# EXISTENCE AND UNIQUENESS THEOREMS FOR SOME SEMI-LINEAR EQUATIONS ON LOCALLY FINITE GRAPHS

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ABSTRACT. We study some semi-linear equations for the $(m, p)$-Laplacian operator on locally finite weighted graphs. We prove existence of weak solutions for all $m \in \mathbb{N}$ and $p \in (1, +\infty)$ via a variational method already known in the literature by exploiting the continuity properties of the energy functionals involved. When $m = 1$, we also establish a uniqueness result in the spirit of the Brezis–Strauss Theorem. We finally provide some applications of our main results by dealing with some Yamabe-type and Kazdan–Warner-type equations on locally finite weighted graphs.

## 1. INTRODUCTION

1.1. **Framework.** When dealing with PDEs coming from the Euler–Lagrange equations of some energy functional, existence and multiplicity results of weak solutions are usually achieved via the so-called Variational Method.

In the recent years, this approach has been employed by many authors in order to deal with a large variety of interesting PDEs on graphs, see [6–9, 11–24, 26–30] and the references therein. Of particular interest for the scopes of the present paper is the work [20], where the authors proved existence of weak solutions for a Yamabe-type equation on locally finite weighted graphs via the celebrated Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1].

The main aim of this note is twofold. On one hand, by exploiting some ideas developed in [10, 25] in the context of Carnot groups we prove the existence of weak solutions for a Yamabe-type equation on locally finite weighted graphs. Our result is similar to the one of [20] but holds under a different set of assumptions. On the other hand, we adapt the strategy of [4] developed in the Euclidean setting to establish a uniqueness result for the weak solutions of Yamabe-type equations on locally finite weighted graphs in the spirit of the celebrated Brezis–Strauss Theorem [5].

1.2. **Main notation.** Before stating our main results, we need to recall some notation, see Section 2 for the precise definitions.

Given $G = (V, E)$ a locally finite non-oriented graph, the *vertex boundary* $\partial\Omega$ and the *vertex interior* $\Omega^\circ$ of a connected subgraph $\Omega \subset V$ are defined as

$$\partial\Omega = \{x \in \Omega : \exists y \notin \Omega \text{ such that } xy \in E\}, \qquad \Omega^\circ = \Omega \setminus \partial\Omega.$$

We say that $\Omega$ is *bounded* if it is a bounded subset of $V$ with respect to the usual *vertex distance* $\mathsf{d}\colon V \times V \to [0, +\infty)$.

Once a symmetric *weight function* $w\colon V \times V \to [0, \infty)$ is given, we can define the *Laplacian* of a function $u\colon V \to \mathbb{R}$ as

$$\Delta u(x) = \frac{1}{\mathfrak{m}(x)} \sum_{y \in V} w_{xy}(u(y) - u(x)) \quad \text{for } x \in V, \tag{1.1}$$

where $\mathfrak{m}\colon V \to [0, +\infty)$ is the measure function

$$\mathfrak{m}(x) = \sum_{y \in V} w_{xy} \quad \text{for all } x \in V. \tag{1.2}$$

The *gradient form* associated to the Laplacian operator is the bilinear symmetric form

$$\Gamma(u, v)(x) = \frac{1}{2\mathfrak{m}(x)} \sum_{y \in V} w_{xy}(u(y) - u(x))(v(y) - v(x)), \quad x \in V,$$

defined for any couple of functions $u, v\colon V \to \mathbb{R}$. As a consequence, the *slope* of the function $u\colon V \to \mathbb{R}$ is given by

$$|\nabla u|(x) = \sqrt{\Gamma(u, u)(x)} = \left( \frac{1}{2\mathfrak{m}(x)} \sum_{y \in V} w_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}} \quad \text{for } x \in V.$$

Note that $|\Gamma(u, v)| \leq |\nabla u| |\nabla v|$ for any couple of functions $u, v\colon V \to \mathbb{R}$. In analogy with the Euclidean framework, for any $m \in \mathbb{N}$ we recursively define the *m-slope* of the function $u$ as

$$|\nabla^m u| = \begin{cases} |\nabla(\Delta^{\frac{m-1}{2}} u)| & \text{if } m \text{ is odd}, \\ |\Delta^{\frac{m}{2}} u| & \text{if } m \text{ is even}, \end{cases}$$

where $|\Delta^{\frac{m}{2}} u|$ denotes the usual absolute value of the function $\Delta^{\frac{m}{2}} u$. The natural operator associated to the Sobolev spaces $(W_0^{m,p}(\Omega), \|\cdot\|_{W_0^{m,p}(\Omega)})$ (see (2.1) below for the precise definition) is the $(m, p)$-*Laplacian operator*

$$\mathcal{L}_{m,p}\colon W_0^{m,p}(\Omega) \to L^p(\Omega)$$

defined in the distributional sense for all $u \in W_0^{m,p}(\Omega)$ as

$$\int_\Omega \mathcal{L}_{m,p} u \, \varphi \, \mathrm{d}\mathfrak{m} = \begin{cases} \displaystyle\int_\Omega |\nabla^m u|^{p-2} \, \Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \varphi) \, \mathrm{d}\mathfrak{m} & \text{if } m \text{ is odd}, \\ \displaystyle\int_\Omega |\nabla^m u|^{p-2} \, \Delta^{\frac{m}{2}} u \, \Delta^{\frac{m}{2}} \varphi \, \mathrm{d}\mathfrak{m} & \text{if } m \text{ is even}, \end{cases} \tag{1.3}$$

whenever $\varphi \in W_0^{m,p}(\Omega)$.

