Micro-slip-induced multiplicative plasticity: existence of energy minimizers

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Abstract

To account for material slips at microscopic scale, we take deformations as SBV functions φ, which are orientation-preserving outside a jump set taken to be two-dimensional and rectifiable. For their distributional derivative $F = D\varphi$ we admit the common multiplicative decomposition $F = F^e F^p$ into so-called elastic and plastic factors. Then, we consider a polyconvex energy with respect to F^e, augmented by the measure $|\mathrm{curl}\, F^p|$. For this type of energy we prove existence of minimizers in the space of SBV maps with appropriate constraints such as the one avoiding interpenetration of matter. Our analysis rests on a representation of the slip system in terms of currents with both $\mathbb{Z}^3$ and $\mathbb{R}^3$ valued multiplicity. The first choice is particularly significant in periodic crystalline materials at a lattice level, while the latter covers a more general setting and requires to account for an energy extra term involving the slip boundary size.

Key words: Plasticity; dislocations; energy minimization; geometric measure theory; currents; calculus of variations.

1 Introduction

We consider the energy of bodies undergoing irrecoverable strain that emerges from the cumulative effects of internal slips, and investigate the existence of its minimizers in the space of special bounded variation functions. The picture is particularly appropriate for (periodic) crystalline bodies, where slips are associated with dislocations, but we can consider it as appropriate even for those amorphous bodies where plasticity occurs as a consequence of internal slips among grains of various nature. In both cases, in large strain setting a multiplicative decomposition of the deformation gradient into so-called *elastic* and *plastic* factors plays a key descriptive role. Here we intend it in terms of measures by considering deformations as special bounded variation functions, which preserve the local spatial orientation outside their jump set.

At macroscopic scale, the irrecoverable strain characterizing what we call *plasticity* emerges after loading-unloading processes and has multiple specific sources, all concerning irreversible rearrangements of matter at microscopic spatial scales. Examples in different classes of materials are void or inclusion growth, localized micro-crackings that favor shear over surfaces, molecular entanglement or disentanglement, relative slips among neighboring crystalline grains, slips inside single crystals. This list is not exhaustive but we settle it because in the present paper we limit our attention to slip mechanisms and we assume that they develop over a two-dimensional rectifiable set S. The choice includes slip planes in single crystals, each being often, but not always, a close-packed plane, e.g., (111) in face-centered cubic crystals, (0001) in hexagonal ones [12, p. 3]. The boundary line between the slipping portion of a crystal and the rest is a *dislocation*, a line defect which can be variously detected essentially because the pertinent atomic-level disturbances are so severe as to determine scattering not consonant with that of the remaining crystal [44]. Dislocations interact with each other at finite distance; the Γ-limit of pertinent non-local energies justifies field-type representation of dislocation densities [18], [10], [11]. In a bottom-to-top view, we can also start by looking at atomic scale from which we may progressively move on the basis of Kohn-Sham's density functional theory—based on an approximation of the ground-state energy of an interacting inhomogeneous electron gas in a static potential—up to a large scale simulations of dislocations [45]. Intended as line defects, they are rather ubiquitous. We can find structures that we can appropriately consider to be dislocations even in bubble rafts, which are soap bubbles arranged in a crystalline form, or in crystallized colloids. We also find line defects in complex materials like liquid crystals. However, although in the nematic phase they are properly disclinations because they are just characterized by a discontinuity in the inclination of rod-like molecules composing nematics, in the smectic phase we can recognize the presence of proper dislocations between neighboring nematic layers [1] (such a case, however, does not fall within our setting because for nematics we need at least an additional descriptor of the material morphology, namely the local direction of the pertinent stick molecules, not considered here because we deal just with gross-scale deformations). In crystals, the Burgers vector indicates the units of lattice translations altered by the presence of a dislocation [27], [12], [44]. Edge dislocations can be seen as due to a single atomic plane insertion midway in the crystal to distort nearby planes of atoms; in this case the Burgers vector indicates a single step on a lattice walk around the planar inclusion [27], [12]. Screw dislocations emerge ideally from a process in which we cut through half-plane a crystal and slip the two faces of the cut by a lattice vector. It forms a structure in which a walk on the lattice determines a helix and the resulting Burgers vector is parallel to the dislocation line [12]. In both cases we may individuate a slip surface parallel to the Burgers vector. Essentially, this is an ideal picture dating back to V. Volterra. Real dislocations commonly have mixed nature, given by a combination of the two mechanisms. Also, we may record partial dislocations (distinguished into Frank's and Shockley's families, the former ones with sessile character, the latter with glissile nature), jogs, because a dislocation line is rarely uniformly straight, kinks, even junctions when dislocations meet in their walk through a crystal, a walk that can develop gliding along a plane containing also the Burgers vector, or climbing when they encounter vacancies in the crystal lattice. In polycrystals, dislocations may go out grains pinning at inter-granular interstices where they may favor or obstruct grain rearrangements [15]. When grains slip we could think of a Burgers vector measuring the relative shift. In fact, although the Burgers vector emerged in the analysis of crystals, we may define it even in amorphous solids [38], a circumstance suggesting also to think in general of linear defects moving into differentiable manifolds [34].

Here we look at the Burgers vector in a more abstract way by considering a multiplicity for the current associated with the deformation map. First we consider a $\mathbb{Z}^3$-valued multiplicity Θ, taking into account the possibility that the deformation graph may wrap around the line defect. Such a function generalizes the concept of relative translation $b \in \mathbb{Z}^3$ between the upper and lower surface in the slip plane S that one finds in Volterra's picture of dislocations in crystals. The boundary of S is the dislocation loop Γ with Burgers vector b. We will assume that Θ lies in the approximate tangent plane to the 2-rectifiable set corresponding to the current $\bar{S}$, in agreement with the slipping mechanism we aim at describing. This scheme is suitable essentially for periodic crystals with lattice based on $\mathbb{Z}^m$, $m = 1, 2, 3$, although here we develop the analysis in three-dimensional ambient space. When we look at quasi-periodic crystals—also called *quasicrystals* although quasi-periodicity in the spatial arrangement of atoms has been included in the definition of crystals by the International Union of Crystallography—we see that the notion of dislocation makes sense [25]; also, thicker linear defects called *metadislocations* appear [22], [23]. In quasicrystals, the pertinent (standard, in the sense of lattice-based) Burgers vector of dislocations has a dimension that may even double the ambient one. In fact, a quasi-periodic lattice can be viewed as the projection of a periodic atomic array in a given space over an incommensurate subspace with lower dimension (e.g., a three-dimensional quasi-periodic atomic array is the projection of a six-dimensional periodic lattice over an incommensurate $3D$-subspace) [50], [51]. If we would analyze the circumstance, we should consider the multiplicity Θ of a current associated with the so-called phason field, which describes the atomic shifts necessary to ensure lattice quasi-periodicity in the physical space [36]. We do not consider here this case. However, although we keep in mind the classical case of periodic crystals, our treatment applies also to those bodies described, at level of morphology, just by the region they occupy in the physical space (Cauchy's bodies, in short), specifically those characterized by microscopic slips, which cumulatively determine irrecoverable finite strain.

At macroscopic continuum scale, in order to represent such an effect we commonly accept the multiplicative decomposition of the deformation gradient F, into so-called *elastic* (F^e) and *plastic* (F^p) factors above mentioned, namely

$$F = F^e F^p \tag{1.1}$$

(the first authors introducing the decomposition are those of references [7], [26], [5], [6], [28], and we tend to accept a suggestion in reference [35], calling it the Kröner-Lee decomposition).

Let Ω be a fit region in the three-dimensional point space, endowed with piecewise Lipschitz boundary, a region that we take as a reference shape for a continuous body. At every $x \in \Omega$, F^p maps tangent vectors to Ω at x onto a linear space where, at least pictorially, we think to represent the local rearrangement of matter in a small neighborhood of x. Then, F^e maps that linear space onto the tangent space of a configuration considered deformed with respect to Ω. This last mapping represents only crowding and shearing of material elements; it does not involve any structural irreversible change in the matter. Per se, F is compatible with a deformation, i.e., $F = D\varphi$, with $\varphi : \Omega \to \mathbb{R}^3$, which we consider to be orientation preserving, while in general F^e and F^p are not compatible, i.e., unless we are in very special conditions, e.g., a deck of sliding cards, we cannot write φ as a composition of two maps, one of elastic nature (to be definite in some way), the other which plastic character (see the detailed analyses on crystal lattices developed in references [13], [41], [42], [43]). By varying x in Ω, the union of all linear spaces reached by $F^p(x)$ is not (or better, *not necessarily*) the tangent bundle of some intermediate configuration. Rather, we find it more correct to speak of *intermediate spaces*, which visualize the ideal decomposition of recoverable strain from irreversible rearrangements of matter depicted by the product $F^e F^p$.

This view is also compatible with a scheme in which plastic rearrangements can be described through a multiplicity of reference shapes, a parameterized family of configurations with infinitesimal generator a volume-preserving vector field, a type of horizontal variation (although special because it has to be a material isomorphism), as proposed in reference [35]. In this setting, a mechanical dissipation inequality written relatively to such changes allows us to describe from a unique invariance requirement all pertinent rules [35] (in fact, we can also depict changes in the reference shape through variations of the pertinent metrics to which, under appropriate conditions, the emergence of associated configurational forces occurs [33]). However, here we do not tackle the problem of describing plastic flows. We just consider equilibrium along a deformation allowing micro-slips over two-dimensional rectifiable sets.

For this reason, we consider φ to be a special function of bounded variation, namely $\varphi \in SBV(\Omega, \mathbb{R}^3)$ assumed to preserve orientation and to avoid self-penetration of matter; its jumps occur over the set S already mentioned. As such the distributional derivative $F = D\varphi$ of φ is a measure compatible with the multiplicative decomposition (1.1), as shown for single crystal slips in reference [46] and further analyzed in terms of lattice-to-continuum limit [47]. We take the plastic factor F^p to be an $\mathbb{R}^{3\times 3}$-valued bounded measure in Ω, which decomposes as

$$F^p = a(x)I\mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma})$$

where I is the 3×3 identity matrix, $\mathscr{L}^3$ is the Lebesgue measure, and $a(x)$ is a measurable function in Ω satisfying

$$C^{-1} \leq a(x) \leq C \qquad \forall\, x \in \Omega$$

for some given real constant $C > 1$. The presence of a accounts for possible plastic volume changes. In reference [46] just the case $a = 1$ is considered and the last addendum in the structure of F^p is the Schmidt tensor associated with the crystal slip system (a picture in terms of SBV functions can be naturally considered for elastic microcracked bodies [32]), while here we substitute that tensor with $\hat{F}(\bar{S}, \overline{\Gamma})$, which is a tensor valued rectifiable measure supported by a 2-rectifiable set in such a way that the dislocation measure $\operatorname{curl} F^p = \operatorname{curl} \hat{F}(\bar{S}, \overline{\Gamma})$ is supported on a 1-rectifiable set.

More precisely, the measure $\hat{F}(\bar{S}, \overline{\Gamma})$ corresponds to a $\mathbb{Z}^3$-valued rectifiable current $\bar{S}$ with boundary $\overline{\Gamma}$ that describes the dislocation measure $\operatorname{curl} F^p$ at a first glance. Then, to consider a microscopic level, we also take the measure $\hat{F}(\bar{S}, \overline{\Gamma})$ to be associated with a $\mathbb{R}^3$-valued multiplicity Θ.

In our picture the elastic factor F^e is a tensor-valued summable field

$$F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$$

where $|F^p|$ is the total variation of F^p, and the minors of F^e are required to satisfy some integrability assumptions with respect to the Lebesgue measure.

The multiplicative decomposition (1.1) implies that the jump set $S(\varphi)$ identifies the 2-rectifiable set corresponding to the current $\bar{S}$. As a consequence, prescribing Θ in the approximate tangent space of the jump set is tantamount to say that the deformation jump $\varphi^+ - \varphi^- \in \mathbb{R}^3$ is $\mathscr{H}^2$-a.e. tangent to the set $S(\varphi)$.

In this setting we first consider the energy given by

$$\mathscr{F}_{p,s}(\varphi) := \int_\Omega \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \right) dx + |\mathrm{curl}\, F^p|(\Omega), \tag{1.2}$$

where $M(F^e)$ is the vector with entries all minors of the elastic factor F^e, and $p > 1$, $s > 0$ are real exponents. We prove for $\mathscr{F}_{p,s}$ existence of minimizers in the SBV space, under Dirichlet-type boundary conditions, after choosing $\mathbb{Z}^3$-valued multiplicities for the related currents.

Then, we consider an energy variant given by

$$\widetilde{\mathscr{F}}_{p,s}(\varphi) := \int_\Omega \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \right) dx + |\mathrm{curl}\, F^p|(\Omega) + \mathbf{S}(\overline{\Gamma}), \tag{1.3}$$

where $\mathbf{S}(\overline{\Gamma})$ is the size of a line-defect-supported current, and prove existence of its minimizers under the same boundary conditions but considering $\mathbb{R}^3$-valued multiplicity of currents; this last choice imposes the boundary current $\overline{\Gamma} = \partial \bar{S}$ to be with bounded size. $\mathbf{S}(\overline{\Gamma})$ is a line energy; it can be justified by the limit of functionals with singular kernels under appropriate conditions [10], [21].

Our results apply also in the more general case in which the energy dependence on F^e is through a density which is a convex function of the F^e minors.

