4^n copies K_n^i of K, translated and rotated by a multiple of the rotation $R = R_2$ (which generates all the R_i), *i.e.* $K_n^i = x_n^i + R \circ \ldots \circ R(K)$. To prove this we can use relation (3.3) and an induction argument: since

$$C_1 = 3(K - x^0) = \bigcup_{i=1}^{4} 3(S_i(K) - x^0) = \bigcup_{i=1}^{4} R_i(K) + 3s_i - 3x^0 ,$$

for $n = 1$ we have $K_1^i = R_i(K) + 3s_i - 3x^0$; then, iterating the same procedure, we get all the sets K_n^i

$$C_n = 3^n(K - x^0) = \bigcup_{i_1, \ldots, i_n = 1}^{4} 3^n(S_{i_1} \circ \ldots \circ S_{i_n}(K) - x^0) = \bigcup_{i=1}^{4^n} K_n^i .$$

Consider now two of these copies, K_n^i and K_n^j. By construction, they are either contiguous with one terminal point in common or disjoint, and in this case their distance has to be greater than three times the distance between $S_1(K)$ and $S_3(K)$.
Then, if we choose δ small enough, for every n the open ball B_δ, with center in the origin of $\mathbb{R}^2$, intersects at most two of the sets K_n^i.
We notice that the number of possible mutual configurations of the pair of isometric copies of K, if their intersection is not empty, is finite, since $R^6 = \mathrm{Id}$.
Hence, $C_n \cap B_\delta = (K_n^{i_n} \cup K_n^{j_n}) \cap B_\delta$, for suitable indices i_n and j_n; moreover, possibly passing to a subsequence, we can suppose that the sets $D_n = K_n^{i_n} \cup K_n^{j_n}$ differ only by translations, that is, $C_n \cap B_\delta = (p_n + D) \cap B_\delta$, where the set D is a connected union of at most two copies of K. Passing again to a subsequence, we can also suppose that the points p_n converge to a point $p \in \mathbb{R}^2$. Otherwise, if $p_n \to \infty$, we would have $C_n \cap B_\delta = \emptyset$, for n large enough, and hence $\mu(B_{3^n\delta}(x^0)) = 0$, in contradiction with the hypothesis that x^0 belongs to the support of μ. Now, if $(p + D) \cap B_\delta = \emptyset$, by the convergence in the sense of Kuratowski of the sets $p_n + D$ to $p + D$, for n large enough we would have $C_n \cap B_{\delta/2} = (p_n + D) \cap B_{\delta/2} = \emptyset$, and again x^0 would not belong to K.
We notice now that, by the rescaling property of the Hausdorff measure $\mathcal{H}^\alpha$, we have

$$\mu_{x^0,1/3^n} = \frac{\mathcal{H}^\alpha \llcorner C_n}{3^{n\alpha}} \ ,$$

and then

$$\frac{\mu_{x^0,1/3^n} \llcorner B_\delta}{\mu_{x^0,1/3^n}(B_\delta)} = \frac{\mathcal{H}^\alpha \llcorner (C_n \cap B_\delta)}{\mathcal{H}^\alpha (C_n \cap B_\delta)} \ .$$

By the previous argument, the right hand side of the above equality weakly converges to the measure

$$\frac{\mathcal{H}^\alpha \llcorner ((p + D) \cap B_\delta)}{\mathcal{H}^\alpha ((p + D) \cap B_\delta)} \ .$$

Rescaling the sets C_n of a factor $1/\delta$ in order to obtain a blow-up in the unit ball, and recalling Definition 2.5, we finally get a measure ν in $\mathrm{Tan}(\mu, x^0)$, which, up to a multiplicative factor, coincides with $\mathcal{H}^\alpha$ restricted to the intersection of the unit ball with at most two copies of K.
Suppose that the tangent space $T_\mu(x^0)$ is one dimensional: then, by Lemma 2.6, the measure ν would be a product $\mathcal{H}^1 \times \sigma$ and this is clearly impossible. On the other hand, the dimension of $T_\mu(x^0)$ cannot be two, by an upper estimate for the dimension of the tangent space that we are going to prove in the next section, Theorem 4.2. ∎

4. An estimate for the dimension and a rectifiability theorem

In the first part of the section we show some estimates from above for the dimension of T_μ. Then we deduce, from a criterion proved by Preiss in [59], a rectifiability theorem based

on tangent spaces T_μ. Our result is in some sense more general than Theorem 2.3, since it also includes the case of a measure μ given by a sum of Hausdorff measures of different dimensions, each one concentrated on a rectifiable set.
We begin with the following lemma, where the usual notation $\theta^*_\alpha(\mu,x)$ is adopted for the α-dimensional upper density of μ at the point x, that is

$$\theta^*_\alpha(\mu,x) = \operatorname*{limsup}_{\rho\to 0^+} \frac{\mu(B_\rho(x))}{\omega_\alpha \rho^\alpha} ,$$

where $\omega_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$, being $\Gamma(t) = \int_0^{+\infty} s^{t-1}e^{-s}\,ds$ the Euler function.
Later on, we will also use the α-dimensional lower density $\theta_{*\alpha}(\mu,x)$, which is analogously defined, replacing the limsup with the liminf.

Lemma 4.1. *Let α be a real number, $0 \le \alpha \le n$, and let $E \subseteq \mathbb{R}^n$ be a set where*

$$\theta^*_\alpha(\mu,x) = +\infty .$$

Then, for μ-a.e. $x \in E$, the dimension of $T_\mu(x)$ cannot be larger than the integer part of α.

Proof. We claim that the following condition is satisfied when $x \in E$ and $t \in (0,1)$:

$$\operatorname*{limsup}_{\rho\to 0^+} \frac{\mu(B_{t\rho}(x))}{\mu(B_\rho(x))} \ge t^\alpha . \tag{4.1}$$

Suppose by contradiction that there exist $x \in E$, $t \in (0,1)$ and $\overline{\rho} > 0$ such that

$$\mu(B_{t\rho}(x)) \le t^\alpha \mu(B_\rho(x)) \qquad \forall \rho \in (0,\overline{\rho}] .$$

Then in particular

$$\mu(B_{t^n\overline{\rho}}(x)) \le t^{n\alpha} \mu(B_{\overline{\rho}}(x)) \qquad \forall n \in \mathbb{N} .$$

If for every $\rho \in (0,\overline{\rho}]$ we choose an integer n such that ρ belongs to the interval $(t^{n+1}\overline{\rho}, t^n\overline{\rho}]$, we get

$$\mu(B_\rho(x)) \le \mu(B_{t^n\overline{\rho}}(x)) \le t^{n\alpha}\mu(B_{\overline{\rho}}(x)) \le [(t\overline{\rho})^{-\alpha}\mu(B_{\overline{\rho}}(x))]\rho^\alpha .$$

But, since $[(t\overline{\rho})^{-\alpha}\mu(B_{\overline{\rho}}(x))]$ does not depend on the choice of $\rho \in (0,\overline{\rho}]$, the above inequality contradicts the hypothesis on $\theta^*_\alpha(\mu,x)$, and therefore (4.1) holds.
Let us denote by Q^n_ρ an open cube of $\mathbb{R}^n$ centered at the origin with sides of semi-length ρ, either orthogonal or parallel to $T_\mu(x)$. Then there exists a constant $c(n) < 1$ such that

$$\overline{Q^n_{c(n)}} \subseteq B .$$

By (4.1), for μ-a.e. $x \in E$ and for every $t \in (0,1)$, we can choose a sequence of positive numbers ρ_i tending to zero such that the corresponding blown-up measures $\frac{\mu_{x,\rho_i}}{\mu(B_{\rho_i}(x))}$ converge to a measure $\nu \in \operatorname{Tan}(\mu,x)$ satisfying both the inequality

$$\nu\left(\overline{B_t}\right) \ge t^\alpha$$

and the product decomposition (2.4).
Let $d(x)$ be the dimension of $T_\mu(x)$. Since $\nu(Q^n_{c(n)}) \leq \nu(B) \leq 1$, if we take into account the representation of ν, we get

$$[2c(n)]^{d(x)}\sigma(Q^{n-d(x)}_{c(n)}) \leq 1 \ . \tag{4.2}$$

On the other hand, since $\overline{B_t} \subseteq \overline{Q^n_t}$,

$$(2t)^{d(x)}\sigma(Q^{n-d(x)}_t) = \nu\left(\overline{Q^n_t}\right) \geq \nu\left(\overline{B_t}\right) \geq t^\alpha \ . \tag{4.3}$$

Finally, if we choose $t < c(n)$, putting together (4.2) and (4.3), we get

$$t^{d(x)-\alpha} \geq [c(n)]^{d(x)} \ ,$$

which implies $d(x) \leq \alpha$, because t can be chosen arbitrarily small. Since $d(x)$ is an integer, we have obtained the result. ■

The following theorem is a straightforward consequence of the Lemma 4.1.

Theorem 4.2. *Let E be any subset of $\mathbb{R}^n$. Then*

$$\dim T_\mu(x) \leq \mathcal{H}\text{-}\dim(E) \qquad \text{for } \mu\text{-a.e. } x \in E \ . \tag{4.4}$$

Proof. If $\alpha > \mathcal{H}\text{-}\dim(E)$, then $\theta^*_\alpha(\mu, x) = +\infty$ for μ-a.e. $x \in E$. Therefore Lemma 4.1 immediately yields the estimate (4.4). ■

A nice property of tangent spaces in dimension one is given in the next lemma.

Lemma 4.3. *Let μ be a positive Radon measure on $\mathbb{R}$, and let μ^s be the singular part of μ with respect to the Lebesgue measure. Then*

$$T_\mu(x) = \{0\} \qquad \text{for } \mu^s\text{-a.e. } x \ .$$

Proof. Let $\mu = \mu^a + \mu^s$, where μ^a is the absolutely continuous part of μ, and let g be the density of μ^a with respect to $\mathcal{H}^1$. Let Φ be a tangent field. Since the distributional derivative of $\Phi\mu$ is a Radon measure, there exists a function f with bounded variation such that $\Phi\mu = f\mathcal{H}^1$ (recall Example 4 of Section 3). Then Φ must be zero μ^s-almost everywhere, because

$$\Phi\mu^a + \Phi\mu^s = f\mathcal{H}^1 \quad \Longrightarrow \quad \Phi\mu^s = (f - \Phi g)\mathcal{H}^1 \ .$$

■

Remark 4.4. The analogous statement for a measure μ on $\mathbb{R}^n$ would be

$$\dim T_\mu(x) < n \qquad \text{for } \mu^s\text{-a.e. } x \ ,$$

where μ^s now denotes the singular part of μ with respect to $\mathcal{H}^n$. It is not clear whether this conjecture is true or not, because of the remarkable difference between the one-dimensional and the n-dimensional case (see also Example 2 of Section 3). We point out that this difference, essentially due to the presence of the divergence operator in Definition 2.1, rules out the use of slicing type techniques in treating the spaces T_μ. Moreover, also the blow-up techniques, which are often useful, may fail, since the set $\mathrm{Tan}(\mu,x)$ can be sometimes very wild; O'Neil [57] has recently constructed a measure μ for which , at μ-almost every point x, every measure belongs to $\mathrm{Tan}(\mu,x)$.

We now pass to the rectifiability result.
We denote by $\mathcal{R}_k$ the class of all measures of type $\theta\mathcal{H}^k \llcorner E$, with k an integer, E a countably $\mathcal{H}^k$-rectifiable subset of $\mathbb{R}^n$, and θ a positive function in $\mathcal{H}^k_{\mathrm{loc}}(E)$.

Theorem 4.5. *Let k be an integer, $k \le n$, and let E_k be the set of the points $x \in \mathbb{R}^n$ such that*

(i) the dimension of $T_\mu(x)$ is equal to k;

(ii) $\theta_{*k}(\mu,x)$ is positive and finite;

(iii) the "doubling condition" holds at x, i.e.

$$\limsup_{\rho\to 0^+} \frac{\mu(B_{2\rho}(x))}{\mu(B_\rho(x))} < +\infty \ .$$

Then the measure $\mu \llcorner E_k$ belongs to the class $\mathcal{R}_k$.

The proof of this theorem is based on the following rectifiability criterion of Preiss [59].

Theorem 4.6. *Let σ be a measure on $\mathbb{R}^n$ satisfying, for σ-a.e. x, these two conditions:*
(A) if we set $\tau = 1 - 2^{-k-6}$ and

$$E_\rho(x) = \left\{ z \in B_\rho(x) \text{ s.t. } \exists\, s \in (0,\rho) : \frac{\sigma(B_s(z))}{s^k} \le \tau \frac{\sigma(B_\rho(x))}{\rho^k} \right\} \ ,$$

we have

$$\liminf_{\rho\to 0^+} \frac{\sigma(E_\rho(x))}{\sigma(B_\rho(x))} = 0 \ ;$$

(B) if we set $\kappa = 8^{-k-9}k^{-4}$, and let $G_{n,k}$ denote the class of k-dimensional linear subspaces of $\mathbb{R}^n$, and

$$F_\rho(x) = \sup_{V\in G_{n,k}} \left\{ \inf_{z\in(x+V)\cap B_\rho(x)} \frac{\sigma(B_{\kappa\rho}(z))}{\sigma(B_\rho(x))} \right\} \ ,$$

we have

$$\liminf_{\rho\to 0^+} F_\rho(x) > 0 \ .$$

Then $\sigma \in \mathcal{R}_k$.

We are now in a position to prove Theorem 4.5.

Proof. We set $\sigma = \mu \llcorner E_k$, and we show that, under the hypotheses (i), (ii), and (iii), both (A) and (B) are satisfied σ-almost everywhere. It is enough to verify that such conditions hold for μ-a.e. x which is a density point of the set E_k with respect to μ. Then, since at such points x we have

$$\lim_{\rho\to 0} \frac{\sigma(B_\rho(x))}{\mu(B_\rho(x))} = 1 \ ,$$

we will replace σ with μ.
Let us begin by checking condition (A). We define, for positive constants ε and c, the sets

$$E(\varepsilon, c) = \{z : \frac{\mu(B_r(z))}{r^k} \geq c \quad \forall r \in (0,\varepsilon)\}$$

and

$$\widetilde{E}(\varepsilon, c) = E(\varepsilon, \tau c) \setminus \bigcup_{n=1}^{\infty} E\left(\frac{\varepsilon}{n}, c\right) \ .$$

By (ii), for μ-a.e. $x \in E_k$, we can choose $t > 0$ such that

$$\tau t < \theta_{*k}(\mu, x) < t \ .$$

Then there exists $s > 0$ such that x belongs to $\widetilde{E}(s,t)$. We can also suppose that x is a density point of $\widetilde{E}(s,t)$, or, equivalently,

$$\lim_{\rho\to 0} \frac{\mu\left(\widetilde{E}^c(s,t) \cap B_\rho(x)\right)}{\mu(B_\rho(x))} = 0 \ . \tag{4.5}$$

Now let us verify that (4.5) implies that the property (A) holds at the point x, and consequently that it is satisfied μ-almost everywhere. Since $x \in \widetilde{E}(s,t)$, we have by definition that $x \notin \bigcup_n E\,(s/n, t)$, and therefore for every $n \in \mathbb{N}$ there exists $\rho_n \in (0, s/n)$ such that

$$\frac{\mu(B_{\rho_n}(x))}{\rho_n^k} \leq t \ .$$

For such a sequence ρ_n, it can be easily verified the inclusion $E_{\rho_n}(x) \subseteq [\widetilde{E}^c(s,t) \cap B_{\rho_n}(x)]$. Indeed, if we suppose by contradiction that there exists $z \in E_{\rho_n}(x) \cap \widetilde{E}(s,t)$, we could select $s_n \in (0, \rho_n)$ such that

$$\tau t \leq \frac{\mu(B_{s_n}(z))}{s_n^k} \leq \frac{\mu(B_{\rho_n}(x))}{\rho_n^k} \ ,$$

against the choice of the sequence ρ_n. Then we have

$$\liminf_{\rho\to 0^+} \frac{\mu(E_\rho(x))}{\mu(B_\rho(x))} \le \liminf_{\rho_n\to 0^+} \frac{\mu(E_{\rho_n}(x))}{\mu(B_{\rho_n}(x))} \le \liminf_{\rho_n\to 0^+} \frac{\mu({}^c\widetilde{E}(s,t)\cap B_{\rho_n}(x))}{\mu(B_{\rho_n}(x))} = 0 \ .$$

We now verify condition (B). First we observe that it is enough to check (B) at any point x where the decomposition property (2.4) holds. For every $\rho > 0$, let $z_\rho \in [x + T_\mu(x)] \cap B_\rho(x)$. We claim that

$$\liminf_{\rho\to 0^+} \frac{\mu(B_{\kappa\rho}(z_\rho))}{\mu(B_\rho(x))} > 0 \ . \tag{4.6}$$

Indeed, we can choose a sequence ρ_i of positive numbers tending to zero such that the following three conditions hold:

$$\lim_{i\to+\infty} \frac{\mu(B_{\kappa\rho_i}(z_{\rho_i}))}{\mu(B_{\rho_i}(x))} = \liminf_{\rho\to 0^+} \frac{\mu(B_{\kappa\rho}(z_\rho))}{\mu(B_\rho(x))} \ ;$$

$$\lim_{i\to+\infty} \frac{\mu_{x,\rho_i}}{\mu(B_{\rho_i}(x))} = \nu \in \mathrm{Tan}(\mu,x) \ ;$$

$$\lim_{i\to+\infty} \frac{z_{\rho_i} - x}{\rho_i} = z \in \overline{B} \cap T_\mu(x) \ .$$

Then by the above conditions and by the semicontinuity properties of the weak convergence of measures, we get the estimate

$$\liminf_{\rho\to 0^+} \frac{\mu(B_{\kappa\rho}(z_\rho))}{\mu(B_\rho(x))} \ge \nu(B_\kappa(z)\cap B) \ . \tag{4.7}$$

Now let us make the crucial remark that the doubling condition assumed by hypothesis implies

$$0 \in \mathrm{spt}\,\nu \qquad \forall \nu \in \mathrm{Tan}(\mu,x) \ . \tag{4.8}$$

Indeed (iii) can be reformulated as

$$\liminf_{\rho\to 0^+} \frac{\mu(B_{\rho/2}(x))}{\mu(B_\rho(x))} > 0 \ ,$$

and so, by an iteration argument, we obtain

$$\nu\left(\overline{B_{2^{-k}}}\right) \ge \liminf_{\rho\to 0^+} \frac{\mu(B_{\rho/2^k}(x))}{\mu(B_\rho(x))} > 0 \qquad \forall k \in \mathbb{N} \ ,$$

which gives (4.8).
Then if we choose $z' \in B \cap T_\mu(x)$ with $B_{\frac{\kappa}{2}}(z') \subseteq B_\kappa(z)$, using (4.8) we have

$$\nu(B_\kappa(z)\cap B) \ge \nu(B_{\frac{\kappa}{2}}(z')\cap B) = \nu((-z' + B_{\frac{\kappa}{2}}(z'))\cap B) > 0 \ . \tag{4.9}$$

Thus, combining (4.7) and (4.9), we get (4.6). Finally, if we observe that $T_\mu(x) \in G_{n,k}$ for μ-a.e. $x \in E_k$, we arrive at the conclusion that (B) is satisfied μ-almost everywhere. ∎

III. Mean curvature of a measure and related variational problems*

1. Introduction

The notion of mean curvature of a manifold is an important tool for many problems in Analysis and in Geometry. When we deal with the classical case of a smooth manifold, the definition of mean curvature can be given in terms of second order derivatives of parametrizations, or in terms of first derivatives of tangent fields. However, when we deal with minimization problems involving manifolds as unknown, minimizing sequences of classical smooth manifolds may tend to more irregular objects, so that several theories have been developed to take into account singularities, and to define for these weak objects too some generalized notions of tangential derivatives and mean curvature. This is for instance the case of the theory of rectifiable and general varifolds, for which we refer to the treatises [41], [64].

Here, we try to consider as generalized manifolds even weaker objects as measures. The definition of tangent space associated to a measure μ has been given in the paper [18], together with the corresponding notion of tangential operator D_μ; these tools revealed crucial for some shape optimization problems, considered in [19].

In this paper, in a similar way, we give the definition of generalized mean curvature $H(\mu)$ associated to a measure μ, and we prove a lower semicontinuity result for a class of functionals of the form

$$F(\mu) = \int_{\mathbb{R}^n} f(x, \mu, H(\mu))\, d\mu\ .$$

For instance, the classical

$$F(M) = \int_M (1 + |h(M)|^2)\, d\mathcal{H}^k\ ,$$

where $h(M)$ is the mean curvature vector of the smooth k-manifold M, becomes in this weaker framework

$$F(\mu) = \int_{\mathbb{R}^n} (1 + |h(\mu)|^2)\, d\mu\ ,$$

where the generalized pointwise curvature $h(\mu)$ is defined as the absolute continuous part of $H(\mu)$ with respect to μ. By means of a suitable notion of boundary $\partial\mu$ of a measure μ we shall be able to consider minimization problems with "Dirichlet boundary conditions"

$$\min\{F(\mu)\ :\ \partial\mu = \lambda\}\ ,$$

* Paper in memory of E. De Giorgi, by G. Bouchitté, G. Buttazzo, and I. Fragalà, appeared on the Annali della Scuola Normale Superiore di Pisa (1997) [16].

where λ is a given measure. In the smooth case this would correspond to the problem

$$\min\{F(M) \;:\; \partial M = \Gamma\;,\; \nu_M = \gamma\}\;,$$

where Γ is a prescribed boundary, and γ an assigned normal field.

We give here a brief outline of the paper.

In Section 2 we introduce the notion of curvature of a measure and the space $\mathcal{M}_{BC}$ of measures with bounded curvature; after showing some examples, we prove a monotonicity lemma for measures with bounded curvature whose tangent space has constant dimension.

In Section 3 we study the lower semicontinuity for a class of functionals depending on curvatures, and we show how this problem is related to the lower semicontinuity of the dimension of tangent spaces with respect to the weak convergence of measures.

The study of the lower semicontinuity for the dimension of tangent spaces is treated in Section 4 under two different groups of assumptions: the former is similar to known results about k-varifolds, while the latter involves a more interesting and rather curious condition on the logarithm of the densities.

Section 5 is devoted to some examples and applications.

2. The curvature of a measure

Let μ be a positive and finite Borel regular measure on $\mathbb{R}^n$. In the following the notation $\mathcal{M}^+$ will be used to denote the class of positive and finite Borel regular measures on $\mathbb{R}^n$, while $\mathcal{M}^d$ will denote the space of Borel regular $\mathbb{R}^d$-valued measures ν with finite total variation $|\nu|$ on $\mathbb{R}^n$ (in the case $d=1$ we simply write $\mathcal{M}$ instead of $\mathcal{M}^1$). Then, since throughout the paper we are dealing with measures on $\mathbb{R}^n$, the domain of integration is always omitted whenever it coincides with all $\mathbb{R}^n$.

We are going to define a notion of curvature for μ. To this aim we look at the classical divergence theorem for a C^2 k-manifold $M \subset \mathbb{R}^n$ with boundary ∂M and for a vector field $X \in C^1(M, \mathbb{R}^n)$:

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^k = -\int_{\partial M} X\cdot\eta \, d\mathcal{H}^k - \int_M X\cdot h \, d\mathcal{H}^k\;, \tag{2.1}$$

where h is the mean curvature vector of M, and η is the inward co-normal unit vector of ∂M (see for example [64]).

In view of (2.1) it seems reasonable to give the following definition for the curvature of a measure μ. We recall that, following Bouchitté, Buttazzo and Seppecher [18], the tangent space to μ can be defined, for μ-a.e. x, as

$$T_\mu(x) := \mu - \operatorname{ess}\bigcup\{\Phi(x) : \Phi \in X_\mu\}\;, \tag{2.2}$$

where

$$X_\mu := \{\Phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ |\Phi| \in L^1_\mu \ , \ \operatorname{div}(\Phi\mu) \in \mathcal{M}\} \ .$$

For μ-a.e. x we denote by $P_\mu(x,\cdot) : \mathbb{R}^n \to T_\mu(x)$ the orthogonal projection of $\mathbb{R}^n$ on $T_\mu(x)$.

Definition 2.1. *The curvature of μ is defined as the vector-valued distribution*

$$H(\mu) := \operatorname{div}(P_\mu \mu) \ .$$

In other words $H(\mu)$ is defined by

$$\langle H(\mu), X\rangle = -\int \operatorname{div}_\mu X \, d\mu \qquad \forall X \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \ , \tag{2.3}$$

where $\operatorname{div}_\mu X = \sum_{i=1}^n (P_\mu(\nabla X^i))_i$.

It is now quite natural to consider the class of all measures μ such that $H(\mu)$ is a vector measure.

Definition 2.2. *We denote by $\mathcal{M}_{BC}$, which stays for measures with bounded curvature, the class*

$$\mathcal{M}_{BC} := \left\{\mu \in \mathcal{M}^+ \ : \ H(\mu) \in \mathcal{M}^n\right\} \ .$$

Remark 2.3. The curvature $H(\mu)$ of a measure $\mu \in \mathcal{M}_{BC}$ is not necessarily absolutely continuous with respect to μ (see for instance Examples 2.5 and 2.7 below). In particular, by the Radon-Nikodym theorem, when μ belongs to $\mathcal{M}_{BC}$, we can write

$$H(\mu) = h(\mu)\mu + \partial\mu \ ,$$

where $h(\mu) \in L^1_\mu(\mathbb{R}^n, \mathbb{R}^n)$ is the density of $H(\mu)$ with respect to μ, and $\partial\mu$ is the singular part of $H(\mu)$ with respect to μ. In the sequel the notation $h(\mu)$ will be always used to indicate the density $\frac{dH(\mu)}{d\mu}$; we shall call $h(\mu)$ the *pointwise curvature* of μ and $\partial\mu$ the *boundary* of μ.

Remark 2.4. Definition 2.2 seems to be a quite natural generalization of the notion of function with bounded variation. Actually one immediately sees that if u is a real function defined on an open subset Ω of $\mathbb{R}^n$ with a Lipschitz boundary, u belongs to $BV(\Omega)$ if and only if the measure $\mu = u\mathcal{L}^n \llcorner \Omega$ belongs to $\mathcal{M}_{BC}$ and in this case the curvature $H(\mu)$ coincides with the measure $Du - u\nu\mathcal{H}^{n-1}\llcorner\partial\Omega$, where Du represents the gradient of u in the sense of distributions and ν is the exterior normal unit vector to $\partial\Omega$. However, due to the nonlinearity of the mapping $\mu \mapsto H(\mu)$, the natural metric on $\mathcal{M}_{BC}$ defined by

$$\delta(\mu,\nu) := |\mu - \nu| + |H(\mu) - H(\nu)| \ ,$$

cannot be associated to a norm.

Example 2.5. In the case $\mu = \mathcal{H}^k \llcorner M$, with M a C^2 k-manifold with boundary in $\mathbb{R}^n$, by (2.1) one immediately obtains

$$H(\mu) = \eta \mathcal{H}^{k-1} \llcorner \partial M + h\, \mathcal{H}^k \llcorner M \ ,$$

where h denotes the mean curvature vector of M and η the co-normal unit vector of ∂M.

Example 2.6. When the tangent space to μ defined in (2.2) is reduced to zero μ-almost everywhere, $H(\mu)$ is of course zero. This is for instance the case if μ is a finite sum of Dirac masses, or else $\mu = \mathcal{H}^\alpha \llcorner C$ with C an α-dimensional Cantor set contained in $[0,1]$ with $\mathcal{H}^\alpha(C) \in (0, +\infty)$ (see [42]).

Example 2.7. Let us consider two cases in which there are corner points, which give rise to a "boundary effect". Let μ_1 and μ_2 be the restrictions of the one-dimensional Hausdorff measure $\mathcal{H}^1$ on the structures represented respectively in Figures 1 and 2 below.

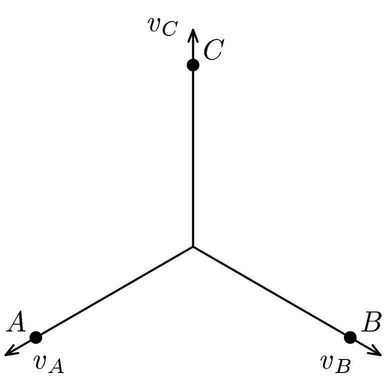

Figure 1

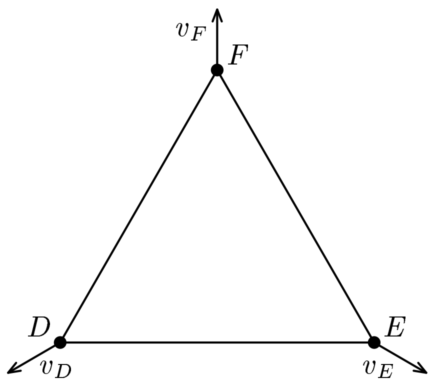

Figure 2

In both cases the boundary effect consists in the sum of three oriented Dirac masses. In fact it can be easily verified, by using directly Definition 2.1, that we have

$$H(\mu_1) = v_A \delta_A + v_B \delta_B + v_C \delta_C$$

and

$$H(\mu_2) = v_D \delta_D + v_E \delta_E + v_F \delta_F \ ,$$

with $|v_A| = |v_B| = |v_C| = 1$ and $|v_D| = |v_E| = |v_F| = \sqrt{3}$.

Example 2.8. Let $V = \underline{v}(M,\theta)$ be a k-rectifiable varifold, and let $\mu_V = \theta \mathcal{H}^k \llcorner M$ be the associated weight measure. A function $h \in L^1_{\mu_V}(M, \mathbb{R}^n)$ is said to be the generalized mean curvature of V if

$$\int \operatorname{div}_M X \, d\mu_V = - \int X \cdot h \, d\mu_V \qquad \forall X \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n) \; .$$

Moreover it is well known that the left hand side is equal to the first variation of mass

$$\langle \delta V, X \rangle := \frac{d}{dt} \underline{M}(\varphi_{t_\sharp} V)_{|t=0} \; ,$$

where $\underline{M}(\varphi_{t_\sharp} V)$ denotes the mass of the image varifold $\varphi_{t_\sharp} V$, being φ_t a one-parameter group of smooth diffeomorphisms of $\mathbb{R}^n$ into itself with X as initial velocity (see [64] for the details). Now, let us suppose that the first variation of V is a bounded measure. In this case T_{μ_V} coincides with the approximate tangent space to μ_V (see [42] for a proof). Then μ_V belongs to $\mathcal{M}_{BC}$, $H(\mu_V)$ is equal to δV, and V has generalized mean curvature h if and only if $H(\mu_V)$ is absolutely continuous with respect to μ_V, with density h.

The following lemma is a useful result for the measures with bounded curvature whose tangent space has constant dimension. It is proved with the same technique used in the literature for the case of rectifiable varifolds.

Lemma 2.9. *If $\mu \in \mathcal{M}_{BC}$ is such that $\dim T_\mu(x) = k$ for μ-a.e. x, then the k-dimensional density of μ at x*

$$\theta_k(\mu, x) := \lim_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\omega_k \rho^k}$$

exists and it is finite μ-a.e. (here $\omega_k = \pi^{k/2} \Gamma(1 + k/2)$, being $\Gamma(t) = \int_o^{+\infty} s^{t-1} \mathrm{e}^{-s} \, ds$ the Euler function). If moreover θ_k is strictly positive μ-a.e., then there exists a k-rectifiable subset $M \subset \mathbb{R}^n$ such that

$$\mu = \theta_k \; \mathcal{H}^k \llcorner M \; .$$

Proof. For any fixed $\xi \in \mathbb{R}^n$, we choose in (2.3) a vector field $X \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$ of the form

$$X(x) = \gamma(r)(x - \xi) \; ,$$

where $r = |x - \xi|$, and $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth nonincreasing function such that for a positive ρ we have $\gamma(r) = 0$ for $r \geq \rho$ and $\gamma(r) = 1$ for $r \leq \rho/2$. We have

$$DX(x) = \gamma(r) \operatorname{Id} + r\gamma'(r) \left(\frac{x - \xi}{r} \otimes \frac{x - \xi}{r} \right) \; ;$$

thus, if we compute $\mathrm{div}_\mu X$ taking into account the assumption on the dimension of T_μ, we get

$$\mathrm{div}_\mu X(x) = k\gamma(r) + r\gamma'(r)\left(1 - |D^\perp r|^2\right) \ ,$$

where $D^\perp r = Dr - D_\mu r$. Substituting this into (2.3), we obtain

$$\langle H(\mu), X\rangle = -k\int \gamma(r)\,d\mu - \int r\gamma'(r)\,d\mu + \int r\gamma'(r)|D^\perp r|^2\,d\mu \ .$$

In particular we can take $\gamma(r) = \Phi(r/\rho)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a smooth nonincreasing function such that $\Phi(t) = 0$ for $t \geq 1$ and $\Phi(t) = 0$ for $t \leq 0$. Then the last equality can be rewritten as

$$\begin{aligned}\langle H(\mu), X\rangle &= -k\int \Phi\left(\frac{r}{\rho}\right)\,d\mu + \int \rho\frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)\,d\mu - \int \rho\frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)|D^\perp r|^2\,d\mu \\ &= -k\int \Phi\left(\frac{r}{\rho}\right)\,d\mu + \rho\frac{\partial}{\partial\rho}\int \Phi\left(\frac{r}{\rho}\right)\,d\mu - \rho\frac{\partial}{\partial\rho}\int \Phi\left(\frac{r}{\rho}\right)|D^\perp r|^2\,d\mu \ .\end{aligned}$$

Now we let Φ increase to the characteristic function of $(-\infty, 1)$, so that $\Phi(r/\rho)$ increases to $\chi_{B_\rho(\xi)}(r)$; we obtain

$$\langle H(\mu), \chi_{B_\rho(\xi)}(r)(x-\xi)\rangle = -k\mu(B_\rho(\xi)) + \rho\frac{\partial}{\partial\rho}\mu(B_\rho(\xi)) - \rho\frac{\partial}{\partial\rho}\int_{B_\rho(\xi)} |D^\perp r|^2\,d\mu \ ,$$

and dividing by ρ^{k+1}:

$$\begin{aligned}\frac{1}{\rho^k}\left\langle H(\mu), \chi_{B_\rho(\xi)}(r)\frac{(x-\xi)}{\rho}\right\rangle &= -\frac{k}{\rho^{k+1}}\mu(B_\rho(\xi)) + \frac{1}{\rho^k}\frac{\partial}{\partial\rho}\mu(B_\rho(\xi)) - \frac{\partial}{\partial\rho}\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^k}\,d\mu \\ &= \frac{\partial}{\partial\rho}\left[\frac{\mu(B_\rho(\xi))}{\rho^k}\right] - \frac{\partial}{\partial\rho}\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^k}\,d\mu \ .\end{aligned}$$

Setting $f(\xi,\rho) := \mu(B_\rho(\xi))/\rho^k$ we have

$$\frac{\partial}{\partial\rho}f(\xi,\rho) \geq -\frac{1}{\rho^k}|H(\mu)|(B_\rho(\xi)) \ ,$$

which implies

$$\frac{1}{f(\xi,\rho)}\frac{\partial}{\partial\rho}f(\xi,\rho) \geq -\frac{|H(\mu)|(B_\rho(\xi))}{\mu(B_\rho(\xi))} \ . \tag{2.4}$$

If we choose ξ as a density point of the total variation measure of $H(\mu)$ with respect to μ, the limit as $\rho \to 0$ of the right hand side of the above inequality exists and is finite, so that μ-a.e. we have

$$\frac{1}{f(\xi,\rho)}\frac{\partial}{\partial\rho}f(\xi,\rho) \geq -c(\xi)$$

for a suitable nonnegative constant $c = c(\xi)$. This means that, for μ-a.e. $\xi \in \mathbb{R}^n$ the function

$$\rho \mapsto \exp[c(\xi)\rho]f(\xi,\rho)$$

is non-decreasing, and in particular that for μ-a.e. $\xi \in \mathbb{R}^n$ the function $f(\xi,\cdot)$ admits a finite limit when ρ tends to zero.

The last statement follows from known rectifialility results, which can be found for instance on a fundamental paper by Preiss [59], and on a more recent paper by Fragalà and Mantegazza [42].

■

3. Lower semicontinuous functionals of curvatures

In view of considering minimum problems involving curvatures, we study in this paragraph the lower semicontinuity of functionals $F : \mathcal{M}_{BC} \to \mathrm{IR} \cup \{+\infty\}$ of the form

$$F(\mu) = \begin{cases} J(\mu, H(\mu)) & \text{if } \mu \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

where J is defined on the product space $\mathcal{M}_{BC} \times \mathcal{M}^n$ and $\mathcal{A}$ is a subclass of measures in $\mathcal{M}_{BC}$. If J is coercive and weakly lower semicontinuous with respect to the weak convergence of measures, the existence of a minimum for J follows straightforward by direct methods of the calculus of variations as soon as the set $\{(\mu, H(\mu)) \,:\, \mu \in \mathcal{A}\}$ is weakly closed in $\mathcal{M}_{BC} \times \mathcal{M}^n$. Since this property does not hold if we choose $\mathcal{A} = \mathcal{M}_{BC}$ (see Example 3.6), the main point of this section will be to exhibit some suitable subclasses $\mathcal{A}$ for which the direct methods can be applied.
As a preliminary tool, we need an upper semicontinuity theorem for the dimension of the tangent space; actually we prove that if a sequence of measures $\{\mu_h\}_h$ is bounded in $\mathcal{M}_{BC}$, when we pass to the weak limit, the dimension of the tangent space cannot decrease. The proof is obtained by using standard convex analysis arguments; it is based on the following lemma.

Lemma 3.1. *Let $\{\mu_h\}_h$ be a sequence in $\mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$. Let d be a positive integer and $\{f_h\}_h$ be a sequence belonging to $L^p_{\mu_h}(\mathrm{IR}^n, \mathrm{IR}^d)$ for some $p \in [1, +\infty)$. Suppose there exists a positive constant M such that*

$$\|f_h\|_{L^p_{\mu_h}} \le M \qquad \text{for every } h\ .$$

Then the sequence $\{f_h \mu_h\}_h$ is bounded in $\mathcal{M}^d$, and the weak limit of any convergent subsequence is absolutely continuous with respect to μ, with a density f in $L^p_\mu(\mathrm{IR}^n, \mathrm{IR}^d)$. Moreover for every convex and lower semicontinuous function $\Phi : \mathrm{IR}^d \to [0, +\infty)$ we have

$$\liminf_{h \to +\infty} \int \Phi(f_h)\, d\mu_h \ge \int \Phi(f)\, d\mu\ .$$

Proof. By Hölder inequality, we have

$$\int |f_h|\, d\mu_h \le \left(\int |f_h|^p\, d\mu_h \right)^{1/p} (\mu_h(\mathrm{IR}^n))^{1/p'}\ ,$$

so that the sequence $\{f_h\, \mu_h\}_h$ is uniformly bounded in variation, hence weakly relatively compact in $\mathcal{M}^d$. Possibly passing to a subsequence, we can assume that $\{f_h\, \mu_h\}_h \rightharpoonup \nu$. Applying this convergence to a test function $\varphi \in C^0_c(\mathrm{IR}^n, \mathrm{IR}^d)$ and using Fenchel's inequality, we get

$$\begin{aligned} \liminf_{h \to +\infty} \frac{1}{p} \int |f_h|^p d\mu_h &\ge \liminf_{h \to +\infty} \left(\int f_h\, \varphi\, d\mu_h \; - \frac{1}{p'} \int |\varphi|^{p'} d\mu_h \right) \\ &\ge < \nu, \varphi > - \frac{1}{p'} \int |\varphi|^p d\mu\ . \end{aligned}$$

As the left hand side of the above inequality is bounded, we deduce that

$$\sup \left\{ < \nu, \varphi > : \ \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}^d) \ , \ \|\varphi\|_{L^{p'}} \leq 1 \right\} < +\infty \ .$$

Thus ν, extended to $L_\mu^{p'}(\mathbb{R}^n, \mathbb{R}^d)$, can be identified with an element f belonging to $L_\mu^p(\mathbb{R}^n, \mathbb{R}^d)$ (*i.e.* $\nu = f\,\mu$).
To obtain the last statement, we write

$$\int \Phi(f_h)\, d\mu_h = \int \Psi(f_h, 1)\, d\mu_h \ ,$$

where $\Psi : \mathbb{R}^d \times \mathbb{R} \to [0, +\infty)$ is the one homogeneous convex function defined by

$$\Psi(z,t) := \begin{cases} t\,\Phi(z/t) & \text{if } t > 0 \\ \Phi^\infty(z) & \text{if } t = 0 \\ +\infty & \text{if } t < 0 \ . \end{cases}$$

Here $\Phi^\infty(z) = \lim\limits_{t \to 0} t\,\Phi(z/t)$ denotes the recession functional of Φ. By the lower semicontinuity of Φ, we obtain that Ψ is lower semicontinuous with respect to the pair (z,t). Then, by Lemma 3.2 below:

$$\begin{aligned} \liminf_{h \to +\infty} \int \Phi(f_h)\, d\mu_h &= \liminf_{h \to +\infty} \int \Psi(f_h, \mu_h)\, d\mu_h \\ &\geq \int \Psi(f, \mu)\, d\mu = \int \Phi(f) d\mu \ . \end{aligned}$$

■

Lemma 3.2. *Let $\Psi(x,z) : \mathbb{R}^n \times \mathbb{R}^d \to [0, +\infty)$ be a lower semicontinuous function such that, for every $x \in \mathbb{R}^n$, $\Psi(x, \cdot)$ is convex and homogeneous of degree 1. Then the functional defined by*

$$\mathcal{M}^d \ni \lambda \mapsto \int \Psi(x, \lambda) := \int \Psi\left(x, \frac{d\lambda}{d|\lambda|}\right)\ d|\lambda| \ ,$$

is weakly lower semicontinuous. Note that $\int \Psi(x,\lambda) = \int \Psi\big(x, \frac{d\lambda}{d\theta}\big)\ d\theta$ for every λ such that $\lambda \ll \theta$.