The $(m, p)$-Laplacian $\mathcal{L}_{m,p} u$ can be explicitly computed at any point of $\Omega$. In particular, $\mathcal{L}_{1,p}$ is the *p-Laplacian operator*, given by

$$\Delta_p u(x) = \frac{1}{\mathfrak{m}(x)} \sum_{y \in \Omega} \left( |\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x) \right) w_{xy}(u(y) - u(x)), \quad x \in \Omega, \tag{1.4}$$

for all $u \in W_0^{1,p}(\Omega)$. When $p = 2$, we recover the usual Laplacian operator defined in (1.1).

1.3. **Main results.** We are now ready to state our main results. Our first main theorem is the following existence result for a Yamabe-type equation for the $(m, p)$-Laplacian operator on locally finite weighted graphs.

**Theorem 1.1.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $m \in \mathbb{N}$, $p \in (1, +\infty)$ and $q \in [p - 1, +\infty)$. Let $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

$$|f(x, t)| \leq a(x) + b(x) |t|^q \quad \text{for every } (x, t) \in \Omega \times \mathbb{R} \tag{1.5}$$

*for some non-negative $a, b \in L^1(\Omega)$ with $\|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)} > 0$. There exists*

$$\Lambda = \Lambda(m, p, q, \|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)}) > 0 \tag{1.6}$$

*such that the Yamabe-type problem*

$$\begin{cases} \mathcal{L}_{m,p} u = \lambda f(x, u) & \text{in } \Omega^\circ \\ |\nabla^j u| = 0 & \text{on } \partial\Omega, \quad 0 \leq j \leq m - 1, \end{cases} \tag{1.7}$$

*admits at least one non-trivial solution $u_\lambda \in W_0^{m,p}(\Omega)$ for every $0 < \lambda < \Lambda$.*

We observe that the growth condition of the function $f$ assumed in (1.5) of Theorem 1.1 is different from the one assumed in [20, Theorem 3]. In particular, we do not assume that $f(x, 0) = 0$ for all $x \in \Omega$. We also underline that the existence threshold (1.6) depends uniquely on the growth of the function $f$ and not on the first eigenvalue of the $(m, p)$-Laplacian, as instead it happens in [20, Theorem 3].

Our second main result is the following uniqueness theorem for a Yamabe-type equation for the $p$-Laplacian operator on locally finite weighted graphs in the spirit of the famous Brezis–Strauss Theorem, see [4, 5].

**Theorem 1.2.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $p \in [1, +\infty)$ and let $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(x, 0) = 0$ and $t \mapsto g(x, t)$ is non-decreasing for all $x \in \Omega$. If $f_1, f_2 \in L^1(\Omega)$, $h \in L^1(\partial\Omega)$ and $u_1, u_2 \in W^{1,p}(\Omega)$ solve the problems*

$$\begin{cases} -\Delta_p u_i + g(x, u_i) = f_i & \text{in } \Omega^\circ \\ u_i = h & \text{on } \partial\Omega \end{cases} \quad \text{for } i = 1, 2,$$

*then*

$$\int_\Omega |g(x, u_1) - g(x, u_2)| \, d\mathfrak{m} \leq \int_\Omega |f_1 - f_2| \, d\mathfrak{m}. \tag{1.8}$$

*As a consequence, for every $f \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$ the problem*

$$\begin{cases} -\Delta_p u + g(x, u) = f & \text{in } \Omega^\circ \\ u = h & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

*admits at most one solution $u \in W^{1,p}(\Omega)$.*

By combining Theorem 1.1 and Theorem 1.2, we get the following well-posedness result for a Yamabe-type problem for the $p$-Laplacian on locally finite weighted graphs.

**Proposition 1.3.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $p \in (1, +\infty)$, $q \in [p-1, +\infty)$ and $a, b \in L^1(\Omega)$ with $\inf_\Omega b \geq 0$. There exists*

$$\Lambda = \Lambda(m, p, q, \|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)}) > 0$$

*such that the Yamabe-type problem*

$$\begin{cases} -\Delta_p u + b|u|^{q-1}u = a & in \ \Omega^\circ \\ u = 0 & on \ \partial\Omega \end{cases} \tag{1.10}$$

*has a unique solution $u \in W_0^{1,p}(\Omega)$.*

1.4. **Organization of the paper.** The structure of the paper is the following. In Section 2 we recall the preliminary definitions and notions needed in the paper. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2 respectively. Finally, in Section 5 we provide some applications of our main results, along with the proof of Proposition 1.3.

## 2. Preliminaries

In this section, we introduce the main notation and some preliminary results we will need in the sequel of the paper.