2 Background material

2.1 Special functions of bounded variation

A real valued summable function $v \in L^1(\Omega)$ is said to be *of bounded variation* if the distributional derivative Dv is a finite $\mathbb{R}^3$-valued measure in Ω. In this case, the function v is approximately differentiable $\mathscr{L}^3$-a.e. in Ω, and the approximate gradient ∇v agrees with the density of the Radon-Nikodym derivative of Dv with respect to the Lebesgue measure $\mathscr{L}^3$. Therefore, the decomposition $Dv = \nabla v\, \mathscr{L}^3 + D^s v$ holds, where the component $D^s v$ is singular with respect to $\mathscr{L}^3$. Also, the *jump set* $S(v)$ of v is a countably 2-rectifiable subset of Ω that agrees $\mathscr{H}^2$-essentially with the complement of v Lebesgue's set, where $\mathscr{H}^2$ is the two-dimensional Hausdorff measure. If, in addition, the singular component $D^s v$ is concentrated on the jump set $S(v)$, we say that v is a *special function of bounded variation*, and write in short $v \in SBV(\Omega)$. In this case, we find $D^s v = D^J v$, with $D^J v = (v^+ - v^-)\nu\, \mathscr{H}^2 \llcorner S(v)$, where $v^\pm$ are the one-sided limits at points in the jump set $S(v)$ with respect to the given unit normal ν to $S(v)$.

A vector field $u : \Omega \to \mathbb{R}^3$ belongs to the class $SBV(\Omega, \mathbb{R}^3)$ if all its components u^j are in $SBV(\Omega)$. Therefore, the distributional derivative Du belongs to the class $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ of *matrix-valued bounded Radon measures.* It decomposes as $Du = \nabla u \mathscr{L}^3 + D^J u$, where the approximate gradient ∇u belongs to $L^1(\Omega, \mathbb{R}^{3\times 3})$. The jump component correspondingly reads as $D^J u = (u^+ - u^-) \otimes \nu \mathscr{H}^2 \llcorner S(u)$, where the jump set $S(u) := \cup_{j=1}^3 S(u^j)$ is oriented by the unit normal ν and the one-sided limits $u^\pm$ are defined componentwise. Therefore, the total variation $|Du|(B)$ of Du reads

$$|Du|(B) = \int_B |\nabla u| \, dx + \int_{B \cap S(u)} |u^+ - u^-| \, d\mathscr{H}^2$$

for each Borel set $B \subset \Omega$ (the treatise [3] offers an accurate analysis of SBV functions properties).

2.2 Compatibility condition

We take two isomorphic copies of $\mathbb{R}^3$, say $\tilde{\mathbb{R}}^3$ and $\mathbb{R}^3$, with the isomorphism being just an identification. We select Ω in $\mathbb{R}^3$ and consider it as a reference configuration. For $x \in \Omega$, we select in a neighborhood of it a basis $\{\mathbf{e}_A\}$, where capital letters indicate coordinates in the reference configuration. Orientation preserving differentiable maps φ select deformed shapes with respect to Ω in the other copy of $\mathbb{R}^3$, endowed with basis $\{\tilde{\mathbf{e}}_i\}$, with a convention that the lower-case indices indicate coordinates in the deformed shape.

Here and below F is a linear operator mapping at each $x \in \Omega$ the tangent space $T_x\Omega$ into $\mathbb{R}^3$, so that we write $F(x) \in \mathrm{Hom}(T_x\Omega, \mathbb{R}^3)$, intending F of the form $F = F_A^i \tilde{\mathbf{e}}_i \otimes \mathbf{e}^A$.

Take $\psi \in C_c^1(\Omega, \mathbb{R}^{3\times 3})$ as a tensor valued field with components ψ_A^i. We set $\mathrm{curl}\, \psi \in C_c^0(\Omega, \mathbb{R}^{3\times 3})$ as the tensor valued field $\nabla \times \psi$ with components $(\mathrm{curl}\, \psi)_A^i := (\epsilon_A{}^B{}_C (\partial \psi_B^i)^C)^\intercal$ where ϵ, with components ϵ_{ACB}, is the Levi-Civita alternating symbol.

The measure $\mathrm{curl}\, F$ is defined in a distributional sense for any $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ of the form above as

$$\langle \mathrm{curl}\, F, \psi \rangle := \langle F, \mathrm{curl}\, \psi \rangle = \sum_{i,A=1}^{3} \langle F_A^i, (\mathrm{curl}\, \psi)_A^i \rangle \,, \quad \psi \in C_c^1(\Omega, \mathbb{R}^{3\times 3}) \,.$$

If $F = Du$ for some $u \in BV(\Omega, \mathbb{R}^3)$, using that $\mathrm{div}(\mathrm{curl}\, \psi) = 0$ for each $\psi \in C_c^2(\Omega, \mathbb{R}^{3\times 3})$, it turns out that F satisfies the *compatibility condition*

$$\mathrm{curl}\, F = 0 \,. \tag{2.1}$$

Actually, the inverse implication holds, too. In fact, a result by M. Miranda [37] yields that any $\mathbb{R}^3$-valued distribution T in Ω, with distributional derivative

DT a finite measure in $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$, can be represented as $T = u\mathscr{L}^3$ for some $u \in L^1(\Omega, \mathbb{R}^3)$ [48]. Moreover, since the domain Ω is simply-connected, any measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ satisfying the compatibility condition (2.1) is equal to the derivative DT of a distribution T. Therefore, $F = Du$ with $u \in BV(\Omega, \mathbb{R}^3)$. In particular, if F is absolutely continuous with density in $L^q(\Omega, \mathbb{R}^{3\times 3})$ for some $q \geq 1$, it turns out that $F = \nabla u \mathscr{L}^3$ for some Sobolev vector field $u \in W^{1,q}(\Omega, \mathbb{R}^3)$.

3 Slip planes

Consider Ω to be occupied only by a single crystal endowed with a slip plane S (this assumption holds only in this section). Assume that the slip activates along a deformation $\varphi : \Omega \to \mathbb{R}^3$ so that φ jumps across the slip and is per se a SBV map with distributional derivative

$$D\varphi = \nabla\varphi \mathscr{L}^3 + b \otimes \nu \mathscr{H}^2 \llcorner S \tag{3.1}$$

where $b \in \mathbb{R}^3$ is the Burgers vector. The standard assumption that physically admissible deformations be orientation preserving is tantamount to impose $\det \nabla\varphi > 0$ a.e. in Ω. Therefore, since $\operatorname{curl} D\varphi = 0$, we get

$$\operatorname{curl}(\nabla\varphi \mathscr{L}^3) = -b \otimes \tau \mathscr{H}^1 \llcorner \Gamma \tag{3.2}$$

where $\Gamma = \partial S$ indicates the dislocation associated with the slip plane, oriented by τ. The multiplicative decomposition (1.1) of $F = D\varphi$ holds true and we have (at crystal scale)

$$F^e = \nabla\varphi\,, \qquad F^p = I\mathscr{L}^3 + (\nabla\varphi)^{-1}(b \otimes \nu)\mathscr{H}^2 \llcorner S\,. \tag{3.3}$$

We intend a measure $F^e\mathscr{L}^3 \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ to be associated with the elastic strain. For the specific case of a single slip, if we can imagine to *fix* a slipped configuration of the crystal, we could even think of F^e as the gradient of a differentiable, orientation preserving map φ^e defined over the slipped shape, so that $\operatorname{curl} F^e = (\operatorname{curl} \nabla\varphi^e)\mathscr{L}^3 = 0$. In general it is not so. In fact, as already recalled, at every $x \in \Omega$, the linear operator F^p maps the tangent space $T_x\Omega$ onto a linear space, say $\mathscr{L}_x$, which is isomorphic to $\mathbb{R}^k$, with k selected into $\{1, 2, 3\}$. In principle, varying x in Ω, the $\mathscr{L}_x$ also varies. Not necessarily the union of all $\mathscr{L}_x$, as x ranges in Ω, is the tangent bundle of some *intermediate configuration* reached from Ω by means of some deformation. Consequently, in general, although F is compatible, in the sense that $F = D\varphi$, its elastic and plastic factors F^e and F^p are incompatible, i.e., $\operatorname{curl}(F^e\mathscr{L}^3) \neq 0$ and $\operatorname{curl} F^p \neq 0$. In other words, in general we cannot write φ as a composition $\varphi = \varphi^e \circ \varphi^p : \Omega \to \mathbb{R}^3$, where, e.g., $\varphi^p : \Omega \to \mathbb{R}^3$ is an SBV-function and $\varphi^e : \varphi^p(\Omega) \to \mathbb{R}^3$ a "smooth" elastic deformation. However, assuming F^e invertible, using that $\operatorname{curl} F = 0$, and writing $F^p = (F^e)^{-1}F$, their incompatibilities are related by the following formula (see reference [46])

$$\operatorname{curl} F^p = (\det F) \operatorname{curl} [(F^e)^{-1}] \, F^T \tag{3.4}$$

where $\operatorname{curl} [(F^e)^{-1}]$ is computed in the deformed configuration $\varphi(\Omega)$.

In the presence of a finite number N of smooth dislocation loops within the considered crystal, we can write

$$F^p = I\mathscr{L}^3 + \sum_{h=1}^{N} b_h \otimes \nu_h \mathscr{H}^2 \llcorner S_h$$

where S_h is a smooth flat surface in Ω with boundary $\Gamma_h = \partial S_h$ a smooth, simple, and closed curve in Ω, and ν_h a smooth unit normal to S_h. Also, the Burgers vector $b_h \in \mathbb{Z}^3$ is constant. Then, the pertinent dislocation density tensor is

$$\operatorname{curl} F^p = \sum_{h=1}^{N} b_h \otimes \tau_h \mathscr{H}^1 \llcorner \Gamma_h$$

where τ_h is a tangent unit vector orienting the closed curve Γ_h in a consistent way with respect to the orientation induced by ν_h on S_h (see also [40] and [46]).

4 Dislocations and rectifiable currents

Both measures previously considered, namely

$$\mu_S := \sum_{h=1}^{N} b_h \otimes \nu_h \mathscr{H}^2 \llcorner S_h \quad \text{and} \quad \mu_\Gamma := \sum_{h=1}^{N} b_h \otimes \tau_h \mathscr{H}^1 \llcorner \Gamma_h \, , \tag{4.1}$$

can be seen as triplets of integer multiplicity (in short i.m.) rectifiable currents of dimension $k = 2$ and $k = 1$, respectively, each triplet living on the same k-rectifiable set, in such a way that the equality $\operatorname{curl} \mu_S = \mu_\Gamma$ reduces to a boundary condition in the sense of currents. To discuss the issue, first we fix some general notions.

4.1 Integer rectifiable currents

If $U \subset \mathbb{R}^n$ is an open set, and $k = 0, \ldots, n$, we denote by $\mathscr{D}_k(U)$ the strong dual of the space of compactly supported smooth k-forms $\mathscr{D}^k(U)$, whence $\mathscr{D}_0(U)$ is the class of distributions in U. For any $T \in \mathscr{D}_k(U)$, we define its *mass* $\mathbf{M}(T)$

as

$$\mathbf{M}(T) := \sup\{\langle T, \omega\rangle \mid \omega \in \mathscr{D}^k(U)\,,\ \|\omega\| \leq 1\}$$

and (for $k \geq 1$) its *boundary* as the $(k-1)$-current ∂T defined by the relation

$$\langle \partial T, \eta\rangle := \langle T, \mathrm{d}\eta\rangle, \qquad \forall \eta \in \mathscr{D}^{k-1}(U)$$

where $\mathrm{d}\eta$ is the differential of η. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathscr{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h, \omega\rangle = \langle T, \omega\rangle\,, \qquad \forall\, \omega \in \mathscr{D}^k(U)\,.$$

If $T_h \rightharpoonup T$, by lower semicontinuity we also have

$$\mathbf{M}(T) \leq \liminf_{h\to\infty} \mathbf{M}(T_h)\,.$$

For $k \geq 1$, a k-current T with finite mass is called *rectifiable* if

$$\langle T, \omega\rangle = \int_{\mathscr{M}} \theta\, \langle \omega, \xi\rangle\, \mathrm{d}\mathscr{H}^k\,, \qquad \forall\, \omega \in \mathscr{D}^k(U)\,,$$

with $\mathscr{M}$ a k-rectifiable set in U, $\xi : \mathscr{M} \to \Lambda^k \mathbb{R}^n$ a $\mathscr{H}^k \llcorner \mathscr{M}$-measurable function such that $\xi(x)$ is a simple unit k-vector orienting the approximate tangent space to $\mathscr{M}$ at $\mathscr{H}^k$-a.e. $x \in \mathscr{M}$, and $\theta : \mathscr{M} \to [0, +\infty)$ a $\mathscr{H}^k \llcorner \mathscr{M}$-summable and non-negative function. Therefore, we get $\mathbf{M}(T) = \int_{\mathscr{M}} \theta\, d\mathscr{H}^k < \infty$ and the short-hand notation $T = [\![\, \mathscr{M}, \xi, \theta \,]\!]$ is commonly adopted.

In addition, if the multiplicity function θ is integer-valued, the current T is called *i.m. rectifiable* and the corresponding class is denoted by $\mathscr{R}_k(U)$.