Proof. We refer to Reshetnyak [60]. ■

Remark 3.3. The assumption $\Phi \geq 0$ in Lemma 3.1 can be dropped if we assume that the sequence $\{\mu_h\}_h$ converge tightly to μ. Indeed, we may substitute Φ by $(\Phi - g)$ where g is an affine minorant of Φ and then notice that the tight convergence of $\{\mu_h\}_h$ implies

$$\lim_{h \to +\infty} \int g(f_h) d\mu_h = \int g(f) d\mu \ .$$

We can now state the upper semicontinuity result for the dimension of T_μ.

Theorem 3.4. *Let $\{\mu_h\}_h$ be a bounded sequence in $\mathcal{M}_{BC}$, i.e. there exist positive constants C_1 and C_2 such that*

$$|\mu_h| \leq C_1 \;, \qquad |H(\mu_h)| \leq C_2 \; . \tag{3.2}$$

Up to subsequences, let μ be the weak limit of $\{\mu_h\}_h$ and let $f\mu$, with $f \in L^\infty_\mu(\mathbb{R}^n)$, be the weak limit of $\{(\dim T_{\mu_h})\mu_h\}_h$. Then we have

$$\dim T_\mu(x) \geq f(x) \qquad \text{for } \mu\text{-a.e. } x \; . \tag{3.3}$$

Proof. We apply Lemma 3.1, by taking $p = +\infty$, and as f_h the functions $P_h : \mathbb{R}^n \to \mathbb{R}^{n^2}$ which map μ_h-a.e. point $x \in \mathbb{R}^n$ into the matrix of the orthogonal projection of $\mathbb{R}^n$ on $T_{\mu_h}(x)$. By the first part of Lemma 3.1, if we pass to a subsequence we can suppose that

$$P_h\mu_h \rightharpoonup A\mu \; , \tag{3.4}$$

with A belonging to $L^\infty_\mu(\mathbb{R}^n, \mathbb{R}^{n^2})$. Now, let us denote by $\mathcal{S}_1(n \times n)$ the space of symmetric matrices M with spectrum $\sigma(M)$ contained in the closed interval $[0,1]$. We take as a function $\Phi : \mathbb{R}^{n^2} \to \mathbb{R} \cup \{+\infty\}$ the function

$$\Phi(M) = \begin{cases} 0 & \text{if } M \in \mathcal{S}_1(n \times n) \\ +\infty & \text{otherwise.} \end{cases}$$

Since by the second part of Lemma 3.1, we must have $\Phi(A) = 0$ μ-a.e., we get that A takes its values in the symmetric matrices with the spectrum contained in $[0,1]$, so that

$$\operatorname{rank}(A) \geq \operatorname{tr}(A) \qquad \mu\text{-a.e.} \tag{3.5}$$

Next, let us observe that (3.4) yields the weak convergence of $H(\mu_h)$ to $\operatorname{div} A\mu$ in the sense of distributions. Then by the hypothesis (3.2) on the boundedness of the curvatures in $\mathcal{M}^n$, we must also have the weak convergence of the measures $H(\mu_h)$ to $\operatorname{div} A\mu$. In particular $\operatorname{div} A\mu$ turns out to be a measure, and this gives the crucial information

$$\operatorname{rank}(A) \leq \dim T_\mu \qquad \mu\text{-a.e.} \tag{3.6}$$

Moreover, by taking the trace in (3.4), we obtain

$$(\dim T_{\mu_h})\mu_h \rightharpoonup (\operatorname{tr} A)\mu \; . \tag{3.7}$$

Taking into account (3.5), (3.6), and (3.7), we get (3.3); indeed we have

$$(\dim T_\mu)\mu \geq (\operatorname{rank} A)\mu \geq (\operatorname{tr} A)\mu = f\mu \; . \qquad \blacksquare$$

Remark 3.5. In particular, when $\dim T_\mu(x) = k$ for μ_h-a.e. x, with k a fixed integer, by the above theorem we get that the dimension of T_μ cannot be smaller than k on a set of positive measure μ.

Example 3.6. When the limit measure μ is identically zero, Theorem 3.4 gives evidently no information. In fact it is possible to approximate the zero measure with a sequence of measures bounded in $\mathcal{M}_{BC}$ and having a tangent space of positive dimension. For instance, we can take in $\mathbb{R}^2$

$$\mu_h = \mathcal{H}^1 \llcorner \partial B_{r_h} \ ,$$

where B_{r_h} are the balls of center the origin and radius r_h tending to zero. Then μ_h strongly converges to zero, the dimension of T_{μ_h} is equal one μ_h-a.e., and the total variation of $H(\mu_h)$ is equal 2π for every h.

Example 3.7. In (3.3) we can have the strict inequality $\dim T_\mu(x) > f(x)$ for μ-a.e. x. This happens for instance if we take in $\mathbb{R}$

$$\mu_h = \frac{1}{h} \sum_{i=1}^{h} \delta_{i/h} \ ,$$

where $\delta_{i/h}$ is the Dirac mass at i/h. Then μ_h have zero curvature and weakly converge to the Lebesgue measure on $[0,1]$, which has a one-dimensional tangent space.

We are now in a position to state our result of weak convergence for the curvatures. We deduce it as a direct consequence of Theorem 3.4.

Theorem 3.8. *Let $\{\mu_h\}_h$ be a bounded sequence in $\mathcal{M}_{BC}$ such that $\{\mu_h\}_h$ weakly converge to μ and $\{(\dim T_{\mu_h})\mu_h\}_h$ weakly converge to $f\mu$. Then the condition*

$$\dim T_\mu(x) \le f(x) \qquad \mu\text{-a.e.} \tag{3.8}$$

is necessary and sufficient to have

$$P_{\mu_h}\mu_h \rightharpoonup P_\mu \mu \ . \tag{3.9}$$

In this case we have also

$$H(\mu_h) \rightharpoonup H(\mu) \ . \tag{3.10}$$

Proof. By the same proof as in Theorem 3.4, after passing to a subsequence, we can suppose that $P_{\mu_h}\mu_h \rightharpoonup A\mu$, where the function A belongs to $L^\infty_\mu(\mathbb{R}^n, \mathbb{R}^{n^2})$, takes values in $\mathcal{S}_1(n\times n)$, and satisfies

$$(\dim T_\mu)\mu \ge (\operatorname{rank} A)\mu \ge (\operatorname{tr} A)\mu = f\mu \ .$$

Then, by assumption (3.8) we infer

$$\dim T_\mu = \operatorname{rank} A = \operatorname{tr} A \qquad \mu\text{-a.e.}$$

In particular, if we recall that the columns of the matrix A belong to X_μ and that $\sigma(A) \subset [0,1]$, the above equality means that, for μ-a.e. x, $A(x)$ is the orthogonal projection on $T_\mu(x)$ *i.e.* $A = P_\mu$. Then, since the selected subsequence was arbitrary, we get (3.9). Finally (3.10) follows, because the sequence $\{H(\mu_h)\}_h$ is bounded in $\mathcal{M}^n$ and, by (3.9), converges to $H(\mu)$ in the distributional sense.
The converse implication is trivial, since (3.9) implies

$$(\operatorname{tr} P_{\mu_h})\mu_h \rightharpoonup (\operatorname{tr} P_\mu)\mu = f\mu \ ,$$

so that $f(x) = \dim T_\mu(x)$ for μ-a.e. x. ∎

Remark 3.9. Theorem 3.8 reduces the problem of weak convergence of curvatures to the check of the lower semicontinuity condition (3.8) for the dimension of the tangent space. Such condition seems to play a crucial role and brings to the fore a quite involved geometrical problem: does it exist any criterion to decide whether (3.8) is satisfied or not? Section 4 partially answers to this question, by providing some conditions which are sufficient to have (3.8).

A relevant consequence of Theorem 3.8 is the lower semicontinuity of a large class of functionals of the form (3.1).

Corollary 3.10. *Let J be a functional defined on the product space $\mathcal{M}^+ \times \mathcal{M}^n$ and let $\mathcal{A}$ be a subclass of measures in $\mathcal{M}_{BC}$. Assume that*

(i) J is weakly lower semicontinuous and coercive i.e.

$$\lim_{|\lambda|+|\mu|\to+\infty} J(\lambda,\mu) = +\infty \ ,$$

(ii) For every $M > 0$, the set $\{(\mu,\nu) \mid \mu \in \mathcal{A} \ , \ |H(\mu)| \le M \ , \ \nu \ge \dim(T_\mu)\,\mu\}$ is weakly closed in $\mathcal{M}_{BC} \times \mathcal{M}^+$.

Then the functional $F(\mu)$ defined by (3.1) is weakly lower semicontinuous and achieves its infimum over $\mathcal{A}$.

Proof. Let $\{\mu_h\}_h \subset \mathcal{A}$ be a sequence such that $\mu_h \rightharpoonup \mu$ and set $\nu_h := \dim(T_{\mu_h})\,\mu_h$. Possibly passing to a subsequence, we can assume that $\liminf_{h\to+\infty} F(\mu_h) = \lim_{h\to+\infty} F(\mu_h) < +\infty$. Then by the coercivity assumption i) and by Lemma 3.1 we can find $C > 0$ and $f \in L^\infty_\mu$ such that

$$|\mu_h| + |H((\mu_h)| \le C \qquad , \qquad \nu_h \rightharpoonup f\,\mu \ .$$

Hence we can use assumption ii) to obtain that μ still belongs to the class $\mathcal{A}$ and that the inequality $f \geq \dim(T_\mu)$ holds μ-almost everywhere. Theorem 3.8 leads to conclude that $H(\mu_h)$ does converge weakly to $H(\mu)$ in $\mathcal{M}^n$ and then, by the lower semicontinuity of J, we have

$$\begin{aligned}\liminf_{h\to+\infty} F(\mu_h) \geq &\liminf_{h\to+\infty} J(\mu_h, H(\mu_h)) \\ &\geq J(\mu, H(\mu)) \;=\; F(\mu)\ .\end{aligned}$$

Finally, by condition i), F is coercive and therefore achieves its infimum. ∎

4. Lower semicontinuity of the dimension of tangent space

As pointed out in the previous section, the dimension of the tangent space for measures in the space $\mathcal{M}_{BC}$ cannot decrease under a uniform control of the curvatures and, in order to obtain the weak convergence of the curvatures, we need only to check the lower semicontinuity of the dimension of T_μ (see condition (3.8)). To this aim we will use as a main tool Lemma 2.9 and the monotonicity property for measures in $\mathcal{M}_{BC}$.
Since it seems difficult to extend this monotonicity property to measures with varying dimension, we focus here our attention on the case of a sequence of measures $\{\mu_h\}_h$ whose tangent space has constant dimension k. As noticed in Remark 3.5, we are able in this case to prove that the dimension of the limit measure μ must be greater than or equal to k.
A very useful criterion to have the opposite inequality is given in Lemma 4.1. This lemma is deduced from a result proved in [42]; we use the standard notation $\theta^*_k(\mu,x)$ to indicate the k-dimensional upper density of μ at the point x, given by

$$\theta^*_k(\mu,x) := \limsup_{\rho\to 0} \frac{\mu\big(B_\rho(x)\big)}{\omega_k \rho^k}\ .$$

Lemma 4.1. *Let k be an integer, $0 \leq k \leq n$, and let E be a Borel subset of $\mathbb{R}^n$. Then*

$$\theta^*_k(\mu,x) > 0 \quad \mu\text{-a.e. on } E \implies \dim T_\mu(x) \leq k \quad \mu\text{-a.e. on } E\ .$$

Proof. For any $\alpha \in (k, k+1)$, we have $\theta^*_\alpha(\mu,x) = +\infty$ μ-a.e. on E. Then it is sufficient to apply Lemma 4.1 of [42] to deduce that we have $\dim T_\mu \leq [\alpha] = k$ for μ-a.e. $x \in E$. ∎

Another preliminary lemma deals with a generalization of known results about the Hardy-Littlewood maximal operator. Given two measures λ and μ in $\mathcal{M}^+$, we denote by $M(\lambda,\mu)$ the maximal function of λ with respect to μ, defined by

$$M(\lambda,\mu)(x) := \sup_{\rho>0} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))}\ .$$

Lemma 4.2. *Given λ, $\mu \in \mathcal{M}^+$, the following facts hold:*

i) there exists a constant $c = c(n)$ such that for every $t > 0$

$$\mu\left(\{x \ : \ M(\lambda,\mu)(x) > t\}\right) \leq \frac{c(n)|\lambda|}{t} \ ;$$

ii) if $\lambda \ll \mu$, with $\frac{d\lambda}{d\mu} \in L^p_\mu(\mathbb{R}^n)$ for some $p > 1$, then $M(\lambda,\mu) \in L^p_\mu(\mathbb{R}^n)$; moreover there exists a positive constant $c = c(n,p)$ such that

$$\|M(\lambda,\mu)\|_{L^p_\mu} \leq c(n,p) \left\| \frac{d\lambda}{d\mu} \right\|_{L^p_\mu} .$$

Proof. We refer to the book by Mattila [50], Theorem 2.19. ■

We are now in a position to state the main theorem of the section. Let us fix some integer $k \in [0,n]$ and denote by $\mathcal{M}_k$ the subset of measures $\mu \in \mathcal{M}_{BC}$ such that $\dim T_\mu(x) = k$ for μ-a.e. x.

Theorem 4.3. *Let $\{\mu_h\}_h$ be a sequence in $\mathcal{M}_k$ such that*

$$\mu_h \rightharpoonup \mu \ , \qquad \sup_h |H(\mu_h)| = C < +\infty.$$

Let θ_h and θ denote the k-dimensional (upper) density of μ_h and μ at x. Then

$$\liminf_{h\to+\infty} \ \|(\theta_h)^{-1}\|_{L^\infty_{\mu_h}} \ \geq \ \|\theta^{-1}\|_{L^\infty_\mu} \ .$$

If moreover μ_h satisfies $H(\mu_h) \ll \mu_h$ and $\sup_h \|h(\mu_h)\|_{L^p_{\mu_h}} < +\infty$ for some $p > 1$, then

$$\liminf_{h\to+\infty} \int (\log\theta_h)^-\, d\mu_h \ \geq \ \int (\log\theta)^- d\mu \ .$$

Proof. We set

$$f_h(x,\rho) := \frac{\mu_h(B_\rho(x))}{\omega_k \rho^k} \quad , \quad M_h := M(|H(\mu_h)|, \mu_h)$$

and we observe that, by (2.4) (see proof of Lemma 2.9) and by the definition of M_h, we have

$$\frac{1}{f_h(x,\rho)} \frac{\partial}{\partial\rho} f_h(x,\rho) \geq -\frac{|H(\mu_h)|(B_\rho(x))}{\mu_h(B_\rho(x))} \geq -M_h(x) \ .$$

Hence, by monotonicity, for every $\rho > 0$ and for every integer h, the following inequality holds μ_h-a.e.

$$\theta_h(x) \leq \exp(M_h(x)\rho)\, f_h(x,\rho) \ . \tag{4.1}$$

First, let us prove the statement about the lower semicontinuity of $\|\theta^{-1}\|_{L^\infty_\mu}$. Fix $\alpha > 0$. It is enough to prove that if $\theta_h \geq \alpha$ holds μ_h-a.e. (for every h), then $\theta \geq \alpha$ μ-a.e. Let B_{ht} be the subset of $\mathbb{R}^n$ where M_h is greater than t. By Lemma 4.2 (i)

$$\mu_h(B_{ht}) \leq \frac{c(n)}{t}|H(\mu_h)| \leq \frac{c(n)C}{t} \ .$$

We set

$$N := \bigcap_{m=1}^{\infty} \bigcup_{h=1}^{\infty} \left[\operatorname{int} \bigcap_{l \geq h} B_{lm} \right]$$

It is easy to check that $\mu(N) = 0$. In fact for every $m \in \mathbb{N}$ we have

$$\mu(N) \leq \mu \left\{ \bigcup_{h=1}^{\infty} \left[\operatorname{int} \bigcap_{l \geq h} B_{lm} \right] \right\} = \lim_{h \to +\infty} \mu \left\{ \operatorname{int} \bigcap_{l \geq h} B_{lm} \right\} \leq$$

$$\leq \liminf_{h \to +\infty} \mu_h \left\{ \bigcap_{l \geq h} B_{lm} \right\} \leq \liminf_{h \to +\infty} \mu_h(B_{lm}) \leq \frac{c(n)C}{m} \ .$$

Let us prove now that $\theta \geq \alpha$ outside N. If $x \notin N$, we can find $\overline{m} \in \mathbb{N}$, a sequence of points $\{x_h\}_h$, and a sequence of integers $\{l(h)\}_h$, with $l(h) \geq h$, such that $|x - x_h| \leq 1/h$ and $x_h \notin B_{l(h)\overline{m}}$. In particular $M_{l(h)}(x_h) \leq \overline{m}$; thus if we take into account (4.1) we get

$$\alpha \leq \theta_{l(h)}(x_h) \ \leq \ \exp(\overline{m}\rho) \ f_{l(h)}(x_h, \rho) \qquad \forall \rho > 0 \ , \forall h \ . \tag{4.2}$$

Now, since the inclusion $\overline{B}_\rho(x_h) \subset \overline{B}_{\rho'}(x)$ holds eventually for every $\rho' > \rho$, the weak convergence of $\{\mu_h\}_h$ to μ implies that

$$\limsup_{h \to +\infty} \mu_h(\overline{B}_\rho(x_h)) \leq \mu(\overline{B}_\rho(x)) \ .$$

Passing to the limsup in (4.2) and observing that the equality $\mu(B_\rho(x)) = \mu(\overline{B}_\rho(x))$ holds for a dense subset of values of ρ, we infer that $\dfrac{\mu(B_\rho(x))}{\omega_k \rho^k} \geq \alpha$ for every ρ. The conclusion follows by taking the limsup as ρ tends to 0.

Now, let us prove the second part of the thesis.

Let us assume that the sequence $\{(\log \theta_h(x))^-\}_h$ is bounded in $L^1_{\mu_h}$ (otherwise there is nothing to prove). Then θ_h is strictly positive μ_h-a.e. and by (4.2) we have

$$(\log \theta_h(x))^- \geq (\log f_h)^-(x, \rho) - M_h(x)\rho \ .$$

Integrating this inequality with respect to μ_h, we obtain for every $\rho > 0$

$$\begin{aligned} \liminf_{h \to +\infty} \int (\log \theta_h)^- \, d\mu_h &\geq \liminf_{h \to +\infty} \int (\log f_h)^- \, d\mu_h \ - \ \rho \limsup_{h \to +\infty} \int M_h(x) \, d\mu_h \\ &= (\mathrm{I}) - (\mathrm{II}). \end{aligned} \tag{4.3}$$

Let us consider separately terms (I) and (II). Term (II) can be estimated by using Hölder inequality with some $p > 1$, Lemma 4.2 (ii), and the boundedness of the sequence $\{\mu_h\}_h$:

$$\rho \int M_h(x)\varphi(x)\, d\mu_h \leq \rho \int M_h(x)\, d\mu_h \leq \rho \|M_h\|_{L^p_{\mu_h}} |\mu_h|^{1/p'}$$
$$\leq \rho c(p) \|h(\mu_h)\|_{L^p_{\mu_h}} |\mu_h|^{1/p'} \leq \Lambda \rho \ ,$$

where Λ is a suitable positive constant.
As for term (I), we claim that, except possibly for ρ in a countable subset, we have

$$\liminf_{h\to+\infty} \int (\log f_h(x,\rho))^- \, d\mu_h \geq \int (\log f(x,\rho))^- \, d\mu \ , \tag{4.4}$$

where $f(x,\rho) := \dfrac{\mu(B_\rho(x))}{\omega_k \rho^k}$. Then, putting together the estimates of terms (I) and (II), (4.3) becomes

$$\liminf_{h\to+\infty} \int (\log \theta_h)^- \, d\mu_h \geq \int (\log f(x,\rho))^- \, d\mu \ - \Lambda\rho \ .$$

In view of the definition of θ, we have $\liminf\limits_{\rho\to 0} (\log f(x,\rho))^- \geq (\log \theta(x))^-$. We conclude by letting ρ tend to 0 and by using Fatou's lemma

$$\liminf_{h\to+\infty} \int (\log \theta_h)^- \, d\mu_h \geq \liminf_{\rho\to 0} \int (\log f(x,\rho))^- \, d\mu \geq \int (\log \theta(x))^- \, d\mu \ .$$

It remains to prove claim (4.4). To this aim, we note that we can choose ρ such that we have $\mu \otimes \mu(\{(x,y) \ : \ |x-y| = \rho\})$ equal to zero. Then, observing that the product measures $\{\mu_h \otimes \mu_h\}_h$ weakly converge to $\mu \otimes \mu$ and using Fubini's Theorem, we infer that, for every continuous compactly supported test function φ

$$\lim_{h\to+\infty} \int \mu_h(B_\rho(x))\varphi(x)\, d\mu_h = \lim_{h\to+\infty} \int\int_{\{|x-y|<\rho\}} \varphi(x)\, d(\mu_h \otimes \mu_h)$$
$$= \int\int_{\{|x-y|<\rho\}} \varphi(x)\, d(\mu \otimes \mu) = \int \mu(B_\rho(x))\varphi(x)\, d\mu \ .$$

In other words, we have the weak convergence of the sequence $\{f_h(\cdot,\rho)\, \mu_h\}_h$ to $f(\cdot,\rho)\, \mu$. Then we can apply Lemma 3.1 with $f_h(x) := f_h(x,\rho)$, $f(x) := f(x,\rho)$ and the convex function $\Phi(t) := (\log t)^-$. Thus claim (4.4) holds, and this achieves our proof. ∎

By using Theorem 3.8, Lemma 4.1, and Theorem 4.3, we immediately deduce the following corollary.

Corollary 4.4. *Let $\{\mu_h\}_h$ be a sequence in $\mathcal{M}_k$ weakly converging to μ. Assume that for some positive constants C and α, one of the following conditions holds*

(i) $$\theta_h \geq \alpha \quad \mu_h\text{-a.e.}, \quad |H(\mu_h)| \leq C$$

(ii) $$\int (\log \theta_h)^- \, d\mu_h \leq C, \quad \|h(\mu_h)\|_{L^p_{\mu_h}} \leq C \quad (H(\mu_h) \ll \mu_h) \ .$$

Then

$$\mu \in \mathcal{M}_k \quad , \quad P_{\mu_h}\mu_h \rightharpoonup P_\mu \mu \quad , \quad H(\mu_h) \rightharpoonup H(\mu) \ .$$

Remark 4.5. If condition (i) holds, then, by Theorem 4.3, also the k-dimensional density θ of μ will satisfy the inequality $\theta \geq \alpha$ μ-a.e.

5. Some examples and applications

It may be interesting to apply our results to prove the existence of a solution for minimum problems involving measures. For instance the problem of minimizing some functionals on a certain class of measures arises in a quite natural way in the framework of shape optimization (see [19]). Here we give, as a simple example, an existence result for a minimum problem on $\mathcal{M}_{BC}$ under a " Dirichlet boundary condition".
We denote by $\mathcal{M}_{k,\alpha}$ the subset of $\mathcal{M}_k$ given by all measures $\mu \in \mathcal{M}_k$ with k-dimensional density greater than or equal to α.

Lemma 5.1. *Let $\Phi : \mathbb{R}^n \to [0,+\infty)$ be a convex, lower semicontinuous, and proper function satisfying $\lim\limits_{|z|\to\infty} (\Phi(z)/|z|) = +\infty$. Let us fix an integer k, $\alpha > 0$ and $\lambda \in \mathcal{M}^n$ such that the set $\{\mu \in \mathcal{M}_{k,\alpha} : \ \partial\mu = \lambda\}$ is not empty. Then the minimum problem*

$$(\mathcal{P}) \qquad \min\left\{ \int (1 + \Phi(h(\mu))\, d\mu \ : \ \mu \in \mathcal{M}_{k,\alpha} \ , \ \partial\mu = \lambda \right\}$$

admits a solution.

Proof. We apply Corollary 3.10, with

$$J(\mu,\nu) := \begin{cases} \int (1 + \Phi(f))\, d\mu & \text{if } \nu = \ f\,\mu + \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

The functional J is coercive on $\mathcal{M}^+ \times \mathcal{M}^n$. In order to check that it is also weakly lower semicontinuous, it is enough to apply Lemma 3.2 writing J as $J(\mu,\nu) = \int \Psi(\nu - \lambda, \mu)$ where Ψ satisfies $\Psi(z,t) = t\,\Phi(z/t) + t$ if $t > 0$, $\Psi(0,0) = 0$ and $\Psi(z,t) = +\infty$ otherwise. Since $J(\mu, H(\mu)) < +\infty$ implies that $\partial\mu = \lambda$, we only need to check that, taking $\mathcal{A} = \mathcal{M}_{k,\alpha}$, assumption ii) of Corollary 3.10 is satisfied. This is a straightforward consequence of Corollary 4.4 and Remark 4.5. ∎

Remark 5.2. The minimum problem mentioned in the introduction is obtained by taking $\Phi(z) = |z|^2$ in the lemma above. We think that in this case the assumption that all competing measures must have a given constant dimension for their tangent space is superfluous; in other words, we believe that for every measure λ the minimum problem

$$\min\left\{ \int (1 + |h(\mu)|^2)\, d\mu \ : \ \partial\mu = \lambda \right\}$$

admits a solution, but at this stage the proof still seems rather difficult.

Remark 5.3. It is possible to prove that Lemma 5.1 still holds if we replace the admissible class $\mathcal{M}_{k,\alpha}$ by the set of measures μ in $\mathcal{M}_k$ such that the k-dimensional density $\theta_k(\mu, x)$ is a positive integer. In this case $\theta_k(\mu, x)$ is called the *integer multiplicity* of μ (see [64]).
In particular, when $k = 1$ and $\lambda = v_A \delta_A + v_B \delta_B$, the minimum problem $(\mathcal{P})$ becomes a weak version of the problem of finding the optimal curve, having endpoints A, B and prescribed tangent vectors v_A, v_B, which minimizes a functional depending on the pointwise curvature (see [37], [49]). However, we do not impose here any topological constraint as connection: thus, in some particular cases, when A and B are far enough, we may get solutions given by two disjoint loops starting from the points A and B.

IV. BV functions with respect to a measure and relaxation of metric integral functionals *

1. Introduction

In this paper we define the space of bounded variation functions with respect to a Radon measure μ, and we study some of its properties. Our approach is inspired by [18], where the Sobolev-type spaces $W_\mu^{1,p}$ associated with a measure μ are introduced, and the relaxation of integral functionals on $W_\mu^{1,p}$ is studied for $p > 1$. We focus our attention on the relaxation in L_μ^1 of integral functionals with respect to μ, where the integrand is a sub-linear function φ. Following the geometric approach proposed in [7], where $\mathbb{R}^n$ is viewed as a Banach space endowed with a Finsler metric, one can look at the integrand φ as a metric: this leads to give, for any $u \in L_\mu^1(\mathbb{R}^n)$, a natural distributional definition of (μ, φ)-total variation $|D_\mu u|_\varphi$. The consistency of our definition with the usual notion of $|Du|$ when μ is the Lebesgue measure and φ is the euclidean metric is proved in Section 3 (see Proposition 3.1). We next introduce the space $BV_{\mu,\varphi}$ as the class of all functions $u \in L_\mu^1(\mathbb{R}^n)$ with $|D_\mu u|_\varphi < +\infty$.

In Section 4 we enlighten the interest of our (μ, φ)-total variation by showing a quite general coarea formula in L_μ^1. When applied to particular measures μ, this formula clearly encompasses previous generalizations (see [41], [9], [43], [63]). We stress that our proof technique, suggested by the use of distributional definitions, is different from the classical methods based on the approximation with smooth functions [39]; namely we use a commutation argument from [21] between supremum and integral. As a corollary, we show that the chain rule holds for functions in the Sobolev space $W_\mu^{1,1}$.

In Section 5 we show that $|D_\mu u|_\varphi$ and $BV_{\mu,\varphi}$ coincide respectively with the relaxation on L_μ^1 of an integral functional, and with its domain of finiteness (see Theorem 5.1); furthermore, under suitable regularity assumptions, we prove an integral representation theorem for $|D_\mu u|_\varphi$ on the Sobolev space $W_\mu^{1,1}$ (which is a strict subspace of $BV_{\mu,\varphi}$).

The problem of extending this type of result out of $W_\mu^{1,1}$ is quite delicate (see [20] for the integral representation in the classical space $BV(\Omega)$), and is studied in Section 6. Here we provide some sufficient conditions in order to have an integral representation for $|D_\mu u|_\varphi$ on $BV_{\mu,\varphi}$; we also give a counterexample showing that such a representation does not hold for a general measure μ.

---

* Paper by G. Bellettini, G. Bouchitté, and I. Fragalà, to appear on Convex Analysis (1999) [13].

2. Notation

For a positive integer d, let $\mathcal{R}^d$ be the class of all $\mathbb{R}^d$-valued Borel measures with finite total variation on $\mathbb{R}^n$; when $d = 1$, we simply denote by $\mathcal{R}$ the space of signed Borel measures with finite total variation on $\mathbb{R}^n$, and we let $\mathcal{R}_+$ be the subclass of $\mathcal{R}$ given by all positive and finite Borel measures.
For $\mu \in \mathcal{R}_+$ we denote by $\mu \llcorner E$ the restriction measure of μ to a μ-measurable subset E of $\mathbb{R}^n$, and by $\operatorname{spt}\mu$ the support of μ. Whenever dealing with integrals with respect to μ on $\mathbb{R}^n$, we omit the integration domain. For any $k \in [0, N)$, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure and $\mathcal{L}^N$ is the Lebesgue measure. If α is a vector-valued measure with finite total variation, the polar decomposition of α is given by $\alpha = \theta|\alpha|$, where $|\alpha|$ denotes the positive total variation measure of α, and the density $\theta := \frac{d\alpha}{d|\alpha|}$ has unitary norm $|\alpha|$-a.e. ; by writing $\alpha \ll \mu$, we mean that $|\alpha|$ is absolutely continuous with respect to μ. For $p = 1$ or $p = +\infty$, we set $L^p_\mu := L^p(\mathbb{R}^n, d\mu)$ and $(L^p_\mu)^N := (L^p(\mathbb{R}^n, d\mu))^N$; the subscript μ is omitted when $\mu = \mathcal{L}^N$. The spaces of continuous functions, of continuous vector fields, and of continuous vector fields vanishing at infinity on $\mathbb{R}^n$ are denoted respectively by $\mathcal{C}$, $\mathcal{C}^N$, $\mathcal{C}^N_0$. Unless otherwise specified, the symbol of duality $\langle \cdot, \cdot \rangle$ is used for the pairing between L^∞_μ and L^1_μ, while the euclidean norm and scalar product between two vectors z and z' of $\mathbb{R}^n$ are denoted by $|\cdot|$ and $z \cdot z'$. For a subset E of $\mathbb{R}^n$ we denote by χ_E the characteristic function of E.
We set

$$X_\mu := \{\sigma \in (L^\infty_\mu)^N \ : \ \operatorname{div}(\sigma\mu) \in L^\infty_\mu\} \ ; \tag{2.1}$$

in (2.1) we call $\operatorname{div}(\sigma\mu)$ the distribution whose action on a test function $\psi \in \mathcal{D} := \mathcal{C}^\infty_c(\mathbb{R}^n)$ is given by

$$\langle \operatorname{div}(\sigma\mu), \psi \rangle_{(\mathcal{D}', \mathcal{D})} := - \int \sigma \cdot \nabla\psi \, d\mu \ . \tag{2.2}$$

In other words, an element σ of $(L^\infty_\mu)^N$ belongs to X_μ if and only if there exists a constant $C \in [0, +\infty)$ such that $|\int \sigma \cdot \nabla\psi| \, d\mu \ \le \ C \, \|\psi\|_{L^1_\mu}$ for every $\psi \in \mathcal{D}$. The explicit expression of $\operatorname{div}(\sigma\mu)$ for a special choice of μ is given in (3.5).
Similarly to [18, Section 2], we define the tangent space to μ at a point $x \in \mathbb{R}^n$ as

$$T_\mu(x) := \mu - \operatorname{ess} \bigcup \{\sigma(x) \ : \ \sigma \in X_\mu\} \ ;$$

in particular, for any $\sigma \in X_\mu$ we have $\sigma(x) \in T_\mu(x)$ for μ-a.e. x.
We always assume that $T_\mu(x)$ is not reduced to zero μ-a.e. , and we set

$$T\mu := \{(x, z) \ : \ x \in \mathbb{R}^n \, , \ z \in T_\mu(x)\} \ .$$

When $\mu = \mathcal{L}^N \llcorner \Omega$, where Ω is an open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary, we have $T_\mu(x) = \mathbb{R}^n$ for μ-a.e. x, while for $\mu = \mathcal{H}^k \llcorner M$, where M is a Lipschitz k-manifold, T_μ coincides μ-a.e. with the usual tangent space to M. For further properties of T_μ we refer to [42]. For $\psi \in \mathcal{D}$ and μ-a.e. $x \in \mathbb{R}^n$, the symbol $\nabla_\mu \psi(x)$ stands for $P_\mu(x)[\nabla\psi(x)]$, where $P_\mu(x)[\,\cdot\,]$ is the orthogonal projection of $\mathbb{R}^n$ onto $T_\mu(x)$.
A Sobolev-type space $W^{1,1}_\mu$ can be defined, following [18], as the completion of $\mathcal{D}$ with respect to the norm $\|\psi\|_{L^1_\mu} + \|\nabla_\mu \psi\|_{(L^1_\mu)^N}$. Thus any function $u \in W^{1,1}_\mu$ admits a *tangential gradient* $\nabla_\mu u \in (L^1_\mu)^N$, and the following integration by parts formula holds:

$$
-\langle \operatorname{div}(\sigma\mu), u\rangle = \int \sigma \cdot \nabla_\mu u \, d\mu \ , \qquad \sigma \in X_\mu \ , \ u \in W^{1,1}_\mu \ . \tag{2.3}
$$

For the proof of (2.3), which is the same as in the case $p > 1$, we refer to [18].
If $(\eta_i)_{i\in I}$ is an arbitrary family of μ-measurable closed-valued multifunctions from $\mathbb{R}^n$ to $\mathbb{R}^n$, we denote by $\mu - \operatorname{ess\,sup}_{i\in I}\eta_i$ the multifunction η from $\mathbb{R}^n$ to $\mathbb{R}^n$ characterized (up to μ-negligible sets) by the two properties (see [68]):

(i) $\eta_i(x) \subseteq \eta(x)$ for μ-a.e. x , $\forall i \in I$;
(ii) η is minimal with respecto to inclusion μ-a.e. (*i.e.* for any other μ-measurable and closed-valued multifunction θ such that, for every $i \in I$, $\eta_i(x) \subseteq \theta(x)$ for μ-a.e. x, there holds $\eta(x) \subseteq \theta(x)$ for μ-a.e. x).

We say that a function $\varphi : T\mu \to [0, +\infty)$ is a metric integrand, and we write $\varphi \in \mathcal{M}$, if the following conditions are satisfied:

$$
\text{for any } z \in \mathbb{R}^n \ , \text{ the map } x \mapsto \varphi(x, P_\mu(x)[z]) \text{ is } \mu\text{-measurable on } \mathbb{R}^n \ ; \tag{2.4}
$$

$$
\text{for } \mu\text{-a.e. } x \in \mathbb{R}^n \ , \text{ the map } z \mapsto \varphi(x,z) \text{ is convex on } T_\mu(x) \ ; \tag{2.5}
$$

$$
\text{there exists } C > 0 \text{ such that } \varphi(x,z) \le C|z| \ \text{ for } (x,z) \in T\mu \ ; \tag{2.6}
$$

$$
\varphi(x, tz) = t\varphi(x,z) \text{ for } (x,z) \in T\mu \ , \ t > 0 \ . \tag{2.7}
$$

We associate with φ the dual metric φ^o defined on the dual bundle $T^*\mu$ of $T\mu$ as

$$
\varphi^o(x, z^*) := \sup\{z \cdot z^* \ : \ z \in T_\mu(x) \ , \ \varphi(x,z) \le 1\} \ .
$$

Notice that $z \cdot z^* \le \varphi(x,z)\,\varphi^o(x,z^*)$ for any $z \in T_\mu(x)$, $z^* \in T^*_\mu(x)$, and that $\varphi^{oo} = \varphi$. Furthermore, the metric φ^o still enjoys properties (2.4), (2.5), and (2.7) on $T^*\mu$ (see for instance [27]).

Finally, we recall that, if J is a proper functional defined on a Banach space X with values in $\mathbb{R} \cup \{+\infty\}$, the relaxed functional $\overline{J}$ of J is defined as the greatest lower semicontinuous functional less than or equal to J on X (see [26]), while the Fenchel conjugate functional J^* of J is defined on the dual space of X (see [61]) by $J^*(x^*) := \sup\{\langle x \,,\, x^*\rangle - J(x) \ : \ x \in X\}$.

3. The (μ,φ)-total variation

Some generalized notions of bounded variation functions have been proposed in the literature: for instance the theory of perimeters on a smooth manifold has been studied in [52], while more recently the class of BV functions over rectifiable currents has been introduced in [9]. Our definition reads as follows.

Let $\mu \in \mathcal{R}_+$ and $\varphi \in \mathcal{M}$. For every function $u \in L^1_\mu$ we define the (μ,φ)-total variation $|D_\mu u|_\varphi$ of u as

$$|D_\mu u|_\varphi := \sup\{-\langle \mathrm{div}(\sigma\mu), u\rangle \ : \sigma \in X_\mu\,,\ \varphi^o(x,\sigma(x)) \leq 1 \text{ for } \mu\text{-a.e. } x\} \tag{3.1}$$

and we set

$$BV_{\mu,\varphi} := \{u \in L^1_\mu \ : \ |D_\mu u|_\varphi < +\infty\}\ .$$

In particular, if E is a μ-measurable subset of $\mathbb{R}^n$ such that $\chi_E \in L^1_\mu$, we define the (μ,φ)-perimeter $P_{\mu,\varphi}(E)$ of E as $P_{\mu,\varphi}(E) := |D_\mu \chi_E|_\varphi$.

It immediately follows that the functionals $u \mapsto |D_\mu u|_\varphi$ and $E \mapsto P_{\mu,\varphi}(E)$ are lower semicontinuous with respect to the convergence in L^1_μ. Notice also that, whenever φ satisfies a coercivity-type condition of the form

$$\varphi(x,z) \geq C^{-1}|z|\ , \qquad (x,z) \in T\mu\ , \tag{3.2}$$

the space $BV_{\mu,\varphi}$ is independent of φ; in particular, when $\varphi(x,z) = |z|$, $BV_{\mu,\varphi}$ will be denoted by BV_μ.

Let us now show that, if $\mu = \mathcal{L}^N \llcorner \Omega$ and $\varphi(x,z) = |z|$, then $|D_\mu u|_\varphi$ and $BV_{\mu,\varphi}$ reduce to the usual notions $|Du|(\Omega)$ and $BV(\Omega)$, respectively. To this aim, we recall (see [8], Theorem 1.9 with $p = N$ and $q = N/(N-1)$) that, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $u \in BV(\Omega)$, and $\sigma \in (L^\infty(\Omega))^N$ with $\mathrm{div}\,\sigma \in L^N(\Omega)$, then it is possible to define, in a natural way, a real valued measure (σ, Du) satisfying $\int_\Omega (\sigma, Du) \leq \|\sigma\|_{L^\infty(\Omega)}|Du|(\Omega)$, and a trace $[\sigma\cdot\nu] \in L^\infty(\partial\Omega)$ on $\partial\Omega$ of the normal component of σ, such that the following Gauss-Green formula holds:

$$\int_\Omega u\,\mathrm{div}\,\sigma\,dx + \int_\Omega (\sigma, Du) = \int_{\partial\Omega} [\sigma\cdot\nu]u\,d\mathcal{H}^{N-1} \tag{3.3}$$

(here ν is the outward unit normal to $\partial\Omega$).

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, let $\mu := \mathcal{L}^N \llcorner \Omega$ and $\varphi(x,z) := |z|$. Then $|D_\mu u|_\varphi = |Du|(\Omega)$ for every $u \in L^1_\mu = L^1(\Omega)$, hence $BV_{\mu,\varphi} = BV(\Omega)$.*

Proof. Given $\sigma \in (L^\infty(\Omega))^N$ with $\mathrm{div}\,\sigma \in L^\infty(\Omega)$, and $\psi \in \mathcal{D}$, (3.3) yields:

$$\langle \mathrm{div}(\sigma\mu), \psi\rangle_{(\mathcal{D}',\mathcal{D})} = -\int_\Omega \sigma\cdot\nabla\psi\,dx = \int_\Omega \psi\,\mathrm{div}\,\sigma\,dx - \int_{\partial\Omega} [\sigma\cdot\nu]\psi\,d\mathcal{H}^{N-1}\ .$$

In order to have $\sigma \in X_\mu$, the right hand side member needs to be controlled by the norm of ψ in $L^1(\Omega)$. This yields $[\sigma \cdot \nu] = 0$ $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$. Therefore

$$X_\mu = \left\{\sigma \in (L^\infty(\Omega))^N \ : \ \operatorname{div}\sigma \in L^\infty(\Omega)\ ,\ [\sigma\cdot\nu] = 0 \ \ \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega\right\} \ . \tag{3.4}$$

The inclusion $(\mathcal{C}^1_c(\Omega))^N \subseteq X_\mu$ and the fact that $\varphi^o(x,z) = |z|$ imply that $|D_\mu u|_\varphi \geq |Du|(\Omega)$, hence $BV_{\mu,\varphi} \subseteq BV(\Omega)$. It remains to prove that $|D_\mu u|_\varphi \leq |Du|(\Omega)$. For any $u \in BV(\Omega)$ and $\sigma \in X_\mu$, by (3.3) and (3.4) we have

$$-\langle \operatorname{div}(\sigma\mu), u\rangle = -\int_\Omega u \operatorname{div}\sigma\, dx = \int_\Omega (\sigma, Du) \ .$$

Hence, if $\|\sigma\|_{(L^\infty(\Omega))^N} \leq 1$, we get $-\langle \operatorname{div}(\sigma\mu), u\rangle \leq |Du|(\Omega)$. Passing to the supremum over σ, we deduce $|D_\mu u|_\varphi \leq |Du|(\Omega)$. As a consequence, $BV_{\mu,\varphi} = BV(\Omega)$. ∎

Example 3.2. A natural case to be considered is when $\mu = a\mathcal{H}^k \llcorner M$, where M is a smooth connected k-manifold and φ is a continuous metric integrand on the tangent bundle to M. If the density a is a positive function in $L^\infty_\mu \cap \mathcal{C}^1$ with $\nabla \log a \in (L^\infty_\mu)^N$, and σ is a $\mathcal{C}^1$ tangent field to M, one can easily check, using the divergence theorem on a smooth manifold (see for instance [64]), that $\sigma \in X_\mu$, as

$$\operatorname{div}(\sigma\mu) = (\operatorname{div}_\mu \sigma + \sigma\cdot\nabla \log a)\mu \ , \tag{3.5}$$

where $\operatorname{div}_\mu \sigma := \sum_{i=1}^N (\nabla_\mu \sigma^i)_i$. Moreover, essentially due to the smoothness of $\operatorname{spt}\mu$, we will show in Section 6 that, as in the classical case $\mu = \mathcal{L}^N \llcorner \Omega$, there exists a vector valued measure $D_\mu u$ which allows to give an integral representation of the (μ,φ)-total variation on the whole space $BV_{\mu,\varphi}$.