2.1. **Non-oriented graphs.** Let $V$ be a non-empty set and let $E \subset V \times V$. We write

$$x \sim y \iff xy = (x, y) \in E.$$

We will always assume that

$$xy \in E \iff yx \in E.$$

We say that the couple $G = (V, E)$ is a *non-oriented graph* with *vertices $V$* and *edges $E$*.

The non-oriented graph $G$ is *locally finite* if

$$\#\{y \in V : xy \in E\} < +\infty \quad \text{for all } x \in V,$$

that is, each vertex in $V$ belongs to a finite number of edges in $E$.

Given $n \in \mathbb{N}$, a *path* on $G$ is any finite sequence of vertices $\{x_k\}_{k=1,\dots,n} \subset V$ such that

$$x_k x_{k+1} \in E \quad \text{for all } k = 1, \dots, n-1.$$

The *length* of a path on $G$ is the number of edges in the path. We say that $G$ is *connected* if, for any two vertices $x, y \in V$, there is a path connecting $x$ and $y$. If $G$ is connected, then the function $\mathsf{d} \colon V \times V \to [0, +\infty)$ given by

$$\mathsf{d}(x, y) = \min\{n \in \mathbb{N}_0 : x \text{ and } y \text{ can be connected by a path of length } n\},$$

for $x, y \in V$, is a distance on $V$. As a consequence, any connected locally finite non-oriented graph has at most countable many vertices.

Let $G = (V, E)$ be a locally finite non-oriented graph. A *weight* on $G$ is a function $w \colon V \times V \to [0, +\infty)$, $w(x, y) = w_{xy}$ for $x, y \in V$, such that

$$w_{xy} = w_{yx} \quad \text{and} \quad w_{xy} > 0 \iff xy \in E$$

for all $x, y \in V$. We conclude this section by pointing out that the function $\mathfrak{m} \colon V \to [0, +\infty)$ defined in (1.2) can be interpreted as a measure on the graph by simply setting

$$\int_V u \, d\mathfrak{m} = \int_V u(x) \, d\mathfrak{m}(x) = \sum_{y \in V} u(x) \, \mathfrak{m}(x) \in [0, +\infty]$$

for any function $u \colon V \to [0, +\infty)$.

## 2.2. Sobolev spaces on bounded domains.

Let $G = (V, E)$ be a weighted locally finite graph and let $\Omega \subset V$ be a bounded domain. Note that the integral

$$\int_\Omega u \, d\mathfrak{m} = \int_\Omega u(x) \, d\mathfrak{m}(x) = \sum_{x \in \Omega} u(x) \, \mathfrak{m}(x)$$

of a function $u \colon \Omega \to \mathbb{R}$ is well defined, since $\Omega$ is a finite set. Let $p \in [1, +\infty]$ and $m \in \mathbb{N}_0$. The *Sobolev space* $W^{m,p}(\Omega)$ is the set of all functions $u \colon \Omega \to \mathbb{R}$ such that

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{k=0}^{m} \|\nabla^k u\|_{L^p(\Omega)} < +\infty. \tag{2.1}$$

When $m = 0$, this space is simply the *Lebesgue space* $L^p(\Omega)$. Since $\Omega$ is a finite set, the Banach space $(W^{m,p}(\Omega), \| \cdot \|_{W^{m,p}(\Omega)})$ is finite dimensional and, actually, coincides with the set of all real-valued functions on $\Omega$.

For $m \in \mathbb{N}$, we define

$$C_0^m(\Omega) := \left\{ u \colon \Omega \to \mathbb{R} : |\nabla^k u| = 0 \text{ on } \partial\Omega \text{ for all } 0 \le k \le m - 1 \right\} \tag{2.2}$$

and we let $W_0^{m,p}(\Omega)$ be the completion of $C_0^m(\Omega)$ with respect to the Sobolev norm (2.1). The following result is proved in [20, Theorem 7].

**Theorem 2.1** (Sobolev embedding). *Let $G = (V, E)$ be a locally finite graph and let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $m \in \mathbb{N}$ and $p \in [1, +\infty)$. The space $W_0^{m,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $q \in [1, +\infty]$, i.e. there exists a constant $C_{m,p} > 0$, depending only on $m$, $p$ and $\Omega$, such that*

$$\|u\|_{L^q(\Omega)} \le C_{m,p} \|\nabla^m u\|_{L^p(\Omega)} \tag{2.3}$$

*for all $q \in [1, +\infty]$ and $u \in W_0^{m,p}(\Omega)$.*

By Theorem 2.1, the space $(W_0^{m,p}(\Omega), \| \cdot \|_{W_0^{m,p}(\Omega)})$ is a finite dimensional Banach space, where

$$\|u\|_{W_0^{m,p}(\Omega)} = \|\nabla^m u\|_{L^p(\Omega)} \tag{2.4}$$

is a norm on $W_0^{m,p}(\Omega)$ equivalent to the norm (2.1). Since $\Omega$ is a finite set, the Banach space $(W_0^{m,p}(\Omega), \| \cdot \|_{W_0^{m,p}(\Omega)})$ is finite dimensional and, actually, coincides with the set $C_0^m(\Omega)$ defined in (2.2).