Currents in $\mathscr{R}_k(U)$ generalize the action given by integration of k-forms on smooth oriented k-surfaces $\mathscr{M}$, where one takes $\theta \equiv 1$. Their relevance in the calculus of variations relies of Federer-Fleming's compactness theorem [14], stating that if a sequence $\{T_h\} \subset \mathscr{R}_k(U)$ satisfies $\sup_h \mathbf{M}(T_h) < \infty$ and $\sup_h \mathbf{M}((\partial T_h) \llcorner U) < \infty$, there exists $T \in \mathscr{R}_k(U)$ and a (not relabeled) subsequence of $\{T_h\}$ such that $T_h \rightharpoonup T$ weakly in $\mathscr{D}_k(U)$. As a consequence, if $T \in \mathscr{R}_k(U)$ satisfies $\mathbf{M}((\partial T) \llcorner U) < \infty$, the boundary rectifiability theorem states that $\partial T \in \mathscr{R}_{k-1}(U)$.

An extended treatment of currents is in the treatise [20] (see also [21]).

4.2 $\mathbb{Z}^m$-valued rectifiable currents

Let $m \in \mathbb{N}^+$ and $k = 1, \ldots, n$. In this paper, a *$\mathbb{Z}^m$-valued rectifiable k-current* T in U is defined by a triplet $(\mathscr{M}, \xi, \Theta)$, where $\mathscr{M}$ and ξ are as above, but $\Theta : \mathscr{M} \to \mathbb{Z}^m$ is a $\mathbb{Z}^m$-valued $\mathscr{H}^k \llcorner \mathscr{M}$-summable multiplicity function. More precisely, setting $\Theta = (\theta^1, \dots, \theta^m)$, we see $\overline{T} = (T^1, \dots, T^m)$ as an *ordered m-tuple of i.m. rectifiable currents* $T^j \in \mathcal{R}_k(U)$, with $T^j = [\![\mathcal{M}, \sigma^j \xi, \sigma^j \theta^j]\!]$ for $j = 1, \dots, m$, where $\sigma^j = 0$ if $\theta^j = 0$ and $\sigma^j = \theta^j/|\theta^j|$ otherwise. Denote by $\overline{\omega} = (\omega_1, \dots, \omega_m)$ an ordered m-tuple of k-forms $\omega_j \in \mathcal{D}^k(U)$, in short $\overline{\omega} \in [\mathcal{D}^k(U)]^m$, and by $[\mathcal{R}_k(U)]^m$ the class of $\mathbb{Z}^m$-valued rectifiable k-currents $\overline{T}$ as above. The action of $\overline{T}$ on $\overline{\omega}$ is defined through its components by

$$\langle \overline{T}, \overline{\omega} \rangle := \sum_{j=1}^{m} \langle T^j, \omega_j \rangle \,.$$

Differently from, e.g., reference [9], we are not dealing with rectifiable k-currents $\hat{T}$ with coefficients in $\mathbb{Z}^m$, in short $\hat{T} \in \mathcal{R}_k(U, \mathbb{Z}^m)$, with action on a form $\omega \in \mathcal{D}^k(U)$ defined by $\langle \hat{T}, \omega \rangle := \int_{\mathcal{M}} \Theta \, \langle \omega, \xi \rangle \mathrm{d}\mathcal{H}^k$ for some triplet $(\mathcal{M}, \xi, \Theta)$ as above. In order to recover $\hat{T}$ from $\overline{T} = (T^1, \dots, T^m)$, it suffices to observe that $\langle \hat{T}, \omega \rangle = \sum_{j=1}^m \langle T^j, \omega \rangle e_j$, where $(e_1, \dots, e_m)$ is the canonical basis in $\mathbb{R}^m$.

Remark 4.1 *If $T^j \in \mathcal{R}_k(U)$ for $j = 1, \dots, m$ we find a current $\overline{T} \in [\mathcal{R}_k(U)]^m$ with components $\overline{T} = (T^1, \dots, T^m)$. In fact, letting $T^j = [\![\mathcal{M}_j, \xi_j, \theta_j]\!]$, we choose $\mathcal{M}$ as the set of points x in $\hat{\mathcal{M}} := \bigcup_{j=1}^m \mathcal{M}_j$ with unitary k-dimensional density Θ^k, namely $\Theta^k(\hat{\mathcal{M}}, x) = 1$. Then, we equip $\mathcal{M}$ with an orientation ξ. Eventually, it suffices to define the multiplicity $\Theta = (\theta^1, \dots, \theta^m)$ as follows: for $x \in \mathcal{M}$ and $j = 1, \dots, m$, if $\Theta^k(\mathcal{M}^j, x) = 0$ we let $\theta^j(x) = 0$, whereas if $\Theta^k(\mathcal{M}^j, x) = 1$ we let $\theta^j(x) = \pm\theta_j(x)$, according to the sign in the equality $\xi_j(x) = \pm\xi(x)$.*

The *weak convergence* $\overline{T}_h \rightharpoonup \overline{T}$ in the class $[\mathcal{R}_k(U)]^m$ is defined by components through the formula $\langle \overline{T}_h, \overline{\omega} \rangle \to \langle \overline{T}, \overline{\omega} \rangle$ for each $\overline{\omega} \in [\mathcal{D}^k(U)]^m$. In a similar way, the *boundary* of a current $\overline{T} \in [\mathcal{R}_k(U)]^m$ is defined by the formula $\langle \partial\overline{T}, \overline{\omega} \rangle := \langle \overline{T}, \mathrm{d}\overline{\omega} \rangle$ for any $\overline{\omega} \in [\mathcal{D}^{k-1}(U)]^m$, where $\mathrm{d}\overline{\omega} := (\mathrm{d}\omega_1, \dots, \mathrm{d}\omega_m)$ is in $[\mathcal{D}^k(U)]^m$. We also define the *mass* $\mathbf{M}(\overline{T}) := \sum_{j=1}^m \mathbf{M}(T^j) < \infty$ and the *boundary mass* $\mathbf{M}((\partial\overline{T}) \llcorner U) := \sum_{j=1}^m \mathbf{M}((\partial T^j) \llcorner U)$ if $\overline{T} = (T^1, \dots, T^m)$ as above.

If $\overline{T} \in [\mathcal{R}_k(\Omega)]^m$ satisfies $\mathbf{M}((\partial\overline{T}) \llcorner \Omega) < \infty$, on account of the previous remark, the boundary rectifiability theorem yields $\partial\overline{T} \in [\mathcal{R}_{k-1}(\Omega)]^m$.

In a similar way, if a sequence $\{\overline{T}_h\} \subset [\mathcal{R}_k(\Omega)]^m$ satisfies $\sup_h \mathbf{M}(\overline{T}_h) < \infty$ and $\sup_h \mathbf{M}((\partial\overline{T}_h) \llcorner \Omega) < \infty$, by using Federer-Fleming's compactness theorem and a diagonal argument, we can find a current $\overline{T} \in [\mathcal{R}_k(\Omega)]^m$ and a (not relabeled) subsequence of $\{\overline{T}_h\}$ such that $\overline{T}_h \rightharpoonup \overline{T}$.

Consider $n = m = 3$, so that $U = \Omega$. Vector fields ϕ in $C_c^\infty(\Omega, \mathbb{R}^3)$ agree with 0-forms $\overline{\omega}$ in $[\mathscr{D}^1(\Omega)]^3$, say $\overline{\omega} = \omega_\phi^{(0)}$. A 1-form $\overline{\omega} \in [\mathscr{D}^1(\Omega)]^3$ is identified by a tensor valued field $\psi = \psi_{jA}\mathbf{e}^j \tilde{\otimes} \mathbf{e}_A$ in $C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ by letting $\omega_j = \sum_{A=1}^3 \psi_{jA}\mathrm{d}x^A$ for $j = 1, 2, 3$. In this case, we write $\overline{\omega} = \omega_\psi^{(1)}$. In a similar way, a 2-form $\overline{\omega} \in [\mathscr{D}^2(\Omega)]^3$ is identified by a tensor valued field $\zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ with components ζ_{jA} by letting $\omega_j = \sum_{A=1}^3 (-1)^{A-1}\zeta_{jA}\widehat{\mathrm{d}x^A}$, where $\mathrm{d}x^A \wedge \widehat{\mathrm{d}x^A} = (-1)^{A-1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. In this case, we write $\overline{\omega} = \omega_\zeta^{(2)}$. Finally, a 3-form $\overline{\omega} \in [\mathscr{D}^3(\Omega)]^3$ is identified by a covector field $\eta \in C_c^\infty(\Omega, \mathbb{R}^3)$ with $\eta = (\eta_1, \eta_2, \eta_3)$, by letting $\omega_j = \eta_j \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, and we write $\overline{\omega} = \omega_\eta^{(3)}$. With this notation, we have:

$$\mathrm{d}\omega_\phi^{(0)} = \omega_{\nabla\phi}^{(1)}\,, \qquad \mathrm{d}\omega_\psi^{(1)} = \omega_{\mathrm{curl}\,\psi}^{(2)}\,, \qquad \mathrm{d}\omega_\zeta^{(2)} = \omega_{\mathrm{div}\zeta}^{(3)} \tag{4.2}$$

and hence the identities $\mathrm{curl}\,\nabla\phi = 0$ and $\mathrm{div}(\mathrm{curl}\,\psi) = 0$ turn out to be equivalent to the closure relations $\mathrm{d} \circ \mathrm{d} = \mathrm{d}^2 = 0$ for 0-forms and 1-forms, respectively.

Example 1 *For $k = 2$, a current $\overline{T} = \overline{T}_S \in [\mathscr{R}_2(\Omega)]^3$ is naturally associated with the measure μ_S defined in* (4.1)*. Assuming for the sake of simplicity that $\mathscr{H}^2(S_{h_1} \cap S_{h_2}) = 0$ for $1 \le h_1 < h_2 \le N$, it suffices to take $\mathscr{M} = \cup_{h=1}^N S_h$ and define $\xi \equiv *\nu_h$ and $\Theta \equiv b_h$ on each S_h, where $*$ is the Hodge operator in $\mathbb{R}^3$. Similarly, for $k = 1$, a current $\overline{T} = \overline{T}_\Gamma \in [\mathscr{R}_1(\Omega)]^3$ is naturally associated with the measure μ_Γ in* (4.1)*. By assuming again $\mathscr{H}^1(\Gamma_{h_1} \cap \Gamma_{h_2}) = 0$ for $1 \le h_1 < h_2 \le N$, it suffices to take $\mathscr{M} = \cup_{h=1}^N \Gamma_h$, setting $\xi \equiv \tau_h$ and $\Theta \equiv b_h$ on each Γ_h. We also notice that*

$$\mathrm{curl}\,\mu_S = \mu_\Gamma \quad \iff \quad \partial\overline{T}_S = \overline{T}_\Gamma\,.$$

Since $\langle \mu_\Gamma, \psi \rangle = \langle \overline{T}_\Gamma, \omega_\psi^{(1)} \rangle$ and $\langle \mu_S, \zeta \rangle = \langle \overline{T}_S, \omega_\zeta^{(2)} \rangle$, it suffices, in fact, to recall that $\langle \partial\overline{T}_S, \omega_\psi^{(1)} \rangle = \langle \overline{T}_S, d\omega_\psi^{(1)} \rangle$ and to use the second formula in (4.2)*. Therefore, the implication $\Rightarrow$ readily follows, whereas the reverse, namely $\Leftarrow$, holds true by a standard density argument based on the dominated convergence theorem. Finally, the closure relation $d \circ d = 0$ yields that $\partial\overline{T}_\Gamma = 0$ if $\mu_\Gamma = \mathrm{curl}\,\mu_S$. On account of identities* (4.2)*, the null-boundary property $\partial\overline{T}_\Gamma = 0$ is equivalent to the requirement that μ_Γ is a divergence-free dislocation measure (see reference [9]).*

For a given current $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$, with a slight abuse of notation we let $F = F(\bar{S})$ denote the tensor valued distribution in Ω acting on test functions $\zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ as

$$\langle F, \zeta \rangle := \langle \bar{S}, \omega_\zeta^{(2)} \rangle\,.$$

Since $\mathbf{M}(\bar{S}) < \infty$, the distribution F can be extended to a measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ with total variation bounded by the mass of $\bar{S}$, namely $|F|(\Omega) \le$

