GEOMETRICALLY CONSTRAINED WALLS IN TWO DIMENSIONS

M. MORINI & V. SLASTIKOV

Abstract. We address the effect of extreme geometry on a non-convex variational problem, motivated by studies on magnetic domain walls trapped by thin necks. The recent analytical results of [15] revealed a variety of magnetic structures in three-dimensional ferromagnets depending on the size of the constriction. The main purpose of this paper is to study geometrically constrained walls in two dimensions. The analysis turns out to be significantly more challenging and requires the use of different techniques. In particular, the purely variational point of view of [15] cannot be adopted in the present setting and is here replaced by a PDE approach.

Existence of local minimizers representing geometrically constrained walls is proven under suitable symmetry assumptions on the domains and an asymptotic characterization of the wall profile is given. The limiting behavior, which depends critically on the scaling of length and height of the neck, turns out to be more complex than in the higher-dimensional case and a richer variety of regimes is shown to exist.

1. Introduction

The interest towards *geometrically constrained walls* is motivated by studies of magnetoresistance properties of thin films and multilayers with magnetic point contacts or pinholes (see [2, 9, 13, 14, 16, 21]) and related applications in magnetic storage devices.

It was first noted by Bruno [2] that for a dumbbel-shaped uniaxial ferromagnet with a very small constriction (see Figure 1), the thin neck will be the preferred location for a *domain wall*–

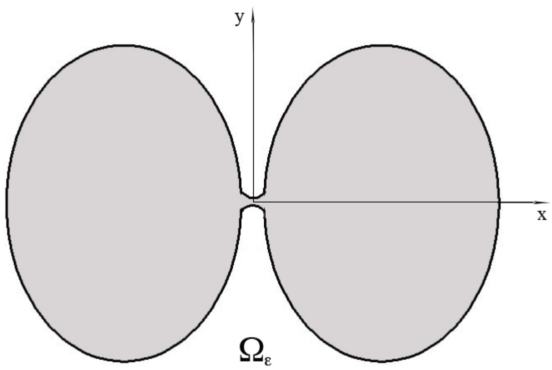

Figure 1. A dumbbell shaped domain with a small neck.

a transition layer between two regions of (almost) constant magnetization. More precisely, he considers a thin ferromagnetic film with cross section along the xy-plane given by a domain as in Figure 1. The ferromagnet is uniaxial with preferred directions $m = (0, \pm 1, 0)$ and, for the sake of simplification, the magnetization m is allowed to vary only in the yz-plane (this assumption corresponds to the case of a film with perpendicular anisotropy); i.e.

$$m = (0, \cos u, \sin u),$$

with u representing the angle between m and the y-axis. Assuming in addition that the magnetostatic interaction can be ignored (these modeling hypotheses are customary in the analysis of

Bloch walls), stable magnetic structures will be described by the local minimizers of a non-convex energy of the form

$$F_\varepsilon(u) := \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2\, dxdy + \int_{\Omega_\varepsilon} W(u)\, dxdy\,, \tag{1.1}$$

where $\Omega_\varepsilon \subset \mathbb{R}^2$ is the region in Figure 1 and W is a double-well potential. Here, ε is a small scale parameter related to the size of the neck, as it will be specified later. Bruno's essential observation is that when the size of the constriction is sufficiently small, the impact of the geometry of the neck on the structure of the wall profile becomes dominant and produces a limiting behavior that is independent of the material parameters (whence the name of *geometrically constrained* walls). However, his analysis is largely formal and based on the ansatz that the wall profile is one-dimensional; i.e., $u = u(x)$ so that (1.1) reduces to a 1D variational problem. This last assumption is really restrictive and subsequent formal analysis and simulations [16] confirmed the existence of two and three-dimensional profiles for geometrically constrained walls in thin films and thin wires, respectively.

A rigorous mathematical study of geometrically constrained walls has been eventually undertaken in [15], but only in the three-dimensional case, that is, assuming that Ω_ε is a rotationally symmetric domain obtained by rotating a two-dimensional region as in Figure 1 about the x-axis. In that paper, the authors construct a suitable family $\{u_\varepsilon\}$ of non-constant local minimizers of the reduced micromagnetic energy (1.1) and then investigate their asymptotic behavior using variational methods. The analysis in [15] reveals a variety of magnetic structures depending on the size of the constriction and shows, in particular, that Bruno's 1D ansatz is correct only in one specific regime.

The main goal of this work is to analytically investigate the original two-dimensional problem considered by Bruno. We confirm some conjectures about wall profiles stated in [15] but we also show that planar geometrically constrained walls have a more complex behavior than three dimensional ones, and display a richer variety of regimes. Moreover, the mathematical analysis turns out to be more challenging and requires the use of different techniques.

Before describing our result in more detail, we mention that there exists an extensive mathematical literature devoted to the study of nonlinear partial differential equations with different types of boundary conditions in singularly perturbed domains (see for instance [8, 10, 11, 12, 4, 19]). We refer to the introduction of [15] for a brief description of some of these papers. Here, we mention the work by Jimbo, which is closer in spirit to the analysis performed in our paper: In the series of articles [10, 11, 12], using PDE methods, he studies the asymptotic bahavior of the solutions to certain semilinear elliptic equations in dumbbell-shaped domains, in the case of a neck having fixed length and shrinking in radial (or vertical) direction. However, the situation we consider here (a neck shrinking in both directions) is significantly more complicated and requires a more intricate analysis. In some sense, the case considered by Jimbo is similar in our notation to the regime of a subcritical thin neck (see below), where the limiting behavior is one-dimensional.

We now describe our results in more detail, referring to the next sections for the precise statements. We assume that the varying domains Ω_ε are symmetric with respect to the y-axis and are given by the disjoint union of two bulk domains (whose shape is fixed) with a small neck N_ε shrinking both in the horizontal and the vertical direction as $\varepsilon \to 0^+$. More precisely,

$$N_\varepsilon = \{(\varepsilon x, \delta y) : \, (x, y) \in N\}\,, \tag{1.2}$$

where N is the unscaled neck given by

$$N = \{(x, y) : \, x \in (-1, 1)\,,\ -f_2(x) < y < f_1(x)\}\,,$$

for some positive even functions f_1 and f_2. (The regularity assumptions on f_1, f_2 and on the bulk domains are not relevant for the present discussion and will be specified in the next sections of the paper.) We also assume, without loss of generality, the normalization condition $f_1(1) = f_2(1) = \frac{1}{2}$, so that the opening of the unscaled neck is 1. Finally, we take $W(u)$ to be an even double-well potential with wells located at -1 and 1, a model example being $W(u) = (u^2 - 1)^2$.

We address the two following main issues:

(i) existence of non-constant local minimizers representing geometrically constrained walls;
(ii) description of their asymptotic profile, depending on the geometry of the neck.

The proof of the existence is based on the observation that if $\{u_\varepsilon\}$ is any family of non-constant critical points such that u_ε approaches 1 on the right-hand bulk part of Ω_ε and -1 on the left-hand bulk part of Ω_ε as $\varepsilon \to 0^+$, then u_ε is in fact a local minimizer for ε sufficiently small. Indeed, by a careful estimate of the Poincaré constants in the singularly perturbed domains Ω_ε (Lemma 2.2), we can show that the second variation of the energy becomes positive definite for ε small enough (Proposition 2.3). The construction of the family $\{u_\varepsilon\}$ of critical points with the desired properties is done through a minimization procedure and uses the symmetry of Ω_ε (Proposition 2.1 and the definition in formula (2.7)). We finally remark that the non-convexity of Ω_ε is necessary to the existence of stable non-constant solutions, due to the result proved in [3].

Concerning the second issue, the main feature of geometrically constrained walls is their strong dependence on the geometry of the neck. There are two major length scales describing the neck: δ, corresponding to the height of the neck, and ε, corresponding to its length (see (1.2)). Throughout the paper, we regard δ as an ε-dependent parameter; i.e., $\delta = \delta(\varepsilon)$. Depending on how δ scales with respect to ε as $\varepsilon \to 0^+$, we can identify several regimes that correspond to different asymptotic behaviors of the wall profiles:

(1) the *thin neck* regime, corresponding to $\delta/\varepsilon \to 0$; this further subdivides into three sub-regimes:
   (a) the *subcritical thin neck* regime, corresponding to $\delta|\ln\delta|/\varepsilon \to 0$;
   (b) the *critical thin neck* regime, corresponding to $\delta|\ln\delta|/\varepsilon \to \ell$, with $\ell \in (0, +\infty)$;
   (c) the *supercritical thin neck* regime, corresponding to $\delta|\ln\delta|/\varepsilon \to \infty$;
(2) the *normal neck* regime, corresponding to $\delta/\varepsilon \to \ell$, with $\ell \in (0, +\infty)$;
(3) the *thick neck* regime, corresponding to $\delta/\varepsilon \to \infty$.

We point out here an important difference to the higher dimensional case considered in [15], where the asymptotic behavior depends only on the limit of δ/ε and thus only three regimes (thin, normal, and thick neck) are observed. Hence, in two-dimensions we have a richer variety of regimes that we describe below in more detail.

To better explain the difference between our results and those proved in higher dimensions, we start by recalling the analysis of the normal neck regime $\delta/\varepsilon \to \ell$, with $\ell \in (0, +\infty)$, performed in [15]. In order to characterize the asymptotic profile, the authors show that the constructed local minimizers $\{u_\varepsilon\}$, suitably rescaled, converge to a minimizer of an appropriate asymptotic variational problem. More precisely, setting $v_\varepsilon(x, y, z) := u_\varepsilon(\varepsilon x, \varepsilon y, \varepsilon z)$, they show that $\{v_\varepsilon\}$ converge to the unique minimizer v of the Dirichlet energy over the unbounded limiting domain $\Omega_\infty := \lim \frac{1}{\varepsilon}\Omega_\varepsilon$, under the "boundary" conditions $v \approx 1$ at $+\infty$ and $v \approx -1$ at $-\infty$ (these conditions are suitably incorporated in the definition of the functional space, over which the energy is minimized). In particular, the wall profile turns out to be three-dimensional and not confined inside the neck. The main tool used to get compactness is a scale invariant Poincaré inequality, which fails to be true in two-dimensions.

The two-dimensional *normal neck regime* presents some significant differences. The rescaled profiles $u_\varepsilon(\varepsilon x, \varepsilon y)$ converge to 0. In order to get a non-trivial limit we have to rescale also the dependent variable by a logarithmic factor (this reflects the slow decay of the fundamental solution to the Laplace's equation in two-dimensions). Precisely, we set $v_\varepsilon(x, y) := |\ln\varepsilon| u_\varepsilon(\varepsilon x, \varepsilon y)$ and we show that $\{v_\varepsilon\}$ converge (in a suitable sense) to the *unique* solution v of the problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_\infty, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_\infty, \\ \dfrac{v(x,y)}{\ln|(x,y)|} \to 1 & \text{as } |(x,y)| \to +\infty \text{ with } x > 1, \\ \dfrac{v(x,y)}{\ln|(x,y)|} \to -1 & \text{as } |(x,y)| \to +\infty \text{ with } x < -1, \\ v(0,0) = 0\,, \end{cases} \tag{1.3}$$

where $\Omega_\infty := \lim \frac{1}{\varepsilon}\Omega_\varepsilon$ is the unbounded limiting domain (see Figure 9) and is given by the union of the unscaled neck with the two half planes $\{x > 1\}$ and $\{x < -1\}$. Here (and in the other regimes) the potential W enters only in prescribing the conditions at infinity through the location of the wells, while its specific form does not affect the limiting problem. Note also that the asymptotic profile is two-dimensional. It is determined only by the geometry of the neck and spreads well into the bulk regions, as in the case treated in [15] (see Figure 2). However, it is easy to see that the Dirichlet energy of v is infinite, and thus problem (1.3) is not variational. This explains why the purely variational methods of [15] do not apply here.

Our approach is rather based on the use of the Maximum Principle and on PDE methods. We construct lower and upper bounds that asymptotically match at infinity, giving a quite accurate (almost explicit) description of the local minimizers u_ε. Such bounds provide us, in particular, with good L^∞-estimates for u_ε on any bounded subdomain and allow us to capture the asymptotic behavior at infinity. This information is enough to pass to the limit and obtain a well defined asymptotic problem for geometrically constrained walls. Finally, the uniqueness of the solution to (1.3) is nontrivial and can be established using some arguments from Potential Theory.

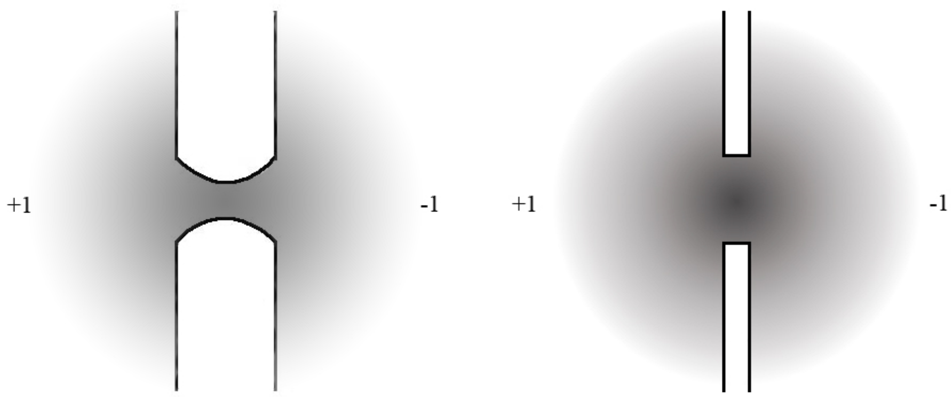

FIGURE 2. The normal (on the left) and thick (on the right) neck regimes. The shaded region corresponds to the domain wall.

We now describe the other regimes. We show that in the *subcritical thin neck* regime the wall profile is essentially one-dimensional and asymptotically confined inside the neck (see Figure 4). Indeed, the approximate profiles of geometrically constrained walls in this case are given by

$$u_\varepsilon(x,y) \approx \begin{cases} -1 & \text{to the left of the neck} \\ v(\frac{x}{\varepsilon}) & \text{inside the neck} \\ 1 & \text{to the right of the neck} \end{cases}$$

where v solves the one-dimensional problem

$$\min\left\{\frac{1}{2}\int_{-1}^{1}(f_1+f_2)(v')^2\,dx:\ v(-1)=-1,\ v(1)=1\right\}.$$

The boundary conditions $v(\pm 1)=\pm 1$ clearly indicate that (asymptotically) the whole transition takes place in N_ε. This is the only regime where Bruno's one-dimensional ansatz [2] turns out to be justified.

As for a *thick neck* regime, we prove that the profile is almost constant inside the neck and the transition happens exclusively in the bulk regions (see Figure 2). Since the geometry of the neck does not play any role, we assume for simplicity that $f_1=f_2\equiv\frac{1}{2}$. The (approximate) profiles of the geometrically constrained walls turn out to be given by

$$u_\varepsilon(x,y)\approx\begin{cases}\frac{1}{|\ln\delta|}v(\frac{x+\varepsilon}{\delta},\frac{y}{\delta}) & \text{to the left of the neck}\\ 0 & \text{inside the neck}\\ \frac{1}{|\ln\delta|}v(\frac{x-\varepsilon}{\delta},\frac{y}{\delta}) & \text{to the right of the neck,}\end{cases}$$

where v is the *unique* solution to

$$\begin{cases}\Delta v=0 & \text{in }\Omega_\infty,\\ \partial_\nu v=0 & \text{on }\partial\Omega_\infty,\\ \dfrac{v(x,y)}{\ln|(x,y)|}\to 1 & \text{as }|(x,y)|\to+\infty\text{ with }x>0,\\ \dfrac{v(x,y)}{\ln|(x,y)|}\to -1 & \text{as }|(x,y)|\to+\infty\text{ with }x<0,\\ v(0,0)=0\,,\end{cases}$$

defined on the limiting domain $\Omega_\infty:=\lim\frac{1}{\delta}\Omega_\varepsilon=\mathbb{R}^2\setminus\{(0,y):\ |y|\geq\frac{1}{2}\}$ (see Figure 18). Note that the geometry of neck and the potential W do not appear in the limiting problem. As before, the proof is based on the construction of lower and upper bounds and on the use of tools from Potential Theory.

Although the methods, the rescaling, and the limiting problems are different, all the results considered so far are somewhat similar in spirit to those obtained in [15] for the corresponding three-dimensional regimes. There are however no analogs of the *critical and supercritical thin neck* regimes in 3D and they really capture the essence of the two-dimensional problem. Let us explain this in more detail.

In the *critical thin neck* regime ($\frac{\delta|\ln\delta|}{\varepsilon}\to\ell$) a nontrivial behavior is observed both inside and outside of the neck. This is related to the fact that the energy contribution due to the transition occurring inside the neck is of the same order as the one coming from the bulk regions. This suggests that the interplay between these two contributions is more subtle than in the previous cases (or in 3D). The (approximate) wall profiles in the bulk region to the right of the neck; i.e., for $x\geq\varepsilon$ are given by

$$u_\varepsilon(x,y)\approx\frac{1}{|\ln\delta|}w^+\Big(\frac{x-\varepsilon}{\delta},\frac{y}{\delta}\Big)+\frac{\pi\,m_{f_1f_2}}{\pi\,m_{f_1f_2}+2\ell}\,,\tag{1.4}$$

where w^+ is the *unique* solution to the problem

$$\begin{cases} \Delta w^+ = 0 & \text{in } \Omega^+_\infty, \\ \partial_\nu w^+ = 0 & \text{on } \partial\Omega^+_\infty, \\ \dfrac{w^+(x,y)}{\ln|(x,y)|} \to \dfrac{2\ell}{\pi\, m_{f_1 f_2} + 2\ell} & \text{as } |(x,y)| \to +\infty \text{ with } x > 0, \\ \dfrac{w^+(x,y)}{x} \to \dfrac{2\ell\pi}{\pi\, m_{f_1 f_2} + 2\ell} & \text{uniformly in } y \text{ as } x \to -\infty, \\ w^+(0,0) = 0\,, \end{cases} \tag{1.5}$$

with $\Omega^+_\infty := \lim \frac{1}{\delta}(\Omega_\varepsilon - (\varepsilon, 0)) = \big\{(x,y) : x \le 0\,, -\frac{1}{2} < y < \frac{1}{2}\big\} \cup \{(x,y) : x > 0\}$ (see figure 3) and

$$m_{f_1 f_2} := \int_{-1}^{1} \frac{1}{f_1 + f_2}\, dx\,.$$

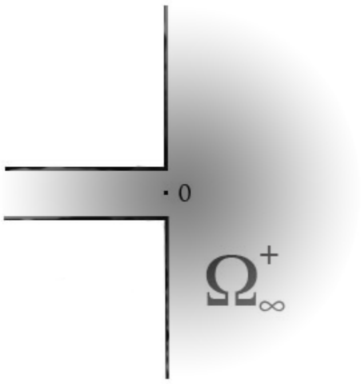

FIGURE 3. The limiting domain Ω^+_∞.

The profiles for $x \le -\varepsilon$ are related to the above approximate profiles by an odd reflection with respect to the x-variable.

Note that in (1.5) the geometry is "linearized" and the shape of the neck "weakly" affects the limiting bulk behavior only through the constant $m_{f_1 f_2}$ appearing in the conditions at infinity. However, if we denote by w^+_ℓ the solution to (1.5), then the family $\{w^+_\ell\}_{\ell > 0}$ is universal; i.e., it is independent of the geometry of the neck, the bulk regions, and the specific form of W.

We can actually capture a stronger dependence on the geometry, by zooming at the neck as we did in the subcritical regime. By doing this, we discover that the approximate profiles in N_ε are given by

$$u_\varepsilon(x,y) \approx v\Big(\frac{x}{\varepsilon}\Big),$$

where v solves one-dimensional problem

$$\min\left\{ \frac{1}{2}\int_{-1}^{1} (f_1 + f_2)(\theta')^2\, dx : \theta(-1) = -\frac{\pi\, m_{f_1 f_2}}{\pi\, m_{f_1 f_2} + 2\ell}\,, \theta(1) = \frac{\pi\, m_{f_1 f_2}}{\pi\, m_{f_1 f_2} + 2\ell} \right\}.$$

Hence, the shape of v is strongly affected by f_1 and f_2. Note that the wall profile v inside the neck **does not** reach -1 and 1 at the ends (as $-1 < \theta(-1) < \theta(1) < 1$). This is related to the fact that part of the transition occurs also in the bulk regions, as described by (1.4) and (1.5). From the mathematical point of view, the analysis of the critical regime is the most challenging. It is still based on the construction of refined lower and upper bounds, but here there is an additional technical problem: due to the non-trivial interplay between neck and bulk regions, one has to carefully link the bounds obtained inside to those constructed outside of the neck. The difficulty is overcome by estimating the oscillation of the local minimizers in N_ε through a delicate Harnack inequality argument (see Propositions 4.8 and 4.20).

There is an interesting observation about problem (1.5): the two conditions at $\pm\infty$ are not independent. It turns out that the logarithmic behavior of $w^+|_{\{x>0\}}$ and the particular form of $\Omega^+_\infty \cap \{x < 0\}$, together with at most linear growth at $-\infty$, uniquely determine the asymptotic behavior of $w|_{\{x<0\}}$. More precisely, if w is harmonic in Ω^+_∞ with homogeneous Neumann boundary conditions, $w|_{\{x>0\}}(x,y) \approx m \ln |(x,y)|$ for $|(x,y)|$ large and $w|_{\{x<0\}}$ grows at most linearly, then $w|_{\{x<0\}} \approx cx$ for $|x|$ large, with $c = c(m)$ uniquely (and explicitly) determined by m (see Proposition 4.14). This observation will be crucial in order to capture the behavior at $-\infty$ of the function w^+ appearing in (1.4).

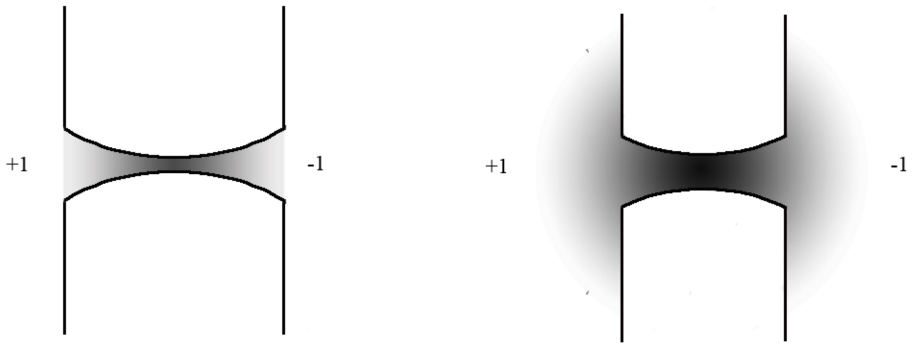

FIGURE 4. The subcritical (on the left) and critical (on the right) thin neck regimes.

The *supercritical thin neck* regime can be derived as a limit of the critical regime when $\ell \to +\infty$: We show that $u_\varepsilon \approx 0$ inside the neck, while in the bulk region u_ε is approximately given by (1.4) and (1.5), with ℓ formally replaced by $+\infty$.

We finally mention that for all regimes we are also able to compute the exact limits of the rescaled energies $|\ln\delta|F_\varepsilon(u_\varepsilon, N_\varepsilon)$ and $|\ln\delta|F_\varepsilon(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon)$, that is, both inside and outside of the neck (see the precise statements of the theorems in the next sections).

We conclude this introduction by remarking that we do not attempt to classify or characterize all geometrically constrained walls. We prove the existence of a family of symmetric local minimizers corresponding to geometrically constrained walls and study their limiting behavior. It is natural to ask whether or not there are other non-constant local minimizers besides the ones constructed in Proposition 2.1. Moreover, such a construction makes crucial use of the symmetry of the domain. The extension of the present analysis to more general non-symmetric domains will be the subject of future investigation.

The paper is organized as follows. In Section 2 we formulate the problem and construct the family of geometrically constrained walls that will be studied throughout the paper. The remaining sections are devoted to the study of their asymptotic behavior in the various regimes. In Section 3 we consider the case of a normal neck. In Section 4 we treat the thin neck regime: for the sake of presentation we start by considering in Subsection 4.1 the case of a flat neck, since the proofs are more transparent and yet contain most of the essential ideas. The extension to general non-flat necks is achieved in Subsection 4.2. Here, the constructions become more intricate and, in fact, it seems very hard to provide matching lower and upper bounds in the neck. Nevertheless, we are able to describe the limiting behavior at infinity thanks to Proposition 4.14, whose proof makes use of the more precise bounds constructed in the flat case. In Section 5 we obtain the asymptotic behavior in the thick neck regime. Finally, in the Appendix we collect some auxiliary results used throughout the paper.

2. Construction of the geometrically constrained walls

In this section we give the precise formulation of the problem and we construct the family of critical points representing the geometrically constrained walls, which will be investigated throughout the paper. We also show that under some additional regularity assumptions on the domain such critical points are local minimizers.

We start by describing the limiting domain. The right part of the limiting domain, denoted Ω^r, is a Lipschitz domain satisfying the following properties (see Figure 5):

(O1): the origin $(0,0)$ belongs to $\partial\Omega^r$;
(O2): Ω^r lies in the right half-plane $x > 0$;
(O3): $\partial\Omega^r \cap B_{r_0}(0,0)$ is of class $C^{1,\gamma}$ for some $\gamma \in (0,1)$ and for some $r_0 > 0$.

The left part, denoted by Ω^l, is obtained by reflecting Ω^r with respect to the y-axis, i.e.,

$$\Omega^l = \mathcal{R}\Omega^r := \{(-x,y) : (x,y) \in \Omega^r\}.$$ [1]

The limiting set is then given by

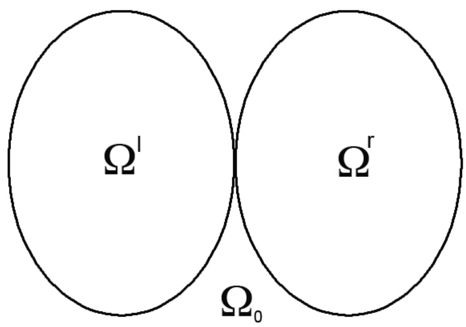

Figure 5. The limiting set Ω_0.

$$\Omega_0 := \Omega^l \cup \Omega^r .$$

The profile of the neck after rescaling is described by two Lipschitz even functions $f_1, f_2 : [-1,1] \mapsto (0,+\infty)$ and by the two small parameters $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$, which determine the scaling the length and height of the neck, respectively. *Throughout the paper, δ will always be considered as depending on ε, even though, for notational convenience, we will often omit to explicitly write such a dependence.* We also assume, without loss of generality, that the normalization condition

$$f_1(1) = f_2(1) = \frac{1}{2}$$

holds, so that the opening of the unscaled neck equals 1. To describe the ε-domain we distinguish the two cases $\lim_{\varepsilon\to 0^+} \delta/\varepsilon < +\infty$ and $\lim_{\varepsilon\to 0^+} \delta/\varepsilon = +\infty$. In the former case, which includes the normal and thin neck regimes, we set

$$\Omega_\varepsilon = \Omega^l_\varepsilon \cup N_\varepsilon \cup \Omega^r_\varepsilon, \tag{2.1}$$

where

$$\Omega^r_\varepsilon := \Omega^r + (\varepsilon - c\delta^{1+\gamma}, 0), \qquad \Omega^l_\varepsilon := \mathcal{R}\Omega^r_\varepsilon \tag{2.2}$$

[1]Here and in the following $\mathcal{R}$ denotes the reflection with respect to the y-axis.

and

$$N_\varepsilon := \Big\{(x,y)\,:\,|x|\le\varepsilon,\,-\delta f_2\Big(\frac{x}{\varepsilon}\Big)<y<\delta f_1\Big(\frac{x}{\varepsilon}\Big)\Big\}$$

(see Figure 6).

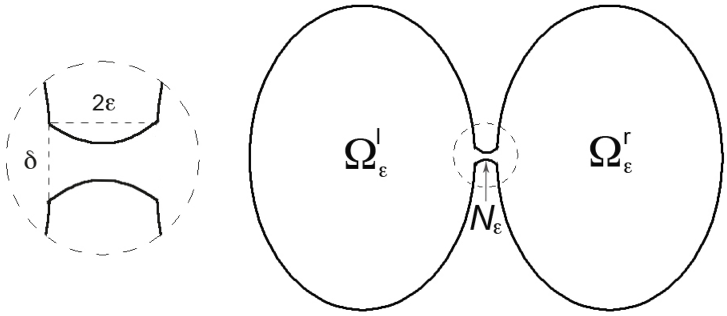

FIGURE 6. The dumbbell-shaped set Ω_ε in the case $\lim_{\varepsilon\to0}\delta/\varepsilon<+\infty$.

The exponent γ in (2.2) is the one appearing in condition (O3) and the constant $c>0$ is big enough so that the neck N_ε is "well attached" to the bulk part of the domain; more precisely, c is chosen in such a way that the segment $\{(x,y)\,:\,x=\varepsilon,-\delta f_2(1)\le y\le\delta f_1(1)\}$ lies in Ω_ε^r. Note that this is possible for δ small enough thanks to assumption (O3). We may also assume without loss of generality that the neck meets only the $C^{1,\gamma}$ part of $\partial\Omega^r\cup\partial\Omega^l$ (this is a smallness condition on δ). Observe that if the boundary of Ω^r is flat (and vertical) in a neighborhood of the origin, then there is no need of the correction $(c\delta^{1+\gamma},0)$ and we may simply take $\Omega_\varepsilon^r:=\Omega^r+(\varepsilon,0)$.

In the case of a thick neck, i.e. when $\lim_{\varepsilon\to0^+}\delta/\varepsilon=+\infty$, we always assume the boundary of Ω^r to be flat (and vertical) in a neighborhood of the origin and we take in (2.1) $\Omega_\varepsilon^r:=\Omega^r+(\varepsilon,0)$, $\Omega_\varepsilon^l:=\mathcal{R}\Omega_\varepsilon^r$. Moreover, as explained in the introduction, we also assume N_ε to be flat; i.e., we take $f_1=f_2\equiv\frac12$ (see Figure 7). Finally, note that in all cases Ω_ε is a Lipschitz domain.

It will often be enough to work separately on the right half of Ω_ε; i.e., on

$$\Omega_\varepsilon^+:=\Omega_\varepsilon\cap\{x>0\}\qquad\text{and}\qquad N_\varepsilon^+:=N_\varepsilon\cap\{x>0\}\,.$$

The corresponding result for

$$\Omega_\varepsilon^-:=\mathcal{R}\Omega_\varepsilon^+=\Omega_\varepsilon\cap\{x<0\}\qquad\text{and}\qquad N_\varepsilon^-:=\mathcal{R}N_\varepsilon^+=N_\varepsilon\cap\{x<0\}$$

will be then deduced by symmetry. We also set

$$\Gamma_{0,\varepsilon}:=\{(0,y):-\delta f_2(0)\le y\le\delta f_1(0)\}=\partial\Omega_\varepsilon^+\cap\{x=0\}\,.$$

We view geometrically constrained walls as suitable critical points in Ω_ε of the energy functional

$$F(u,\Omega):=\frac12\int_\Omega|\nabla u|^2\,dxdy+\int_\Omega W(u)\,dxdy\,,$$

defined for all $u\in H^1(\Omega)$, where Ω is an open subset of $\mathbb{R}^2$. Throughout the paper $W:\mathbb{R}\to[0,+\infty)$ is a double-well potential with the following properties:

(W1) W is of class C^1 and even; i.e. $W(u)=W(-u)$;
(W2) $W^{-1}(0)=\{-1,1\}$;
(W3) $W'>0$ in $(-1,0)$ and $W'<0$ in $(0,1)$;
(W4) W is twice differentiable at -1 and 1, with $W''(-1)=W''(1)>0$.

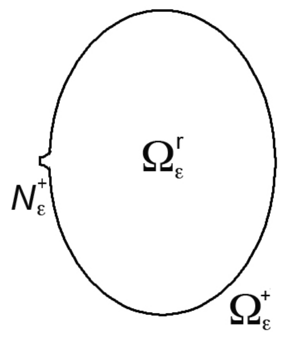

FIGURE 7. The domain Ω^+_ε in the case $\lim_{\varepsilon\to 0}\delta/\varepsilon < +\infty$.