## 3. Proof of Theorem 1.1

In this section, we prove our first main result following the strategy outlined in [10]. Given $\lambda > 0$, we define

$$\Phi(u) = \|u\|_{W_0^{m,p}(\Omega)}, \qquad \Psi_\lambda(u) = \lambda \int_\Omega F(x, u) \, d\mathfrak{m}, \tag{3.1}$$

for all $u \in W_0^{m,p}(\Omega)$, where

$$F(x,t) = \int_0^t f(x,\tau)\,\mathrm{d}\tau \quad \text{for all } t \in \mathbb{R}. \tag{3.2}$$

Note that, thanks to the assumption in (1.5), the functional $\Psi_\lambda$ is well defined and (strongly) continuous on $W_0^{m,p}(\Omega)$. Indeed, we can estimate

$$|F(x,t)| \leq \int_0^{|t|} |f(x,\tau)|\,\mathrm{d}\tau \leq \int_0^{|t|} a(x) + b(x)\,|\tau|^q\,\mathrm{d}\tau = a(x)\,|t| + b(x)\,\frac{|t|^{1+q}}{1+q}$$

for all $(x,t) \in \Omega \times \mathbb{R}$, so that

$$|\Psi_\lambda(u)| \leq \lambda \left( \|a\|_{L^1(\Omega)} \|u\|_{L^\infty(\Omega)} + \|b\|_{L^1(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}^{1+q}}{1+q} \right)$$

which is finite for all $u \in W_0^{m,p}(\Omega)$ by Theorem 2.1. In addition, if $(u_n)_{n\in\mathbb{N}} \subset W_0^{m,p}(\Omega)$ is converging to some $u \in W_0^{m,p}(\Omega)$, then $u_n \to u$ in $L^\infty(\Omega)$ as $n \to +\infty$ and thus

$$\lim_{n\to+\infty} \Psi_\lambda(u_n) = \lim_{n\to+\infty} \sum_{x\in\Omega} F(x, u_n(x))\,\mathfrak{m}(x) = \sum_{x\in\Omega} F(x, u(x))\,\mathfrak{m}(x) = \Psi_\lambda(u)$$

by the continuity of the function $t \mapsto F(x,t)$ for $x \in \Omega$ fixed.

The following two results are proved in [10, Lemma 3.2 and Lemma 3.3] respectively for the case $p = 2$. Here we reproduce the proofs in our setting in the more general case $p \in (1, +\infty)$ for the reader's ease.

**Lemma 3.1.** *Let $p \in (1, +\infty)$ and $\lambda > 0$. If*

$$\limsup_{\varepsilon\to0+} \frac{\displaystyle\sup_{u\in\Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u\in\Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon} < \varrho^{p-1} \tag{3.3}$$

*for some $\varrho > 0$, then*

$$\inf_{\sigma<\varrho} \frac{\displaystyle\sup_{u\in\Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u\in\Phi^{-1}([0,\sigma])} \Psi_\lambda(u)}{\varrho^p - \sigma^p} < \frac{1}{p}. \tag{3.4}$$

*Proof.* Let $\varepsilon \in (0, \varrho)$ and note that

$$\lim_{\varepsilon\to0+} \frac{\varepsilon}{\varrho^p - (\varrho-\varepsilon)^p} = \frac{1}{p\varrho^{p-1}}.$$

Therefore, in virtue of (3.3), we get that

$$\limsup_{\varepsilon\to0+} \frac{\displaystyle\sup_{u\in\Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u\in\Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varrho^p - (\varrho-\varepsilon)^p}$$

$$= \limsup_{\varepsilon\to0+} \frac{\displaystyle\sup_{u\in\Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u\in\Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon} \cdot \frac{\varepsilon}{\varrho^p - (\varrho-\varepsilon)^p}$$

$$= \frac{1}{p\varrho^{p-1}} \limsup_{\varepsilon\to0+} \frac{\displaystyle\sup_{u\in\Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u\in\Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon} < \frac{1}{p}.$$

Thus we can find $\bar{\varepsilon} \in (0, \varrho)$ such that

$$\frac{\displaystyle\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho - \bar{\varepsilon}])} \Psi_\lambda(u)}{\varrho^p - (\varrho - \bar{\varepsilon})^p} < \frac{1}{p}$$

and so $\bar{\sigma} = \varrho - \bar{\varepsilon} < \varrho$ gives

$$\inf_{\sigma < \varrho} \frac{\displaystyle\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\sigma])} \Psi_\lambda(u)}{\varrho^p - \sigma^p} < \frac{\displaystyle\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\bar{\sigma}])} \Psi_\lambda(u)}{\varrho^p - \bar{\sigma}^p} < \frac{1}{p}$$

proving (3.4). The proof is complete. $\qquad\square$

**Lemma 3.2.** *Let $p \in (1, +\infty)$ and $\lambda > 0$. If (3.4) holds for some $\varrho > 0$, then*

$$\inf_{u \in \Phi^{-1}([0,\varrho))} \frac{\displaystyle\sup_{v \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(v) - \Psi_\lambda(u)}{\varrho^p - \|u\|^p_{W_0^{m,p}(\Omega)}} < \frac{1}{p}. \tag{3.5}$$