Example 3.3. We point out that the choice

$$\mu := \left[\pi^{-\frac{N}{2}}(\lambda_1\cdot\ldots\cdot\lambda_N)^{-\frac{1}{2}} \exp\left(-\sum_{i=1}^N x_i^2/\lambda_i\right)\right]\mathcal{L}^N \ , \quad \varphi(z) := \left(\sum_{i=1}^N \eta_i z_i^2\right)^{1/2} \ ,$$

where λ_i, η_i, $i = 1,\ldots,N$, are suitable positive real numbers, could relate, when $N \to +\infty$, the space $BV_{\mu,\varphi}$ to a theory of perimeters in infinite-dimensional Hilbert spaces (see [36] and [50]).

4. Coarea formula

Several generalizations of the classical coarea-formula for BV functions [39, Section 5.5] have been proposed in the literature (see for instance [41, 3.2.22], [9, Proposition 2.13], [43, Theorem 2.3.5] , [63, Theorem 17.1]). We now present a coarea-type formula, which holds on L^1_μ for any $\mu \in \mathcal{R}_+$ and any $\varphi \in \mathcal{M}$, provided one adopts the distributional definition of (μ,φ)-total variation introduced in the previous section. As a consequence we obtain the stability of $BV_{\mu,\varphi}$ under composition (see Corollary 4.2 below). We let $\{u > t\} := \{x \in \mathbb{R}^n \ : u(x) > t\}$.

Theorem 4.1. *Let $\mu \in \mathcal{R}_+$ and $\varphi \in \mathcal{M}$. Then for any $u \in L^1_\mu$, the map $t \mapsto P_{\mu,\varphi}(\{u > t\})$ is Lebesgue-measurable and the following coarea-type formula holds:*

$$|D_\mu u|_\varphi = \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ . \tag{4.1}$$

Proof. Define

$$\mathcal{K} := \{\sigma \in X_\mu \ : \ \varphi^o(x, \sigma(x)) \leq 1 \ \text{ for } \mu\text{-a.e. } x\} \ . \tag{4.2}$$

Let us fix $u \in L^1_\mu$ and define for every $\sigma \in \mathcal{K}$

$$f_\sigma(t) \ := \ -\langle \chi_{\{u>t\}}, \operatorname{div}(\sigma\mu)\rangle \ , \qquad t \in \mathbb{R} \ .$$

The function $t \mapsto \mu\big(\{u > t\}\big)$ is bounded and non-increasing on $\mathbb{R}$, hence continuous at all $t \in \mathbb{R} \setminus D$, with D at most countable. Then f_σ is also continuous on $\mathbb{R} \setminus D$ for any $\sigma \in X_\mu$. Applying the Lindelöf's Theorem to the family of continuous functions f_σ on $\mathbb{R}\setminus D$, there exists a countable sequence $\{\sigma_n\} \subset \mathcal{K}$ such that

$$\sup\{f_\sigma(t) : \sigma \in \mathcal{K}\} = \mathcal{L}^1 - \operatorname{ess\,sup}\{f_\sigma(t) : \sigma \in \mathcal{K}\} = \sup_n f_{\sigma_n}(t) \ , \qquad \forall t \in \mathbb{R} \setminus D \ . \tag{4.3}$$

Therefore

$$P_{\mu,\varphi}(\{u > t\}) = \sup_n f_{\sigma_n}(t) \ , \qquad \mathcal{L}^1\text{−a.e. } t \in \mathbb{R} \ , \tag{4.4}$$

which entails the measurability statement of the theorem.
Notice now that, for any $\sigma \in X_\mu$, there holds

$$\begin{aligned} -\langle u, \operatorname{div}(\sigma\mu)\rangle &= -\langle u^+, \operatorname{div}(\sigma\mu)\rangle + \langle u^-, \operatorname{div}(\sigma\mu)\rangle \\ &= -\int_0^{+\infty} \langle \chi_{\{u>t\}}, \operatorname{div}(\sigma\mu)\rangle\, dt + \int_{-\infty}^0 \langle 1 - \chi_{\{u>t\}}, \operatorname{div}(\sigma\mu)\rangle\, dt \\ &= \int_{\mathbb{R}} f_\sigma(t)\, dt \ , \end{aligned} \tag{4.5}$$

where we used Fubini's theorem and the fact that $\langle 1, \operatorname{div}(\sigma\mu)\rangle = 0$ for any $\sigma \in X_\mu$.
If we pass to the supremum over $\sigma \in \mathcal{K}$ in (4.5), using (4.3) and (4.4), we get

$$|D_\mu u|_\varphi \leq \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ , \qquad u \in L^1_\mu \ .$$

It remains to prove the difficult part of (4.1), namely

$$|D_\mu u|_\varphi \geq \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ , \qquad u \in L^1_\mu \ . \tag{4.6}$$

First, we observe that to prove (4.6) it is enough to verify that

$$\int \varphi(x,\nabla_\mu u)\,d\mu \ge \int_{\mathbb{R}} P_{\mu,\varphi}(\{u>t\})\,dt\ , \qquad u\in\mathcal{D}\ . \tag{4.7}$$

Indeed, by Theorem 5.1 below, given $u\in L^1_\mu$ there exists a sequence $\{u_n\}\subseteq\mathcal{D}$ converging to u in L^1_μ such that

$$|D_\mu u|_\varphi = \lim_{n\to+\infty}\int \varphi(x,\nabla_\mu u_n)\,d\mu\ .$$

Possibly passing to a subsequence (still denoted by $\{u_n\}$), we have $\chi_{\{u_n>t\}}\to\chi_{\{u>t\}}$ in L^1_μ for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$. Then applying (4.7) to every u_n and passing to the limit as $n\to+\infty$, by the L^1_μ-lower semicontinuity of the (μ,φ)-perimeter and Fatou's Lemma we have

$$\begin{aligned}\int_{\mathbb{R}} P_{\mu,\varphi}(\{u>t\})\,dt &\le \int_{\mathbb{R}} \liminf_{n\to+\infty} P_{\mu,\varphi}(\{u_n>t\})\,dt \le \liminf_{n\to+\infty}\int_{\mathbb{R}} P_{\mu,\varphi}(\{u_n>t\})\,dt\\ &\le \lim_{n\to+\infty}\int\varphi(x,\nabla_\mu u_n)\,d\mu = |D_\mu u|_\varphi\ ,\end{aligned}$$

that is (4.6).
We are thus reduced to prove (4.7). Let $u\in\mathcal{D}$. We introduce the following subset $\mathcal{C}(u)$ of $L^1(\mathbb{R})$:

$$\mathcal{C}(u) := \left\{\sum_{i=1}^m \alpha_i\, f_{\sigma_i}\ ,\ \sigma_i\in\mathcal{K}\,,\ \alpha_i\in\mathcal{D}(\mathbb{R};[0,1])\,,\ \sum_{i=1}^m\alpha_i = 1\,,\ m\in\mathbb{N}\right\}\ .$$

We claim that

$$\int\varphi(x,\nabla_\mu u)\,d\mu \ge \sup_{g\in\mathcal{C}(u)}\int_{\mathbb{R}} g\,dt\ . \tag{4.8}$$

Indeed, let $g=\sum_{i=1}^m\alpha_i f_{\sigma_i}\in\mathcal{C}(u)$, where $\sigma_i\in\mathcal{K}$, $\alpha_i\in\mathcal{D}(\mathbb{R};[0,1])$, and $\sum_{i=1}^m\alpha_i=1$. Define $A_i(t):=\int_{-\infty}^t\alpha_i(s)\,ds$ for $t\in\mathbb{R}$, $i=1,\dots,m$. Then we have the chain rule identity $\nabla_\mu(A_i\circ u)=\alpha_i(u)\nabla_\mu u$. Using Fubini's theorem and noticing that $\varphi(x,\nabla_\mu u(x))\ge\nabla_\mu u(x)\cdot\sigma(x)$ for μ-a.e. x whenever σ belongs to $\mathcal{K}$, we get

$$\begin{aligned}\int_{\mathbb{R}} g(t)\,dt &= \int_{\mathbb{R}} -\langle\chi_{\{u>t\}},\sum_{i=1}^m\alpha_i(t)\,\mathrm{div}(\sigma_i\mu)\rangle\,dt = \sum_{i=1}^m -\langle A_i\circ u,\mathrm{div}(\sigma_i\mu)\rangle\\ &= \sum_{i=1}^m\int\nabla_\mu(A_i\circ u)\cdot\sigma_i\,d\mu = \sum_{i=1}^m\int\alpha_i(u)\,\nabla_\mu u\cdot\sigma_i\,d\mu\\ &\le \sum_{i=1}^m\int\alpha_i(u)\,\varphi(x,\nabla_\mu u)\,d\mu = \int\varphi(x,\nabla_\mu u)\,d\mu\ ,\end{aligned}$$

which proves the claim (4.8).

By construction, the family $\mathcal{C}(u)$ enjoys the following property (stability by partitions of unity): for $l \in \mathbb{N}$, if $g_1, \dots, g_l$ belong to $\mathcal{C}(u)$ and $\beta_1, \dots, \beta_l$ are functions in $\mathcal{D}(\mathbb{R};[0,1])$ with $\sum_{j=1}^{l} \beta_j = 1$, then there holds $\sum_{j=1}^{l} \beta_j g_j \in \mathcal{C}(u)$. This enables us to apply an argument about commutation between supremum and integral (see [21, Theorem 1]) which entails

$$\sup_{g \in \mathcal{C}(u)} \int_{\mathbb{R}} g \, dt = \int_{\mathbb{R}} \left(\mathcal{L}^1 - \operatorname*{ess\,sup}_{g \in \mathcal{C}(u)} g \right) dt .$$

Since $\mathcal{C}(u)$ contains $\{f_\sigma : \sigma \in \mathcal{K}\}$, by (4.8) and (4.4) we deduce

$$\begin{aligned} \int \varphi(x, \nabla_\mu u) \, d\mu &\geq \sup_{g \in \mathcal{C}(u)} \int_{\mathbb{R}} g \, dt = \int_{\mathbb{R}} \left(\mathcal{L}^1 - \operatorname*{ess\,sup}_{g \in \mathcal{C}(u)} g \right) dt \\ &\geq \int_{\mathbb{R}} \left(\mathcal{L}^1 - \operatorname*{ess\,sup}_{\sigma \in \mathcal{K}} f_\sigma \right) dt . \end{aligned} \tag{4.9}$$

Finally, one can check directly by the definition of μ-essential supremum that the choice of the family $\{\sigma_n\} \subset \mathcal{K}$ in (4.3) entails

$$P_{\mu,\varphi}(\{u > t\}) = \mathcal{L}^1 - \operatorname*{ess\,sup}_{\sigma \in \mathcal{K}} f_\sigma . \tag{4.10}$$

From (4.9) and (4.10) we get (4.7), and this completes the proof of (4.1). ■

Corollary 4.2. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then for every $u \in BV_{\mu,\varphi}$, it holds $A \circ u \in BV_{\mu,\varphi}$. Moreover if A is monotone non-decreasing, we can write*

$$|D(A \circ u)|_{\mu,\varphi} = \int_{\mathbb{R}} a(t) \, P_{\mu,\varphi}(\{u > t\}) \, dt , \tag{4.11}$$

where $a(t)$ denotes the a.e. defined derivative of A.

Proof. Let us assume first that A is monotone non-decreasing. Set $B(t) := \sup\{s : A(s) < t\}$; then $B(t)$ is a non-decreasing function in $BV_{loc}(\mathbb{R})$ such that $B \circ A(t) = t$ and $\{A \circ u > t\} = \{u > B(t)\}$ for a.e.t. Applying (4.1) to $A(u)$, we deduce

$$|D(A \circ u)|_{\mu,\varphi} = \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > B(t)\}) \, dt .$$

We deduce (4.11) by noticing that the functions $f(s) := \int_{-\infty}^{A(s)} P_{\mu,\varphi}(\{u > B(t)\}) \, dt$ and $g(s) := \int_{-\infty}^{s} P_{\mu,\varphi}(\{u > t\}) a(t) \, dt$ are absolutely continuous, non-decreasing and have a.e. the same derivative. Therefore they coincide and have the same limit as $s \to +\infty$ that is

$$\int_{\mathbb{R}} P_{\mu,\varphi}(\{u > B(t)\}) \, dt = \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\}) a(t) \, dt$$

Since $a(t)$ is bounded, it follows that for $u \in BV_{\mu,\varphi}$, $|D(A \circ u)|_{\mu,\varphi} < +\infty$ and so $A \circ u$ belongs to $BV_{\mu,\varphi}$. This implication can be extended to the general case, by writting $A \circ u = (A_+ \circ u) - (A_- \circ u)$ where A_+ and A_- denote primitives of the positive and of the negative parts of $a(t)$. ■

Remark 4.3. In light of Section 5, the left hand side of (4.11) can be written in an integral form with respect to μ when $u \in W^{1,1}_\mu$. Indeed the chain rule $\nabla_\mu(A \circ u) = a(u)\nabla_\mu u$ applies, see Theorem 5.4. Thus, due to Theorem 5.3, when μ and φ satisfy suitable regularity assumptions (see Theorem 5.7), (4.11) becomes :

$$\int \varphi(x, \nabla_\mu u)\, a(u)\, d\mu = \int_{\mathbb{R}} a(t)\, P_{\mu,\varphi}(\{u > t\})\, dt\ .$$

5. Relaxation on $BV_{\mu,\varphi}$ and integral representation on $W^{1,1}_\mu$

Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let $J : L^1_\mu \to [0, +\infty]$ be the functional defined by

$$J(u) := \begin{cases} \displaystyle\int \varphi(x, \nabla_\mu u)\, d\mu & \text{if } u \in \mathcal{D} \\ +\infty & \text{if } u \in L^1_\mu \setminus \mathcal{D}\ . \end{cases} \tag{5.1}$$

In this section, we consider the relaxed functional of J defined by

$$\overline{J}(u) = \inf \left\{ \liminf_{n \to +\infty} J(u_n)\ :\ u_n \to u \text{ in } L^1_\mu \right\}\ .$$

The next theorem relates $\overline{J}$ to the distributional notion (3.1) of (μ, φ)-total variation.

Theorem 5.1. *Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let J be defined as in (5.1). Then for every $u \in L^1_\mu$ we have $\overline{J}(u) = |D_\mu u|_\varphi$; in particular $BV_{\mu,\varphi}$ is the finiteness domain of $\overline{J}$.*

Proof. It is useful to extend φ to a metric integrand Φ defined on $\mathbb{R}^n \times \mathbb{R}^n$ by setting

$$\Phi(x, z) := \varphi(x, P_\mu(x)[z])\ . \tag{5.2}$$

For any $z \in \mathbb{R}^n$, the map $x \mapsto \Phi(x, z)$ is μ-measurable, and Φ satisfies (2.5), (2.6), and (2.7) on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, if we identify $T^*\mu$ with $T\mu$ through the canonical scalar product on $\mathbb{R}^n$ and we use the homogeneity of Φ, it turns out that the Fenchel conjugate $\Phi^*(x, \cdot)$ of $\Phi(x, \cdot)$ is

$$\Phi^*(x, z^*) = \begin{cases} 0 & \text{if } z^* \in T_\mu(x)\ ,\ \varphi^o(x, z^*) \le 1 \\ +\infty & \text{otherwise} \end{cases}\ , \qquad (x, z^*) \in \mathbb{R}^n \times \mathbb{R}^n\ . \tag{5.3}$$

In terms of the integrand Φ, the functional (5.1) can be written as $J(u) = \int \Phi(x, \nabla u)\, d\mu$ if $u \in \mathcal{D}$, and $J(u) = +\infty$ if $u \in L^1_\mu \setminus \mathcal{D}$. Then we can use such expression for J in order to compute $\overline{J}$: this allows to closely follow the proof of [18, Theorem 3.1], to which we refer for the details. Since J is convex, there holds $\overline{J} = J^{**}$ [62, Theorem 5], where J^{**} is defined as the double Fenchel transform of J in the duality between L^1_μ and L^∞_μ.

Let A be the densely defined linear operator from L^1_μ to $(L^1_\mu)^N$ given by $A(u) = \nabla u$ with domain $D(A) := \mathcal{D}$; denoting by A^* the adjoint of A, we have $D(A^*) = X_\mu \subseteq (L^\infty_\mu)^N$, $A^* : D(A^*) \ni \sigma \mapsto -\operatorname{div}(\sigma\mu) \in L^\infty_\mu$. Hence, by [18, Theorem 5.1] we have

$$J^*(v) = \inf\left\{ \int \Phi^*(x,\sigma)\, d\mu \ :\ \sigma \in X_\mu \ ,\ A^*\sigma = v \right\} \ ,\quad v \in L^\infty_\mu \ .$$

Arguing exactly as in the proof of Theorem 3.1 of [18], we get

$$J^{**}(u) = \sup\left\{ -\langle \operatorname{div}(\sigma\mu), u\rangle - \int \Phi^*(x,\sigma)\, d\mu \ :\ \sigma \in X_\mu \right\} \ ,\quad u \in L^1_\mu \ .$$

Therefore taking into account (5.3), and recalling (4.2), we have for every $u \in L^1_\mu$

$$\overline{J}(u) = \sup\left\{ -\langle \operatorname{div}(\sigma\mu), u\rangle \ : \sigma \in \mathcal{K} \right\} = |D_\mu u|_\varphi \ .$$

■

Remark 5.2. Notice that $\overline{J} = \overline{J'}$ for any functional J' with $\overline{J} \le J' \le J$ on L^1_μ. This in particular applies for

$$J'(u) := \begin{cases} \displaystyle\int \varphi(x, \nabla_\mu u)\, d\mu & \text{if } u \in W^{1,1}_\mu \\ +\infty & \text{if } u \in L^1_\mu \setminus W^{1,1}_\mu \ . \end{cases}$$

Indeed $J' \le J$. Moreover, since $J'(u) \ge \int \sigma \cdot \nabla_\mu u\, d\mu = -\langle \operatorname{div}(\sigma\mu), u\rangle$ for any $u \in W^{1,1}_\mu$ and $\sigma \in \mathcal{K}$, passing to the supremum over $\sigma \in \mathcal{K}$ and using Theorem 5.1 we get $J' \ge \overline{J}$.

We next give an integral representation result for the (μ,φ)-total variation on the Sobolev space $W^{1,1}_\mu$. Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$ and $\mathcal{K}$ be given by (4.2). We define a new integrand $h : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ by

$$h(x,z) := \mu - \operatorname*{ess\,sup}_{\sigma \in \mathcal{K}} \{\sigma(x) \cdot z\} \ . \tag{5.4}$$

It is easy to check that the restriction of h to $T\mu$ belongs to the class $\mathcal{M}$; in view of Lemma 5.5 below, this restriction coincides with φ under suitable regularity assumptions.

Theorem 5.3. *Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let h be defined by (5.4). Then the following representation formula of the (μ,φ)-total variation holds:*

$$|D_\mu u|_\varphi = \int h(x, \nabla_\mu u)\, d\mu \ ,\quad u \in W^{1,1}_\mu \ . \tag{5.5}$$

Proof. By applying Theorem 5.1 and using (2.3) , we get

$$\overline{J}(u) = \sup_{\sigma\in\mathcal{K}} \{-\langle \operatorname{div}(\sigma\mu), u\rangle\} = \sup_{\sigma\in\mathcal{K}} \left\{ \int \sigma\cdot\nabla_\mu u\, d\mu \right\} , \qquad u \in W^{1,1}_\mu . \tag{5.6}$$

We notice that the subset $S(u)$ of L^1_μ defined by $S(u) := \{\sigma\cdot\nabla_\mu u \ : \ \sigma\in\mathcal{K}\}$ is stable by smooth partitions of unity. Indeed let $\sigma_1, \sigma_2, \dots, \sigma_l$ be elements of $\mathcal{K}$ and let $\alpha_1, \alpha_2, \dots, \alpha_l$ belong to $\mathcal{D}$, with $\alpha_i \geq 0$ and $\sum_{i=1}^l \alpha_i = 1$. By the convexity of $\varphi^o(x,\cdot)$, $\sigma := \sum_{i=1}^l \alpha_i\,\sigma_i$ still satisfies $\varphi^o(x,\sigma(x)) \leq 1$ μ-a.e. ; moreover, since $\operatorname{div}(\sigma\,\mu) = \sum_{i=1}^l [\alpha_i \operatorname{div}(\sigma_i\,\mu) + (\nabla\alpha_i\cdot\sigma_i)\,\mu]$, we have also $\sigma\in X_\mu$. Applying the commutation result between supremum and integral proved in [21, Theorem 1], and taking into account (5.6) and (5.4), we conclude, for $u\in W^{1,1}_\mu$,

$$\overline{J}(u) = \int \mu - \operatorname*{ess\,sup}_{\sigma\in\mathcal{K}} \{\sigma\cdot\nabla_\mu u\}\ d\mu = \int h(x,\nabla_\mu u)\, d\mu ,$$

that is (5.5). ∎

As a consequence of Theorem 5.3, we get the following chain rule formula on $W^{1,1}_\mu$.

Theorem 5.4. *Let $\mu\in\mathcal{R}_+$ and $u\in W^{1,1}_\mu$. Then for any $\mathcal{L}^1$- negligible set $N\subset\mathbb{R}$ it holds*

$$\nabla_\mu u = 0 \qquad \mu\text{-a.e. on } u^{-1}(N) ; \tag{5.7}$$

moreover, for any $a\in L^\infty(\mathbb{R})$, setting $A(t) := \int_0^t a(s)\,ds$, $A\circ u$ belongs to $W^{1,1}_\mu$, and the following chain rule holds:

$$\nabla_\mu(A\circ u) = a(u)\nabla_\mu u \qquad \mu\text{-a.e.} \tag{5.8}$$

Proof. We first prove that $A\circ u$ belongs to $W^{1,1}_\mu$ and that (5.8) holds under the assumption $A\in\mathcal{C}^\infty(\mathbb{R})$. Let $\{u_n\}\subset\mathcal{D}$ be a sequence converging to u in $W^{1,1}_\mu$. Then $A\circ u_n\in\mathcal{D}$ for every n, and projecting onto the tangent space to μ the usual chain rule, we get $\nabla_\mu(A\circ u_n) = a(u_n)\nabla_\mu u_n$; since $a(u_n)\nabla_\mu u_n \to a(u)\nabla_\mu u$ in measure μ, and since a is bounded, by Vitali's convergence criterion we get that $\nabla_\mu(A\circ u_n)\to a(u)\nabla_\mu u$ in L^1_μ. This implies by definition that $A\circ u\in W^{1,1}_\mu$ and that $\nabla_\mu(A\circ u) = a(u)\nabla_\mu u$.
Let $\varphi(x,z) := |z|$, and let h be the function defined in (5.4) associated with φ. Then, using Theorem 5.3, we get

$$\begin{aligned} \int a(u)h(x,\nabla_\mu u)\,d\mu &= \int h(x,\nabla_\mu(A\circ u))\,d\mu \\ &= |D(A\circ u)|_{\mu,\varphi} = \int_{\mathbb{R}} a(t)P_{\mu,\varphi}(\{u>t\})\,dt . \end{aligned} \tag{5.9}$$

One can check that (5.9) still holds if we replace the $\mathcal{C}^\infty$ function a by the characteristic function of a Borel subset $N \subset \mathbb{R}$: indeed N can be approximated from the exterior and from the interior respectively by open and compact sets, which in turn can be approximated respectively by an increasing or a decreasing sequence of smooth functions.
In particular, let us take $a = \chi_N$ in (5.9), where $N \subset \mathbb{R}$ is $\mathcal{L}^1$-negligible. We get

$$\int_{u^{-1}(N)} h(x, \nabla_\mu u)\, d\mu = \int_N P_{\mu,\varphi}(\{u > t\})\, dt = 0 \ .$$

Then, recalling the definition (5.4) of h, it follows that $\nabla_\mu u(x) \in T_\mu^\perp(x)$ for μ-a.e. $x \in u^{-1}(N)$. As by construction $\nabla_\mu u(x) \in T_\mu(x)$ for μ-a.e. x, this implies (5.7).
It remains to prove that (5.8) holds for any $a \in L^\infty(\mathbb{R})$. Let $\{a_n\}$ be a sequence of $\mathcal{C}^\infty$ functions bounded in $L^\infty(\mathbb{R})$ and converging to a $\mathcal{L}^1$-a.e. . Then $A_n \circ u \to A \circ u$ in L^1_μ. Let N be a negligible subset of $\mathbb{R}$ such that $a_n(t) \to a(t)$ for every $t \in \mathbb{R} \setminus N$. We have $a_n(u) \to a(u)$ μ-a.e. on $\mathbb{R}^n \setminus u^{-1}(N)$, which implies by (5.7) that $a_n(u)\nabla_\mu u \to a(u)\nabla_\mu u$ μ-a.e. Since we know that $\nabla_\mu(A_n \circ u) = a_n(u)\nabla_\mu u$ for every n, it follows from the completeness of $W^{1,1}_\mu$ that the sequence $\{A \circ u_n\}$ converges in $W^{1,1}_\mu$, hence its limit is $A \circ u$ and (5.8) holds. ■

Our aim now is to compare the integrands h and φ. We will adopt the notation $h_1 \preceq h_2$ (or equivalently $h_1(x,z) \preceq h_2(x,z)$) and $h_1 \equiv h_2$ to denote the following relations between metric integrands of the class $\mathcal{M}$:

$$\begin{aligned} h_1 \preceq h_2 \quad &\Longleftrightarrow \quad h_1(x,z) \le h_2(x,z) \text{ for all } x \in \mathbb{R}^n \setminus E, \text{ with } \mu(E) = 0 \ , \ z \in T_\mu(x) \ ; \\ h_1 \equiv h_2 \quad &\Longleftrightarrow \quad h_1 \preceq h_2 \text{ and } h_2 \preceq h_1 \ . \end{aligned}$$

Next, we associate with φ a lower semicontinuous regularization (with respect to μ) by setting

$$\overline{\varphi}_\mu(x,z) := \sup\Big\{\phi(x)\cdot z \ : \ \phi \in \mathcal{C}^N \ , \ \phi(x)\cdot z \preceq \varphi(x,z) \ , \ P_\mu \circ \phi \in X_\mu\Big\} \ . \tag{5.10}$$

It is easy to check that $\overline{\varphi}_\mu$ belongs to the class $\mathcal{M}$ when restricted to $T\mu$ (note however that the equality $\overline{\varphi}_\mu = +\infty$ will occur mostly on the complement of $T\mu$).
The following comparison result holds.

Lemma 5.5. *Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let h, $\overline{\varphi}_\mu$ be defined by (5.4) and (5.10) respectively. Then*

$$\overline{\varphi}_\mu \ \preceq \ h \ \preceq \ \varphi \ . \tag{5.11}$$

Proof. Let $\mathcal{K}$ be defined by (4.2) and set

$$\begin{aligned} \mathcal{C}_\mu &:= \big\{P_\mu \circ \phi \ : \ \phi \in \mathcal{C}^N, \ P_\mu \circ \phi \in X_\mu \ , \ \phi(x)\cdot z \preceq \varphi(x,z)\big\} \\ \mathcal{K}_\mu &:= \big\{\sigma \in (L^\infty_\mu)^N \ : \ \sigma(x) \in T_\mu(x) \ , \ \varphi^o(x,\sigma(x)) \le 1 \ \mu\text{-a.e. } x\big\} \end{aligned}$$

Note that a field σ in $\mathcal{K}_\mu$ is not necessarily in X_μ, since in the definition of $\mathcal{K}_\mu$ we have skipped the (non-local) condition on $\mathrm{div}(\sigma\,\mu)$.
Since $\mathcal{C}_\mu \subseteq \mathcal{K} \subseteq \mathcal{K}_\mu$, we have, for μ-a.e. x in $\mathbb{R}^n$ and for all $z \in T_\mu(x)$:

$$\begin{aligned}\overline{\varphi}_\mu(x,z) &= \mu - \operatorname*{ess\,sup}_{\sigma\in\mathcal{C}_\mu}\{\sigma(x)\cdot z\} \le \mu - \operatorname*{ess\,sup}_{\sigma\in\mathcal{K}}\{\sigma(x)\cdot z\} = h(x,z) \\ &\le \mu - \operatorname*{ess\,sup}_{\sigma\in\mathcal{K}_\mu}\{\sigma(x)\cdot z\} = \sup\{z\cdot z^* \ : \ z^*\in T_\mu(x)\,,\ \varphi^o(x,z^*)\le 1\}\ . \\ &= \varphi(x, P_\mu(x)[z])\end{aligned}$$

This completes the proof of (5.11). ∎

Remark 5.6. The following examples, in which we have $\overline{\varphi}_\mu \equiv h$, show that the equivalence $h \equiv \varphi$ may not hold, due to the lack of regularity either of φ or of μ.

i) Let $N=1$, let $\mu = \mathcal{L}^1 \llcorner (0,1)$, let $F \subset (0,1)$ be a closed set with empty interior such that $\mathcal{L}^1(F) > 0$ and let us set

$$\varphi(x,z) := \begin{cases} |z| & \text{if } x\in F \\ 0 & \text{if } x \notin F\ . \end{cases}$$

Then one can check that $\mathcal{K}$ is reduced to $\{0\}$, hence $h\equiv 0$, while φ is by definition non-zero on a set of positive measure μ.

ii) Let $N=1$, let $\varphi(x,z) = |z|$, let F be as in the example i), and let $\mu = a(x)\mathcal{L}^1\llcorner(0,1)$, where

$$a(x) := \begin{cases} 2 & \text{if } x\in F \\ 1 & \text{if } x \in (0,1)\setminus F. \end{cases}$$

For any $\sigma\in\mathcal{K}$, we have $|a\sigma|\le 1$ on $(0,1)$: indeed $a\sigma$ is continuous, because $\frac{d}{dx}(a\sigma) \in L^\infty(0,1)$, and it holds $|a\sigma|\le 1$ on the dense set $(0,1)\setminus F$. Hence $|\sigma| \le a^{-1}$ μ-a.e. for any $\sigma\in\mathcal{K}$; moreover, for any constant $\lambda \le 1$, the function $\sigma(x) := \lambda\,(a(x))^{-1}$ belongs to $\mathcal{K}$. By definition (5.4) it follows that $h(x,z) = a(x)^{-1}|z|$, and in particular that we do not have $h\equiv\varphi$.

In order to avoid the pathological-type behaviours described in Remark 5.6, we are led to introduce the following assumption on μ:

$$\mathcal{V}_\mu := \{\phi \in \mathcal{C}_0^N \ : \ P_\mu \circ \phi \in X_\mu\ \} \text{ is dense in } \mathcal{C}_0^N\ . \tag{5.12}$$

This condition will also be useful in Section 6, and it can be shown that it is related to the regularity of the mean curvature vector $H(\mu) := \mathrm{div}(P_\mu \mu)$ of μ [16].

Theorem 5.7. *Let $\mu \in \mathcal{R}_+$ satisfy (5.12), and let $\varphi \in \mathcal{M}$ be a lower semicontinuous metric on $T\mu$ such that (3.2) holds. Then $\overline{\varphi}_\mu \equiv \varphi$. In particular, it holds*

$$|D_\mu u|_\varphi = \int \varphi(x, \nabla_\mu u)\, d\mu \ , \qquad u \in W^{1,1}_\mu \ .$$

Proof. By Lemma 5.5, it is enough to show that $\varphi \preceq \overline{\varphi}_\mu$. Let $t < 1$, and let $\phi \in \mathcal{C}^N$ such that $\phi(x) \cdot z \preceq t\varphi(x,z)$. Then by (5.12) there exists a sequence $\{\phi_n\} \subseteq \mathcal{V}_\mu$ which converges uniformly to ϕ. For n large enough, we have $\|\phi_n - \phi\|_\infty \leq (1-t)C^{-1}$, where C is the positive constant appearing in (3.2); thus, there exists $\bar{n}$ such that, for $n > \bar{n}$,

$$\begin{aligned} \phi_n(x) \cdot z &\preceq \|\phi_n - \phi\|_\infty |z| + |\phi(x) \cdot z| \\ &\preceq (1-t)C^{-1}|z| + t\varphi(x,z) \preceq \varphi(x,z) \ . \end{aligned}$$

By definition (5.10), we deduce

$$\phi(x) \cdot z = \lim_{n \to +\infty} \phi_n(x) \cdot z \ \preceq \ \overline{\varphi}_\mu(x,z) \ ;$$

then, using also the lower semicontinuity assumption on φ, we get (see for instance [26, Lemma 2.2.3])

$$t\varphi(x,z) \ = \ \sup\Big\{\phi(x) \cdot z \ : \ \phi \in \mathcal{C}^N \ , \ \phi(x) \cdot z \ \leq \ t\varphi(x,z)\Big\} \ \preceq \ \overline{\varphi}_\mu(x,z) \ .$$

Since $t < 1$ was arbitrary, it follows $\varphi \ \preceq \ \overline{\varphi}_\mu$. ∎

6. Integral representation on $BV_{\mu,\varphi}$

The natural generalization of Theorem 5.3 would be an integral representation for the (μ,φ)-total variation ot the type

$$|D_\mu u|_\varphi = \int \mathrm{h}(x, D_\mu u) \ , \qquad u \in BV_{\mu,\varphi} \ , \tag{6.1}$$

for some metric integrand h and some element $D_\mu u \in \mathcal{R}^N$. Here and in the following, the notation must be intended in the usual sense of integration theory, namely

$$\int \mathrm{h}(x, D_\mu u) = \int \mathrm{h}(x, \nu^u_\mu(x))\, d|D_\mu u| \ ,$$

where $|D_\mu u|$ is the total variation measure of $D_\mu u$, and ν^u_μ is the density of $D_\mu u$ with respect to $|D_\mu u|$.

In order to have (6.1), we ask whether it is possible to find a vector measure $D_\mu u$ which yields such an integral representation. This question turns out to be quite delicate, and we will prove in Example 6.4 that it has, in general, a negative answer. In fact, as we will show, this problem is directly related to the possibility of finding a closed extension $\overline{T}$ of the (densely defined) linear operator T defined by

$$T : L^1_\mu \to \mathcal{R}^N \quad , \quad D(T) := W^{1,1}_\mu \quad , \quad Tu := (\nabla_\mu u)\,\mu \ ,$$

where $\mathcal{R}^N$ is endowed with the weak star topology. Recall that $\overline{T}$, if it exists, is unique and its graph coincides with the closure $\overline{G(T)}$ in $L^1_\mu \times \mathcal{R}^N$ of the graph $G(T)$ of T. Unfortunately, the closability of T is not satisfied for a general measure μ (see again Example 6.4 below, where the closure of the graph of T fails to be a graph). The following lemma gives a necessary and sufficient condition on μ for the closability of T.

Lemma 6.1. *Let $\mu \in \mathcal{R}_+$. Then T is closable if and only if (5.12) holds. In this case, $\overline{T}u$ is the vector-valued measure determined by*

$$\langle \overline{T}u, \phi\rangle_{(\mathcal{R}^N, \mathcal{C}^N_0)} := -\langle u, \operatorname{div}[(P_\mu \phi)\mu]\rangle \ , \qquad \phi \in \mathcal{V}_\mu \ , \tag{6.2}$$

and we have $BV_\mu \subseteq D(\overline{T})$.

Proof. Recalling that $\mathcal{R}^N$ is endowed with the weak star topology, the adjoint operator $T^* : \mathcal{C}^N_0 \to L^\infty_\mu$ has domain $D(T^*) = \mathcal{V}_\mu$ and is defined by $T^*\phi := -\operatorname{div}[(P_\mu\phi)\mu]$ for all $\phi \in \mathcal{V}_\mu$. Since $\overline{G(T)} = \left[\mathcal{J}(G(T^*))\right]^\perp$, where $\mathcal{J} : L^1_\mu \times \mathcal{R}^N \ni (u,\lambda) \to (-\lambda, u) \in \mathcal{R}^N \times L^1_\mu$ [22, II.6], we have

$$\overline{G(T)} = \Big\{(u,\lambda) \in L^1_\mu \times \mathcal{R}^N \ : \ \langle \lambda, \phi\rangle_{(\mathcal{R}^N, \mathcal{C}^N_0)} := -\langle u, \operatorname{div}[(P_\mu\phi)\mu]\rangle \ , \ \phi \in \mathcal{V}_\mu\Big\} . \tag{6.3}$$

Thus, $\overline{G(T)}$ is a graph if and only if the linear subspace of $\mathcal{R}^N$ given by $\{\lambda \in \mathcal{R}^N \ : \ (0,\lambda) \in \overline{G(T)}\} = \mathcal{V}_\mu^\perp$ reduces to $\{0\}$, that is if and only if $\mathcal{V}_\mu$ is dense in $\mathcal{C}^N_0$. In this case, $\overline{G(T)}$ coincides with the graph of the operator $\overline{T}$ given by (6.2). Whenever $u \in BV_\mu$, applying (3.1) (with $\varphi(x,z) = |z|$), we find that there exists a positive constant C such that

$$|\langle u, \operatorname{div}[(P_\mu\phi)\mu]\rangle| \le \ C\,\|P_\mu\phi\|_{(L^\infty_\mu)^N} \ \le C\|\phi\|_{(L^\infty_\mu)^N} \ , \qquad \phi \in \mathcal{V}_\mu \ .$$

Therefore $BV_\mu \subseteq D(\overline{T})$. ∎

Given $\mu \in \mathcal{R}_+$ satisfying (5.12) and $u \in BV_\mu$, we set $D_\mu u := \overline{T}u$. Notice that $D_\mu u = (\nabla_\mu u)\,\mu$ whenever $u \in W^{1,1}_\mu$. Then we can derive the following

Lemma 6.2. *Let $\mu \in \mathcal{R}_+$ satisfy (5.12) and let $\{u_n\}$ be a sequence in $W^{1,1}_\mu$ such that*

$$u_n \to u \ \ in\ L^1_\mu \quad , \quad \sup_n \int |\nabla_\mu u_n| \, d\mu \ < +\infty \ .$$

Then $u \in BV_\mu$ and $(\nabla_\mu u_n)\,\mu \to D_\mu u$ weakly star in $\mathcal{R}^N$.

Proof. The sequence $\{\big(u_n, (\nabla_\mu u_n)\,\mu\big)\}$ is precompact in $L^1_\mu \times \mathcal{R}^N$ and, by definition of $\overline{T}$, any of its cluster points (u, λ) belongs to $G(\overline{T})$. Therefore the whole sequence converges to $(u, \overline{T}u)$. It remains to check that u belongs to BV_μ, which follows from the lower semicontinuity of the (μ,φ)-total variation when $\varphi(x,z) = |z|$ and from Remark 5.2. We have indeed

$$|D_\mu u|_\varphi \leq \liminf_{n\to+\infty} |D_\mu u_n|_\varphi = \liminf_{n\to+\infty} \int |\nabla_\mu u_n| \, d\mu < +\infty \ .$$

∎

Let $\mu \in \mathcal{R}_+$ satisfy (5.12). We are going to prove that, for any smooth and coercive $\varphi \in \mathcal{M}$, (6.1) holds with $\mathrm{h} = \varphi$ provided the following additional regularity assumption on μ is fulfilled:

$$\begin{aligned} &\sup\Big\{\langle u, \mathrm{div}\,\big[(P_\mu\phi)\mu\big]\rangle \ : \ \phi \in \mathcal{V}_\mu \ , \ |\phi| \leq 1\Big\} = \\ &\sup\Big\{\langle u, \mathrm{div}(\sigma\mu)\rangle \ : \ \sigma \in X_\mu \ , \ |\sigma| \leq 1 \ \mu\text{-a.e.}\Big\} \ . \end{aligned} \tag{6.4}$$

Notice that conditions (5.12) and (6.4) are both satisfied when we take $\mu = a\mathcal{H}^k \llcorner M$ as in Example 3.2. Indeed in that case (5.12) is satisfied because $\mathcal{V}_\mu$ contains $\mathcal{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ (which happens whenever μ has mean curvature in $(L^\infty_\mu)^N$, see also Remark 5.6), while (6.4) can be checked by using local coordinates and approximation by convolution.