A model case is of course represented by $W(u) = (u^2-1)^2$. Note that (W1), (W2), and (W4) imply the existence of a constant $\mu > 0$ such that

$$\begin{aligned} &W'(u) \geq \mu(u-1) \quad \text{and} \quad W(u) \geq \mu(u-1)^2 \qquad \text{for } u \in (0,1)\,, \\ &W'(u) \leq \mu(u+1) \quad \text{and} \quad W(u) \geq \mu(u+1)^2 \qquad \text{for } u \in (-1,0)\,. \end{aligned} \tag{2.3}$$

The critical points representing the geometrically constrained walls are constructed through the following minimization procedure.

Proposition 2.1. *For $\varepsilon > 0$ sufficiently small let w_ε be a minimizer of the following problem*

$$\min\{F(v,\Omega^+_\varepsilon) : v \in H^1(\Omega^+_\varepsilon)\,, v = 0 \ \text{on}\ \Gamma_{0,\varepsilon}\}\,.$$

Then

$$\text{either} \quad 0 < w_\varepsilon \leq 1 \quad \text{or} \quad -1 \leq w_\varepsilon < 0 \quad \text{in}\ \Omega^+_\varepsilon\,. \tag{2.4}$$

In all cases

$$\||w_\varepsilon| - 1\|_{L^2(\Omega^+_\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0^+\,. \tag{2.5}$$

Moreover,

$$F(w_\varepsilon,\Omega^+_\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0^+\,.$$

Proof. First note that if w_ε is a minimizer then so is $|w_\varepsilon|$. Using assumption (W2) on W and a truncation argument, it is readily seen that $0 \leq |w_\varepsilon| \leq 1$. Hence, $|w_\varepsilon|$ solves the Euler-Lagrange equation $\Delta|w_\varepsilon| = W'(|w_\varepsilon|)$ in Ω^+_ε and, by standard interior elliptic regularity, $|w_\varepsilon| \in C^{1,\alpha}_{loc}(\Omega^+_\varepsilon)$ for all $\alpha \in (0,1)$. Moreover, $\Delta|w_\varepsilon| \leq 0$ in Ω^+_ε thanks to (W3) and a standard application of the Strong Maximum Principle yields $|w_\varepsilon| > 0$ in Ω^+_ε. Since w_ε is a Sobolev function, the alternative (2.4) holds.

As

$$F(|w_\varepsilon|,\Omega^+_\varepsilon) \geq \int_{\Omega^+_\varepsilon} W(|w_\varepsilon|)\,dxdy \geq \mu \int_{\Omega^+_\varepsilon} (|w_\varepsilon|-1)^2\,dxdy$$

by (2.3), in order to obtain (2.5) it is enough to show that $F(|w_\varepsilon|,\Omega^+_\varepsilon) \to 0$ as $\varepsilon \to 0^+$. To this purpose, we exhibit an admissible family of test functions $\{z_\varepsilon\}_\varepsilon$ for which $F(z_\varepsilon,\Omega^+_\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Set $M := \max\{\|f_1\|_\infty, \|f_2\|_\infty\}$ and consider the domain $U^+_\varepsilon := \widetilde{N}_\varepsilon \cup \{x > \varepsilon/2\}$, where $\widetilde{N}_\varepsilon := \{(x,y) : 0 < x \leq \varepsilon/2\,, |y| < M\delta\}$. Clearly $\Omega^+_\varepsilon \subset U^+_\varepsilon$. Fix $\alpha \in (0,1)$ and define

$$z_\varepsilon(x,y) := \begin{cases} 0 & \text{if } (x,y) \in \widetilde{N}_\varepsilon \cup B_{M\delta}(\varepsilon/2,0), \\ \frac{1}{|\ln(M\delta^{1-\alpha})|} \ln \frac{|(x-\varepsilon/2,y)|}{M\delta} & \text{if } (x,y) \in \Big(B_{\delta^\alpha}(\frac{\varepsilon}{2},0) \setminus B_{M\delta}(\frac{\varepsilon}{2},0)\Big) \cap U^+_\varepsilon, \\ 1 & \text{otherwise in } U^+_\varepsilon. \end{cases}$$

As $\mathcal{L}^2\Big(\widetilde{N}_\varepsilon \cup (B_{\delta^\alpha}(\varepsilon/2,0)\cap\{x>\varepsilon/2\})\Big) = \delta\varepsilon M + \frac{\pi}{2}\delta^{2\alpha}$ and

$$\frac{1}{2}\int_{U_\varepsilon} |\nabla z_\varepsilon|^2\,dxdy = \frac{\pi}{2|\ln(M\delta^{1-\alpha})|}\,,$$

we have

$$(\delta\varepsilon M + \tfrac{\pi}{2}\delta^{2\alpha})\max_{[0,1]} W + \frac{\pi}{2|\ln(M\delta^{1-\alpha})|} \geq F(z_\varepsilon, U_\varepsilon^+) \geq F(z_\varepsilon, \Omega_\varepsilon^+). \tag{2.6}$$

Since the right-hand side of the above inequality tends to zero, the proof of the proposition is concluded. □

We are ready to define the critical points representing the geometrically constrained walls. For every $\varepsilon>0$ sufficiently small we set

$$u_\varepsilon(x,y) := \begin{cases} w_\varepsilon(x,y) & \text{if } (x,y)\in\Omega_\varepsilon^+,\\ -w_\varepsilon(-x,y) & \text{if } (x,y)\in\Omega_\varepsilon^-,\end{cases} \tag{2.7}$$

where $\{w_\varepsilon\}$ is the family of minimizers constructed in Proposition 2.1, which satisfies the first inequality in (2.4). Hence,

$$u_\varepsilon > 0 \quad \text{in } \Omega_\varepsilon^+\,, \qquad u_\varepsilon<0 \quad \text{in } \Omega_\varepsilon^-\,, \tag{2.8}$$

and u_ε is a weak solution to the Neumann problem

$$\begin{cases} \Delta u_\varepsilon = W'(u_\varepsilon) & \text{in } \Omega_\varepsilon,\\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{2.9}$$

Moreover,

$$\|u_\varepsilon - 1\|_{L^2(\Omega_\varepsilon^+)} \to 0\,, \quad \|u_\varepsilon+1\|_{L^2(\Omega_\varepsilon^-)}\to 0\,, \quad \text{and} \quad F(u_\varepsilon,\Omega_\varepsilon)\to 0 \qquad \text{as } \varepsilon\to 0^+. \tag{2.10}$$

We will show that if $\partial\Omega^r$ is sufficiently regular, then the critical points defined in (2.7) are in fact local minimizers for ε small enough. To this aim, we need a preliminary lemma dealing with a precise estimate of the Poincaré constant in suitable subdomains of Ω_ε^+.

Lemma 2.2. *Given $\eta\in(0,r_0)$ (r_0 being the constant appearing in hypothesis (O3)), let $\Omega_{\varepsilon,\eta}^+ := \Omega_\varepsilon^+\cap B_\eta(\varepsilon,0)$. Then, there exists $C>0$ independent of η such that for $\eta\in(0,r_0)$ we have*

$$\int_{\Omega_{\varepsilon,\eta}^+} |\nabla\varphi|^2\,dxdy \geq \frac{C}{\eta^2}\int_{\Omega_{\varepsilon,\eta}^+}|\varphi|^2\,dxdy \qquad \text{for all } \varphi\in H^1(\Omega_{\varepsilon,\eta}^+),\ \int_{\Omega_{\varepsilon,\eta}^+}\varphi\,dxdy = 0\,, \tag{2.11}$$

provided that ε is small enough.

Proof. We give the proof assuming for simplicity that $\partial\Omega^r$ is straight (and vertical) in $B_{r_0}(0,0)$. In fact there is no loss of generality, since one may reduce to this case by applying a local diffeomorphism. Assume also

$$\lim_{\varepsilon\to0}\frac{\delta}{\varepsilon} < +\infty\,. \tag{2.12}$$

We start by recalling the well-known Poincaré Inequality on half balls:

$$\int_{\Omega^r\cap B_\eta(0,0)} |\nabla\varphi|^2\,dxdy \geq \frac{C}{\eta^2}\int_{\Omega^r\cap B_\eta(0,0)} \varphi^2\,dxdy \tag{2.13}$$

for all $\varphi\in H^1(\Omega^r\cap B_\eta(0,0))$ such $\int_{\Omega^r\cap B_\eta(0,0)}\varphi\,dxdy=0$, with $C>0$ a universal constant.

We split the remaining part of the proof into three steps.

Step 1. Set $R_\varepsilon := (0,\varepsilon)\times(-\delta,0)$. We claim that

$$\int_{R_\varepsilon}|\nabla\varphi|^2\,dxdy \geq \frac{1}{\varepsilon^2}\int_{R_\varepsilon}|\varphi|^2\,dxdy \qquad \text{for all } \varphi\in H^1(R_\varepsilon) \text{ s.t. } \varphi=0 \text{ on } \{x=\varepsilon\}. \tag{2.14}$$

Indeed, if φ is of class C^1, then for all $(x,y)\in R_\varepsilon$ we may write $\varphi(x,y) = -\int_x^\varepsilon \frac{\partial\varphi}{\partial x}(s,y)\,ds$ and, in turn, by Hölder Inequality we get $\varphi^2(x,y)\leq\varepsilon\int_0^\varepsilon|\nabla\varphi(s,y)|^2\,ds$. Integrating the last inequality with respect to x and y, (2.14) follows for φ of class C^1. The general case is obtained by approximation.

Step 2. We claim the existence of a constant $C_1 > 0$ independent of ε such that

$$\int_{N_\varepsilon^+} |\nabla\varphi|^2\,dxdy \geq \frac{C_1}{\varepsilon^2}\int_{N_\varepsilon^+} |\varphi|^2\,dxdy \qquad \text{for all } \varphi \in H^1(N_\varepsilon^+) \text{ s.t. } \varphi = 0 \text{ on } \{x = \varepsilon\}. \tag{2.15}$$

Indeed, let $\Phi_\varepsilon : R_\varepsilon \to N_\varepsilon^+$ be the bi-Lipschitz diffeomorphism defined as

$$(x, y) \mapsto \left(x, [f_1(\tfrac{x}{\varepsilon}) + f_2(\tfrac{x}{\varepsilon})]y + \delta f_1(\tfrac{x}{\varepsilon})\right) .$$

Using (2.12), one may check that

$$0 < c_1 \leq \det D\Phi_\varepsilon \leq c_2 \qquad \text{and} \qquad |D\Phi_\varepsilon| \leq c_2\,, \tag{2.16}$$

with c_1 and c_2 independent of ε. Fix $\varphi \in H^1(N_\varepsilon^+)$ such that $\varphi = 0$ on $\{x = \varepsilon\}$. Since $\varphi \circ \Phi_\varepsilon \in H^1(R_\varepsilon)$, by the previous step and a change of variable we have

$$\begin{aligned}\int_{N_\varepsilon^+} \frac{D\Phi_\varepsilon D\Phi_\varepsilon^{\mathrm{T}}}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1}\nabla\varphi\cdot\nabla\varphi\,dxdy &= \int_{R_\varepsilon} |\nabla(\varphi\circ\Phi_\varepsilon)|^2\,dxdy \\ &\geq \frac{1}{\varepsilon^2}\int_{R_\varepsilon} |\varphi\circ\Phi_\varepsilon|^2\,dxdy = \frac{1}{\varepsilon^2}\int_{N_\varepsilon^+} \frac{1}{\det D\Phi_\varepsilon}\circ\Phi_\varepsilon^{-1}|\varphi|^2\,dxdy\,.\end{aligned}$$

Recalling (2.16), the claim follows.
Step 3. Note that $\widetilde{\Omega}_\varepsilon^+ := \Omega_{\varepsilon,\eta}^+ - (\varepsilon, 0) = (\Omega^r \cap B_\eta(0,0)) \cup (N_\varepsilon^+ - (\varepsilon, 0))$. Set

$$\lambda_\varepsilon := \min\left\{\int_{\widetilde{\Omega}_\varepsilon^+} |\nabla\varphi|^2\,dxdy : \|\varphi\|_{L^2(\widetilde{\Omega}_\varepsilon^+)} = 1 \text{ and } \int_{\widetilde{\Omega}_\varepsilon^+} \varphi\,dxdy = 0\right\}. \tag{2.17}$$

Clearly, in order to conclude the proof of the lemma it is enough to show that

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon \geq \frac{C}{\eta^2}\,, \tag{2.18}$$

where C is the constant appearing in (2.13). To this aim fix any subsequence $\varepsilon_n \to 0^+$ and assume, without loss of generality, that $\lim_n \lambda_{\varepsilon_n} < +\infty$. Let $\{\varphi_n\}$ be the corresponding minimizers in (2.17). Since $\sup_n \|\varphi_n\|^2_{H^1(\Omega^r\cap B_\eta(0,0))} \leq \sup_n \lambda_{\varepsilon_n} + 1 < +\infty$, there exist φ and a subsequence (not relabelled) such that

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly in } H^1(\Omega^r \cap B_\eta(0,0)). \tag{2.19}$$

We claim that

$$\int_{\Omega^r\cap B_\eta(0,0)} \varphi\,dxdy = 0 \quad \text{and} \quad \int_{\Omega^r\cap B_\eta(0,0)} \varphi^2\,dxdy = 1\,. \tag{2.20}$$

To this aim, extend $\varphi_n|_{\Omega^r\cap B_\eta(0,0)}$ to a function $\tilde\varphi_n \in H^1(\mathbb{R}^2)$ in such a way that

$$\|\tilde\varphi_n\|_{H^1(\mathbb{R}^2)} \leq C'\|\varphi_n\|_{H^1(\widetilde{\Omega}_{\varepsilon_n}^+)}\,, \tag{2.21}$$

with C' independent of n. Note that this is possible due to the regularity of $\partial\Omega^r$. Fix $p > 2$. Then,

$$\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} \tilde\varphi_n^2\,dxdy \leq \Big(\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} \tilde\varphi_n^p\,dxdy\Big)^{\frac{2}{p}} |N_{\varepsilon_n}^+|^{1-\frac{2}{p}} \leq c_p\|\varphi_n\|^2_{H^1(\widetilde{\Omega}_{\varepsilon_n}^+)}|N_{\varepsilon_n}^+|^{1-\frac{2}{p}} \to 0\,, \tag{2.22}$$

where we used the imbedding of $H^1(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ and (2.21). Moreover,

$$\begin{aligned}\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} |\nabla\varphi_n|^2\,dxdy &\geq \frac{1}{2}\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} |\nabla(\varphi_n - \tilde\varphi_n)|^2\,dxdy - \int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} |\nabla\tilde\varphi_n|^2\,dxdy \\ &\geq \frac{C_1}{\varepsilon_n^2}\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} |\varphi_n - \tilde\varphi_n|^2\,dxdy - C_2\,,\end{aligned} \tag{2.23}$$

where in the last inequality we have used (2.15) and the fact that $\sup_n \|\tilde\varphi_n\|^2_{H^1(\mathbb{R}^2)} \leq C_2 < +\infty$ by (2.21). Since the left-hand side of (2.23) is bounded, recalling (2.22), we deduce

$$\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} \varphi_n^2\,dxdy \to 0 \quad \text{and} \quad \int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} \varphi_n\,dxdy \leq \Big(\int_{N_{\varepsilon_n}^+ - (\varepsilon_n,0)} \tilde\varphi_n^2\,dxdy\Big)^{\frac{1}{2}} |N_{\varepsilon_n}^+|^{\frac{1}{2}} \to 0\,.$$

Thus, claim (2.20) follows observing that $\int_{\Omega^r\cap B_\eta(0,0)}\varphi_n\,dxdy = -\int_{N^+_{\varepsilon_n}-(\varepsilon_n,0)}\varphi_n\,dxdy$, $\int_{\Omega^r\cap B_\eta(0,0)}\varphi_n^2\,dxdy = 1-\int_{N^+_{\varepsilon_n}-(\varepsilon_n,0)}\varphi_n^2\,dxdy$, and recalling (2.19).

By lower semicontinuity we then have

$$\lim_{n\to\infty}\lambda_{\varepsilon_n} \geq \liminf_{n\to\infty}\int_{\Omega^r\cap B_\eta(0,0)}|\nabla\varphi_n|^2\,dxdy \geq \int_{\Omega^r\cap B_\eta(0,0)}|\nabla\varphi|^2\,dxdy \geq \frac{C}{\eta^2}\,,$$

where the last inequality is a consequence of (2.13) and (2.20). This concludes the proof of (2.18) and, in turn, of the lemma under assumption (2.12). If (2.12) does not hold; i.e., if we are in thick neck regime, then the proof is similar and in fact easier since in this case the neck is assumed to be flat. We leave the details to the reader. □

We are in a position to show that under additional regularity assumptions on W and $\partial\Omega^r$ the critical points $\{u_\varepsilon\}$ are in fact local minimizers.

Proposition 2.3. *In addition to the standing hypotheses, assume that the potential W is of class C^2 and $\partial\Omega^r$ is of class $C^{1,\gamma}$. Then, the critical points defined in* (2.7) *are isolated local minimizers in the following sense: there exists $\beta>0$ independent of ε such that*

$$F(u_\varepsilon,\Omega_\varepsilon) < F(v,\Omega_\varepsilon) \qquad \textit{for all } v\in H^1(\Omega_\varepsilon) \textit{ s.t. } 0<\|u_\varepsilon - v\|_{L^\infty(\Omega_\varepsilon)}\leq\beta,$$

provided that ε is small enough.

Proof. Given $u\in H^1(\Omega_\varepsilon)$, $\Omega\subset\Omega_\varepsilon$, and $\varphi\in H^1(\Omega)$, we may consider the second variation of the functional $F(\cdot,\Omega)$ at u with respect to the direction φ, defined as

$$\partial^2F(u,\Omega)[\varphi] := \frac{d^2}{dt^2}F(u+t\varphi,\Omega)|_{t=0} = \int_\Omega|\nabla\varphi|^2\,dxdx + \int_\Omega W''(u)\varphi^2\,dxdy\,.$$

Step 1. We claim the existence of $\beta_0>0$, independent of ε, such that

$$\partial^2F(u_\varepsilon,\Omega^+_\varepsilon)[\varphi] > \beta_0\|\varphi\|^2_{L^2(\Omega^+_\varepsilon)} \qquad \text{for all } \varphi\in H^1(\Omega^+_\varepsilon),\text{ with } \varphi\neq 0, \tag{2.24}$$

provided that ε is small enough. To this aim, fix $\eta\in(0,r_0)$ and for all $\tau\in(0,\eta/2]$ set $A_\varepsilon(\tau) := \Omega^+_\varepsilon\cap\{|(x-\varepsilon,y)|>\tau|\}$. Then, by (2.9), (2.10), and standard regularity estimates (here we are using the $C^{1,\gamma}$ regularity of $\partial\Omega^r$) we get, in particular,

$$\|u_\varepsilon - 1\|_{L^\infty(A_\varepsilon(\tau))}\to 0\,.$$

By condition (W4) on the potential W, we then have

$$W''(u_\varepsilon) > \frac{W''(1)}{2} > 0 \qquad \text{in } A_\varepsilon(\tau), \tag{2.25}$$

provided that ε is sufficiently small.

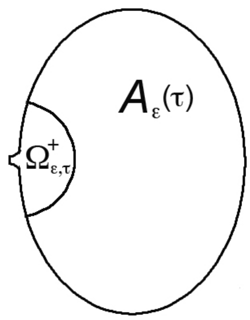

FIGURE 8. *The sets $\Omega^+_{\varepsilon,\tau}$ and $A_\varepsilon(\tau)$.*

Set

$$\overline{\varphi} := \fint_{\Omega^+_{\varepsilon,\eta}} \varphi \, dxdy \,,$$

where $\Omega^+_{\varepsilon,\eta}$ is the set defined in Lemma 2.2. Using the Poincaré inequality (2.11), for ε sufficiently small and for $\alpha > 0$ (to be chosen later) we have

$$\begin{aligned}
\partial^2 F(u_\varepsilon, \Omega^+_{\varepsilon,\eta})[\varphi] &\geq \frac{C}{\eta^2} \int_{\Omega^+_{\varepsilon,\eta}} |\varphi - \overline{\varphi}|^2 \, dxdy + \int_{\Omega^+_{\varepsilon,\eta}} W''(u_\varepsilon)\varphi^2 \, dxdy \\
&= \frac{C}{\eta^2} \int_{\Omega^+_{\varepsilon,\eta}} |\varphi - \overline{\varphi}|^2 \, dxdy + \int_{\Omega^+_{\varepsilon,\eta}} W''(u_\varepsilon)|\varphi - \overline{\varphi}|^2 \, dxdy \\
&+ 2\overline{\varphi} \int_{\Omega^+_{\varepsilon,\eta}} W''(u_\varepsilon)(\varphi - \overline{\varphi}) + \overline{\varphi}^2 \int_{\Omega^+_{\varepsilon,\eta}} W''(u_\varepsilon) \, dxdy \\
&\geq \left(\frac{C}{\eta^2} - (1+\alpha)|W''|_{L^\infty(0,1)} \right) \int_{\Omega^+_{\varepsilon,\eta}} |\varphi - \overline{\varphi}|^2 \, dxdy \\
&+ \overline{\varphi}^2 \left(\int_{\Omega^+_{\varepsilon,\eta}} W''(u_\varepsilon) \, dxdy - \frac{1}{\alpha} \int_{\Omega^+_{\varepsilon,\eta}} |W''(u_\varepsilon)| \, dxdy \right) . \qquad (2.26)
\end{aligned}$$

Note that

$$|\Omega^+_{\varepsilon,\tau}| \leq c\tau^2 \quad \text{and} \quad |\Omega^+_{\varepsilon,\eta} \cap A_\varepsilon(\tau)| \geq \frac{\eta^2}{c} \qquad (2.27)$$

for some constant $c > 1$ independent of η and τ, provided that ε is sufficiently small. Therefore, choosing $\tau^2 := \eta^2 W''(1)/(4c^2 |W''|_{L^\infty(0,1)})$, also by (2.25), we have

$$\begin{aligned}
\int_{\Omega^+_{\varepsilon,\eta}} W''(u_\varepsilon) \, dxdy &= \int_{\Omega_{\varepsilon,\eta} \cap A_\varepsilon(\tau)} W''(u_\varepsilon) \, dxdy + \int_{\Omega^+_{\varepsilon,\tau}} W''(u_\varepsilon) \, dxdy \\
&\geq \frac{W''(1)}{2c}\eta^2 - c|W''|_{L^\infty(0,1)}\tau^2 = \frac{W''(1)}{4c}\eta^2 \,.
\end{aligned}$$

Hence, recalling (2.26) and the first inequality in (2.27) (with τ replaced by η), we conclude that

$$\begin{aligned}
\partial^2 F(u_\varepsilon, \Omega^+_{\varepsilon,\eta})[\varphi] &\geq \left(\frac{C}{\eta^2} - (1+\alpha)|W''|_{L^\infty(0,1)} \right) \int_{\Omega^+_{\varepsilon,\eta}} |\varphi - \overline{\varphi}|^2 \, dxdy \\
&+ \eta^2 \left(\frac{W''(1)}{4c} - \frac{c}{\alpha}|W''|_{L^\infty(0,1)} \right) \overline{\varphi}^2 \\
&\geq \min\left\{ \frac{C}{\eta^2} - (1+\alpha)|W''|_{L^\infty(0,1)}, \frac{\eta^2}{|\Omega^+_{\varepsilon,\eta}|} \left(\frac{W''(1)}{4c} - \frac{c}{\alpha}|W''|_{L^\infty(0,1)} \right) \right\} \|\varphi\|^2_{L^2(\Omega^+_{\varepsilon,\eta})} \,. \qquad (2.28)
\end{aligned}$$

It is now clear that we can choose α so large and η so small that the constant multiplying $\|\varphi\|^2_{L^2(\Omega^+_{\varepsilon,\eta})}$ in the above inequality is positive. Finally, note that by (2.25)

$$\partial^2 F(u_\varepsilon, A_\varepsilon(\eta))[\varphi] \geq \int_{A_\varepsilon(\eta)} W''(u_\varepsilon)\varphi^2 \, dxdy > \frac{W''(1)}{2} \|\varphi\|^2_{L^2(A_\varepsilon(\eta))} \,. \qquad (2.29)$$

Collecting (2.28) and (2.29), we obtain (2.24).
Step 2. By (2.24) and the analogous inequality in Ω^-_ε, we have

$$\partial^2 F(u_\varepsilon, \Omega_\varepsilon)[\varphi] > \beta_0 \|\varphi\|^2_{L^2(\Omega_\varepsilon)} \qquad \text{for all } \varphi \in H^1(\Omega_\varepsilon), \text{ with } \varphi \neq 0. \qquad (2.30)$$

Since W'' is of class C^2, we may find $\beta \in (0,1)$ so that

$$|W''(t+s) - W''(t)| \leq \frac{\beta_0}{2} \qquad \text{for all } t \in [-1,1] \text{ and } s \in [-\beta, \beta]. \qquad (2.31)$$

Fix $v \in H^1(\Omega_\varepsilon)$ such that $0 < \|u_\varepsilon - v\|_{L^\infty(\Omega_\varepsilon)} \leq \beta$ and set $f(t) := F(u_\varepsilon + t(v - u_\varepsilon), \Omega_\varepsilon)$. Then, for $t \in (0,1)$ we have

$$f''(t) = \partial^2 F(u_\varepsilon + t(v - u_\varepsilon), \Omega_\varepsilon)[v - u_\varepsilon]$$

$$\geq \partial^2 F(u_\varepsilon, \Omega_\varepsilon)[v-u_\varepsilon] - \int_{\Omega_\varepsilon} |W''(u_\varepsilon + t(v-u_\varepsilon)) - W''(u_\varepsilon)|(v-u_\varepsilon)^2 \, dxdy$$
$$\geq \beta_0 \|u_\varepsilon - v\|^2_{L^2(\Omega_\varepsilon)} - \frac{\beta_0}{2}\|u_\varepsilon - v\|^2_{L^2(\Omega_\varepsilon)} = \frac{\beta_0}{2}\|u_\varepsilon - v\|^2_{L^2(\Omega_\varepsilon)},$$

where in the last inequality we have used (2.30) and (2.31). Hence, also by the fact that $f'(0) = 0$ due to the criticality of u_ε, we deduce

$$F(v, \Omega_\varepsilon) = f(1) = f(0) + \int_0^1 (1-t) f''(t) \, dt$$
$$\geq F(u_\varepsilon, \Omega_\varepsilon) + \frac{\beta_0}{2}\|u_\varepsilon - v\|^2_{L^2(\Omega_\varepsilon)} \int_0^1 (1-t)\, dt = F(u_\varepsilon, \Omega_\varepsilon) + \frac{\beta_0}{4}\|u_\varepsilon - v\|^2_{L^2(\Omega_\varepsilon)},$$

which concludes the proof of the proposition. □

3. The normal neck

We start with the case where the length and the width of the neck scale with the same order. We call this a *normal neck*.

Throughout the section we assume that $\delta = \delta(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = \ell \qquad 0 < \ell < +\infty. \tag{3.1}$$

Before stating the main result we need to introduce the domain Ω_∞, where the asymptotic (rescaled) wall profile will be defined. Roughly speaking it is given by the limit of the rescaled sets $\frac{1}{\varepsilon}\Omega_\varepsilon$ and consists of the union of two half planes (the limits of the rescaled bulk domains) and the rescaled neck. More precisely,

$$\Omega_\infty := \Omega^l_\infty \cup N_\infty \cup \Omega^r_\infty,$$

where $\Omega^l_\infty := \{(x,y) : x < -1\}$, $\Omega^r_\infty := \{(x,y) : x > 1\}$, and $N_\infty := \{(x,y) : |x| \leq 1, -\ell f_2(x) < y < \ell f_1(x)\}$ (see Figure 9 below).

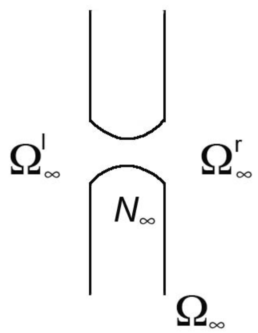

Figure 9. The limiting set Ω_∞.

We are ready to state the main theorem of this section.

Theorem 3.1. *Assume that* (3.1) *holds. Let* $\{u_\varepsilon\}$ *be the family of minimizing geometrically constrained walls defined in* (2.7) *and* $\{v_\varepsilon\}$ *the corresponding rescaled profiles defined by*

$$v_\varepsilon(x,y) := |\ln \varepsilon| u_\varepsilon(\varepsilon x, \varepsilon y)$$

for $(x,y) \in \frac{1}{\varepsilon}\Omega_\varepsilon$*. Then, for every* $p \geq 1$ *we have* $v_\varepsilon \to v$ *in* $W^{2,p}_{loc}(\Omega_\infty)$ *as* $\varepsilon \to 0^+$*, where* v *is the unique solution to the following problem:*

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_\infty, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_\infty, \\ \dfrac{v(x,y)}{\ln|(x,y)|} \to 1 & \text{as } |(x,y)| \to +\infty \text{ with } x > 1, \\ \dfrac{v(x,y)}{\ln|(x,y)|} \to -1 & \text{as } |(x,y)| \to +\infty \text{ with } x < 1, \\ v(0,0) = 0\,. \end{cases} \tag{3.2}$$

Moreover, $\nabla v_\varepsilon \chi_{\frac{1}{\varepsilon}\Omega_\varepsilon} \to \nabla v \chi_{\Omega_\infty}$ *in* $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ *as* $\varepsilon \to 0^+$*. Finally,*

$$\lim_{\varepsilon\to 0^+} |\ln\varepsilon| F(u_\varepsilon, \Omega_\varepsilon) = \pi\,. \tag{3.3}$$

Remark 3.2. Note that the local convergence of $\{v_\varepsilon\}$ to v stated in the theorem is well defined. Indeed, since $\mathbb{R}^2 \setminus \frac{1}{\varepsilon}\Omega_\varepsilon \to \mathbb{R}^2 \setminus \Omega_\infty$ in the Hausdorff metric, it follows that for every $\Omega' \subset\subset \Omega_\infty$ we have $\Omega' \subset\subset \frac{1}{\varepsilon}\Omega_\varepsilon$ for ε sufficiently small.

Remark 3.3. The theorem states that the rescaled profiles of the minimizing geometrically constrained walls and the energy display a universal behavior, which depends only on the shape of the rescaled neck. In particular, it is independent of Ω^l, Ω^r, and the particular shape of the double-well potential W.

Proof of Theorem 3.1. By symmetry it is enough to investigate the asymptotic behavior of $v_\varepsilon|_{\Omega^+_\varepsilon}$. The study is based on the construction of suitable subsolutions and supersolutions.

To this aim, it is convenient to introduce a conformal map, which straightens the boundary of Ω^r near the origin. More precisely, we may find $\eta \in (0, r_0)$ (see (O3)) and $\Phi = (\Phi_1, \Phi_2)\colon B_\eta(0,0) \to \mathbb{R}^2$ of class $C^{1,\gamma}$ with the following properties:

(a) $\Phi(0,0) = (0,0)$, $D\Phi(0,0) = Id$;
(b) Φ is conformal in $B_\eta(0,0) \cap \Omega^r$;
(c) Φ maps $\partial\Omega^r \cap B_\eta(0,0)$ into a vertical segment contained in the y-axis;
(d) $\det D\Phi \geq \frac{1}{2}$ in $B_\eta(0,0)$.