*Proof.* In virtue of (3.4), we can find $\bar{\sigma} \in (0, \varrho)$ such that

$$\sup_{u \in \Phi^{-1}([0,\bar{\sigma}])} \Psi_\lambda(u) > \sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \frac{1}{p}(\varrho^p - \bar{\sigma}^p).$$

Since the functional $\Psi_\lambda$ is continuous on $W_0^{m,p}(\Omega)$, we can find $\bar{u} \in W_0^{m,p}(\Omega)$ with $\|\bar{u}\|_{W_0^{m,p}(\Omega)} = \bar{\sigma}$ such that

$$\sup_{u \in \Phi^{-1}([0,\bar{\sigma}])} \Psi_\lambda(u) = \sup_{\|u\|_{W_0^{m,p}(\Omega)} = \bar{\sigma}} \Psi_\lambda(u) = \Psi_\lambda(\bar{u})$$

and so

$$\Psi_\lambda(\bar{u}) > \sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \frac{1}{p}(\varrho^p - \bar{\sigma}^p).$$

We thus conclude that

$$\inf_{u \in \Phi^{-1}([0,\varrho))} \frac{\displaystyle\sup_{v \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(v) - \Psi_\lambda(u)}{\varrho^p - \|u\|^p_{W_0^{m,p}(\Omega)}} < \frac{\displaystyle\sup_{v \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(v) - \Psi_\lambda(\bar{u})}{\varrho^p - \|\bar{u}\|^p_{W_0^{m,p}(\Omega)}} < \frac{1}{p}$$

proving (3.5). The proof is complete. $\qquad\square$

We are now ready to prove our first main result, in analogy with [10, Theorem 3.1].

*Proof of Theorem 1.1.* Let $\lambda > 0$ and consider the energy functional $\mathcal{E}_\lambda \colon W_0^{m,p}(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{E}_\lambda(u) = \frac{\Phi(u)^p}{p} - \Psi_\lambda(u) \quad \text{for all } u \in W_0^{m,p}(\Omega),$$

where $\Phi$ and $\Psi_\lambda$ are as in (3.1). By the growth condition (1.5) and Theorem 2.1, we have that $\mathcal{E}_\lambda \in C^1(W_0^{m,p}(\Omega); \mathbb{R})$, with derivative at $u \in W_0^{m,p}(\Omega)$ given by

$$\mathcal{E}'_\lambda(u)[\varphi] = \begin{cases} \displaystyle\int_\Omega |\nabla^m u|^{p-2}\, \Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \varphi)\, d\mathfrak{m} - \lambda \int_\Omega f(x, u)\, \varphi\, d\mathfrak{m} & \text{if } m \text{ is odd}, \\[4mm] \displaystyle\int_\Omega |\nabla^m u|^{p-2}\, \Delta^{\frac{m}{2}} u\, \Delta^{\frac{m}{2}} \varphi\, d\mathfrak{m} - \lambda \int_\Omega f(x, u)\, \varphi\, d\mathfrak{m} & \text{if } m \text{ is even}, \end{cases}$$

for any $\varphi \in W_0^{m,p}(\Omega)$. In particular, the solutions of the problem (1.7) are exactly the critical points of the functional $\mathcal{E}_\lambda$. Now let $\varrho > 0$ to be fixed later. Since $\mathcal{E}_\lambda$ is a continuous functional on $W_0^{m,p}(\Omega)$, there exists $u_{\lambda,\varrho} \in \Phi^{-1}([0,\varrho])$ such that

$$\mathcal{E}_\lambda(u_{\lambda,\varrho}) = \inf_{u \in \Phi^{-1}([0,\varrho])} \mathcal{E}_\lambda(u). \tag{3.6}$$

To conclude the proof, we just need to show that $\|u_{\lambda,\varrho}\|_{W_0^{m,p}(\Omega)} < \varrho$. To this aim, for $\varepsilon \in (0, \varrho)$ we consider

$$\Lambda(\varrho, \varepsilon) = \frac{\displaystyle\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon}.$$

Recalling the definition of $\Psi_\lambda$ in (3.1), we have

$$\Lambda(\varrho, \varepsilon) = \frac{1}{\varepsilon} \left( \sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u) \right)$$

$$\leq \frac{\lambda}{\varepsilon} \sup_{u \in \Phi^{-1}([0,1])} \int_\Omega \left| \int_{(\varrho-\varepsilon)u(x)}^{\varrho u(x)} |f(x,t)| \, \mathrm{d}t \right| \mathrm{d}\mathfrak{m}(x).$$