Theorem 6.3. *Let $\mu \in \mathcal{R}_+$ such that (5.12) and (6.4) hold. Assume that $\varphi \in \mathcal{M}$ satisfies the coercivity condition (3.2) and admits a continuous extension on the whole $\mathbb{R}^N \times \mathbb{R}^N$. Then*

$$|D_\mu u|_\varphi = \int \varphi(x, D_\mu u) \ , \qquad u \in BV_\mu \ . \tag{6.5}$$

Proof. Denote by $G(u)$ the right hand side of (6.5). Let us first show that $|D_\mu u|_\varphi \geq G(u)$. We can assume that $|D_\mu u|_\varphi$ is finite. By Theorem 5.1, there exists a sequence $\{u_n\} \subset \mathcal{D}$ such that $u_n \to u \in L^1_\mu$ and $\int \varphi(x, \nabla_\mu u_n)\, d\mu \to |D_\mu u|_\varphi$. By Lemma (6.2), we deduce that $u \in BV_\mu$ and $(\nabla_\mu u_n)\,\mu \to D_\mu u$ weakly star in $\mathcal{R}^N$. By the continuity of (the extension of) φ, we can apply Reshetnyak's lower semicontinuity theorem [60, Theorem 2], which gives the claimed inequality:

$$|D_\mu u|_\varphi = \lim_{n\to+\infty} \int \varphi(x, \nabla_\mu u_n)\, d\mu \geq \int \varphi(x, D_\mu u) = G(u) \ .$$

In order to show the converse inequality, let us fix $u \in BV_\mu$ and divide the proof into in two steps. As a first step, we show that (6.5) holds when $\varphi(x,z) := |z|$. Indeed, by (6.4), (6.2), and (5.12), we get:

$$\begin{aligned}|D_\mu u|_{\varphi=|\cdot|} &= \sup\Big\{\langle u, \operatorname{div}(\sigma\mu)\rangle \ : \ \sigma \in X_\mu \ , |\sigma| \le 1 \ \mu\text{-a.e.}\Big\} \\ &= \sup\Big\{\langle u, \operatorname{div}\left[(P_\mu\phi)\mu\right]\rangle \ : \ \phi \in \mathcal{V}_\mu \ , \ |\phi| \le 1\Big\} = \int |D_\mu u| \ .\end{aligned}$$

As a second step, we pass to consider the case of a general metric $\varphi \in \mathcal{M}$ as in the assumptions. By the first step, we can choose a suitable sequence $\{u_n\} \subset \mathcal{D}$ such that $u_n \to u \in L^1_\mu$ and $\int |\nabla_\mu u_n|\, d\mu \to \int |D_\mu u|$. By the lower semicontinuity of the (μ,φ)-total variation, and by Reshetnyak's continuity theorem [60, Theorem 3], we infer

$$|D_\mu u|_\varphi \le \liminf_{n\to+\infty} \int \varphi(x, \nabla_\mu u_n)\, d\mu = \int \varphi(x, D_\mu u) = G(u) \ .$$

■

Finally, let us give an example of measure μ for which condition (5.12) is not satisfied (and in fact the closure of the graph of the operator T is not a graph). The measure μ under consideration is the one-dimensional Hausdorff measure on a Lipschitz (but not $\mathcal{C}^1$) curve S in $\mathbb{R}^2$. We find that, if E is a set whose boundary (in S) meets the singular part of S, the explicit expression for the (μ,φ)-perimeter of E (see formula (6.6)) is not compatible with an integral representation of the form $|D_\mu u|_\varphi = \int \varphi(x, \alpha_u)$, for a vector-valued measure α_u. Indeed, in order to have such an integral representation, the map $\varphi \mapsto |D_\mu u|_\varphi$ needs to be linear for any $u \in BV_{\mu,\varphi}$, while (6.6) shows that this is not the case for $u = \chi_E$.

Example 6.4. Let $N = 2$ and $\mu := \mathcal{H}^1 \llcorner S$, where

$$S := S_+ \cup S_- \ , \qquad S_+ := \{(t,t) \ : \ t \in [0,1]\} \ , \qquad S_- := \{(t,-t) \ : \ t \in [-1,0]\} \ .$$

Clearly the tangent space $T_\mu(x)$ is μ-a.e. one-dimensional and it is determined by the direction $\nu(x) := \nu_\pm$ respectively for $x \in S_\pm \setminus \{O\}$, where O is the origin of $\mathbb{R}^2$ and $\nu_\pm := (1/\sqrt{2}, \pm 1/\sqrt{2})$. Let $\varphi \in \mathcal{M}$ not depend explicitly on x, *i.e.* $\varphi(x,z) = \varphi(z)$ for $x \in S \setminus \{O\}$ and $z \in \mathbb{R}\nu(x)$. We claim that, if we take $E := S_+$, there holds

$$P_{\mu,\varphi}(E) = \min\{\varphi(\nu_+)\, , \varphi(\nu_-)\} \ . \tag{6.6}$$

Let us prove (6.6). Let $\gamma : I := [-\sqrt{2}, \sqrt{2}] \to S$ be a parametrization by arc-length of $S \setminus \{O\}$ such that $\gamma(\pm\sqrt{2}) := A^\pm := (\pm 1, 1)$ and $\gamma(0) = O$. A measurable vector field σ such that $\sigma(x) \in T_\mu(x)$ for μ-a.e. x can be written as

$$\sigma(x) = \tilde{\sigma}(\gamma^{-1}(x))\, \nu(x) \ , \qquad x \in S \setminus \{O\} \ ,$$

where $\tilde{\sigma}$ is a scalar function defined $\mathcal{L}^1$-a.e. on I. One can check that $\sigma \in X_\mu$ if and only if $\tilde{\sigma}$ belongs to $W_0^{1,\infty}(I)$ and in that case $\mathrm{div}(\sigma\mu) = (\tilde{\sigma}' \circ \gamma^{-1})\,\mu$ where $\tilde{\sigma}'$ coincides with the $\mathcal{L}^1$-a.e. defined derivative of the Lipschitz function $\tilde{\sigma}$.
Let $u = \chi_E$. Recalling (3.1), letting $\tilde{u}$ such that $u = \tilde{u} \circ \gamma^{-1}$, and integrating by parts, we get

$$\begin{aligned}
P_{\mu,\varphi}(E) :&= \sup\Big\{ -\langle \mathrm{div}(\sigma\mu), u\rangle \ :\ \sigma \in X_\mu\,,\ \varphi^o(\sigma(x)) \le 1 \text{ for } \mu\text{-a.e. } x\Big\} \\
&= \sup\Big\{ -\int_{-\sqrt{2}}^{\sqrt{2}} \tilde{u}(s)\,\tilde{\sigma}'(s)\,ds \ :\ \tilde{\sigma} \in W_0^{1,\infty}(I),\ \varphi^o(\tilde{\sigma}(s)\nu) \le 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in I\Big\} \\
&= \sup\Big\{\tilde{\sigma}(0) :\ \tilde{\sigma} \in W_0^{1,\infty}(I)\,,\ \varphi^o(\tilde{\sigma}\,\nu) \le 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in I\Big\} \\
&= \sup\Big\{\lambda \in \mathbb{R}\ :\ \max\{\varphi^o(\lambda\nu_+), \varphi^o(\lambda\nu_-)\} \le 1\Big\} \\
&= \min\Big\{\varphi(\nu_+)\,,\ \varphi(\nu_-)\Big\}\ .
\end{aligned}$$

We conclude the paper with three further observations concerning Example 6.4.

i) The formula (6.6) seems rather disconcerting. The key point is that the condition $\sigma \in X_\mu$ allows the existence of tangent fields σ whose norm is continuous but whose direction jumps at the corner O. Computing the concentration of energy at O for a given u leads to individuate the optimal jump of σ.

ii) By Theorem 5.1, we have $P_{\mu,\varphi}(E) = \overline{J}(\chi_E)$, hence there exists a sequence $\{u_n\} \subset W_\mu^{1,1}$ converging to χ_E in L_μ^1 such that $P_{\mu,\varphi}(E) = \lim_{n\to+\infty} J(u_n)$. For instance we can take $u_n := v_n \circ \gamma^{-1}$, where $\{v_n\}$ is any sequence of non-decreasing functions in $W^{1,1}(I)$ such that $v_n(-\sqrt{2}) = 0$, $v_n(\sqrt{2}) = 1$, $v_h \to \chi_E \circ \gamma$ $\mathcal{L}^1$-a.e. on I and $v_n(0) = 1$ if $\varphi(\nu_+) \ge \varphi(\nu_-)$ and $v_n(0) = 0$ otherwise.

iii) The closure in $\mathcal{C}_0^2$ of the subspace $\mathcal{V}_\mu$ associated with μ defined in (5.12) is given here by

$$\overline{\mathcal{V}}_\mu = \big\{\phi \in \mathcal{C}_0^2\ :\ \phi(O) \in \mathbb{R}(1,0)\big\}\ . \tag{6.7}$$

This can be derived from the fact that, if we ask that $\phi \in X_\mu$ when ϕ is a tangent vector field in $(\mathcal{C}^1(S \setminus \{O\}))^2$, we find the necessary condition $\phi(O^+)\cdot\nu_+ = \phi(O^-)\cdot\nu_-$, where $\phi(O^\pm) := \lim_{S_\pm \ni x \to O} \phi(x)$. Thus, by (6.7), $\overline{\mathcal{V}}_\mu$ is a strict closed subspace of $\mathcal{C}_0^2$ and therefore the operator T is not closable, or equivalently $\overline{G(T)}$ is not a graph. In fact, one can check by using (6.3) and (6.7) that $(\chi_E, (t, 1/\sqrt{2})\,\delta_O)$ belongs to $\overline{G(T)}$ for any real t.

V. Convergence of Sobolev spaces on varying manifolds *

1. Introduction

In several optimization problems arising in Differential Geometry and in Mechanics, one has to find the "best manifold" according to a given criterion. This is the case for instance of minimal surfaces, in which the criterion is simply the area of the manifold, or of problems of Willmore type, where the cost is as integral involving a function of the mean curvature.
More in general, one can consider optimal control problems where the control variable is a manifold, and the associated state variables are functions defined on it. For instance, given a smooth closed curve Γ in $\mathbb{R}^n$ and an unitary vector field η on it, we want to find a surface S having Γ as a boundary and η as a co-normal vector field, and minimizing an integral functional J with respect to the two-dimensional measure $\mathcal{H}^2$ on S:

$$J(S,u_S) := \int_S (1+|u_S|^2)\, d\mathcal{H}^2 \ , \tag{1.1}$$

where the function u_S is supposed to solve a state equation of the kind

$$-\Delta_S u + u = g \ \text{ on } S \ , \qquad u \in H^1(S) \ . \tag{1.2}$$

Here g is a continuous prescribed source term compactly supported in $\mathbb{R}^n$, and Δ_S is the Laplace-Beltrami operator on S.
To our knowledge, no general method exists in the literature to deal with such a kind of problem; actually in the last years, much attention has been paid to optimization problems where the control variable belongs to some class of open subsets of $\mathbb{R}^n$ [35, 66, 25], but little has been done in the case of low-dimensional structures. The aim of this paper is to propose a general approach to this new and yet quite natural class of optimal control problems.
Our method starts from the ideas of some recent papers [18, 16], in which it has been shown that a fruitful way to describe k-manifolds in $\mathbb{R}^n$ is to identify them with the overlying k-dimensional Hausdorff measures. Indeed, the weak setting of measures seems to offer the advantage of good compacteness properties, useful to apply the so-called *direct method* of the Calculus of Variations: it is a classical tool when dealing with variational problems, and it requires to find a topology on the class of admissible controls, which makes the cost functional coercive and lower semicontinuous as well [26].
Identifying manifolds to measures, a quite natural notion of convergence can be adopted for a sequence of controls $\{\mu_h\}$, namely, the weak * convergence in the sense of measures. Furthermore,

* Paper by G. Bouchitté, G. Buttazzo and I. Fragalà, submitted (1999) [17].

if μ belongs to the class $\mathcal{M}^+$ of all positive and finite Borel measures on $\mathbb{R}^n$, it is possible to define its *tangent bundle* T_μ and consequently the *Sobolev spaces* $H^{1,p}_\mu(\Omega)$ on an open subset Ω of $\mathbb{R}^n$: thus, a state equation such as (1.2) corresponds, in our weak setting of measures, to the Euler-Lagrange equation for a variational problem posed on the space $H^{1,2}_\mu(\mathbb{R}^n)$, where $\mu := \mathcal{H}^2 \llcorner S$.

On the other hand, (1.1) can be turned into an integral with respect to μ, that is $J(\mu, u_\mu) := \int_{\mathbb{R}^n}(1+|u_\mu|^2)\, d\mu$, where μ must be concentrated over a surface: more in general we can consider, for μ belonging to some subclass of $\mathcal{M}^+$, a cost functional $J(\mu, u_\mu)$ coercive with respect to the weak * convergence on the class of admissible controls, and lower semicontinuous with respect to the convergence $u_{\mu_h}\mu_h \rightharpoonup u_\mu \mu$, that we adopt for the state variables.

In this situation, the crucial point in order to get the existence of a solution is then to establish, on some weakly closed subclass of $\mathcal{M}^+$, the continuity of the application $\mu \mapsto u_\mu \mu$ with respect to the weak * convergence of measures, being u_μ the solution to the variational problem which provides the state equation.

Herein we deal with this problem, working on the class of measures with a tangent space of constant dimension k, with bounded *mean curvature* (see [16]), and with a positive k-dimensional density μ-a.e.: we stress that, as shown in [42], such measures are embedded in the class of k-rectifiable varifolds [64].

The contents are organized as follows.

Some notation used throughout the paper, and a brief recall of some preliminary notions, such as the tangent bundle, the mean curvature, and the Sobolev spaces $H^{1,p}_\mu(\Omega)$ associated with a positive measure μ, are given in Section 2. The difference between a new definition of weak Sobolev spaces $W^{1,p}_\mu(\Omega)$ and the strong spaces $H^{1,p}_\mu(\Omega)$ is here emphasized. This phenomenon, well known in the case when $\mu = \mathcal{L}^n$, is the source of a gap in the definition of energy integrals, and becomes crucial when dealing with varying measures.

Section 3 is devoted to some relevant improvements of previous results [16]: it is concerned with the convergence of tangent spaces and curvatures, and it is divided into three subsections. In subsection 3.1, we introduce and study the weak and the strong convergence of functions with respect to a given sequence of measures $\mu_h \rightharpoonup \mu$. The main result of subsection 3.2 is the equivalence between the weak and the strong convergence for the projecting operators P_{μ_h} onto the tangent spaces of a sequence of measures μ_h: as a consequence, we obtain a strong-weak convergence theorem for the projectors and the curvatures of a sequence of measures $\{\mu_h\}$ satisfying suitable assumptions (roughly speaking, they must have upper bounded mean curvatures, and lower bounded densities). In order to enforce this result for studying of the convergence of the Sobolev spaces $H^{1,p}_{\mu_h}(\Omega)$, that is the Γ-convergence of the associated energies, some additional assumptions will be needed on the measures μ_h, and in particular on their curvatures. Therefore, in subsection 3.3, we prove the stability of some subclasses of $\mathcal{M}^+$, and in particular the closedness of a suitable class $\mathcal{A}_\lambda$ of measures having a *prescribed boundary*, see Lemma 3.15.

We also establish a relation holding on the class $\mathcal{A}_\lambda$ between the weak convergence of measures and the Kuratowski convergence of their supports, which entails as a corollary the stability of the topological connectedness of the supports, see Theorem 3.17.
In Section 3, we study in detail the convergence of energies with respect to measures which vary in the class $\mathcal{A}_\lambda$ introduced in subsection 3.3, possibly imposing a Dirichlet condition on the fixed boundary of μ. More precisely, in subsections 4.1 we obtain, as a natural generalization of a relaxation result proved in [18], an upper bound for the limit energy, which holds on the whole $\mathcal{M}^+$. In subsection 4.2, we prove a lower bound for the limit energy in terms of the weak Sobolev space $W^{1,p}_\mu(\Omega)$ associated with the limit measure μ; such result holds on the class $\mathcal{A}_\lambda$ introduced in subsection 3.3. Subsection 4.3 contains several enlightening examples showing that, as it happens for special domains Ω in the well known case when μ is the Lebesgue measure, $H^{1,p}_\mu(\Omega)$ may be strictly contained into $W^{1,p}_\mu(\Omega)$, even when $\Omega = \mathbb{R}^n$. Therefore, a gap phenomenon may occur between the upper and lower bounds found respectively in subsections 4.1 and 4.2. When this is the case, considering suitable approximating sequences $\{\mu_h\}$, we arrive at the conclusion that it is not possible to establish in general which of the two bounds do coincides with the limit energy. In subsection 4.4, we present a conjecture for the convergence of energies defined on the weak Sobolev spaces.
Finally in Section 5, we illustrate, in a more precise and detailed way than above, the perspective of applying our approach in order to obtain the existence of a solution for optimization problems with manifolds as controls.

2. Notation and preliminaries

We give here, for the reader's convenience, a list of the notation used throughout the paper, and a brief recall of some preliminary notions such as the definitions of tangent space, mean curvature and Sobolev spaces associated with a measure.

$\mathcal{M}^+$	the class of positive and finite Borel regular measures on $\mathbb{R}^n$;
$\|\lambda\|$	the total variation measure of a possibly vector-valued measure λ;
$\mathcal{M}^d$	the class of Borel regular $\mathbb{R}^d$-valued measures λ on $\mathbb{R}^n$ with $\|\lambda\|(\mathbb{R}^n) < +\infty$ (d is omitted when $d=1$);
$Y(\mathbb{R}^n, \mu; \mathbb{R}^d)$	the class of Young measures λ on $\mathbb{R}^n \times \mathbb{R}^d$ with $\lambda(B \times \mathbb{R}^n) = \mu(B)$ for every Borel subset B of $\mathbb{R}^n$;
$\operatorname{spt} \lambda$	the support of the measure λ ;
$[\lambda]$	the barycenter of a measure λ $\big([\lambda] := \int z \, d\lambda(z)\big)$;

$\llcorner$	the symbol for the restriction of a measure or of a distribution;
$\mathcal{C}_0(\mathbb{R}^d)$	the space of continuous functions vanishing at infinity on $\mathbb{R}^d$;
$\langle\ ,\ \rangle$	the duality pairing between $\mathcal{C}_0(\mathbb{R}^d)$ and $\mathcal{M}^d$ (unless otherwise specified);
$\mathcal{L}^n$	the Lebesgue measure on $\mathbb{R}^n$;
$\mathcal{H}^k$	the k-dimensional Hausdorff measure on $\mathbb{R}^n$;
$\theta_k(\mu,x)$	the k-dimensional density of μ at x $\left(\theta_k(\mu,x) := \lim_{\rho\to 0^+} \frac{\mu(B_\rho(x))}{\omega_k \rho^k}\right.$, being $B_\rho(x)$ the ball of radius ρ and center x, and ω_k the $\mathcal{L}^k$ measure of the unit ball in $\left.\mathbb{R}^k\right)$;
δ_P	the Dirac delta at the point P;
$(L^p_\mu(\Omega))^d$	the space $L^p(\Omega,\mu;\mathbb{R}^d)$ (Ω is omitted when $\Omega=\mathbb{R}^n$) ;
$(\mathcal{D}(\Omega))^d$	the space of test functions $\psi\in\mathcal{C}^\infty_0(\Omega;\mathbb{R}^d)$ (Ω is omitted when $\Omega=\mathbb{R}^n$);
$(\mathcal{D}'(\Omega))^d$	the space of $\mathbb{R}^d$-valued distributions on $\mathbb{R}^n$ (Ω is omitted when $\Omega=\mathbb{R}^n$);
χ_E	the indicator function taking the values 0 on E and $+\infty$ on its complementary;
$\cdot$	the symbol for the scalar product in $\mathbb{R}^n$ or in $\mathbb{R}^{n^2}$ (for $A\in\mathbb{R}^{n^2}$, $v\in\mathbb{R}^n$, and $\alpha\in\mathbb{R}$, we write Av and αA to denote respectively the vector $A_{ij}v_j$ and the matrix αA_{ij}).

Given a measure $\mu\in\mathcal{M}^+$, we set

$$X_\mu := \left\{\Phi\in(L^1_\mu)^n \ : \ \mathrm{div}(\Phi\mu)\in\mathcal{M}\right\} ,$$

where $\mathrm{div}(\Phi\mu)$ is the distribution

$$\langle \mathrm{div}(\Phi\mu),\psi\rangle_{(\mathcal{D}',\mathcal{D})} := -\int \Phi\cdot\nabla\psi\, d\mu , \qquad \forall\psi\in\mathcal{D} .$$

Then the *tangent space* $T_\mu(x)$ can be defined for μ-a.e. $x\in\mathbb{R}^n$ as the μ-measurable multifunction

$$T_\mu(x) := \mu - \mathrm{ess}\bigcup\{\Phi(x) \ : \ \Phi\in X_\mu\} ,$$

characterized by (see for instance [21, Section 2])

(i) for every $\Phi\in X_\mu$, $\Phi(x)\in T_\mu(x)$ for μ-a.e. x;

(ii) $T_\mu(x)$ is the minimal (with respect to inclusion μ-a.e. x) closed valued and μ-measurable multifunction verifying (i).

This notion of tangent space to a measure has been introduced (in a slightly different version) in [18], and has been studied and compared with other definitions in [42]. It is easy to see that, for μ-a.e. $x\in\mathbb{R}^n$, $T_\mu(x)$ is a linear subspace of $\mathbb{R}^n$. We denote by $\dim T_\mu(x)$ its dimension, which may be a non-constant function of x, and by $P_\mu(x)$ the orthogonal projection of $\mathbb{R}^n$ onto $T_\mu(x)$. The tangential gradient of $\psi\in\mathcal{D}$ with respect to μ is given, for μ-a.e. $x\in\mathbb{R}^n$, by

$$\nabla_\mu\psi(x) := P_\mu(x)\nabla\psi(x) .$$

The *mean curvature* $H(\mu)$ of a measure $\mu\in\mathcal{M}^+$ has been introduced in [16] as the distribution $\mathrm{div}(P_\mu\mu)$, *i.e.*

$$\langle H(\mu),X\rangle_{((\mathcal{D}')^n,\mathcal{D}^n)} := -\int \mathrm{div}_\mu X\, d\mu , \qquad \forall X\in\mathcal{D}^n ,$$

where $\operatorname{div}_\mu X := \sum_{i=1}^n (\nabla_\mu X^i)_i$. The class of all measures $\mu \in \mathcal{M}^+$ such that $H(\mu) \in \mathcal{M}^n$ will be denoted by $\mathcal{M}_{BC}$, which stays for *measures with bounded curvature* (see [16] for more details and examples). For $\mu \in \mathcal{M}_{\rm BC}$, the Radon-Nykodim Theorem yields $H(\mu) = (h_\mu)\mu + \partial\mu$, where h_μ belongs to $(L^1_\mu)^n$ and $\partial\mu$ is a measure in $\mathcal{M}^n$, singular with respect to μ, which will be called *the boundary of* μ. In the case $\mu = \mathcal{H}^k \llcorner S$, where S is a smooth k-dimensional manifold with boundary ∂S, one can check by using the divergence theorem on a smooth manifold (see for instance [64]) that h_μ coincides with the vector mean curvature of S, and $\partial\mu$ with the measure $\nu\mathcal{H}^{k-1}\llcorner\partial S$, where ν is the inner unit conormal vector to ∂S (that is ν is tangent to S and normal to ∂S).

Let $p \in [1+\infty]$, let p' be the conjugate exponent of p, and set

$$\begin{aligned} X^{p'}_\mu &:= \left\{ \Phi \in (L^{p'}_\mu)^n \ : \ \operatorname{div}(\Phi\mu) \in L^{p'}_\mu \right\} \ ; \\ T^p_\mu(x) &:= \mu - \operatorname{ess} \bigcup \left\{ \Phi(x) \ : \ \Phi \in X^{p'}_\mu \right\} \ ; \\ P^p_\mu(x) &:= \text{ the orthogonal projection of } \mathbb{R}^n \text{ onto } T^p_\mu(x). \end{aligned}$$

Notice that the notion of T^p_μ may *a priori* depend on p, even if this does not happen for all 'usual' measures [18, 42] (see also Remark 3.14 (iv)); thus we will omit the indication of p in the notation $P^p_\mu(x)$.

Let Ω be an open subset of $\mathbb{R}^n$. In order to define the *Sobolev space* $H^{1,p}_\mu(\Omega)$, we can localize on Ω the procedure followed in [18], by considering the linear operator $A : D(A) \subset L^p_\mu(\Omega) \to (L^p_\mu(\Omega))^n$ defined by

$$D(A) := \mathcal{D} \ , \qquad A\psi(x) := P_\mu(x)\nabla\psi(x) \ .$$

Then A is closable (*cf.* [18, Proposition 2.1]) and, by definition, $H^{1,p}_\mu(\Omega)$ is the domain of its unique closed extension $\overline{A}$, *i.e.* the completion of $\mathcal{D}$ with respect to the norm $\|\psi\|_{L^p_\mu(\Omega)} + \|A\psi\|_{(L^p_\mu(\Omega))^n}$. It turns out to be a reflexive Banach space for $p \in (1,+\infty)$. We set $\nabla_\mu u := \overline{A}u$ for every $u \in H^{1,p}_\mu(\Omega)$.

In a similar way, if we change the domain of the operator A into $\mathcal{D}(\Omega)$, we obtain the *Dirichlet space* $H^{1,p}_{\mu,0}(\Omega)$ given be the functions of $H^{1,p}_\mu(\Omega)$ vanishing on the complement of Ω.

Let $\mu \in \mathcal{M}_{BC}$ satisfy

$$h_\mu \in (L^\infty_\mu)^n \ , \qquad \mu\big(\operatorname{spt}(\partial\mu)\big) = 0 \ ; \tag{2.1}$$

notice that (2.1) is fulfilled whenever μ belongs to the class $\mathcal{A}_\lambda$ which will be introduced in subsection 3.3.

For any function $u \in L^p_\mu(\Omega)$, we define its distributional μ-tangential gradient $D_\mu u \in (\mathcal{D}'(\Omega))^n$, as

$$D_\mu u := \operatorname{div}(uP_\mu\mu)\llcorner\Omega_\mu - uh_\mu \ , \tag{2.2}$$

where $\Omega_\mu := \Omega \setminus \mathrm{spt}(\partial\mu)$. In particular, when $\mu = \mathcal{L}^n$, $D_\mu u$ reduces to the usual notion of distributional gradient of u.
We then define the Sobolev space $W^{1,p}_\mu(\Omega)$ as

$$W^{1,p}_\mu(\Omega) := \left\{ u \in L^p_\mu(\Omega) \ : \ D_\mu u \in (L^p_\mu(\Omega))^n \right\}$$

(observe that, by (2.1), $D_\mu u \in (L^p_\mu(\Omega))^n$ if and only if $\mathrm{div}(uP_\mu\mu) \llcorner \Omega_\mu \in (L^p_\mu(\Omega))^n$); the corresponding weak Dirichlet space is then given by

$$W^{1,p}_{\mu,0}(\Omega) := \left\{ u \in L^p_\mu(\Omega) \ : \ \tilde{u} \in W^{1,p}_\mu(\mathbb{R}^n) \right\} ,$$

being $\tilde{u}$ the extension to zero of u outside Ω.
The next proposition shows that $W^{1,p}_\mu(\Omega)$ is a Banach space and that (2.2) extends on $W^{1,p}_\mu(\Omega)$ the previous definition of $\nabla_\mu u$ given for $u \in H^{1,p}_\mu(\Omega)$.

Proposition 2.1. *Let $\mu \in \mathcal{M}_{BC}$ with $h_\mu \in (L^\infty_\mu)^n$. Then:*

(i) $W^{1,p}_\mu(\Omega)$ is a Banach space endowed with the norm $\|u\|_{L^p_\mu(\Omega)} + \|D_\mu u\|_{L^p_\mu(\Omega)}$;

(ii) $H^{1,p}_\mu(\Omega)$ is a (closed) subspace of $W^{1,p}_\mu(\Omega)$, and $D_\mu u = \nabla_\mu u$ for any $u \in H^{1,p}_\mu(\Omega)$.

Proof. If $\{u_h\}$ is a Cauchy sequence in $L^p_\mu(\Omega)$, there exists $u \in L^p_\mu(\Omega)$ such that $u_h \to u$ in $L^p_\mu(\Omega)$, and this yields $D_\mu u_h = \mathrm{div}(u_h P_\mu \mu) - u_h h_\mu \rightharpoonup \mathrm{div}(uP_\mu\mu) - uh_\mu = D_\mu u$ in $(\mathcal{D}'(\Omega_\mu))^n$. If $\{D_\mu u_h\}$ is also a Cauchy sequence in $(L^p_\mu(\Omega))^n$, there exists $v \in (L^p_\mu(\Omega))^n$ such that $D_\mu u_h \to v$ in $(L^p_\mu(\Omega))^n$. By the uniqueness of the limit in $(\mathcal{D}'(\Omega_\mu))^n$, and taking into account that $\Omega \setminus \Omega_\mu$ is μ-negligible, it follows that $v = D_\mu u$ μ-a.e. on Ω.
Let us now prove the second part of the statement. For any $u \in H^{1,p}_\mu(\Omega)$, by definition, there exists $\{u_h\} \subset \mathcal{D}$ such that $u_h \to u$ in $L^p_\mu(\Omega)$ and $\nabla_\mu u_h \to \nabla_\mu u$ in $(L^p_\mu(\Omega))^n$. For any h, since u_h are smooth, there holds $\mathrm{div}(u_h P_\mu \mu) \llcorner \Omega_\mu = \big(\nabla_\mu u_h + u_h\, h_\mu\big)\mu$. Since $\mathrm{div}(u_h P_\mu\mu) \llcorner \Omega_\mu \rightharpoonup \mathrm{div}(uP_\mu\mu) \llcorner \Omega_\mu$ in $(\mathcal{D}'(\Omega_\mu))^n$, and $\big(\nabla_\mu u_h + u_h\, h_\mu\big) \to \big(\nabla_\mu u + u\, h_\mu\big)$ in $(L^p_\mu(\Omega))^n$, by the uniqueness of the limit in $(\mathcal{D}'(\Omega_\mu))^n$, $\mathrm{div}(uP_\mu\mu) \llcorner \Omega_\mu = \nabla_\mu u + u\, h(\mu)$. ■

3. Convergence of tangent spaces and curvatures

Let $\{\mu_h\}$ be a sequence in $\mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$. To the aim of studying the behaviour of the corresponding tangent spaces T_{μ_h}, we introduce in subsection 3.1 the notions of *weak* and *strong* convergence of f_h to f with respect to the pair (μ_h, μ), being $f_h \in (L^1_{\mu_h})^d$ and $f \in (L^1_\mu)^d$. Then in subsection 3.2 we prove, using an extremality argument from Convex Analysis, that the strong and the weak convergence are equivalent for the matrices P_{μ_h} of orthogonal projection onto T_{μ_h}. This theorem, as well as the stability of some subclasses of $\mathcal{M}_{\mathrm{BC}}$ presented in subsection 3.3, is a relevant improvement of previous results (see [16]).

- 3.1 Weak and strong convergence with respect to measures

We recall that any Young measure $\nu \in Y(\mathbb{R}^n, \mu; \mathbb{R}^d)$ can be described by a family of probabilities $(\nu_x)_{x\in\mathbb{R}^n}$ on $\mathbb{R}^d$, usually called the *disintegration* of ν, characterized by $\nu = \int (\delta_x \otimes \nu_x)\, d\mu(x)$, *i.e.*

$$\int_{\mathbb{R}^n\times\mathbb{R}^d} \varphi(x,z)\, d\nu(x,z) = \int \langle \varphi(x,\cdot), \nu_x\rangle\, d\mu(x)$$

for every ν-integrable function φ on $\mathbb{R}^n \times \mathbb{R}^d$. For a brief survey about Young measures, we refer for instance to [69], while a proof of the disintegration formula can be found in [38].

Definition 3.1. *Let $\mu_h, \mu \in \mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$, and let f_h, $f : \mathbb{R}^n \to \mathbb{R}^d$ be respectively μ_h and μ-integrable. We say that f_h weakly converges to f with respect to the pair (μ_h, μ), and we write $f_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} f$, if $f_h\mu_h \rightharpoonup f\mu$ in $\mathcal{M}^d$.*

When $f_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} f$, it is quite natural to consider, as in the general varifolds theory [64], the measures $\nu_h \in Y(\mathbb{R}^n, \mu_h; \mathbb{R}^d)$ associated with f_h, that is

$$\langle \nu_h, \varphi\rangle := \int \varphi(x, f_h(x))\, d\mu_h(x)\ , \qquad \forall \varphi \in \mathcal{C}_0(\mathbb{R}^n \times \mathbb{R}^d)\ .$$

Since $\{\nu_h\}$ is bounded, a subsequence $\{\nu_{h_k}\}$ weakly converges to a measure ν on the product space $\mathbb{R}^n \times \mathbb{R}^d$. It turns out that μ is the image of ν by the projection map of $\mathbb{R}^n \times \mathbb{R}^d$ onto $\mathbb{R}^n$; indeed, for any test function $\psi \in \mathcal{C}_0(\mathbb{R}^n)$ we have $\langle \lambda_h, \psi\rangle = \langle \mu_h, \psi\rangle$. Therefore the disintegration of ν yields

$$\lim_{k\to+\infty} \int \varphi(x, f_{h_k}(x))\, d\mu_{h_k} = \int \langle \varphi(x,\cdot), \nu_x\rangle\, d\mu(x)\ , \qquad \forall \varphi \in \mathcal{C}_0(\mathbb{R}^n \times \mathbb{R}^d)\ .$$

We then say that $\{f_{h_k}\}$ weakly converges to the family of probabilities (ν_x) *in the sense of Young measures* with respect to (μ_{h_k}, μ), and we write $f_{h_k} \stackrel{Y(\mu_{h_k},\mu)}{\longrightarrow} (\nu_x)$.

We are mainly interested in the case where the family of probabilities (ν_x) does not depend on the choice of the subsequence and is a family of Dirac masses, which means, roughly speaking, that we have no oscillations of the sequence $\{f_h\}$ with respect to $\{\mu_h\}$. Thus we give the following definition.

Definition 3.2. *Let μ_h, μ, f_h, f be as in Definition 3.1. We say that f_h strongly converges to f with respect to the pair (μ_h, μ), and we write $f_h \stackrel{s(\mu_h,\mu)}{\longrightarrow} f$, if*

$$\lim_{h\to+\infty} \int \varphi(x, f_h(x))\, d\mu_h = \int \varphi(x, f(x))\, d\mu(x)\ , \qquad \forall \varphi \in \mathcal{C}_0(\mathbb{R}^n \times \mathbb{R}^d)\ .$$

Remark 3.3. Notice that, when $\mu_h = \mu$ for every h, the strong convergence of f_h to f with respect to (μ, μ) reduces to the usual convergence in $Y(\mathbb{R}^n, \mu; \mathbb{R}^d)$ of the measures carried by the graphs of f_h to the measure carried by the graph of f. In particular, see for instance [69, Proposition 1], the strong convergence of f_h to f with respect to (μ, μ) occurs if and only if f_h converges to f in measure (with respect to μ).

The following proposition allows us to enlarge the class of admissible test functions φ when $f_h \stackrel{Y(\mu_h,\mu)}{\longrightarrow} (\nu_x)$.

Proposition 3.4. *Let $f_h \in (L^1_{\mu_h})^d$, with $f_h \stackrel{Y(\mu_h,\mu)}{\longrightarrow} (\nu_x)$. Then*
(i) for every non-negative lower semicontinuous function φ on $\mathbb{R}^n \times \mathbb{R}^d$ we have

$$\liminf_{h\to+\infty} \int \varphi(x, f_h(x))\, d\mu_h \geq \int \langle \varphi(x,\cdot), \nu_x \rangle\, d\mu(x) \ ;$$

(ii) assume in addition that $\{f_h\}$ is uniformly bounded in the $(L^p_{\mu_h})^d$-norm , i.e. $\|f_h\|_{(L^p_{\mu_h})^d} \leq M$ for some $p > 1$ and some positive constant M; then, for any continuous function φ on $\mathbb{R}^n \times \mathbb{R}^d$ satisfying a growth condition of the type $|\varphi(x,z)| \leq C(1 + |z|^q)$ with $q < p$, we have

$$\lim_{h\to+\infty} \int \varphi(x, f_h(x))\, d\mu_h = \int \langle \varphi(x,\cdot), \nu_x \rangle\, d\mu(x) \ .$$

(In particular, $\int \langle |z|^p, \nu_x \rangle\, d\mu < +\infty$.)

Proof. (i) Let $\{\varphi_k\}$ be an increasing sequence of compactly supported continuous functions on $\mathbb{R}^n \times \mathbb{R}^d$ such that $\varphi = \sup\limits_k \varphi_k$; by monotone convergence, we get

$$\int \langle \varphi(x,\cdot), \nu_x \rangle\, d\mu(x) = \lim_{k\to+\infty} \int \langle \varphi_k(x,\cdot), \nu_x \rangle\, d\mu(x) =$$
$$= \sup_k \lim_{h\to+\infty} \int \varphi_k(x, f_h(x))\, d\mu_h(x) \leq \liminf_{h\to+\infty} \int \varphi(x, f_h(x))\, d\mu_h(x) \ .$$

(ii) We may assume without loss of generality that φ is non-negative, as the statement can be proved separately for $\varphi^+ := \max\{\varphi, 0\}$ and $\varphi^- := -\min\{\varphi, 0\}$. Then by (i) it is enough to show the inequality

$$\limsup_{h\to+\infty} \int \varphi(x, f_h(x))\, d\mu_h \leq \int \langle \varphi(x,\cdot), \nu_x \rangle\, d\mu(x) \ .$$

For any positive R, set $A_h^R := \{x \in \mathbb{R}^n \ : \ \varphi(x, f_h(x)) < R\}$ and $B_h^R := \mathbb{R}^n \setminus A_h^R$. Then we can write

$$\int \varphi(x, f_h(x))\, d\mu_h = \int_{A_h^R} \varphi(x, f_h(x))\, d\mu_h + \int_{B_h^R} \varphi(x, f_h(x))\, d\mu_h := \mathrm{I}_h^R + \mathrm{II}_h^R \ .$$

We claim that $\limsup_{R\to+\infty}\limsup_{h\to+\infty}(\mathrm{II}_h^R) = 0$. Indeed, by the growth condition on φ, the uniform bound on the $(L^p_{\mu_h})^d$-norm of f_h, the Hölder inequality, and the hypothesis $p > q$, we get, for a suitable positive constant C':

$$R\mu_h(B_h^R) \le \mathrm{II}_h^R \le C\int_{B_h^R}(1+|f_h|^q)\,d\mu_h \le C\Big\{\mu_h(B_h^R)+\mu_h(B_h^R)^{1-q/p}\Big[\int|f_h|^p\,d\mu_h\Big]^{q/p}\Big\} \le C'.$$

Hence,

$$\mathrm{II}_h^R \le C\Big[\mu_h(B_h^R)+M^q\mu_h(B_h^R)^{1-q/p}\Big] \le C\Big[\frac{C'}{R}+M^q\Big(\frac{C'}{R}\Big)^{1-q/p}\Big]\ ,$$

which gives the assertion, as $p > q$. Now it is sufficient to prove that

$$\limsup_{R\to+\infty}\ \limsup_{h\to+\infty}\ (\mathrm{I}_h^R) \le \int\langle\varphi(x,\cdot),\nu_x\rangle\,d\mu(x)\ . \tag{3.1}$$

To this aim, let θ be a continuous function with compact support on $\mathbb{R}^+$, with values in $[0,1]$, and satisfying $\theta(t) = t$ for $t \in [0,1]$. Set $\varphi^R(x,z) := R\theta\left(\frac{\varphi(x,z)}{R}\right)$. One has $\mathrm{I}_h^R = \int\varphi^R(x,f_h(x))\,d\mu_h(x)$; thus, since $f_h \overset{Y(\mu_h,\mu)}{\longrightarrow} (\nu_x)$, and φ^R is admissible as a test function in the Young convergence, for every R it holds

$$\limsup_{h\to+\infty}(\mathrm{I}_h^R) \le \int\langle\varphi^R(x,\cdot),\nu_x\rangle\,d\mu(x)\ . \tag{3.2}$$

Now notice that

$$\begin{aligned}|\langle\varphi^R(x,\cdot),\nu_x\rangle| &\le C\int(1+|z|^q)\,d\nu_x(z) \le C\Big\{1+\Big[\int|z|^p\,d\nu_x(z)\Big]^{q/p}\Big\} \le \\ &\le C\Big\{1+\liminf_{h\to+\infty}\Big[\int|f_h|^p\,d\mu_h\Big]^{q/p}\Big\} \le C(1+M^q)\ ,\end{aligned}$$

hence (3.1) follows from (3.2) by dominated convergence. ■

The next result will be useful to prove the strong convergence of the projecting operators. We prove it under the assumption $\|f_h\|_{(L^p_{\mu_h})^d} \le M$ for some $p > 1$, but we stress that this is just a way for simplifying the proof; actually, such hypothesis will be verified, with $p = +\infty$, in the framework of Theorem 3.10.

Lemma 3.5. *Let $\{\mu_h\} \subset \mathcal{M}^+$, with $\mu_h \rightharpoonup \mu$, and let $f_h \in (L^p_{\mu_h})^d$, with $\|f_h\|_{(L^p_{\mu_h})^d} \le M$, for some $p > 1$. Assume that $f_h \overset{w(\mu_h,\mu)}{\longrightarrow} f$ and $f_h \overset{Y(\mu_h,\mu)}{\longrightarrow} (\nu_x)$. Then $[\nu_x] = f(x)$ for μ-a.e. x. Moreover, if it exists a closed subset C of $\mathbb{R}^d$ such that, for every h, $f_h(x) \in C$ for μ_h-a.e. x , then $\mathrm{spt}(\nu_x) \subseteq C$ for μ-a.e. x.*

Proof. By Proposition 3.4 (i) and by the assumption $\|f_h\|_{(L^p_{\mu_h})^d} \le M$, we have:

$$\int\langle|z|^p,\nu_x(z)\rangle\,d\mu(x) \le \liminf_{h\to+\infty}\int|f_h(x)|^p\,d\mu_h(x) \le M^p\ .$$

Thus, the barycenter $[\nu_x]$ exists finite for μ-a.e. x. Take now $\varphi(x,z) = \psi(x)z^* \cdot z$, where $\psi \in \mathcal{C}_0(\mathbb{R}^n)$ and z^* is fixed in $\mathbb{R}^n$. Notice that, since $p > 1$, by Proposition 3.4 (ii) φ is an admissible test function in the Young convergence. Therefore we get

$$\int \psi(x)\langle z^* \cdot z, \nu_x(z)\rangle \, d\mu(x) = \lim_{h\to+\infty} \int \psi(x)[z^* \cdot f_h(x)] \, d\mu_h(x) = \int \psi(x)[z^* \cdot f(x)] \, d\mu(x) \ .$$

Since $\psi \in \mathcal{C}_0(\mathbb{R}^n)$ is arbitrary, we conclude that $[\nu_x] = f(x)$ for μ-a.e. x.
Assume now that f_h belongs, for μ_h-a.e. x, to a closed subset C of $\mathbb{R}^d$. Taking as a test function $\varphi(x,z) = \alpha(x)\beta(z)$, where $\alpha \in \mathcal{C}_0(\mathbb{R}^n)$ and $\beta \in \mathcal{C}_0(\mathbb{R}^d)$ with $\beta = 0$ on C, we get

$$\int \alpha(x)\langle \beta(z), \nu_x(z)\rangle \, d\mu(x) = \lim_{h\to+\infty} \int \alpha(x)\beta(f_h(x)) \, d\mu_h(x) = 0 \ .$$

Since α is arbitrary in $\mathcal{C}_0(\mathbb{R}^n)$, we deduce that, for μ-a.e. $x \in \mathbb{R}^n$, it is $\langle \beta, \nu_x\rangle = 0$ for any $\beta \in \mathcal{C}_0(\mathbb{R}^d)$ vanishing over C. ■

Definition 3.6. *Let $\{\mu_h\} \subseteq \mathcal{M}^+$, $\mu_h \rightharpoonup \mu$. Let $(f_h, g_h) \in (L^1_{\mu_h})^d \times (L^1_{\mu_h})^m$, and let $(f,g) \in (L^1_\mu)^d \times (L^1_\mu)^m$. We say that $\{(f_h, g_h)\}$ strongly-weakly converges to (f,g) with respect to (μ_h, μ), and we write $(f_h, g_h) \overset{sw(\mu_h,\mu)}{\longrightarrow} (f,g)$, if $f_h \overset{s(\mu_h,\mu)}{\longrightarrow} f$ and $g_h \overset{w(\mu_h,\mu)}{\longrightarrow} g$.*

Remark 3.7. When $(f_h, g_h) \overset{sw(\mu_h,\mu)}{\longrightarrow} (f,g)$, passing to a subsequence there exists a family of probabilities (λ_x) on $\mathbb{R}^d \times \mathbb{R}^m$ such that $(f_{h_k}, g_{h_k}) \overset{Y(\mu_{h_k},\mu)}{\longrightarrow} (\lambda_x)$ and $\lambda_x = \delta_{f(x)} \otimes \nu_x$, where (ν_x) is a family of probabilities on $\mathbb{R}^m$ with $[\nu_x] = g(x)$ for μ-a.e. x.