The existence of such maps is guaranteed by the regularity assumption (O3) and by a theorem due to Pommerenke (see [17]). We also consider the translated map $\Phi_\varepsilon = (\Phi_{\varepsilon,1}, \Phi_{\varepsilon,2})$ defined as $\Phi_\varepsilon(\cdot,\cdot) := \Phi(\cdot - \varepsilon + c\delta^{1+\gamma}, \cdot)$. For all $0 \leq \rho_0 < \rho_1$ (with ρ_1 sufficiently small) we may consider the sets

$$\begin{aligned} A(\rho_0,\rho_1) &:= \{(x,y) \in B_\eta(0,0) : \rho_0 < |\Phi| < \rho_1,\ \Phi_1 > 0\} \subset \Omega^r\,, \\ A_\varepsilon(\rho_0,\rho_1) &:= (\varepsilon - c\delta^{1+\gamma}, 0) + A(\rho_0,\rho_1) \\ &= \{(x,y) \in B_\eta(\varepsilon - c\delta^{1+\gamma}, 0) : \rho_0 < |\Phi_\varepsilon| < \rho_1,\ \Phi_{\varepsilon,1} > 0\} \subset \Omega^r_\varepsilon\,. \end{aligned}$$

Step 1. (Lower bounds) We start by constructing a family of functions that will provide a lower bound for u_ε in a suitable subregion of Ω^r_ε. An entirely similar construction will provide an upper bound in the corresponding subregion of Ω^l_ε. To this aim, fix $0 < \rho_0 < \rho_1$ such that $\overline{A_\varepsilon(\rho_0,\rho_1)} \subset\subset B_\eta(\varepsilon - c\delta^{1+\gamma}, 0) \cap \{x > \varepsilon - c\delta^{1+\gamma}\}$ (see Figure 10). Then, by (2.9), (2.10), and standard regularity estimates (here we are using the $C^{1,\gamma}$ regularity of $\partial\Omega^r \cap B_\eta(0,0)$) we get, in particular,

$$\eta_\varepsilon := \|u_\varepsilon - 1\|_{C^{1,\gamma}(\overline{A_\varepsilon(\rho_0,\rho_1)})} \to 0\,. \tag{3.4}$$

Set

$$\Gamma_\varepsilon := \{|\Phi_\varepsilon| = \rho_0, \Phi_{\varepsilon,1} > 0\} \tag{3.5}$$

and choose $c_1 > 0$ such that

$$c_1 > \max\{\ell, 1\}\,. \tag{3.6}$$

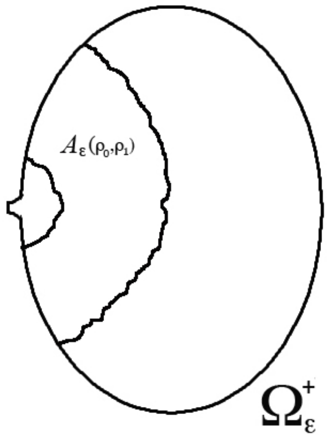

FIGURE 10. The annuli where the convergence is uniform.

Note that

$$\frac{\overline{A_\varepsilon(c_1\varepsilon,\rho_0)}}{\varepsilon} \to \{x \geq 1\} \setminus B_{c_1}(1,0) \qquad \text{locally in the Hausdorff metric.}$$

Let $w_\varepsilon : A_\varepsilon(c_1\varepsilon,\rho_0) \to [0,+\infty)$ be defined as

$$w_\varepsilon := (1-\eta_\varepsilon)\Big(1 - \frac{\ln\frac{|\Phi_\varepsilon|}{\rho_0}}{\ln\frac{c_1\varepsilon}{\rho_0}}\Big). \tag{3.7}$$

Using the conformal character of Φ_ε, one can see that w_ε satisfies:

$$\begin{cases} \Delta w_\varepsilon = 0 & \text{in } A_\varepsilon(c_1\varepsilon,\rho_0)\,, \\ w_\varepsilon = 0 & \text{on } \{|\Phi_\varepsilon| = c_1\varepsilon,\ \Phi_{\varepsilon,1} > 0\}\,, \\ w_\varepsilon = 1-\eta_\varepsilon & \text{on } \Gamma_\varepsilon\,, \\ \partial_\nu w_\varepsilon = 0 & \text{on } \partial A_\varepsilon(c_1\varepsilon,\rho_0) \cap \partial\Omega^r_\varepsilon\,. \end{cases}$$

On the other hand, recalling (2.9), the fact that $W'(u_\varepsilon) \leq 0$ in $A_\varepsilon(c_1\varepsilon,\rho_0)$ by assumption (W3), and the definition of η_ε, we deduce

$$\begin{cases} \Delta u_\varepsilon \leq 0 & \text{in } A_\varepsilon(c_1\varepsilon,\rho_0)\,, \\ u_\varepsilon \geq 0 & \text{on } \{|\Phi_\varepsilon| = c_1\varepsilon,\ \Phi_{\varepsilon,1} > 0\}\,, \\ u_\varepsilon \geq 1-\eta_\varepsilon & \text{on } \Gamma_\varepsilon\,, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial A_\varepsilon(c_1\varepsilon,\rho_0) \cap \partial\Omega^r_\varepsilon\,. \end{cases}$$

The weak comparison principle stated in Proposition 6.1 now gives

$$w_\varepsilon \leq u_\varepsilon \qquad \text{in } A_\varepsilon(c_1\varepsilon,\rho_0)\,. \tag{3.8}$$

Moreover, using the expansion

$$|\Phi_\varepsilon(\varepsilon x, \varepsilon y)| = \varepsilon|(x-1,y)| + o(\varepsilon) \tag{3.9}$$

and (3.4), one can easily check that

$$|\ln\varepsilon| w_\varepsilon(\varepsilon\,\cdot,\varepsilon\,\cdot) \to \ln\frac{|(\cdot-1,\cdot)|}{c_1} \quad \text{uniformly on the compact subsets of } \{x>1\} \setminus \overline{B_{c_1}(1,0)}. \tag{3.10}$$

Step 2. (Upper bounds) We will construct a supersolution in the region

$$N_\varepsilon^+ \cup A_\varepsilon(0,\rho_0)$$

(see Figure 10). The construction of the comparison function will be performed separately in

$$C_\varepsilon := N_\varepsilon^+ \cup A_\varepsilon(0, c_1\varepsilon) \quad \text{and} \quad A_\varepsilon(c_1\varepsilon, \rho_0)\,.$$

Here c_1 and ρ_0 are the same constants chosen in the previous step. Set $C_\infty := N_\infty^+ \cup (B_{c_1}(1,0) \cap \{x > 1\})$, see figure 11. Using the definition of the sets $A_\varepsilon(0, c_1\varepsilon)$ and the properties of the map Φ_ε, it is easy to see that

$$\frac{\overline{C}_\varepsilon}{\varepsilon} \to \overline{C}_\infty \qquad \text{in the Hausdorff metric.}$$

The convergence is in fact much stronger as it will be specified later. Let $\tilde{z}$ be the solution to

$$\begin{cases} \Delta \tilde{z} = 0 & \text{in } C_\infty\,, \\ \tilde{z} = 0 & \text{on } \partial C_\infty \cap \{x = 0\} =: \sigma_0\,, \\ \tilde{z} = 1 & \text{on } \partial B_{c_1}(1,0) \cap \{x \geq 1\} =: \sigma_1\,, \\ \partial_\nu \tilde{z} = 0 & \text{on } \partial C_\infty \setminus (\sigma_0 \cup \sigma_1)\,. \end{cases}$$

Since $\tilde{z}$ achieves its maximum on σ_1, Hopf's Lemma, combined with a reflection argument to

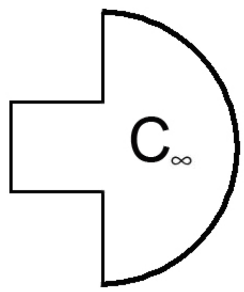

FIGURE 11. Limiting subdomain C_∞.

remove the corner singularities at $(1, c_1)$ and $(1, -c_1)$, implies that $\partial_\nu \tilde{z}$ is strictly positive on σ_1. The same reflection argument shows that $\partial_\nu \tilde{z}$ is of class C^∞ up to the end-points of σ_1. Hence, there exists $a > 0$ such that

$$\partial_\nu \tilde{z} > a > 0 \quad \text{on } \sigma_1\,. \tag{3.11}$$

Let σ_0^ε and σ_1^ε denote the portions of $\partial(C_\varepsilon/\varepsilon)$ approximating σ_0 and σ_1, respectively. We may now consider the solution $\tilde{z}_\varepsilon$ to the following problem:

$$\begin{cases} \Delta \tilde{z}_\varepsilon = -\varepsilon & \text{in } \frac{C_\varepsilon}{\varepsilon}\,, \\ \tilde{z}_\varepsilon = 0 & \text{on } \sigma_0^\varepsilon\,, \\ \tilde{z}_\varepsilon = 1 & \text{on } \sigma_1^\varepsilon\,, \\ \partial_\nu \tilde{z}_\varepsilon = 0 & \text{on } \frac{\partial C_\varepsilon}{\varepsilon} \setminus (\sigma_0^\varepsilon \cup \sigma_1^\varepsilon)\,. \end{cases} \tag{3.12}$$

In order to study the asymptotic behavior of $\tilde{z}_\varepsilon$ we need to better exploit the convergence of $C_\varepsilon/\varepsilon$: From the definition of N_ε^+, the assumption (O3), and the properties of the map Φ_ε, one can easily check that there exist bilipschitz transformations $\Psi_\varepsilon : C_\infty \to C_\varepsilon/\varepsilon$ such that Ψ_ε and Ψ_ε^{-1} converge to the identity map in the $W^{1,\infty}$-norm. Moreover, Ψ_ε can be taken conformal (and in fact equal to $\frac{1}{\varepsilon}\Phi_\varepsilon^{-1}(\varepsilon\,\cdot, \varepsilon\,\cdot))$ in a neighborhood $\mathcal{U}$ of σ_1, with $C^{1,\gamma}$-norms bounded by a constant independent of ε. Hence, the functions $\hat{z}_\varepsilon := \tilde{z}_\varepsilon \circ \Psi_\varepsilon$ satisfy

$$\begin{cases} \operatorname{div}(A_\varepsilon \hat{z}_\varepsilon) = g_\varepsilon & \text{in } C_\infty\,, \\ \hat{z}_\varepsilon = 0 & \text{on } \sigma_0\,, \\ \hat{z}_\varepsilon = 1 & \text{on } \sigma_1\,, \\ \langle A_\varepsilon \nabla \hat{z}_\varepsilon, \nu \rangle = 0 & \text{on } \partial C_\infty \setminus (\sigma_0 \cup \sigma_1)\,, \end{cases}$$

where $A_\varepsilon \to Id$ uniformly in C_∞ with equibounded $C^{0,\gamma}$-norms in $\mathcal{U} \cap C_\infty$ and $g_\varepsilon \to 0$ in $C^{0,\gamma}(\mathcal{U} \cap C_\infty)$. Standard elliptic estimates now imply that $\hat{z}_\varepsilon \to \tilde{z}$ strongly in $H^1(C_\infty)$ and in $C^{2,\sigma}(\mathcal{U} \cap C_\infty)$ for every $\sigma \in (0, \gamma)$. Note that in order to prove the last statement a (local) reflection argument in needed as before to remove the corner singularites at the intersection points of σ_1 with $\{y = 1\}$. This is possible due to the fact that the matrices A_ε are of the form $\lambda_\varepsilon Id$ for a suitable coefficient λ_ε of class $C^{0,\gamma}$ in $\mathcal{U}$. Recalling that $\hat{z}_\varepsilon := \tilde{z}_\varepsilon \circ \Psi_\varepsilon$ and taking into account (3.11) we deduce that for all ε small enough

$$\partial_\nu \tilde{z}_\varepsilon > a > 0 \quad \text{on } \sigma_\varepsilon\,. \tag{3.13}$$

In order to construct the supersolution in $A_\varepsilon(c_1\varepsilon, r_0)$ we consider first the following auxiliary problem

$$\begin{cases} \Delta h_\varepsilon = 2\mu h_\varepsilon & \text{in } B_{\rho_0}(0,0) \setminus B_{c_1\varepsilon}(0,0)\,, \\ h_\varepsilon = -1 & \text{on } \partial B_{c_1\varepsilon}(0,0)\,, \\ h_\varepsilon = 0 & \text{on } \partial B_{\rho_0}(0,0)\,. \end{cases} \tag{3.14}$$

A straightforward application of Proposition 6.1 yields $-1 \le h_\varepsilon \le 0$. Moreover, the solution is necessarily radial and can be written in the form

$$h_\varepsilon(x,y) = r_\varepsilon(\sqrt{2\mu}|(x,y)|)\,, \tag{3.15}$$

where r_ε solves the corresponding ODE:

$$\begin{cases} r_\varepsilon''(t) + \frac{r_\varepsilon'(t)}{t} - r_\varepsilon(t) = 0 & \text{in } (c_1\varepsilon\sqrt{2\mu}, \rho_0\sqrt{2\mu})\,, \\ r_\varepsilon(c_1\varepsilon\sqrt{2\mu}) = -1\,, \\ r_\varepsilon(\rho_0\sqrt{2\mu}) = 0\,. \end{cases} \tag{3.16}$$

The general solution to the ODE takes the form $r_\varepsilon(t) = aI_0(t) + bK_0(t)$, where I_0 and K_0 are the zero-order modified Bessel functions of first and second kind, respectively (see [1]). After some elementary calculations, one finds

$$r_\varepsilon(t) = -\frac{I_0(\rho_0\sqrt{2\mu})}{I_0(\rho_0\sqrt{2\mu})K_0(c_1\varepsilon\sqrt{2\mu}) - K_0(\rho_0\sqrt{2\mu})I_0(c_1\varepsilon\sqrt{2\mu})}\Big(-\frac{K_0(\rho_0\sqrt{2\mu})}{I_0(\rho_0\sqrt{2\mu})}I_0(t) + K_0(t)\Big)\,. \tag{3.17}$$

We are now ready to define

$$z_\varepsilon(x,y) := \begin{cases} \dfrac{m}{|\ln\varepsilon|}\tilde{z}_\varepsilon\Big(\dfrac{x}{\varepsilon}, \dfrac{y}{\varepsilon}\Big) & \text{in } C_\varepsilon\,, \\ \Big(1 - \dfrac{m}{|\ln\varepsilon|}\Big)h_\varepsilon(\Phi_\varepsilon(x,y)) + 1 & \text{in } A_\varepsilon(c_1\varepsilon, \rho_0)\,, \end{cases} \tag{3.18}$$

where the constant $m > 0$ will be chosen below. Using (3.15), (3.17), the identities

$$I_0'(t) = I_1(t) \qquad \text{and} \qquad K_0'(t) = -K_1(t)\,, \tag{3.19}$$

together with the fact that

$$I_0(0) = 1\,, \qquad\qquad \frac{K_0(t)}{|\ln t|} \to 1 \quad \text{as } t \to 0^+\,, \tag{3.20}$$

and

$$I_1(t) = O(t)\,, \qquad\qquad tK_1(t) \to 1 \quad \text{as } t \to 0^+\,, \tag{3.21}$$

one can check that for all $M > c_1$

$$\frac{\lambda_1}{\varepsilon|\ln\varepsilon|} \le |\nabla h_\varepsilon| \le \frac{\lambda_2}{\varepsilon|\ln\varepsilon|} \quad \text{in } A_\varepsilon(c_1\varepsilon, M\varepsilon)\,, \tag{3.22}$$

with λ_1, $\lambda_2 > 0$ depending only on M (and independent of ε). Here, I_1 and K_1 denote the first-order modified Bessel functions of first and second kind, respectively.

On the other hand, denoting by ν_{C_ε} the outer unit normal to C_ε along $\varepsilon\sigma_1^\varepsilon$, by (3.13) we have

$$\frac{m}{\varepsilon|\ln\varepsilon|}\nabla\tilde{z}_\varepsilon\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\cdot\nu_{C_\varepsilon}(x,y) > \frac{ma}{\varepsilon|\ln\varepsilon|} \quad \text{for } (x,y)\in\varepsilon\sigma_1^\varepsilon\,.$$

Therefore, we may choose $m>0$ so big that

$$\frac{m}{\varepsilon|\ln\varepsilon|}\nabla\tilde{z}_\varepsilon\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\cdot\nu_{C_\varepsilon}(x,y) > \left(1-\frac{m}{|\ln\varepsilon|}\right)\partial_{\nu_{C_\varepsilon}}(h_\varepsilon\circ\Phi_\varepsilon)(x,y) \quad \text{for } (x,y)\in\varepsilon\sigma_1^\varepsilon \tag{3.23}$$

for ε small enough. By (3.12) and (3.18) we have

$$\Delta z_\varepsilon = -\frac{m}{\varepsilon|\ln\varepsilon|} < \mu(z_\varepsilon-1) \quad \text{in } C_\varepsilon \text{ for } \varepsilon \text{ small enough.} \tag{3.24}$$

Moreover, recalling that Φ_ε is conformal, we can write

$$\Delta z_\varepsilon = \det D\Phi_\varepsilon\,(\Delta h_\varepsilon)\circ\Phi_\varepsilon = \det D\Phi_\varepsilon\,2\mu(z_\varepsilon-1)\le\mu(z_\varepsilon-1) \quad \text{in } A_\varepsilon(c_1\varepsilon,\rho_0)\,, \tag{3.25}$$

where in the last inequality we used the fact that $\det D\Phi_\varepsilon \ge \frac{1}{2}$ (by assumption (d) on the map Φ) and $z_\varepsilon - 1\le 0$. Thus, we can summarize (3.23), (3.24), and (3.25) as follows:

$$\begin{cases}\Delta z_\varepsilon \le \mu(z_\varepsilon-1) & \text{in the sense of distributions in } N_\varepsilon^+\cup A_\varepsilon(0,\rho_0),\\ z_\varepsilon = 0 & \text{on } \varepsilon\sigma_0^\varepsilon\,,\\ z_\varepsilon = 1 & \text{on } \Gamma_\varepsilon\,,\\ \partial_\nu z_\varepsilon = 0 & \text{on the remaining part of } \partial(N_\varepsilon^+\cup A_\varepsilon(0,\rho_0))\,,\end{cases} \tag{3.26}$$

where Γ_ε is defined in (3.5). Recall now that by (2.3) and (2.9), we have

$$\begin{cases}\Delta u_\varepsilon \ge \mu(u_\varepsilon-1) & \text{in } N_\varepsilon^+\cup A_\varepsilon(0,\rho_0),\\ u_\varepsilon = 0 & \text{on } \varepsilon\sigma_0^\varepsilon\,,\\ u_\varepsilon \le 1 & \text{on } \Gamma_\varepsilon\,,\\ \partial_\nu z_\varepsilon = 0 & \text{on the remaining part of } \partial(N_\varepsilon^+\cup A_\varepsilon(0,\rho_0))\,.\end{cases} \tag{3.27}$$

The weak comparison principle stated in Proposition 6.1 eventually yields

$$u_\varepsilon \le z_\varepsilon \quad \text{in } N_\varepsilon^+\cup A_\varepsilon(0,\rho_0)\,. \tag{3.28}$$

In order to study the pointwise behavior of z_ε, note that by $(3.19)_2$ for every $t>1$ we have

$$K_0(\varepsilon t)-K_0(\varepsilon) = -\int_\varepsilon^{\varepsilon t}K_1(s)\,ds = -\int_1^t(K_1(\varepsilon z)\varepsilon z)\frac{1}{z}\,dz\,.$$

On the other hand, by $(3.21)_2$ for every $\eta\in(0,1)$ we have

$$(1-\eta)\ln t \le \int_1^t(K_1(\varepsilon z)\varepsilon z)\frac{1}{z}\,dz \le (1+\eta)\ln t\,,$$

provided that ε is small enough. We deduce that for $t>1$

$$K_0(\varepsilon)-K_0(\varepsilon t)\to\ln t \qquad \text{as } \varepsilon\to 0.$$

Hence, using also (3.20) and the expansion (3.9), after a lengthy but straightforward calculation it can be checked that

$$|\ln\varepsilon|z_\varepsilon(\varepsilon x,\varepsilon y)\to m+\ln\frac{|(x-1,y)|}{c_1} \quad \text{for } (x,y)\in\{x>1\}\setminus\overline{B_{c_1}(1,0)} \text{ as } \varepsilon\to 0. \tag{3.29}$$

Moreover, taking into account (3.22), we may conclude that the above convergence is in fact uniform on the compact subsets of $\{x>1\}\setminus\overline{B_{c_1}(1,0)}$.

Step 3. (Limit of the energy) By (3.28), we have $u_\varepsilon \le \frac{m}{|\ln\varepsilon|}$ on $\{|\Phi_\varepsilon| = c_1\varepsilon\,,\ \Phi_{\varepsilon,1} > 0\}$. Hence, also by (3.4),

$$\begin{aligned}
\liminf_{\varepsilon\to 0} |\ln\varepsilon| F(u_\varepsilon, \Omega_\varepsilon^+) &\ge \liminf_{\varepsilon\to 0} \frac{|\ln\varepsilon|}{2} \int_{A_\varepsilon(c_1\varepsilon,\rho_0)} |\nabla u_\varepsilon|^2 dxdy \\
&\ge \lim_{\varepsilon\to 0} |\ln\varepsilon| \min\Big\{ \frac{1}{2}\int_{A_\varepsilon(c_1\varepsilon,\rho_0)} |\nabla u|^2 dxdy : \ u \le \tfrac{m}{|\ln\varepsilon|} \text{ on } \{|\Phi_\varepsilon| = c_1\varepsilon\,,\ \Phi_{\varepsilon,1} > 0\}\,, \\
&\qquad u \ge 1-\eta_\varepsilon \text{ on } \{|\Phi_\varepsilon| = \rho_0\,,\ \Phi_{\varepsilon,1} > 0\} \Big\} \\
&= \lim_{\varepsilon\to 0} |\ln\varepsilon| \min\Big\{ \frac{1}{2}\int_{A_\varepsilon(c_1\varepsilon,\rho_0)} |\nabla u|^2 dxdy : \ u = \tfrac{m}{|\ln\varepsilon|} \text{ on } \{|\Phi_\varepsilon| = c_1\varepsilon\,,\ \Phi_{\varepsilon,1} > 0\}\,, \\
&\qquad u = 1-\eta_\varepsilon \text{ on } \{|\Phi_\varepsilon| = \rho_0\,,\ \Phi_{\varepsilon,1} > 0\} \Big\}\,, \qquad (3.30)
\end{aligned}$$

where the last equality easily follows by a truncation argument. The unique minimizer of the last minimum problem is given by $z^\varepsilon := w^\varepsilon \circ \Phi_\varepsilon$, with

$$w^\varepsilon(x,y) := 1-\eta_\varepsilon + \frac{1-\eta_\varepsilon - \frac{m}{|\ln\varepsilon|}}{\ln\frac{\rho_0}{c_1\varepsilon}} \ln\frac{|(x,y)|}{\rho_0}\,.$$

By computing explicitly its energy, we deduce from (3.4) and (4.65) that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} |\ln\varepsilon| F(u_\varepsilon,\Omega_\varepsilon) &= 2\liminf_{\varepsilon\to 0} |\ln\varepsilon| F(u_\varepsilon,\Omega_\varepsilon^+) \ge \liminf_{\varepsilon\to 0} |\ln\varepsilon| \int_{A_\varepsilon(c_1\varepsilon,\rho_0)} |\nabla z^\varepsilon|^2 dxdy \\
&= \liminf_{\varepsilon\to 0} |\ln\varepsilon| \int_{\{c_1\varepsilon<|(x,y)|<\rho_0,\, x>0\}} |\nabla w^\varepsilon|^2 dxdy = \lim_{\varepsilon\to 0} |\ln\varepsilon| \pi \frac{\Big(1-\eta_\varepsilon - \frac{m}{|\ln\varepsilon|}\Big)^2}{\ln\frac{\rho_0}{c_1\varepsilon}} = \pi\,.
\end{aligned}$$

Note that the second equality follows from the conformal character of Φ_ε. For the opposite inequality, observe that by (2.6) we obtain

$$\limsup_{\varepsilon\to 0} |\ln\varepsilon| F(u_\varepsilon,\Omega_\varepsilon) = 2\limsup_{\varepsilon\to 0} |\ln\varepsilon| F(u_\varepsilon,\Omega_\varepsilon^+) \le \frac{\pi}{1-\alpha}\,,$$

where α is any number in $(0,1)$. Hence, (3.3) is established.

Step 4. (Asymptotic behavior) By (3.28) and the final part of Step 2, recalling that $u_\varepsilon \ge 0$ in Ω_ε^+, we conclude that $\{v_\varepsilon \chi_{\Omega_\varepsilon^+/\varepsilon}\}$ is locally equibounded in $L^\infty(\mathbb{R}^2)$. By symmetry, the same holds for $\{v_\varepsilon \chi_{\Omega_\varepsilon/\varepsilon}\}$. Since

$$\begin{cases} \Delta v_\varepsilon = \varepsilon^2 |\ln\varepsilon| W'(\frac{v_\varepsilon}{|\ln\varepsilon|}) & \text{in } \frac{\Omega_\varepsilon}{\varepsilon}\,, \\ \partial_\nu v_\varepsilon = 0 & \text{on } \frac{\partial\Omega_\varepsilon}{\varepsilon}\,, \end{cases}$$

where the $W'(v_\varepsilon/|\ln\varepsilon|)$'s are in turn equibounded in the L^∞-norm, we may apply Proposition 6.2 to obtain the existence of a harmonic function v, with $v(0,0) = 0$ and $\partial_\nu v = 0$ on $\partial\Omega_\infty$, and a subsequence (not relabelled) such that $v_\varepsilon \to v$ in $W^{2,p}_{loc}(\Omega_\infty)$ for every $p \ge 1$ and $\nabla v_\varepsilon \chi_{\Omega_\varepsilon/\varepsilon} \to \nabla v \chi_{\Omega_\infty}$ in $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$. Moreover, by (3.8), (3.10), (3.28), and (3.29) we obtain

$$\ln\frac{|(x-1,y)|}{c_1} \le v(x,y) \le m + \ln\frac{|(x-1,y)|}{c_1}$$

for $(x,y) \in \{x>1\} \setminus \overline{B_{c_1}(1,0)}$. Hence,

$$\frac{v(x,y)}{\ln|(x,y)|} \to 1 \qquad \text{as } |(x,y)| \to +\infty \text{ with } x > 1,$$

and by symmetry we conclude that v solves (3.2).

Step 5. (Uniqueness) In order to conclude the proof we are left with showing that problem (3.2) admits a unique solution. To this aim let v_1 and v_2 be two solutions. Then $w := v_1 - v_2$ satisfies

$$\begin{cases} \Delta w = 0 & \text{in } \Omega_\infty, \\ \partial_\nu w = 0 & \text{on } \partial\Omega_\infty, \\ \dfrac{w(x,y)}{\ln|(x,y)|} \to 0 & \text{as } |(x,y)| \to +\infty, \\ w(0,0) = 0\,. \end{cases}$$

Let w^+ denote the restriction of w to the half-space $\{x > 1\}$. Reflecting w^+ with respect to $\{x = 1\}$ we obtain a $\tilde{w}$ that is harmonic in $\mathbb{R}^2 \setminus \{(x,y) : x = 1, -\frac{\ell}{2} \le y \le \frac{\ell}{2}\}$ and satisfies

$$\lim_{|(x,y)|\to+\infty} \frac{\tilde{w}(x,y)}{\ln|(x,y)|} = 0\,.$$

Consider the function $z := \tilde{w} \circ K_\ell$, where K_ℓ is the Kelvin transform

$$(x,y) \mapsto \frac{\frac{\ell}{2}(x+1,y)}{|(x+1,y)|^2}\,.$$

Then z is harmonic in $B_1(0,0) \setminus \{(0,0)\}$ and satisfies

$$\lim_{(x,y)\to 0} \frac{z(x,y)}{\ln|(x,y)|} = 0\,.$$

It is well known that under these circumstances $(0,0)$ is a removable singularity. In particular z is bounded; that is, $\tilde{w}$ is bounded. Arguing similarly for the restriction of w to $\{x < -1\}$, we conclude that w is bounded in Ω_∞. By the Riemann Mapping Theorem there exists a conformal mapping Ψ that maps the infinite strip $\mathcal{R} := (-1,1) \times \mathbb{R}$ onto Ω_∞. Hence $w \circ \Psi$ in bounded and harmonic in $\mathcal{R}$ and satisfies a homogeneous Neumann condition on $\partial\mathcal{R}$. By reflecting $w \circ \Psi$ infinitely many times, we obtain a bounded entire harmonic function. By Liouville Theorem, we conclude that w is constant and thus $w \equiv 0$.

□

4. The thin neck

The section is devoted to the asymptotic analysis of the family $\{u_\varepsilon\}$ when the height δ of the neck converges to zero faster than the width ε; i.e.,

$$\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon} = 0. \tag{4.1}$$

In fact, unlike the higher dimensional case (see [15]), we need to further distinguish three subregimes according to the following cases:

$$\lim_{\varepsilon\to 0^+} \frac{\delta|\ln\delta|}{\varepsilon} = 0\,, \quad \lim_{\varepsilon\to 0^+} \frac{\delta|\ln\delta|}{\varepsilon} = \ell \in (0,+\infty)\,, \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{\delta|\ln\delta|}{\varepsilon} = +\infty\,.$$

We will refer to them as the *subcritical*, the *critical*, and the *supercritical* thin neck regime, respectively.

In addition to the standing assumptions stated in Section 2, throughout the section we assume that

$$f_1\,,\, f_2 \in C^{1,1}([-1,1])\,,$$

and, without loss of generality, that the normalization condition

$$f_1(1) = f_2(1) = \frac{1}{2}$$

holds. We now state the main results of this section. We start by considering the subcritical case. We will show that in the limit the whole transition from -1 to 1 takes place in the neck . This is done by rescaling the transition profiles u_ε to the fixed domain

$$N := \{(x,y) : |x| < 1, -f_2(x) < y < f_1(y)\} . \tag{4.2}$$

Theorem 4.1 (Subcritical thin neck)**.** *Assume that* (4.1) *holds and that*

$$\lim_{\varepsilon\to 0^+} \frac{\delta|\ln\delta|}{\varepsilon} = 0 . \tag{4.3}$$

Let $\{u_\varepsilon\}$ be the family of minimizing geometrically constrained walls constructed in Proposition 2.1 and $\{v_\varepsilon\}$ the family of rescaled profiles defined by

$$v_\varepsilon(x,y) := u_\varepsilon(\varepsilon x, \delta y) . \tag{4.4}$$

Then $v_\varepsilon \to v$ in $H^1(N)$ and in $L^\infty(N)$, where $v(x,y) := \hat{v}(x)$ with $\hat{v}$ being the unique solution to the one-dimensional problem

$$\min\left\{\frac{1}{2}\int_{-1}^{1}(f_1+f_2)(\theta')^2\,dx : \theta \in H^1(-1,1), \theta(-1) = -1\,, \theta(1) = 1\right\} . \tag{4.5}$$

Moreover,

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\delta} F(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\delta} F(u_\varepsilon, N_\varepsilon) = 2\left(\int_{-1}^{1} \frac{1}{f_1+f_2}\,dx\right)^{-1} . \tag{4.6}$$

Remark 4.2. Also in this case the rescaled profiles v_ε display a universal behavior that depends only on the shape of the neck. The one-dimendional nature of the limiting wall profile is related to the fact that the height of the varying necks is much smaller than their width. The boundary conditions satisfied by $\hat{v}$ show that the complete transition from -1 to 1 asymptotically takes place in the neck. Equation (4.6) reflects the same fact from the energy view point.

We move now to the critical case. Differently from the previous regime, the transition partly occurs also outside the neck. In order to describe the limiting behavior of the family $\{u_\varepsilon\}$ we need to use three different magnifying glasses: the first one, centered at the origin, will catch the asymptotic one-dimensional behavior of the profiles inside the neck through the same scaling considered in Theorem 4.1. The second and the third one act with a dilation of order $\frac{1}{\delta}$ about the points $(\varepsilon, 0)$ and $(-\varepsilon, 0)$, respectively, and deliver the limiting behavior of the transition profiles outside the neck (in the right and in the left bulk regions of the varying domains, respectively). In order to state the next result, we set

$$m_{f_1 f_2} := \int_{-1}^{1} \frac{1}{f_1+f_2}\,dx . \tag{4.7}$$

Theorem 4.3 (Critical thin neck)**.** *Assume that* (4.1) *holds and that*

$$\lim_{\varepsilon\to 0^+} \frac{\delta|\ln\delta|}{\varepsilon} = \ell \in (0,+\infty) . \tag{4.8}$$

Let $\{u_\varepsilon\}$ be the family of minimizing geometrically constrained walls constructed in Proposition 2.1. Then the following statements hold true.