Thanks to the growth condition (1.5), we can estimate

$$\frac{\lambda}{\varepsilon} \int_\Omega \left| \int_{(\varrho-\varepsilon)u(x)}^{\varrho u(x)} |f(x,t)| \, \mathrm{d}t \right| \mathrm{d}\mathfrak{m}(x)$$

$$\leq \frac{\lambda}{\varepsilon} \int_\Omega \varepsilon \, a(x)|u(x)| + b(x) \left( \frac{\varrho^{q+1} - (\varrho-\varepsilon)^{q+1}}{q+1} \right) |u(x)|^{q+1} \, \mathrm{d}\mathfrak{m}(x)$$

$$\leq \lambda \|u\|_{L^\infty(\Omega)} \|a\|_{L^1(\Omega)} + \frac{\lambda \|u\|_{L^\infty(\Omega)}^{q+1} \|b\|_{L^1(\Omega)}}{q+1} \left( \frac{\varrho^{q+1} - (\varrho-\varepsilon)^{q+1}}{\varepsilon} \right)$$

for all $u \in W_0^{m,p}(\Omega)$. Thus, by the embedding inequality (2.3), we get

$$\Lambda(\varrho, \varepsilon) \leq \lambda \, C_{m,p} \|a\|_{L^1(\Omega)} + \frac{\lambda \, C_{m,p}^{q+1} \|b\|_{L^1(\Omega)}}{q+1} \left( \frac{\varrho^{q+1} - (\varrho-\varepsilon)^{q+1}}{\varepsilon} \right)$$

and so

$$\limsup_{\varepsilon \to 0+} \Lambda(\varrho, \varepsilon) \leq \lambda \left( C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q \right).$$

We now define

$$\lambda_\varrho = \frac{\varrho^{p-1}}{C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q} \in (0, +\infty) \tag{3.7}$$

and, consequently,

$$\Lambda = \sup_{\varrho > 0} \lambda_\varrho \in (0, +\infty)$$

(note that $\Lambda < +\infty$ is ensured by the fact that $q \geq p-1$). Now fix $\lambda < \Lambda$ and choose the parameter $\varrho > 0$ in such a way that $\lambda < \lambda_\varrho < \Lambda$. This choice implies that

$$\limsup_{\varepsilon \to 0+} \Lambda(\varrho, \varepsilon) \leq \lambda \left( C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q \right)$$

$$< \lambda_\varrho \left( C_{m,p} \|a\|_{L^1(\Omega)} + C_{m,p}^{q+1} \|b\|_{L^1(\Omega)} \varrho^q \right) < \varrho^{p-1},$$

so that

$$\limsup_{\varepsilon \to 0+} \frac{\displaystyle\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\varrho-\varepsilon])} \Psi_\lambda(u)}{\varepsilon} < \varrho^{p-1}.$$

We can now apply Lemma 3.1 to get that

$$\inf_{\sigma < \varrho} \frac{\displaystyle\sup_{u \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(u) - \sup_{u \in \Phi^{-1}([0,\sigma])} \Psi_\lambda(u)}{\varrho^p - \sigma^p} < \frac{1}{p}$$

and so, by Lemma 3.2, we infer that

$$\inf_{u \in \Phi^{-1}([0,\varrho))} \frac{\displaystyle\sup_{v \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(v) - \Psi_\lambda(u)}{\varrho^p - \|u\|_{W_0^{m,p}(\Omega)}^p} < \frac{1}{p}.$$

The above inequality implies that there exists $w_{\lambda,\varrho} \in \Phi^{-1}([0,\varrho))$ such that

$$\sup_{v \in \Phi^{-1}([0,\varrho])} \Psi_\lambda(v) < \Psi_\lambda(w_{\lambda,\varrho}) + \frac{\varrho^p - \|w_{\lambda,\varrho}\|_{W_0^{m,p}(\Omega)}^p}{p}.$$

Now, if by contradiction we assume that $\|u_{\lambda,\varrho}\|_{W_0^{m,p}(\Omega)} = \varrho$, then the previous inequality implies that

$$\Psi_\lambda(u_{\lambda,\varrho}) < \Psi_\lambda(w_{\lambda,\varrho}) + \frac{\varrho^p - \|w_{\lambda,\varrho}\|_{W_0^{m,p}(\Omega)}^p}{p}$$

which is equivalent to $\mathcal{E}_\lambda(u_{\lambda,\varrho}) > \mathcal{E}_\lambda(w_{\lambda,\varrho})$, contradicting (3.6). The proof is complete. $\square$

**Remark 3.3** (The precise value of $\Lambda$ in Theorem 1.1)**.** Note that the above proof allows to give a precise value to the existence threshold $\Lambda > 0$ in Theorem 1.1. Indeed, one just need to find the maximal value of the function defined in (3.7), which is explicitly computable in term of $p$, $q$, $\|a\|_{L^1(\Omega)}$, $\|b\|_{L^1(\Omega)}$ and $C_{m,p}$. In particular, in the limiting case $q = p - 1$, one has

$$\Lambda = \lim_{\varrho \to +\infty} \frac{\varrho^{p-1}}{C_{m,p}\|a\|_{L^1(\Omega)} + C_{m,p}^p \|b\|_{L^1(\Omega)}\, \varrho^{p-1}} = \frac{1}{C_{m,p}^p \|b\|_{L^1(\Omega)}},$$

which does not depend on $\|a\|_{L^1(\Omega)}$.

## 4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. The overall strategy is to adapt the line developed in [4, Appendix B] for the Euclidean setting to the present framework. Note that [4] is focused on the case $p = 2$ only. Nonetheless, exploiting the explicit expression (1.4) of the $p$-Laplacian, we are able to extend the approach of [4] also to the case $p \neq 2$.

We begin with the following result, analogous to [4, Lemma B.1].