$\mathbf{M}(\bar{S})$. Moreover, by using the notation (S, ξ_S, Θ_S) for $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$, and choosing the unit normal ν_S to S as a covector $\nu_S = (\nu_{S1}, \nu_{S2}, \nu_{S3})$ in such a way that $\xi_S = *\nu_S^\sharp$, with $\nu_S^\sharp$ the vector associated with the covector ν_S by the metric in Ω, we have $(-1)^{A-1}\langle \phi\, \widehat{\mathrm{d}x^A}, \xi_S \rangle = \phi\, \nu_{SA}$ for each $A = 1, 2, 3$ and $\phi \in C_c^\infty(\Omega)$. Therefore, the component F_A^j acts on bounded and continuous functions $\phi \in C_b(\Omega)$ as

$$\langle F_A^j, \phi \rangle = \int_\Omega \phi\, \mathrm{d}F_A^j = \int_S \Theta_S^j \nu_{SA}\, \phi\, d\mathscr{H}^2$$

and hence we can write

$$F = F(\bar{S}) = \Theta_S \otimes \nu_S\, \mathscr{H}^2 \llcorner S\, .$$

4.4 Confinement condition

A *confinement condition* for dislocations has been discussed in reference [30]. We can translate it within the setting discussed here by requiring that the current $\bar{S}$ has compact support contained in Ω. By looking at the corresponding measure, this property becomes

$$\mathrm{spt}\, F(\bar{S}) \subset \Omega\, .$$

In addition, if the boundary current $\partial \bar{S}$ has finite mass, there exists a current $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ with support contained in Ω such that $\partial \bar{S} = \overline{\Gamma}$. By adopting the notation $(\Gamma, \tau_\Gamma, \Theta_\Gamma)$, as above it turns out that the tensor valued distribution $F = F(\overline{\Gamma})$ in Ω acting on test functions $\psi \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$ as

$$\langle F, \psi \rangle := \langle \overline{\Gamma}, \omega_\psi^{(1)} \rangle$$

can be extended to a measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$, with total variation bounded by the mass of $\overline{\Gamma}$, $|F|(\Omega) \le \mathbf{M}(\overline{\Gamma})$, and actually

$$F(\overline{\Gamma}) = \Theta_\Gamma \otimes \tau_\Gamma\, \mathscr{H}^1 \llcorner \Gamma\, .$$

Moreover, the boundary condition $\partial \bar{S} = \overline{\Gamma}$ is equivalent to

$$\langle \mathrm{curl}\, F(\bar{S}), \psi \rangle = \langle \bar{S}, \mathrm{d}\omega_\psi^{(1)} \rangle = \langle \overline{\Gamma}, \omega_\psi^{(1)} \rangle \qquad \forall\, \psi \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})\, .$$

We thus have for all $\phi \in C_b(\Omega)$ and $A, j = 1, 2, 3$

$$\langle (\mathrm{curl}\, F(\bar{S}))_A^j, \phi \rangle = \int_\Omega \phi\, \mathrm{dcurl}\, F(\bar{S})_A^j = \int_\Gamma \Theta_\Gamma^j\, \phi\, \tau_{\Gamma A}\, \mathrm{d}\mathscr{H}^1\, ,$$

which is

$$\mathrm{curl}\, F(\bar{S}) = \Theta_\Gamma \otimes \tau_\Gamma\, \mathscr{H}^1 \llcorner \Gamma\, .$$

Finally, the support condition on $\overline{\Gamma}$ is equivalent to the confinement condition

$$\operatorname{spt}(\operatorname{curl} F(\bar{S})) \subset \Omega\,.$$

4.5 Tangency condition

When dislocations glide, the Burgers vector b is parallel to the slip plane. When they climb the geometry involved is not so simple. When material grains relatively move, e.g. in polycrystalline bodies, we might also accept at least in approximate sense a tangency condition of the (averaged) Burgers vector of the line defect forest at the inter-granular interstices. In the generalized sense adopted here, we consider a tangency condition by assuming that the multiplicity Θ_S is orthogonal to the unit normal ν_S, i.e., $\Theta_S(x) \bullet \nu_S(x) = 0$ at $\mathscr{H}^2$-a.e. $x \in S$, where $\bullet$ denotes the scalar product in $\mathbb{R}^3$. By taking $\zeta_{jA} = \delta_{jA}\,\phi(x)$, with δ_{jA} the Kronecker delta, for some $\phi \in C_c^\infty(\Omega)$, we find

$$\langle \bar{S}, \omega_\zeta^{(2)} \rangle = \int_S (\Theta_S \bullet \nu_S)\,\phi\,\mathrm{d}\mathscr{H}^2$$

and hence, in terms of currents, the *tangency condition* reads

$$\langle \bar{S}, \overline{\omega}_\phi \rangle = 0 \qquad \forall\, \phi \in C_c^\infty(\Omega) \tag{4.3}$$

where $\overline{\omega}_\phi = (\omega_{1\phi}, \omega_{2\phi}, \omega_{3\phi})$, with $\omega_{j\phi} := (-1)^{j-1}\delta_{jA}\phi(x)\widehat{\mathrm{d}x^A}$ for $j = 1, 2, 3$.

If the condition (4.3) holds, $\Theta_S(x) \bullet \nu_S(x) = 0$ at $\mathscr{H}^2$-a.e. $x \in S$.

In the smooth case, if we assume S to be a flat surface contained in the slip plane with a constant Burgers vector $\Theta_S \equiv b$, the multiplicity Θ_Γ of the boundary current $\overline{\Gamma} = \partial\overline{S}$ is tangential to the osculating plane to the dislocation loop Γ. However, for currents $\overline{S} \in [\mathscr{R}_2(\Omega)]^3$ associated with a smooth surface S with multiplicity Θ_S, in general the tangency condition (4.3) does not imply a geometric property concerning the multiplicity Θ_Γ of the dislocation loop Γ.

4.6 Plastic deformations with rectifiable dislocations

Definition 4.1 *We call* generalized slip surface *any $\mathbb{Z}^3$-valued current $\bar{S}$ in $[\mathscr{R}_2(\Omega)]^3$ satisfying the confinement condition* $\operatorname{spt}\bar{S} \subset \Omega$*, the tangency condition* (4.3)*, and such that the boundary current $\overline{\Gamma} := \partial\bar{S}$ has finite mass. The $\mathbb{Z}^3$-valued current $\overline{\Gamma} = [\mathscr{R}_1(\Omega)]^3$ is called a* rectifiable dislocation *in* Ω*, and we write $\overline{\Gamma} \in disl(\Omega)$.*

As we have seen, $\partial\overline{\Gamma} = 0$ and $\operatorname{spt}\overline{\Gamma} \subset \Omega$ for every $\overline{\Gamma} \in \operatorname{disl}(\Omega)$. In addition, since in general no energy contribution is associated to the slip surface, as a constitutive condition we require that

$$\mathbf{M}(\bar{S}) \leq c \cdot \mathbf{M}(\overline{\Gamma})^2 \tag{4.4}$$

for some fixed real constant $c > 0$. This bound holds true when $\overline{\Gamma}$ is associated with a dislocation loop Γ strictly contained in Ω and lying in a slip plane, and $\Theta_\Gamma \equiv b$ for some Burgers vector tangential to the slip plane. Choose $\overline{S}$ as the 2-current generated by the triplet (S, ξ, b), where S is the flat surface in Ω with boundary Γ, the 2-vector ξ being chosen in accordance to the orientation τ_Γ of $\overline{\Gamma}$. In this case, the inequality (4.4) holds true with c equal to square of the isoperimetric constant in $\mathbb{R}^2$.

Definition 4.2 *A tensor-valued bounded measure $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is said to be* associated with a generalized slip surface $\bar{S}$ with rectifiable dislocation $\overline{\Gamma}$, *writing $F = \hat{F}(\bar{S}, \overline{\Gamma})$, if*

$$\langle F, \zeta \rangle = \langle \bar{S}, \omega^{(2)}_\zeta \rangle \qquad \forall\, \zeta \in C^\infty_c(\Omega, \mathbb{R}^{3\times 3})$$

for some generalized slip surface $\bar{S}$ with dislocation $\overline{\Gamma} = \partial\bar{S}$ in disl(Ω).

As a consequence, we identify the plastic tensor of a dislocation:

Definition 4.3 *A tensor-valued measure $F^p \in \mathscr{M}(\Omega, \mathbb{R}^{3\times 3})$ is called a* plastic deformation tensor with generalized slip surface $\bar{S}$ and rectifiable dislocation $\overline{\Gamma}$ *if*

$$F^p = a(x) I\, \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma})$$

where $\hat{F}(\bar{S}, \overline{\Gamma})$ is associated with a generalized slip surface $\bar{S}$ with rectifiable dislocation $\overline{\Gamma}$. Moreover, $a(x)$ is a Borel function in Ω satisfying

$$C^{-1} \leq a(x) \leq C \qquad \forall\, x \in \Omega \tag{4.5}$$

for some given real constant $C > 1$.

With these assumptions, the dislocation density tensor $\operatorname{curl} F^p$ agrees with $\operatorname{curl} \hat{F}(\bar{S}, \overline{\Gamma})$ and hence we can identify it by means of the rectifiable dislocation $\overline{\Gamma}$ in $\operatorname{disl}(\Omega)$ through the formula

$$\langle \operatorname{curl} F^p, \psi \rangle = \langle \overline{\Gamma}, \omega^{(1)}_\psi \rangle \qquad \forall\, \psi \in C^\infty_c(\Omega, \mathbb{R}^{3\times 3})\ . \tag{4.6}$$

Eventually, for any given plastic deformation as above, due to the bound (4.4) we get

$$|\hat{F}(\bar{S}, \overline{\Gamma})|(\Omega) \leq c \cdot \mathbf{M}(\overline{\Gamma})^2\,, \qquad 2^{-1}\mathbf{M}(\overline{\Gamma}) \leq |\operatorname{curl} F^p|(\Omega) \leq \mathbf{M}(\overline{\Gamma})\,.$$

Weakly converging sequences of generalized slip surfaces with bounded masses preserve the tangency condition.

Proposition 4.1 *Let $\{\bar{S}_h\}_h \subset [\mathscr{R}_2(\Omega)]^3$ be a sequence of generalized slip surfaces satisfying $\cup_h \mathrm{spt}\, \bar{S}_h \subset \mathscr{K}$ for some compact set $\mathscr{K} \subset \Omega$ and*

$$\sup_h \Big(\mathbf{M}(\bar{S}_h) + \mathbf{M}(\partial \bar{S}_h) \Big) < \infty \,.$$

Then, there exists a (not relabeled) subsequence and a generalized slip surface $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ such that $\bar{S}_h \rightharpoonup \bar{S}$ weakly in $[\mathscr{D}_2(\Omega)]^3$ and $\mathrm{spt}\, \bar{S} \subset \mathscr{K}$.

Proof. Due to the validity of Federer-Fleming's compactness theorem, we only have to check that the limit current $\bar{S}$ satisfies the tangency condition. We have seen that such a geometric condition is equivalent to the identity (4.3), whereas the weak convergence $\bar{S}_h \rightharpoonup \bar{S}$ implies that $\langle \bar{S}_h, \overline{\omega}_\phi \rangle \to \langle \bar{S}, \overline{\omega}_\phi \rangle$ for every $\phi \in C^\infty_c(\Omega)$, whence property (4.3) is preserved, as required. ■

Another question to be investigated is a stability of the corresponding tangency condition concerning the deformation map $\varphi \in SBV(\Omega, \mathbb{R}^3)$, namely, that the jump of φ is tangential to the approximate tangent space to $S(\varphi)$ at $\mathscr{H}^2 \llcorner S(\varphi)$-a.e. point. To this purpose, we observe that this tangency condition is equivalent to the following property:

$$\int_{S(\varphi)} \phi\, (\varphi^+ - \varphi^-) \bullet \nu \, d\mathscr{H}^2 = 0 \qquad \forall\, \phi \in C^\infty_c(\Omega)\,.$$

Proposition 4.2 *Let $p > 1$ and $\{\varphi_h\} \subset SBV(\Omega, \mathbb{R}^3)$ satisfy*

$$\sup_h \Big(\|\varphi_h\|_\infty + \int_\Omega |\nabla \varphi_h|^p \, dx + \mathscr{H}^2(S(\varphi_h)) \Big) < \infty$$

where each φ_h satisfies the tangency condition. Then, there exists a (not relabeled) subsequence and a vector field $\varphi \in SBV(\Omega, \mathbb{R}^3)$ that satisfies the tangency condition and is such that $\varphi_h \to \varphi$ in $L^1(\Omega, \mathbb{R}^3)$, $\nabla \varphi_h \rightharpoonup \nabla \varphi$ weakly in $L^p(\Omega, \mathbb{R}^{3\times 3})$, and $(\varphi^+_h - \varphi^-_h) \otimes \nu_h \mathscr{H}^2 \llcorner S(\varphi_h)$ weakly converges in the sense of measures to $(\varphi^+ - \varphi^-) \otimes \nu \mathscr{H}^2 \llcorner S(\varphi) \otimes \nu$. More generally, the tangency condition holds true for any weak limit point φ.