(i) *Let $\{v_\varepsilon\}$ the family of rescaled profiles defined by* (4.4)*. Then $v_\varepsilon \to v$ in $H^1(N)$ and in $L^\infty(N)$, where $v(x,y) := \hat{v}(x)$ with $\hat{v}$ being the unique solution to the one-dimensional problem*

$$\min\left\{\frac{1}{2}\int_{-1}^{1}(f_1+f_2)(\theta')^2\,dx : \theta \in H^1(-1,1)\,,\right.$$
$$\left.\theta(-1) = -\frac{\pi\, m_{f_1 f_2}}{\pi\, m_{f_1 f_2} + 2\ell}\,, \theta(1) = \frac{\pi\, m_{f_1 f_2}}{\pi\, m_{f_1 f_2} + 2\ell}\right\} . \tag{4.9}$$

Moreover,

$$\lim_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, N_\varepsilon) = \frac{2\ell\pi^2\, m_{f_1 f_2}}{\big(\pi\, m_{f_1 f_2} + 2\ell\big)^2}\,. \tag{4.10}$$

(ii) *Define*

$$w_\varepsilon^+(x,y) := |\ln\delta| u_\varepsilon(\delta x + \varepsilon, \delta y) \quad \textit{for } (x,y)\in \widetilde{\Omega}_\varepsilon^+ := \tfrac{\Omega_\varepsilon^+ - (\varepsilon,0)}{\delta}\,. \tag{4.11}$$

Then, setting $c_\varepsilon := u_\varepsilon(\varepsilon, 0)$,

$$c_\varepsilon \to \frac{\pi\, m_{f_1 f_2}}{\pi\, m_{f_1 f_2} + 2\ell} \quad \textit{as } \varepsilon\to 0^+ \tag{4.12}$$

and the functions $w_\varepsilon^+ - c_\varepsilon|\ln\delta|$ *converge in* $W^{2,p}_{loc}(\Omega_\infty^+)$ *for every* $p\ge 1$ *to the unique solution* w^+ *of the problem*

$$\begin{cases} \Delta w^+ = 0 & \textit{in } \Omega_\infty^+,\\ \partial_\nu w^+ = 0 & \textit{on } \partial\Omega_\infty^+,\\ \dfrac{w^+(x,y)}{\ln|(x,y)|} \to \dfrac{2\ell}{\pi\, m_{f_1 f_2} + 2\ell} & \textit{as } |(x,y)|\to +\infty \textit{ with } x>0,\\ \dfrac{w^+(x,y)}{x} \to \dfrac{2\ell\pi}{\pi\, m_{f_1 f_2} + 2\ell} & \textit{uniformly in } y \textit{ as } x\to -\infty,\\ w^+(0,0) = 0\,, \end{cases} \tag{4.13}$$

where

$$\Omega_\infty^+ := \big\{(x,y) :\, x\le 0\,, -\tfrac12 < y < \tfrac12\big\} \cup \{(x,y) :\, x>0\}\,. \tag{4.14}$$

Moreover, $\nabla w_\varepsilon^+ \chi_{\widetilde{\Omega}_\varepsilon^+} \to \nabla w^+ \chi_{\Omega_\infty^+}$ *in* $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$.

The asymptotic behavior of $w_\varepsilon^-(x,y) := |\ln\delta| u_\varepsilon(\delta x - \varepsilon, \delta y) = -w_\varepsilon^+(-x,y)$ *follows by symmetry.*

(iii) *We have*

$$\lim_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon\setminus N_\varepsilon) = \frac{4\pi\ell^2}{\big(\pi\, m_{f_1 f_2} + 2\ell\big)^2}\,. \tag{4.15}$$

Remark 4.4. Since $\mathbb{R}^2\setminus\widetilde{\Omega}_\varepsilon^+ \to \mathbb{R}^2\setminus\Omega_\infty^+$ in the Hausdorff metric, the local convergence of $w_\varepsilon^+ - c_\varepsilon|\ln\delta|$ to w^+ stated in the theorem is well defined (see Remark 3.2).

Remark 4.5. Note that problem (4.9) is the same as (4.5) except for the boundary conditions $\theta(1) = -\theta(-1)\in(0,1)$, which show that only a part of the transition occurs inside the neck. As before, the one-dimensional limiting profile described by (4.9) is determined by the shape of the neck itself. Note also that in (4.13) the geometry is "linearized" and the shape of the neck "weakly" affects the limiting bulk behavior only through the constant $m_{f_1 f_2}$ appearing in the conditions at infinity. However, if we denote by w_ℓ^+ the solution to (4.13), then the family $\{w_\ell^+\}_{\ell>0}$ is universal; i.e., it is indipendent of Ω^r, f_1, f_2, and W. We finally remark that the two conditions at infinity in (4.13) are not independent, as shown by Proposition 4.14 below.

Finally, in the supercritical case the whole transition takes place in the bulk. We apply a dilation of order $1/\delta$ about the points $(\varepsilon,0)$ and $(-\varepsilon,0)$ to capture the asymptotic behavior of the transition profiles in Ω_ε^+ and Ω_ε^-, respectively.

Theorem 4.6 (Supercritical thin neck)**.** *Assume that* (4.1) *holds and that*

$$\lim_{\varepsilon\to 0^+} \frac{\delta|\ln\delta|}{\varepsilon} = +\infty\,. \tag{4.16}$$

Let $\{u_\varepsilon\}$ *be the family of minimizing geometrically constrained walls constructed in Proposition 2.1. Then the following statements hold true.*

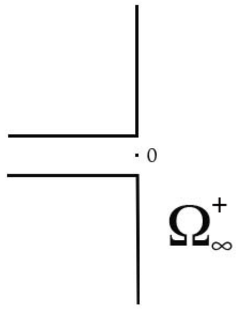

FIGURE 12. The limiting set Ω^+_∞.

(i) *There exist positive constants* $0 < c_1 < c_2 < +\infty$ *such that*

$$c_1 \frac{\varepsilon}{\delta|\ln\delta|} \le \|u_\varepsilon\|_{L^\infty(N_\varepsilon)} \le c_2 \frac{\varepsilon}{\delta|\ln\delta|}\,. \tag{4.17}$$

(ii) *Let* w^+_ε *and* $\widetilde{\Omega}^+_\varepsilon$ *be as in* (4.11). *Then, setting* $c_\varepsilon := u_\varepsilon(\varepsilon, 0)$, *we have* $c_\varepsilon \simeq \frac{\varepsilon}{\delta|\ln\delta|} \to 0$ *and the functions* $w^+_\varepsilon - c_\varepsilon|\ln\delta|$ *converge in* $W^{2,p}_{loc}(\Omega^+_\infty)$ *for every* $p \ge 1$ *to the unique solution* w^+ *of the problem*

$$\begin{cases} \Delta w^+ = 0 & \text{in } \Omega^+_\infty, \\ \partial_\nu w^+ = 0 & \text{on } \partial\Omega^+_\infty, \\ \dfrac{w^+(x,y)}{\ln|(x,y)|} \to 1 & \text{as } |(x,y)| \to +\infty \text{ with } x > 0, \\ \dfrac{w^+(x,y)}{x} \to \pi & \text{uniformly in } y \text{ as } x \to -\infty, \\ w^+(0,0) = 0\,, \end{cases} \tag{4.18}$$

where Ω^+_∞ *is defined as in* (4.14). *Moreover,* $\nabla w^+_\varepsilon \chi_{\widetilde{\Omega}^+_\varepsilon} \to \nabla w^+ \chi_{\Omega^+_\infty}$ *in* $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$.

The asymptotic behavior of $w^-_\varepsilon(x,y) := |\ln\delta| u_\varepsilon(\delta x - \varepsilon, \delta y) = -w^+_\varepsilon(-x,y)$ *follows by symmetry.*

(iii) *We have*

$$\lim_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon) = \lim_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = \pi\,. \tag{4.19}$$

Regarding the local convergence stated in the theorem see Remark 4.4.

Remark 4.7. The behavior of the rescaled profiles $w^\pm_\varepsilon$ in the above theorem is universal; i.e., it independent of the shape of the neck, of the bulk, and of the potential.

Note also that by (4.16) and (4.17) the L^∞-norm of u_ε in the neck N_ε vanishes in the limit: this makes precise the statement that no transition occurs inside the neck. In particular,

$$v_\varepsilon \to 0 \qquad \text{in } H^1(N), \tag{4.20}$$

with v_ε defined as in (4.4). The limit (4.19) reflects the same fact from the energy point of view. Finally, observe that (4.18), (4.19), and (4.20) can be obtained formally by letting $\ell \to +\infty$ in (4.13), (4.15), and (4.9) respectively.

Since the proofs of the theorems are quite technical, in order to better convey the main new ideas we assume throughout thr section that $\partial\Omega^r$ (and, in turn, $\partial\Omega^l$) is flat in a neighborhood of the origin. More precisely, hypothesis (O3) is replaced by:

$$\text{there exists } r_0 > 0 \text{ such that } \partial\Omega^r \cap B_{2r_0}(0,0) \text{ is flat (and vertical).} \tag{4.21}$$

In particular, we may take

$$\Omega^r_\varepsilon = \Omega^r + (\varepsilon, 0)\,,$$

as remarked in Section 2. The general case can be then reduced to this one by using suitable conformal mappings, as shown in Section 3. We leave the details to the reader.

We deal first with the critical thin neck case, which requires a finer analysis. Again for the sake of clarity, we start by considering a flat thin neck; i.e., we assume that $f_1 \equiv f_2 \equiv 1/2$, so that

$$N_\varepsilon := \left\{ (x, y) : |x| \le \varepsilon\,,\ |y| < \frac{\delta}{2} \right\}.$$

The main changes needed to treat the non-flat case are collected in a separate subsection.

4.1. **The flat thin neck.** Throughout the subsection we assume (4.1), (4.21), and

$$f_1 \equiv f_2 \equiv \frac{1}{2}\,. \tag{4.22}$$

We start by estimating the oscillation of u_ε in a δ-neighborhood of the point $(\varepsilon, 0)$. This will be useful in linking the limiting behavior of u_ε in the bulk with the one in the neck. The crucial estimate is provided by the following proposition, whose proof is based on a careful use of Harnack Inequality.

Proposition 4.8. *For $\delta \le x_0 \le \varepsilon - \delta$ let $Q_\delta(x_0, 0)$ denote the square with center at $(x_0, 0)$ and side length δ. Then*

$$\operatorname*{osc}_{Q_\delta(x_0,0)} u_\varepsilon \le C \left(\frac{\delta}{x_0 + \delta} \inf_{Q_\delta(x_0,0)} u_\varepsilon + \delta^2 \right),$$

with $C > 0$ independent of ε and x_0.

Proof. Setting $c_\varepsilon := \inf_{Q_\delta(x_0,0)} u_\varepsilon$, by (2.8) we have $c_\varepsilon > 0$. Let $(x_1, y_1) \in \overline{Q_\delta(x_0, 0)}$ be such that $u_\varepsilon(x_1, y_1) = c_\varepsilon$. Since by (2.8), (2.9), and hypothesis (W3) on W the function u_ε is superharmonic in Ω^+_ε, it easily follows from the Maximum Principle that $u_\varepsilon \ge c_\varepsilon$ in $\Omega^+_\varepsilon \cap \{x > x_1\}$. Thus, setting

$$\theta_\varepsilon(x, y) := \frac{c_\varepsilon x}{x_0 + \delta} \qquad \text{and} \qquad M^+_\varepsilon := N^+_\varepsilon \cap \{x < x_0 + \delta\}\,,$$

we have that $\theta_\varepsilon \le u_\varepsilon$ on the vertical part of ∂M^+_ε and $\partial_\nu \theta_\varepsilon = \partial_\nu u_\varepsilon = 0$ on the remaining part of ∂M^+_ε. Recalling that u_ε is superharmonic, it follows from the Maximum Principle that

$$u_\varepsilon - \theta_\varepsilon \ge 0 \qquad \text{in } M^+_\varepsilon. \tag{4.23}$$

Let $\hat u_\varepsilon$ be the function defined in $Q_{2\delta}(x_0, 0)$ by reflecting u_ε with respect to $\{y = \delta/2\}$ and $\{y = -\delta/2\}$. Note that $\hat u_\varepsilon$ solves the equation $\Delta \hat u_\varepsilon = W'(\hat u_\varepsilon)$ in $Q_{2\delta}(x_0, 0)$ and, by (4.23), $\hat u_\varepsilon - \theta_\varepsilon \ge 0$ in the same region. Set $w_\varepsilon := \hat u_\varepsilon - \theta_\varepsilon$ and $g_\varepsilon := W'(\hat u_\varepsilon) \le 0$. Since $\Delta w_\varepsilon = g_\varepsilon$ and $w_\varepsilon \ge 0$ in $Q_{2\delta}(x_0, 0)$, we may now apply the Harnack Inequality for nonhomogeneous elliptic equations (see for instance [20]) to deduce the existence of a constant K independent of ε and x_0 such that

$$\sup_{Q_\delta(x_0,0)} w_\varepsilon \le K \left(\inf_{Q_\delta(x_0,0)} w_\varepsilon + \delta \|g_\varepsilon\|_{L^2(Q_{2\delta}(x_0,0))} \right) \le K \left(\inf_{Q_\delta(x_0,0)} w_\varepsilon + 2 \max_{[0,1]} |W'| \delta^2 \right), \tag{4.24}$$

where the last inequality easily follows from the definition of g_ε and the fact that $0 < \hat u_\varepsilon \le 1$. Observe now that

$$\operatorname*{osc}_{Q_\delta(x_0,0)} u_\varepsilon = \sup_{Q_\delta(x_0,0)} u_\varepsilon - c_\varepsilon \le \sup_{Q_\delta(x_0,0)} w_\varepsilon + \sup_{Q_\delta(x_0,0)} \theta_\varepsilon - c_\varepsilon \le \sup_{Q_\delta(x_0,0)} w_\varepsilon\,. \tag{4.25}$$

Moreover, since $x_1 \ge x_0 - \delta/2$, we have

$$\begin{aligned} \inf_{Q_\delta(x_0,0)} w_\varepsilon = \inf_{Q_\delta(x_0,0)} (u_\varepsilon - \theta_\varepsilon) &\le c_\varepsilon - \theta_\varepsilon(x_1, y_1) = c_\varepsilon - \frac{c_\varepsilon x_1}{x_0 + \delta} \\ &\le c_\varepsilon - \frac{c_\varepsilon (x_0 - \delta/2)}{x_0 + \delta} = \frac{3\delta}{2(x_0 + \delta)} \inf_{Q_\delta(x_0,0)} u_\varepsilon\,. \end{aligned} \tag{4.26}$$

The thesis of the proposition now follows by combining (4.24), (4.25), and (4.26). □

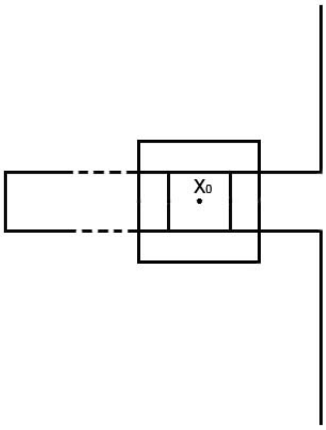

FIGURE 13. The sets $Q_\delta(x_0,0)$ and $Q_{2\delta}(x_0,0)$.

The following lemmas provide useful lower and upper bounds.

Lemma 4.9 (Lower bound)**.** *Let $r_0>0$ be as in (4.21) and set*

$$\eta_\varepsilon := \max\{1-u_\varepsilon(x,y) : |(x-\varepsilon,y)| = r_0,\, x\geq\varepsilon\}\,. \tag{4.27}$$

Define

$$\begin{aligned}
&\widetilde{N}_\varepsilon^+ := \{(x,y) : |y|<\delta/2,\, 0<x<\varepsilon-\delta\}\,,\\
&R_\varepsilon := \{(x,y) : |y|<\delta/2,\, |x-\varepsilon+\delta/2|\leq\delta/2\}\cup\left(\overline{B_\delta(\varepsilon,0)}\cap\{x>\varepsilon\}\right)\,,\\
&A_\varepsilon(\delta,r_0) := \{(x,y) : \delta<|(x-\varepsilon,y)|<r_0\,, x>\varepsilon\}\,, \text{ and}\\
&R := \mathrm{int}(R_\varepsilon-(\varepsilon,0))/\delta = \{(x,y) : |y|<1/2,\, -1<x\leq 0\}\cup(B_1(0,0)\cap\{x>0\})\,.
\end{aligned}$$

The following statements hold true.

(i) *Assume that either (4.8) or (4.16) holds. For m_1, $m_2>0$ define*

$$u_\varepsilon^{lo}(x,y) := \begin{cases} \dfrac{m_1 x}{\delta|\ln\delta|} & \text{for } (x,y)\in\widetilde{N}_\varepsilon^+,\\[2mm] \dfrac{m_2}{|\ln\delta|}\xi\left(\dfrac{x-\varepsilon}{\delta},\dfrac{y}{\delta}\right)+\dfrac{m_1}{\delta|\ln\delta|}(\varepsilon-\delta) & \text{for } (x,y)\in R_\varepsilon,\\[2mm] \dfrac{1-\eta_\varepsilon-\left(\frac{m_1}{\delta|\ln\delta|}(\varepsilon-\delta)+\frac{m_2}{|\ln\delta|}\right)}{\ln\frac{r_0}{\delta}}\ln\dfrac{|(x-\varepsilon,y)|}{r_0}+(1-\eta_\varepsilon) & \text{for } (x,y)\in A_\varepsilon(\delta,r_0), \end{cases} \tag{4.28}$$

where ξ is the solution to the following problem:

$$\begin{cases} \Delta\xi=0 & \text{in } R,\\ \xi=0 & \text{on } \partial R\cap\{x=-1\},\\ \xi=1 & \text{on } \partial B_1(0,0)\cap\{x>0\},\\ \partial_\nu\xi=0 & \text{on the remaining part of } \partial R. \end{cases}$$

Then, there exist m_1, $m_2>0$ independent of ε such that $u_\varepsilon^{lo}\leq u_\varepsilon$ in $\widetilde{N}_\varepsilon^+\cup R_\varepsilon\cup A_\varepsilon(\delta,r_0)$ for ε small enough.

(ii) *Assume that (4.3) holds. Then, for any $m_1\in(0,1)$ there exists $m_2>0$ independent of ε such that the function u_ε^{lo} defined as in (4.28), with $m_1/(\delta|\ln\delta|)$ replaced by m_1/ε, is a lower bound for u_ε in $\widetilde{N}_\varepsilon^+\cup R_\varepsilon\cup A_\varepsilon(\delta,r_0)$, provided that ε is small enough.*

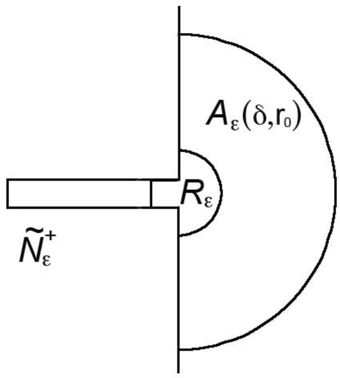

FIGURE 14. The construction of the lower bound.

Proof. Arguing as for (3.4), we have

$$\eta_\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0. \tag{4.29}$$

Since u_ε is superharmonic in Ω_ε^+ and

$$\begin{cases} u_\varepsilon^{lo} \le u_\varepsilon & \text{on } \{x=0\} \cup (\partial B_{r_0}(\varepsilon,0) \cap \{x > \varepsilon\}), \\ \partial_\nu u_\varepsilon^{lo} = \partial_\nu u_\varepsilon = 0 & \text{on the remaining of } \partial\left(\widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0)\right), \end{cases}$$

by the weak comparison principle (see Proposition 6.1) it will be enough to show the existence of m_1, $m_2 > 0$ independent of ε such that u_ε^{lo} is subharmonic. In turn, as u_ε^{lo} is harmonic in the interior of each of the three subregions $\widetilde{N}_\varepsilon^+$, R_ε, and $A_\varepsilon(\delta, r_0)$, it will suffice to find suitable m_1, $m_2 > 0$ such that

$$\partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{lo}|_{\widetilde{N}_\varepsilon^+}) \le \partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{lo}|_{R_\varepsilon}) \quad \text{on } \partial\widetilde{N}_\varepsilon^+ \cap \partial R_\varepsilon \tag{4.30}$$

and

$$\partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{lo}|_{R_\varepsilon}) \le \partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{lo}|_{A_\varepsilon(\delta,r_0)}) \quad \text{on } \partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0), \tag{4.31}$$

where $\nu_{\widetilde{N}_\varepsilon^+}$ and ν_{R_ε} denote the outer unit normal vectors to $\widetilde{N}_\varepsilon^+$ and R_ε, respectively. By (4.8) or (4.16), taking into account (4.29), we may find $m_0 > 0$ such that on $\partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0)$

$$\partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{lo}|_{A_\varepsilon(\delta,r_0)}) = \frac{1 - \eta_\varepsilon - \left(\frac{m_1}{\delta|\ln\delta|}(\varepsilon - \delta) + \frac{m_2}{|\ln\delta|}\right)}{\delta \ln\frac{r_0}{\delta}} \ge \frac{1}{2\delta|\ln\delta|} \quad \text{for } m_1 \le m_0, \tag{4.32}$$

provided that ε is small enough.
By reflecting ξ with respect to $\{y = 1/2\}$ and $\{y = -1/2\}$ we can remove the corner singularities at $(-1, 1/2)$ and $(-1, -1/2)$ and apply Hopf Lemma to deduce that $-\partial_{\nu_R}\xi$ is positive and smooth on $\partial R \cap \{x = -1\}$. Thus,

$$0 < a_0 \le -\partial_{\nu_R}\xi \le a_1 \qquad \text{on } \partial R \cap \{x = -1\}, \tag{4.33}$$

where $a_0 := -\max_{\partial R\cap\{x=-1\}} \partial_{\nu_R}\xi$ and $a_1 := -\min_{\partial R\cap\{x=-1\}} \partial_{\nu_R}\xi$. Similarly, Hopf Lemma and a reflection argument with respect to the y-aixs to remove the corner singularities at $\partial B_1(0,0)\cap\{x = 0\}$ yields

$$0 < b_0 \le \partial_{\nu_R}\xi \le b_1 \qquad \text{on } \partial B_1(0,0) \cap \{x \ge 0\}, \tag{4.34}$$

with $b_0 := \min_{\partial B_1(0,0)\cap\{x\ge 0\}} \partial_{\nu_R}\xi$ and $b_1 := \max_{\partial B_1(0,0)\cap\{x\ge 0\}} \partial_{\nu_R}\xi$. Choosing $m_2 > 0$ so that $b_1 m_2 \le 1/2$, by (4.32), (4.34), and the definition of u_ε^{lo} we have

$$\partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{lo}|_{R_\varepsilon}) \le \frac{b_1 m_2}{\delta|\ln\delta|} \le \frac{1}{2\delta|\ln\delta|} \le \partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{lo}|_{A_\varepsilon(\delta,r_0)}) \quad \text{on } \partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0)\,;$$

that is, (4.31). Once m_2 is fixed, in view of (4.33) the choice $m_1 := \min\{m_0, a_0 m_2\}$ suffices to guarantee (4.30). This concludes the proof of part (i) of the statement.

The proof of the second part is similar. Let $m_1 \in (0,1)$. Then,

$$\partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{lo}|_{A_\varepsilon(\delta,r_0)}) = \frac{1-\eta_\varepsilon - \left(\frac{m_1}{\varepsilon}(\varepsilon-\delta) + \frac{m_2}{|\ln\delta|}\right)}{\delta \ln\frac{r_0}{\delta}} \geq \frac{1-m_1}{2\delta|\ln\delta|}$$

on $\partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0)$, provided that ε is small enough. Choosing m_2 so small that the quantity $b_1 m_2$ (with b_1 as in (4.34)) is smaller than $(1-m_1)/2$, we get (4.31). Finally, noticing that $\partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{lo}|_{\widetilde{N}_\varepsilon^+}) = m_1/\varepsilon$ and $\partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{lo}|_{R_\varepsilon}) \geq a_0 m_2/(\delta|\ln\delta|)$ on $\partial\widetilde{N}_\varepsilon^+ \cap \partial R_\varepsilon$, condition (4.30) is satisfied for ε small enough, thanks to (4.3). □

Lemma 4.10 (Upper bound). *Let $r_0 > 0$, η_ε, $\widetilde{N}_\varepsilon^+$, R_ε, $A_\varepsilon(\delta, r_0)$, and R be as in Lemma 4.9, and assume that either (4.8) or (4.16) holds. Let μ be the constant appearing in (2.3) and for M_1, $M_2 > 0$ define*

$$u_\varepsilon^{up}(x,y) := \begin{cases} -\mu x^2 + \dfrac{M_1 x}{\delta|\ln\delta|} & \text{for } (x,y) \in \widetilde{N}_\varepsilon^+, \\ \dfrac{M_2}{|\ln\delta|}\xi_\varepsilon\left(\dfrac{x-\varepsilon}{\delta}, \dfrac{y}{\delta}\right) - \mu(\varepsilon-\delta)^2 + \dfrac{M_1}{\delta|\ln\delta|}(\varepsilon-\delta) & \text{for } (x,y) \in R_\varepsilon, \\ \left(1 + \mu(\varepsilon-\delta)^2 - \dfrac{M_1}{\delta|\ln\delta|}(\varepsilon-\delta) - \dfrac{M_2}{|\ln\delta|}\right) h_\varepsilon(x,y) + 1 & \text{for } (x,y) \in A_\varepsilon(\delta, r_0), \end{cases} \tag{4.35}$$

where ξ_ε is the solution to

$$\begin{cases} \Delta\xi_\varepsilon = -\delta & \text{in } R, \\ \xi_\varepsilon = 0 & \text{on } \partial R \cap \{x = -1\}, \\ \xi_\varepsilon = 1 & \text{on } \partial B_1(0,0) \cap \{x > 0\}, \\ \partial_\nu \xi_\varepsilon = 0 & \text{on the remaining part of } \partial R, \end{cases}$$

while

$$h_\varepsilon(x,y) := r_\varepsilon(\sqrt{\mu}|(x-\varepsilon, y)|), \tag{4.36}$$

and

$$r_\varepsilon(t) := -\frac{I_0(r_0\sqrt{\mu})}{I_0(r_0\sqrt{\mu})K_0(\delta\sqrt{\mu}) - K_0(r_0\sqrt{\mu})I_0(\delta\sqrt{\mu})}\left(-\frac{K_0(r_0\sqrt{\mu})}{I_0(r_0\sqrt{\mu})}I_0(t) + K_0(t)\right). \tag{4.37}$$

Here, I_0 and K_0 are the zero-order modified Bessel functions. Then, there exist M_1, $M_2 > 0$ independent of ε such that $u_\varepsilon \leq u_\varepsilon^{up}$ in $\widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0)$ for ε small enough.

Proof. We will show that for a suitable choice of M_1 and $M_2 > 0$ the function u_ε^{up} satisfies

$$\begin{cases} \Delta u_\varepsilon^{up} \leq \mu(u_\varepsilon^{up} - 1) & \text{in } \widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0), \\ u_\varepsilon^{up} = u_\varepsilon = 0 & \text{on } \{x = 0\} \cap \{|y| \leq \delta/2\}, \\ u_\varepsilon^{up} = 1 \geq u_\varepsilon & \text{on } \partial B_{r_0}(\varepsilon, 0) \cap \{x > \varepsilon\}, \\ \partial_\nu u_\varepsilon^{up} = \partial_\nu u_\varepsilon = 0 & \text{on the remaining of } \partial\left(\widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0)\right) \end{cases} \tag{4.38}$$

for ε small enough. Recalling that $\Delta u_\varepsilon = W'(u_\varepsilon) \geq \mu(u_\varepsilon - 1)$ by the first inequality in (2.3), the conclusion will then follow from Proposition 6.1.

Since $u_\varepsilon^{up} > 0$ in $\widetilde{N}_\varepsilon^+$ for ε suffiently small, we have $\Delta u_\varepsilon^{up} = -2\mu < -\mu < \mu(u_\varepsilon^{up} - 1)$ in $\widetilde{N}_\varepsilon^+$. Moreover, $\Delta u_\varepsilon^{up} = -M_2/(\delta|\ln\delta|) < \mu(u_\varepsilon^{up} - 1)$ in R_ε for ε small enough. Finally, note that r_ε is the solution to (3.16), with $c_1\varepsilon$ and 2μ replaced by δ and μ, respectively. Hence, $\Delta h_\varepsilon = \mu h_\varepsilon$ or, equivalently, $\Delta u_\varepsilon^{up} = \mu(u_\varepsilon^{up} - 1)$ in $A_\varepsilon(\delta, r_0)$. This shows that the first inequality in (4.38) holds separately in each subregion. Again from (3.16), we have $h_\varepsilon = -1$ on $\partial B_\delta(\varepsilon, 0) \cap \{x > \varepsilon\}$, $h_\varepsilon = 0$ on $\partial B_{r_0}(\varepsilon, 0) \cap \{x > \varepsilon\}$, and $\partial_\nu h_\varepsilon = 0$ on the remaining part of $\partial A_\varepsilon(\delta, r_0)$. From this we easily deduce that all the boundary conditions in (4.38) are satisfied. It remains to show that the variational inequality $\Delta u_\varepsilon^{up} \le \mu(u_\varepsilon^{up}-1)$ holds globally in $\widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0)$. In view of the preceeding observations, it suffices to choose M_1 and M_2 so that

$$\partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{up}|_{\widetilde{N}_\varepsilon^+}) \ge \partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{up}|_{R_\varepsilon}) \quad \text{on } \partial\widetilde{N}_\varepsilon^+ \cap \partial R_\varepsilon \tag{4.39}$$

and

$$\partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{up}|_{R_\varepsilon}) \ge \partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{up}|_{A_\varepsilon(\delta,r_0)}) \quad \text{on } \partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0). \tag{4.40}$$

To this purpose, let ξ be the function introduced in Lemma 4.9 and a_0, a_1, b_0, and b_1 the quantities appearing in (4.33) and (4.34). A continuity argument similar to the one used for (3.13) (in fact easier, since here the ξ_ε's are defined in the fixed domain R) shows that

$$0 < \frac{a_0}{2} < -\partial_{\nu_R}\xi_\varepsilon < 2a_1 \qquad \text{on } \partial R \cap \{x = -1\} \tag{4.41}$$

and

$$0 < \frac{b_0}{2} < \partial_{\nu_R}\xi_\varepsilon < 2b_1 \qquad \text{on } \partial B_1(0,0) \cap \{x \ge 0\}, \tag{4.42}$$

provided that ε is sufficiently small. Moreover, arguing as for (3.22), we have

$$\frac{\lambda_1}{\delta|\ln\delta|} \le \partial_{\nu_{R_\varepsilon}} h_\varepsilon \le \frac{\lambda_2}{\delta|\ln\delta|} \quad \text{on } \partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0), \tag{4.43}$$

with λ_1, $\lambda_2 > 0$ independent of ε.

We now assume (4.8). In this case, we choose M_1 and M_2 so that

$$2a_1 M_2 < M_1 \quad \text{and} \quad 0 < \left(1 - \frac{M_1}{\ell}\right)\lambda_2 < \frac{b_0 M_2}{2}\,. \tag{4.44}$$

Then, for ε small enough we have

$$\partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{up}|_{\widetilde{N}_\varepsilon^+}) > \frac{2a_1M_2}{\delta|\ln\delta|} > \partial_{\nu_{\widetilde{N}_\varepsilon^+}}(u_\varepsilon^{up}|_{R_\varepsilon}) \quad \text{on } \partial\widetilde{N}_\varepsilon^+ \cap \partial R_\varepsilon,$$

where the first inequality follows from $(4.44)_1$ and the definition $u_\varepsilon^{up}|_{\widetilde{N}_\varepsilon^+}$, while the second inequality is a consequence of (4.41) and the definition of $u_\varepsilon^{up}|_{R_\varepsilon}$. Thus, we have checked (4.39). In a similar manner, using (4.8), (4.42), (4.43), the second inequality in (4.44), and the definitions of $u_\varepsilon^{up}|_{R_\varepsilon}$ and $u_\varepsilon^{up}|_{A_\varepsilon(\delta,r_0)}$ we can check that also (4.40) is satisfied for ε small enough. Hence, the lemma is proved under the assumption that (4.8) holds.