**Lemma 4.1.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $p \in [1, +\infty)$ and $f \in L^1(\Omega)$. If $u \in W_0^{1,p}(\Omega)$ is a solution of the problem*

$$\begin{cases} -\Delta_p u = f & in\ \Omega^\circ \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{4.1}$$

*then*

$$\int_\Omega f\, H(u)\, d\mathfrak{m} \geq 0$$

*for every non-decreasing locally Lipschitz function* $H\colon \mathbb{R} \to \mathbb{R}$ *such that* $H(0) = 0$.

*Proof.* We start by observing that $H(u) \in W_0^{1,p}(\Omega)$. Indeed, $H(v) \in C_0^0(\Omega)$ for all $v \in C_0^0(\Omega)$ with $|\nabla H(v)| \leq L|\nabla v|$ on $\Omega$, where $L = \mathrm{Lip}(H, [-c,c])$, $c = \|v\|_{L^\infty(\Omega)}$. Using $H(u)$ as a test function in (4.1), we get

$$\int_\Omega f\, H(u)\, d\mathfrak{m} = -\int_\Omega \Delta_p u\, H(u)\, d\mathfrak{m} = \int_\Omega |\nabla u|^{p-2}\, \Gamma(u, H(u))\, d\mathfrak{m} \geq 0,$$

because

$$\Gamma(u, H(u))(x) = \frac{1}{2\mathfrak{m}(x)} \sum_{y \in \Omega} w_{xy}(u(y) - u(x))(H(u(y)) - H(u(x))) \geq 0$$

since $H$ is non-decreasing. The proof is complete. $\qquad\square$

As a consequence, and in analogy with [4, Proposition B.2], from Lemma 4.1 we deduce the following result.

**Corollary 4.2.** *Let* $G = (V, E)$ *be a weighted locally finite graph. Let* $\Omega \subset V$ *be a bounded domain such that* $\Omega^\circ \neq \varnothing$ *and* $\partial\Omega \neq \varnothing$. *Let* $p \in [1, +\infty)$, $M > 0$ *and* $f \in L^1(\Omega)$. *If* $u \in W_0^{1,p}(\Omega)$ *is a solution of the problem*

$$\begin{cases} -\Delta_p u = f & in\ \Omega^\circ \\ u = 0 & on\ \partial\Omega, \end{cases}$$

*then*

$$\int_{\Omega \cap \{u \geq M\}} f\, d\mathfrak{m} \geq 0, \quad \int_{\Omega \cap \{u \leq -M\}} f\, d\mathfrak{m} \geq 0.$$

*In particular,*

$$\int_{\Omega \cap \{|u| \geq M\}} f\, \mathrm{sgn}(u)\, d\mathfrak{m} \geq 0,$$

*where* $\mathrm{sgn}\colon \mathbb{R} \to \mathbb{R}$ *is the* sign function *defined by* $\mathrm{sgn}(t) = \frac{t}{|t|}$ *for* $t \neq 0$ *and* $\mathrm{sgn}(0) = 0$.

*Proof.* For every $n \in \mathbb{N}$ such that $n > \frac{1}{M}$, we let $H_n\colon \mathbb{R} \to \mathbb{R}$ be the function

$$H_n(t) = \begin{cases} 0 & \text{for } t \leq M - \frac{1}{n} \\ nt - nM + 1 & \text{for } M - \frac{1}{n} < t < M \\ 1 & \text{for } t \geq M. \end{cases}$$

Since $H_n$ is Lipschitz, non-decreasing and such that $H_n(0) = 0$, by Lemma 4.1 we get that

$$\int_\Omega f\, H_n(u)\, d\mathfrak{m} \geq 0.$$

Passing to the limit as $n \to +\infty$, we find that

$$\int_{\Omega \cap \{u \geq M\}} f\, d\mathfrak{m} \geq 0,$$

as desired. The conclusion thus follows by linearity. $\qquad\square$

We are now ready to prove our second main result, in analogy with [4, Corollary B.1].

*Proof of Theorem 1.2.* The function $v = u_1 - u_2 \in W_0^{1,p}(\Omega)$ solves the problem

$$\begin{cases} -\Delta_p v = F & \text{in } \Omega^\circ \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

with $F = f_1 - f_2 - g(x, u_1) + g(x, u_2) \in L^1(\Omega)$. By Corollary 4.2, we have that

$$\int_\Omega F \operatorname{sgn}(v) \, d\mathfrak{m} \geq 0,$$

which is equivalent to

$$\int_\Omega (g(x, u_1) - g(x, u_2)) \operatorname{sgn}(u_1 - u_2) \, d\mathfrak{m} \leq \int_\Omega (f_1 - f_2) \operatorname{sgn}(u_1 - u_2) \, d\mathfrak{m}$$

and (1.8) immediately follows. As a consequence, if $f_1 = f_2$ then also $g(x, u_1) = g(x, u_2)$ and thus $F = 0$ in (4.2). Therefore $\Delta_p v = 0$ in $\Omega$ and thus

$$\sup_\Omega |v| \leq C_{1,p} \int_\Omega |\nabla v|^p \, d\mathfrak{m} = -C_{1,p} \int_\Omega v \, \Delta_p v \, d\mathfrak{m} = 0$$

by Theorem 2.1 and (1.3), so that $u_1 = u_2$. The proof is complete. $\qquad\square$

## 5. Applications

In this last section we briefly discuss some applications of our main results.

