CRYSTALLINITY OF THE HOMOGENIZED ENERGY DENSITY OF PERIODIC LATTICE SYSTEMS

ANTONIN CHAMBOLLE AND LEONARD KREUTZ

Abstract. We study the homogenized energy densities of periodic ferromagnetic Ising systems. We prove that, for finite range interactions, the homogenized energy density, identifying the effective limit, is crystalline, i.e. its Wulff crystal is a polytope, for which we can (exponentially) bound the number of vertices. This is achieved by deriving a dual representation of the energy density through a finite cell formula. This formula also allows easy numerical computations: we show a few experiments where we compute periodic patterns which minimize the anisotropy of the surface tension.

1. Introduction

The study of discrete interfacial energies has attracted widespread attention in the mathematical community over last decades, with applications in various contexts such as computer vision [7], crystallization problems [8], fracture mechanics [6, 18, 33], or statistical physics [38, 39]. To give examples, in computer vision the understanding of these energies allows to investigate functional correctness of segmentation algorithms [22]. Whereas for crystallization problems it gives fluctuation estimates on the macroscopic shape of the crystal cluster of ground state configurations [27, 31, 32, 37].

In this work, we consider energies defined on discrete periodic sets $\mathcal{L} \subset \mathbb{R}^d$ and corresponding Ising systems. We refer to [1, 12, 20, 21, 29, 34, 35, 36] for an abundant literature on the derivation of continuum limits of such systems and their effective behavior. More precisely, we consider $\mathcal{L}$ satisfying the following two conditions (see Figure 1)

(i) (Discreteness) There exists $c > 0$ such that $\mathrm{dist}(x, \mathcal{L} \setminus \{x\}) \geq c$ for all $x \in \mathcal{L}$;
(ii) (Periodicity) There exists $T \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$, it holds that $\mathcal{L} + Tz = \mathcal{L}$;

To each function $u : \mathcal{L} \to \{0, 1\}$ and each $A \subset \mathbb{R}^d$ we associate an energy

$$E(u, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ , \tag{1}$$

where $(z)^+$ denotes the positive part of $z \in \mathbb{R}$, $c_{i,j} \colon \mathcal{L} \times \mathcal{L} \to [0, +\infty)$ are T-periodic, that is $c_{i+Tz,j+Tz} = c_{i,j}$ for all $i, j \in \mathcal{L}$ and $z \in \mathbb{Z}^d$ and satisfy the following decay assumption

(iii) (Decay of interactions) For all $i \in \mathcal{L}$ there holds

$$\sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty .$$

2010 *Mathematics Subject Classification.* 35B27, 49J45, 82B20, 82D40.
Key words and phrases. Γ-convergence, Ising system, Crystallinity, Wulff Shape.

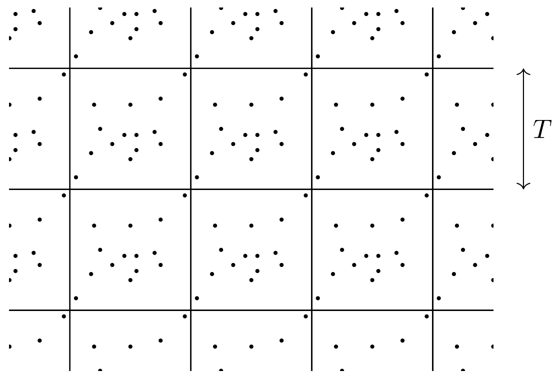

FIGURE 1. An example of the set $\mathcal{L}$

Assuming conditions (i)-(iii) (and some additional coercivity assumption) ensures that the asymptotic behavior of (1) is well described (in a variational sense) by a continuum perimeter energy. More precisely, let us introduce a scaling parameter $\varepsilon > 0$. We consider the scaled energies

$$E_\varepsilon(u) = \sum_{i,j\in\varepsilon\mathcal{L}} \varepsilon^{d-1} c^\varepsilon_{i,j}(u(i)-u(j))^+ \,,$$

where $c^\varepsilon_{i,j} = c_{i/\varepsilon,j/\varepsilon}$ and $u\colon \varepsilon\mathcal{L}\to\{0,1\}$. By identifying u with its piecewise constant interpolation taking the value $u(i)$ on the Voronoi cell centered at $i\in\varepsilon\mathcal{L}$ we may regard the energies as defined on $L^1_{\mathrm{loc}}(\mathbb{R}^d,\{0,1\})$. Integral representation results [2, 3, 17] then guarantee that the energies E_ε Γ-converge (see [11, 28] for an introduction to that subject) with respect to the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-topology to a continuum energy of the form

$$E_0(u) = \int_{\partial^*\{u=1\}} \varphi(\nu_u(x))\,\mathrm{d}\mathcal{H}^{d-1} \quad u\in BV_{\mathrm{loc}}(\mathbb{R}^d;\{0,1\})\,.$$