We now assume (4.16). Note that, by (4.43), for any M_1, $M_2 > 0$

$$\partial_{\nu_{R_\varepsilon}}(u_\varepsilon^{up}|_{A_\varepsilon(\delta,r_0)}) \le \frac{2\lambda_2}{\delta|\ln\delta|} \quad \text{on } \partial R_\varepsilon \cap \partial A_\varepsilon(\delta, r_0), \tag{4.45}$$

provided that ε is sufficiently small. Choosing M_1, M_2 so that

$$\frac{M_2 b_0}{2} \ge 2\lambda_2 \qquad \text{and} \qquad M_1 \ge 2a_1 M_2$$

and using (4.41), (4.42), and (4.45) we can check as before that both (4.39) and (4.40) are satisfied for ε sufficiently small.

□

Remark 4.11. One can show that

$$0 \le \xi_\varepsilon \le 1 + \delta \qquad \text{in } R. \tag{4.46}$$

Indeed, the lower bound follows at once from the comparison principle, while for the upper bound one may consider the function $\zeta_\varepsilon(x,y) := -\delta x^2 + 1 + \delta$: we have $\Delta\zeta_\varepsilon = -2\delta < -\delta$ in R, $\zeta_\varepsilon \ge \xi_\varepsilon$ on the vertical part of ∂R, and $\partial_\nu\zeta_\varepsilon = \partial_\nu\xi_\varepsilon = 0$ on the remaining part of ∂R. Hence, Proposition 6.1 yields $\xi_\varepsilon \le \zeta_\varepsilon$ in R.

Remark 4.12. A careful inspection of the proof of the previous lemma shows that the choice of the parameters M_1 and M_2 entails the further property that

$$0 < u_\varepsilon^{up} < 1 \qquad \text{in } \widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0). \tag{4.47}$$

Indeed, when (4.8) holds, condition (4.44) guarantees that

$$0 < u_\varepsilon^{up}|_{\widetilde{N}_\varepsilon^+} < \frac{1 + \frac{M_1}{\ell}}{2} < 1 \qquad \text{for } \varepsilon \text{ small enough.} \tag{4.48}$$

In turn, the constant multiplying h_ε in the definition of $u_\varepsilon^{up}|_{A_\varepsilon(\delta,r_0)}$ is positive and strictly smaller than 1. Since $-1 \le h_\varepsilon \le 0$, it follows that $u_\varepsilon^{up}|_{A_\varepsilon(\delta,r_0)} < 1$. Hence, taking into account also the definition of u_ε^{up} in R_ε and (4.46), inequality (4.47) follows. When (4.16) holds, the verification of (4.47) is even easier. Even more is true in this case: one may check that $u_\varepsilon^{up} \le c\varepsilon/(\delta|\ln\delta|)$ in $\widetilde{N}_\varepsilon^+ \cup R_\varepsilon$ for some constant $c > 0$ independent of ε. We leave the details to the reader.

Corollary 4.13. *Assume that either* (4.8) *or* (4.16) *holds. If ε is small enough then for all $0 < x \le \varepsilon$ we have*

$$\operatorname*{osc}_{B_\delta(x,0)\cap\Omega_\varepsilon^+} u_\varepsilon \le \frac{C}{|\ln\delta|}\,, \tag{4.49}$$

with $C > 0$ independent of ε and x.

Proof. Throughout the proof C denotes a positive constant independent of ε and x, which may vary from line to line.
Step 1. We claim that for ε small enough and for all $\delta \le x \le \varepsilon - \delta$ we have

$$\operatorname*{osc}_{Q_\delta(x,0)} u_\varepsilon \le \frac{C}{|\ln\delta|}\,, \tag{4.50}$$

with $C > 0$ independent of ε and x. Indeed, fix any such x. By Lemma 4.10 and the definition of u_ε^{up} we have

$$\inf_{Q_\delta(x,0)} u_\varepsilon \le \min_{Q_\delta(x,0)} u_\varepsilon^{up} \le \frac{M_1 x}{\delta|\ln\delta|}\,.$$

Hence, by Proposition 4.8 we have

$$\operatorname*{osc}_{Q_\delta(x,0)} u_\varepsilon \le C\left(\frac{\delta}{x+\delta}\inf_{Q_\delta(x,0)} u_\varepsilon + \delta^2\right) \le C\left(\frac{M_1 x}{(x+\delta)|\ln\delta|} + \delta^2\right) \le \frac{2CM_1}{|\ln\delta|}$$

for ε is small enough. Thus, the claim is proved.
Step 2. We claim that (4.49) holds whenever $B_\delta(x,0) \subset \{0 \le x \le \varepsilon - \delta/2\}$. Indeed, if $B_\delta(x,0) \subset \{\delta/2 \le x \le \varepsilon - \delta/2\}$, estimate (4.49) follows immediately from the previous step since $B_\delta(x,0)\cap\Omega_\varepsilon^+$ can be covered up with at most 2 squares of the kind considered in (4.50). If $B_\delta(x,0) \subset \{x \le 3\delta\}$, then (4.49) follows from the fact that $0 \le u_\varepsilon \le u_\varepsilon^{up}$, so that

$$\operatorname*{osc}_{B_\delta(x,0)} u_\varepsilon \le \operatorname*{osc}_{\widetilde{N}_\varepsilon^+\cap\{x\le 3\delta\}} u_\varepsilon^{up} \le \frac{C}{|\ln\delta|}\,,$$

where the last inequality is a direct consequence of the definition of u_ε^{up} in $\widetilde{N}_\varepsilon^+$. The claim follows observing that if $B_\delta(x,0) \subset \{0 \le x \le \varepsilon - \delta/2\} \setminus \{\delta/2 \le x \le \varepsilon - \delta/2\}$, then $B_\delta(x,0) \subset \{x \le 3\delta\}$.
Step 3. Set $m_\varepsilon := \min_{\{|y|\le\delta/2\}} u_\varepsilon(\varepsilon - \delta, y)$. By (4.50),

$$0 < m_\varepsilon \le u_\varepsilon(\varepsilon - \delta, y) \le m_\varepsilon + \frac{C}{|\ln\delta|} \tag{4.51}$$

for $|y| \le \delta/2$. Moreover, the comparison principle and the superharmonicity of u_ε yield

$$u_\varepsilon \ge m_\varepsilon \qquad \text{in } \Omega_\varepsilon^+ \cap \{x \ge \varepsilon - \delta\}. \tag{4.52}$$

Finally, with the same notation introduced in Lemma 4.10 (see (4.35)), consider

$$\tilde{u}_\varepsilon^{up}(x,y) := \begin{cases} \dfrac{M}{|\ln\delta|}\xi_\varepsilon\left(\dfrac{x-\varepsilon}{\delta},\dfrac{y}{\delta}\right) + m_\varepsilon + \dfrac{C}{|\ln\delta|} & \text{for } (x,y)\in R_\varepsilon,\\ \left(1-m_\varepsilon-\dfrac{C+M}{|\ln\delta|}\right)h_\varepsilon(x,y)+1 & \text{for } (x,y)\in A_\varepsilon(\delta,r_0), \end{cases}$$

where C is the constant appearing in (4.51) and M satisfies

$$\frac{Mb_0}{2} > 2\lambda_2\,,$$

with b_0 and λ_2 as in (4.42) and (4.43), respectively. Recalling (4.51), we can argue as in Lemma 4.10 to deduce that $\tilde{u}_\varepsilon^{up}$ is an upper bound for u_ε in $R_\varepsilon\cup A_\varepsilon(\delta,r_0)$. Summarizing, also by (4.52) we have

$$0 < m_\varepsilon \le u_\varepsilon \le \tilde{u}_\varepsilon^{up} \qquad \text{in } R_\varepsilon\cup A_\varepsilon(\delta,r_0). \tag{4.53}$$

Since $\operatorname{osc}\tilde{u}_\varepsilon^{up} \le K/|\ln\delta|$ in R_ε for some $K>0$ independent of ε (here we used also (4.46)), we deduce from (4.51) and (4.53) that

$$\operatorname*{osc}_{R_\varepsilon} u_\varepsilon \le \frac{C}{|\ln\delta|}\,,$$

with $C>0$ independent of ε. The conclusion follows by combining the previous estimate with the one provided in Step 2. □

We are now in a position to prove Theorem 4.3, under the particular assumption (4.22).

Proof of Theorem 4.3 (the flat neck case). We split the proof into several steps. We start by examining the behavior of the local minimizers in the neck.

Step 1. (energy bounds in the neck) Note that by (2.6) we have

$$\int_{\Omega_\varepsilon^+}|\nabla u_\varepsilon|^2\,dxdy \le 2F(u_\varepsilon,\Omega_\varepsilon^+) \le \frac{C}{|\ln\delta|} \tag{4.54}$$

for some constant $C>0$ independent of ε. Considering the function v_ε defined in (4.4), it follows

$$\begin{aligned}\int_{N_\varepsilon}|\nabla u_\varepsilon|^2\,dxdy &= 2\int_{N_\varepsilon^+}|\nabla u_\varepsilon|^2\,dxdy\\ &= 2\int_{N_\varepsilon^+}\left[\frac{1}{\varepsilon^2}\left|\partial_x v_\varepsilon\Big(\frac{x}{\varepsilon},\frac{y}{\delta}\Big)\right|^2 + \frac{1}{\delta^2}\left|\partial_y v_\varepsilon\Big(\frac{x}{\varepsilon},\frac{y}{\delta}\Big)\right|^2\right]dxdy\\ &= 2\int_{N^+}\left[\frac{\delta}{\varepsilon}|\partial_x v_\varepsilon(x,y)|^2 + \frac{\varepsilon}{\delta}|\partial_y v_\varepsilon(x,y)|^2\right]dxdy \le \frac{C}{|\ln\delta|}\,,\end{aligned} \tag{4.55}$$

where $N^+ := N\cap\{x>0\}$, with N defined in (4.2). Multiplying both sides of the last inequality by ε/δ and recalling (4.8), we obtain

$$\int_N\left[|\partial_x v_\varepsilon(x,y)|^2 + \frac{\varepsilon^2}{\delta^2}|\partial_y v_\varepsilon(x,y)|^2\right]dxdy = 2\int_{N^+}\left[|\partial_x v_\varepsilon(x,y)|^2 + \frac{\varepsilon^2}{\delta^2}|\partial_y v_\varepsilon(x,y)|^2\right]dxdy \le C \tag{4.56}$$

for some constant $C>0$ independent of ε. Since $\varepsilon/\delta\to\infty$ as $\varepsilon\to0$, by (4.56) we easily deduce that v_ε is bounded in $H^1(N)$ and any weak limit point v is one-dimensional; that is, $v(x,y)=\hat{v}(x)$ for some odd function $\hat{v}\in H^1(-1,1)$. We will show that $\hat{v}$ is independent of the subsequence and solves (4.9). To this aim, let v_ε be a (not relabelled) subsequence such that

$$v_\varepsilon \rightharpoonup v \qquad \text{weakly in } H^1(N) \tag{4.57}$$

for some one-dimensional v of the form

$$v(x,y) = \hat{v}(x) \qquad \text{with } \hat{v}\in H^1(-1,1) \text{ an odd function.} \tag{4.58}$$

We claim that

$$v_\varepsilon \to v \qquad \text{in } L^\infty(N). \tag{4.59}$$

Indeed, by (4.57) we may find $\bar{y} \in (-\frac{1}{2}, \frac{1}{2})$ such that $v_\varepsilon(\cdot, \bar{y}) \rightharpoonup \hat{v}$ weakly in $H^1(-1,1)$ and, thus,

$$v_\varepsilon(\cdot, \bar{y}) \to \hat{v} \qquad \text{in } L^\infty(-1,1). \tag{4.60}$$

Moreover, by (4.49) for all $0 < x \le 1$ we have

$$\operatorname*{osc}_{y\in(-\frac{1}{2},\frac{1}{2})} v_\varepsilon(x,\cdot) \le \frac{C}{|\ln\delta|} \tag{4.61}$$

for some $C > 0$ independent of ε. Hence, combining (4.60) and (4.61) we deduce that $v_\varepsilon \to v$ in $L^\infty(N^+)$ and, in turn, by symmetry that (4.59) holds. From (4.8), (4.55), (4.57), and (4.58) we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, N_\varepsilon^+) &\ge \liminf_{\varepsilon\to 0} |\ln\delta| \frac{1}{2}\int_{N_\varepsilon^+} |\nabla u_\varepsilon|^2 \, dxdy \\
&= \liminf_{\varepsilon\to 0} \frac{1}{2}\int_{N^+} \left[\frac{\delta|\ln\delta|}{\varepsilon}|\partial_x v_\varepsilon(x,y)|^2 + \frac{\varepsilon|\ln\delta|}{\delta}|\partial_y v_\varepsilon(x,y)|^2\right] dxdy \\
&\ge \frac{1}{2}\int_{N^+} \ell|\partial_x v|^2 dxdy = \frac{1}{2}\ell\int_0^1 |\hat{v}'|^2\, dx \ge \frac{1}{2}\ell\hat{v}(1)^2 \,,
\end{aligned} \tag{4.62}$$

where the last inequality follows easily from Jensen's inequality and the fact that $\hat{v}(0) = 0$. We finally remark that by (4.49) (with $x = \varepsilon$), (4.59), (4.58), and the definition of v_ε, setting $c_\varepsilon := u_\varepsilon(\varepsilon, 0) = v_\varepsilon(1,0)$ we have

$$c_\varepsilon - \frac{C}{|\ln\delta|} \le u_\varepsilon \le c_\varepsilon + \frac{C}{|\ln\delta|} \quad \text{in } \overline{B_\delta}(\varepsilon,0)\cap\Omega_\varepsilon^+, \quad c_\varepsilon \to \hat{v}(1) \quad \text{as } \varepsilon\to 0, \tag{4.63}$$

with $C > 0$ independent of ε.

Step 2. (energy bounds in the bulk) Recall that by (4.27) and (4.29) we have $1 - \eta_\varepsilon \le u_\varepsilon \le 1$ on $\partial B_{r_0}(\varepsilon, 0) \cap \{x > \varepsilon\}$, where

$$\eta_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, by Lemma 4.10 and (4.48) we have that $c_\varepsilon = u_\varepsilon(\varepsilon,0) < (1 + M_1/\ell)/2 < 1$ for ε small enough. In particular,

$$c_\varepsilon + \frac{C}{|\ln\delta|} < 1 - \eta_\varepsilon \qquad \text{for } \varepsilon \text{ small enough.} \tag{4.64}$$

Thus, also by (4.63), we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+\setminus N_\varepsilon^+) \ge \liminf_{\varepsilon\to 0} \frac{|\ln\delta|}{2}\int_{\{\delta<|(x-\varepsilon,y)|<r_0,\, x>\varepsilon\}} |\nabla u_\varepsilon|^2 dxdy \\
&\ge \lim_{\varepsilon\to 0} |\ln\delta| \min\Bigg\{\frac{1}{2}\int_{\{\delta<|(x-\varepsilon,y)|<r_0,\, x>\varepsilon\}} |\nabla u|^2 dxdy : \, u \le c_\varepsilon + \tfrac{C}{|\ln\delta|} \text{ on } \partial B_\delta(\varepsilon,0)\cap\{x>\varepsilon\}, \\
&\qquad\qquad u \ge 1-\eta_\varepsilon \text{ on } \partial B_{r_0}(\varepsilon,0)\cap\{x>\varepsilon\}\Bigg\} \\
&= \lim_{\varepsilon\to 0} |\ln\delta| \min\Bigg\{\frac{1}{2}\int_{\{\delta<|(x-\varepsilon,y)|<r_0,\, x>\varepsilon\}} |\nabla u|^2 dxdy : \, u = c_\varepsilon + \tfrac{C}{|\ln\delta|} \text{ on } \partial B_\delta(\varepsilon,0)\cap\{x>\varepsilon\}, \\
&\qquad\qquad u = 1-\eta_\varepsilon \text{ on } \partial B_{r_0}(\varepsilon,0)\cap\{x>\varepsilon\}\Bigg\} \,,
\end{aligned} \tag{4.65}$$

where the last equality easily follows by (4.64) and a standard truncation argument. The unique minimizer of the last minimization problem is given by

$$\tilde{u}_\varepsilon(x,y) = 1 - \eta_\varepsilon + \frac{1 - \eta_\varepsilon - c_\varepsilon - \frac{C}{|\ln\delta|}}{\ln\frac{r_0}{\delta}} \ln\frac{|(x-\varepsilon,y)|}{r_0}\,.$$

By computing explicitly its energy, we deduce from (4.63), (4.29), and (4.65) that

$$\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+ \setminus N_\varepsilon^+) \geq \lim_{\varepsilon\to 0} \frac{|\ln\delta|}{2} \int_{\{\delta<|(x-\varepsilon,y)|<r_0,\, x>\varepsilon\}} |\nabla \tilde{u}_\varepsilon|^2 dxdy$$

$$= \lim_{\varepsilon\to 0} |\ln\delta| \frac{1}{2}\pi \frac{\left(1-\eta_\varepsilon - c_\varepsilon - \frac{C}{|\ln\delta|}\right)^2}{\ln\frac{r_0}{\delta}} = \frac{1}{2}\pi(1-\hat{v}(1))^2 . \tag{4.66}$$

Step 3. (asymptotic behavior in the neck and limit of the energy) By (4.62) and (4.66) we have

$$\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) \geq \frac{1}{2}\ell\hat{v}(1)^2 + \frac{1}{2}\pi(1-\hat{v}(1))^2 \geq \frac{\pi\ell}{2(\pi+\ell)} , \tag{4.67}$$

where the last inequality follows from the fact that

$$\frac{1}{2}\ell t^2 + \frac{1}{2}\pi(1-t)^2 > \frac{\pi\ell}{2(\pi+\ell)} \qquad \text{for } t \neq \frac{\pi}{\pi+\ell} . \tag{4.68}$$

On the other hand, for any fixed $\alpha \in (0,1)$ we may consider the test functions z_ε defined as

$$z_\varepsilon(x,y) := \begin{cases} \dfrac{\pi x}{(\pi+\ell)\varepsilon} & \text{in } N_\varepsilon^+, \\ \dfrac{\pi}{\pi+\ell} & \text{in } \{|(x-\varepsilon,y)| \leq \delta,\, x>\varepsilon\}, \\ \dfrac{\ell}{\pi+\ell}\dfrac{1}{|\ln\delta^{1-\alpha}|}\ln\dfrac{|(x-\varepsilon,y)|}{\delta^\alpha} + 1 & \text{in } \{\delta < |(x-\varepsilon,y)| < \delta^\alpha,\, x>\varepsilon\}, \\ 1 & \text{otherwise in } \Omega_\varepsilon^+. \end{cases}$$

Taking into account the minimality of u_ε, we have

$$\limsup_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) \leq \limsup_{\varepsilon\to 0} |\ln\delta| F(z_\varepsilon, \Omega_\varepsilon^+)$$

$$\leq \lim_{\varepsilon\to 0} |\ln\delta| \left(\frac{1}{2}\int_{\Omega_\varepsilon^+} |\nabla z_\varepsilon|^2\, dxdy + \mathcal{L}^2\left((N_\varepsilon^+ \cup B_{\delta^\alpha}(\varepsilon,0)) \cap \{x>\varepsilon\}\right) \max_{[0,1]} W \right)$$

$$= \lim_{\varepsilon\to 0} |\ln\delta| \frac{1}{2}\int_{\Omega_\varepsilon^+} |\nabla z_\varepsilon|^2\, dxdy = \frac{1}{2}\frac{\pi\ell}{(\pi+\ell)^2}\left(\pi + \frac{\ell}{1-\alpha}\right) , \tag{4.69}$$

where the last equality follows by explicit computation of the Dirichlet energy of z_ε.

Combining (4.67) and (4.69), since α can be chosen arbitrarily close to 0, we conclude

$$\lim_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) = \frac{1}{2}\ell\hat{v}(1)^2 + \frac{1}{2}\pi(1-\hat{v}(1))^2 = \frac{\pi\ell}{2(\pi+\ell)} , \tag{4.70}$$

which, in turn, yields

$$\hat{v}(1) = \frac{\pi}{\pi+\ell} \tag{4.71}$$

thanks to (4.68). Note that the last equality and $(4.63)_2$ imply (4.12). Moreover, the limit in (4.70) is independent of the selected subsequence and thus the full sequence converges. Now, combining (4.62), (4.66), (4.70), and (4.71) one deduces that

$$\lim_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, N_\varepsilon) = 2\lim_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, N_\varepsilon^+)$$

$$= \ell\int_0^1 |\hat{v}'|^2\, dx = \frac{1}{2}\ell\int_{-1}^1 |\hat{v}'|^2\, dx = \ell\hat{v}(1)^2 = \frac{\pi^2\ell}{(\pi+\ell)^2} \tag{4.72}$$

and

$$\lim_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon \setminus N_\varepsilon) = 2\lim_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+ \setminus N_\varepsilon^+) = \pi(1-\hat{v}(1))^2 = \frac{\pi\ell^2}{(\pi+\ell)^2} . \tag{4.73}$$

Note that the fourth equality in (4.72) (together with the fact that $\hat{v}$ is odd) implies that $\hat{v}$ is a linear function. Hence, recalling (4.71), $\hat{v}$ does not depend on the selected subsequence and solves problem (4.9) under the particular assumption (4.22). In turn, equations (4.72) and (4.73) hold for the full sequence and prove (4.10) and (4.15), respectively.

Note that by (4.55)

$$\int_N \left[(\partial_x v_\varepsilon)^2 + \frac{\varepsilon^2}{\delta^2}(\partial_y v_\varepsilon)^2\right] dxdy = \frac{\varepsilon}{\delta}\int_{N_\varepsilon} |\nabla u_\varepsilon|^2\, dxdy\,.$$

Hence, using also (4.8) and (4.72), we have

$$\begin{aligned}\limsup_{\varepsilon\to0^+}\int_N |\nabla v_\varepsilon|^2\, dxdy &\le \limsup_{\varepsilon\to0^+}\int_N \left[(\partial_x v_\varepsilon)^2 + \frac{\varepsilon^2}{\delta^2}(\partial_y v_\varepsilon)^2\right] dxdy = \frac{1}{\ell}\lim_{\varepsilon\to0^+}|\ln\delta|\int_{N_\varepsilon}|\nabla u_\varepsilon|^2\, dxdy\\ &\le \frac{1}{\ell}\lim_{\varepsilon\to0^+}|\ln\delta| F(u_\varepsilon, N_\varepsilon) = \frac{\pi^2}{(\pi+\ell)^2} = \int_N |\nabla v|^2\, dxdy\,.\end{aligned}$$

Recalling (4.57), we have shown that

$$v_\varepsilon \to v \qquad \text{strongly in } H^1(N).$$

Step 4. (lower and upper bounds in the neck) Taking into account (4.63) and the fact that u_ε is superharmonic in N_ε^+, we have that the linear function

$$n_\varepsilon^{lo}(x,y) := \left(\frac{c_\varepsilon}{\varepsilon} - \frac{C}{\varepsilon|\ln\delta|}\right)x \tag{4.74}$$

is a lower bound in the neck. Set $M := \max_{[0,1]}|W'|$ and consider the function

$$n_\varepsilon^{up}(x,y) := -Mx^2 + \left(\frac{c_\varepsilon}{\varepsilon} + \frac{C}{\varepsilon|\ln\delta|} + M\varepsilon\right)x\,. \tag{4.75}$$

Since $\Delta n_\varepsilon^{up} = -2M$, $\Delta u_\varepsilon = W'(u_\varepsilon) > -2M$, $n_\varepsilon^{up}(0,y) = 0$, $n_\varepsilon^{up} = c_\varepsilon + \frac{C}{|\ln\delta|} \ge u_\varepsilon$ on $\partial N_\varepsilon^+\cap\{x=\varepsilon\}$ by (4.63), and $\partial_\nu n_\varepsilon^{up} = 0$ on the remaining part of ∂N_ε^+, it follows from Proposition 6.1 that n_ε^{up} is an upper bound for u_ε in N_ε^+. Summarizing,

$$n_\varepsilon^{lo} \le u_\varepsilon \le n_\varepsilon^{up} \qquad \text{in } N_\varepsilon^+. \tag{4.76}$$

Step 5. (lower and upper bounds in the bulk) We will construct upper and lower bounds in the annulus $A_\varepsilon(\delta, r_0) := \{(x,y) :\, \delta < |(x-\varepsilon, y)| < r_0,\, x > \varepsilon\}$.

Let b_ε^{lo} be the harmonic function defined by

$$b_\varepsilon^{lo}(x,y) := 1 - \eta_\varepsilon + \frac{1 - \eta_\varepsilon - c_\varepsilon + \frac{C}{|\ln\delta|}}{\ln\frac{r_0}{\delta}}\ln\frac{|(x-\varepsilon,y)|}{r_0}\,. \tag{4.77}$$

Since u_ε is superharmonic in $A_\varepsilon(\delta,r_0)$, $b_\varepsilon^{lo} \le u_\varepsilon$ on $\partial A_\varepsilon(\delta,r_0)\cap\Omega_\varepsilon^+$ by (4.27) and (4.63), and $\partial_\nu u_\varepsilon = \partial_\nu b_\varepsilon^{lo} = 0$ on the remaining part of $\partial A_\varepsilon(\delta,r_0)$, the weak comparison principle (Proposition 6.1) implies that b_ε^{lo} is a lower bound for u_ε in $A_\varepsilon(\delta,r_0)$.

Now, let h_ε be the function defined in (4.36) and set

$$b_\varepsilon^{up} := \left(1 - c_\varepsilon - \frac{C}{|\ln\delta|}\right)h_\varepsilon + 1\,. \tag{4.78}$$

Then, arguing as in the proof of Lemma 4.10, we have

$$\begin{cases}\Delta b_\varepsilon^{up} = \mu(b_\varepsilon^{up} - 1) & \text{in } A_\varepsilon(\delta,r_0),\\ b_\varepsilon^{up} = c_\varepsilon + \frac{C}{|\ln\delta|} & \text{on } \partial B_\delta(\varepsilon,0)\cap\{x>\varepsilon\},\\ b_\varepsilon^{up} = 1 & \text{on } \partial B_{r_0}(\varepsilon,0)\cap\{x>\varepsilon\},\\ \partial_\nu b_\varepsilon^{up} = 0 & \text{on the remaining of } \partial A_\varepsilon(\delta,r_0).\end{cases} \tag{4.79}$$

Since $\Delta u_\varepsilon = W'(u_\varepsilon) \ge \mu(u_\varepsilon - 1)$ by the first inequality in (2.3), recalling (4.27) and (4.63), we may apply Proposition 6.1 to deduce that b_ε^{up} is an upper bound for u_ε in $A_\varepsilon(\delta,r_0)$. Summarizing,

$$b_\varepsilon^{lo} \le u_\varepsilon \le b_\varepsilon^{up} \qquad \text{in } A_\varepsilon(\delta,r_0). \tag{4.80}$$

Step 6. (asymptotic behavior of $\{w_\varepsilon^+\}$) Let w_ε^+ be the function introduced in (4.11). Notice that (4.76) is equivalent to saying that for $-\frac{\varepsilon}{\delta} < x < 0$ and $|y| < \frac{1}{2}$ we have

$$|\ln\delta| n_\varepsilon^{lo}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta| \le w_\varepsilon^+(x,y) - c_\varepsilon|\ln\delta| \le |\ln\delta| n_\varepsilon^{up}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta|\,. \tag{4.81}$$

Analogously, by (4.80), for $1 < |(x,y)| < \frac{r_0}{\delta}$ with $x > 0$ we have

$$|\ln\delta| b_\varepsilon^{lo}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta| \le w_\varepsilon^+(x,y) - c_\varepsilon|\ln\delta| \le |\ln\delta| b_\varepsilon^{up}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta|\,. \tag{4.82}$$

Finally, note that (4.63) implies

$$-C \le w_\varepsilon^+ - c_\varepsilon|\ln\delta| \le C \qquad \text{in } \overline{B_1(0,0)}. \tag{4.83}$$

Moreover, using (4.8), (4.29), $(4.63)_2$, (4.71), (4.74), and (4.75), after some elementary calculations we get

$$\begin{aligned} &|\ln\delta| n_\varepsilon^{lo}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta| \to \frac{\pi\ell x}{\pi+\ell} - C \quad \text{as } \varepsilon\to 0,\\ &|\ln\delta| n_\varepsilon^{up}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta| \to \frac{\pi\ell x}{\pi+\ell} + C \quad \text{as } \varepsilon\to 0 \end{aligned} \tag{4.84}$$

for $(x,y)\in\Omega_\infty^+\cap\{x<0\}$, and

$$|\ln\delta| b_\varepsilon^{lo}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta| \to \frac{\ell}{\pi+\ell}\ln|(x,y)| - C \quad \text{as } \varepsilon\to 0^+ \tag{4.85}$$

for $(x,y)\in\{x>0\}\setminus\overline{B_1(0,0)}$. Moreover, it can be easily checked that convergence in (4.84) and (4.85) is in fact uniform on the compact subsets of $\{x>0\}\setminus\overline{B_1(0,0)}$. By the same facts and by (4.36), (4.37), and (4.78) (using the asymptotic expansions of Bessel functions as in the proof of (3.29)) after some lengthy but straigthforward computations we also deduce

$$|\ln\delta| b_\varepsilon^{up}(\delta x+\varepsilon, \delta y) - c_\varepsilon|\ln\delta| \to \frac{\ell}{\pi+\ell}\ln|(x,y)| + C \quad \text{as } \varepsilon\to 0 \tag{4.86}$$

for $(x,y)\in\{x>0\}\setminus\overline{B_1(0,0)}$. Again, the convergence is in fact uniform on the compact subsets of $\{x>0\}\setminus\overline{B_1(0,0)}$ (this can be checked arguing as at the end of Step 2 in the proof of Theorem 3.1). In particular, the lower and upper bounds in (4.81), (4.82), and (4.83) have locally equibounded L^∞-norms, thus, the same holds for $\hat w_\varepsilon^+ := w_\varepsilon^+ - c_\varepsilon|\ln\delta|$. Since

$$\begin{cases} \Delta\hat w_\varepsilon^+ = \delta^2|\ln\delta| W'\left(\frac{w_\varepsilon^+}{|\ln\delta|}\right) & \text{in } \widetilde\Omega_\varepsilon^+\,,\\ \partial_\nu\hat w_\varepsilon^+ = 0 & \text{on } \partial\widetilde\Omega_\varepsilon^+\,, \end{cases}$$

where $\widetilde\Omega_\varepsilon^+$ is defined in (4.11). Since the $W'(w_\varepsilon^+/|\ln\delta|)$'s are in turn equibounded in the L^∞-norm, we may apply Proposition 6.2 to deduce the existence of a harmonic function w^+, with $\partial_\nu\hat w = 0$ on $\partial\Omega_\infty^+$, and a subsequence (not relabelled) such that $w_\varepsilon^+ - c_\varepsilon|\ln\delta| \to w^+$ in $W^{2,p}_{loc}(\Omega_\infty^+)$ for every $p\ge 1$ and $\nabla w_\varepsilon^+\chi_{\widetilde\Omega_\varepsilon^+} \to \nabla w^+\chi_{\Omega_\infty^+}$ in $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$. As $w_\varepsilon^+(0,0) - c_\varepsilon|\ln\delta| = 0$ by the choice of c_ε, we have $w^+(0,0) = 0$. Moreover, by (4.81), (4.82), (4.84), (4.85), and (4.86) we deduce that

$$\frac{\pi\ell x}{\pi+\ell} - C \le w^+(x,y) \le \frac{\pi\ell x}{\pi+\ell} + C \quad \text{for } (x,y)\in\Omega_\infty^+\cap\{x<0\}$$

and

$$\frac{\ell}{\pi+\ell}\ln|(x,y)| - C \le w^+(x,y) \le \frac{\ell}{\pi+\ell}\ln|(x,y)| + C \quad \text{for } (x,y)\in\{x>0\}\setminus\overline{B_1}(0,0).$$

Hence, w^+ is a solution to (4.13). The conclusion of the theorem follows from the fact that problem (4.13) admits a unique solution, as proved in Proposition 4.14 below. □

The following proposition deals with the uniqueness part of Theorem 4.3.