Proof. As before, due to the validity of the compactness theorem in SBV, we only have to check that if a (not relabeled) subsequence of $\{\varphi_h\}$ weakly converges to φ in the BV-sense, then the tangency condition is preserved. For this purpose, we observe that the current $[\![\, \partial SG\varphi^j_h \,]\!]$ carried by the subgraph of the j-th component of φ_h weakly converges to the current $[\![\, \partial SG\varphi^j \,]\!]$ carried by the subgraph of the j-th component of φ, for $j = 1, 2, 3$. On the other hand, for each function $\phi \in C_c^\infty(\Omega)$ we get

$$(-1)^A \langle [\![\partial SG\varphi^j]\!], \mathrm{d}\phi(x)\widehat{\mathrm{d}x^A} \rangle = \int_\Omega (\partial_A \phi(x) \varphi^j(x) + \phi(x) \partial_A \varphi^j(x))\, \mathrm{d}x \\ + \int_{S(\varphi)} \phi(x)(\varphi^{j+}(x) - \varphi^{j-}(x)) \nu_A(x)\, \mathrm{d}\mathscr{H}^2$$

and a similar formula holds true for φ_h^j. As a consequence of the weak convergence $\nabla\varphi_h \rightharpoonup \nabla\varphi$ in $L^p(\Omega, \mathbb{R}^{3\times 3})$ and of the identity

$$\lim_{h\to\infty} \langle [\![\partial SG\varphi_h^j]\!], d\phi(x)\widehat{dx^A} \rangle = \langle [\![\partial SG\varphi^j]\!], d\phi(x)\widehat{dx^A} \rangle \qquad \forall\, A, j = 1, 2, 3$$

we infer

$$\lim_{h\to\infty} \int_{S(\varphi_h)} \phi\, (\varphi_h^+ - \varphi_h^-) \bullet \nu_h\, d\mathscr{H}^2 = \int_{S(\varphi)} \phi\, (\varphi^+ - \varphi^-) \bullet \nu\, d\mathscr{H}^2$$

for each $\phi \in C_c^\infty(\Omega)$, which yields the stability of the tangency condition. ■

4.8 Lack of stability of the bound

The inequality (4.4) is not preserved by the weak convergence in the sense of currents. Namely, if $\{\bar{S}_h\} \subset [\mathscr{R}_2(\Omega)]^3$ satisfies

$$\mathbf{M}(\bar{S}_h) \leq c \cdot \mathbf{M}(\overline{\Gamma}_h)^2 < \widetilde{c} < \infty \qquad \forall\, h$$

for some fixed real constants $c, \widetilde{c} > 0$, where $\partial\bar{S}_h = \overline{\Gamma}_h \in [\mathscr{R}_1(\Omega)]^3$ and $\operatorname{spt} \bar{S}_h \subset \mathscr{K}$ for each h and for some given compact set $\mathscr{K} \subset \Omega$, by Federer-Fleming's compactness theorem, possibly passing to a (not relabeled) subsequence, it turns out that $\{\bar{S}_h\}$ weakly converges to some current $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ and $\{\overline{\Gamma}_h\}$ to some current $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ such that $\partial\bar{S} = \overline{\Gamma}$ and $\operatorname{spt} \bar{S} \subset \mathscr{K}$. By lower semicontinuity of the mass, we have $\mathbf{M}(\bar{S}) < \widetilde{c}$ and $\mathbf{M}(\overline{\Gamma})^2 < \widetilde{c}$. However, in general the weak limit currents $\bar{S}$ and $\overline{\Gamma}$ fail to satisfy the inequality (4.4).

Notice that the bound (4.4) is preserved if e.g. we assume that the compact set $\mathscr{K}$ giving the confinement condition is a convex (or star-shaped) subset of a 2-dimensional affine plane of $\mathbb{R}^3$. In this case, in fact, by a cone construction it turns out that for each $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ with $\operatorname{spt} \overline{\Gamma} \subset \mathscr{K}$, there is a unique current $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ satisfying $\partial\bar{S} = \overline{\Gamma}$ and $\operatorname{spt} \bar{S} \subset \mathscr{K}$. Therefore, as we said before, the bound (4.4) holds true with c equal to the square of the isoperimetric constant in $\mathbb{R}^2$.

5 The elastic factor F^e

We thus assume that $F \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ satisfies the compatibility condition $\operatorname{curl} F = 0$ and a multiplicative decomposition $F = F^e F^p$, where $F^p \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is the plastic factor previously defined.

As to the elastic factor F^e, we assume that it is summable as a function of x, namely $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$, and is such that $\det F^e > 0$ a.e. in Ω. Since the total variation of F^p decomposes as

$$|F^p| = \sqrt{3}\, a(x) \mathscr{L}^3 + |\Theta_S \otimes \nu_S|\, \mathscr{H}^2 \llcorner S$$

it turns out that $F = D\varphi$ for some function $\varphi \in SBV(\Omega, \mathbb{R}^3)$ satisfying

$$D\varphi = \nabla\varphi\, \mathscr{L}^3 + (\varphi^+ - \varphi^-) \otimes \nu \mathscr{H}^2 \llcorner S(\varphi)$$

where the absolutely continuous and jump components are strictly related to the elastic and plastic factors, respectively.

In fact, by the assumption (4.5) we have $\nabla\varphi(x) = a(x)\, F^e(x)$ for a.e. $x \in \Omega$, where the L^1-norms of $\nabla\varphi$ and F^e are comparable, namely

$$C^{-1}\, \|F^e\|_{L^1(\Omega)} \leq \|\nabla\varphi\|_{L^1(\Omega)} \leq C\, \|F^e\|_{L^1(\Omega)}\,.$$

More specifically, we admit the possibility of plastic changes in volume (the so-called Bell's effect, after James Bell). When $a(x)$ is identically 1, we recover the traditional choice of considering just volume-preserving plastic strain.

Also, from now on, the Lebesgue spaces and pertinent norms are referred with respect to the Lebesgue measure $\mathscr{L}^3$, when not otherwise specified.

Furthermore, the jump set $S(\varphi)$ of φ agrees with the set S corresponding to the generalized slip surface $\bar{S}$, under the assumption that $|\Theta_S| > 0$ on S. Therefore, choosing the unit normal $\nu = \nu_S$, the requirement that φ jump $\varphi^+ - \varphi^- \in \mathbb{R}^3$ is tangent to the jump set $S(\varphi)$ at $\mathscr{H}^2$-a.e. point can be described in terms of the multiplicative decomposition. In fact, for $\mathscr{H}^2$-a.e. $x \in S$, both the unit vectors

$$v(x) = \frac{\Theta_S(x)}{|\Theta_S(x)|}\,, \qquad w(x) = \frac{\varphi^+(x) - \varphi^-(x)}{|\varphi^+(x) - \varphi^-(x)|}$$

lie in the approximate tangent plane to S at x. Therefore, there exists a unique rotation matrix $R(x) \in SO(3)$ with rotation axis oriented by the unit normal $\nu_S(x)$ such that $w(x) = R(x)v(x)$. We thus define F^e on $\mathscr{H}^2$-a.e. point x in the $\mathscr{L}^3$-negligible set $S = S(\varphi)$ as

$$F^e(x) = \frac{|\varphi^+(x) - \varphi^-(x)|}{|\Theta_S(x)|}\, R(x)$$

so that condition $\det F^e > 0$ is preserved and the multiplicative decomposition implies that $\mathscr{H}^2$-a.e. on S

$$(\varphi^+ - \varphi^-) \otimes \nu = F^e\,(\Theta_S \otimes \nu_S) \quad \Longrightarrow \quad \varphi^+ - \varphi^- = F^e \Theta_S\,. \tag{5.1}$$

A closure theorem in SBV

In the sequel, if $G \in \mathbb{R}^{3\times 3}$, we write $M(G)$ for the list $(G, \operatorname{cof} G, \det G) \in \mathbb{R}^{19}$, where $\operatorname{cof} G$ is the cofactor matrix. We require a summability condition on the function $M(F^e)$, actually on $M(\nabla\varphi)$. In the SBV setting, the weak L^1 convergence of the minors holds true as a consequence of the closure theorem proven in reference [16]. Here it reads as follows:

Theorem 5.1 *Let $\{u_h\}$ a sequence of functions from $SBV(\Omega, \mathbb{R}^3)$ converging in $L^1(\Omega, \mathbb{R}^3)$ to a summable function $u : \Omega \to \mathbb{R}^3$. Assume that for some real exponents $p \geq 2$, $q \geq p/(p-1)$, and $r > 1$*

$$\sup_h \Big\{ \|u_h\|_\infty + \int_\Omega \Big(|\nabla u_h|^p + |\operatorname{cof} \nabla u_h|^q + |\det \nabla u_h|^r \Big)\, dx + \mathscr{H}^2(S(u_h)) \Big\} < \infty\,.$$

Then, $u \in SBV(\Omega, \mathbb{R}^3)$, the sequence $\mathscr{H}^2 \llcorner S(u_h)$ weakly converges in Ω to a measure μ greater than $\mathscr{H}^2 \llcorner S(u)$, and $\{M(\nabla u_h)\}$ converges to $M(\nabla u)$ weakly in $L^1(\Omega, \mathbb{R}^{19})$.

In reference [40], certain weak regularity properties on φ are assumed outside the fixed dislocation loop Γ, in order to obtain the closure property in the minimization process. However, the identity (3.2) implies that the gradient $\nabla\varphi$ fails to be in L^2 around the dislocation loop Γ. The same problem would emerge in our treatment if we rely on Theorem 5.1. To avoid it, we follow a different approach, that is based on Federer-Fleming's closure theorem for the *currents G_φ carried by the graph* of maps $\varphi : \Omega \to \mathbb{R}^3$. We point out that a similar approach is followed in the second main result presented in reference [40], where the authors assume that the elastic deformation is a *Cartesian map* from V into $\mathbb{R}^3$ for each open set V contained in $\Omega \setminus \Gamma$, where Γ is the fixed dislocation loop.

Since we deal with the multiplicative decomposition $F = F^e F^p$ with general plastic deformation tensors, we shall require a bound on the mass of the boundary current ∂G_φ in terms of the total variation of the measure naturally associated with the generalized slip surface of the plastic strain.

Let $u \in L^1(\Omega, \mathbb{R}^3)$ be an $\mathscr{L}^3$-a.e. approximately differentiable map (as e.g. an SBV vector field). The map u has a Lusin representative on the subset $\widetilde{\Omega}$ of Lebesgue points pertaining to both u and ∇u, where $\mathscr{L}^3(\Omega \setminus \widetilde{\Omega}) = 0$. We say that $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ if $M(\nabla u) \in L^1(\Omega, \mathbb{R}^{19})$.

The *graph* of a map $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is defined by

$$\mathscr{G}_u := \Big\{(x,y) \in \Omega \times \mathbb{R}^3 \mid x \in \widetilde{\Omega}\,,\ y = \widetilde{u}(x)\Big\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u. It turns out that $\mathscr{G}_u$ is a 3-rectifiable set of $U = \Omega \times \mathbb{R}^3$, with $\mathscr{H}^3(\mathscr{G}_u) < \infty$. The approximate tangent 3-plane at $(x, \widetilde{u}(x))$ is generated by the vectors $\mathbf{t}_A(x) = (e_A, \partial_A u(x)) \in \mathbb{R}^{3+3}$, for $A = 1, 2, 3$, where $\partial_A u$ is the A-th column vector of the gradient matrix ∇u, and $\nabla u(x)$ is the Lebesgue value of ∇u at $x \in \widetilde{\Omega}$. Therefore, the unit 3-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \mathbf{t}_3(x)}{|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \mathbf{t}_3(x)|}$$

provides an orientation to the graph $\mathscr{G}_u$, and the current $G_u = [\![\mathscr{G}_u, \xi, 1]\!]$ carried by the graph of u is i.m. rectifiable in $\mathscr{R}_3(\Omega \times \mathbb{R}^3)$, with mass

$$\mathbf{M}(G_u) = \mathscr{H}^3(\mathscr{G}_u) = \int_\Omega \sqrt{1 + |M(\nabla u)|^2}\, d\mathscr{L}^3 < \infty\ .$$

By Stokes theorem, if u is of class C^2 we have

$$\langle \partial G_u, \eta \rangle = \langle G_u, d\eta \rangle = \int_{\mathscr{G}_u} d\eta = \int_{\partial \mathscr{G}_u} \eta = 0$$

for every 2-form $\eta \in \mathscr{D}^2(\Omega \times \mathbb{R}^3)$, which is tantamount to write the null-boundary condition

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^3 = 0\,. \tag{5.2}$$

This property holds also, by approximation, for Sobolev maps in $W^{1,3}(\Omega, \mathbb{R}^3)$. It defines the class of Cartesian maps $u \in \mathrm{cart}^1(\Omega, \mathbb{R}^3)$. However, in general, the boundary ∂G_u does not vanish and may not have finite mass.

On the other hand, if $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is such that ∂G_u has finite mass in $\Omega \times \mathbb{R}^3$, the boundary rectifiability theorem yields that $\partial G_u \in \mathscr{R}_2(\Omega \times \mathbb{R}^3)$, i.e., the boundary current is supported by a 2-rectifiable set in $\Omega \times \mathbb{R}^3$, and actually $u \in SBV(\Omega, \mathbb{R}^3)$, with

$$\mathscr{H}^2(S(u)) \leq \mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^3) < \infty\ .$$

Federer-Fleming's compactness theorem grants the *weak convergence of minors* [20, Vol. I, Sec. 3.3.2].

Theorem 5.2 *Let $\{u_h\}$ be a sequence in $\mathscr{A}^1(\Omega, \mathbb{R}^3)$, $u \in L^1(\Omega, \mathbb{R}^3)$ an a.e. approximately differentiable map, and $v \in L^1(\Omega, \mathbb{R}^{19})$. Assume that $u_h \to u$ strongly in $L^1(\Omega, \mathbb{R}^3)$ and that $M(\nabla u_h) \rightharpoonup v$ weakly in $L^1(\Omega, \mathbb{R}^{19})$. If in addition*

$$\sup_h \mathbf{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^3) < \infty \tag{5.3}$$

then $u \in \mathscr{A}^1(\Omega, \mathbb{R}^N)$ and $v(x) = M(\nabla u(x))$ for $\mathscr{L}^3$-a.e $x \in \Omega$. Moreover, we have that $G_{u_h} \rightharpoonup G_u$ weakly in $\mathscr{D}_3(\Omega \times \mathbb{R}^3)$, whence by lower semicontinuity

$$\begin{aligned} \mathbf{M}(G_u) &\le \liminf_{h\to\infty} \mathbf{M}(G_{u_h}) < \infty \\ \mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^3) &\le \liminf_{h\to\infty} \mathbf{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^3) < \infty\,. \end{aligned}$$

The boundary mass equi-boundedness (i.e., the estimate (5.3)) is automatically satisfied by sequences of Cartesian maps $\{u_h\} \subset \mathrm{cart}^1(\Omega, \mathbb{R}^3)$, i.e., those for which the condition (5.2) holds true. In this case the limit u is a Cartesian map too, since the weak convergence in terms of currents preserves condition (5.2). This is the special version of Theorem 5.2 applied in the second existence result in reference [40], with Ω replaced by the open sets $V \subset \Omega \setminus \Gamma$.