Remark 3.8. If $(f_h, g_h) \in L^p_{\mu_h} \times L^q_{\mu_h}$ strongly-weakly converge to (f,g) with respect to (μ_h, μ), then the product sequence $\{f_h g_h\}$ converges to fg weakly with respect to (μ_h, μ), provided $\frac{1}{p} + \frac{1}{q} < 1$, and $\|f_h\|_{L^p_{\mu_h}} \leq C$, $\|g_h\|_{L^q_{\mu_h}} \leq C$ for some positive constant C. This can be easily checked by using Proposition 3.4 (ii) and Remark 3.7. Arguing by components, the same holds if (f_h, g_h) take values into $\mathbb{R}^{n^2} \times \mathbb{R}^n$ or into $\mathbb{R} \times \mathbb{R}^{n^2}$, and the product $f_h g_h$ is intended respectively as the application of the matrix f_h to the vector g_h, or as the multiplication of the the matrix g_h by the scalar f_h.

- 3.2 Strong convergence of the projecting operators

We begin by a preliminary lemma, proved for completeness, which provides a useful criterion to establish the non-oscillation of a sequence $\{f_h\}$ with respect to $\{\mu_h\}$. For a convex subset K of $\mathbb{R}^d$, we denote by K^{ext} the set of its extremal points.

Lemma 3.9. *Let ν be a probability measure on $\mathbb{R}^d$, with $\int_{\mathbb{R}^d} |z|\,d\nu(z) < +\infty$. If there exists a closed convex subset K of $\mathbb{R}^d$ such that $\operatorname{spt}\nu \subseteq K$ and $[\nu] \in K^{\text{ext}}$, then $\nu = \delta_{[\nu]}$.*

Proof. Suppose by contradiction there exists a measurable subset A of $\mathbb{R}^n$ with $0 < \nu(A) < 1$. Set $t := \nu(A)$, $B := \mathbb{R}^n \setminus A$, $\nu_A := \nu \llcorner A$, and $\nu_B := \nu \llcorner B$. Then $t[\nu_A] + (1-t)[\nu_B] = [\nu]$. By assumption $\operatorname{spt}(\nu_A)$ and $\operatorname{spt}(\nu_B)$ are contained in K; thus, by Jensen's inequality, $[\nu_A]$ and $[\nu_B]$ belong to K. Since $[\nu] \in K^{\text{ext}}$, it then must be $t = 0$ or $t = 1$, in contradiction with the choice of A. ∎

We are now in a position to prove the main result of strong convergence for the projecting operators.

Theorem 3.10. *Let $\{\mu_h\} \subseteq \mathcal{M}^+$, $\mu_h \rightharpoonup \mu$. Then*

$$P_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} P_\mu \implies P_{\mu_h} \overset{s(\mu_h,\mu)}{\longrightarrow} P_\mu \,. \tag{3.3}$$

Proof. Since $P_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} P_\mu$, passing to a subsequence, we have $P_{\mu_{h_k}} \overset{Y(\mu_{h_k},\mu)}{\longrightarrow} (\nu_x)$ for a suitable measure $\nu \in Y(\mathbb{R}^n, \mu; \mathbb{R}^{n^2})$. By Lemma 3.5, we have, for μ-a.e. x, $[\nu_x] = P_\mu(x)$ and $\operatorname{spt}(\nu_x) \subseteq K$, where $K \subset \mathbb{R}^{n^2}$ is the closed convex set of all symmetric matrices A whose spectrum $\sigma(A)$ is contained in the interval $[0,1]$. By Lemma 3.9, it is then enough to show that $P_\mu(x)$ belongs to K^{ext} for μ-a.e. x; indeed this yields $\nu_x = \delta_{P_\mu(x)}$ for μ-a.e. x, thus (ν_x) does not depend on the choice of the subsequence and we have $P_{\mu_h} \overset{s(\mu_h,\mu)}{\longrightarrow} P_\mu$. Let us show that $K^{\text{ext}} = \{A \in K \,:\, \sigma(A) \subseteq \{0,1\}\}$. In order to prove the inclusion $K^{\text{ext}} \subseteq \{A \in K \,:\, \sigma(A) \subseteq \{0,1\}\}$, assume that $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \notin \{0,1\}$. Let $2\lambda_1 = \alpha_1 + \beta_1$, with $\alpha_1, \beta_1 \in (0,1)$, $\alpha_1 \neq \beta_1$, and set $B := \operatorname{diag}(\alpha_1, \dots, \lambda_n)$, $C := \operatorname{diag}(\beta_1, \dots, \lambda_n)$. We then have $A = \frac{1}{2}B + \frac{1}{2}C$, with B, $C \in K$, $B \neq C$. So $A \notin K^{\text{ext}}$. Conversely, assume that $A \in K$, $\sigma(A) \subseteq \{0,1\}$, and $A = tB + (1-t)C$, for some $B, C \in K$, $t \in (0,1)$. If $(Av, v) = \lambda$ (with $\lambda \in \{0,1\}$), we have $t(Bv,v) + (1-t)(Cv,v) = \lambda$, which implies (in both cases $\lambda = 0$ or $\lambda = 1$) $(Bv,v) = (Cv,v) = \lambda$. Then $B = C = A$ and $A \in K^{\text{ext}}$. ∎

The following corollaries follow from the coupling between the above theorem and some results about the weak convergence of the projecting operators proved in [16].

Corollary 3.11. *Let $\{\mu_h\} \subset \mathcal{M}_{\text{BC}}$ with $\mu_h \rightharpoonup \mu$, $|H(\mu_h)|(\mathbb{R}^n) \leq C$ for some positive constant C, and $\dim T_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} \dim T_\mu$. Then $P_{\mu_h} \overset{s(\mu_h,\mu)}{\longrightarrow} P_\mu$.*

Proof. It follows from Theorem 3.10 above and Theorem 3.8 of [16]. ∎

Corollary 3.12. *Let* $\{\mu_h\} \subseteq \mathcal{M}_k := \{\mu \in \mathcal{M}_{\rm BC} \ : \dim T_\mu = k \ \mu\text{-a.e.} \}$, *with* $\mu_h \rightharpoonup \mu$. *Assume that for some positive constants* C *and* c, *one of the following conditions holds*

$$(i) \qquad \theta_h \geq c \quad \mu_h\text{-a.e.}, \quad |H(\mu_h)|(\mathbb{R}^n) \leq C$$

$$(ii) \qquad \int (\log \theta_h)^- \, d\mu_h \leq C, \quad H(\mu_h) \ll \mu_h \ , \quad \|h_{\mu_h}\|_{L^p_{\mu_h}} \leq C \quad , \ p > 1 \ .$$

Then $P_{\mu_h} \overset{s(\mu_h,\mu)}{\longrightarrow} P_\mu$; *furthermore, in case (ii),* $H(\mu) \ll \mu$ *and* $(P_{\mu_h}, h_{\mu_h}) \overset{sw(\mu_h,\mu)}{\longrightarrow} (P_\mu, h_\mu)$.

Proof. The strong convergence of the projecting operators follows from Theorem 3.10 above and Corollary 4.4 of [16]. To prove the weak convergence of the curvatures in case (ii), notice that, by the assumption $\|h_{\mu_h}\|_{L^p_{\mu_h}} \leq C$, with $p > 1$, the sequence $\{H(\mu_h)\}$ converges, up to subsequences, to a measure absolutely continuous with respect to μ (see for instance Lemma 3.1 of [16]). On the other hand, by Corollary 4.4 of [16], $H(\mu_h) \rightharpoonup H(\mu)$. Therefore $H(\mu) \ll \mu$, and $h_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} h_\mu$. ■

In the following proposition, we show that the strong-weak convergence result stated in the above corollary yields the lower semicontinuity of functionals depending on (P_μ, h_μ) which are possibly not convex with respect to P_μ.

Proposition 3.13. *Under the same assumptions of Corollary 3.12, case (ii), for every lower semicontinuous integrand* $\varphi : \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R}^n \to [0, +\infty]$ *such that* $\varphi(x, s, \cdot)$ *is convex on* $\mathbb{R}^n$, *we have*

$$\int \varphi(x, P_\mu(x), h_\mu(x)) \, d\mu(x) \leq \liminf_{h \to +\infty} \int \varphi(x, P_{\mu_h}(x), h_{\mu_h}(x)) \, d\mu_h(x) \ .$$

Proof. One can extract a subsequence $(P_{\mu_{h_k}}, h_{\mu_{h_k}})$ such that

$$\liminf_{h \to +\infty} \int \varphi(x, P_{\mu_h}(x), h_{\mu_h}(x)) \, d\mu_h(x) = \lim_{k \to +\infty} \int \varphi(x, P_{\mu_{h_k}}(x), h_{\mu_{h_k}}(x)) \, d\mu_{h_k}(x) \ .$$

Possibly passing to a subsequence again, there exists a family of probabilities (λ_x) on $\mathbb{R}^{n^2} \times \mathbb{R}^n$ such that $(P_{\mu_{h_k}}, h_{\mu_{h_k}}) \overset{Y(\mu_{h_k},\mu)}{\longrightarrow} (\lambda_x)$. Furthermore (see Remark 3.7) (λ_x) can be decomposed as $\lambda_x = \delta_{P_\mu(x)} \otimes \nu_x$, where (ν_x) is a family of probabilities on $\mathbb{R}^n$ with $[\nu_x] = h_\mu(x)$ for μ-a.e. x. Then, by Proposition 3.4 (i) and by Jensen's inequality, we get

$$\begin{aligned} &\lim_{k \to +\infty} \int \varphi(x, P_{\mu_{h_k}}(x), h_{\mu_{h_k}}(x)) \, d\mu_{h_k}(x) \geq \int \langle \varphi(x, P_\mu(x), \cdot), \nu_x \rangle \, d\mu(x) \\ &\geq \int \varphi(x, P_\mu(x), [\nu_x]) \, d\mu(x) = \int \varphi(x, P_\mu(x), h_\mu(x)) \, d\mu \ . \end{aligned}$$

■

- 3.3 Stability of some subclasses of $\mathcal{M}_{BC}$

Having in mind applications to optimal control theory where the admissible measures μ are supported by some k-dimensional manifold having a prescribed boundary with an assigned co-normal field (see Section 5), we consider a fixed measure $\lambda \in \mathcal{M}^n$, supported on a $(k-1)$-rectifiable subset Γ of $\mathbb{R}^n$, and we define the class $\mathcal{A}_\lambda$ of all measures $\mu \in \mathcal{M}_{BC}$ such that:

$$\dim T_\mu(x) = k \quad \text{for } \mu\text{-a.e. } x, \tag{3.4}$$

$$\theta_k(\mu, x) \geq c \quad \text{for } \mu\text{-a.e. } x, \tag{3.5}$$

$$\|h_\mu\|_{(L^\infty_\mu)^n} \leq C \ , \quad \partial\mu = \lambda, \tag{3.6}$$

$$P_\mu(x) h_\mu(x) = 0 \quad \text{for } \mu\text{-a.e. } x, \tag{3.7}$$

where $k \leq n$ is a positive integer, and c, C are fixed positive constants. We assume that λ, c, and C are chosen in order to have $\mathcal{A}_\lambda \neq \emptyset$.

Remark 3.14. (i) If $\lambda = \eta \mathcal{H}^{k-1} \llcorner \Gamma$, with $|\eta| = 1$, and if $c \leq 1$, the class $\mathcal{A}_\lambda$ contains all the measures μ of the type $\mu = \mathcal{H}^k \llcorner M$, where M is a smooth k-dimensional manifold with boundary Γ, co-normal field η, and scalar mean curvature in modulus less than or equal to C.

(ii) All the measures $\mu \in \mathcal{A}_\lambda$ are k-rectifiable varifolds, *i.e.* are of the type $\mu = \theta \mathcal{H}^k \llcorner M$, where M is a k-rectifiable subset of $\mathbb{R}^n$ [64, Chapter 3], and θ is a $\mathcal{H}^k$-integrable function on M (see the rectifiability criterion proved in [42, Theorem 4.5]).

(iii) Condition (3.7), in the case of a measure μ of the type (i) multiplied by a smooth density, means that the tangential gradient of $\theta_k(\mu, \cdot)$ vanishes μ-a.e.

(iv) For any $\mu \in \mathcal{A}_\lambda$, the tangent space T^p_μ does not depend on the exponent p. Actually, for any $p \in [1, +\infty]$, there holds $T^p_\mu = T_\mu$. Indeed, since $\operatorname{div}(P_\mu \mu) \llcorner \Omega_\mu \in (L^\infty_\mu)^n$, and since $\mu(\Gamma) = 0$, the columns of the matrix P_μ belongs to $X^{p'}_\mu$. Therefore $T_\mu \subseteq T^p_\mu$; the converse implication is trivial.

In the next lemma, we prove the closedness of $\mathcal{A}_\lambda$ in $\mathcal{M}^+$. Observe that condition (3.7) is not needed for the validity of such result, as well as for that of Theorem 3.17 below; however, (3.7) will become crucial in the next section (see in particular Theorem 4.3).

Lemma 3.15. *The class $\mathcal{A}_\lambda$ is weakly closed in $\mathcal{M}^+$.*

Proof. Let $\{\mu_h\} \subset \mathcal{A}_\lambda$, with $\mu_h \rightharpoonup \mu$, and let us prove that $\mu \in \mathcal{A}_\lambda$. Possibly passing to a subsequence, we have $H(\mu_h) \rightharpoonup \nu$. Then, by [16, Corollary 4.4 and Remark 4.5], μ verifies (3.4) and (3.5), and $\nu = H(\mu)$. It remains to prove that μ verifies (3.6) and (3.7) as well . Since $\|h_{\mu_h}\|_{(L^\infty_{\mu_h})^n} \leq C$ for every h, the weak limit of $H(\mu_h)$ is of the type $\nu = f\mu + \lambda$, with

$\|f\|_{(L^\infty_\mu)^n} \leq C$. Hence $H(\mu) = f\mu + \lambda$, and, since $\lambda \perp \mu$ (see Remark 3.14 (ii)), this yields $f = h_\mu$, that is (3.6). Finally, let us prove (3.7). We know that $h_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} h_\mu$, and, by Corollary 3.12, that $P_{\mu_h} \overset{s(\mu_h,\mu)}{\longrightarrow} P_\mu$. Then, by Remark 3.8, $P_{\mu_h} h_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} P_\mu h_\mu$. Since by assumption we have $P_{\mu_h} h_{\mu_h} = 0$ μ_h-a.e. for every h, we obtain $P_\mu h_\mu = 0$ μ-a.e. x. ■

Remark 3.16. By a closedness result which holds for the class of integer varifolds [64], the above lemma still holds if condition (3.5) is replaced by $\theta_k(\mu, x) \in \mathbb{N}$ for μ-a.e.x.

The stability of topological properties of manifolds under some convergence notion is a quite interesting and difficult problem (see for instance [65]). Here we present a stability result for the connectedness of the support of measures belonging to the class $\mathcal{A}_\lambda$.

Theorem 3.17. *Let $\{\mu_h\} \subset \mathcal{A}_\lambda$, $\mu_h \rightharpoonup \mu$, and assume that for every h the set $S_h := \mathrm{spt}(\mu_h)$ is contained into a fixed ball of $\mathbb{R}^n$. Then S_h converge in the Hausdorff metric to the support of μ. In particular, if every S_h has a number of connected components smaller than a fixed integer N, the same holds for the support of μ.*

Proof. Since by hypothesis S_h are all contained into a ball of $\mathbb{R}^n$, possibly passing to a subsequence we can assume that S_h converge in the Hausdorff metric to a compact set S. Let us prove that $S := \mathrm{spt}\,\mu$, independently from the choice of the subsequence. The inclusion $\supseteq$ is an immediate consequence of the weak convergence of μ_h to μ. Conversely, let $x_0 \in S$, and let us prove that x_0 belongs to $\mathrm{spt}\,\mu$. If $x_0 \in \mathrm{spt}(\partial\mu)$, then $x_0 \in \mathrm{spt}(H(\mu))$, and this easily implies, by the definition of $H(\mu)$, that $x_0 \in \mathrm{spt}\,\mu$. So we can assume without loss of generality that $x_0 \notin \mathrm{spt}(\partial\mu)$; then, since $\mathrm{spt}(\partial\mu)$ is compact, there exists $R > 0$ such that $\mathrm{dist}(x, \mathrm{spt}(\partial\mu)) = 2R$. Let $\{x_h\}$ be a sequence converging to x_0, with $x_h \in S_h$ for every h, and satisfying $\mathrm{dist}(x_h, \mathrm{spt}(\partial\mu)) > R$ for every $h > \bar{h}$. Setting

$$M_h^R(x_h) := \sup_{\rho \in (0,R)} \frac{|H(\mu_h)|(B_\rho(x_h))}{\mu_h(B_\rho(x_h))} ,$$

and using a monotonicity formula proved in [16, (4.1)], we infer

$$\mu_h(B_\rho(x_h)) \geq \rho^k \theta_k(\mu_h, x_h) \exp\big(-M_h^R(x_h)\rho\big) , \qquad \forall h \in \mathbb{N} ,\ \forall \rho \in (0,R) . \tag{3.8}$$

By the definition of M_h^R, (3.6), and the fact that $B_\rho(x_h) \cap \mathrm{spt}(\partial\mu) = \emptyset$ for every $\rho \in (0,R)$ and $h > \bar{h}$, we deduce

$$M_h^R(x_h) \leq C + \frac{|\lambda|(B_\rho(x_h))}{\mu_h(B_\rho(x_h))} = C , \qquad \forall h > \bar{h} ,\forall \rho \in (0,R) .$$

Thus, using also (3.5), (3.8) implies

$$\mu_h(B_\rho(x_h)) \geq c\rho^k \exp(-\rho) , \qquad \forall h > \bar{h} ,\ \forall \rho \in (0,R) . \tag{3.9}$$

Notice now that

$$\mu\big(\overline{B}_\rho(x_0)\big) \geq \limsup_{h\to+\infty} \mu_h\big(\overline{B}_\rho(x_h)\big) \ , \qquad \forall\rho>0 \ ; \tag{3.10}$$

indeed the same inequality holds, by the weak convergence of μ_h to μ, if the radius ρ at the left hand side is replaced by any $\rho' > \rho$. Therefore, using (3.9) and (3.10), and taking into account that the equality $\mu\big(\overline{B}_\rho(x_0)\big) = \mu\big(B_\rho(x_0)\big)$ holds for a dense subset of radius ρ, we get

$$\mu\big(B_\rho(x_0)\big) > 0 \ , \qquad \forall\rho\in(0,R) \ ,$$

which yields $x_0 \in \operatorname{spt}\mu$ as required.
The second part of the statement is then an easy consequence of the equality $S = \operatorname{spt}\mu$. Indeed, arguing separately on each of the N sequences of the (possibly empty) N connected components of S_h, we can assume without loss of generality that $N = 1$: thus, it is easy to prove that the limit in the Hausdorff metric of a sequence of connected sets is necessarily connected (see for instance [40, Theorem 3.18]). ■

Remark 3.18. The assumption (3.6) is not necessary for the validity of the above theorem when $k = 1$; in fact, in that case, the result can be easily deduced using (3.5) and a well-known result of semicontinuity for the one-dimensional Hausdorff measure (see for instance [40, Theorem 3.18]). On the contrary, it is easy to see that the number of connected components of the support may increase, when $k > 1$, if dealing with measures μ_h such that h_{μ_h} is not bounded in $L^\infty_{\mu_h}$ or such that $\partial\mu_h$ is not fixed: consider respectively the example where $n = 3$ and the measures μ_h are given by $\mathcal{H}^2$ over the surfaces S_h drawn in Figure 3.1a, and the example where $n = 2$ and the measures μ_h are given by $\mathcal{L}^2$ over the open sets A_h drawn in Figure 3.1b (notice in both cases the difference between the limit of $\operatorname{spt}(\mu_h)$ in the Hausdorff distance and the support of the limit measure).

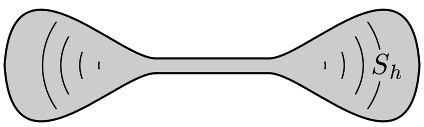

Figure 3.1a

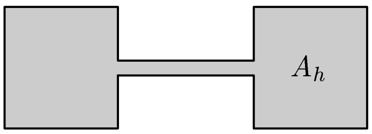

Figure 3.1b

As already noticed in the remark above, the case of one dimensional measures is quite special; actually, in the next proposition we show that, when $k = 1$ and $|H(\mu_h)| \leq C$, a statement stronger than Theorem 3.17 holds. Following [40], we say that a subset of $\mathbb{R}^n$ is a *continuum* if it is compact and connected.

Proposition 3.19. *Let $\mu_h = \theta_h \mathcal{H}^1 \llcorner S_h$ be a sequence in $\mathcal{M}_{BC}$, with S_h continua contained into a fixed ball of $\mathbb{R}^n$, $\theta_h \geq c > 0$, and $|H(\mu_h)| \leq C < +\infty$. If $\mu_h \rightharpoonup \mu$, then μ is of the form $\mu = \theta \mathcal{H}^1 \llcorner S$, where S is a continuum and $\theta \geq c$.*

Proof. By [16, Corollary 4.4], we know that μ is of the form $\mu = \theta \mathcal{H}^1 \llcorner S_1$, with $\theta \geq c$ μ-a.e., and S_1 1-rectifiable. Moreover, by Theorem 3.17, we know that $S_h = \mathrm{spt}(\mu_h)$ converge in the Hausdorff distance to a continuum S which coincides with the support of μ. Thus, $S = \overline{S}_1$, and the proof is concluded if we show that $\mathcal{H}^1(S \setminus S_1) = 0$. To that aim, let us prove that the following inequality holds in the sense of measures:

$$\mu \geq c \mathcal{H}^1 \llcorner S \ , \tag{3.11}$$

which clearly yields $\mathcal{H}^1(S \setminus S_1) = 0$. To obtain (3.11), it is enough to show that

$$\mu(A) \geq c \mathcal{H}^1(S \cap A) \ , \tag{3.12}$$

for any open set A such that $\mu(\partial A) = 0$. Indeed this implies by inner approximation that the same inequality holds as well for any open set B (and hence for any Borel subset of $\mathbb{R}^n$), as

$$\begin{aligned} \mu(B) &= \sup \ \big\{ \mu(A) \ : \ A \text{ open } \subset\subset B \ , \ \partial\mu(A) = 0 \big\} \\ &\geq c \sup \ \big\{ \mathcal{H}^1(S \cap A) \ : \ A \text{ open } \subset\subset B \ , \ \partial\mu(A) = 0 \big\} = c \mathcal{H}^1(S \cap B) \ . \end{aligned}$$

Let us prove (3.12). Since $\mu_h \rightharpoonup \mu$, and $\mu(\partial A) = 0$, taking also into account that $\theta_h \geq c$, we have

$$\mu(A) \geq \limsup_{h \to +\infty} \mu_h(A) \geq c \limsup_{h \to +\infty} \mathcal{H}^1(S_h \cap A) \ .$$

Thus, to get (3.12) it is enough to show that

$$\liminf_{h \to +\infty} \mathcal{H}^1(S_h \cap A) \geq \mathcal{H}^1(S \cap A) \ . \tag{3.13}$$

Let $A' \subset\subset A$ and let K be a compact set such that $A' \subset K \subset A$. Possibly passing to subsequences, we can assume that $\liminf\limits_{h \to +\infty} \mathcal{H}^1(S_h \cap K) = \lim\limits_{h \to +\infty} \mathcal{H}^1(S_h \cap K)$, and that $S_h \cap K$ converge in the Hausdorff metric to a compact set F, which turns out to contain $S \cap A'$. Then, applying [40, Theorem 3.18], we get

$$\liminf_{h \to +\infty} \mathcal{H}^1(S_h \cap A) \geq \liminf_{h \to +\infty} \mathcal{H}^1(S_h \cap K) \geq \mathcal{H}^1(F) \geq \mathcal{H}^1(S \cap A') \ .$$

By the arbitrariness of A', (3.13) holds, and the proof is achieved. ■

4. Convergence of energies with respect to varying measures

Let $\{\mu_h\} \subset \mathcal{M}^+$, let $p \in (1, +\infty)$, and let Ω be an open subset of $\mathbb{R}^n$. Consider a sequence of energy functionals with respect to μ_h, defined on $L^p_{\mu_h}(\Omega)$ by

$$F_h(u) := \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\, d\mu_h & \text{if } u \in \mathcal{D} \qquad (\textit{ variant}\text{: if } u \in \mathcal{D}(\Omega) \text{ }) \\ +\infty & \text{otherwise} \end{cases}, \tag{4.1}$$

where the integrand $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is assumed to satisfy the following assumptions:

$$\forall\, z \in \mathbb{R}^n, \text{ the function } f(\cdot, z) \text{ is continuous;} \tag{4.2}$$

$$\forall\, x \in \Omega, \text{ the function } f(x, \cdot) \text{ is convex ;} \tag{4.3}$$

$$\forall\, (x, z) \in \Omega \times \mathbb{R}^n,\ \lambda |z|^p \leq f(x, z) \leq \Lambda(1 + |z|^p) \text{ for some } \Lambda \geq \lambda > 0. \tag{4.4}$$

Assuming that $\mu_h \rightharpoonup \mu$ in $\mathcal{M}^+$, our aim is to study the behaviour of the sequence of functionals $\{F_h\}$ in the sense of Γ-convergence. To this purpose, notice that the domain $L^p_{\mu_h}(\Omega)$ of F_h varies according to the sequence $\{\mu_h\}$. Thus, the first need is to establish a notion of convergence for a sequence of functions $u_h \in L^p_{\mu_h}(\Omega)$ to an element u of $L^p_\mu(\Omega)$: we adopt the most natural one, namely, we say that u_h converge to u whenever $u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u$ (where u_h and u are extended to zero out of Ω).
We then say that F_h Γ-converge to $\mathcal{F}$ if both Γ- liminf and the Γ- limsup of F_h agree with $\mathcal{F}$ on $L^p_\mu(\Omega)$; they are defined respectively by [34]

$$\Gamma\text{-}\liminf_{h \to +\infty} F_h(u) := \inf \Big\{ \liminf_{h \to +\infty} F_h(u_h) \ :\ u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u \Big\}, \qquad \forall u \in L^p_\mu(\Omega) \ ;$$

$$\Gamma\text{-}\limsup_{h \to +\infty} F_h(u) := \inf \Big\{ \limsup_{h \to +\infty} F_h(u_h) \ :\ u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u \Big\}, \qquad \forall u \in L^p_\mu(\Omega) \ .$$

In subsections 4.1 and 4.2 we find respectively an upper bound F for the $\Gamma\text{-}\limsup_{h \to +\infty} F_h$ and a lower bound $\widetilde{F}$ for the $\Gamma\text{-}\liminf_{h \to +\infty} F_h$; subsection 4.3 is devoted to show, by way of some enlightening examples, that a gap phenomenon may occur between F and $\widetilde{F}$; consequently, in subsection 4.4 we consider the Γ-convergence of a sequence obtained by suitably enlarging the domain of the functionals F_h.

- 4.1 Upper bound for the Γ- limsup of F_h

The following theorem is a straightforward generalization to the case of varying measures of a result proved in [18, Theorem 3.1].

Theorem 4.1. *Let* $\{\mu_h\} \subset \mathcal{M}^+$, *with* $\mu_h \rightharpoonup \mu$. *Set*

$$F(u) := \begin{cases} \displaystyle\int_\Omega f_\mu(x, \nabla_\mu u)\, d\mu & \text{if } u \in H^{1,p}_\mu(\Omega) \\ +\infty & \text{otherwise} \end{cases} , \tag{4.5}$$

where

$$f_\mu(x,z) := \inf\{f(x, z+\xi) \ : \ \xi \in T^p_\mu(x)^\perp\} , \qquad \forall (x,z) \in \Omega \times \mathbb{R}^n . \tag{4.6}$$

Then

$$\Gamma\text{-}\limsup_{h\to+\infty} F_h(u) \le F(u) , \qquad \forall u \in L^p_\mu(\Omega) . \tag{4.7}$$

Proof. We can assume without loss of generality that $u \in H^{1,p}_\mu(\Omega)$, as otherwise (4.7) is obviously true. Whenever $u \in \mathcal{D}$, the weak convergence of μ_h to μ yields

$$\limsup_{h\to+\infty} F_h(u) = \limsup_{h\to+\infty} \int_\Omega f(x, \nabla u)\, d\mu_h = \int_\Omega f(x, \nabla u)\, d\mu .$$

Now, for any $u \in H^{1,p}_\mu(\Omega)$, we can find, by [18, Theorem 3.1], a sequence $\{u_k\} \subset \mathcal{D}$ converging to u in $L^p_\mu(\Omega)$ and satisfying

$$\limsup_{k\to+\infty} \int_\Omega f(x, \nabla u_k)\, d\mu \le \int_\Omega f_\mu(x, \nabla_\mu u)\, d\mu = F(u) \tag{4.8}$$

(this was actually proved on $\mathbb{R}^n$, but the same holds as well if $\mathbb{R}^n$ is replaced by any open domain Ω). Hence for such a sequence $\{u_k\}$ we obtain

$$\limsup_{k\to+\infty} \limsup_{h\to+\infty} F_h(u_k) \le F(u) . \tag{4.9}$$

Taking into account that $u_k\mu_h \underset{h\to+\infty}{\rightharpoonup} u_k\mu \underset{k\to+\infty}{\rightharpoonup} u\mu$, we can take a diagonal sequence $(h, k(h))$ such that $u_{k(h)} \overset{w(\mu_h,\mu)}{\longrightarrow} u$; moreover, by (4.9), we can extract $k'(h)$ from $k(h)$ in order that $\limsup_{h\to+\infty} F_h(u_{k'(h)}) \le F(u)$. Therefore, (4.7) is proved by taking $\{u_{k'(h)}\}$ as approximating sequence. ■

Remark 4.2. In the variant with Dirichlet condition, that is when the domain of the functionals F_h is the space $\mathcal{D}(\Omega)$, Theorem 4.1 still holds provided $H^{1,p}_\mu(\Omega)$ is replaced by $H^{1,p}_{\mu,0}(\Omega)$ in the definition (4.5) of F. Indeed, a relaxation result fully analogous to [18, Theorem 3.1] provides, for any $u \in H^{1,p}_{\mu,0}(\Omega)$, the existence of a sequence $\{u_k\} \subset \mathcal{D}(\Omega)$ satisfying (4.8).

- 4.2 Lower bound for the Γ- liminf of F_h

In this subsection we show that, when dealing with the Γ- liminf of the functionals F_h (even if the measures μ_h belong to the class $\mathcal{A}_\lambda$ introduced in subsection 3.3), one falls in a natural way upon the Sobolev space $W^{1,p}_\mu(\Omega)$ associated with the limit measure μ.
The main result of this subsection is the following.

Theorem 4.3. *Let* $\{\mu_h\} \subset \mathcal{A}_\lambda$, *with* $\mu_h \rightharpoonup \mu$. *Set*

$$\widetilde{F}(u) := \begin{cases} \displaystyle\int_\Omega f_\mu(x, D_\mu u)\, d\mu & \text{if } u \in W^{1,p}_\mu(\Omega) \\ +\infty & \text{otherwise} \end{cases} , \tag{4.10}$$

where f_μ *is defined by (4.6). Then*

$$\Gamma\text{-}\liminf_{h\to+\infty} F_h(u) \geq \widetilde{F}(u) , \qquad \forall u \in L^p_\mu(\Omega) . \tag{4.11}$$

Proof. We can assume without loss of generality that $u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u$ with $\{F_h(u_h)\}$ bounded. We claim that this entails

$$u \in W^{1,p}_\mu(\Omega) ; \qquad \nabla u_h \overset{w(\mu_h,\mu)}{\longrightarrow} D_\mu u + \xi , \quad \text{with } \ \xi(x) \in T^p_\mu(x)^\perp \ \text{ for } \mu\text{-a.e. } x . \tag{4.12}$$

Observe first that, if (4.12) holds, the required inequality $\liminf\limits_{h\to+\infty} F_h(u_h) \geq \widetilde{F}(u)$ follows; indeed, applying Proposition 3.4 (i) with $\varphi = f$, and recalling (4.6), we get

$$\liminf_{h\to+\infty} F_h(u_h) = \liminf_{h\to+\infty} \int_\Omega f(x, \nabla u_h)\, d\mu_h \geq \int_\Omega f(x, D_\mu u + \xi)\, d\mu \geq \int_\Omega f_\mu(x, D_\mu u)\, d\mu = \widetilde{F}(u) .$$

Thus, it is enough to prove (4.12). Lemma 3.15 implies that $\mu \in \mathcal{A}_\lambda$; since $\|u_h\|_{L^p_{\mu_h}(\Omega)}$ is uniformly bounded with $p > 1$, we have $u \in L^p_\mu(\Omega)$. Let us show that $\operatorname{div}(uP_\mu\mu) \llcorner \Omega_\mu \in (L^p_\mu(\Omega))^n$, which entails $u \in W^{1,p}_\mu(\Omega)$. By Corollary 3.12 and Remark 3.6, we have $u_h P_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} uP_\mu$, which yields $\operatorname{div}(u_h P_{\mu_h}\mu_h) \llcorner \Omega_\mu \rightharpoonup \operatorname{div}(uP_\mu\mu) \llcorner \Omega_\mu$ in $(\mathcal{D}'(\Omega_\mu))^n$. On the other hand, since by hypothesis $u_h \in H^{1,p}_{\mu_h}(\Omega)$, it can be easily proved by a density argument that $\operatorname{div}(u_h P_{\mu_h}\mu_h) \llcorner \Omega_\mu = (\nabla_{\mu_h} u_h + u_h h_{\mu_h})\mu_h$ (notice indeed that by (3.6) $\operatorname{spt}(\partial\mu_h) = \operatorname{spt}(\partial\mu)$ for every h). Thus, $(\nabla_{\mu_h} u_h + u_h h_{\mu_h})\mu_h$ converge to $\operatorname{div}(uP_\mu\mu)$ in $(\mathcal{D}'(\Omega_\mu))^n$.
Since $F_h(u_h)$ and $\|h_{\mu_h}\|_{(L^\infty_{\mu_h})^n}$ are bounded, we have (possibly passing to subsequences)

$$\nabla u_h \overset{w(\mu_h,\mu)}{\longrightarrow} v , \quad u_h h_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} \eta , \quad (\nabla_{\mu_h} u_h + u_h h_{\mu_h})\mu_h \overset{w(\mu_h,\mu)}{\longrightarrow} f , \tag{4.13}$$

with $v, f, \eta \in (L^p_\mu(\Omega))^n$. By the uniqueness of the limit in $(\mathcal{D}'(\Omega_\mu))^n$, we deduce that $\operatorname{div}(uP_\mu\mu) \llcorner \Omega_\mu$ equals f, and in particular belongs to $(L^p_\mu(\Omega))^n$.
Finally, let us prove that $\nabla_{\mu_h} u_h \overset{w(\mu_h,\mu)}{\longrightarrow} D_\mu u + \xi$, with $\xi(x) \in T^p_\mu(x)^\perp$ μ-a.e.. To that aim, we apply the weak-strong convergence result of subsection 3.2 to the sequences $(\nabla u_h, P_{\mu_h})$, and $(u_h h_{\mu_h}, P_{\mu_h})$, which yields respectively, by Corollary 3.12, Remark 3.8, and (3.7), $\nabla_{\mu_h} u_h \overset{w(\mu_h,\mu)}{\longrightarrow} P_\mu v$ and $P_\mu \eta = 0$ μ-a.e. x. Thus, by (4.13), $\operatorname{div}(uP_\mu\mu) \llcorner \Omega_\mu = (P_\mu v + \eta)\mu$. Finally, since $u \in W^{1,p}_\mu(\Omega)$, we have by definition $\operatorname{div}(uP_\mu\mu) \llcorner \Omega_\mu = (D_\mu u + uh_\mu)\mu$, hence $(P_\mu v - D_\mu u)\mu = (uh_\mu - \eta)\mu$, which yields $P_\mu v = D_\mu u$ μ-a.e. as $T_\mu \cap T_\mu^\perp = \{0\}$. ∎

Remark 4.4. In the variant with Dirichlet condition, one can easily obtain, by minor modifications in the proof, that Theorem 4.3 still holds if $W^{1,p}_\mu(\Omega)$ is replaced by $W^{1,p}_{\mu,0}(\Omega)$ in the definition of (4.10).

- 4.3 The gap phenomenon between F and $\widetilde{F}$

If $\{\mu_h\} \subset \mathcal{A}_\lambda$, with $\mu_h \rightharpoonup \mu$, coupling Theorems 4.1 and 4.3 we obtain

$$\widetilde{F}(u) \leq \Gamma\text{-}\liminf F_h(u) \leq \Gamma\text{-}\limsup F_h(u) \leq F(u) , \qquad \forall\, u \in L^p_\mu(\Omega) . \tag{4.14}$$

In particular, (4.14) yields the Γ-convergence of the sequence $\{F_h\}$ to F whenever $W^{1,p}_\mu(\Omega)$ coincides with $H^{1,p}_\mu(\Omega)$ (by Remarks 4.2 and 4.4, the same holds for the variant with Dirichlet condition, whenever $H^{1,p}_{\mu,0}(\Omega) = W^{1,p}_{\mu,0}(\Omega)$).
Thus, the following questions arise in a natural way:

- does the equality $W^{1,p}_\mu(\Omega) = H^{1,p}_\mu(\Omega)$ hold on the class $\mathcal{A}_\lambda$?
- if this is not the case, is it possible to establish that F_h Γ-converge to any of the upper bound F and the lower bound $\widetilde{F}$?

The examples below prove that, in general, the first question has a negative answer, as the strict inclusion $H^{1,p}_\mu(\Omega) \subset W^{1,p}_\mu(\Omega)$ may hold. In the case when μ is the Lebesgue measure on an open subset $A \subset \mathbb{R}^n$, we recover two well-known cases where the difference between the two spaces is due to the lack of regularity either of $\partial\Omega$, or of $\partial A (= \mathrm{spt}(\partial\mu))$ (*cf.* Examples 4.5 and 4.6); for measures supported by lower dimensional sets, we show that the strict inclusion $H^{1,p}_\mu(\Omega) \subset W^{1,p}_\mu(\Omega)$ may hold even for measures μ without boundary, and taking $\Omega = \mathbb{R}^n$. This is due to "*doubling phenomenons*" either for the support of μ, or for its multiplicity (*cf.* Examples 4.7 and 4.8).
Turning to the second question, it is obvious that, if the sequence $\{\mu_h\}$ is identically equal to μ, one obtains the upper bound F as a Γ-limit. The following examples show that, for a suitable sequence $\{\mu_h\}$, one may obtain the lower bound $\widetilde{F}$ as well. Moreover, it seems that the measures μ_h can be constructed with the same boundary as μ only in the examples when the support of μ has a codimension higher than 1.

Example 4.5. Let $n = 2$, $\mu = \mathcal{L}^2 \llcorner B_1(0)$, $\Omega := \mathbb{R}^2 \setminus F$, where $F := \{y = 0\,,\ x \geq 0\}$. Then $H^{1,p}_\mu(\Omega) = H^{1,p}\big(B \setminus F\big)$, while $W^{1,p}_\mu(\Omega) = W^{1,p}\big(B \setminus F\big)$. Thus it is well known that $H^{1,p}_\mu(\Omega) \neq W^{1,p}_\mu(\Omega)$. Moreover, taking as approximating sequence $\mu_h = \mathcal{L}^2 \llcorner B_h$, where an element of the sequence of open sets B_h is drawn in Figure 4.1b, we have $F_h \xrightarrow{\Gamma} \widetilde{F}$.