Proposition 4.14. *Let α, $\beta > 0$ and consider the set*

$$\Omega^+_\infty := \{(x,y) : x \le 0, |y| < \tfrac{\alpha}{2}\} \cup \{(x,y) : x > 0\}.$$

Then, the problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega^+_\infty, \\ \partial_\nu w = 0 & \text{on } \partial\Omega^+_\infty, \\ \dfrac{w(x,y)}{\ln|(x,y)|} \to \beta & \text{as } |(x,y)| \to +\infty \text{ with } x > 0, \\ w \text{ grows at most linearly in } \Omega^+_\infty \cap \{x<0\}, \\ w(0,0) = 0 \end{cases} \tag{4.87}$$

admits a unique solution. Moreover,

$$\frac{w(x,y)}{x} \to \frac{\pi\beta}{\alpha} \qquad \text{uniformly in } y \text{ as } x \to -\infty. \tag{4.88}$$

Remark 4.15. The somewhat surprising fact about the previous statement is that the logarithmic behavior of $w|_{\{x>0\}}$ at infinity and the one-dimensional geometry of the domain in $\{x < 0\}$ uniquely determine the asymptotic behavior of $w|_{\{x<0\}}$.

Proof. By an obvious rescaling argument, we may assume without loss of generality that $\alpha = 1$.

The argument is similar to the one presented in Step 4 of the proof of Theorem 3.1. Let w_1 and w_2 be two solutions. Then $\hat{w} := w_1 - w_2$ satisfies

$$\begin{cases} \Delta \hat{w} = 0 & \text{in } \Omega^+_\infty, \\ \partial_\nu \hat{w} = 0 & \text{on } \partial\Omega^+_\infty, \\ \dfrac{\hat{w}(x,y)}{\ln|(x,y)|} \to 0 & \text{as } |(x,y)| \to +\infty, \text{ with } x > 0, \\ w \text{ grows at most linearly in } \Omega^+_\infty \cap \{x<0\}, \\ \hat{w}(0,0) = 0\,. \end{cases}$$

We split the remaining part of the proof into several steps.

Step 1. We claim that $\hat{w}^+ := \hat{w}|_{\{x>0\}}$ is bounded. Indeed, reflecting $\hat{w}^+$ with respect to $\{x = 0\}$ we obtain a function (still denoted by $\hat{w}^+$)that is harmonic in $\mathbb{R}^2 \setminus \{(x,y) : x = 0, |y| \le \frac{1}{2}\}$ and satisfies

$$\lim_{|(x,y)|\to+\infty} \frac{\hat{w}^+(x,y)}{\ln|(x,y)|} = 0\,.$$

By the same Kelvin transform argument used in Step 4 of the proof of Theorem 3.1, we deduce that $\hat{w}^+$ is bounded. Hence, the claim follows.

Step 2. We now consider the restriction $\hat{w}^- := \hat{w}|_{\Omega^+_\infty \cap \{x<0\}}$. By repeated reflections about the lines $\{y = \frac{n}{2}\}$, $n \in \mathbb{Z} \setminus \{0\}$, we obtain a function (still denoted by $\hat{w}^-$) that is harmonic in the half plane $\{x < 0\}$ and grows at most linearly; i.e., there exist a, $b > 0$ such that

$$|\hat{w}^-(x,y)| \le a|x| + b \tag{4.89}$$

for all $x < 0$ and $y \in \mathbb{R}$. Finally, observe that by construction $\hat{w}^-(x,\cdot)$ and, in turn, $\frac{\partial \hat{w}^-}{\partial x}(x,\cdot)$ are 2-periodic functions for all $x < 0$.

Step 3. We claim that $\nabla\hat{w}^-$ and $\nabla^2\hat{w}^-$ are bounded in $\{x < 0\}$.

Indeed, recall that by the so called Cauchy's Estimates (see [18, Theorem 10.25]) , there exists $C > 0$ such that if $B_r(x,y) \subset \{x < 0\}$, then

$$|\nabla\hat{w}^-(x,y)| \le \frac{C}{r} \operatorname*{osc}_{B_r(x,y)} \hat{w}^- \quad \text{and} \quad |\nabla^2\hat{w}^-(x,y)| \le \frac{C}{r^2} \operatorname*{osc}_{B_r(x,y)} \hat{w}^-\,. \tag{4.90}$$

By (4.89) we deduce that

$$\operatorname*{osc}_{B_{|x|}(x,y)} \hat{w}^- \leq 2a|x| + b\,.$$

We may now apply (4.90) with $r := |x|$ to establish the claim.

Step 4. By the previous step, the function $\frac{\partial \hat{w}^-}{\partial x}$ is bounded in $\{x < 0\}$ and extends continuously to $\{x = 0\}$. Set $f(y) := \frac{\partial \hat{w}^-}{\partial x}(0, y)$. Then f is a continuous 2-periodic function. We claim that $\frac{\partial \hat{w}^-}{\partial x}$ coincides with the *Poisson integral of f*; i.e.,

$$\frac{\partial \hat{w}^-}{\partial x}(x, y) = \frac{|x|}{\pi} \int_{\mathbb{R}} \frac{f(t)}{|y-t|^2 + x^2}\, dt \tag{4.91}$$

for all $x < 0$ and $y \in \mathbb{R}$.

To this aim, denote by z the function defined by the right-hand side of (4.91). Then, it can be checked (see [7, Theorem 14]) that z is bounded, harmonic in $\{x < 0\}$, and continuous up to the boundary $\{x = 0\}$, with $z(0, \cdot) = f$. Hence, setting $u(x, y) := -\frac{\partial \hat{w}^-}{\partial x}(-x, y) + z(-x, y)$ for $x > 0$, we extend $\frac{\partial \hat{w}^-}{\partial x} - z$ to a bounded entire harmonic function that vanishes on $\{x = 0\}$. By Liuoville's Theorem we deduce that $\frac{\partial \hat{w}^-}{\partial x} - z = 0$ in $\{x < 0\}$, thereby proving (4.91).

Step 5. Set $\ell_1 := \frac{1}{2}\int_0^2 f(t)\, dt$. We claim that there exists $C > 0$ such that

$$\Big|\frac{\partial \hat{w}^-}{\partial x}(x, y) - \ell_1\Big| \leq \frac{C}{x^2} \tag{4.92}$$

for all $x < 0$ and $y \in \mathbb{R}$. Indeed, by (4.91) and by changing variables twice, we get

$$\begin{aligned}\frac{\partial \hat{w}^-}{\partial x}(x, y) - \ell_1 &= \frac{|x|}{\pi} \int_{\mathbb{R}} \frac{f(t) - \ell_1}{|y-t|^2 + x^2}\, dt \\ &= \frac{|x|}{\pi} \int_{\mathbb{R}} \frac{f(y+s) - \ell_1}{s^2 + x^2}\, ds \\ &= \frac{1}{\pi|x|} \int_{\mathbb{R}} \frac{f(y+s) - \ell_1}{(\frac{s}{|x|})^2 + 1}\, ds \\ &= \frac{1}{\pi} \int_{R} \frac{f(y+|x|r) - \ell_1}{r^2 + 1}\, dr\,. \end{aligned}\tag{4.93}$$

Now recall that f is 2-periodic. In turn, by the definition of ℓ_1, the function

$$F(t) := \int_0^t (f(s) - \ell_1)\, ds$$

is 2-periodic and, thus, bounded. We may then integrate by parts in (4.93) to obtain

$$\begin{aligned}\frac{\partial \hat{w}^-}{\partial x}(x, y) - \ell_1 &= \frac{2}{\pi|x|} \int_{\mathbb{R}} \frac{r}{(r^2+1)^2} F(y + |x|r)\, dr \\ &= \frac{2}{\pi|x|} \int_{\mathbb{R}} \frac{r}{(r^2+1)^2} (F(y + |x|r) - \ell_2)\, dr\,, \end{aligned}\tag{4.94}$$

where we set $\ell_2 := \frac{1}{2}\int_0^2 F(t)\, dt$ and we used the fact that $\int_{\mathbb{R}} \frac{r}{(r^2+1)^2}\, dr = 0$. Defining

$$G(t) := \int_0^t (F(s) - \ell_2)\, ds\,,$$

we have as before that G is a 2-periodic function. Thus, an integration by parts in (4.94) yields

$$\frac{\partial \hat{w}^-}{\partial x}(x, y) - \ell_1 = \frac{2}{\pi|x|^2} \int_{\mathbb{R}} \frac{1 - 3r^2}{(1+r^2)^3} G(y + |x|r)\, dr\,.$$

Hence, (4.92) follows with $C := \frac{2\|G\|_\infty}{\pi} \int_{\mathbb{R}} \frac{|1-3r^2|}{(1+r^2)^3}\, dr$.

Step 6. We claim that $\hat{w}^- - \ell_1 x$ is bounded in $\Omega^+_\infty \cap \{x < 0\}$. Indeed, for $x \leq -1$ by (4.92) we have

$$|\hat{w}^-(x,y) - \ell_1 x| \leq |\hat{w}^-(-1,y) + \ell_1| + \int_x^{-1} \Big| \frac{\partial \hat{w}^-}{\partial x}(s,y) - \ell_1 \Big| \, ds$$
$$\leq \sup_{|y|\leq \frac{1}{2}} |\hat{w}^-(-1,y) + \ell_1| + \int_{-\infty}^{-1} \frac{C}{|s|^2} \, ds \, ,$$

from which the claim follows.

Step 7. By Step 1 and Step 6 we conclude that $\hat{w}$ is bounded above if $\ell_1 \geq 0$ or bounded below if $\ell_1 \leq 0$ (or bounded if $\ell_1 = 0$). Assume without loss of generality that $\ell_1 \geq 0$. Then, by the Riemann Mapping Theorem there exists a conformal mapping Ψ that maps the infinite strip $\mathcal{R} := (-1,1) \times \mathbb{R}$ onto Ω^+_∞. Hence $\hat{w} \circ \Psi$ is bounded above and harmonic in $\mathcal{R}$ and satisfies a homogeneous Neumann condition on $\partial\mathcal{R}$. By repeated reflections of $\hat{w} \circ \Psi$ with respect to the lines $\{y = n\}$, $n \in \mathbb{Z} \setminus \{0\}$, we obtain an entire harmonic function that is bounded above. Liouville's Theorem and the fact that $\hat{w}(0,0) = 0$ imply $\hat{w} \circ \Psi \equiv 0$, which concludes the proof of the uniqueness.

Step 8. In order to show (4.88) (with $\alpha = 1$), it is now enough to exhibit $\beta_0 > 0$ and a function w_{β_0} that solves (4.87) and satisifes (4.88) for $\beta = \beta_0$. Indeed, for any $\beta > 0$ the unique solution to (4.87) would then be given by $(\beta/\beta_0) w_{\beta_0}$, which obviously satisfies (4.88). Observe now that, setting $\beta_0 := \ell/(\ell + \pi)$, the corresponding solution w_{β_0} coincides with the function w^+ constructed in the last step of the proof of Theorem 4.3. □

Proof of Theorem 4.1. (the flat neck case) Define

$$z_\varepsilon(x,y) := \begin{cases} \frac{x}{\varepsilon} & \text{if } x \in (0,\varepsilon), \\ 1 & \text{otherwise.} \end{cases}$$

Then, by minimality of u_ε we have

$$F(u_\varepsilon, \Omega_\varepsilon) = 2F(u_\varepsilon, \Omega^+_\varepsilon) \leq 2F(z_\varepsilon, \Omega^+_\varepsilon) \leq \frac{\delta}{\varepsilon} + 2\delta\varepsilon \max_{[0,1]} W \, . \tag{4.95}$$

Arguing as in Step 1 of the proof of Theorem 4.3, we obtain

$$\int_N \left[|\partial_x v_\varepsilon(x,y)|^2 + \frac{\varepsilon^2}{\delta^2} |\partial_y v_\varepsilon(x,y)|^2 \right] dxdy = \frac{\varepsilon}{\delta} \int_{N_\varepsilon} |\nabla u_\varepsilon|^2 \, dxdy \leq C \, ,$$

where the last inequality (with $C > 0$ independent of ε) follows from (4.95). Hence, passing to a (not relabelled) subsequence, we may assume that

$$v_\varepsilon \rightharpoonup v \qquad \text{weakly in } H^1(N)$$

for some one-dimensional v of the form

$$v(x,y) = \hat{v}(x) \qquad \text{with } \hat{v} \in H^1(-1,1) \text{ an odd function.}$$

Arguing as for (4.62), we deduce

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\delta} F(u_\varepsilon, N^+_\varepsilon) \geq \frac{1}{2} \int_0^1 |\hat{v}'|^2 \, dx \geq \frac{1}{2} \hat{v}(1)^2 \, . \tag{4.96}$$

We claim that

$$\hat{v}(1) = 1 \, . \tag{4.97}$$

To this aim, it is enough to show that $\hat{v}(1) \geq 1$. This follows from the lower bound constructed in Lemma 4.9-(ii), since the constant M_1 can be chosen arbitrarily close to 1.
Combining (4.96) and (4.97), we get

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\delta} F(u_\varepsilon, N^+_\varepsilon) \geq \frac{1}{2} \, .$$

Recalling (4.95), we conclude

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta}F(u_\varepsilon, N_\varepsilon^+) = \lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta}F(u_\varepsilon, \Omega_\varepsilon^+) = \frac{1}{2}\,,$$

which proves (4.6). In turn, the inequalities in (4.96) are, in fact, equalities, which implies that $\hat{v}$ is linear and solves problem (4.5). The strong convergence of v_ε to v can be deduced as in the proof of Theorem 4.3. This concludes the proof of the theorem in the flat case. □

Proof of Theorem 4.6. (the flat neck case). The first statement of the theorem follows from the lower and upper bounds constructed in Lemmas 4.9-(i) and 4.10, respectively.
In particular,

$$c_\varepsilon - \frac{C}{|\ln\delta|} \le u_\varepsilon \le c_\varepsilon + \frac{C}{|\ln\delta|} \quad \text{in } \overline{B_\delta}(\varepsilon,0)\cap\Omega_\varepsilon^+, \quad c_\varepsilon\to 0 \quad \text{as } \varepsilon\to 0,$$

with $C>0$ independent of ε. Having this, statement (ii) can be proven repeating word for word Steps 2–6 of the proof of Theorem 4.3, with $\ell=+\infty$.

□

4.2. **The non-flat thin neck.** In this subsection we prove Theorems 4.1, 4.3 and 4.6 in the general case. Here, the main technical difficulty is in the construction of the lower and upper bounds, which requires more refined elliptic estimates. These estimates are provided in the next two lemmas.

Lemma 4.16. *Let $f_{1,\varepsilon}$, $f_{2,\varepsilon}\in C^{1,1}([0,1])$ satisfy*

$$f'_{i,\varepsilon}(0) = f'_{i,\varepsilon}(1) = 0 \tag{4.98}$$

and

$$\|f_{i,\varepsilon} - c_i\|_{C^{0,1}([0,1])} \le K\frac{\delta}{\varepsilon} \tag{4.99}$$

for some positive constants c_1, c_2, and K independent of ε. Consider the domain $U_\varepsilon := \{(x,y) : 0<x<1\,, -f_{2,\varepsilon}(x) < y < f_{1,\varepsilon}(x)\}$ and let $A_\varepsilon : U_\varepsilon \to \mathbb{M}^{2\times 2}_{sym}$ be a matrix-valued function of class $C^{0,1}$ such that

$$\sup_\varepsilon \|A_\varepsilon\|_{C^{0,1}(U_\varepsilon;\mathbb{M}^{2\times2}_{sym})} \le K < +\infty\,, \qquad \tfrac{1}{2}|\xi|^2 \le A_\varepsilon(x,y)\xi\cdot\xi \le 2|\xi|^2 \quad \text{for all } \xi\in\mathbb{R}^2,$$

and $A_\varepsilon(x,y) = I$ in an η-neighborhood of each corner point of ∂U_ε, where I is the identity matrix. Let ζ_ε be the solution to the problem

$$\begin{cases} \operatorname{div}(A_\varepsilon\nabla\zeta_\varepsilon) = g_\varepsilon & \text{in } U_\varepsilon,\\ \zeta_\varepsilon = 0 & \text{on } \{x=0\},\\ \zeta_\varepsilon = 0 & \text{on } \{x=1\},\\ A_\varepsilon\nabla\zeta_\varepsilon\cdot\nu = n_\varepsilon & \text{on the remaining part of } \partial U_\varepsilon, \end{cases}$$

where $\|g_\varepsilon\|_{L^p(U_\varepsilon)} \le K\frac{\delta}{\varepsilon}$, $p>2$, and $\|n_\varepsilon\|_{C^{0,1}(\partial U_\varepsilon)} \le K\frac{\delta}{\varepsilon}$. Assume also that n_ε vanishes at the corner points of ∂U_ε. Then, $\|\zeta_\varepsilon\|_{C^1(\overline{U}_\varepsilon)} \le C\frac{\delta}{\varepsilon}$, with $C>0$ depending only on c_1, c_2, K, and η.

Proof. Consider the diffeomorphism $\Phi : U_\varepsilon \to R := (0,1)\times(-1,0)$,

$$\Phi_\varepsilon(x,y) := \left(x, \frac{y - f_{1,\varepsilon}(x)}{f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x)}\right)\,,$$

and set $z_\varepsilon := \zeta_\varepsilon\circ\Phi_\varepsilon^{-1}$. Elementary calculations show that z_ε solves

$$\begin{cases} \operatorname{div}(B_\varepsilon\nabla z_\varepsilon) = h_\varepsilon & \text{in } R,\\ z_\varepsilon = 0 & \text{on } \{x=0\},\\ z_\varepsilon = 0 & \text{on } \{x=1\},\\ B_\varepsilon\nabla z_\varepsilon\cdot\nu = p_\varepsilon & \text{on the remaining part } \Gamma \text{ of } \partial R, \end{cases}$$

where $B_\varepsilon := \frac{D\Phi_\varepsilon A_\varepsilon D\Phi_\varepsilon^T}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1}$,

$$h_\varepsilon := \frac{g_\varepsilon}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1}, \quad \text{and} \quad p_\varepsilon := n_\varepsilon \circ \Phi_\varepsilon^{-1} J^1_{\Phi_\varepsilon^{-1}}. \tag{4.100}$$

In the above formula $J^1_{\Phi_\varepsilon^{-1}}$ denotes the 1-dimensional Jacobian of Φ_ε^{-1} defined at every point of ∂R as

$$J^1_{\Phi_\varepsilon^{-1}} := \frac{|(D\Phi_\varepsilon)^T \circ \Phi_\varepsilon^{-1}[\nu]|}{\det D\Phi_\varepsilon^{-1} \circ \Phi_\varepsilon^{-1}}.$$

Moreover, whenever $A_\varepsilon = I$ (in particular, near the corners) an explicit computation gives

$$B_\varepsilon(x,y) = \begin{pmatrix} f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x) & -f'_{1,\varepsilon}(x) - y(f'_{1,\varepsilon}(x) + f'_{2,\varepsilon}(x)) \\ -f'_{1,\varepsilon}(x) - y(f'_{1,\varepsilon}(x) + f'_{2,\varepsilon}(x)) & \frac{1+\left[f'_{1,\varepsilon}(x)+y\left(f'_{1,\varepsilon}(x)+f'_{2,\varepsilon}(x)\right)\right]^2}{f_{1,\varepsilon}(x)+f_{2,\varepsilon}(x)} \end{pmatrix}. \tag{4.101}$$

By standard elliptic estimates we get that

$$\|z_\varepsilon\|_{H^1(R)} \le C(\|h_\varepsilon\|_{L^2(R)} + \|p_\varepsilon\|_{L^2(\Gamma)}), \tag{4.102}$$

where C depends only on the ellipticity constants of B_ε that, in turn, are independent of ε. Let R' be the subset of R obtained by removing the four balls of radius $\eta/2$ centered at the corners. By classical L^p estimates, we have

$$\|z_\varepsilon\|_{W^{2,p}(R')} \le C(\|z_\varepsilon\|_{H^1(R)} + \|h_\varepsilon\|_{L^p(R)} + \|p_\varepsilon\|_{W^{1,1-\frac{1}{p}}(\Gamma)}) \tag{4.103}$$

$$\le C(\|z_\varepsilon\|_{H^1(R)} + \|h_\varepsilon\|_{L^p(R)} + \|p_\varepsilon\|_{C^{0,1}(\Gamma)}), \tag{4.104}$$

with C depending only on η, the ellipticity constants of B_ε, and on the $C^{0,1}$-norms of its coefficients, which, in turn, are determined only by c_1, c_2, and K.

We now want to perform a similar estimate near the corners. We just show it for the origin, since the argument is the same for the remaining corners. Let $\hat{z}$ be the odd extension of z given by

$$\hat{z}_\varepsilon(x,y) := \begin{cases} z_\varepsilon(x,y) & \text{in } R, \\ -z_\varepsilon(-x,y) & \text{in } (-1,0)\times(-1,0). \end{cases}$$

In a similar way, we denote the odd extensions of h_ε, p_ε, and the coefficients of B_ε^{ij} of the matrix B_ε by $\hat{h}_\varepsilon$, $\hat{p}_\varepsilon$, and $\hat{B}_\varepsilon^{ij}$, respectively. Setting $S := R \cup ((-1,0]\times(-1,0))$, it is easy to check that $\hat{z}$ satisfies

$$\begin{cases} \operatorname{div}(C_\varepsilon \nabla \hat{z}_\varepsilon) = \hat{h}_\varepsilon & \text{in } S, \\ C_\varepsilon \nabla \hat{z}_\varepsilon \cdot \nu = \hat{p}_\varepsilon & \text{on } \{y=0\}, \end{cases}$$

where $C_\varepsilon := B_\varepsilon$ in R and

$$C_\varepsilon := \begin{pmatrix} -\hat{B}_\varepsilon^{11} & \hat{B}_\varepsilon^{12} \\ \hat{B}_\varepsilon^{12} & -\hat{B}_\varepsilon^{22} \end{pmatrix} \qquad \text{in } (-1,0)\times(-1,0).$$

Note that the coefficients of the matrix C_ε are Lipschitz continuous in $B_\eta(0,0)$, due to the fact that by (4.98) and (4.101) the coefficient B_ε^{12} vanishes on $B_\eta(0,0)\cap\{x=0\}$. Similarly, recalling (4.100) and the fact that n_ε vanishes at the corner points of ∂U_ε, we have that $\hat{p}_\varepsilon$ is Lipschitz continuous. Hence, we may apply the standard L^p elliptic estimates in $B_\eta(0,0)\cap S$ to obtain

$$\begin{aligned} \|z_\varepsilon\|_{W^{2,p}(B_{\eta/2}(0,0)\cap R)} \le \|\hat{z}_\varepsilon\|_{W^{2,p}(B_{\eta/2}(0,0)\cap S)} &\le C(\|\hat{z}_\varepsilon\|_{H^1(B_\eta(0,0)\cap S)} + \|\hat{h}_\varepsilon\|_{L^p(B_\eta(0,0)\cap S)} \\ &\quad + \|\hat{p}_\varepsilon\|_{W^{1,1-\frac{1}{p}}(-\eta,\eta)}) \\ &\le C(\|z_\varepsilon\|_{H^1(R)} + \|h_\varepsilon\|_{L^p(R)} + \|p_\varepsilon\|_{C^{0,1}([0,1])}). \end{aligned}$$

where the constant C depends only on the ellipticity constants of B_ε, the $C^{0,1}$-norms of its coefficients, and η. By (4.100), (4.102), and the assumptions on g_ε and n_ε, we easily deduce that the right-hand sides of the previous inequality and of (4.103) are bounded by $C\frac{\delta}{\varepsilon}$, for some constant

C depending only on c_1, c_2, K, and η. Thus, the conclusion follows from the Sobolev Embedding Theorem, recalling that $\zeta_\varepsilon = z_\varepsilon \circ \Phi_\varepsilon$. □

Lemma 4.17. *Fix $x_0 \in [0, \frac{\varepsilon}{\delta} - 2]$ and consider the curvilinear polygon*

$$N_{x_0,\varepsilon} := \tag{4.105}$$
$$\{(x,y) : x \in (-\tfrac{\varepsilon}{\delta}, \tfrac{\varepsilon}{\delta}),\ -f_2(\tfrac{\delta}{\varepsilon}x) < y < f_1(\tfrac{\delta}{\varepsilon}x)\} \cap \{(x,y) : x_0 + g_{1,\varepsilon}(y) < x < x_0 + 1 + g_{2,\varepsilon}(y)\},$$

where $\|g_{i,\varepsilon}\|_{C^{1,1}(\mathbb{R})} \le K\frac{\delta}{\varepsilon}$, with $K > 0$ independent of ε. Assume also that the angles at the four corners of $N_{x_0,\varepsilon}$ are equal to $\pi/2$ and that the curves $\{x = g_{i,\varepsilon}(y)\}$ are straight lines in an η-neighborhood of the corner points, for some η sufficiently small and independent of ε.
Let θ_ε be the solution to the problem

$$\begin{cases} \Delta\theta_\varepsilon = g_\varepsilon & \text{in } N_{x_0,\varepsilon}, \\ \theta_\varepsilon = 0 & \text{on } \{x = x_0 + g_{1,\varepsilon}(y)\}, \\ \theta_\varepsilon = 1 & \text{on } \{x = x_0 + 1 + g_{2,\varepsilon}(y)\}, \\ \partial_\nu\theta_\varepsilon = 0 & \text{on the remaining part of } \partial N_{x_0,\varepsilon}, \end{cases}$$

where $\|g_\varepsilon\|_{L^p(N_{x_0,\varepsilon})} \le K\frac{\delta}{\varepsilon}$, with $p > 2$. Finally, define $\theta(x,y) := x - x_0$. Then, $\|\theta_\varepsilon - \theta\|_{C^1} \le C\frac{\delta}{\varepsilon}$, where the constant $C > 0$ depends only on K, η, and not on x_0 and ε.

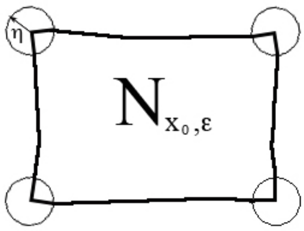

FIGURE 15. The domain $N_{x_0,\varepsilon}$.

Proof. Thanks to the assumptions on the domains $N_{x_0,\varepsilon}$, we may find diffeomorphisms $\Phi_\varepsilon : N_{x_0,\varepsilon} \to U_\varepsilon := \Phi_\varepsilon(N_{x_0,\varepsilon})$ of class $C^{1,1}$ with the following properties:

(i) $U_\varepsilon := \{(x,y) : 0 < x < 1, -f_{2,\varepsilon}(x) < y < f_{1,\varepsilon}(x)\}$ for some $f_{1,\varepsilon},\ f_{2,\varepsilon} \in C^{1,1}([0,1])$ satisfying
$$f'_{i,\varepsilon}(0) = f'_{i,\varepsilon}(1) = 0 \tag{4.106}$$
and $\|f_{i,\varepsilon} - f_{i,\varepsilon}(0)\|_{C^{1,1}([0,1])} \le C\frac{\delta}{\varepsilon}$, with $C > 0$ independent of ε;

(ii) $\|\Phi_\varepsilon - (Id - (x_0, 0))\|_{C^{1,1}} \le C\frac{\delta}{\varepsilon}$ as $\varepsilon \to 0$;

(iii) Φ_ε is coincides with a rotation in the η-neighborhood of the corner points.

We set $\zeta_\varepsilon(x,y) := \theta_\varepsilon \circ \Phi_\varepsilon^{-1}(x,y) - (x - x_0)$. Then, ζ_ε solves the problem

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla\zeta_\varepsilon) = \tilde{g}_\varepsilon & \text{in } U_\varepsilon, \\ \zeta_\varepsilon = 0 & \text{on } \{x = 0\}, \\ \zeta_\varepsilon = 0 & \text{on } \{x = 1\}, \\ A_\varepsilon \nabla\zeta_\varepsilon \cdot \nu = n_\varepsilon & \text{on the remaining part of } \partial U_\varepsilon, \end{cases}$$

where $A_\varepsilon := \frac{D\Phi_\varepsilon D\Phi_\varepsilon^T}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1}$, $\tilde{g}_\varepsilon := \frac{g_\varepsilon}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1} - \operatorname{div}(A_\varepsilon(1,0))$, and

$$n_\varepsilon := A_\varepsilon(1,0) \cdot \left(\frac{f'_{i,\varepsilon}}{\sqrt{1+(f'_{i,\varepsilon})^2}}, (-1)^i \frac{1}{\sqrt{1+(f'_{i,\varepsilon})^2}} \right) \qquad \text{on } \{(x,y) : y = f'_{i,\varepsilon}(x)\}.$$

It is clear that $\|A_\varepsilon - I\|_{C^{0,1}} \to 0$ as $\varepsilon \to 0$ (where I is the identity matrix), $\|\tilde{g}_\varepsilon\|_{L^p(U_\varepsilon)} \le C\frac{\delta}{\varepsilon}$, and $\|n_\varepsilon\|_{C^{0,1}} \le C\frac{\delta}{\varepsilon}$. Moreover, using the fact that A_ε is diagonal in a neighborhood of the corner points of U_ε and taking into account (4.106), we also deduce that n_ε vanishes at the corner points of U_ε. Hence, we may apply Lemma 4.16 to conclude that $\|\zeta_\varepsilon\|_{C^1(\overline{U}_\varepsilon)} \le C\frac{\delta}{\varepsilon}$. In turn, the same holds for $\theta_\varepsilon - \theta \circ \Phi_\varepsilon = \zeta_\varepsilon \circ \Phi_\varepsilon$. The conclusion follows now easily, recalling property (ii) stated at the beginning of the proof. ☐

The next lemma provides the main building block for the construction of the lower bound in the neck. We use the following notation: For $x \in \mathbb{R}$ the symbol $[x]$ stands for the integer part of x.

Lemma 4.18. *For every ε sufficiently small, there exists $g_\varepsilon \in C^{1,1}(\mathbb{R})$ with the following properties:*

(i) $\|g_\varepsilon\|_{C^{1,1}} \le K\frac{\delta}{\varepsilon}$, *with K independent of ε;*

(ii) *setting*

$$\widehat{N}_\varepsilon^+ := \{(x,y) : 0 < x < \tfrac{\varepsilon}{\delta}, -f_2(\tfrac{\delta}{\varepsilon}x) < y < f_1(\tfrac{\delta}{\varepsilon}x)\} \cap \{(x,y) : 0 < x < [\tfrac{\varepsilon}{\delta}] - 1 + g_\varepsilon(y)\},$$

there exists θ_ε satisfying

$$\begin{cases} \Delta\theta_\varepsilon \ge 0 & \text{in } \widehat{N}_\varepsilon^+, \\ \theta_\varepsilon(0,y) = 0, \\ \theta_\varepsilon([\tfrac{\varepsilon}{\delta}] - 1 + g_\varepsilon(y), y) = \text{const} =: d_\varepsilon, \\ \partial_\nu\theta_\varepsilon = 0 & \text{on the remaining part of } \partial\widehat{N}_\varepsilon^+, \end{cases} \tag{4.107}$$

and

$$m_1 \le \frac{\partial\theta_\varepsilon}{\partial x} \le m_2, \tag{4.108}$$

with $0 < m_1 < m_2$ independent of ε. Moreover, θ_ε is asymptotically one-dimensional as $\varepsilon \to 0$; more precisely,

$$\left|\frac{\partial\theta_\varepsilon}{\partial y}\right| \le m_3\frac{\delta}{\varepsilon}, \tag{4.109}$$

with $m_3 > 0$ independent of ε. Finally, for any $\eta > 0$ we have

$$(m_1 - \eta)x \le \theta_\varepsilon(x,y) \le (m_2 + \eta)x \quad \text{for } (x,y) \in \widehat{N}_\varepsilon^+, \tag{4.110}$$

provided that ε is sufficiently small.