5.3 The graph boundary of the deformation

In order to apply the closure theorem 5.2, we need to ensure that the SBV deformation $\varphi : \Omega \to \mathbb{R}^3$ belongs to the class $\mathscr{A}^1(\Omega, \mathbb{R}^3)$. Now, the multiplicative decomposition $D\varphi = F^e F^p$ gives

$$\begin{aligned} \nabla\varphi(x) &= a(x)\, F^e(x) \\ \mathrm{cof}\, \nabla\varphi(x) &= a(x)^2 \mathrm{cof}\, F^e(x) \\ \det \nabla\varphi(x) &= a(x)^3 \det F^e(x) \end{aligned} \tag{5.4}$$

a.e. in Ω. Therefore, on account of the bounds (4.5), it suffices to require that

$$M(F^e) \in L^1(\Omega, \mathbb{R}^{19})\,.$$

The boundary current ∂G_φ describes 'vertical' parts in the graph of φ. The latter may represent shear bands in this setting, as they indicate fractures in the elastic-brittle case. Here, we find it physically reasonable to require that the projection on Ω falls within the 2-rectifiable set S corresponding to the singular component $\hat{F}(\bar{S},\overline{\Gamma})$ of the plastic factor F^p in the multiplicative decomposition $D\varphi = F^e F^p$. This condition generalizes the requirement in reference [40] on the summability of distributional determinant and adjoints of $\nabla\varphi$ outside a given disclination loop Γ. It is enclosed in the bound:

$$\mathbf{M}((\partial G_\varphi) \llcorner V \times \mathbb{R}^3) \leq c_1 \cdot |\hat{F}(\bar{S},\overline{\Gamma})|(V) \tag{5.5}$$

for each open set $V \subset \Omega$ and for some absolute constant $c_1 > 0$. Notice that, on account of the ansatz (4.4), and recalling that $\mathbf{M}(\overline{\Gamma}) \leq 2|\mathrm{curl}\, F^p|(\Omega)$, the latter bound implies the inequality

$$\mathbf{M}((\partial G_\varphi) \llcorner \Omega \times \mathbb{R}^3) \leq 4c_1 c \cdot |\mathrm{curl}\, F^p|(\Omega)^2 \,. \tag{5.6}$$

Proposition 5.1 *With the previous assumptions, the inequality* (5.6) *only depends on the minors* $M(F^e)$ *of the elastic factor and on the total variation of the dislocation measure* $\mathrm{curl}\, F^p$.

Proof. If $u \in \mathscr{A}^1(\Omega,\mathbb{R}^3)$, condition $\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^3) < \infty$ is equivalent to a bound for all $A, j = 1, 2, 3$ of the quantities

$$\sup\langle \partial G_u, \phi(x,y)\, \widehat{\mathrm{d}x^A}\rangle\,, \quad \sup\langle \partial G_u, \phi(x,y)\, \mathrm{d}x^A \wedge \mathrm{d}y^j\rangle\,, \quad \sup\langle \partial G_u, \phi(x,y)\, \widehat{\mathrm{d}y^j}\rangle$$

where each supremum is taken among all test functions $\phi \in C_c^\infty(\Omega\times\mathbb{R}^3)$ such that $\|\phi\|_\infty \leq 1$. For "horizontal" 2-forms $\phi(x,y)\, \widehat{\mathrm{d}x^A}$ we have

$$\langle \partial G_u, \phi(x,y)\, \widehat{\mathrm{d}x^A}\rangle = (-1)^{A-1} \int_\Omega \partial_A[\phi(x,u(x))]\, \mathrm{d}x$$

and, e.g., for "vertical" 2-forms $\phi(x,y)\, \widehat{\mathrm{d}y^j}$

$$\langle \partial G_u, \phi(x,y)\, \widehat{\mathrm{d}y^j}\rangle = (-1)^{j-1} \sum_{A=1}^{3} \int_\Omega \partial_A[\phi(x,u(x))]\, (\mathrm{adj}\ \nabla u(x))^j_A \, \mathrm{d}x \,.$$

The Laplace formulas imply

$$\begin{aligned}(-1)^{j-1} \sum_{A=1}^{3} \partial_A[\phi(x,u)](\mathrm{adj}\ \nabla u)^j_A = {} & \sum_{A=1}^{3} (-1)^{A-1} \frac{\partial\phi}{\partial x_A}(x,u)\, M^{\bar{j}}_{\bar{A}}(\nabla u) \\ & + (-1)^{j-1} \frac{\partial\phi}{\partial y_j}(x,u)\ \det\nabla u\end{aligned}$$

where $M^{\bar{j}}_{\bar{A}}(G)$ indicates the 2×2-minor of G obtained by deleting the j-th raw and A-th column (compare [20, Vol. I, Sec. 3.3.2]). Therefore, by condition (4.5) and formulas (5.4) it turns out that the left-hand side of inequality (5.6) only depends on $M(F^e)$, as required. ■

Besides imposing that the deformation φ be orientation-preserving (i.e., $\det F > 0$ a.e. on Ω), we also accept self-contact between distant portions of the boundary but avoid at the same time self-penetration of the matter. In order to guarantee this behavior, in 1987 P. Ciarlet and J. Nečas [8] proposed the introduction of the additional constraint:

$$\int_{\Omega'} \det \nabla\varphi(x)\, dx \le \mathscr{L}^3(\widetilde{\varphi}(\widetilde{\Omega}'))$$

for any sub-domain Ω' of Ω, where $\widetilde{\Omega}'$ is intersection of Ω' with the domain $\widetilde{\Omega}$ of the Lebesgue's representative $\widetilde{\varphi}$ of φ.

In 1989, M. Giaquinta, G. Modica, and J. Souček weakened such a constraint [20, Vol. II, Sec. 2.3.2]. Their version reads

$$\int_{\Omega} f(x, \varphi(x)) \, \det \nabla\varphi(x)\, dx \le \int_{\mathbb{R}^3} \sup_{x\in\Omega} f(x,y)\, dy \quad \forall\, f \in C_c^\infty(\Omega \times \mathbb{R}^3)\,, \quad f \ge 0\,. \tag{5.7}$$

We adopt it here. Again by the identities (5.4), it turns out that (5.7) is essentially a property of the elastic factor F^e. We also point out that this condition is preserved by the weak convergence $G_{\varphi_h} \rightharpoonup G_\varphi$.

6 Existence of minimizers

What we have discussed so far deals with kinematics and allows us to define a class of physically admissible competitors minimizing the energy (1.2) and related variants that can be analyzed in the same way.

6.1 The admissible class

The class $\mathscr{A} = \mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega)$ of admissible competitors depends on

- the reference body shape $\Omega \subset \mathbb{R}^3$, taken to be open, simply connected, and endowed with a surface like Lipschitz boundary $\partial\Omega$,
- positive constants M, C, c_1, with $C > 1$, and
- a compact set $\mathscr{K}$ contained in Ω related to the confinement condition.

A map $\varphi : \Omega \to \mathbb{R}^3$ belongs to the admissible class $\mathscr{A}$ of *elastic-plastic deformations with rectifiable dislocations* provided that the properties listed below hold true.

(1) $\varphi \in SBV(\Omega, \mathbb{R}^3)$ satisfies $\|\varphi\|_\infty \leq M$ for some fixed constant $M > 0$ and the tangency conditon: the jump of φ is tangential to the approximate tangent plane at $\mathscr{H}^2$-a.e. point in the slip set $S(\varphi)$.
(2) The plastic factor F^p in the multiplicative decomposition $D\varphi = F^e F^p$ belongs to $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$. It indicates the effects of slips over a generalized slip surface represented by the current $\bar{S}$ associated with a rectifiable dislocation $\overline{\Gamma}$ (see Definition 4.3).
(3) $\bar{S}$ is supported over some given compact set $\mathscr{K}$, namely $\operatorname{spt} \bar{S} \subset \mathscr{K}$, and the bounding constant $C > 1$ for the term $a(x)$ in (4.5) is fixed.
(4) The elastic factor F^e in the multiplicative decomposition belongs to $L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$ and is such that $M(F^e) \in L^1(\Omega, \mathbb{R}^{19})$ and $\det F^e > 0$ a.e. in Ω.
(5) The boundary of the graph current G_φ satisfies the mass bound (5.5) for each open set $V \subset \Omega$ and for some absolute constant $c_1 > 0$.
(6) Condition (5.7) avoiding self-penetration of matter holds.

Due to the lack of stability of the bound (4.4), as explained in Sec. 4.8, we are led to introduce the following

Definition 6.1 *Given some real constant $c > 0$, we denote by $\widetilde{\mathscr{A}}_{M,C,c_1,\mathscr{K},c}(\Omega)$ the subclass of maps φ in $\mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega)$ such that the bound (4.4) on the mass of $\bar{S}$ in terms of the mass of $\overline{\Gamma}$ holds.*

6.2 The energy functional

As already recalled in the Introduction, we consider a homogeneous material admitting an energy that is polyconvex with respect to the elastic factor F^e and includes weakly non-local effects encoded by $\operatorname{curl} F^p$. Its simplest form reads

$$\mathscr{F}_{p,s}(\varphi) := \int_\Omega \Big(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \Big)\, dx + |\operatorname{curl} F^p|(\Omega)$$

where $D\varphi = F^e F^p$ as above, while $p > 1$ and $s > 0$ are real exponents.

Essentially, the analysis we propose does not change if we replace the integrand depending on F^e with, e.g., a non-negative convex function f on $\mathbb{R}^{19}$ such that

$$f(M(G)) \geq c_2 \cdot (|M(G)|^p + |\det G|^{-s})$$

for all $G \in \mathbb{R}^{3\times 3}$, where $c_2 > 0$ is a real constant.

If $\varphi \in \mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega)$, the slip set $S(\varphi)$ is $\mathscr{H}^2$-essentially contained in the given compact subset $\mathscr{K}$ of Ω, whence the restriction of φ to the open set $\Omega \setminus \mathscr{K}$ is a Sobolev map. Therefore, if φ has bounded energy, $\mathscr{F}_{p,s}(\varphi) < \infty$, it turns out that $\varphi_{|\Omega\setminus\mathscr{K}} \in W^{1,p}(\Omega \setminus \mathscr{K}, \mathbb{R}^3)$. We thus may impose a Dirichlet-type condition by choosing a function γ in the trace space $W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ and requiring that the equality $\mathrm{Tr}(\varphi) = \gamma$ holds $\mathscr{H}^2$-a.e. in $\partial\Omega$, where $\mathrm{Tr}(\varphi)$ is the trace of φ on the boundary of Ω. We thus let

$$\begin{aligned} \mathscr{A}_\gamma &:= \{\varphi \in \mathscr{A}_{M,C,c_1,\mathscr{K}}(\Omega) \mid \mathscr{F}_{p,s}(\varphi) < \infty\,, \quad \mathrm{Tr}(\varphi) = \gamma\} \\ \widetilde{\mathscr{A}_\gamma} &:= \{\varphi \in \widetilde{\mathscr{A}}_{M,C,c_1,\mathscr{K},c}(\Omega) \mid \mathscr{F}_{p,s}(\varphi) < \infty\,, \quad \mathrm{Tr}(\varphi) = \gamma\}\,. \end{aligned}$$

The absence of dislocations amounts to the condition $|\mathrm{curl}\, F^p|(\Omega) = 0$. In this case, the bound (4.4) reduces F^p to $a(x)\, I\, \mathscr{L}^3$, so that the deformation φ is a Sobolev map in $W^{1,p}(\Omega, \mathbb{R}^3)$.

Suitable choices of the boundary term γ force the occurrence of defects in this setting when we impose constraints on the energy derivative with respect to $M(F^e)$; pertinent specific examples, expressed in the formal language adopted here, are in reference [19].

The presence of $\mathrm{curl}\, F^p$ in the energy is a way to account for geometrically necessary dislocations, which can be detected by orientation imaging microscopy [31]. Numerical simulations accounting for the energetic weight of $\mathrm{curl}\, F^p$ corroborate the interpretation (see, e.g., references [24], [17], [29]).