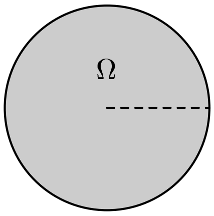

Figure 4.1a

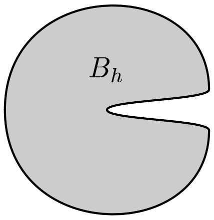

Figure 4.1b

Example 4.6. Let $n = 2$, let $\mu = \mathcal{L}^2 \llcorner E$, where $E := [0,1]^2 \cup [-1,0]^2$, and let $\Omega = \mathbb{R}^2$. Then $H^{1,p}_\mu(\mathbb{R}^2) = H^{1,p}(E)$, while $W^{1,p}_\mu(\mathbb{R}^2) = W^{1,p}(E)$. Since, for $p > 2$, the elements of $H^{1,p}(E)$ are continuous, $H^{1,p}_\mu(\mathbb{R}^2)$ is strictly contained into $W^{1,p}_\mu(\mathbb{R}^2)$. Moreover, if $\mu_h = \mathcal{L}^2 \llcorner E_h$ is an approximating sequence where E_h are chosen as in Figure 4.2b, we obtain $F_h \xrightarrow{\Gamma} \widetilde{F}$, whereas, E_h are chosen as in Figure 4.2c, we obtain $F_h \xrightarrow{\Gamma} F$.

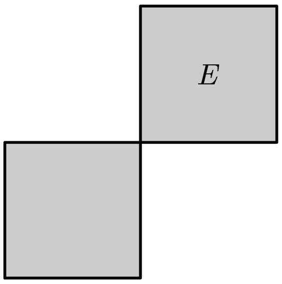

Figure 4.2a

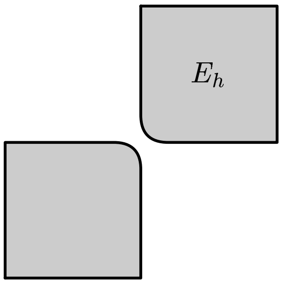

Figure 4.2b

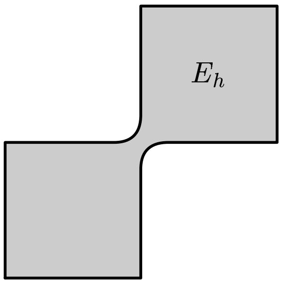

Figure 4.2c

Example 4.7. Let n be arbitrary, let μ be the measure $\mathcal{H}^1$ over a closed curve S having a double point P as drawn in Figure 4.3a, and let $\Omega = \mathbb{R}^n$. Then, any function $u \in H^{1,p}_\mu(\mathbb{R}^n)$ is continuous at P: indeed, it can be approximated by a sequence of smooth functions $\{u_h\}$ with $\{\nabla u_h\}$ uniformly bounded in the $L^p_\mu(\mathbb{R}^n)$-norm. On the other hand, decomposing S as the union of two smooth manifolds S' and S'', and integrating by parts separately on S' and S'', one can easily see that the condition $u \in W^{1,p}_\mu(\mathbb{R}^n)$ means that u belongs to $W^{1,p}(S)$ when seen as a function of the arc-parameter on S; in particular, u does not need not to be continuous at P. When $n \geq 3$, an approximating sequence $\mu_h = \mathcal{H}^1 \llcorner S_h$ which leads to $F_h \xrightarrow{\Gamma} \widetilde{F}$, can be constructed by 'detaching' the two branches meeting at the singular point of μ as drawn in Figure 4.3b.

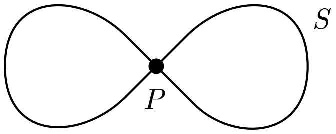

Figure 4.3a

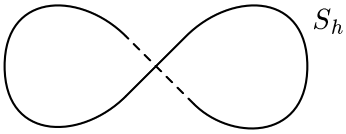

Figure 4.3b

Example 4.8. Let $n=2$, let μ be the measure $\mathcal{H}^1$ over the curve S drawn in Figure 4.4a, weighted by a double density on the central portion S'', and let $\Omega = \mathrm{I\!R}^2$. Integrating by parts separately on the manifolds S', S'', and S''', one can check that the function u identically equal to 1 on S'', and going smoothly from 0 to 2 along an oriented parametrization of each of the loops S' and S''', belongs to the space $W^{1,p}_{\mu}(\mathrm{I\!R}^2)$. But it cannot belong to $H^{1,p}_{\mu}(\mathrm{I\!R}^2)$, because, similarly as in Example 4.7, this would entail the continuity of u on S, which is clearly violated at the points P_1 and P_2. An approximating sequence $\{\mu_h\}$ which leads to $F_h \xrightarrow{\Gamma} \widetilde{F}$, is given by the measure $\mathcal{H}^1$ (with multiplicity equal one) over a sequence of curves S_h like those drawn in Figure 4.4b.

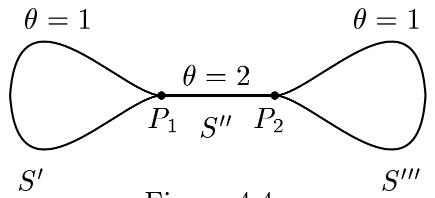

Figure 4.4a

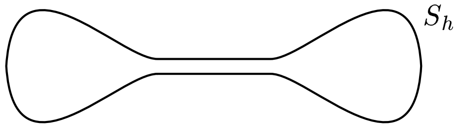

Figure 4.4b

- 4.4 A conjecture for the convergence of weak Sobolev spaces

In order to avoid the gap phenomenon which occurs, as shown in subsection 4.3, when dealing with the Γ-convergence of the energies F_h, we are led to reformulate the problem extending the domain of F_h to the weak Sobolev space $W^{1,p}_{\mu_h}(\Omega)$. For any $u \in L^p_{\mu_h}(\Omega)$, let us define

$$(4.15) \qquad \widetilde{F}_h(u) := \begin{cases} \displaystyle\int_\Omega f(x, D_{\mu_h} u)\, d\mu_h & \text{if } u \in W^{1,p}_{\mu_h}(\Omega) \qquad (\textit{ variant}: \text{ if } u \in W^{1,p}_{\mu_h,0}(\Omega)\ ) \\ +\infty & \text{otherwise} \end{cases},$$

where the integrand f is still assumed to satisfy (4.2)-(4.4).
By minor modifications in the proof of Theorem 4.3, we get

$$(4.16) \qquad \Gamma\text{-}\liminf_{h\to+\infty} \widetilde{F}_h(u) \geq \widetilde{F}(u)\ , \qquad \forall u \in L^p_\mu(\Omega)\ ,$$

where the functional $\widetilde{F}$ is defined by (4.10).
It seems reasonable to conjecture that, when $\partial\mu_h = \partial\mu$, then $\widetilde{F}_h$ do Γ-converge to $\widetilde{F}$, namely that the following inequality holds:

$$(4.17) \qquad \Gamma\text{-}\limsup_{h\to+\infty} \widetilde{F}_h(u) \leq \widetilde{F}(u)\ , \qquad \forall u \in L^p_\mu(\Omega)\ .$$

The proof of (4.17) seems to be concerned with a quite delicate approximation problem for which at present we do not have a general solution (see Example4.10 below).

Remark 4.9. We stress that (4.17) is satisfied for all the examples of subsection 4.3 in which the boundary of μ_h do not vary, while (see Figure 4.2c) it can be false when such condition is violated. We also observe that the requirement $\partial\mu = \lambda$ in the definition of $\mathcal{A}_\lambda$ cannot be weakened by simply asking that $\partial\mu$ has a fixed prescribed support. Indeed, notice first that the class of measures thus obtained is not closed: for instance, the boundary of the measure $\mathcal{H}^1$ concentrated over any of the curves S_h drawn in Figure 4.5 is supported at the point P, but the limit measure μ has a zero boundary. As a consequence of this phenomenon, the Γ- liminf of $\widetilde{F}_h$ may be strictly smaller than $\widetilde{F}$; indeed, it is enough to take a sequence of functions u_h which go smoothly from 0 to 1 along an oriented parametrization of each S_h: then $\widetilde{F}_h(u_h)$ is bounded, but $u_h \stackrel{w(\mu_h,\mu)}{\longrightarrow} u$, with $u \notin W^{1,p}_\mu(\mathrm{I\!R}^3)$.

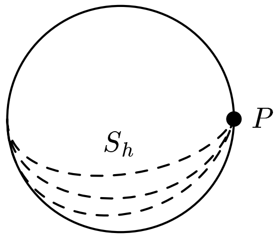

Figure 4.5

Example 4.10. We end this section with a case where (4.17) holds. Given two points a and b in $\mathrm{I\!R}^n$, and a positive constant C, consider the class of measures

$$
\begin{aligned}
\mathcal{A}(a,b,C) := \Big\{ \mu = \gamma^\sharp(\mathcal{L}^1 \llcorner (0,L)) \ :\ L \in [0,+\infty),\ \gamma : [0,L] \to \mathrm{I\!R}^n\ ,\\
\gamma(0) = a,\ \gamma(1) = b\ ,\ |\gamma'| = 1,\ |\gamma''| \le C \Big\}\ ,
\end{aligned} \tag{4.18}
$$

which all have the form $\mu = \theta(x)\mathcal{H}^1 \llcorner S$, where S is a curve joining the points a and b, and $\theta(x)$ is an integer multiplicity. Then it is possible to prove that this class is weakly closed and for all sequences $\mu_h \rightharpoonup \mu$, with $\mu_h \in \mathcal{A}(a,b,C)$, the condition (4.17) holds.

5. A perspective for optimization problems with manifolds as controls

Let us consider an optimization problem where the control variable is a connected k-dimensional manifold S (embedded in a fixed compact set K of $\mathrm{I\!R}^n$), with a prescribed compact boundary Γ, and an assigned co-normal vector field $\nu_\Gamma = \eta$ on it. In addition, we impose that the length of the mean curvature vector $h(S)$ is smaller than a fixed positive constant C.
As a state equation, we take

$$
-\Delta_S u + u = g \quad \text{on } S\ , \qquad u \in W^{1,2}(S) \quad (\text{or } u \in W^{1,2}_0(S))\ ,
$$

where Δ_S is the Laplace-Beltrami operator on S, and the source term g is a given continuous function compactly supported on $\mathbb{R}^n$.
As a cost functional, we take

$$J(S,u) := \int_S j(x,u)\, d\mathcal{H}^k \ ,$$

being $j(x,z)$ continuous in x and convex in z, with $j(x,z) \geq 1$ on $\mathbb{R}^n \times \mathbb{R}^n$.
Summarizing, we are looking for a solution of the variational problem

$$(\mathcal{P}) \ \inf \Big\{ \int_S j(x,u_S)\, d\mathcal{H}^k \ : \ \partial S = \Gamma \ , \quad \nu_s = \eta \text{ on } \Gamma \ , \quad |h(S)| \leq C \ , \quad -\Delta_S u_S + u_S = g \text{ on } S \Big\} \ .$$

In order to deal with such a kind of problem, we formulate it in a weaker form.
To that aim, we associate with every manifold S a measure μ supported on S, enlarging by this way the admissible competitors to the class of all measures μ having a connected support contained into K, and belonging to the class $\mathcal{A}_\lambda$ described by (3.4)-(3.7), where we take $\lambda = \alpha \mathcal{H}^{k-1} \llcorner \Gamma$ (being α a prescribed bounded density on Γ).
Thus, the weak formulation of $(\mathcal{P})$ reads as

$$(\widetilde{\mathcal{P}}) \qquad \begin{aligned} \inf \Big\{ \int_B j(x,u_\mu)\, d\mu \quad : \quad & \operatorname{spt}\mu \text{ connected } \subseteq K \ , \quad \dim T_\mu = k \ , \quad \theta_k(\mu) \geq c \ , \\ & \|h_\mu\|_{(L^\infty_\mu)^n} \leq C \ , \quad P_\mu h_\mu = 0 \ , \quad \partial\mu = \alpha \mathcal{H}^{k-1} \llcorner \Gamma \ , \\ & -\operatorname{div}\big((D_\mu u_\mu)\mu\big) + u_\mu = g\mu \ \text{ in } \ \mathcal{D}' \Big\} \ , \end{aligned}$$

where now the state variable u belongs to the Sobolev space $W^{1,2}_\mu(\mathbb{R}^n)$ (or to $W^{1,2}_{\mu,0}(\mathbb{R}^n \setminus \Gamma)$).
In order to get the existence of a solution to problem $(\widetilde{\mathcal{P}})$ via the direct method of the Calculus of Variations [26], we need to check first that the class of admissible measures is weakly sequentially closed, and then that the functional we want to minimize is weakly sequentially lower semicontinuous on such class.
The results proved in subsection 3.3 prove that the former requirement is satisfied (see Lemma 3.15 and Theorem 3.17).
The latter requirement is satisfied as well, if we establish that

$$(5.1) \qquad \mu_h \rightharpoonup \mu \quad \Longrightarrow \quad u_{\mu_h} \overset{w(\mu_h,\mu)}{\longrightarrow} u_\mu \ ;$$

indeed, if (5.1) holds, the weak lower semicontinuity of $\mu \mapsto \int_B j(x,u_\mu)\, d\mu$ follows from Proposition 3.4 (which can be applied by the assumptions made on the integrand j).
Notice now that u is a solution to the state equation if and only if $u \in \operatorname{argmin} G(\mu,\cdot)$, where the functional $G(\mu,\cdot)$ is defined on L^2_μ by

$$G(\mu,u) := \begin{cases} \displaystyle\int_{\mathbb{R}^n} \frac{1}{2}\big(|u|^2 + |D_\mu u|^2\big)\, d\mu - \int_{\mathbb{R}^n} g\, u\, d\mu & \text{if } u \in W^{1,2}_\mu(\mathbb{R}^n) \ \ (\text{or } u \in W^{1,2}_{\mu,0}(\mathbb{R}^n \setminus \Gamma)) \ , \\ +\infty & \text{otherwise} \ . \end{cases}$$

Notice also that, whenever $\mu_h \rightharpoonup \mu$, the functionals $G(\mu_h,\cdot)$ are equicoercive with respect to the convergence $u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u$, because, for any real t, $\{u\,\mu_h \,:\, u \in L^p_{\mu_h}\,,\ G(\mu_h,u) \le t\}$ is contained in a bounded subset K_t, which is weakly * compact in $\mathcal{M}$. Hence, (5.1) is satisfied if the sequence $G(\mu_h,\cdot)$ Γ-converges to the functional $G(\mu,\cdot)$ with respect to the convergence $u_h \overset{w(\mu_h,\mu)}{\longrightarrow} u$ (see [34, Theorem 7.8]). Furthermore, since the term $\int_{\mathbb{R}^n} gu\,d\mu$ is continuous with respect to such convergence, the Γ-convergence of the sequence $G(\mu_h,\cdot)$ is equivalent to the Γ-convergence of the sequence $\widetilde{F}_h$ conjectured in subsection 4.4 (see (4.17)). Thus, postulating this conjecture, the existence of a solution for the optimal control problem $(\widetilde{\mathcal{P}})$ would follow via the direct method of the Calculus of Variations.

Example 5.1. In the case $k=1$, the optimization problem $(\widetilde{\mathcal{P}})$ above admits a solution if we restrict μ to vary in the class $\mathcal{A}(a,b,C)$ defined by (4.18).

Example 5.2. The existence of a solution for problem $(\widetilde{\mathcal{P}})$ could be applied to find optimal truss-like structures: consider the case when $k=1$ and $\lambda = \sum_{i=1}^N \delta_{P_i} v_i$, being P_i and v_i respectively N fixed points and vectors in $\mathbb{R}^n$, as for instance in Figure 4.6. As a reasonable cost functional, we take for instance $J(S,u) = \int_S (1+u^2)\,d\mathcal{H}^1$, which penalizes the length of S and the displacement on S as well. Proving the validity of (4.17) on $\mathcal{A}_\lambda$ would then imply, as a straightforward application of the direct method, the existence of an optimal structure $\mu = \theta(x)\mathcal{H}^1 \llcorner S$, with S rectifiable and $\theta(x) \ge 1$.
An alternative approach to such a kind of problem, independent of the validity of (4.17), could be to show that, for every admissible structure μ, there exists another admissible structure μ' which has a lower cost and whose singularities are of the kind "triple junction" like in Figure 4.7, with a density constant in a neighbourhood of each triple point, and the number of triple points *a priori* bounded. For these type of structures, it is easy to check that the strong and the weak Sobolev spaces coincide, so that the arguments described in this section and in the previous one could provide the existence of an optimum.

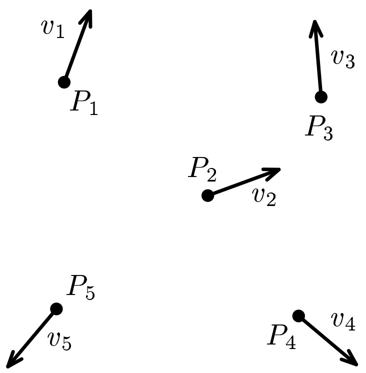

Figure 4.6

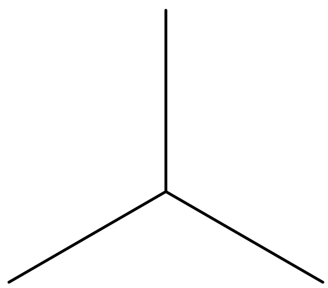
Figure 4.7

VI. Homogenization of thin structures by two-scale method with respect to measures *

1. Introduction

For a given periodic Radon measure μ on $\mathbb{R}^n$, we study the asymptotic behaviour of a sequence of functionals of the form

$$J_\varepsilon(u) := \int_\Omega j\left(\frac{x}{\varepsilon}, \nabla u\right)\, d\mu_\varepsilon\ , \qquad u \in \mathcal{C}_0^1(\Omega)\ , \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, the integrand $j = j(y,z)$ is assumed to be periodic μ-measurable in y and convex with a p-growth condition in z, ε is a positive parameter tending to zero, and μ_ε is the rescaled measure $\mu_\varepsilon(B) := \varepsilon^n \mu\big(\frac{B}{\varepsilon}\big)$.
We think of μ as the Hausdorff measure $\mathcal{H}^k$ on a k-dimensional periodic domain of $\mathbb{R}^n$: when the small parameter ε tends to zero, we recover by this way the classical framework of homogenization on perforated domains (when $k = n$), or on reticulated thin elastic structures (when $k < n$) (see [15] for an introduction to the subject). Actually, for an assigned continuous bounded function f on Ω, the minimum problem

$$\inf\left\{ J_\varepsilon(u) - \int_\Omega f u\, d\mu_\varepsilon \ :\ u \in \mathcal{C}_0^1(\Omega) \right\} \tag{$\mathcal{P}_\varepsilon$}$$

is the variational counterpart of an elliptic equation posed, with boundary conditions, on an ε-periodic domain. For instance, in the case when μ is the Lebesgue measure on a perforated domain, the choice of a quadratic energy density $j(y,z) = \frac{1}{2}\sum_{1\le i,j\le n} a_{i,j}(y) z_i z_j$, where $\mathcal{A}(y) = a_{i,j}(y)$ is a symmetric matrix of periodic and bounded coefficients, has been intensively studied in the literature (see [28, 55]). More precisely, the classical setting corresponds to the choice $\mu = \mathcal{L}^n \llcorner A$, where A is the complement of a periodic system of ♣ holes ♠ ; denoting by A_ε the set εA, occupied by the material, and by T_ε the collection of ♣ holes ♠ $A \setminus A_\varepsilon$, the minimum problem $(\mathcal{P}_\varepsilon)$ corresponds in this case to the system

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(\frac{x}{\varepsilon})\nabla u_\varepsilon\big) = f & \text{in } A_\varepsilon\ , \\ u_\varepsilon = 0 & \text{on } \partial\Omega\ , \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial T_\varepsilon\ . \end{cases} \tag{$\mathcal{S}_\varepsilon$}$$

The behaviour of $(\mathcal{S}_\varepsilon)$ when $\varepsilon \to 0$ is well-known and gives rise to a limit problem of the same type, $-\operatorname{div}(\mathcal{A}_{eff}\nabla u) = f$; the constant matrix $\mathcal{A}_{eff}$ represents the physical characteristics

* Preliminary version of a paper by G. Bouchitté and I. Fragalà (1999).

of an homogeneized material which macroscopically behaves like the inhomogeneous perforated body as $\varepsilon \to 0$.

The same is the essence of homogenization in the case of thin periodic structures, except that another parameter δ must be considered, which corresponds to the thickness of the region occupied by the material; thus, a double passage to the limit is needed, as the small parameters ε and δ ♣ ♠ both tend to zero [31, 10, 29, 32]. For instance, the case of a one-dimensional ε-periodic structure εS, reinforced by some material distributed along bars of cross-section δ, corresponds in our framework to the choice $\mu = \mathcal{H}^1 \llcorner S$, and possibly to $\mu_\delta = |S_\delta|^{-1} \mathcal{L}^n \llcorner S_\delta$, being $S_\delta := \{x \in \mathbb{R}^n \, : \, \mathrm{dist}(x, S) \leq \delta\}$.

The classical approach to this kind of problems is based on the possibility of defining a suitable extension of the solution u_ε of each elliptic problem such as $(\mathcal{S}_\varepsilon)$ to the whole $\mathbb{R}^n$ [67, 30, 48]; this assumption strongly depends on the shape of the domain under consideration, so that the proof technique must be ♣ adapted ♠ in each case.

Here we adopt a different point of view, in order to study the Γ-convergence of the functionals J_ε defined in (1.1). This seems to be an unifying approach for structures of any dimension, and makes use of the tangential calculus with respect to a positive Radon measure μ which has been developed in some recent papers [18, 16]. In particular, we ♣ introduce a notion of ♠ two-scale convergence with respect to measures, which generalizes the method proposed in [56, 3, 2]; we prove a structure result for the two-scale possible limits of a sequence $\{(u_\varepsilon, \nabla u_\varepsilon)\}$, when $u_\varepsilon \in \mathcal{C}^1_0(\Omega)$ satisfy the uniform estimate $\int (|u_\varepsilon|^p + |\nabla u_\varepsilon|^p)\, d\mu_\varepsilon \leq M$ (see Theorem 5.1). This allows us to deduce the homogeneized energy density associated with J_ε in terms of a local unit-cell problem (see Theorem 6.2).

Our main assumptions involve some concepts of *connectedness* on the measure μ. They can be formulated by using the corresponding periodic Sobolev spaces $H^{1,p}_\mu(\mathbf{T})$ on the n-torus $\mathbf{T}$, and are related to the connectedness notions for an open domain of $\mathbb{R}^n$ discussed in [70] as an alternative to the extension operator technique.

The case when μ is 'non-connected' is not treated within; neverthless, we point out that in such a situation the two-parameter variational integrals involved in the above described case of fattened thin structures, may give rise to a non-local limit energy. This phenomenon deserves to our advice further investigation, as it throws some light on the non-indifference in letting tend to zero first the periodicity or the thickness parameter.

A short outline of the paper follows.

In Section 2 we introduce the new notion of two-scale convergence with respect to measures, and some related results are given, which will be useful for the proof of the main homogenization theorem.

In Section 3 we develop the theory of the periodic Sobolev spaces associated with a measure μ; this allows in particular to formulate in a natural way some different notions of connectedness

for μ, which are briefly discussed in Section 4.
Section 5 is devoted to the statement and proof of the main structure theorem for a two-scale limit of a sequence of gradients; the homogenization result is then deduced in Section 6.
♣ Finally in Section 7 we study the convergence of a sequence of two-parameter variational integrals. We first show that, when μ is connected, the limit energy is independent of the order we let the two parameters ε and δ tend to zero. We conclude by pointing out that the same is not true if μ has several connected components; in fact in this case a non-local term may appear in the limit energy, according how we perform the limit process. ♠

2. Two-scale convergence with respect to a measure

We fix some preliminary notation. Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary and let Y be the unitary cube of $\mathbb{R}^n$. We will always assume that μ is a positive Y-periodic Radon measure on $\mathbb{R}^n$ with $\mu(\partial Y) = 0$; notice that the latter condition is not restrictive since Y may be replaced by any translated cell $y_0 + Y$. For any $\varepsilon > 0$, we denote by μ_ε the measure defined by

$$\int_\Omega \varphi(x)\, d\mu_\varepsilon(x) = \varepsilon^n \int_{\Omega/\varepsilon} \varphi(\varepsilon x)\, d\mu(x) \qquad \forall \varphi \in \mathcal{C}_0(\Omega)\ , \tag{2.1}$$

being $\mathcal{C}_0(\Omega)$ the space of continuous and compactly supported functions on Ω.
It is easy to check, using (2.1) and the periodicity of μ, that

$$\mu_\varepsilon \rightharpoonup \mu(Y)\, \mathcal{L}^n \tag{2.2}$$

in the vague topology of measures. To have $\mu_\varepsilon \rightharpoonup \mathcal{L}^n$, we assume in the following the not restrictive condition $\mu(Y) = 1$. We also set for brevity $m := (\mathcal{L}^n \llcorner \Omega) \otimes (\mu \llcorner Y)$.
We call $\mathbf{T}$ the n-dimensional torus $\mathbb{R}^n/\mathbf{Z}^n$, and throughout the paper we identify functions on $\mathbf{T}$ with Y-periodic functions on $\mathbb{R}^n$. In particular, by writing $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))$, we mean that $\varphi = \varphi(x,y)$ is smooth in both its variables $(x,y) \in \Omega \times \mathbb{R}^n$, compactly supported in x and Y-periodic in y. We can now set the following

Definition 2.1. *Let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$ and $u_0 \in L^p_m(\Omega \times \mathbf{T})$, for some $p \geq 1$. We say that $\{u_\varepsilon\}$ two-scale converges to u_0 (with respect to μ and as $\varepsilon \to 0$), and we write $u_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u_0$, if*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \varphi\big(x, \frac{x}{\varepsilon}\big)\, d\mu_\varepsilon(x) = \int_{\Omega \times Y} u_0(x,y) \varphi(x,y)\, dm(x,y) \quad \forall \varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))\ . \tag{2.3}$$

Remark 2.2. Note that, if $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0$, then $u_\varepsilon \mu_\varepsilon \rightharpoonup u\mathcal{L}^n$, where $u(x) := \int_Y u_0(x,y)\,d\mu(y)$ (choose in Definition 2.1 a test function $\varphi(x,y) = \psi(x) \in \mathcal{D}(\Omega)$).
On the other hand, similarly as in the well-known case where $\mu_\varepsilon = \mathcal{L}^n$ for every ε, Definition 2.1 retains more information on the sequence $\{u_\varepsilon \mu_\varepsilon\}$ than the pure weak convergence in the sense of measures: it also takes into account the 'oscillations' which have the same frequency as the test functions $\varphi(x, \frac{x}{\varepsilon})$. For instance, in the simplest case when $u_\varepsilon = u(x, \frac{x}{\varepsilon})$, with u continuous on $\Omega \times \mathbb{R}^n$ and Y-periodic in its second variable, we have $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u$. This follows from the convergence

$$v\big(x, \frac{x}{\varepsilon}\big)\mu_\varepsilon \rightharpoonup \Big\{ \int_Y v(x,y)\,d\mu(y) \Big\} \mathcal{L}^n \ , \tag{2.4}$$

holding for any function v continuous on $\Omega \times \mathbb{R}^n$ and Y-periodic in y.

In view of the following compactness property, the two-scale convergence is a good notion for a sequence $\{u_\varepsilon\}$ which satisfies an uniform bound of the type $\int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon \le M$, with $p > 1$. Throughout this section, the symbols $\|\cdot\|_{p,\mu_\varepsilon}$ and $\|\cdot\|_{p,m}$ will be used respectively for the $L^p_{\mu_\varepsilon}(\Omega)$ and the $L^p_m(\Omega \times Y)$ norms.

Proposition 2.3. *Let $p > 1$, and let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$ satisfy $\int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon \le M$. Then there exists a subsequence u_{ε_k} such that $u_{\varepsilon_k} \rightharpoonup\!\!\rightharpoonup u_0$, with $u_0 \in L^p_m(\Omega \times Y)$; in particular, $u_{\varepsilon_k}\mu_{\varepsilon_k} \rightharpoonup u\mathcal{L}^n$, where $u(x) := \int_Y u_0(x,y)\,d\mu(y)$.*

Proof. Let $(B\,, \|\cdot\|_B)$ be the Banach space of all continuous functions on $\Omega \times Y$ vanishing at the boundary ♣ of $\Omega \times Y$ ♠ , endowed with the uniform norm. For any $\varphi \in B$ we have:

$$\|\varphi\big(x, \frac{x}{\varepsilon}\big)\|_{p',\mu_\varepsilon} \ \le \ C\|\varphi\|_B \ , \tag{2.5}$$

$$\lim_{\varepsilon \to 0} \|\varphi\big(x, \frac{x}{\varepsilon}\big)\|_{p',\mu_\varepsilon} \ = \ \|\varphi(x,y)\|_{p',m} \ . \tag{2.6}$$

♣ The first inequality follows from the boundedness of μ_ε (as Ω is relatively compact with $\mathcal{L}^n(\partial\Omega) = 0$, we have by (2.2) $\mu_\varepsilon(\Omega) \to \mathcal{L}^n(\Omega)$ as $\varepsilon \to 0$). To prove (2.6), we simply apply (2.4) with $v(x,y) = |\varphi(x,y)|^{p'}$. ♠ Now let us consider for any $\varepsilon > 0$ the linear operator T_ε defined on B by

$$\langle T_\varepsilon, \varphi \rangle := \int_\Omega u_\varepsilon(x) \varphi\big(x, \frac{x}{\varepsilon}\big)\, d\mu_\varepsilon \ ,$$

where we think of φ as implicitly extended by Y-periodicity on $\Omega \times \mathbb{R}^n$. By the Hölder inequality, the assumption on u_ε and (2.5), we have

$$\begin{aligned} |\langle T_\varepsilon, \varphi \rangle| \le \|u_\varepsilon\|_{p,\mu_\varepsilon} \|\varphi\big(x, \frac{x}{\varepsilon}\big)\|_{p',\mu_\varepsilon} &\le M^{1/p} \|\varphi\big(x, \frac{x}{\varepsilon}\big)\|_{p',\mu_\varepsilon} \\ &\le M^{1/p} C \|\varphi\|_B \ . \end{aligned} \tag{2.7}$$

Thus T_ε is a bounded sequence in the space B' of finite Borel measures and, possibly passing to a subsequence, we can assume that T_ε weakly star converges to an element T_0 of B'. Then, using (2.6) to pass to the limit as $\varepsilon \to 0$ in (2.7), we get

$$|\langle T_0, \varphi \rangle| \leq M^{1/p} \|\varphi(x,y)\|_{p',m} \qquad \forall \varphi \in B \ .$$

Hence, by the density of B in $L^{p'}_m(\Omega \times Y)$, T_0 may be represented by an element $u_0 \in L^p_m(\Omega \times Y)$ satisfying

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \varphi\big(x, \frac{x}{\varepsilon}\big)\, d\mu_\varepsilon = \int_{\Omega \times Y} u_0(x,y) \varphi(x,y)\, dm \qquad \forall \varphi \in B \ .$$

Finally, taking into account Remark 2.2, the second part of the statement immediately follows from the two-scale convergence of u_{ε_k} to u_0.

■

Remark 2.4. Under the assumptions of Proposition 2.3, the space of admissible test functions in the two-scale convergence can be considerably enlarged. Indeed, the same proof of Proposition 2.3 works whenever the space of continuous functions vanishing at the boundary of $\Omega \times Y$ is replaced by another separable Banach space B, dense in $L^{p'}_m(\Omega \times Y)$, such that (2.5) and (2.6) hold. For instance, all these conditions are satisfied if we take $B = \mathcal{C}(\overline{\Omega}, L^p_\mu(\mathbf{T}))$ (a detailed proof can be derived with minor changes from [3, Corollary 5.4]); in particular, the convergence (2.4) holds for any $v \in \mathcal{C}(\overline{\Omega}, L^p_\mu(\mathbf{T}))$.

The two-scale convergence enjoys the following lower semicontinuity property when dealing with convex periodic integrands.

Proposition 2.5. *Let $v_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega; \mathbb{R}^d)$ two-scale converge (by components) to $v_0 \in L^p_m(\Omega \times Y; \mathbb{R}^d)$, $p > 1$; let $j = j(y,z)$ be a function on $\mathbb{R}^n \times \mathbb{R}^d$ μ-measurable and Y-periodic in y, convex in z and satisfying, for some positive constants C, c, the estimate*

$$c|z|^p \leq j(y,z) \leq C\,(1 + |z|^p) \qquad \forall (y,z) \in \mathbb{R}^n \times \mathbb{R}^d \ .$$

Then

$$\liminf_{\varepsilon \to 0} \int_\Omega j\big(\frac{x}{\varepsilon}, v_\varepsilon\big)\, d\mu_\varepsilon \geq \int_{\Omega \times Y} j(y, v_0(x,y))\, dm \ .$$

Proof. For any function $\varphi \in \mathcal{D}(\Omega \times Y)$, the Fenchel's inequality gives

$$\int_\Omega j\big(\frac{x}{\varepsilon}, v_\varepsilon\big)\, d\mu_\varepsilon \geq \int_\Omega v_\varepsilon(x) \varphi\big(x, \frac{x}{\varepsilon}\big)\, d\mu_\varepsilon - \int_\Omega j^*\Big(\frac{x}{\varepsilon}, \varphi\big(x, \frac{x}{\varepsilon}\big)\Big)\, d\mu_\varepsilon \ ,$$

where $j^*(y,\cdot)$ denotes the conjugate functional of $j(y,\cdot)$, and φ is extended by Y-periodicity on $\Omega \times \mathbb{R}^n$. Since $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v_0$, recalling Remark 2.4, passing to the limit as $\varepsilon \to 0$ in the above inequality yields

$$\liminf_{\varepsilon\to 0} \int_\Omega j\big(\frac{x}{\varepsilon}, v_\varepsilon\big)\, d\mu_\varepsilon \geq \int_{\Omega\times Y} v_0(x,y)\varphi(x,y)\, dm - \int_{\Omega\times Y} j^*\Big(y, \varphi(x,y)\Big)\, dm \ . \tag{2.8}$$

Taking the supremum of the right hand side of (2.8) when φ varies in $\mathcal{D}(\Omega\times Y)$, one gets, using a classical localization argument and the convexity assumption on j:

$$\liminf_{\varepsilon\to 0} \int_\Omega j\big(\frac{x}{\varepsilon}, v_\varepsilon\big)\, d\mu_\varepsilon \ \geq \ \int_{\Omega\times Y} j^{**}(y, v_0(x,y))\, dm \ = \int_{\Omega\times Y} j(y, v_0(x,y))\, dm.$$

■

Applying the above proposition with $j(y,z) = |z|^p$, we get

$$\liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon \geq \int_{\Omega\times Y} |u_0|^p \, dm \ , \tag{2.9}$$

whenever $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$ two-scale converge to $u_0 \in L^p(\Omega\times Y)$ $(p>1)$.

Definition 2.6. *Let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$ and $u_0 \in L^p_m(\Omega\times \mathbf{T})$, for some $p > 1$. We say that $\{u_\varepsilon\}$ two-scale strongly converges to u_0 (with respect to μ and as $\varepsilon \to 0$), and we write $u_\varepsilon \twoheadrightarrow u_0$, if*

$$u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0 \quad \text{and} \quad \limsup_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon \leq \int_{\Omega\times Y} |u_0|^p \, dm \ .$$

In view of (2.9), when $u_\varepsilon \twoheadrightarrow u_0$, the $L^p_{\mu_\varepsilon}(\Omega)$-norm of u_ε converge to the $L^p_m(\Omega\times Y)$-norm of u_0. This means that all the oscillations of the sequence $\{u_\varepsilon \mu_\varepsilon\}$ are captured by the two-scale limit, namely, they are in resonance with those of the test functions $\varphi(x, \frac{x}{\varepsilon})$.

Example 2.7. Let $u_\varepsilon = u_0(x, \frac{x}{\varepsilon})$, where $u_0(x,y)$ is given by $u_0(x,y) = u(x)w(y)$, with $u \in L^p(\Omega)$ and $w \in L^\infty(\mathbf{T})$. Then it is easy to check that $u_\varepsilon \twoheadrightarrow u_0$. In fact this conclusion holds also for more general u_0, in particular for $u_0 \in C(\overline{\Omega}; L^p_\mu(\mathbf{T}))$.

Proposition 2.8. *Let $p > 1$, and let $u_\varepsilon \in L^p_{\mu_\varepsilon}(\Omega)$, $v_\varepsilon \in L^{p'}_{\mu_\varepsilon}(\Omega)$ satisfy*

$$u_\varepsilon \twoheadrightarrow u_0 \ , \quad u_0 \in L^p_m(\Omega\times Y) \ , \tag{2.10}$$

$$v_\varepsilon \rightharpoonup\!\!\rightharpoonup v_0 \ , \quad v_0 \in L^{p'}_m(\Omega\times Y) \ , \quad \int_\Omega |v_\varepsilon|^{p'} \, d\mu_\varepsilon \leq M \ . \tag{2.11}$$

Then

$$u_\varepsilon v_\varepsilon \mu_\varepsilon \rightharpoonup u_0 v_0 m \tag{2.12}$$

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon - u_0\big(x,\frac{x}{\varepsilon}\big)|^p\, d\mu_\varepsilon = 0 \quad \textit{whenever } u_0 \in \mathcal{C}(\Omega\times \mathbf{T})\ . \tag{2.13}$$

Proof. Let $\{\varphi_h\} \subset \mathcal{D}(\Omega\times Y)$ be a sequence of smooth functions converging to u_0 in $L^p_m(\Omega\times Y)$ and extended by Y-periodicity on $\Omega\times \mathbb{R}^n$. For any test function $\psi\in C_0(\Omega)$, it holds

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon v_\varepsilon \psi\, d\mu_\varepsilon = \lim_{h\to+\infty}\lim_{\varepsilon\to 0}\Big\{\int_\Omega \big[u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\big]v_\varepsilon\psi\, d\mu_\varepsilon + \int_\Omega \varphi_h\big(x,\frac{x}{\varepsilon}\big)v_\varepsilon\psi\, d\mu_\varepsilon\Big\}\ .$$

By (2.11) and the choice of $\{\varphi_h\}$ we have

$$\begin{aligned}\lim_{h\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega \varphi_h\big(x,\frac{x}{\varepsilon}\big)v_\varepsilon\psi\, d\mu_\varepsilon &= \lim_{h\to+\infty}\int_{\Omega\times Y}\varphi_h(x,y)v_0(x,y)\psi(x)\, dm\\ &= \int_{\Omega\times Y} u_0(x,y)v_0(x,y)\psi(x)\, dm\end{aligned}\ .$$

To prove (2.12), it is then enough to show that

$$\lim_{h\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega \big[u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\big]v_\varepsilon\psi\, d\mu_\varepsilon = 0\ . \tag{2.14}$$

Applying the Hölder inequality and assumption (2.11), we deduce

$$\Big|\int_\Omega \big[u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\big]v_\varepsilon\psi\, d\mu_\varepsilon\Big| \le M^{1/p'}\|\psi\|_\infty \|u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\|_{p,\mu_\varepsilon}\ .$$

The Clarkson inequalities give, respectively for $p\ge 2$ and for $p\le 2$:

$$\begin{cases}\|u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\|^p_{p,\mu_\varepsilon} \le 2^p\Big\{\frac12\|u_\varepsilon\|^p_{p,\mu_\varepsilon} + \frac12\|\varphi_h\big(x,\frac{x}{\varepsilon}\big)\|^p_{p,\mu_\varepsilon} - \Big\|\frac{u_\varepsilon+\varphi_h\big(x,\frac{x}{\varepsilon}\big)}{2}\Big\|^p_{p,\mu_\varepsilon}\Big\}\\[2mm] \|u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\|^{p'}_{p,\mu_\varepsilon} \le 2^{p-1}\Big\{\Big[\frac12\|u_\varepsilon\|^p_{p,\mu_\varepsilon} + \frac12\|\varphi_h\big(x,\frac{x}{\varepsilon}\big)\|^p_{p,\mu_\varepsilon}\Big]^{\frac{1}{p-1}} - \Big\|\frac{u_\varepsilon+\varphi_h\big(x,\frac{x}{\varepsilon}\big)}{2}\Big\|^{p'}_{p,\mu_\varepsilon}\Big\}\end{cases}$$

Using (2.10) and the smoothness of each φ_h, we infer

$$\begin{cases}\limsup\limits_{\varepsilon\to 0}\|u_\varepsilon - \varphi_h\big(x,\frac{x}{\varepsilon}\big)\|^p_{p,\mu_\varepsilon} \le 2^p\Big\{\frac12\|u_0\|^p_{p,m} + \frac12\|\varphi_h\|^p_{p,m} - \Big\|\frac{u_0+\varphi_h}{2}\Big\|^p_{p,m}\Big\}\\[2mm] \limsup\limits_{\varepsilon\to 0}\|u_0 - \varphi_h\|^{p'}_{p,m} \le 2^{p-1}\Big\{\Big[\frac12\|u_0\|^p_{p,m} + \frac12\|\varphi_h\|^p_{p,m}\Big]^{\frac{1}{p-1}} - \Big\|\frac{u_0+\varphi_h}{2}\Big\|^{p'}_{p,m}\Big\}\end{cases}$$

Passing now to the limsup as $h\to+\infty$, by the choice of the sequence $\{\varphi_h\}$ both the right hand sides of the above inequalities tend to zero, so that (2.14) is proved.
To obtain (2.13), it is enough to write the Clarkson inequalities with φ_h replaced by u_0 and to observe that, being u_0 globally continuous and Y-periodic in its second variable, the right hand sides converge to zero as $\varepsilon\to 0$. ∎

3. The periodic Sobolev spaces $\S(\mathbf{T})$

First, let us give a brief recall about the Sobolev spaces $\S$ introduced in [18].
Let $\mathcal{D} := \mathcal{C}_0^\infty(\mathbb{R}^n)$, and $\mathcal{D}'$ the space of distributions on $\mathbb{R}^n$. Let $p, p' \in [1,+\infty]$ be fixed conjugate exponents. For any $q \geq 1$, let $L^q_\mu := L^q_\mu(\mathbb{R}^n)$, $(L^q_\mu)^n := L^q_\mu(\mathbb{R}^n;\mathbb{R}^n)$, and similarly for $L^q_{\mu,\mathrm{loc}}$, $(L^q_{\mu,\mathrm{loc}})^n$. The notation $\|\cdot\|_{q,\mu}$ will be used throughout the section for the usual norms of L^q_μ and $(L^q_\mu)^n$, as well as of $L^q_\mu(\mathbf{T})$ and $(L^q_\mu(\mathbf{T}))^n$ (implicitly assuming in this case, whenever unambiguous, that the integrals are made over Y).
We recall that, for any $\sigma \in (L^1_{\mu,\mathrm{loc}})^n$, $\mathrm{div}(\sigma\mu) \in \mathcal{D}'$ is defined by

$$\langle \mathrm{div}(\sigma\mu), \psi\rangle_{(\mathcal{D}',\mathcal{D})} := -\int_{\mathbb{R}^n} \sigma \cdot \nabla\psi \, d\mu \qquad \forall \psi \in \mathcal{D} .$$

Whenever $\mathrm{div}(\sigma\mu)$ is a measure absolutely continuous with respect to μ with a density belonging to $L^{p'}_\mu$, we write for brevity $\mathrm{div}(\sigma\mu) \in L^{p'}_\mu$, and we denote by $\mathrm{div}_\mu \sigma$ the derivative $\frac{\partial}{\partial\mu}\mathrm{div}(\sigma\mu)$.
It will be useful in the following to notice that there holds

$$\mathrm{div}(\psi\sigma\mu) = \psi\,\mathrm{div}(\sigma\mu) + (\sigma\cdot\nabla\psi)\mu \qquad \forall\psi \in \mathcal{D},\ \forall\sigma \in (L^1_{\mu,\mathrm{loc}})^n . \tag{3.1}$$

We set $X^{p'}_\mu$ the class of all vector fields Φ which are tangent to μ, given by

$$X^{p'}_\mu := \left\{ \Phi \in (L^{p'}_\mu)^n \ : \ \mathrm{div}(\Phi\mu) \in L^{p'}_\mu \right\} ,$$

and $T^p_\mu(x)$ the tangent space of μ at x, obtained as

$$T^p_\mu(x) := \mu - \mathrm{ess}\bigcup \left\{ \Phi(x) \ : \ \Phi \in X^{p'}_\mu \right\} , \qquad x \in \mathbb{R}^n .$$

For details on the meaning of the μ-essential union and on the properties of the multifunction T^p_μ, we refer to [18, 42]; in particular, at least for usual measures, $T^p_\mu(x)$ does not depend on the exponent p, and it will be denoted simply by $T_\mu(x)$.
The Sobolev space $\S = \S(\mathbb{R}^n)$ is defined as the completion of $\mathcal{D}$ with respect to the norm $\|\psi\|_{1,p,\mu} := \|\psi\|_{p,\mu} + \|\nabla_\mu\psi\|_{p,\mu}$, where $\nabla_\mu\psi(x) := P_\mu(x)[\nabla\psi(x)]$ is the orthogonal projection $P_\mu(x)$ of $\nabla\psi(x)$ onto $T_\mu(x)$. In other words, it turns out that the linear operator $A : D(A) \subset L^p_\mu \to (L^p_\mu)^n$ defined by $D(A) = \mathcal{D}$ and $A\psi = \nabla_\mu\psi$ is closable ([18, Proposition 2.1]), and the domain of its unique closed extension $\overline{A}$ is called $\S$. By definition $\S$ is a Banach space, and it is reflexive when $p \in (1,+\infty)$. For any $u \in \S$, we set $\nabla_\mu u := \overline{A}u$. Then a density argument gives straightforward the following useful integration by parts:

$$\int_{\mathbb{R}^n} \nabla_\mu u \cdot \Phi \, d\mu = -\int_{\mathbb{R}^n} u \ \mathrm{div}_\mu \Phi \, d\mu \qquad \forall u \in \S,\ \forall \Phi \in X^{p'}_\mu . \tag{3.2}$$

Notice the adjoint operator $A^* : (L^{p'}_\mu)^n \to L^{p'}_\mu$ equals $-\operatorname{div}(P_\mu \sigma\mu)$ when applied to any vector field σ belonging to its domain $D(A^*)$, given by

$$D(A^*) = \left\{ \sigma \in (L^{p'}_\mu)^n \ : \ \operatorname{div}(P_\mu \sigma \mu) \in L^{p'}_\mu \right\} = \left\{ \sigma \in (L^{p'}_\mu)^n \ : \ P_\mu \sigma \in X^{p'}_\mu \right\} =: Y^{p'}_\mu \ .$$

In particular, when $p > 1$, the following characterization by duality of § can be deduced from Lemma 3.1 below:

$$\S = \left\{ u \in L^p_\mu \text{ s.t. } \exists C > 0 \ : \ |\langle u, \operatorname{div}(P_\mu \sigma \mu)\rangle| \leq C \|\sigma\|_{p',\mu} \quad \forall \sigma \in Y^{p'}_\mu \right\} . \tag{3.3}$$

Lemma 3.1. *Let V, W be two Banach spaces, with W reflexive. Assume that $T : D(T) \subset V \to W$ is a linear and closable operator with dense domain. Then*

$$D(\overline{T}) = \left\{ v \in V \textit{ s.t. } \exists C > 0 \ : \ |\langle v, T^* w'\rangle_{(V,V')}| \leq C \|w'\|_{W'} \ \forall w' \in D(T^*) \right\} . \tag{3.4}$$

The proof of this lemma is an immediate consequence of the equality $\overline{T} = T^{**}$, which holds by standard arguments of Functional Analysis [22].