Proof. The idea is to split $\widehat{N}_\varepsilon^+$ into the (disjoint) union of domains of the type considered in Lemma 4.17. More precisely, for every $k = 0, \dots, [\frac{\varepsilon}{\delta}] - 2$ we construct functions $g_{k,\varepsilon}$ such that the domain $N_{k,\varepsilon}$ defined as in (4.105), with $x_0 = k$ and $g_{1,\varepsilon}$, $g_{2,\varepsilon}$ replaced by $g_{k-1,\varepsilon}$, $g_{k,\varepsilon}$, respectively, satisfies the assumptions of Lemma 4.17. Note that this is possible since the derivatives of the functions $x \mapsto f_i(\frac{\delta}{\varepsilon}x)$ are $\frac{\delta}{\varepsilon}$-small in the $C^{0,1}$-norm. Note also that we are using the convention $g_{-1,\varepsilon} = 0$, which is admissible since $f'_1(0) = f'_2(0) = 0$. Finally, we may construct the domains in such a way that the corners points of $N_{k,\varepsilon}$ lie on the lines $\{x = k\}$ and $\{x = k+1\}$ (see the picture below).

Let $\theta_{k,\varepsilon}$ be the solution to the following problem

$$\begin{cases} \Delta\theta_{k,\varepsilon} = 0 & \text{in } N_{k,\varepsilon}, \\ \theta_{k,\varepsilon} = 0 & \text{on } \{x = k + g_{k-1,\varepsilon}(y)\}, \\ \theta_{k,\varepsilon} = 1 & \text{on } \{x = k + 1 + g_{k,\varepsilon}(y)\}, \\ \partial_\nu\theta_{k,\varepsilon} = 0 & \text{on the remaining part of } \partial N_{k,\varepsilon}. \end{cases}$$

FIGURE 16. The domains $N_{k,\varepsilon}$.

Then, by Lemma 4.17 we have that

$$1 - C\frac{\delta}{\varepsilon} \le \frac{\partial \theta_{k,\varepsilon}}{\partial x} \le 1 + C\frac{\delta}{\varepsilon}\,, \qquad \left|\frac{\partial \theta_{k,\varepsilon}}{\partial y}\right| \le C\frac{\delta}{\varepsilon} \tag{4.111}$$

and

$$1-C\frac{\delta}{\varepsilon} \le \partial_{\nu_{k,\varepsilon}}\theta_{k,\varepsilon} \le 1+C\frac{\delta}{\varepsilon}\,, \quad 1-C\frac{\delta}{\varepsilon} \le -\partial_{\nu_{k+1,\varepsilon}}\theta_{k+1,\varepsilon} \le 1+C\frac{\delta}{\varepsilon} \quad \text{ on } \partial N_{k,\varepsilon}\cap\partial N_{k+1,\varepsilon}, \tag{4.112}$$

for some $C > 0$ independent of k and ε, where $\nu_{k,\varepsilon}$ and $\nu_{k+1,\varepsilon}$ denote the outer unit normals to $\partial N_{k,\varepsilon}$ and $\partial N_{k+1,\varepsilon}$, respectively. Note that in the last estimate we also used the fact that $\|g_{k,\varepsilon}\|_{C^{1,1}} \le K\frac{\delta}{\varepsilon}$ so that the normal derivatives appearing in (4.112) are $\frac{\delta}{\varepsilon}$-close to derivative in the x-direction. For every k, let

$$\lambda_k := \left(\frac{1 + C\frac{\delta}{\varepsilon}}{1 - C\frac{\delta}{\varepsilon}}\right)^k$$

and define θ_ε as

$$\theta_\varepsilon := \theta_{0,\varepsilon} \quad \text{in } N_{0,\varepsilon}\,, \qquad \theta_\varepsilon := \lambda_k\theta_{k,\varepsilon} + \sum_{j=0}^{k-1}\lambda_j \quad \text{in } N_{k,\varepsilon} \text{ for } k \ge 1.$$

Note that θ_ε is continuous. Moreover, by construction and by (4.111), estimates (4.108) and (4.109) hold for ε small enough, with

$$m_1 := \frac{1}{2}\,, \qquad m_2 := 1 + \lim_{\varepsilon\to 0+}\left(\frac{1 + C\frac{\delta}{\varepsilon}}{1 - C\frac{\delta}{\varepsilon}}\right)^{[\frac{\varepsilon}{\delta}]-2} = 1 + \mathrm{e}^{2C}\,, \qquad \text{and} \qquad m_3 := Cm_2\,.$$

The function θ_ε clearly satisifes the last three equalities in (4.107). Hence, it remains to show that $\Delta\theta_\varepsilon \ge 0$ in the sense of distributions. To this aim, since θ_ε is harmonic in each domain $N_{k,\varepsilon}$, it is enough to check that for every $k \ge 1$

$$-\lambda_k\partial_{\nu_{k,\varepsilon}}\theta_{k,\varepsilon} \ge \lambda_{k-1}\partial_{\nu_{k-1,\varepsilon}}\theta_{k-1,\varepsilon} \qquad \text{on } \partial N_{k-1,\varepsilon}\cap\partial N_{k,\varepsilon}. \tag{4.113}$$

But the last inequality follows immediately from the definition of λ_k and estimate (4.112).
Finally, (4.110) follows easily from (4.108) and (4.109). □

We now show the corresponding construction for the upper bound in the neck.

Lemma 4.19. *For every ε sufficiently small, there exists $g_\varepsilon \in C^{1,1}(\mathbb{R})$ with the following properties:*

(i) $\|g_\varepsilon\|_{C^{1,1}} \le K\frac{\delta}{\varepsilon}$*, with K independent of ε;*

(ii) *defining $\widehat{N}_\varepsilon^+$ as in Lemma 4.18, for all $M > 0$ there exists $\zeta_{\varepsilon,M}$ satisfying*

$$\begin{cases} \Delta\zeta_{\varepsilon,M} \le -\frac{2\mu\delta^2|\ln\delta|}{M} & \text{in } \widehat{N}_\varepsilon^+,\\ \zeta_{\varepsilon,M}(0,y) = 0\,, \\ \zeta_{\varepsilon,M}(\left[\frac{\varepsilon}{\delta}\right] - 1 + g_\varepsilon(y), y) = \text{const} =: d_{\varepsilon,M}\,, \\ \partial_\nu\zeta_{\varepsilon,M} = 0 & \text{on the remaining part of } \partial\widehat{N}_\varepsilon^+, \end{cases} \tag{4.114}$$

where μ is defined in (2.3)*, and*

$$n_1 \le \frac{\partial \zeta_{\varepsilon,M}}{\partial x} \le n_2\,, \tag{4.115}$$

with $0 < n_1 < n_2$ independent of ε, μ, and M. Moreover, $\zeta_{\varepsilon,M}$ is asymptotically one-dimensional as $\varepsilon \to 0$; more precisely,

$$\left|\frac{\partial \zeta_{\varepsilon,M}}{\partial y}\right| \le n_3 \frac{\delta}{\varepsilon}\,, \tag{4.116}$$

with $n_3 > 0$ independent of ε and M. Finally, for any $\eta > 0$ we have

$$(n_1 - \eta)x \le \zeta_{\varepsilon,M}(x,y) \le (n_2 + \eta)x\,, \tag{4.117}$$

provided that ε is sufficiently small.

Proof. The argument is essentially the same as the one presented in the proof of the previous lemma, with the only difference that the function $\theta_{k,\varepsilon}$ must be replaced by the function $\zeta_{k,\varepsilon}$ that solves the problem

$$\begin{cases} \Delta \zeta_{k,\varepsilon} = -\frac{2\mu\delta^2|\ln\delta|}{M} & \text{in } \Omega_{k,\varepsilon}, \\ \zeta_{k,\varepsilon} = 0 & \text{on } \{x = k + g_{k-1,\varepsilon}(y)\}, \\ \zeta_{k,\varepsilon} = 1 & \text{on } \{x = k + 1 + g_{k,\varepsilon}(y)\}, \\ \partial_\nu \zeta_{k,\varepsilon} = 0 & \text{on the remaining part of } \partial\Omega_{k,\varepsilon}, \end{cases}$$

and the definition of the λ_k is now given by

$$\lambda_k := \left(\frac{1 - C\frac{\delta}{\varepsilon}}{1 + C\frac{\delta}{\varepsilon}}\right)^k .$$

Arguing as before, one can show that the estimates (4.115), (4.116) and (4.117) hold with $n_1 = \mathrm{e}^{-2C}/2$, $n_2 = 2$ and $n_3 = C$. Similarly, one can show that (4.114) holds. We only mention the fact that crucial inequality to be checked is now (4.113) with the reversed sign. □

Given $x_0 \in (\delta, \varepsilon - 3\delta)$ and $\eta > 0$ (sufficiently small), we set

$$S_\eta(x_0) := \{(x,y) : |x - x_0| < \tfrac{\eta}{2}, -\delta f_2(\tfrac{x}{\varepsilon}) < y < \delta f_1(\tfrac{x}{\varepsilon})\}\,.$$

Proposition 4.20. *For $\delta \le x_0 \le \varepsilon - 3\delta$ let $S_{\frac{\delta}{2}}(x_0)$ be defined as above. Then*

$$\operatorname*{osc}_{S_{\frac{\delta}{2}}(x_0)} u_\varepsilon \le C\left(\frac{\delta}{x_0 + \delta} \inf_{S_{\frac{\delta}{2}}(x_0)} u_\varepsilon + \delta^2\right),$$

with $C > 0$ independent of ε and x_0.

Proof. Set $c_\varepsilon := \inf_{S_{\frac{\delta}{2}}(x_0)} u_\varepsilon$. By (2.8) we have $c_\varepsilon > 0$. Let $(x_1, y_1) \in \overline{S_{\frac{\delta}{2}}(x_0)}$ be such that $u_\varepsilon(x_1, y_1) = c_\varepsilon$. Since u_ε is superharmonic in Ω_ε^+, it easily follows from the Maximum Principle that $u_\varepsilon \ge c_\varepsilon$ in $\Omega_\varepsilon^+ \cap \{x > x_1\}$. Let θ_ε be defined as in Lemma 4.18. Let k_ε be the largest integer such that $\frac{\delta}{2} + k_\varepsilon\delta \le x_0$. Replacing θ_ε by the function (still denoted by θ_ε)

$$\begin{cases} \theta_\varepsilon(\frac{1}{2} + k_\varepsilon - \frac{x_0}{\delta} + x, y) & \text{if } x \ge \frac{x_0}{\delta} - \frac{1}{2} - k_\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

we may assume that θ_ε is harmonic in the region $\frac{1}{\delta}S_{\frac{3}{4}\delta}(x_0)$. Moreover, by (4.110) we have $\frac{m_1}{2}x \le \delta\theta_\varepsilon(\frac{x}{\delta}, \frac{y}{\delta}) \le 2m_2 x$ for $(x,y) \in N_\varepsilon^+$, $x \ge \delta$, and for ε sufficiently small. In particular, we may find λ_ε, with

$$\frac{c_\varepsilon}{2m_2(x_0 + \delta)} \le \lambda_\varepsilon \le \frac{2c_\varepsilon}{m_1(x_0 + \delta)}\,, \tag{4.118}$$

such that, setting $\tilde{\theta}_\varepsilon(x,y) := \lambda_\varepsilon \delta \theta_\varepsilon(\frac{x}{\delta}, \frac{y}{\delta})$, we have

$$c_\varepsilon = \max\{\tilde{\theta}_\varepsilon(x_0+\delta, y) \,:\, -\delta f_2(\tfrac{x_0+\delta}{\varepsilon}) \le y \le \delta f_1(\tfrac{x_0+\delta}{\varepsilon})\}\,. \tag{4.119}$$

Recalling (4.108) and (4.109), we deduce

$$|\nabla \tilde{\theta}_\varepsilon| \le 2m_2\lambda_\varepsilon \tag{4.120}$$

for ε small enough. Set $M^+_\varepsilon = N^+_\varepsilon \cap \{x < x_0 + \delta\}$. We clearly have that $\tilde{\theta}_\varepsilon \le u_\varepsilon$ on the vertical part of ∂M^+_ε and $\partial_\nu \theta_\varepsilon = 0$ on the remaining part of ∂M^+_ε. Since $\tilde{\theta}_\varepsilon$ is subharmonic (and u_ε is superharmonic), it follows from the Maximum Principle that

$$w_\varepsilon := u_\varepsilon - \tilde{\theta}_\varepsilon \ge 0 \qquad \text{in } M^+_\varepsilon. \tag{4.121}$$

Let $\Phi_\varepsilon \,:\, S_{\frac{3}{4}\delta}(x_0) \to R_\delta := (x_0 - \frac{3}{8}\delta, x_0 + \frac{3}{8}\delta) \times (-\delta, 0)$, $\Phi_\varepsilon(x,y) := \Big(x, \frac{y - \delta f_1(\frac{x}{\varepsilon})}{f_1(\frac{x}{\varepsilon}) + f_2(\frac{x}{\varepsilon})}\Big)$, and note that the function $\tilde{w}_\varepsilon := w_\varepsilon \circ \Phi_\varepsilon^{-1}$ solves

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla \tilde{w}_\varepsilon) = g_\varepsilon & \text{in } R_\delta\,, \\ A_\varepsilon \nabla \tilde{w}_\varepsilon \cdot \nu = 0 & \text{on } \{y = -\delta\} \cup \{y = 0\}\,, \end{cases}$$

where $A_\varepsilon := \frac{D\Phi_\varepsilon (D\Phi_\varepsilon)^T}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1}$ and $g_\varepsilon := \frac{W'(u_\varepsilon)}{\det D\Phi_\varepsilon} \circ \Phi_\varepsilon^{-1}$. Denote by $\widehat{w}_\varepsilon$, $\widehat{A}^{ij}_\varepsilon$, and $\widehat{g}_\varepsilon$ the extensions of $\tilde{w}_\varepsilon$, A^{ij}_ε, and g_ε, respectively, to the domain $R'_\delta := (x_0 - \frac{3}{8}\delta, x_0 + \frac{3}{8}\delta) \times (-2\delta, \delta)$ obtained by reflection with respect to $y = -\delta$ and $y = 0$. Then, $\widehat{w}_\varepsilon$ is a non-negative solution to the equation

$$\operatorname{div}(B_\varepsilon \nabla \widehat{w}_\varepsilon) = \widehat{g}_\varepsilon \qquad \text{in } R'_\delta\,,$$

where

$$B_\varepsilon := \begin{pmatrix} \widehat{A}^{11}_\varepsilon & -\widehat{A}^{12}_\varepsilon \\ -\widehat{A}^{12}_\varepsilon & \widehat{A}^{22}_\varepsilon \end{pmatrix}.$$

Note that B_ε is uniformly elliptic with ellipticity constants independent of ε. Since $\Phi_\varepsilon(S_{\frac{\delta}{2}}(x_0)) = (x_0 - \frac{\delta}{4}, x_0 + \frac{\delta}{4}) \times (-\delta, 0)$, we may now apply the Harnack Inequality for nonhomogeneous elliptic equations (see for instance [20]) to deduce the existence of a constant K independent of ε and x_0 such that

$$\sup_{S_{\frac{\delta}{2}}(x_0)} w_\varepsilon = \sup_{\Phi_\varepsilon(S_{\frac{\delta}{2}}(x_0))} \tilde{w}_\varepsilon \le K \left(\inf_{\Phi_\varepsilon(S_{\frac{\delta}{2}}(x_0))} \tilde{w}_\varepsilon + \delta \|\widehat{g}_\varepsilon\|_{L^2(R'_\delta)} \right) \tag{4.122}$$

$$\le K \left(\inf_{S_{\frac{\delta}{2}}(x_0)} w_\varepsilon + C\delta^2 \right)\,, \tag{4.123}$$

where the constant C depends only on C^1 norms of W' and Φ_ε. Observe now that

$$\operatorname*{osc}_{S_{\frac{\delta}{2}}(x_0)} u_\varepsilon = \sup_{S_{\frac{\delta}{2}}(x_0)} u_\varepsilon - c_\varepsilon \le \sup_{S_{\frac{\delta}{2}}(x_0)} w_\varepsilon + \sup_{S_{\frac{\delta}{2}}(x_0)} \tilde{\theta}_\varepsilon - c_\varepsilon \le \sup_{S_{\frac{\delta}{2}}(x_0)} w_\varepsilon\,, \tag{4.124}$$

where in the last inequality we used the fact that $\tilde{\theta}_\varepsilon \le c_\varepsilon$ in M^+_ε, which follows from (4.119), the fact that $\tilde{\theta}_\varepsilon = 0$ on $\{x = 0\}$, and the subharmonicity of $\tilde{\theta}_\varepsilon$.

By (4.119), there exists $-\delta f_2(\frac{x_0+\delta}{\varepsilon}) \le y_2 \le \delta f_1(\frac{x_0+\delta}{\varepsilon})$ such that $\tilde{\theta}_\varepsilon(x_0+\delta, y_2) = c_\varepsilon$. Moreover, using (4.118), (4.120), and the fact that $|(x_0+\delta, y_2) - (x_1, y_1)| \le C\delta$ (with C depending only on f_1 and f_2), we have

$$\inf_{S_{\frac{\delta}{2}}(x_0)} w_\varepsilon = \inf_{S_{\frac{\delta}{2}}(x_0)} (u_\varepsilon - \tilde{\theta}_\varepsilon) \le c_\varepsilon - \tilde{\theta}_\varepsilon(x_1, y_1) = \tilde{\theta}_\varepsilon(x_0+\delta, y_2) - \tilde{\theta}_\varepsilon(x_1, y_1)$$

$$\le 2m_2\lambda_\varepsilon C\delta \le \frac{4C\delta m_2 c_\varepsilon}{m_1(x_0+\delta)} = \frac{4C\delta m_2}{m_1(x_0+\delta)} \inf_{S_{\frac{\delta}{2}}(x_0)} u_\varepsilon\,. \tag{4.125}$$

The thesis of the proposition follows now easily by combining (4.122), (4.124), and (4.125). □

Lemma 4.21 (Lower bound)**.** *Let $r_0 > 0$, η_ε, and $A_\varepsilon(\delta, r_0)$ be as in Lemma 4.9, and let $\widehat{N}_\varepsilon^+$ and θ_ε be as in Lemma 4.18. Define*

$$\widetilde{N}_\varepsilon^+ := \delta \widehat{N}_\varepsilon^+,$$

and denote by R_ε the region of Ω_ε^+ between $\widetilde{N}_\varepsilon^+$ and $A_\varepsilon(\delta, r_0)$. Define also $\widehat{R}_\varepsilon := \frac{1}{\delta} R_\varepsilon - \left[\frac{\varepsilon}{\delta}\right]$. The following statements hold true.

(i) *Assume that either (4.8) or (4.16) holds. For M_1, $M_2 > 0$ define*

$$u_\varepsilon^{lo}(x,y) := \begin{cases} \dfrac{M_1 \theta_\varepsilon(\frac{x}{\delta}, \frac{y}{\delta})}{|\ln \delta|} & \text{for } (x,y) \in \widetilde{N}_\varepsilon^+, \\ \dfrac{M_2}{|\ln \delta|} \xi_\varepsilon \left(\dfrac{x}{\delta} - \left[\dfrac{\varepsilon}{\delta}\right], \dfrac{y}{\delta} \right) + \dfrac{M_1 d_\varepsilon}{|\ln \delta|} & \text{for } (x,y) \in R_\varepsilon, \\ \dfrac{1 - \eta_\varepsilon - \left(\frac{M_1 d_\varepsilon}{|\ln \delta|} + \frac{M_2}{|\ln \delta|} \right)}{\ln \frac{r_0}{\delta}} \ln \dfrac{|(x-\varepsilon, y)|}{r_0} + (1 - \eta_\varepsilon) & \text{for } (x,y) \in A_\varepsilon(\delta, r_0), \end{cases} \tag{4.126}$$

where d_ε is the constant appearing in (4.107) and ξ_ε is the solution to the following problem:

$$\begin{cases} \Delta \xi_\varepsilon = 0 & \text{in } \widehat{R}_\varepsilon, \\ \xi_\varepsilon = 0 & \text{on } \partial \widehat{R}_\varepsilon \cap \partial \widehat{N}_\varepsilon^+, \\ \xi_\varepsilon = 1 & \text{on } \partial B_1(0,0) \cap \{x > 0\}, \\ \partial_\nu \xi_\varepsilon = 0 & \text{on the remaining part of } \partial \widehat{R}_\varepsilon. \end{cases}$$

Then, there exist M_1, $M_2 > 0$ independent of ε such that $u_\varepsilon^{lo} \le u_\varepsilon$ in $\widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0)$ for ε small enough.

(ii) *Assume that (4.3) holds. Then, for any $M_1 \in (0,1)$ there exists $M_2 > 0$ independent of ε such that the function u_ε^{lo} defined as in (4.28), with $M_1/|\ln \delta|$ replaced by $M_1 \delta/\varepsilon$, is a lower bound for u_ε in $\widetilde{N}_\varepsilon^+ \cup R_\varepsilon \cup A_\varepsilon(\delta, r_0)$, provided that ε is small enough.*

Proof. The proof is very similar to the one of Lemma 4.9. Using estimate (4.110), we observe that $\frac{m_1}{2}(\varepsilon/\delta - 1) \le d_\varepsilon \le 2m_2(\varepsilon/\delta - 1)$ for ε small enough. Moreover, the functions ξ_ε are C^1-close to the function v defined in the Lemma 4.9 in a neighborhood of $\partial \widehat{R}_\varepsilon \cap \{x = -1\}$ and $\partial B_1(0,0) \cap \{x > 0\}$. This can be easily proven by using the reflection argument of Lemma 4.17. Hence, in particular, by (4.33) and (4.34) we have

$$0 < a_0/2 \le -\partial_{\nu_{\widehat{R}_\varepsilon}} \xi_\varepsilon \le 2a_1 \text{ on } \partial \widehat{R}_\varepsilon \cap \partial \widehat{N}_\varepsilon^+, \quad 0 < b_0/2 < \partial_{\nu_{\widehat{R}_\varepsilon}} \xi_\varepsilon \le 2b_1 \text{ on } \partial B_1(0,0) \cap \{x \ge 0\}.$$

The rest of the proof goes as in Lemma 4.9. ∎

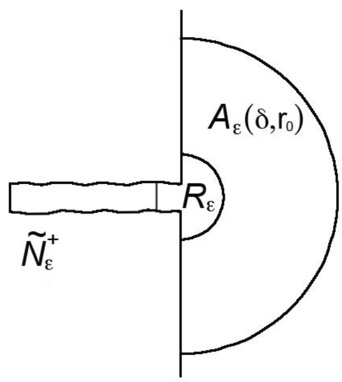

FIGURE 17. The construction of the lower bound.

Lemma 4.22 (Upper bound). *Let $r_0>0$, η_ε, $\widetilde{N}_\varepsilon^+$, $\widehat{N}_\varepsilon^+$ R_ε, $A_\varepsilon(\delta,r_0)$, and $\widehat{R}_\varepsilon$ be as in Lemma 4.21, and assume that either* (4.8) *or* (4.16) *holds. Let μ be the constant appearing in* (2.3). *For $M_1>0$, $M_2>0$ let ζ_{ε,M_1} be the function constructed in Lemma 4.19 and define*

$$u_\varepsilon^{up}(x,y):=\begin{cases}\dfrac{M_1\zeta_{\varepsilon,M_1}(\frac{x}{\delta},\frac{y}{\delta})}{|\ln\delta|} & \text{for } (x,y)\in\widetilde{N}_\varepsilon^+,\\ \dfrac{M_2}{|\ln\delta|}\xi_\varepsilon^{up}\left(\dfrac{x}{\delta}-\left[\dfrac{\varepsilon}{\delta}\right],\dfrac{y}{\delta}\right)+\dfrac{M_1 d_{\varepsilon,M_1}}{|\ln\delta|} & \text{for } (x,y)\in R_\varepsilon,\\ \left(1-\dfrac{M_1 d_{\varepsilon,M_1}}{|\ln\delta|}-\dfrac{M_2}{|\ln\delta|}\right)h_\varepsilon(x,y)+1 & \text{for } (x,y)\in A_\varepsilon(\delta,r_0),\end{cases}\tag{4.127}$$

where ξ_ε^{up} is the solution to

$$\begin{cases}\Delta\xi_\varepsilon^{up}=-\delta & \text{in } \widehat{R}_\varepsilon,\\ \xi_\varepsilon^{up}=0 & \text{on } \partial\widehat{R}_\varepsilon\cap\partial\widehat{N}_\varepsilon^+,\\ \xi_\varepsilon^{up}=1 & \text{on } \partial B_1(0,0)\cap\{x>0\},\\ \partial_\nu\xi_\varepsilon^{up}=0 & \text{on the remaining part of } \partial\widehat{R}_\varepsilon,\end{cases}$$

and $h_\varepsilon(x,y)$ is defined in (4.36). *Then, there exist M_1, $M_2>0$ independent of ε such that $u_\varepsilon\le u_\varepsilon^{up}$ in $\widetilde{N}_\varepsilon^+\cup R_\varepsilon\cup A_\varepsilon(\delta,r_0)$ for ε small enough.*

The proof goes exactly as in Lemma 4.10. For the minor changes, see the proof of Lemma 4.9.

Combining Proposition 4.20 and Lemma 4.22, we can argue as in the proof of Corollary 4.13 to obtain the following corollary.

Corollary 4.23. *Assume that either* (4.8) *or* (4.16) *holds. Choose $\lambda:=\max_{[-1,1]}(f_1+f_2)$. Then for ε small enough and for $0<x\le\varepsilon$ we have*

$$\operatorname*{osc}_{B_{\lambda\delta}(x,0)\cap\Omega_\varepsilon^+}u_\varepsilon\le\frac{C}{|\ln\delta|},\tag{4.128}$$

with $C>0$ independent of ε and x.

We are now in a position to complete the proof of Theorem 4.3 in the non-flat case. We only show the main changes with respect to the proof given in the previous subsection. Here, we underline one subtle point: in the non-flat case we are not able to provide matching lower and upper bounds for u_ε in the neck. Nevertheless, we are able to retrieve the complete limiting behavior at infinity thanks to Proposition 4.14, which is a consequence of the matching lower and upper bounds constructed in the flat case.

We leave the easy changes needed to complete the proofs of Theorems 4.1 and 4.6 to the reader.

Proof of Theorem 4.3 (the non-flat neck case). We split the proof into several steps.
Step 1. (energy bounds in the neck) Arguing as in the corresponding step of the proof in Subsection 4.1, we obtain

$$\int_N\left[|\partial_x v_\varepsilon(x,y)|^2+\frac{\varepsilon^2}{\delta^2}|\partial_y v_\varepsilon(x,y)|^2\right]dxdy=2\int_{N^+}\left[|\partial_x v_\varepsilon(x,y)|^2+\frac{\varepsilon^2}{\delta^2}|\partial_y v_\varepsilon(x,y)|^2\right]dxdy\le C$$

for some constant $C>0$ independent of ε and we may extract a (not relabelled) subsequence such that (4.57) and (4.58) hold. Moreover, using Corollary 4.23 and arguing as in the previous section, one can prove also in this case (4.59) and (4.63).

From (4.8), (4.55), (4.57), and (4.58) we have

$$\begin{aligned}\liminf_{\varepsilon\to0}|\ln\delta|F(u_\varepsilon,N_\varepsilon^+)&\ge\liminf_{\varepsilon\to0}|\ln\delta|\frac12\int_{N_\varepsilon^+}|\nabla u_\varepsilon|^2\,dxdy\\&=\liminf_{\varepsilon\to0}\frac12\int_{N^+}\left[\frac{\delta|\ln\delta|}{\varepsilon}|\partial_x v_\varepsilon(x,y)|^2+\frac{\varepsilon|\ln\delta|}{\delta}|\partial_y v_\varepsilon(x,y)|^2\right]dxdy\end{aligned}$$

$$\begin{aligned}
&\geq \liminf_{\varepsilon\to 0} \frac{\ell}{2}\int_{N^+} |\nabla v_\varepsilon|^2\,dxdy \geq \frac{\ell}{2}\int_{N^+} |\nabla v|^2\,dxdy \\
&= \frac{\ell}{2}\int_0^1 (f_1+f_2)(\hat v')^2\,dx = \frac{\ell}{4}\int_{-1}^1 (f_1+f_2)(\hat v')^2\,dx \\
&\geq \frac{\ell}{4}\min\left\{\int_{-1}^1 (f_1+f_2)(\theta')^2\,dx : \theta\in H^1(-1,1)\,,\theta(1)=-\theta(-1)=\hat v(1)\right\} \\
&= \frac{\ell}{m_{f_1f_2}}\hat v(1)^2\,. \qquad (4.129)
\end{aligned}$$

The last inequality follows from the explicit computation of the minimum problem, observing that the Euler-Lagrange equations together with the boundary conditions give that the minimizer θ_{min} must satisfy

$$\theta'_{min} = \frac{2\hat v(1)}{m_{f_1f_2}}\frac{1}{f_1+f_2} \qquad \text{in } (-1,1),$$

where $m_{f_1f_2}$ is the constant introduced in (4.7).

Step 2. (energy bounds in the bulk) Assume first that $\hat v(1) < 1$. Then by $(4.63)_2$ and (4.29), we have

$$c_\varepsilon + \frac{C}{|\ln\delta|} < 1-\eta_\varepsilon \qquad \text{for } \varepsilon \text{ small enough.}$$

We can now argue exactly as in the corresponding step of the proof in the previous subsection to obtain

$$\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon,\Omega_\varepsilon^+\setminus N_\varepsilon^+) \geq \frac{1}{2}\pi(1-\hat v(1))^2\,. \qquad (4.130)$$

Note that if $\hat v(1)=1$, then (4.130) trivially holds.

Step 3. (asymptotic behavior in the neck and limit of the energy) By (4.129) and (4.130) we have

$$\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon,\Omega_\varepsilon^+) \geq \frac{\ell}{m_{f_1f_2}}\hat v(1)^2 + \frac{1}{2}\pi(1-\hat v(1))^2 \geq \frac{\pi\ell}{m_{f_1f_2}\pi+2\ell}\,, \qquad (4.131)$$

where the last inequality follows from the fact that

$$\frac{\ell}{m_{f_1f_2}}t^2 + \frac{1}{2}\pi(1-t)^2 > \frac{\pi\ell}{m_{f_1f_2}\pi+2\ell} \qquad \text{for } t \neq \frac{m_{f_1f_2}\pi}{m_{f_1f_2}\pi+2\ell}\,. \qquad (4.132)$$

On the other hand, for any fixed $\alpha\in(0,1)$ we may consider the test functions z_ε defined as

$$z_\varepsilon(x,y) := \begin{cases} \dfrac{2}{\varepsilon}\dfrac{\pi}{m_{f_1f_2}\pi+2\ell}\displaystyle\int_0^x \frac{1}{(f_1+f_2)(\frac{s}{\varepsilon})}\,ds & \text{in } N_\varepsilon^+, \\ \dfrac{m_{f_1f_2}\pi}{m_{f_1f_2}\pi+2\ell} & \text{in } \{|(x-\varepsilon,y)|\leq\delta,\, x>\varepsilon\}, \\ \dfrac{2\ell}{m_{f_1f_2}\pi+2\ell}\dfrac{1}{|\ln\delta^{1-\alpha}|}\ln\dfrac{|(x-\varepsilon,y)|}{\delta^\alpha}+1 & \text{in } \{\delta<|(x-\varepsilon,y)|<\delta^\alpha,\, x>\varepsilon\}, \\ 1 & \text{otherwise in } \Omega_\varepsilon^+. \end{cases}$$

Taking into account the minimality of u_ε, we have

$$\begin{aligned}
\limsup_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon,\Omega_\varepsilon^+) &\leq \limsup_{\varepsilon\to 0} |\ln\delta| F(z_\varepsilon,\Omega_\varepsilon^+) \\
&\leq \lim_{\varepsilon\to 0}|\ln\delta|\left(\frac{1}{2}\int_{\Omega_\varepsilon^+}|\nabla z_\varepsilon|^2\,dxdy + \mathcal{L}^2\left((N_\varepsilon^+\cup B_{\delta^\alpha}(\varepsilon,0))\cap\{x>\varepsilon\}\right)\max_{[0,1]}W\right) \\
&= \lim_{\varepsilon\to 0}|\ln\delta|\frac{1}{2}\int_{\Omega_\varepsilon^+}|\nabla z_\varepsilon|^2\,dxdy = \frac{\pi\ell}{(m_{f_1f_2}\pi+2\ell)^2}\left(m_{f_1f_2}\pi+\frac{2\ell}{1-\alpha}\right)\,, \qquad (4.133)
\end{aligned}$$

where the last equality follows by explicit computation of the Dirichlet energy of z_ε.