6.4 Existence theorem

Theorem 6.1 *Take $M, C, c_1, c > 0$, with $C > 1$, $\mathscr{K} \subset \Omega$ a compact set, and $p > 1$, $s > 0$. If for some $\gamma \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ the class $\widetilde{\mathscr{A}_\gamma}$ is non-empty, the functional $\varphi \mapsto \mathscr{F}_{p,s}(\varphi)$ attains a minimum in $\mathscr{A}_\gamma$, i.e., there exists $\varphi_0 \in \mathscr{A}_\gamma$ such that*

$$\mathscr{F}_{p,s}(\varphi_0) = \inf\{\mathscr{F}_{p,s}(\varphi) \mid \varphi \in \widetilde{\mathscr{A}_\gamma}\}\,.$$

Proof. We shall repeatedly extract not relabeled subsequences. Let $\{\varphi_h\} \subset \widetilde{\mathscr{A}_\gamma}$ be a minimizing sequence. Write $D\varphi_h = F^e_h F^p_h$, where

$$F^p_h = a_h(x) I\, \mathscr{L}^3 + \hat{F}(\bar{S}_h, \overline{\Gamma}_h)\,.$$

Then, $\{\overline{\Gamma}_h\} \subset [\mathscr{R}_1(\Omega)]^3$ with $\mathrm{spt}\, \overline{\Gamma}_h \subset \mathscr{K}$, $\partial\overline{\Gamma}_h = 0$, and $\mathbf{M}(\overline{\Gamma}_h) \leq 2\, |\mathrm{curl}\, F^p_h|(\Omega)$ for each h. Moreover, $\{\bar{S}_h\} \subset [\mathscr{R}_2(\Omega)]^3$ with $\mathrm{spt}\, \bar{S}_h \subset \mathscr{K}$, $\partial\bar{S}_h = \overline{\Gamma}_h$, and by the bound (4.4) we get the estimate

$$\mathbf{M}(\bar{S}_h) = |\hat{F}(\bar{S}_h, \overline{\Gamma}_h)|(\Omega) \leq 4c \cdot |\mathrm{curl}\, F_h^p|(\Omega)^2 \qquad \forall\, h\,.$$

Therefore, by Federer-Fleming's compactness theorem we find $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ and $\overline{\Gamma} \in [\mathscr{R}_1(\Omega)]^3$ such that $\operatorname{spt} \bar{S}, \operatorname{spt} \overline{\Gamma} \subset \mathscr{K}$, $\partial \bar{S} = \overline{\Gamma}$, $\partial \overline{\Gamma} = 0$, and a subsequence such that $\bar{S}_h \rightharpoonup \bar{S}$ weakly in $[\mathscr{D}_2(\Omega)]^3$ and $\overline{\Gamma}_h \rightharpoonup \overline{\Gamma}$ weakly in $[\mathscr{D}_1(\Omega)]^3$. Proposition 4.1 implies that the weak limit current $\bar{S}$ satisfies the tangency condition, whence it is a generalized slip surface in our sense.

By (4.5) we may and do assume the existence of a function $a \in L^\infty(\Omega)$ such that $a_h \rightharpoonup a$ weakly in $L^\infty(\Omega)$ and a.e., with $a(x)$ satisfying (4.5). Setting then

$$F^p = a(x)\, I\, \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma}) \tag{6.1}$$

so that $\mathrm{curl}\, \hat{F}(\bar{S}, \overline{\Gamma}) = \mathrm{curl}\, F^p$, we have $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$ and $\mathrm{curl}\, F_h^p \rightharpoonup \mathrm{curl}\, F^p$ weakly as measures in $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$, where the dislocation measure $\mathrm{curl}\, F^p$ is associated to the current $\overline{\Gamma} \in \mathrm{disl}(\Omega)$, and by lower semicontinuity of the total variation

$$|\mathrm{curl}\, F^p|(\Omega) \leq \liminf_{h\to\infty} |\mathrm{curl}\, F_h^p|(\Omega)\,.$$

Moreover, since (5.4) holds true for each h by the multiplicative decomposition, by condition (4.5) and the lower bound $\mathscr{F}_{p,s}(\varphi_h) \geq \|M(\nabla\varphi_h)\|^p_{L^p(\Omega,\mathbb{R}^{19})}$ we get

$$\sup_h \int_\Omega |M(\nabla\varphi_h)|^p\, dx < \infty\,.$$

Since the bounds (4.4) and (5.5) imply the inequality (5.6), we obtain

$$\sup_h \Big(\mathbf{M}(G_{\varphi_h}) + \mathbf{M}((\partial G_{\varphi_h}) \llcorner \Omega \times \mathbb{R}^3) \Big) < \infty\,.$$

Therefore, we can apply Theorem 5.2, which states that, possibly passing to a subsequence, $\varphi_h \to \varphi$ strongly in $L^1(\Omega, \mathbb{R}^3)$ and $M(\nabla\varphi_h) \rightharpoonup M(\nabla\varphi)$ weakly in $L^1(\Omega, \mathbb{R}^{19})$ for some function $\varphi \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ satisfying $\|\varphi\|_\infty \leq M$ and, by lower semicontinuity of the mass with respect to the weak convergence in terms of currents, $\mathbf{M}(G_\varphi) + \mathbf{M}((\partial G_\varphi) \llcorner \Omega \times \mathbb{R}^3) < \infty$.

We thus infer that $\varphi \in SBV(\Omega, \mathbb{R}^3)$. Moreover, since by the multiplicative decomposition $S(\varphi_h) = S_h$ for each h, we get $\sup_h \mathscr{H}^2(S(\varphi_h)) < \infty$ and we can apply Proposition 4.2 to infer that the limit vector field φ satisfies the tangency condition, too.

Since, passing to a subsequence, $a_h(x) \to a(x)$ a.e. in Ω, $M(\nabla\varphi_h) \rightharpoonup M(\nabla\varphi)$ weakly in $L^p(\Omega, \mathbb{R}^{3\times 3})$ and a.e. in Ω, and $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$ as measures, whereas $\varphi_h \to \varphi$ in $L^1(\Omega, \mathbb{R}^3)$, by using the multiplicative decomposition $D\varphi_h = F_h^e F_h^p$ for each h we deduce that the limit deformation φ satisfies itself the multiplicative decomposition $D\varphi = F^e F^p$, where the plastic factor F^p is given by (6.1) and the elastic factor is defined by $F^e(x) := a(x)^{-1}\nabla\varphi(x)$ for a.e. $x \in \Omega$.

Therefore, $M(F_h^e) \rightharpoonup M(F^e)$ weakly in $L^p(\Omega, \mathbb{R}^{19})$ and $F_h^e \to F^e$ a.e. in Ω, so that $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3})$, whereas condition $\sup_h \int_\Omega |\det F_h^e|^{-s}\,\mathrm{d}x < \infty$ implies, by lower semicontinuity, that $\int_\Omega |\det F^e|^{-s}\,\mathrm{d}x < \infty$ and hence that $\det F^e > 0$ a.e. in Ω. In particular, by using the notation (S, ξ_S, Θ_S) for $\bar{S} \in [\mathscr{R}_2(\Omega)]^3$ and assuming that $\Theta_S \in \mathbb{Z}^3 \setminus \{0_{\mathbb{R}^3}\}$ in S, we infer that $\mathscr{H}^2$-essentially $S(\varphi) = S$. We thus have $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$, and by the tangency conditions of both $\bar{S}$ and φ we infer that the relation (5.1) holds true.

The weak convergence of $\varphi_{h|\Omega\setminus\mathscr{K}}$ to $\varphi_{|\Omega\setminus\mathscr{K}}$ in $W^{1,p}(\Omega \setminus \mathscr{K}, \mathbb{R}^3)$ implies the $\mathscr{H}^2$-a.e. convergence $\mathrm{Tr}(\varphi_h) \to \mathrm{Tr}(\varphi)$ of the traces in $\partial\Omega$, whence the limit deformation φ satisfies the prescribed Dirichlet-type condition $\mathrm{Tr}(\varphi) = \gamma$.

Since $G_{\varphi_h} \rightharpoonup G_\varphi$ weakly in $\mathscr{D}_3(\Omega \times \mathbb{R}^3)$, we also infer that the deformation φ satisfies condition (5.7); thus it avoids self-penetration of matter. Moreover, the inequality (5.5) is stable with respect to the weak convergences $G_{\varphi_h} \rightharpoonup G_\varphi$ and $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$, whence it is satisfied by the weak limit current G_φ and measure $\hat{F}(\bar{S}, \overline{\Gamma})$. Therefore, conditions (1)–(6) are satisfied and actually $\varphi \in \mathscr{A}_\gamma$. As already remarked in Sec. 4.8, due to the lack of stability of the bound (4.4) we cannot in general conclude that $\varphi \in \widetilde{\mathscr{A}_\gamma}$.

Since $M(F_h^e) \rightharpoonup M(F^e)$ weakly in $L^1(\Omega, \mathbb{R}^{3\times 3})$ and $\operatorname{curl} F_h^p \rightharpoonup \operatorname{curl} F^p$ weakly as measures, by lower semicontinuity we get

$$\mathscr{F}_{p,s}(\varphi) \leq \liminf_{h\to\infty} \mathscr{F}_{p,s}(\varphi_h)$$

which concludes the proof. ■

7 A more general class of dislocation-type defects

The assumption that the Burgers vectors take value on $\mathbb{Z}^3$ is plausibile from a microscopic point of view. In a continuous theory, i.e., at a macroscopic level, one should replace that condition with a requirement that the $\mathbb{R}^3$-valued Burgers vector b satisfies $|b| \geq c > 0$, where the physical constant c depends on the body microstructure. In a single periodic crystal c is the atomic spacing while for polycrystals it is also related to the grain size (imagine the relative slip of grains in contact); for amorphous materials under loading programs that determine plastic strain through slips, values and meaning of c depend on the material microstructure.

However, such a lower bound cannot be preserved in the minimization process. We thus choose here to work with $\mathbb{R}^3$*-valued size bounded currents*. After introducing the necessary material, an existence theorem similar to Theorem 6.1 is readily proved, provided that a term involving the *size* of the dislocation current is added to the energy functional previously considered (see (1.3)).

For example, in the case of a finite number N of pairwise disjoint dislocation loops Γ_h, see (4.1), the size $\mathbf{S}(\overline{\Gamma})$ of the dislocation current $\overline{\Gamma}$ is the total length

$$\mathbf{S}(\overline{\Gamma}) = \sum_{h=1}^{N} \mathscr{H}^1(\Gamma_h)$$

independently of the Burgers vectors $b_h \in \mathbb{R}^3$.

7.1 Size bounded currents

Let $U \subset \mathbb{R}^n$ an open set and T a rectifiable current in $\mathscr{D}_k(U)$ with finite mass, say $T = [\![\mathscr{M}, \xi, \theta]\!]$. We denote $\operatorname{set}(T)$ the set of points in $\mathscr{M}$ where the k-dimensional density of the measure $\|T\| := \theta\, \mathscr{H}^k \llcorner \mathscr{M}$ is positive, and *size* of T the number $\mathbf{S}(T) := \mathscr{H}^k(\operatorname{set}(T))$.

We say that a rectifiable current T is a *size bounded* one if $\mathbf{S}(T) < \infty$. We indicate by $\mathscr{S}_k(U)$ the corresponding class of size bounded currents.

$T \in \mathscr{S}_k(U)$ implies that $\operatorname{set}(T)$ agrees $\mathscr{H}^k$-essentially with the set of points in $\mathscr{M}$ with positive multiplicity θ. Therefore, an i.m. rectifiable current $T \in \mathscr{R}_k(U)$ is automatically size bounded because $\mathbf{S}(T) \leq \mathbf{M}(T)$, a property that fails to hold in general for currents with real multiplicity.

We also denote by $\mathscr{N}_k(U)$ the class of *normal currents*, those $T \in \mathscr{D}_k(U)$ such that $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner U) < \infty$.

For $T_1, T_2 \in \mathscr{N}_k(U)$ we define a *flat distance* $d(T_1, T_2)$ by

$$d(T_1, T_2) := \inf\{\mathbf{M}(Q) + \mathbf{M}(R) \mid Q \in \mathscr{N}_k(U)\,, \quad R \in \mathscr{N}_{k+1}(U) \\ T_1 - T_2 = Q + \partial R\}\,,$$

where the term R does not appear in the case of top dimension $k = n$.

In general, the flat convergence $d(T_h, T) \to 0$ of normal currents in $\mathscr{N}_k(U)$ implies the weak convergence $T_h \rightharpoonup T$ in $\mathscr{D}_k(U)$. The reverse implication holds true provided that U is a smooth and bounded domain (see reference [49]).

By adapting a result due to Almgren [2, Prop. 2.10], and slightly weakening some assumptions adopted there, a lower semicontinuity result holds true (see the proof in reference [39]).

Theorem 7.1 *Let* $\{T_h\}$, $T \subset \mathscr{S}_k(U) \cap \mathscr{N}_k(U)$ *be such that* $\sup_h \mathbf{N}(T_h) < \infty$ *and* $d(T_h, T) \to 0$. *Then,* $\mathbf{S}(T) \leq \liminf_{h\to\infty} \mathbf{S}(T_h)$.

Also, a closure theorem is valid, as proven in reference [4, Thm. 8.5] referring to the more general setting of currents in metric spaces.

Theorem 7.2 *Let* $\{T_h\} \subset \mathscr{S}_k(U) \cap \mathscr{N}_k(U)$ *be such that*

$$\sup_h (\mathbf{S}(T_h) + \mathbf{N}(T_h)) < \infty .$$

Then, there exists a current $T \in \mathscr{S}_k(U) \cap \mathscr{N}_k(U)$ *and a (not relabeled) subsequence such that* $d(T_h, T) \to 0$.