Let us turn now our attention to Y-periodic functions. Set

$$X^{p'}_\mu(\mathbf{T}) := \left\{ \Phi \in \left(L^{p'}_\mu(\mathbf{T})\right)^n \ : \ \operatorname{div}(\Phi\mu) \in L^{p'}_{\mu,\mathrm{loc}} \right\} \ .$$

Notice that, since $\mu(\partial Y) = 0$, we have

$$\mu - \operatorname{ess}\bigcup \left\{ \Phi(x) \ : \ \Phi \in X^{p'}_\mu(\mathbf{T}) \right\} = T_\mu(x) \qquad \mu\text{-a.e. on } Y \ . \tag{3.5}$$

Let $A_\sharp : D(A_\sharp) \subset L^p_\mu(\mathbf{T}) \to (L^p_\mu(\mathbf{T}))^n$ be the linear operator given by $D(A_\sharp) = \mathcal{C}^\infty(\mathbf{T})$, $A_\sharp \psi = \nabla_\mu \psi$. In order to prove that $A_\sharp$ is closable, we need some preliminary lemmas. For any $\psi \in \mathcal{D}$, we set

$$\psi^\sharp(y) = \sum_{i \in \mathbf{Z}^n} \psi(y - i) \ , \qquad y \in Y \ .$$

The function $\psi^\sharp$ is well-defined as soon as ψ is compactly supported, and it turns out to be Y-periodic by construction.

Lemma 3.2. *For any $v \in L^1_\mu(\mathbf{T})$ and any $\psi \in \mathcal{D}$, there holds $\int_{\mathbb{R}^n} v\psi \, d\mu = \int_Y v\psi^\sharp \, d\mu$.*

Proof. Due to the Y-periodicity of v and μ, we have

$$\int_{\mathbb{R}^n} v\psi \, d\mu = \sum_{i \in \mathbf{Z}^n} \int_{i+Y} v\psi \, d\mu = \sum_{i \in \mathbf{Z}^n} \int_Y v(y)\psi(y-i) \, d\mu(y) \ .$$

The last sum over $i \in \mathbf{Z}^n$ may equivalently be made over the finite set of indices $I_\psi := \{i \in \mathbf{Z}^n \ : \ (Y - i) \cap \operatorname{spt} \psi \neq \emptyset\}$. Thus, it can be passed under the sign of integral and the lemma is proved. ∎

Lemma 3.3. *There exists a sequence $\{\psi_h\} \subset \mathcal{D}$ such that $\psi_h^\sharp \to 1$ in $W^{1,\infty}_\mu(\mathbf{T})$.*

Proof. For any $h \in \mathbb{N}$, take $\psi_h \in \mathcal{C}_0^\infty(\mathbb{R}^n; [0, h^{-n}])$ satisfying

$$\psi_h = h^{-n} \quad \text{on } hY \ , \qquad \psi_h = 0 \quad \text{on } \mathbb{R}^n \setminus (h+1)Y \ ; \tag{3.6}$$

$$|\nabla \psi_h| \leq C h^{-n} \quad \text{on } \mathbb{R}^n \ . \tag{3.7}$$

For any $y \in Y$, we have

$$h^{-n} \mathrm{card}\, I_h(y) \leq \psi_h^\sharp(y) \leq h^{-n} \mathrm{card}\, J_h(y) \ , \tag{3.8}$$

where $I_h(y) = \{i \in \mathbf{Z}^n \ : \ \psi_h(y-i) = h^{-n}\}$, $J_h(y) = \{i \in \mathbf{Z}^n \ : \ \psi_h(y-i) \neq 0\}$.
By (3.6), $\mathrm{card}\, I_h(y) \geq h^n$ and $\mathrm{card}\, J_h(y) \leq (h+1)^n$, thus (3.8) yields $1 \leq \psi_h^\sharp(y) \leq \left(1 + \frac{1}{h}\right)^n$ and we deduce that $\psi_h^\sharp$ converge pointwise to 1 on Y.
On the other hand, by (3.7) we have

$$\sup_{y \in Y} |\nabla \psi_h^\sharp(y)| \leq \sup_{y \in Y} C h^{-n} \left[\mathrm{card}\, J_h(y) - \mathrm{card}\, I_h(y)\right] \leq C h^{-n} \left[(h+1)^n - h^n\right] \ .$$

It follows that $\psi_h^\sharp$ and $\nabla \psi_h^\sharp$ both converge uniformly respectively to 1 and 0 on Y, hence $\psi_h^\sharp$ converge to 1 strongly in $W^{1,\infty}_\mu(\mathbf{T})$. ∎

Lemma 3.4. *For any $u \in \mathcal{C}^\infty(\mathbf{T})$ and any $\Phi \in X^{p'}_\mu(\mathbf{T})$, there holds*

$$\int_Y \nabla_\mu u \cdot \Phi \, d\mu = - \int_Y u \, \mathrm{div}_\mu \, \Phi \, d\mu \ . \tag{3.9}$$

Proof. Take a sequence $\{\psi_h\} \subset \mathcal{D}$ as in Lemma 3.3. For any $u \in \mathcal{C}^\infty(\mathbf{T})$ and for every h, the product function $u\psi_h$ belongs to $\mathcal{D}$. Then, for any $\Phi \in X^{p'}_\mu(\mathbf{T})$, we are allowed to write

$$\int_{\mathbb{R}^n} \nabla_\mu (u\psi_h) \cdot \Phi \, d\mu = - \int_{\mathbb{R}^n} u\psi_h \, \mathrm{div}_\mu \, \Phi \, d\mu \ . \tag{3.10}$$

Using Lemma 3.2, the left hand side of the above equation can be rewritten as

$$\int_{\mathbb{R}^n} \left(\psi_h \nabla_\mu u + u \nabla_\mu \psi_h\right) \cdot \Phi \, d\mu = \int_Y \left(\psi_h^\sharp \nabla_\mu u + u \nabla_\mu \psi_h^\sharp\right) \cdot \Phi \, d\mu \ .$$

By the choice of $\{\psi_h\}$, (3.9) follows from (3.10) when passing to the limit as $h \to +\infty$. ∎

Proposition 3.5. *The operator $A_\sharp$ is closable.*

Proof. We need to show that, if $\{u_h\} \subset \mathcal{C}^\infty(\mathbf{T})$ satisfies $u_h \to 0$ in $L^p_\mu(\mathbf{T})$ and $\nabla_\mu u_h \to v$ in $(L^p_\mu(\mathbf{T}))^n$, then $v = 0$ μ-a.e. on Y. To this aim, it is enough to show that

$$\sup\left\{ \int_Y v \cdot \Phi \, d\mu \ : \ \Phi \in X^{p'}_\mu(\mathbf{T}) \right\} = 0. \tag{3.11}$$

Indeed, $X^{p'}_\mu(\mathbf{T})$ satisfies the following locality property: whenever $\Phi \in X^{p'}_\mu(\mathbf{T})$ and $\psi \in \mathcal{D}(Y)$, the field $\psi\Phi$ (extended by periodicity out of Y) belongs to $X^{p'}_\mu(\mathbf{T})$. Then a commutation argument between supremum and integral can be applied [21, Theorem 1], and, using also (3.5), (3.11) becomes

$$\int_Y \sup\{v \cdot z \ : \ z \in T_\mu(x)\} \, d\mu = 0 \ .$$

This, coupled with the information $v(x) \in T_\mu(x)$ for μ-a.e. x (recall that $v = \lim_h \nabla_\mu u_h$) implies that v vanishes μ-a.e.

It remains to prove (3.11). To this aim, we use Lemma 3.4 and the convergence of $\{u_h\}$ to zero:

$$\int_Y v \cdot \Phi \, d\mu = \lim_{h \to +\infty} \int_Y \nabla_\mu u_h \cdot \Phi \, d\mu = -\int_Y u_h \operatorname{div}_\mu \Phi \, d\mu = 0 \ .$$

■

We can now give the following

Definition 3.6. *Set $\S(\mathbf{T}) := D(\overline{A}_\sharp)$, and $\nabla_\mu u := \overline{A}_\sharp u$ for any $u \in \S(\mathbf{T})$.*

It turns out by this definition that $\S(\mathbf{T})$ is a closed subspace of $H^{1,p}_{\mu,\mathrm{loc}} := \{u \in L^p_{\mu,\mathrm{loc}} \ : \ u\psi \in \S \ \forall \psi \in \mathcal{D}\}$ (in particular it is reflexive for $p \in (1 + \infty)$). Moreover, the integration by parts formula (3.9) can immediately be extended to any $u \in \S(\mathbf{T})$.

Let us consider now the adjoint operator $A^*_\sharp$. We have:

$$D(A^*_\sharp) = \left\{\sigma \in \left(L^{p'}_\mu(\mathbf{T})\right)^n \ : \ \left|\int_Y \sigma \cdot \nabla_\mu u \, d\mu\right| \le C\|u\|_{p',\mu} \ \forall u \in \mathcal{C}^\infty(\mathbf{T})\right\} =: Y^{p'}_\mu(\mathbf{T}) \ .$$

Proposition 3.7. $Y^{p'}_\mu(\mathbf{T}) = \left\{\sigma \in \left(L^{p'}_\mu(\mathbf{T})\right)^n : P_\mu\sigma \in X^{p'}_\mu(\mathbf{T})\right\}$; $A^*_\sharp\sigma = -\operatorname{div}_\mu(P_\mu\sigma)$.

Proof. Let us prove the first assertion. The inclusion $\supseteq$ is an immediate consequence of (3.9). Viceversa, let $\sigma \in Y^{p'}_\mu(\mathbf{T})$, and let us show that the restriction $\operatorname{div}(P_\mu\sigma\mu) \llcorner \Omega$ of $\operatorname{div}(P_\mu\sigma\mu)$ to any bounded open set Ω belongs to $L^{p'}_\mu(\Omega)$. Take $\psi \in \mathcal{D}$ with $\operatorname{spt}\psi \subset \Omega$. Using Lemma 3.2, we get

$$-\langle \operatorname{div}(P_\mu\sigma\mu) \llcorner \Omega, \psi \rangle_{(\mathcal{D},\mathcal{D}')} = \int_{\mathbb{R}^n} P_\mu\sigma \cdot \nabla\psi \, d\mu = \int_Y P_\mu\sigma \cdot \nabla\psi^\sharp \, d\mu = \int_Y \sigma \cdot \nabla_\mu\psi^\sharp \, d\mu \ .$$

Then, since $\sigma \in Y_\mu^{p'}(\mathbf{T})$, there exists $C > 0$ such that

$$(3.12) \qquad |\langle \operatorname{div}(P_\mu \sigma\mu) \llcorner \Omega, \psi\rangle| \leq C\|\psi^\sharp\|_{p,\mu} \qquad \forall \psi \in \mathcal{D}(\Omega) \ .$$

Now, using the boundedness of Ω, we can find a positive constant C' such that

$$(3.13) \qquad \|\psi^\sharp\|_{p,\mu,Y} \leq C'\|\psi\|_{p,\mu,\Omega} \qquad \forall \psi \in \mathcal{D}(\Omega) \ .$$

Indeed, setting $I_\Omega := \big\{ i \in \mathbf{Z}^n \ : \ (Y - i) \cap \Omega \neq \emptyset \big\}$, we obtain

$$\begin{aligned}\int_Y |\psi^\sharp|^p \, d\mu &= \int_Y \Big| \sum_{i \in I_\Omega} \psi(y-i)\Big|^p \, d\mu \leq \int_Y \Big\{ \sum_{i \in I_\Omega} |\psi(y-i)| \Big\}^p \, d\mu \\ &\leq (\operatorname{card} I_\Omega)^{p-1} \sum_{i \in I_\Omega} \int_Y |\psi(y-i)|^p \, d\mu = (\operatorname{card} I_\Omega)^{p-1} \int_\Omega |\psi|^p \, d\mu \ .\end{aligned}$$

Combining (3.12) and (3.13), we deduce that $\operatorname{div}(P_\mu \sigma\mu) \llcorner \Omega$ can be identified with an element of $L_\mu^{p'}(\Omega)$ and the first part of the statement is proved. To obtain the second assertion, we take $u \in \mathcal{C}^\infty(\mathbf{T})$ and $\sigma \in Y_\mu^{p'}(\mathbf{T})$. Then (3.9) yields

$$\int_Y \nabla_\mu u \cdot \sigma \, d\mu = \int_Y \nabla_\mu u \cdot P_\mu \sigma \, d\mu = -\int_Y u \operatorname{div}_\mu(P_\mu \sigma) \, d\mu \ .$$

■

As an immediate consequence of Proposition 3.7 and of Lemma 3.1, we can equivalently define $\S(\mathbf{T})$ as follows:

$$(3.14) \qquad \S(\mathbf{T}) = \Big\{ u \in L_\mu^p(\mathbf{T}) \text{ s.t. } \exists C > 0 : |\langle u, \operatorname{div}(P_\mu \sigma\mu)\rangle| \leq C\|\sigma\|_{p',\mu} \, \forall \sigma \in Y_\mu^{p'}(\mathbf{T}) \Big\}.$$

The next theorem is the periodic version of Theorem 3.1 in [18], to which we refer for some details omitted in the proof.

Proposition 3.8. *Let J be the functional defined on $L_\mu^p(\mathbf{T})$, $(p > 1)$ by*

$$J(u) = \begin{cases} \int_Y j(y, \nabla u) \, d\mu & \text{if } u \in \mathcal{C}^\infty(\mathbf{T}) \\ +\infty & \text{if } u \in L_\mu^p(\mathbf{T}) \setminus \mathcal{C}^\infty(\mathbf{T}), \end{cases}$$

where the integrand $j = j(y,z)$ is μ-measurable in y, convex in z, and satisfies, for some positive constants C, c, the growth condition

$$c|z|^p \leq j(y,z) \leq C\big(1 + |z|^p\big) \qquad \forall (y,z) \in Y \times \mathbb{R}^n \ .$$

Then the relaxed functional of J on L_μ^p is given by

$$\overline{J}(u) = \begin{cases} \int_Y j_\mu(y, \nabla_\mu u) \, d\mu & \text{if } u \in \S(\mathbf{T}) \\ +\infty & \text{if } u \in L_\mu^p(\mathbf{T}) \setminus \S(\mathbf{T}), \end{cases}$$

where

$$(3.15) \qquad j_\mu(y,z) := \inf \big\{ j(y, z + \xi) \ : \ \xi \in [T_\mu(y)]^\perp \big\} \ .$$

Remark 3.9. We stress that $j_\mu(y,z)$ depends only on the component of z along $T_\mu(y)$.

Proof. By convexity, $\overline{J}$ coincides with the bipolar functional J^{**} in the duality $\big(L^p_\mu(\mathbf{T}), L^{p'}_\mu(\mathbf{T})\big)$. Let $B : D(B) \subset L^p_\mu(\mathbf{T}) \to \big(L^p_\mu(T)\big)^n$ be the linear operator defined by $D(B) := \mathcal{C}^\infty(\mathbf{T})$, $Bu = \nabla u$. Set $F_j(u) := \int_Y j(y,u)\, d\mu$. An abstract Convex Analysis lemma for the computation of $(F_j \circ B)^*$ gives (see [18, Theorem 5.1])

$$J^*(v) = (F_j \circ B)^*(v) = \inf \Big\{ \int_Y j^*(y,\Phi) \ : \ B^*\Phi = v \Big\} \,, \qquad v \in L^{p'}_\mu(\mathbf{T}) \ .$$

Now J^{**} equals by definition

$$\begin{aligned} J^{**}(u) &= \sup \big\{ \langle u, v\rangle - J^*(v) \ : \ v \in L^{p'}_\mu(\mathbf{T}) \big\} \\ &= \sup \big\{ \langle u, B^*\Phi\rangle - \int_Y j^*(y,\Phi)\, d\mu \ : \ \Phi \in D(B^*) \big\} \end{aligned}$$

It can be checked, arguing as in the proof of Proposition 3.7, that $D(B^*) = X^{p'}_\mu(\mathbf{T})$ and $B^*\Phi = -\operatorname{div}_\mu \Phi$, hence

$$J^{**}(u) = \sup \Big\{ \int_Y \big[-u \operatorname{div}_\mu \Phi - j^*(y,\Phi) \big]\, d\mu \ : \ \Phi \in X^{p'}_\mu(\mathbf{T}) \Big\} \ .$$

In particular, when $u \not\in \S(\mathbf{T})$, (3.14) and the growth condition satisfied by j^* lead to $J^{**}(u) = +\infty$. On the other hand, if $u \in \S(\mathbf{T})$, we can integrate by parts and we obtain

$$J^{**}(u) = \sup \Big\{ \int_Y \big[\nabla_\mu u \cdot \Phi - j^*(y,\Phi) \big]\, d\mu \ : \ \Phi \in X^{p'}_\mu(\mathbf{T}) \Big\} \ .$$

Applying the same argument of commutation as used in the proof of Proposition 3.5, the supremum can be passed under the sign of integral; thus, taking into account (3.5), we obtain

$$\overline{J}(u) = J^{**}(u) = \int_Y \sup_{z^* \in T_\mu(y)} [\nabla_\mu u \cdot z^* - j^*(y,z^*)] d\mu = \int_Y j_\mu(y, \nabla_\mu u) d\mu \ .$$

∎

4. Connectedness notions for a measure

This short paragraph is devoted to focus the assumptions on μ which will be enforced in the next section to prove the main structure result (see Theorem 5.1).
We propose some different definitions of *connectedness* for a positive Radon measure on $\mathbb{R}^n$: the first pair is related to the connectedness notions proposed in [70] for a periodic subset of $\mathbb{R}^n$; the second one involves a Poincaré-type inequality, and in particular the fourth definition will be

the key for compactness. The notions are all given for a fixed exponent $p \in [1, +\infty]$ and read as follows (C, c are supposed real constants).

- μ is *weakly p-connected on* $\mathbf{T}$ if:

$$(H1) \qquad u \in \S(\mathbf{T}), \quad \nabla_\mu u = 0 \ \mu\text{-a.e.} \ \Rightarrow \ \exists\, c \ : \ u = c \ \mu\text{-a.e.};$$

- μ is *weakly p-connected on* $\mathbb{R}^n$ if:

$$(H2) \qquad u \in H^{1,p}_{\mu,loc}, \quad \nabla_\mu u = 0 \ \mu\text{-a.e.} \ \Rightarrow \ \exists\, c \ : \ u = c \ \mu\text{-a.e.};$$

- μ is *strongly p-connected on* $\mathbf{T}$ if:

$$(H3) \qquad \exists\, C \ : \ \int_Y |u|^p \, d\mu \le C \int_Y |\nabla_\mu u|^p \, d\mu \ , \quad \forall u \in \S(\mathbf{T}) \text{ with } \int_Y u \, d\mu = 0 \, ;$$

- μ is *strongly p-connected on* $\mathbb{R}^n$ if:

$$(H4) \qquad \exists\, C : \int_{kY} |u|^p \, d\mu \le kC \int_{kY} |\nabla_\mu u|^p \, d\mu \ \ \forall k \in \mathbb{N}\,, \forall u \in H^{1,p}_{\mu,loc} \text{ with} \int_{kY} u \, d\mu = 0 \, .$$

Each of these definitions, and even more their possible relations, turns out to be quite delicate. We limit ourselves to summarize here some remarks and examples. Notice that by density it is enough to check $(H1)$-$(H4)$ on smooth functions.
Of course, $(H2) \Longrightarrow (H1)$. In particular, in the special case when μ is the Lebesgue measure on an open periodic subset Q of $\mathbb{R}^n$, the connectedness of Q in the sense of [70, Definition 1.1] is equivalent to $(H1)$.
Notice also that $(H1)$ do depends on p: in the well known-case of the Lebesgue measure concentrated on the black squares of a chess-board, it is satisfied if and only if $p > 2$. If more drastically the support of μ is not connected in the topological sense, then $(H1)$ is obviously false.

Poincaré type-inequalities on manifolds or on weighted Sobolev spaces have recently received a deal of attention for their importance in different areas (see for instance [11, 23, 46]). By $(H4)$, which obviously implies $(H3)$, we require that the Poincaré constant on kY is finite for every $k \in \mathbb{N}$ and further that it is equal to k times the Poincaré constant on Y: by change of variables, this implies in particular that the Poincaré constant for each rescaled measure μ_ε on Y is independent of ε (see (6.4)). We stress in fact that $(H3)$ can hold for a measure which fails to satisfy $(H4)$: this happens for instance when ♣ $n = 3$ and μ is the one-dimensional Hausdorff measure over a periodic network of e_3 parallel fibers. ♠

Obviously, $(H3) \Longrightarrow (H1)$ and $(H4) \Longrightarrow (H2)$, which motivates the choice of the terminology of *weak* and *strong* connectedness (in particular, condition $(H4)$ implies all the others). The other possible relations between the different definitions, or more in general between groups of them, are to our opinion of big interest, but are at present undiscovered.

The global hypothesis $(H4)$ is fulfilled for instance if $n = 2$ and $\mu \llcorner Y$ is the one-dimensional Hausdorff measure over one of the sets:

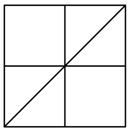

Figure 4.2a

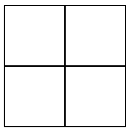

Figure 4.2b

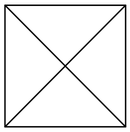

Figure 4.2c

5. The two-scale structure result for a limit of gradients

Proposition 2.3 investigates the two-scale convergence properties of a sequence $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega)$ which satisfies an uniform boundedness estimate in $L^p_{\mu_\varepsilon}$. We can now study what happens when the same condition is also fulfilled for the sequence of gradients $\{\nabla u_\varepsilon\}$. The following structure result holds, whose version in the Lebesgue case has been established in [3]. The proof when μ is a general measure is considerably more delicate; therefore some of its intermediate steps are given before it, in the form of autonomous lemmas.

Theorem 5.1. *Let $p > 1$, let Ω be an open bounded subset of $\mathbb{R}^n$ of class $\mathcal{C}^1$, and let $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega)$ satisfy $\int_\Omega \left(|u_\varepsilon|^p + |\nabla u_\varepsilon|^p\right) d\mu_\varepsilon \leq M$; possibly passing to a subsequence, assume that $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0 \in L^p_m(\Omega \times Y)$ and $\nabla u_\varepsilon \rightharpoonup\!\!\rightharpoonup \chi \in \left(L^p_m(\Omega \times Y)\right)^n$. Then:*

- *(i) if μ satisfies $(H1)$, $u_0(x,y) = u(x)$ (i.e. u_0 is independent of y), where the function u belongs to $W^{1,p}_0(\Omega)$ provided μ satisfies also $(H2)$ and $(H3)$;*
- *(ii) under assumptions $(H2)$ and $(H3)$ on μ, there exists $u_1 \in L^p\left(\Omega, \S(\mathbf{T})\right)$ such that $\chi(x,y) = \nabla u(x) + \nabla_{\mu,y} u_1(x,y) + \xi(y)$, with u as in (i), and $\xi(y) \in \left[T_\mu(y)\right]^\perp$ μ-a.e. on Y; in addition, $\nabla_{\mu_\varepsilon} u_\varepsilon \rightharpoonup\!\!\rightharpoonup \nabla u(x) + \nabla_{\mu,y} u_1(x,y)$.*

Lemma 5.2. *Let $V := \left\{ \mathrm{div}_\mu \Phi \,:\, \Phi \in X^{p'}_\mu(\mathbf{T}) \right\}$. If μ enjoys $(H1)$, then the orthogonal space $V^\perp$ of V in $L^{p'}_\mu(\mathbf{T})$ is given by the constant functions.*

Remark 5.3. A consequence of the above lemma is that the closure of V in $L^{p'}_\mu(\mathbf{T})$ is given by the functions whith zero mean value. A similar result has been obtained in a different framework by Zhikov in the case $p = 2$ (see [70]).

Proof of Lemma 5.2. Consider the functional J on $L^p_\mu(\mathbf{T})$:

$$J(u) = \begin{cases} \frac{1}{p} \int_Y |\nabla_\mu u|^p \, d\mu & \text{if } u \in \S(\mathbf{T}) \,, \\ +\infty & \text{if } u \in L^p_\mu(\mathbf{T}) \setminus \S(\mathbf{T}) \end{cases} .$$

We can write J as $J = F \circ A_\sharp$, where $F(z) := \frac{1}{p} \int |z|^p \, d\mu$, and the linear operator $A_\sharp : D(A_\sharp) \subset L^p_\mu(\mathbf{T}) \to \left(L^p_\mu(\mathbf{T})\right)^n$ is defined as done in Section 3. For the computation of $(F \circ A_\sharp)^*$, we can apply the same lemma already used in the proof of Proposition 3.8 (see [18, Theorem 5.1]), which gives, for any $v \in L^{p'}_\mu(\mathbf{T})$,

$$\begin{aligned} J^*(v) &= \inf \Big\{ \frac{1}{p'} \int_Y |\sigma|^{p'} \, d\mu \;:\; A^*_\sharp \sigma = v \Big\} \\ &= \inf \Big\{ \frac{1}{p'} \int_Y |\sigma|^{p'} \, d\mu \;:\; \sigma \in Y^{p'}_\mu(\mathbf{T}) \,,\; -\mathrm{div}_\mu(P_\mu \sigma) = v \Big\} \end{aligned}$$

Hence V coincides with $\mathrm{dom}(J^*) := \{v \in L^{p'}_\mu(\mathbf{T}) \,:\, J^*(v) < +\infty\}$. Let J^∞ be the recession function of J. It is defined by

$$J^\infty(u) = \lim_{t\to+\infty} \frac{J(tu)}{t} = \begin{cases} +\infty & \text{if } u \notin \S(\mathbf{T}) , \\ \displaystyle\lim_{t\to+\infty} \frac{t^{p-1}}{p} \int_Y |\nabla_\mu u|^p \, d\mu & \text{if } u \in \S(\mathbf{T}). \end{cases}$$

Due to the assumption $(H1)$ on μ, we get

$$J^\infty(u) < +\infty \iff J^\infty(u) = 0 \iff \exists\, c \,:\, u = c \ \mu-\text{a.e. on } Y \, . \tag{5.1}$$

As J is convex lower semicontinuous proper ($J(0) = 0$), by a well-known result in Convex Analysis (see for instance [61, Theorem 13.3]), the recession functional J^∞ of J satisfies

$$J^\infty(u) = \sup\left\{ \int_Y u\, v \, d\mu \,:\, v \in \mathrm{dom}(J^*) \right\} , \qquad u \in L^p_\mu(\mathbf{T});$$

Thus u belongs to the othogonal of $V = \mathrm{dom}(J^*)$ if only if $J^\infty(u) = 0$. The conclusion follows from (5.1). ∎

Lemma 5.4. *Let j satisfy the assumptions of Proposition 3.8, and set*

$$j^{\hom}(z) := \inf\left\{ \int_Y j(y, z + \nabla u(y)) \, d\mu \,:\, u \in \mathcal{C}^\infty(\mathbf{T}) \right\} , \qquad z \in \mathbb{R}^n \, . \tag{5.2}$$

Then:

(i) $j^{\hom}$ *is convex and satisfies a growth condition* $j^{\hom}(z) \le \Lambda\big(1 + |z|^p\big)$ ($\Lambda \in (0, +\infty)$)*;*

(ii) $\big(j^{\hom}\big)^*(z^*) = \inf\left\{ \int_Y j^*(y, \Phi) \, d\mu \,:\, \Phi \in X^{p'}_\mu(\mathbf{T}),\ \mathrm{div}_\mu \Phi = 0,\ \int_Y \Phi \, d\mu = z^* \right\}$*;*

(iii) $j^{\hom}(z) = \inf\left\{ \int_Y j_\mu(y, z_\mu + \nabla_\mu u(y)) \, d\mu \,:\, u \in \S(\mathbf{T}) \right\}$, *where* $z_\mu := P_\mu(y)[z]$*;*

(iv) under the hypotheses $(H2)$ *and (H3) on* μ, $j^{\hom}$ *enjoys a coercivity property* $j^{\hom}(z) \ge \lambda |z|^p$ ($\lambda \in (0, +\infty)$)*, hence in particular* $(j^{\hom})^*$ *is continuous on the whole* $\mathbb{R}^n$*.*

Proof. The convexity and the growth assumption on j give straightforward the corresponding properties for $j^{\hom}$.

To compute $(j^{\hom})^*$ (in the duality $\big((L^p_\mu(\mathbf{T}))^n, (L^{p'}_\mu(\mathbf{T}))^n\big)$), notice that $j^{\hom}$ can be written as

$$j^{\hom} = F_j \nabla \chi_G + \chi_C \ , \tag{5.3}$$

where $F_j(u) = \int_Y j(y, u) \, d\mu$, $\chi_G(u)$ and $\chi_C(u)$ are finite and equal to zero if and only if u belongs respectively to

$$G := \{\nabla\psi \,:\, \psi \in \mathcal{C}^\infty(\mathbf{T})\} \ , \quad C := \{v \in (L^p_\mu(\mathbf{T}))^n \text{ s.t. } \exists\, z \in \mathbb{R}^n \,:\, v = z \ \mu\text{-a.e. on } Y\} \ ,$$

and $(F_j \nabla \chi_G)(u) := \inf\{F_j(u+v) + \chi_G(-v) \;:\; v \in \big(L^p_\mu(\mathbf{T})\big)^n\}$. Being F_j continuous at $0 \in \mathrm{dom}(\chi_G)$, we deduce from (5.3) (see for instance [61, Theorem 16.4])

$$(j^{\mathrm{hom}^*}) = (F_j \nabla \chi_G)^* \nabla \chi_C^* = \big(F_j^* + \chi_G^*\big) \nabla \chi_C^* \,.$$

We have $\chi_G^* = \chi_{G^\perp}$, $\chi_C^* = \chi_{C^\perp}$, with (recall Lemma 3.4)

$$G^\perp = \{\Phi \in X^{p'}_\mu(\mathbf{T}) \;:\; \mathrm{div}_\mu \Phi = 0\}\,, \qquad C^\perp = \Big\{v \in \big(L^{p'}_\mu(\mathbf{T})\big)^n \;:\; \int_Y v\, d\mu = 0\Big\}\,.$$

Thus we get, using [27, Theorem VII.14] for the computation of F_j^*,

$$\begin{aligned}
\big(j^{\mathrm{hom}}\big)^*(z^*) &= \inf\{F_j^*(z^*+\sigma) \;:\; z^*+\sigma \in G^\perp \,,\; \sigma \in C^\perp\} \\
&= \inf\Big\{\int_Y j^*(y, z^*+\sigma)\, d\mu : z^*+\sigma \in X^{p'}_\mu(\mathbf{T}),\, \mathrm{div}_\mu(z^*+\sigma) = 0,\, \int_Y \sigma\, d\mu = 0\Big\} \\
&= \inf\Big\{\int_Y j^*(y, \Phi)\, d\mu \;:\; \Phi \in X^{p'}_\mu(\mathbf{T}) \,,\; \mathrm{div}_\mu \Phi = 0 \,,\; \int_Y \Phi\, d\mu = z^*\Big\}\,.
\end{aligned}$$

The assertion (iii) is a consequence of Proposition 3.8 and of a well-known property of relaxation (see [26, Chapter 1]).

We finally pass to prove (iv). The growth condition from below on j implies

$$j^{\mathrm{hom}}(z) \geq c \inf\Big\{\int_Y |z_\mu + \nabla_\mu u|^p\, d\mu \;:\; u \in \S(\mathbf{T})\Big\} =: c f(z)\,.$$

In fact, by the assumption (H3) on μ, the infimum above is attained on $\S(\mathbf{T})$. Then, we claim that $f(z) > 0$ whatever $z \neq 0$. Indeed, if $f(z) = 0$, there exists $u \in \S(\mathbf{T})$ such that $z_\mu + \nabla_\mu u = 0$ μ-a.e. on Y , or equivalently $\nabla_\mu(z \cdot y + u) = 0$ μ-a.e. on Y. Since the function $y \mapsto z \cdot y + u\,(x)$ belongs to $H^{1,p}_{\mu,\mathrm{loc}}$, the assumption $(H2)$ on μ implies that it must be constant μ-a.e., and this is equivalent, by the periodicity of u, to require that z equals zero. Denoting by m_1 the (strictly positive) minimum of f on the unit ball of $\mathbb{R}^n$, and using the p-homogeneity of f, we deduce that the coercivity property (iv) holds with $\lambda = c m_1$. ■

Lemma 5.5. *Let* $M := \{\sigma \in Y^{p'}_\mu(\mathbf{T}) \;:\; \mathrm{div}_\mu(P_\mu \sigma) = 0\}$. *If* μ *enjoys* $(H3)$, *then the orthogonal space* $M^\perp$ *of* M *in* $\big(L^{p'}_\mu(\mathbf{T})\big)^n$ *is given by* $N := \{\nabla_\mu u \;:\; u \in \S(\mathbf{T})\}$.

Proof. We have $N = R(A_\sharp) :=$ image of $A_\sharp$, and $M = \ker(A_\sharp^*)$, being $A_\sharp^*$ the linear operator defined in Section 3. Therefore $M = N^\perp$ [22, Corollary II.17], and so $M^\perp = \overline{N}$. It remains to prove that N is closed in $\big(L^p_\mu(\mathbf{T})\big)^n$. To this aim, we exploit the assumption $(H3)$ on μ. Let $\{\nabla_\mu u_h\}$ be a sequence of elements of N converging in $\big(L^p_\mu(\mathbf{T})\big)^n$. Then $\{u_h - \int_Y u_h\, d\mu\}$ is a Cauchy sequence in $H^{1,p}_\mu(\mathbf{T})$, since by $(H3)$

$$\Big\|u_h - \int_Y u_h\, d\mu - u_k + \int_Y u_k\, d\mu\Big\|_{p,\mu} \leq \big\|\nabla_\mu u_h - \nabla_\mu u_k\big\|_{p,\mu}\,.$$

By completeness of $\S(\mathbf{T})$, the sequence $\{u_h - \int_Y u_h \, d\mu\}$ converges to some $u \in \S(\mathbf{T})$, so that the limit of $\{\nabla_\mu u_h\}$ equals $\nabla_\mu u$ and in particular it belongs to N. ■

We can now give the

Proof of Theorem 5.1. Let $\Phi \in X^{p'}_\mu(\mathbf{T})$ and $\psi \in \mathcal{D}(\overline{\Omega})$. Using (2.1), one can check that $\operatorname{div}\left(\Phi(\frac{x}{\varepsilon})\mu_\varepsilon\right) = \varepsilon^{-1}(\operatorname{div}_\mu \Phi)\left(\frac{x}{\varepsilon}\right)\mu_\varepsilon$. Then (3.1) gives

$$\operatorname{div}\left(\psi(x)\Phi\left(\frac{x}{\varepsilon}\right)\mu_\varepsilon\right) = \nabla\psi(x)\cdot\Phi\left(\frac{x}{\varepsilon}\right)\mu_\varepsilon + \varepsilon^{-1}\psi(x)(\operatorname{div}_\mu \Phi)\left(\frac{x}{\varepsilon}\right)\mu_\varepsilon \,. \tag{5.4}$$

Since u_ε belong to $\mathcal{D}$ (when extended to zero out of Ω), both sides of the above inequality can be applied to u_ε, and, multiplying by ε, we get

$$\varepsilon\int_\Omega \psi(x)\nabla u_\varepsilon(x)\cdot\Phi\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon = \varepsilon\int_\Omega u_\varepsilon(x)\nabla\psi(x)\cdot\Phi\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon + \int_\Omega u_\varepsilon(x)\psi(x)(\operatorname{div}_\mu \Phi)\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon \,.$$

If we pass to the limit as $\varepsilon \to 0$, taking into account that $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$ are both uniformly bounded in $L^p_{\mu_\varepsilon}(\Omega)$, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi(x)(\operatorname{div}_\mu \Phi)\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon = 0 \,.$$

Now, in view of Remark 2.4, $\varphi(x,y) := \psi(x)\operatorname{div}_\mu\Phi(y)$ is an admissible test function in the two-scale convergence, and we infer

$$\int_\Omega \psi(x)\left[\int_Y u_0(x,y)\operatorname{div}_\mu \Phi(y)\, d\mu(y)\right]dx = 0 \,.$$

By the arbitrariness of $\psi \in \mathcal{D}(\overline{\Omega})$ and of $\Phi \in X^{p'}_\mu(\mathbf{T})$, it follows that, for $\mathcal{L}^n$-a.e. $x\in\Omega$, $u_0(x,\cdot)$ belongs to $V^\perp$, with V defined as in Lemma 5.2. Then by the same lemma we deduce that, for $\mathcal{L}^n$-a.e. $x\in\Omega$, $u_0(x,\cdot)$ is constant μ-a.e. on Y.