Combining (4.131) and (4.133), since α can be chosen arbitrarily close to 0, we conclude

$$\lim_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) = \frac{1}{2}\ell\hat{v}(1)^2 + \frac{1}{2}\pi(1-\hat{v}(1))^2 = \frac{\pi\ell}{m_{f_1f_2}\pi + 2\ell}\,, \tag{4.134}$$

which, in turn, yields

$$\hat{v}(1) = \frac{m_{f_1f_2}\pi}{m_{f_1f_2}\pi + 2\ell} \tag{4.135}$$

thanks to (4.132). Note that the last equality and $(4.63)_2$ imply (4.12). Moreover, the limit in (4.134) is independent of the selected subsequence and thus the full sequence converges. Now, combining (4.96), (4.130), (4.134), and (4.135) one deduces that all the inequalities in (4.96) and (4.130) are in fact equalities and that, in turn, $\hat{v}$ solves (4.9). Hence, $\hat{v}$ does not depend on the selected subsequence. In turn, the equalities in (4.96) and (4.130) hold for the full sequence and prove (4.10) and (4.15), respectively.

The strong convergence in $H^1(N)$ of $\{v_\varepsilon\}$ to v can now be proved as in the corresponding step of the proof in the previous subsection.

Step 4. (lower and upper bounds in the neck) We construct lower and upper bounds for u_ε in the neck. We underline here that differently from the flat case presented in the previous subsection, the construction does not provide matching lower and upper bounds. However, at this point we can take advantage of Proposition 4.14. Given $M > 0$, consider the function $\xi_{\varepsilon,M}$ constructed in Lemma 4.19 and set

$$\lambda_{\varepsilon,M} := \frac{c_\varepsilon + \frac{C}{|\ln\delta|}}{d_{\varepsilon,M}}|\ln\delta|\,, \tag{4.136}$$

where $c_\varepsilon + \frac{C}{|\ln\delta|}$ and $d_{\varepsilon,M}$ are the constants appearing in (4.63) and (4.114), respectively. Note by (4.117) and the definition of $d_{\varepsilon,M}$, we have

$$\frac{n_1}{2}\frac{\varepsilon}{\delta} < d_{\varepsilon,M} < 2n_2\frac{\varepsilon}{\delta}$$

and, in turn, by (4.8), $(4.63)_2$, and (4.136)

$$\frac{\ell}{4n_2}\hat{v}(1) < \frac{\delta|\ln\delta|}{2n_2\varepsilon}\left(c_\varepsilon + \frac{C}{|\ln\delta|}\right) < \lambda_{\varepsilon,M} < \frac{2\delta|\ln\delta|}{n_1\varepsilon}\left(c_\varepsilon + \frac{C}{|\ln\delta|}\right) < \frac{4\ell}{n_1}\hat{v}(1)\,, \tag{4.137}$$

provided that ε is small enough. Thanks to above inequalities, we may choose $M > 0$ (independent of ε) so that

$$\frac{2\mu\lambda_{\varepsilon,M}}{M} > \max_{[0,1]}|W'|\,,$$

where μ is the constant appearing in (2.3). With this choice of M, we set

$$n_\varepsilon^{up}(x,y) := \lambda_{\varepsilon,M}\frac{\xi_{\varepsilon,M}(\frac{x}{\delta},\frac{y}{\delta})}{|\ln\delta|} \qquad \text{for } (x,y)\in\widetilde{N}_\varepsilon^+, \tag{4.138}$$

where $\widetilde{N}_\varepsilon^+$ is defined as in Lemma 4.19. Then, by (4.63), (4.114), and (4.136), we have

$$\begin{cases} \Delta n_\varepsilon^{up} \le -\frac{2\mu\lambda_{\varepsilon,M}}{M} < -\max_{[0,1]}|W'| \le \Delta u_\varepsilon & \text{in } \widetilde{N}_\varepsilon^+,\\ n_\varepsilon^{up} = u_\varepsilon = 0 & \text{on } \partial\widetilde{N}_\varepsilon^+\cap\{x=0\},\\ n_\varepsilon^{up} = c_\varepsilon + \frac{C}{|\ln\delta|} \ge u_\varepsilon & \text{on } \partial\widetilde{N}_\varepsilon^+\cap\Omega_\varepsilon^+,\\ \partial_\nu n_\varepsilon^{up} = \partial_\nu u_\varepsilon = 0 & \text{on the remaining of } \partial\widetilde{N}_\varepsilon^+. \end{cases}$$

Hence, we may apply Proposition 6.1 to deduce that n_ε^{up} is an upper bound for u_ε in $\widetilde{N}_\varepsilon^+$. Now let θ_ε and d_ε as in (4.107). Setting

$$\lambda_\varepsilon := \frac{c_\varepsilon - \frac{C}{|\ln\delta|}}{d_\varepsilon}|\ln\delta|\,,$$

and arguing as before, one can check that λ_ε is bounded by a constant independent of ε and that the function n_ε^{lo}, defined as

$$n_\varepsilon^{lo}(x,y) := \lambda_\varepsilon \frac{\theta_\varepsilon(\frac{x}{\delta}, \frac{y}{\delta})}{|\ln \delta|} \qquad \text{for } (x,y) \in \widetilde{N}_\varepsilon^+,$$

is a lower bound for u_ε in $\widetilde{N}_\varepsilon^+$. Summarazing,

$$n_\varepsilon^{lo} \le u_\varepsilon \le n_\varepsilon^{up} \qquad \text{in } \widetilde{N}_\varepsilon^+. \tag{4.139}$$

Step 5. (lower and upper bounds in the bulk) This step goes exactly as in the previous subsection. We may define b_ε^{lo} and b_ε^{up} as in (4.77) and (4.78), respectively, so that (4.80) holds.
Step 6. (asymptotic behavior in the bulk) We can now argue exactly as in Step 6 of the proof presented in the previous subsection for the flat case, to deduce the existence of a subsequence (not relabeled) such that $w_\varepsilon^+ - c_\varepsilon|\ln\delta| \to w^+$ in $W^{2,p}_{loc}(\Omega_\infty^+)$ for every $p \ge 1$ and $\nabla w_\varepsilon^+ \chi_{\widetilde{\Omega}_\varepsilon^+} \to \nabla w^+ \chi_{\Omega_\infty^+}$ in $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$, where w^+ is a harmonic in Ω_∞^+, $\partial_\nu \hat{w} = 0$ on $\partial\Omega_\infty^+$, $w^+(0,0) = 0$, and

$$\frac{2\ell}{m_{f_1 f_2}\pi + 2\ell} \ln|(x,y)| - C \le w^+(x,y) \le \frac{2\ell}{m_{f_1 f_2}\pi + 2\ell} \ln|(x,y)| + C \quad \text{for } (x,y) \in \{x>0\} \setminus \overline{B_1}(0,0).$$

Moreover, by (4.139), tusing (4.110), (4.117), and the uniform bounds on the constants $\lambda_{\varepsilon,M}$ and λ_ε, one can easily check that the functions $w_\varepsilon^+ - c_\varepsilon|\ln\delta|$ have linear growth in $\widetilde{N}_\varepsilon^+$, uniformly in ε, so that w^+ grows at most linearly in in $\Omega_\infty^+ \cap \{x<0\}$. Summarizing, w^+ satisfies

$$\begin{cases} \Delta w^+ = 0 & \text{in } \Omega_\infty^+, \\ \partial_\nu w^+ = 0 & \text{on } \partial\Omega_\infty^+, \\ \dfrac{w^+(x,y)}{\ln|(x,y)|} \to \dfrac{2\ell}{m_{f_1 f_2}\pi + 2\ell} & \text{as } |(x,y)| \to +\infty \text{ with } x>0, \\ w^+ \text{ grows at most linearly in in } \Omega_\infty^+ \cap \{x<0\}., \\ w^+(0,0) = 0\,. \end{cases}$$

We may now apply Proposition 4.14 to infer that w^+ is the unique solution to (4.13). Thus, the conclusion of the theorem follows. ☐

5. The flat thick neck

This section is devoted to the thick neck regime

$$\lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = +\infty\,. \tag{5.1}$$

Since the geometry of neck will not affect the limiting problem, we only consider for simplicity the case of a flat neck. More precisely, we assume, without loss of generality, that (4.22) holds. The main result is contained in the following theorem.

Theorem 5.1. *Assume that* (5.1) *holds. Let* $\{u_\varepsilon\}$ *be the family of minimizing geometrically constrained walls defined in* (2.7) *and* $\{v_\varepsilon\}$ *the corresponding family of rescaled profiles*

$$v_\varepsilon(x,y) := |\ln\delta| u_\varepsilon(\delta x, \delta y)\,,$$

defined for $(x,y) \in \frac{1}{\delta}\Omega_\varepsilon$. *Set* $\Omega_\infty := \mathbb{R}^2 \setminus \{(0,y) : |y| \geq \frac{1}{2}\}$. *Then* v_ε *converge in* $W^{2,p}_{loc}(\Omega_\infty)$ *for every* $p \geq 1$ *to the unique solution* v *of the following problem:*

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_\infty, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_\infty, \\ \dfrac{v(x,y)}{\ln|(x,y)|} \to 1 & \text{as } |(x,y)| \to +\infty \text{ with } x > 0, \\ \dfrac{v(x,y)}{\ln|(x,y)|} \to -1 & \text{as } |(x,y)| \to +\infty \text{ with } x < 0, \\ v(0,0) = 0\,. \end{cases} \tag{5.2}$$

In fact, $v(0,y) = 0$ *for* $|y| < \frac{1}{2}$. *Moreover,* $\nabla v_\varepsilon \chi_{\frac{1}{\delta}\Omega_\varepsilon} \to \nabla v \chi_{\Omega_\infty}$ *in* $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$. *Finally,*

$$\lim_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon) = \pi\,. \tag{5.3}$$

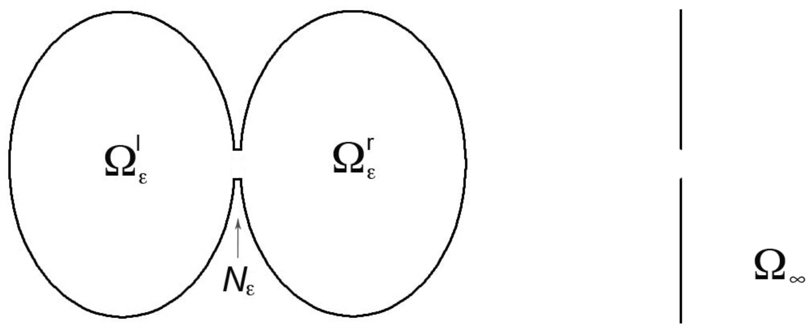

FIGURE 18. The original domain Ω_ε and the rescaled limiting domain Ω_∞.

Remark 5.2. Since $\mathbb{R}^2 \setminus \frac{1}{\delta}\Omega_\varepsilon \to \mathbb{R}^2 \setminus \Omega_\infty$ in the Hausdorff metric, the local convergence of $\{v_\varepsilon\}$ to v stated above is well defined (see Remark 3.2).

Proof. Since the argument is similar to previous cases, we only give a sketch of the proof. It is enough to consider $u_\varepsilon|_{\{x>0\}}$.

Step 1. (lower bounds) We start by constructing the lower bounds. Let η_ε be as in Lemma 4.9 and recall that $\eta_\varepsilon \to 0$. Let r_0, R_ε, and $A_\varepsilon(\delta, r_0)$ be as in Lemma 4.9 (note that now R_ε intersects the half space $\{x < 0\}$). Define

$$u^{lo}_\varepsilon(x,y) := \begin{cases} 0 & \text{if } (x,y) \in R_\varepsilon, \\ \dfrac{1-\eta_\varepsilon}{\ln\frac{r_0}{\delta}} \ln\dfrac{|(x-\varepsilon,y)|}{r_0} + (1-\eta_\varepsilon) & \text{if } (x,y) \in A_\varepsilon(\delta, r_0). \end{cases}$$

Arguing as in the proof of Lemma 4.9 one can easily check that u^{lo}_ε is a lower bound for u_ε in $(R_\varepsilon \cup A_\varepsilon(\delta, r_0)) \cap \Omega^+_\varepsilon$, provided that ε is small enough. *Step 2. (upper bounds)* We now construct suitable upper bounds. Again the construction is very similar to the one in Lemma 4.10. More precisely, let ξ_ε and h_ε be as in Lemma 4.10 and define

$$u^{up}_\varepsilon(x,y) := \begin{cases} \dfrac{M}{|\ln\delta|}\xi_\varepsilon\left(\dfrac{x-\varepsilon}{\delta}, \dfrac{y}{\delta}\right) & \text{if } (x,y) \in R_\varepsilon, \\ \left(1 - \dfrac{M}{|\ln\delta|}\right) h_\varepsilon(x,y) + 1 & \text{if } (x,y) \in A_\varepsilon(\delta, r_0). \end{cases}$$

Arguing as in Lemma 4.10, we can show that there exists $M > 0$ independent of ε such that u^{up}_ε is an upper bound for u_ε in $(R_\varepsilon \cup A_\varepsilon(\delta, r_0)) \cap \Omega^+_\varepsilon$.

Step 3. (limit of the energy).

For $\alpha \in (0,1)$ consider the test functions z_ε defined in Ω_ε^+ as

$$z_\varepsilon(x,y) := \begin{cases} 0 & \text{in } N_\varepsilon^+ \cup \{|(x-\varepsilon,y)| \le \delta,\, x > \varepsilon\}, \\ \dfrac{1}{|\ln \delta^{1-\alpha}|} \ln \dfrac{|(x-\varepsilon,y)|}{\delta^\alpha} + 1 & \text{in } \{\delta < |(x-\varepsilon,y)| < \delta^\alpha,\, x > \varepsilon\}, \\ 1 & \text{otherwise in } \Omega_\varepsilon^+. \end{cases}$$

By the minimality of u_ε and by explicitly computing the Dirichlet energy of z_ε (and estimating the contribution of the potential energy, see (4.69)), we get

$$\limsup_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon) = 2 \limsup_{\varepsilon\to 0^+} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) \le 2 \lim_{\varepsilon\to 0^+} |\ln\delta| F(z_\varepsilon, \Omega_\varepsilon^+) = \frac{\pi}{1-\alpha}\,. \tag{5.4}$$

For the opposite inequality, we observe that

$$\begin{aligned} &\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) \\ &\ge \lim_{\varepsilon\to 0} |\ln\delta| \min\left\{ \frac{1}{2}\int_{A_\varepsilon(\delta,r_0)} |\nabla u|^2 dxdy :\, u \le \tfrac{M}{|\ln\delta|} \text{ on } \partial B_\delta(\varepsilon,0)\cap\{x>\varepsilon\}\,,\right. \\ &\qquad\qquad \left. u \ge 1-\eta_\varepsilon \text{ on } \partial B_{r_0}(\varepsilon,0)\cap\{x>\varepsilon\}\right\} \\ &= \lim_{\varepsilon\to 0} |\ln\delta| \min\left\{ \frac{1}{2}\int_{A_\varepsilon(\delta,r_0)} |\nabla u|^2 dxdy :\, u = \tfrac{M}{|\ln\delta|} \text{ on } \partial B_\delta(\varepsilon,0)\cap\{x>\varepsilon\}\,,\right. \\ &\qquad\qquad \left. u = 1-\eta_\varepsilon \text{ on } \partial B_{r_0}(\varepsilon,0)\cap\{x>\varepsilon\}\right\}\,, \end{aligned}$$

where we used the fact that $u_\varepsilon \le \frac{M}{|\ln\delta|}$ on $\partial B_\delta(\varepsilon,0)\cap\{x>\varepsilon\}$, thanks to the upper bound. The unique minimizer of the last minimum problem is given by

$$\tilde u_\varepsilon(x,y) = (1-\eta_\varepsilon) + \frac{1-\eta_\varepsilon - \frac{M}{|\ln\delta|}}{\ln\frac{r_0}{\delta}} \ln \frac{|(x-\varepsilon,y)|}{r_0}\,.$$

By explicit computation, we deduce that

$$\liminf_{\varepsilon\to 0} |\ln\delta| F(u_\varepsilon, \Omega_\varepsilon^+) \ge \lim_{\varepsilon\to 0} \frac{1}{2}\int_{A_\varepsilon(\delta,r_0)} |\nabla \tilde u_\varepsilon|^2 dxdy = \pi\,.$$

Hence, (5.3) follows from the previous inequality, (5.4), and the arbitrariness of α.

Step 4. (asymptotic behavior) Note that from the previous steps we have that the functions v_ε satisfy

$$|\ln\delta| u_\varepsilon^{lo}(\delta x, \delta y) \le v_\varepsilon(x,y) \le |\ln\delta| u_\varepsilon^{up}(\delta x, \delta y) \qquad \text{for } (x,y) \in (\tfrac{1}{\delta}\Omega_\varepsilon^+) \cap \{|(x-\tfrac{\varepsilon}{\delta}, y)| \le \tfrac{r_0}{\delta}|\}. \tag{5.5}$$

Similar (symmetric) bounds are clearly satisfied in Ω_ε^-. In particular, we deduce that the functions v_ε have locally equibounded L^∞-norms. Since, they satisfy

$$\begin{cases} \Delta v_\varepsilon = \delta^2 |\ln\delta| W'(\frac{v_\varepsilon}{|\ln\delta|}) & \text{in } \frac{1}{\delta}\Omega_\varepsilon\,, \\ \partial_\nu v_\varepsilon = 0 & \text{on } \frac{1}{\delta}\partial\Omega_\varepsilon\,, \end{cases}$$

we may apply Proposition 6.2 to deduce that, up to a subsequence, v_ε converge in $W^{2,p}_{loc}(\Omega_\infty)$ for every $p \ge 1$ to a function v satisiying

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_\infty\,, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_\infty\,, \\ v(0,0) = 0\,. \end{cases}$$

Moreover, $\nabla v_\varepsilon \chi_{\frac{1}{\delta}\Omega_\varepsilon} \to \nabla v \chi_{\Omega_\infty}$ in $L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$.

Note that as $\varepsilon \to 0^+$

$$|\ln\delta| u_\varepsilon^{lo}(\delta x, \delta y) \to \ln|(x,y)| \quad \text{and} \quad |\ln\delta| u_\varepsilon^{up}(\delta x, \delta y) \to \ln|(x,y)| + M$$

for $|(x,y)| > 1$. Recalling (5.5), we conclude that v solves (5.2). In order to conclude the proof we are left with showing that such a problem admits a unique solution. To see this, we may repeat word for word Step 4 of the proof of Theorem 3.1, with the half spaces $\{x > 1\}$ and $\{x < -1\}$ replaced by $\{x > 0\}$ and $\{x < 0\}$, respectively, and with the function K_ℓ replaced by the Kelvin transform $(x,y) \mapsto (x,y)/\left(2|(x,y)|^2\right)$.

□

6. Appendix

In this section we state two auxiliary results that are used throughout the paper and we provide the easy proofs for the reader's convenience.

The first one is a particular instance of the so-called weak comparison principle for elliptic equations.

Proposition 6.1 (Weak comparison principle). *Let $\Omega \subset \mathbb{R}^2$ be a Lispchitz connected open set and let $\Gamma \subset \partial\Omega$ be a relative closed subset with positive $\mathscr{H}^1$-measure. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous nondescreasing function and let u_1, $u_2 \in H^1(\Omega) \cap L^\infty(\Omega)$ be weak solutions to the following system of inequalities:*

$$\begin{cases} \Delta u_1 \geq f(u_1)\,, \quad \Delta u_2 \leq f(u_2) & \text{in } \Omega, \\ \\ \partial_\nu u_1 = \partial_\nu u_2 = 0 & \text{on } \partial\Omega \setminus \Gamma; \end{cases}$$

i.e.,

$$-\int_\Omega \nabla u_1 \cdot \nabla \varphi \, dxdy \geq \int_\Omega f(u_1)\varphi \, dxdy \quad \text{and} \quad -\int_\Omega \nabla u_2 \cdot \nabla \varphi \, dxdy \leq \int_\Omega f(u_2)\varphi \, dxdy \tag{6.1}$$

for all $\varphi \in H^1(\Omega)$, with $\varphi \geq 0$ in Ω and $\varphi = 0$ on Γ. If $u_1 \leq u_2$ on Γ, then $u_1 \leq u_2$ a.e. in Ω.

Proof. Apply (6.1) with $\varphi := \max\{u_1 - u_2, 0\}$ to obtain

$$\int_{\{u_1 > u_2\}} \nabla u_1 \cdot \nabla(u_1 - u_2) \, dxdy \leq -\int_{\{u_1 > u_2\}} f(u_1)(u_1 - u_2) \, dxdy$$

and

$$-\int_{\{u_1 > u_2\}} \nabla u_2 \cdot \nabla(u_1 - u_2) \, dxdy \leq \int_{\{u_1 > u_2\}} f(u_2)(u_1 - u_2) \, dxdy\,.$$

Adding the two inequalities, we have

$$\int_\Omega |\nabla\varphi|^2 \, dxdy = \int_{\{u_1 > u_2\}} |\nabla(u_1 - u_2)|^2 \, dxdy \leq \int_{\{u_1 > u_2\}} [f(u_2) - f(u_1)](u_1 - u_2) \, dxdy\,.$$

As the last integral is nonpositive thanks to the monotonicity of f, we deduce that the first integral vanishes; i.e., $\nabla\varphi$=0 a.e. in Ω. Since Ω is connected and φ vanishes on Γ, we conclude that $\varphi = 0$ a.e. in Ω, which is equivalent to the thesis of the proposition. □

We now deal with the convergence of Neumann problems on varying domains. We recall that, given an open (possibly unbounded) set $\Omega \subset \mathbb{R}^2$ and $f \in L^2_{loc}(\mathbb{R}^2)$, the function u is a weak-solution to the Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

if $u \in H^1(\Omega \cap B_R(0,0))$ for all $R > 0$ and

$$-\int_\Omega \nabla u \cdot \nabla w \, dxdy = \int_\Omega f w \, dxdy$$

for all $w \in H^1(\Omega)$ with bounded support.

Proposition 6.2. *Let $\{\Omega_k\}$ be a sequence of open sets such that*

$$\chi_{\Omega_k} \to \chi_{\Omega_\infty} \text{ in } L^1_{loc}(\mathbb{R}^2) \quad \text{and} \quad \mathbb{R}^2 \setminus \Omega_k \to \mathbb{R}^2 \setminus \Omega_\infty \text{ locally in the Hausdorff sense} \tag{6.2}$$

for a suitable open set Ω_∞, and assume that at least one of the following conditions is satisfied:

(i) *Ω_∞ is locally Lipschitz;*
(ii) *$\Omega_k \subset \Omega_\infty$ for every k.*

Let $\{u_k\}$ be a sequence of functions such that $\{u_k\chi_{\Omega_k}\}$ is locally equibounded in L^∞ and each u_k is a weak solution to the Neumann problem

$$\begin{cases} \Delta u_k = f_k & \text{in } \Omega_k, \\ \partial_\nu u_k = 0 & \text{on } \partial\Omega_k, \end{cases} \tag{6.3}$$

where

$$f_k\chi_{\Omega_k} \to f_\infty\chi_\Omega \qquad \text{in } L^p_{loc}(\mathbb{R}^2) \tag{6.4}$$

for some $p > 2$. Then, up to a subsequence,

$$u_k\chi_{\Omega_k} \to u_\infty\chi_{\Omega_\infty} \quad \text{in } L^q_{loc}(\mathbb{R}^2) \text{ for all } q \in [1, +\infty), \qquad \nabla u_k\chi_{\Omega_k} \to \nabla u_\infty\chi_{\Omega_\infty} \quad \text{in } L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2), \tag{6.5}$$

where u_∞ is a weak solution to

$$\begin{cases} \Delta u_\infty = f_\infty & \text{in } \Omega_\infty, \\ \partial_\nu u_\infty = 0 & \text{on } \partial\Omega_\infty. \end{cases} \tag{6.6}$$

Moreover, $u_k \to u_\infty$ in $W^{2,p}_{loc}(\Omega_\infty)$.

Remark 6.3. The assumptions of the previous theorem can be significantly weakened (see [5], [6], and references therein). Here we decided to give the simplest statement that fits our purposes.

Proof of Proposition 6.2. A standard argument yields that for all $M > 1$ we have

$$\int_{\Omega_k \cap B_M(0,0)} |\nabla u_k|^2 \, dxdy \le C_0 \int_{\Omega_k \cap B_{2M}(0,0)} (|f_k|^2 + |u_k|^2) \, dxdy \tag{6.7}$$

where C_0 is a universal positive constant. In particular, the sequence $\{\nabla u_k\chi_{\Omega_k}\}$ is bounded in $L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2)$. Hence, using also the local L^∞-bounds on the functions u_k and recalling (6.2), we may find a (not relabelled) subsequence and u_∞ such that

$$u_k \to u_\infty \text{ in } L^2_{loc}(\Omega_\infty) \quad \text{and} \quad \nabla u_k\chi_{\Omega_k} \rightharpoonup \nabla u_\infty\chi_{\Omega_\infty} \text{ weakly in } L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2). \tag{6.8}$$

In particular, passing to a further subsequence if necessary, by the first limit in (6.8) we may assume that $u_k\chi_{\Omega_k} \to u_\infty\chi_{\Omega_\infty}$ almost everywhere and, in turn, again by the local L^∞-bounds and by the Dominated Convergence Theorem, we may conclude that the first assertion in (6.5) holds. Fix $w \in H^1(\Omega_\infty)$ with bounded support and assume that (i) is satisfied. Then we may extend w to a function (still denoted by w) in $H^1(\mathbb{R}^2)$ with compact support. Hence, for every k we have

$$-\int_{\Omega_k} \nabla u_k \cdot \nabla w \, dxdy = \int_{\Omega_k} f_k w \, dxdy \, .$$

By (6.4) and (6.8) we may pass to the limit in the above identity and deduce that u_∞ is a weak solution to (6.6). If (ii) holds, then the argument is even easier since we don't need to extend w (and, thus, we don't need to assume the Lipschitz regularity of Ω_∞). Fix $M > 0$ and $\varphi \in C^\infty_c(B_{2M}(0,0); [0,1])$ such that $\varphi = 1$ in $B_M(0,0)$. Choosing $w = \varphi^2 u_k$ as a test function for (6.3), we obtain

$$\int \chi_{\Omega_k}\varphi^2|\nabla u_k|^2 \, dxdy = \int \chi_{\Omega_k} f_k\varphi^2 u_k \, dxdy - 2\int \chi_{\Omega_k} u_k\varphi\nabla u_k \cdot \nabla\varphi \, dxdy \, . \tag{6.9}$$

Analogously, we have

$$\int \chi_{\Omega_\infty}\varphi^2|\nabla u_\infty|^2\,dxdy = \int \chi_{\Omega_\infty} f_\infty \varphi^2 u_\infty\,dxdy - 2\int \chi_{\Omega_\infty} u_\infty \varphi \nabla u_\infty\cdot\nabla\varphi\,dxdy\,. \tag{6.10}$$

By (6.4), the first part of (6.5), and the second limit in (6.8), one deduces that the right-hand side of (6.9) converges to the right-hand side of (6.10). Hence,

$$\limsup_{k\to\infty}\int_{B_M(0,0)}|\chi_{\Omega_k}\nabla u_k|^2\,dxdy \le \lim_{k\to\infty}\int \chi_{\Omega_k}\varphi^2|\nabla u_k|^2\,dxdy = \int \varphi^2|\chi_{\Omega_\infty}\nabla u_\infty|^2\,dxdy$$

for all cut-off functions $\varphi \in C_c^\infty(B_{2M}(0,0);[0,1])$ such that $\varphi = 1$ in $B_M(0,0)$. Choosing a sequence $\{\varphi_n\}$ of cut-off functions such that $\varphi_n \to \chi_{B_M(0,0)}$ in L^1, we deduce

$$\limsup_{k\to\infty}\int_{B_M(0,0)}|\chi_{\Omega_k}\nabla u_k|^2\,dxdy \le \int_{B_M(0,0)}|\chi_{\Omega_\infty}\nabla u_\infty|^2\,dxdy$$

and, in turn, by lower semicontinuity,

$$\lim_{k\to\infty}\int_{B_M(0,0)}|\chi_{\Omega_k}\nabla u_k|^2\,dxdy = \int_{B_M(0,0)}|\chi_{\Omega_\infty}\nabla u_\infty|^2\,dxdy$$

for all $M > 0$. This establishes the second assertion in (6.5). The $W^{2,p}_{loc}$ convergence of $\{u_k\}$ to u_∞ now follows from standard local L^p estimates. □

References

[1] M. Abramowitz, I. Stegun: *Handbook of Mathematical Functions* Dover Publications, New York, 1965.

[2] P. Bruno: *Geometrically constrained magnetic wall.* Phys. Rev. Lett. **83** (1999), 2425–2428

[3] R. Casten, C. Holland: *Instability results for reaction-diffusion equations with Neumann boundary conditions.* J. Differential Equations **27** (1978), 266–273.

[4] J. Casado-Daz, M. Luna-Laynez, F. Murat: *The diffusion equation in a notched beam.* Calc. Var. and Partial Differential Equations **31** (2008), 297–323.

[5] A. Chambolle, F. Doveri: *Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets.* Comm. Partial Differential Equations **22** (1997), 811–840.

[6] G. Dal Maso, F. Ebobisse, M. Ponsiglione: *A stability result for nonlinear Neumann problems under boundary variations.* J. Math. Pures Appl. **82** (2003), 503–532.

[7] L.C. Evans: *Partial Differential Equations* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[8] J.K.Hale, J.Vegas: *A nonlinear parabolic equation with varying domain.*, Arch. Ration. Mech. Anal. **86** (1984), 99–123.

[9] H.D. Chopra, S.Z. Hua: *Ballistic magnetoresistance over 3000% in Ni nanocontacts at room temperature.* Phys. Rev. B **66** (2002), 020403(R)

[10] S. Jimbo: *Singular perturbation of domains and semilinear elliptic equation.* J. Fac. Sci. Univ. Tokyo **35** (1988), 27–76.

[11] S. Jimbo: *Singular perturbation of domains and semilinear elliptic equation 2.* J. Diff. Equat. 75 (1988), 264-289.

[12] S. Jimbo: *Singular perturbation of domains and semilinear elliptic equation 3.* Hokkaido Math. J. **33** (2004), 11–45.

[13] P.-O. Jubert, R. Allenspach, A. Bischof: *Magnetic domain walls in constrained geometries.* Phys. Rev. B **69** (2004), 220410(R)

[14] M. Kläui: *Head-to-head domain walls in magnetic nanostructures.* J. Phys.: Condens. Matter **20** (2008), 313001.

[15] R.V. Kohn, V. Slastikov: *Geometrically constrained walls.* Calc. Var. Partial Differential Equations **28** (2007), 33–57.

[16] V.A. Molyneux, V.V. Osipov, E.V. Ponizovskaya: *Stable two- and three-dimensional geometrically constrained magnetic structures: The action of magnetic fields.* Phys. Review B **65** (2002), 184425.

[17] C. Pommerenke: *Boundary behaviour of conformal maps.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1992.

[18] W. Rudin: *Real and Complex Analysis.* McGraw-Hill Book Co., New York, 1987.

[19] J. Rubistein, M. Schatzman, P. Sternberg: *Ginzburg-Landau model in thin loops with narrow constrictions.* SIAM J. Appl. Math. **64** (2004), 2186–2204.

[20] J. Serrin: *A Harnack inequality for nonlinear equations.* Bull. Amer. Math. Soc. **69** (1963), 481–486.

[21] G. Tatara, Y.-W. Zhao, M. Munoz, N. Garcia: *Domain wall scattering explains 300% ballistic magnetoconductance of nanocontacts.* Phys. Rev. Lett **83** (1999), 2030–2033.