We begin by stating the following result, which shows that the Dirichlet problem in $W_0^{1,2}(\Omega)$ for the Laplcian operator with sufficiently well-behaved non-linearity admits a unique solution.

**Corollary 5.1.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $t \mapsto g(x, t)$ is $C^1$ and non-decreasing with $g(x, 0) = \partial_t g(x, 0) = 0$ for all $x \in \Omega$. Let us set $\bar{f}(x) = g(x, 0)$ for all $x \in \Omega$. There exists $\delta > 0$ with the following property: if $f \in L^2(\Omega)$ with $\|f - \bar{f}\|_{L^2(\Omega)} < \delta$, then the problem*

$$\begin{cases} -\Delta u + g(x, u) = f & \text{in } \Omega^\circ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*admits a unique solution $u \in W_0^{1,2}(\Omega)$.*

The uniqueness part in Corollary 5.1 is clearly immediately achieved by Theorem 1.2, while the existence part follows from the following result, which is inspired by the work [3].

**Lemma 5.2.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $t \mapsto g(x, t)$ is of class $C^1$ with $\partial_t g(x, 0) = 0$ for all $x \in \Omega$. Let us set $\bar{f}(x) = g(x, 0)$ for all $x \in \Omega$. There exist $\delta, \varepsilon > 0$ with the following property: if $f \in L^2(\Omega)$ with $\|f - \bar{f}\|_{L^2(\Omega)} < \delta$, then the problem*

$$\begin{cases} -\Delta u + g(x, u) = f & \text{in } \Omega^\circ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*admits a unique solution $u \in W_0^{1,2}(\Omega)$ with $\|u\|_{W_0^{1,2}(\Omega)} < \varepsilon$.*

*Proof.* Let us consider the function $\mathcal{F}\colon W_0^{1,2}(\Omega) \to L^2(\Omega)$ defined by

$$\mathcal{F}(u) = -\Delta u + g(x, u)$$

for all $u \in W_0^{1,2}(\Omega)$. Note that the map $\mathcal{F}$ is well defined, since $W_0^{1,2}(\Omega) \subset L^\infty(\Omega)$ with continuous embedding by Theorem 2.1 and thus also $x \mapsto g(x, u(x)) \in L^\infty(\Omega)$ by the continuity property of $g$ and by the fact that the number of vertices in $\Omega$ is finite. We additionally note that $\mathcal{F} \in C^1(W_0^{1,2}(\Omega), L^2(\Omega))$. Indeed, the Laplacian $\Delta$ is linear and the map $u \mapsto g(x, u)$ is of class $C^1$ thanks to the continuity properties of $g$. Finally, we observe that the map $\mathcal{F}'(0)\colon W_0^{1,2}(\Omega) \to L^2(\Omega)$ is invertible, since $\mathcal{F}'(0) = -\Delta$ by the assumption that $\partial_t g(x, 0) = 0$ for all $x \in \Omega$. Since $\mathcal{F}(0) = \bar{f}$, the conclusion follows by the Inverse Function Theorem and the proof is complete. $\qquad\square$

Our two main results Theorems 1.1 and 1.2 can be combined in order to achieve the well-posedness of a Yamabe-type problem on bounded domains, namely Proposition 1.3.

*Proof of Proposition 1.3.* The function $g(x, t) = b(x)|t|^{q-1}t$, defined for $(x, t) \in \Omega \times \mathbb{R}$, satisfies the assumptions of Theorem 1.2, so that problem (1.10) admits at most one solution and we just need to deal with the existence issue. If $\|a\|_{L^1(\Omega)} = 0$, then clearly $a = 0$ and thus the null function $u = 0$ is the unique solution of problem (1.10). If $\|a\|_{L^1(\Omega)} > 0$ instead, then we apply Theorem 1.1. Indeed, the function $f(x, t) = a(x) - b(x)|t|^{q-1}t$, defined for $(x, t) \in \Omega \times \mathbb{R}$, satisfies the assumptions of Theorem 1.1, and the conclusion thus follows in virtue of Remark 3.3. $\qquad\square$

We conclude our paper with the following uniqueness result for a Kazdan–Warner-type problem on bounded domains. Its proof is a simple application of Theorem 1.2 and is thus left to the reader.

**Corollary 5.3.** *Let $G = (V, E)$ be a weighted locally finite graph. Let $\Omega \subset V$ be a bounded domain such that $\Omega^\circ \neq \varnothing$ and $\partial\Omega \neq \varnothing$. Let $p \in [1, +\infty)$ and let $\alpha, \beta \in L^1(\Omega)$ be two non-negative functions. For every $f \in L^1(\Omega)$ and $h \in L^1(\partial\Omega)$, the Kazdan–Warner-type problem*

$$\begin{cases} -\Delta_p u + \alpha\, e^{\beta u} = f & in\ \Omega^\circ \\ u = h & on\ \partial\Omega \end{cases} \tag{5.1}$$

*admits at most one solution $u \in W^{1,p}(\Omega)$.*

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