Set $u(x) := u_0(x,y)$, and let us prove that u belongs to $W^{1,p}_0(\Omega)$ under the assumptions $(H2)$ and $(H3)$ on μ. Let ψ, Φ be as above, and assume in addition that $\operatorname{div}_\mu \Phi = 0$. Applying both sides of (5.4) to u_ε (this time without multiplying by ε), it comes

$$-\int_\Omega \psi(x)\nabla u_\varepsilon(x)\cdot\Phi\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon = \int_\Omega u_\varepsilon(x)\nabla\psi(x)\cdot\Phi\left(\frac{x}{\varepsilon}\right)d\mu_\varepsilon \,.$$

Passing to the two-scale limit, we deduce

$$\begin{aligned} -\int_{\Omega\times Y}\psi(x)\chi(x,y)\cdot\Phi(y)\,dm &= \int_{\Omega\times Y} u(x)\nabla\psi(x)\cdot\Phi(y)\,dm \\ &= \left\{\int_\Omega u(x)\nabla\psi(x)\,dx\right\}\cdot\left\{\int_Y \Phi(y)\,d\mu\right\}. \end{aligned} \tag{5.5}$$

By the Hölder inequality, it follows

$$\overline{\Phi}\cdot\int_\Omega u\nabla\psi\,dx \le \|\chi\|_{p,m,\Omega\times Y}\|\Phi\|_{p',\mu,Y}\|\psi\|_{p',\mathcal{L}^n,\Omega}\ , \tag{5.6}$$

where we have set $\overline{\Phi} := \int_Y \Phi(y)\,d\mu(y)$. Consider now the convex subset of $\mathbb{R}^n$

$$K := \Big\{\overline{\Phi}\ :\ \Phi\in X^{p'}_\mu\ ,\ \mathrm{div}_\mu\,\Phi = 0\ ,\ \|\Phi\|_{p',\mu,Y}\le 1\Big\}\ . \tag{5.7}$$

Notice that, by Lemma 5.4 (iii) and the definition of K, we have

$$\mathrm{dom}\,(j^{\mathrm{hom}})^* = \bigcup_{\lambda\in\mathbb{R}}\lambda K\ .$$

If μ satisfies $(H2)$ and $(H3)$, we know from Lemma 5.4 (iv) that $\mathrm{dom}\,(j^{\mathrm{hom}})^* = \mathbb{R}^n$, so K must have a non-empty interior. By the regularity assumption on Ω, it follows from (5.6) that u belongs to $W^{1,p}_0(\Omega)$ (*cf.* [22, Proposition IX.18]).

We can in this case integrate by parts on Ω at the right hand side of (5.5). We get

$$\int_\Omega\psi(x)\Big\{\int_Y\big[\chi(x,y)-\nabla u(x)\big]\cdot\Phi(y)\,d\mu(y)\Big\}\,dx = 0\ .$$

By the arbitrariness of $\psi\in\mathcal{D}(\overline{\Omega})$, it follows that, for $\mathcal{L}^n$-a.e.$x\in\Omega$ and for every vector field $\Phi\in X^{p'}_\mu(\mathbf{T})$ with $\mathrm{div}_\mu\,\Phi=0$, there holds

$$\int_Y\big[\chi(x,y)-\nabla u(x)\big]\cdot\Phi(y)\,d\mu(y) = 0\ .$$

In particular, taking $\Phi = P_\mu\sigma$, with $\sigma\in Y^{p'}_\mu(\mathbf{T})$, we deduce that $P_\mu(\cdot)\big[\chi(x,\cdot)-\nabla u(x)\big]$ belongs to $M^\perp$, where M is defined as in Lemma 5.5. Using such lemma, it then follows that there exists $u_1\in L^p(\Omega,\S(\mathbf{T}))$ such that $P_\mu(y)\big[\chi(x,y)-\nabla u(x)\big] = \nabla_{\mu,y}u_1(x,y)$. Hence $\chi(x,y) = \nabla u(x)+\nabla_{\mu,y}u_1(x,y)+\xi(y)$, with $\xi(y)\in[T_\mu(y)]^\perp$. Finally, we apply Proposition 2.8 to the sequences $P_{\mu_\varepsilon}\in L^{p'}_{\mu_\varepsilon}(\Omega;\mathbb{R}^{n^2})$ and $\nabla u_\varepsilon\in L^p_{\mu_\varepsilon}(\Omega;\mathbb{R}^n)$: we have respectively $P_{\mu_\varepsilon}\twoheadrightarrow P_\mu(y)$ (see Example 2.7) and $\nabla u_\varepsilon \rightharpoonup\!\!\!\rightharpoonup \chi(x,y)$, so that the product $P_{\mu_\varepsilon}(\nabla u_\varepsilon) = \nabla_{\mu_\varepsilon}u_\varepsilon$ two-scale converges to $P_\mu(y)\big(\chi(x,y)\big) = \nabla u(x)+\nabla_{\mu,y}u_1(x,y)$. ■

♣

Remark 5.6. When $(H2)$ does not hold, we can apply all the steps of the above proof except that in this case the convex set K defined by (5.7) has an empty interior. Thus, denoting by M the subspace of $\mathbb{R}^n$ spanned by the relative interior of K, the estimate $z\cdot\nabla u\in L^p(\Omega)$ can be obtained only for $z\in M$. Therefore, in part (i) of Theorem 5.1, the assertion $u\in W^{1,p}_0(\Omega)$ must be replaced by $u\in W^{1,p}_{0,M}(\Omega)$, where

$$W^{1,p}_{0,M}(\Omega) := \big\{u\in L^p(\Omega)\ :\ \forall z\in M\ ,\ \ z\cdot\nabla u\in L^p(\Omega)\ \text{ and }\ u\,(\nu_\Omega\cdot z) = 0 \text{ on } \partial\Omega\big\}\ .$$

♠

6. Homogenization with periodic measures

We can now consider the homogenization problem introduced at the beginning of the paper, namely, the study of the Γ-convergence in the sense of De Giorgi [34] of the sequence of functionals

$$J_\varepsilon(u) = \begin{cases} \int_\Omega j\big(\frac{x}{\varepsilon}, \nabla u\big)\, d\mu_\varepsilon & \text{if } u \in \mathcal{C}_0^1(\Omega) \\ +\infty & \text{if } u \in L^p_\mu(\Omega) \setminus \mathcal{C}_0^1(\Omega) \end{cases} \tag{6.1}$$

Let us fix here the assumptions made on the integrand $j(y,z)$, which however falls into the standard setting already employed in Section 3: j is μ-measurable and Y-periodic in y, convex in z, and satisfies, for suitable positive constants C, c, the p-growth condition (where $p > 1$)

$$c|z|^p \le j(y,z) \le C\big(1 + |z|^p\big) , \qquad (y,z) \in \mathbb{R}^n \times \mathbb{R}^n . \tag{6.2}$$

Naturally, the first step to be done in the study of the asymptotic behaviour of J_ε, is the choice of a suitable convergence on the class of all admissible functions u. At this purpose, note that the role played by u in the expression of $J_\varepsilon(u)$ is that of a function defined μ_ε-almost everywhere; thus, as μ_ε varies with ε, it is convenient to see the convergence of u_ε to u as the weak star convergence of the measure μ_ε to $u\mathcal{L}^n$. Therefore we implicitely extend the definition (6.1) of J_ε to the space $\mathcal{M}$ of Radon measures on $\mathbb{R}^n$ by setting:

$$J_\varepsilon(\lambda) = \begin{cases} \int_\Omega j\big(\frac{x}{\varepsilon}, \nabla u\big)\, d\mu_\varepsilon & \text{if } \lambda = u\mu_\varepsilon \ , \ u \in \mathcal{C}_0^1(\Omega) \ , \\ +\infty & \text{otherwise} . \end{cases} \tag{6.3}$$

The equicoerciveness of (J_ε) on $\mathcal{M}$ obtained in Lemma 6.1 below is related to the strong p-connectness property of μ.

Lemma 6.1. *Let $p > 1$ and μ be a periodic Radon measure satisfying $(H4)$. Then , for any sequence $\{u_\varepsilon\} \in \mathcal{C}_0^1(\Omega)$ such that $J_\varepsilon(u_\varepsilon \mu_\varepsilon)$ is upper bounded, we have that $\sup_\varepsilon \int_\Omega |u_\varepsilon|^p\, d\mu_\varepsilon < +\infty$; therefore $\{u_\varepsilon \mu_\varepsilon\}$ is relatively compact in $\mathcal{M}$ and any of its limit points is absolutely continuous with respect to $\mathcal{L}^n$, with a density $u \in L^p(\Omega)$.*

Proof. By the coerciveness assumption on j, there exists $M' > 0$ such that $\int_\Omega |\nabla u_\varepsilon|^p\, d\mu_\varepsilon \le M'$. Notice now that, for any $\varepsilon > 0$, any integer k, and any $v \in \mathcal{C}_0^1(\varepsilon k Y)$, there holds

$$\int_{\varepsilon k Y} |v|^p\, d\mu_\varepsilon \le (\varepsilon k C)^p \int_{\varepsilon k Y} |\nabla v|^p\, d\mu_\varepsilon \ , \tag{6.4}$$

where C is the positive and finite constant involved in $(H4)$. Indeed, using the definition of μ_ε and the assumption $(H4)$ on μ, we get

$$\begin{aligned} \int_{\varepsilon k Y} |v|^p\, d\mu_\varepsilon &= \varepsilon^n \int_{kY} |v(\varepsilon x)|^p\, d\mu \le \varepsilon^n (kC)^p \int_{kY} \big|\nabla\big(v(\varepsilon x)\big)\big|^p\, d\mu \\ &= \varepsilon^n (\varepsilon k C)^p \int_{kY} |\nabla v(\varepsilon x)|^p\, d\mu = (\varepsilon k C)^p \int_{\varepsilon k Y} |\nabla v|^p\, d\mu_\varepsilon \ . \end{aligned}$$

Since Ω is bounded, for any $\varepsilon > 0$ there exists a positive integer k_ε, with $\varepsilon k_\varepsilon \leq C' < +\infty$, such that $\Omega \subset \varepsilon k_\varepsilon Y$. Applying (6.4) to every $u_\varepsilon \in \mathcal{C}_0^1(\Omega) \subset \mathcal{C}_0^1(\varepsilon k_\varepsilon Y)$, we infer

$$\int_\Omega |u_\varepsilon|^p \, d\mu_\varepsilon = \int_{\varepsilon k_\varepsilon Y} |u_\varepsilon|^p \, d\mu_\varepsilon \leq (C')^p C^p \int_{\varepsilon k_\varepsilon Y} |\nabla u_\varepsilon|^p \, d\mu_\varepsilon \leq (C')^p C^p M' \ . \tag{6.5}$$

Then the weak compactness of $\{u_\varepsilon \mu_\varepsilon\}$ in $\mathcal{M}$, and the fact that any of its limit points is absolutely continuous with respect to $\mathcal{L}^n$, with a density $u \in L^p(\mathbb{R}^n)$, follows straightforward from Proposition 2.3. ■

We can now state our main Γ-convergence theorem which extends to the case of general periodic measures μ the well-known result for $\mu = \mathcal{L}^n$ proved at first time by Marcellini [51].

Theorem 6.2. *Let μ satisfy ♣ $(H2)$ and $(H3)$ ♠, with $p > 1$. Then the sequence $\{J_\varepsilon\}$ defined in (6.3) Γ-converges on $\mathcal{M}$ as $\varepsilon \to 0$ to the homogeneized functional J^{hom} defined by*

$$J^{\text{hom}}(\lambda) = \begin{cases} \int_\Omega j^{\text{hom}}(\nabla u(x)) \, dx & \text{if } \lambda = u\mathcal{L}^n \ , \ u \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise} \ , \end{cases}$$

where the integrand j^{hom} is defined via the unit-cell problem (5.2).

Remark 6.3. ♣ A generalized version of Theorem 6.2 holds when μ does not enjoy $(H2)$. In this case the integrand j^{hom} is still given by formula (5.2), but it is degenerate in the directions $z \in M^\perp$, where M is the linear space defined in Remark 5.6; accordingly, the homogenization result above holds provided $W_0^{1,p}(\Omega)$ is replaced by $W_{0,M}^{1,p}$ (*cf.* Remark 5.6). ♠

Proof of Theorem 6.2. We divide the proof of this theorem into two parts. Recall indeed that the Γ-convergence of J_ε to J consists in the two following assertions:

I) $\inf \left\{ \liminf\limits_{\varepsilon \to 0} J_\varepsilon(\lambda_\varepsilon) \ : \ \lambda_\varepsilon \rightharpoonup \lambda \right\} \geq J^{\text{hom}}(\lambda) \ , \quad \lambda \in \mathcal{M} \ ;$

II) $\inf \left\{ \limsup\limits_{\varepsilon \to 0} J_\varepsilon(\lambda_\varepsilon) \ : \ \lambda_\varepsilon \rightharpoonup \lambda \right\} \leq J^{\text{hom}}(\lambda) \ , \quad \lambda \in \mathcal{M} \ .$

Proof of I). By Lemma 6.1, it is enough to prove I) for $\lambda = u\mathcal{L}^n$, with $u \in W_0^{1,p}(\Omega)$. Let $u_\varepsilon \mu_\varepsilon \rightharpoonup u\mathcal{L}^n$, with $J_\varepsilon(u_\varepsilon \mu_\varepsilon) \leq M$. Then, by Lemma 6.1 and Theorem 5.1 (i), we can assume that $\{u_\varepsilon\}$ two-scale converges to a function $u_0(x,y)$ independent of y, which then must be equal to $u(x)$. Moreover, we know from Theorem 5.1 (ii) that $\nabla u_\varepsilon \rightharpoonup \chi$, where $\chi(x,y) = \nabla u(x) + \nabla_{\mu,y} u_1(x,y) + \xi(y)$, with $u_1 \in L^p(\Omega, \S(\mathbf{T}))$ and $\xi(y) \in \left[T_\mu(y)\right]^\perp$ μ-a.e. Applying Proposition 2.5, definition (3.15) of j_μ and *(iii)* of Lemma 5.4, we obtain

$$\begin{aligned} \liminf_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon \mu_\varepsilon) &\geq \int_{\Omega \times Y} j(y, \chi(x,y)) \, dm = \int_{\Omega \times Y} j\big(y, \nabla u(x) + \nabla_{\mu,y} u_1(x,y) + \xi(y)\big) \, dm \\ &\geq \int_\Omega \left\{ \int_Y \left[j_\mu(y, \nabla u(x) + \nabla_{\mu,y} u_1(x,y) \right] d\mu \right\} dx \geq \int_\Omega j^{\text{hom}}\big(\nabla u(x)\big) \, dx \ . \end{aligned}$$

Proof of II). Let $\lambda = u\mathcal{L}^n$, with $u \in W_0^{1,p}(\Omega)$, otherwise there is nothing to prove. We have to find $\{u_\varepsilon\} \subset \mathcal{C}_0^1(\Omega)$ such that $\limsup_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \leq J^{\hom}(\lambda)$. In view of the density of $\mathcal{D}(\Omega)$ into $W^{1,p}(\Omega)$, and of the continuity of $J^{\hom}$, by a standard diagonalization argument it is not restrictive to assume that $u \in \mathcal{D}(\Omega)$. For an arbitrary function $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))$, consider the sequence $u_\varepsilon \in \mathcal{D}(\Omega)$ defined by

$$u_\varepsilon(x) = u(x) + \varepsilon\varphi\big(x, \frac{x}{\varepsilon}\big) \ .$$

The measures $\{u_\varepsilon \mu_\varepsilon\}$ converge weakly to $u\mathcal{L}^n$, because $\varphi\big(x, \frac{x}{\varepsilon}\big)\mu_\varepsilon \rightharpoonup \Big\{ \int_Y \varphi(x,y)\, d\mu(y) \Big\} \mathcal{L}^n$, and so $\varepsilon\varphi\big(x, \frac{x}{\varepsilon}\big)\mu_\varepsilon \rightharpoonup 0$. We have:

$$\nabla u_\varepsilon(x) = \nabla u(x) + \varepsilon \nabla_x \varphi\big(x, \frac{x}{\varepsilon}\big) + \nabla_y \varphi\big(x, \frac{x}{\varepsilon}\big) \ ;$$

♣ hence, by a well-known lemma concerning convex functions which satisfy a p-growth condition (see for instance [33], 4.2.1.), the following inequality holds for some positive constant C:

$$\big|j(\nabla u_\varepsilon(x)) - j\big(\nabla u(x) + \nabla_y \varphi\big(x, \frac{x}{\varepsilon}\big)\big)\big| \leq C\varepsilon \big|\nabla_x \varphi\big(x, \frac{x}{\varepsilon}\big)\big| \Big\{ 1 + \big|\nabla u(x)\big|^{p-1} + \big|\nabla_y \varphi\big(x, \frac{x}{\varepsilon}\big)\big|^{p-1} \Big\} \ .$$

Integrating over Ω, a quite standard application of the Hölder inequality yields

$$\lim_{\varepsilon\to 0} \int_\Omega j(\nabla u_\varepsilon)\, d\mu_\varepsilon = \lim_{\varepsilon\to 0} \int_\Omega j\big(\nabla u(x) + \nabla_y \varphi\big(x, \frac{x}{\varepsilon}\big)\big)\, d\mu_\varepsilon \ .$$

Hence, to compute the limit of $J_\varepsilon(u_\varepsilon)$, we are reduced to pass to the two-scale limit at the right hand side of the above equality. So we infer ♠

$$\lim_{\varepsilon\to 0} \int_\Omega j\big(\frac{x}{\varepsilon}, \nabla u_\varepsilon\big)\, d\mu_\varepsilon = \int_{\Omega\times Y} j\big(y, \nabla u(x) + \nabla_y \varphi(x,y)\big)\, dm \ .$$

In particular, we deduce

$$\lim_{\varepsilon\to 0} \int_\Omega j\big(\frac{x}{\varepsilon}, \nabla u_\varepsilon\big)\, d\mu_\varepsilon \leq \inf_{\varphi \,\in\, \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))} \Big\{ \int_{\Omega\times Y} j\big(y, \nabla u(x) + \nabla_y \varphi(x,y)\big)\, dm \ .$$

The proof of II) is then concluded provided we show that

$$\begin{aligned} &\inf_{\varphi \,\in\, \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))} \int_\Omega \Big\{ \int_Y j\big(y, \nabla u(x) + \nabla_y \varphi(x,y)\big)\, d\mu(y) \Big\}\, dx = \\ &\int_\Omega \Big\{ \inf_{\varphi \,\in\, \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))} \int_Y j\big(y, \nabla u(x) + \nabla_y \varphi(x,y)\big)\, d\mu(y) \Big\}\, dx \ . \end{aligned} \tag{6.6}$$

Indeed, setting

$$h_\varphi(x) := \int_Y j\big(y, \nabla u(x) + \nabla_y \varphi(x,y)\big)\, d\mu(y) \qquad (\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))) \ ,$$

we derive, by a trivial localization argument,that for all $x \in \Omega$

$$\inf_{\varphi \in \mathcal{D}(\Omega;\mathcal{C}^\infty(\mathbf{T}))} h_\varphi(x) = \inf_{\varphi \in \mathcal{C}^\infty(\mathbf{T})} h_\varphi(x) = j^{hom}(\nabla u(x)) \ .$$

To prove (6.6) let us recall the following commutation argument [21, Theorem 1]: if $\mathcal{H} \subset L^1(\Omega)$ is an *inf-stable family*, in the sense that

$$\tag{6.7} \begin{gathered} \{u_1,\dots,u_N\} \subset \mathcal{H} \ , \quad \{\alpha_1,\dots,\alpha_N\} \subset \mathcal{C}^1(\overline{\Omega};[0,1]), \text{ with } \sum_{i=1}^N \alpha_i = 1 \\ \Downarrow \\ \exists u \in \mathcal{H} \ : \ u \le \sum_{i=1}^N \alpha_i u_i \ , \end{gathered}$$

then

$$\inf_{u\in\mathcal{H}} \int_\Omega u\,dx = \int_\Omega \operatorname*{ess\,inf}_{u\in\mathcal{H}} u \ dx \ .$$

In order to prove (6.6), we apply this principle to the the subfamily of $L^1(\Omega)$ defined by

$$\mathcal{H} := \left\{ h_\varphi \ : \ \varphi \in \mathcal{D}(\Omega;\mathcal{C}^\infty(\mathbf{T})) \right\} \ .$$

To check the inf-stability of $\mathcal{H}$, we consider $\{\varphi_1,\dots,\varphi_N\} \subset \mathcal{D}(\Omega;\mathcal{C}^\infty(\mathbf{T}))$, $\{\alpha_1,\dots,\alpha_N\}$ as in (6.7), and $\overline{\varphi}(x,y) := \sum_{i=1}^N \alpha_i(x)\varphi(x,y)$. Then $\overline{\varphi} \in \mathcal{D}(\Omega;\mathcal{C}^\infty(\mathbf{T}))$, so that $h_{\overline{\varphi}} \in \mathcal{H}$. We claim that $h_{\overline{\varphi}} \le \sum_{i=1}^N \alpha_i h_{\varphi_i}$. Indeed,

$$\begin{aligned} h_{\overline{\varphi}}(x) &:= \int_Y j\big(y,\nabla u(x)+\nabla_y\overline{\varphi}(x,y)\big)\,d\mu(y) = \int_Y j\Big(y,\sum_{i=1}^N \alpha_i(x)[\nabla u(x)+\nabla_y\varphi_i(x,y)]\Big)\,d\mu(y) \\ &\le \sum_{i=1}^N \alpha_i(x)\int_Y j\big(y,\nabla u(x)+\nabla_y\varphi_i(x,y)\big)\,d\mu(y) = \sum_{i=1}^N \alpha_i(x) h_{\varphi_i}(x) \ , \end{aligned}$$

where we used the convexity of j in its second variable. Thus we get

$$\tag{6.8} \inf_{\varphi \in \mathcal{D}(\Omega;\mathcal{C}^\infty(\mathbf{T}))} \int_\Omega h_\varphi(x)\,dx = \int_\Omega \operatorname*{ess\,inf}_{\varphi \in \mathcal{D}(\Omega;\mathcal{C}^\infty(\mathbf{T}))} h_\varphi(x)\,dx \ .$$

On the other hand, $\mathcal{H}$ is contained in the space $\mathcal{C}_0(\Omega)$, which is separable with respect to the uniform norm; hence we can find a sequence $\{h_n\}$ such that $\inf_{h\in\mathcal{H}} h(x) = \inf_n h_n(x)$, for all $x \in \Omega$. It follows from the definition of essential infimum that

$$\tag{6.9} \operatorname*{ess\,inf}_{h\in\mathcal{H}} h(x) = \inf_n h_n(x) = \inf_{h\in\mathcal{H}} h(x) \qquad \mathcal{L}^n\text{-a.e. on } \Omega \ .$$

By (6.8) and (6.9), (6.6) is proved and this achieves our proof. ■

As a consequence of well-known properties of Γ-convergence [34, Theorem 7.8 and Corollary 7.17], we deduce from Lemma 6.1 and Theorem 6.2 the following

Corollary 6.4. ♣ *Under the same hypotheses of Theorem 6.2, let us assume in addition that* μ *is strongly* p*-connected, and* ♠ *consider for any* $\varepsilon > 0$ *the problem*

$$(\mathcal{P}_\varepsilon) \qquad \inf\Big\{J_\varepsilon(u) - \int_\Omega fu\,d\mu_\varepsilon\ ,\quad u \in \mathcal{C}_0^1(\Omega)\Big\}\ ,$$

where J_ε *are defined by (6.1), and* f *is a prescribed function in* $\mathcal{C}(\overline{\Omega})$. *Then* ♣

(i) $\lim_{\varepsilon\to 0}\inf(\mathcal{P}_\varepsilon) = \min(\mathcal{P})$;

(ii) if $\{u_\varepsilon\}$ *is a minimizing sequence for* $(\mathcal{P}_\varepsilon)$, *up to subsequences it holds* $u_\varepsilon\mu_\varepsilon \rightharpoonup u\mathcal{L}^n$, *where* u *solves the limit problem* $(\mathcal{P})$, *given by*

$$(\mathcal{P}) \qquad \min\Big\{J^{\hom}(u) - \int_\Omega fu\,dx\ ,\quad u \in W_0^{1,p}(\Omega)\Big\}\ .\ ♠$$

As usual, in the case of quadratic functionals, the unit cell problem defined in (5.2) is completely determined in terms of n linear problems. Let us assume that the density of J_ε is given by $j(y,z) = \frac{1}{2}\sum_{i,j} a_{ij}(y)z_i\,z_j$, where the coefficients of the matrix $A(y) := a_{ij}(y)$ belong to $L^\infty(\mathbf{T})$ and are such that $\lambda|z|^2 \le j(y,z) \le \Lambda(1+|z|^2)$ for suitable positive constants λ and Λ. Then, we consider the following linear problems on the unit cell:

$$(6.10) \qquad -\operatorname{div}_y\Big[A(y)\Big(e_{i,\mu}(y) + \nabla_\mu\,\chi_i(y)\Big)\Big] = 0\ , \qquad \chi_i \in W_\mu^{1,2}(\mathbf{T})\ , \qquad i = 1,2,\cdots,n\ ,$$

where we have set $e_{i,\mu}(y) := P_\mu(y)e_i$. Accordingly, the homogenized integrand takes the form $j^{\hom}(z) = \frac{1}{2}A^{\hom}z\cdot z$ where the effective matrix $A^{\hom}$ is given by

$$(6.11) \qquad A_{i,j}^{\hom} = \int_Y \big(e_{i,\mu} + \nabla_{\mu,y}\,\chi_i\big)\cdot\big(e_{j,\mu} + \nabla_{\mu,y}\,\chi_j\big)\,d\mu(y)\ ;$$

consequently, the homogenized problem reads as

$$(6.12) \qquad -\operatorname{div}_x\big(A^{\hom}\nabla u(x)\big) = f(x)\ , \qquad u \in W_0^{1,2}(\Omega)\ .$$

In light of our two-scale approach we obtain the following corrector-type result.

Proposition 6.5. *Under the same assumptions of Theorem 6.2, suppose in addition that* j *is quadratic as above and let* $\{u_\varepsilon\}$ *be a minimizing sequence for problem* $(\mathcal{P}_\varepsilon)$. *Then*

$$(6.13) \qquad (u_\varepsilon, \nabla_{\mu_\varepsilon}u_\varepsilon) \rightharpoonup\!\!\rightharpoonup \Big(u(x)\,,\ \sum_i (e_i + \nabla_{\mu,y}\,\chi_i(y))\,\frac{\partial u}{\partial x_i}(x)\Big)\ ,$$

where $u \in W_0^{1,2}(\Omega)$ *and* $\chi_i \in W_\mu^{1,2}(\mathbf{T})$ *are respectively the unique solutions to (6.12) and to (6.10). Moreover, setting* $u_1(x,y) := \sum_i \frac{\partial u}{\partial x_i}(x)\,\chi_i(y)$, *we have*

$$(6.14) \qquad \lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x) - u(x)|^2\,d\mu_\varepsilon = 0$$

$$(6.15) \qquad \lim_{\varepsilon\to 0}\|u_\varepsilon - u - u_1(x,\frac{x}{\varepsilon})\|_{W_{\mu_\varepsilon}^{1,2}(\Omega)} = 0\ .$$

Proof. By Theorem 5.1, up to subsequences $(u_\varepsilon, \nabla_{\mu_\varepsilon} u_\varepsilon)$ two-scale converge to $(u(x), \nabla u(x) + \nabla_{\mu,y} u_1(x,y))$, where $(u, u_1) \in W_0^{1,2}(\Omega) \times L^2(\Omega, W_\mu^{1,2}(\mathbf{T}))$. Let now $(\varphi, \varphi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))$, and multiply by $\varphi(x) + \varphi_1(x, \frac{x}{\varepsilon})$ the Euler equation satisfied by u_ε. Passing to the two-scale limit, we obtain that (u, u_1) satisfy the variational formulation

$$\tag{6.16} \begin{aligned} &\int_{\Omega\times Y} A(y)\,[\nabla u(x) + \nabla_{\mu,y} u_1(x,y)] \cdot [\nabla \varphi(x) + \nabla_{\mu,y}\varphi_1(x,y)]\, dm \\ &= \int_\Omega f(x)\varphi(x)\,dx\ , \qquad \forall (\varphi, \varphi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbf{T}))\ . \end{aligned}$$

By Lax-Milgram's theorem, (6.16) admits a unique solution; so (u, u_1) do not depend on the choice of the subsequence. Now it is easily seen that (6.16) is the variational formulation of the system

$$\begin{cases} -\operatorname{div}_y \Big[A(y)\Big(\nabla u(x) + \nabla_{\mu,y} u_1(x,y)\Big)\Big] = 0 & \text{in } \Omega \times Y \\ -\operatorname{div}_x \Big[\int_Y A(y)\Big(\nabla u(x) + \nabla_{\mu,y} u_1(x,y)\Big)\, d\mu(y)\Big] = f & \text{in } \Omega\ , \end{cases}$$

which is in turn equivalent to the homogenized equations (6.12) and (6.10) via (6.11) and the relation $u_1(x,y) = \sum_i \dfrac{\partial u}{\partial x_i}(x)\,\chi_i(y)$. Thus (6.13) is proved. In order to prove the second part of the statement, we notice first that the corrector result (6.14) follows straightforward from (6.15). Indeed, by (H4), a Poincaré constant independent of ε rules as well for all the measures μ_ε (*cf.* (6.5)). Hence we are done if we show (6.15). First we claim that, since u_ε is a minimizing sequence for $(\mathcal{P}_\varepsilon)$, we have

$$\tag{6.17} \lim_{\varepsilon\to 0} \Big(\int_\Omega A(\frac{x}{\varepsilon}) \big\{\nabla_{\mu_\varepsilon} u_\varepsilon\big\}^2 dx - \int_\Omega f u_\varepsilon dx \Big) = 0\ .$$

Indeed, setting $\alpha_\varepsilon := \int_\Omega A(\frac{x}{\varepsilon})\big\{\nabla_{\mu_\varepsilon} u_\varepsilon\big\}^2 dx$ and $\beta_\varepsilon := \int_\Omega f u_\varepsilon dx$, we have, for every real t and for a suitable sequence $\delta_\varepsilon \to 0$,

$$\frac{1}{2}\alpha_\varepsilon - \beta_\varepsilon = J_\varepsilon(u_\varepsilon) - \int_\Omega f u_\varepsilon\, d\mu_\varepsilon \ \le\ J_\varepsilon(t u_\varepsilon) - \int_\Omega f\,(t u_\varepsilon)\, d\mu_\varepsilon + \delta_\varepsilon = \frac{1}{2} t^2 \alpha_e - t\beta_\varepsilon + \delta_\varepsilon\ .$$

Taking then the infimum with respect to t, we get $\frac{1}{2}\alpha_\varepsilon - \beta_\varepsilon \le -\frac{1}{2}\frac{\beta_\varepsilon^2}{\alpha_\varepsilon} + \delta_\varepsilon$, thus $\dfrac{(\alpha_\varepsilon - \beta_\varepsilon)^2}{\alpha_e} \to 0$, which proves the claim. Now, by (6.17),

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\Omega A(\frac{x}{\varepsilon}) \big\{ \nabla_{\mu_\varepsilon}\big[u_\varepsilon(x) - u(x) - \varepsilon u_1(x, \frac{x}{\varepsilon})\big]\big\}^2\, d\mu_\varepsilon \\ &= \lim_{\varepsilon\to 0} \Big\{ \int_\Omega f(x) u_\varepsilon(x) d\mu_\varepsilon - 2\int_\Omega A(\frac{x}{\varepsilon}) \nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla_{(\mu_\varepsilon)}\big[u(x) + \varepsilon u_1(x, \frac{x}{\varepsilon})\big]\, d\mu_\varepsilon \\ &\qquad + \int_\Omega A(\frac{x}{\varepsilon}) \Big\{ \nabla_{\mu_\varepsilon}\big[u(x) + \varepsilon u_1(x, \frac{x}{\varepsilon})\big]\Big\}^2 d\mu_\varepsilon \Big\} \\ &= \int_\Omega f(x) u(x)\, dx - \int_{\Omega\times Y} A(y) \Big\{ \nabla u(x) + \nabla_{\mu,y} u_1(x,y)\Big\}^2\, dm\ , \end{aligned}$$

where we used the notation $Av^2 := Av \cdot v$. We now observe that, in view of the equality $u_1(x,y) = \sum_i \frac{\partial u}{\partial x_i}(x)\,\chi_i(y)$, it turns out that u_1, $\nabla_x u_1$, and $\nabla_{\mu,y} u_1$ are all 'admissible' in the two-scale convergence (*cf.* Remark 2.4). Hence we are allowed to take $(\varphi, \varphi_1) = (u, u_1)$ as test functions in (6.16): so the last term in the above chain of equalities is zero, and (6.15) follows straightforward using the coercivity of A. ■

Example 6.6. We conclude this section with an example of explicit computation of j^{hom} when j is a quadratic-like energy density of the kind

$$j(z) = \frac{1}{2}\big(a_{11} z_1^2 + 2a_{12} z_1 z_2 + a_{22} z_2^2\big)\ ; \tag{6.18}$$

we suppose the coefficients a_{ij} constant and satisfying the non-degenerate ellipticity condition of Proposition 6.5 above (which in particular applies).

Let us consider the one-dimensional gridwork G in $\mathbb{R}^2$ whose unit cell is represented in Figure 4.2b; the corresponding measure μ will be given by $\mu = \frac{\mathcal{H}^1 \llcorner G}{\mathcal{H}^1(G)}$. To the aim of computing j^{hom}, being j given by (6.18), let us first compute j_μ. For any y belonging to the horizontal bars of G, it is $T_\mu(y)^\perp = \{(0,\xi_2)\ :\ \xi_2 \in \mathbb{R}\}$, so that we have by definition

$$\begin{aligned} j_\mu(y,z) = j_\mu(y,z_\mu) &= \inf_{\xi_2\in\mathbb{R}} j\big(y,(z_1,0)+(0,\xi_2)\big) = \inf_{\xi_2\in\mathbb{R}} \frac{1}{2}\big(a_{11}z_1^2 + 2a_{12}z_1\xi_2 + a_{22}\xi_2^2\big) \\ &= \frac{1}{2}\Big(a_{11} - \frac{a_{12}^2}{a_{22}}\Big) z_1^2 =: \frac{1}{2} q_1 z_1^2\ . \end{aligned}$$

Similarly, when y belongs to the vertical bars of G, we obtain

$$j_\mu(y,z) = \frac{1}{2}\Big(a_{22} - \frac{a_{12}^2}{a_{11}}\Big) z_2^2 =: \frac{1}{2} q_2 z_2^2\ .$$

Note that, by the uniform coercivity of the matrix (a_{ij}), the coefficients $q_1 = \frac{\det A}{a_{2,2}}$ and $q_2 = \frac{\det A}{a_{1,1}}$ are both strictly positive. By Lemma 5.4 (iii), taking for simplicity $Y = [-\frac{1}{2},\frac{1}{2}[^2$, j^{hom} is then given by

$$j^{\mathrm{hom}}(z) = \min\Big\{\frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} q_1\big(z_1+\nabla_1 u\,(y_1,0)\big)^2\,dy_1 + \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} q_2\big(z_2+\nabla_2 u\,(0,y_2)\big)^2\,dy_2\ :\ u\in H^{1,p}_\mu(\mathbf{T})\Big\}\ .$$

Taking into account that any function $u \in H^{1,p}_\mu(\mathbf{T})$ may be represented as $u(y_1,0) = \xi(y_1)$, and $u(0,y_2) = \eta(y_2)$, where ξ and η are periodic functions in $W^{1,p}([-\frac{1}{2},\frac{1}{2}[)$, with $\xi(0) = \eta(0)$, the right hand side of the above equality may be rewritten as

$$\frac{1}{4}\min\Big\{\int_{-\frac{1}{2}}^{\frac{1}{2}} q_1\big(z_1+\xi'(t)\big)^2\,dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} q_2\big(z_2+\eta'(t)\big)^2\,dt\ :\ \xi,\,\eta\in W^{1,p}_{\mathrm{per}}([-\frac{1}{2},\frac{1}{2}[),\ \xi(0)=\eta(0)\Big\} =$$

$$\frac{1}{4}\min\Big\{\int_{-\frac{1}{2}}^{\frac{1}{2}} q_1 z_1^2 + q_1[\xi'(t)]^2\,dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} q_2 z_2^2 + q_2[\eta'(t)]^2\,dt\ :\ \xi,\,\eta\in W^{1,p}_{\mathrm{per}}([-\frac{1}{2},\frac{1}{2}[),\ \xi(0)=\eta(0)\Big\}\ .$$

Clearly the previous minimum is achieved at $\xi = \eta \equiv 0$, thus we conclude that

$$j^{\mathrm{hom}}(z) = \frac{1}{4}\left\{q_1 z_1^2 + q_2 z_2^2\right\} .$$

We recover by this way the same homogenized energy density found in [31]. Analogous calculations may be performed in the case of networks containing also oblique bars, as for instance those whose unit-cell is described by Figures 4.2a and 4.2c.

7. Final remarks on non-connectedness and two-parameter integrals

Our framework of homogenization with periodic measures includes in a natural way the case of thin reinforced structures: they are made of identical cells periodically repeated, in which the material is concentrated along bars of thickness δ, small with respect to the period ε. Thus, it seems natural to apply Theorem 6.2 taking as a measure μ the weak limit, as $\delta \to 0$, of the measures μ_δ which describe the periodic structure of thickness δ.
Another possible way, is to make the homogenization with respect to each μ_δ, and then let δ tend to zero; the explicit computations for this last passage to the limit can be found, for some simple bidimensional or tridimensional structures, possibly containing oblique bars, in [31, 29]. For a quite recent and complete survey about homogenization of reticulated structures, see also [32]. In particular, it has been pointed out in [32, sections III.4 and V.5], that the order chosen to pass to the limit in different parameters is very important. Here we would emphasize the possible consequences of such choice when the measure μ presents properties of non-connectedness. If the homogenized energy was independent of the order of the two limits, the functional obtained as a contemporary limit, as $\varepsilon \to 0$ and $\delta = \delta(\varepsilon) \to 0$ of

$$J_{\delta,\varepsilon}(u) = \int_\Omega j(\ \clubsuit\ \nabla u\ \spadesuit\ )\, d(\mu_\delta)_\varepsilon$$

should not depend on the velocity of convergence to zero of $\delta(\varepsilon)$.
But consider the sequence μ_δ of Y-periodic measures given on Y by the Lebesgue measure with a density a_δ equal respectively to 1 on $Y \setminus F_\delta$ and to $\frac{k}{\delta^2}$ on F_δ, being F_δ a system of cylindrical fibers of very small radius δ, parallel to the three coordinated axes (see Figure 7.1a).

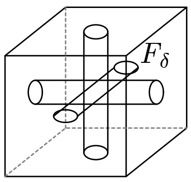

Figure 7.1.a

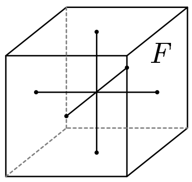

Figure 7.1b

Then $\mu_\delta \rightharpoonup \mu$, where $\mu = \mu' + \mu''$, being the restrictions of μ' and μ'' on Y given respectively by the Lebesgue measure $\mathcal{L}^3 \llcorner Y$ and by $k\pi$ times the one-dimensional measure $\mathcal{H}^1$ over the skeleton F of the fibers (see Figure 7.1b). Notice that μ turns out to be weakly p-connected on $\mathbf{T}$ if and only if $p > 2$; hence, for $p \le 2$, Theorem 5.1 does not apply, and the two-scale limit of a sequence $\{u_\varepsilon\}$ uniformly bounded in $L^p_{\mu_\varepsilon}(\Omega)$ takes *a priori* two different values on the two 'connected components' of μ, *i.e.*

$$u_0(x,y) = \begin{cases} u(x) & \text{if } y \in Y \setminus F \quad , \\ v(x) & \text{if } y \in F \ . \end{cases}$$

This kind of behaviour suggests to consider the measures μ'_δ and μ''_δ given on Y respectively by $\mu_\delta \llcorner (Y \setminus F_\delta)$ and $\mu_\delta \llcorner F_\delta$, and to split the energies $J_{\delta,\varepsilon}$ into

$$J_{\delta,\varepsilon}(u) = \int_\Omega j\big(\frac{x}{\varepsilon}, \nabla u\big)\, d(\mu'_\delta)_\varepsilon + \int_\Omega j\big(\frac{x}{\varepsilon}, \nabla u\big)\, d(\mu''_\delta)_\varepsilon \ . \tag{7.1}$$

Since μ'_δ and μ''_δ weakly converge respectively to μ' and μ'', Theorem 6.2 can now be applied to each of the sequences at the right hand side of (7.1). By this way, with the choice $j(y,z) = \frac{1}{p}|z|^p$, we easily get

$$\Gamma - \lim_{\varepsilon \to 0} J_{\delta(\varepsilon),\varepsilon}(u) \ge \frac{1}{p}\Big\{ \int_\Omega |\nabla u|^p \, dx + k\pi \int_\Omega \Big[\Big|\frac{\partial u}{\partial x_1}\Big|^p + \Big|\frac{\partial u}{\partial x_2}\Big|^p + \Big|\frac{\partial u}{\partial x_3}\Big|^p \Big]\, dx \Big\} \ .$$

Actually, such a lower bound is not optimal, as the right limit energy is given by [14]

$$J(u) = \inf \Big\{ \Phi(u,v) \ : \ v \in W^{1,p}_0(\Omega) \Big\} \ , \qquad u \in W^{1,p}_0(\Omega) \ ,$$

where

$$\Phi(u,v) = \frac{1}{p}\Big\{ \int_\Omega |\nabla u|^p \, dx + k\pi \int_\Omega \Big[\Big|\frac{\partial v}{\partial x_1}\Big|^p + \Big|\frac{\partial v}{\partial x_2}\Big|^p + \Big|\frac{\partial v}{\partial x_3}\Big|^p \Big]\, dx + 6\pi\gamma \int_\Omega |u-v|^p \, dx \Big\} \ .$$

The positive constant γ is given by

$$\gamma = \begin{cases} \lim\limits_{\varepsilon \to 0} \ \varepsilon^{-2} \big| \log\big(\varepsilon\delta(\varepsilon)\big)\big|^{-1} & \text{if } p = 2 \ , \\ \lim\limits_{\varepsilon \to 0} \ \Big|\frac{2-p}{p-1}\Big|^{p-1} \varepsilon^{-p} \delta(\varepsilon)^{2-p} & \text{if } p \ne 2 \ . \end{cases}$$

Thus, $\gamma = +\infty$ if $p > 2$, while γ is finite if $p \le 2$; in particular, for $p \le 2$, a non-local term appears in the limit energy, which expresses the interaction between the two 'connected components' of μ and do depends on the way $\delta(\varepsilon)$ tends to zero.

We address to another forthcoming paper for a systematic treatment, in our framework of measures, of non-local phenomenons due to the lack of connectedness.

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