7.2 $\mathbb{R}^m$*-valued size bounded currents*

In a similar way to the class $[\mathscr{R}_k(U)]^m$, $m \in \mathbb{N}^+$, an $\mathbb{R}^m$*-valued rectifiable* k*-current* $\overline{T}$ in U is defined by a triplet $(\mathscr{M}, \xi, \Theta)$, where $\mathscr{M}$ and ξ are given as above, but $\Theta : \mathscr{M} \to \mathbb{R}^m$ is an $\mathbb{R}^m$-valued $\mathscr{H}^k \llcorner \mathscr{M}$-summable multiplicity function.

Correspondingly, we denote by $\text{set}\,(\overline{T})$ the set of points in $\mathscr{M}$ where the k-dimensional density of the measure $\|\overline{T}\| := |\Theta|\, \mathscr{H}^k \llcorner \mathscr{M}$ is positive, and define $\mathbf{S}(\overline{T}) := \mathscr{H}^k(\text{set}\,(\overline{T}))$.

We call $\overline{T}$ an $\mathbb{R}^m$*-valued size bounded current*, formally writing $\overline{T} \in [\mathscr{S}_k(U)]^m$, when $\mathbf{S}(\overline{T}) < \infty$.

As for the class $[\mathscr{R}_k(U)]^m$, a current $\overline{T} \in [\mathscr{S}_k(U)]^m$ can be seen as an ordered m-tuple $\overline{T} = (T^1, \ldots, T^m)$ of size bounded currents $T^j \in \mathscr{S}_k(U)$, where $\text{set}\,(\overline{T}) = \cup_{j=1}^m \text{set}\,(T^j)$. We also define $\mathbf{N}(\overline{T}) := \mathbf{M}(\overline{T}) + \mathbf{M}((\partial\overline{T}) \llcorner U)$ where, we recall, $\mathbf{M}(\overline{T}) := \sum_{j=1}^m \mathbf{M}(T^j) < \infty$ and $\mathbf{M}((\partial\overline{T}) \llcorner U) := \sum_{j=1}^m \mathbf{M}((\partial T^j) \llcorner U)$ if $\overline{T} = (T^1, \ldots, T^m)$.

Moreover, if $T^j \in \mathscr{S}_k(U)$ for $j = 1, \ldots, m$, we find a current $\overline{T} \in [\mathscr{S}_k(U)]^m$ with components $\overline{T} = (T^1, \ldots, T^m)$. Therefore, if U is a smooth and bounded domain, and a sequence $\{\overline{T}_h\} \subset [\mathscr{S}_k(U)]^m$ satisfies $\sup_h(\mathbf{S}(\overline{T}_h) + \mathbf{N}(\overline{T}_h)) < \infty$, by the compactness and semicontinuity results previously stated we can find a current $\overline{T} \in [\mathscr{S}_k(U)]^m$ and a (not relabeled) subsequence of $\{\overline{T}_h\}$ such that $\overline{T}_h \rightharpoonup \overline{T}$ and also

$$\mathbf{N}(\overline{T}) \leq \liminf_{h\to\infty} \mathbf{N}(\overline{T}_h) , \quad \mathbf{S}(\overline{T}) \leq \liminf_{h\to\infty} \mathbf{S}(\overline{T}_h) .$$

Set $n = m = 3$ and $U = \Omega$.

Definition 7.1 *We call a* generalized slip surface *any* $\mathbb{R}^3$*-valued size bounded current* $\bar{S} \in [\mathscr{S}_2(\Omega)]^3$ *satisfying the confinement condition* $\operatorname{spt} \bar{S} \subset \Omega$*, the tangency condition* (4.3)*, and such that the boundary current* $\overline{\Gamma} := \partial \bar{S}$ *is an* $\mathbb{R}^3$*-valued size bounded current in* $[\mathscr{S}_1(\Omega)]^3$*. The current* $\overline{\Gamma}$ *is called a* size bounded dislocation *in* Ω*, and we write* $\overline{\Gamma} \in \mathrm{s} - \mathrm{disl}(\Omega)$*.*

As before, $\partial\overline{\Gamma} = 0$ and $\operatorname{spt} \overline{\Gamma} \subset \Omega$ for every $\overline{\Gamma} \in \mathrm{s} - \mathrm{disl}(\Omega)$. In addition, as a constitutive condition we require that the bound (4.4) holds and also

$$\mathbf{S}(\bar{S}) \leq c \cdot \mathbf{S}(\overline{\Gamma})^2 \tag{7.1}$$

for some fixed real constant $c > 0$.

Moreover, we see that the tangency condition (4.3) is preserved in the minimization process.

Proposition 7.1 *Let* $\{\bar{S}_h\}_h \subset [\mathscr{S}_2(\Omega)]^3$ *be a sequence of generalized slip surfaces satisfying*

$$\sup_h \Big(\mathbf{N}(\bar{S}_h) + \mathbf{S}(\overline{S}_h) \Big) < \infty \,.$$

Then, there exists a (not relabeled) subsequence and a generalized slip surface $\bar{S} \in [\mathscr{S}_2(\Omega)]^3$ *such that* $\bar{S}_h \rightharpoonup \bar{S}$ *weakly in* $[\mathscr{D}_2(\Omega)]^3$*.*

Proof. On account of the compactness theorem for size bounded currents, we argue exactly as in the proof of Proposition 4.1. ■

Definition 7.2 *A tensor-valued measure* $F^p \in \mathscr{M}(\Omega, \mathbb{R}^{3\times 3})$ *is called a* plastic deformation factor with generalized slip surface $\bar{S}$ and size bounded dislocation $\overline{\Gamma}$ *if*

$$F^p = a(x) I \, \mathscr{L}^3 + \hat{F}(\bar{S}, \overline{\Gamma})$$

where $a(x)$ *is a Borel function in* Ω *satisfying* (4.5) *for some given real constant* $C > 1$*, and*

$$\langle \hat{F}(\bar{S}, \overline{\Gamma}), \zeta \rangle = \langle \bar{S}, \omega_\zeta^{(2)} \rangle \qquad \forall \, \zeta \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$$

for some generalized slip surface $\bar{S}$ *with dislocation* $\overline{\Gamma} = \partial \bar{S}$ *in* $\mathrm{s} - \mathrm{disl}(\Omega)$*.*

Therefore, this time $\operatorname{curl} F^p$ is identified by the size bounded dislocation $\overline{\Gamma}$ in $\mathrm{s} - \mathrm{disl}(\Omega)$ through the formula (4.6).

7.4 Elastic-plastic deformations with size bounded dislocations

Similarly as above, the admissible class $\mathscr{A}^{\mathfrak{s}} = \mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K}}(\Omega)$ of *elastic-plastic deformations with size bounded dislocations* is defined by the maps $\varphi : \Omega \to \mathbb{R}^3$ satisfying the properties (1), (3), (4), (5), and (6) listed in the previous section, but with property (2) replaced by

(2') $F^p \in \mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$ is a plastic deformation factor with generalized slip surface $\bar{S}$ and size bounded dislocation $\overline{\Gamma}$ (see Definition 7.2).

Also, due to the lack of stability of the bounds (4.4) and (7.1) we introduce the following

Definition 7.3 *Given some real constant $c > 0$, we denote by $\widetilde{\mathscr{A}}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K},c}(\Omega)$ the subclass of maps φ in $\mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K}}(\Omega)$ such that both the bounds* (4.4) *and* (7.1) *hold.*

7.5 The energy functional

In order to apply the closure theorem for size bounded currents, this time the energy functional must contain an extra term. Namely, we may define

$$\widetilde{\mathscr{F}}_{p,s}(\varphi) := \int_\Omega \Big(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \Big)\, dx + |\mathrm{curl}\, F^p|(\Omega) + \mathbf{S}(\overline{\Gamma})$$

for some real exponents $p > 1$ and $s > 0$, where in the first term we can take more general integrands as already mentioned above.

7.6 Existence result

As before, we impose a Dirichlet-type condition by choosing a function γ in $W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ and letting

$$\begin{aligned} \mathscr{A}^{\mathfrak{s}}_\gamma &:= \{\varphi \in \mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K}}(\Omega) \mid \widetilde{\mathscr{F}}_{p,s}(\varphi) < \infty\,, \quad \mathrm{Tr}(\varphi) = \gamma\} \\ \widetilde{\mathscr{A}}^{\mathfrak{s}}_\gamma &:= \{\varphi \in \widetilde{\mathscr{A}}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K},c}(\Omega) \mid \widetilde{\mathscr{F}}_{p,s}(\varphi) < \infty\,, \quad \mathrm{Tr}(\varphi) = \gamma\}\,. \end{aligned}$$

The following existence result is proved in a way similar to the one adopted for Theorem 6.1.

Theorem 7.3 *Take $M, C, c_1, c > 0$, with $C > 1$, $\mathscr{K} \subset \Omega$ a compact set, and $p > 1$, $s > 0$. If for some $\gamma \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ the class $\widetilde{\mathscr{A}}^{\mathfrak{s}}_\gamma$ is non-empty, the* *functional* $\varphi \mapsto \widetilde{\mathscr{F}}_{p,s}(\varphi)$ *attains a minimum in* $\widetilde{\mathscr{A}_\gamma^{\mathfrak{s}}}$*, i.e., there exists* $\varphi_0 \in \widetilde{\mathscr{A}_\gamma^{\mathfrak{s}}}$ *such that*

$$\widetilde{\mathscr{F}}_{p,s}(\varphi_0) = \inf\{\widetilde{\mathscr{F}}_{p,s}(\varphi) \mid \varphi \in \widetilde{\mathscr{A}_\gamma^{\mathfrak{s}}}\}\,.$$

Proof. For $\{\varphi_h\} \subset \widetilde{\mathscr{A}_\gamma^{\mathfrak{s}}}$ a minimizing sequence, write $D\varphi_h = F_h^e F_h^p$, where $F_h^p = a_h(x) I\,\mathscr{L}^3 + \hat{F}(\bar{S}_h, \overline{\Gamma}_h)$. Then, $\overline{\Gamma}_h \in [\mathscr{S}_1(\Omega)]^3$, $\operatorname{spt}\overline{\Gamma}_h \subset \mathscr{K}$, $\partial\overline{\Gamma}_h = 0$, and $\mathbf{M}(\overline{\Gamma}_h) \leq 2\,|\operatorname{curl} F_h^p|(\Omega)$ for each h. Moreover, $\bar{S}_h \in [\mathscr{S}_2(\Omega)]^3$ with $\operatorname{spt}\bar{S}_h \subset \mathscr{K}$, $\partial\bar{S}_h = \overline{\Gamma}_h$, and by the bounds (4.4) and (7.1)

$$\mathbf{M}(\bar{S}_h) \leq 4c\cdot|\operatorname{curl} F_h^p|(\Omega)^2\,, \quad \mathbf{S}(\bar{S}_h) \leq c\cdot\mathbf{S}(\overline{\Gamma}_h)^2 \qquad \forall\, h\,.$$

Therefore, by the compactness theorem on size bounded currents we find $\bar{S} \in [\mathscr{S}_2(\Omega)]^3$ and $\overline{\Gamma} \in [\mathscr{S}_1(\Omega)]^3$ such that $\operatorname{spt}\bar{S}, \operatorname{spt}\overline{\Gamma} \subset \mathscr{K}$, $\partial\bar{S} = \overline{\Gamma}$, $\partial\overline{\Gamma} = 0$, and a subsequence such that $\bar{S}_h \rightharpoonup \bar{S}$ weakly in $[\mathscr{D}_2(\Omega)]^3$ and $\overline{\Gamma}_h \rightharpoonup \overline{\Gamma}$ weakly in $[\mathscr{D}_1(\Omega)]^3$. By Proposition 7.1 the weak limit current $\bar{S}$ satisfies the tangency condition, whence it is a generalized slip surface in our sense.

As a consequence of the bounds (4.5), we find again a function $a \in L^\infty(\Omega)$ such that $a_h \rightharpoonup a$ weakly in $L^\infty(\Omega)$ and a.e., with $a(x)$ satisfying (4.5). Then, by setting F^p as in (6.1), so that $\operatorname{curl}\hat{F}(\bar{S}, \overline{\Gamma}) = \operatorname{curl} F^p$, we have $\hat{F}(\bar{S}_h, \overline{\Gamma}_h) \rightharpoonup \hat{F}(\bar{S}, \overline{\Gamma})$ and $\operatorname{curl} F_h^p \rightharpoonup \operatorname{curl} F^p$ weakly as measures in $\mathscr{M}_b(\Omega, \mathbb{R}^{3\times 3})$, where the dislocation measure $\operatorname{curl} F^p$ is associated with the current $\overline{\Gamma} \in \mathrm{s}-\mathrm{disl}(\Omega)$, and by lower semicontinuity of the total variation and size

$$|\operatorname{curl} F^p|(\Omega) \leq \liminf_{h\to\infty} |\operatorname{curl} F_h^p|(\Omega)\,, \quad \mathbf{S}(\overline{\Gamma}) \leq \liminf_{h\to\infty} \mathbf{S}(\overline{\Gamma}_h)\,.$$

By following steps in the proof of Theorem 6.1, we consequently infer that $\varphi \in \mathscr{A}_\gamma^{\mathfrak{s}}$. Finally, since $M(F_h^e) \rightharpoonup M(F^e)$ weakly in $L^1(\Omega, \mathbb{R}^{3\times 3})$, by lower semicontinuity we get

$$\widetilde{\mathscr{F}}_{p,s}(\varphi) \leq \liminf_{h\to\infty} \widetilde{\mathscr{F}}_{p,s}(\varphi_h)$$

which concludes the proof. ■

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