Here, $BV_{\mathrm{loc}}(\mathbb{R}^d;\{0,1\})$ denotes the space of functions with (locally) bounded variation and values in $\{0,1\}$, $\partial^*\{u=1\}$ denotes the reduced boundary of the level set $\{u=1\}$, $\nu_u(x)$ its measure theoretic normal at the point $x\in\partial^*\{u=1\}$, and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, see [4] for the precise definitions of these notions. The energy density $\varphi\colon\mathbb{R}^d\to[0,+\infty)$ can be recovered via the asymptotic cell formula

$$\varphi(\nu) := \lim_{\delta\to0}\lim_{S\to+\infty}\frac{1}{S^{d-1}}\inf\left\{E(u,Q^\nu_S) : u\colon\mathcal{L}\to\{0,1\}, u(i)=u_\nu(i) \text{ on } \mathcal{L}\setminus Q^\nu_{(1-\delta)S}\right\}\,, \qquad (2)$$

where

$$u_\nu(x) = \begin{cases} 1 & \langle x,\nu\rangle\geq 0\,,\\ 0 & \langle x,\nu\rangle < 0\,.\end{cases}$$

Here, Q^ν_S is a cube with side-length S orientation in direction $\nu\in\mathbb{S}^{d-1}$. In the case $\mathcal{L}=\mathbb{Z}^2$, $c_{i,j}=1$ if $|i-j|=1$ and $c_{i,j}=0$ otherwise, we have that $\varphi(\nu)=2\|\nu\|_1$, see Figure 2.

The goal of this article is to investigate the energy density φ. In particular we show, that for finite interaction range $c_{i,j}$, that is there exists $R>0$ such that $c_{i,j}=0$ if $|i-j|>R$, then φ is *crystalline*. This means that the solution to

$$\min\left\{\int_{\partial^* A}\varphi(\nu_A(x))\,\mathrm{d}\mathcal{H}^{d-1} : |A|=1\right\}$$

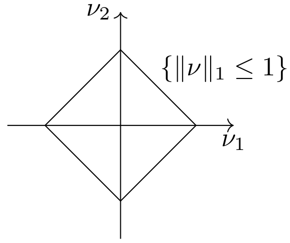

FIGURE 2. The energy density in the case of nearest neighbor interactions on $\mathbb{Z}^2$

is a convex polyhedron [30]. The finite range of interaction is crucial. Indeed, example 2.8 shows that for infinite range interactions this is in general not true. In [15, 16] it is shown that, as the periodicity T of the interactions tends to $+\infty$, it is possible to approximate any norm as surface energy density satisfying suitable growth conditions. We refer to [2] for a random setting where it is shown that an isotropic energy density (and thus non-crystalline) can be obtained in the limit.

The proof of the crystallinity in the case of finite range interactions relies on the following alternative representation result of the density, proven in Proposition 2.6. Namely, we prove that

$$\varphi(\nu) = \frac{1}{T^d} \inf \left\{ E(u, Q_T) : u \colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic} \right\} . \tag{3}$$

This representation formula is reminiscent of the representation formula of the energy density of integral functionals obtained via homogenization of T-periodic integral functionals in $W^{1,p}$ [14]. To motivate this, consider the positively 1-homogeneous extensions of E_ε defined by

$$F_\varepsilon(u) = \sum_{i,j \in \varepsilon\mathcal{L}} \varepsilon^{d-1} c^\varepsilon_{i,j} (u(i) - u(j))^+ ,$$

for $u : \varepsilon\mathcal{L} \to \mathbb{R}$. The Γ-limit F_0 of the above sequence is clearly positively 1-homogeneous and convex as the sequence of functionals satisfies these properties. Thus, F_0 admits an integral representation of the form

$$F_0(u) = \int f_0(\nabla u) \, \mathrm{d}x + \int f_0 \left(\frac{dD_s u}{d|D_s u|} \right) \mathrm{d}|D_s u| \, ,$$

where $f_0 \colon \mathbb{R}^d \to \mathbb{R}^d$ is convex and positively 1-homogeneous, see [13]. (We like to stress however, that this integral representation for the spin energies considered above is not proven in the literature) Here, the important point is that the density of the singular part and the density of absolutely continuous parts agree. In the continuous setting, in [23, 26] it has been shown that for continuous and convex densities, that satisfy a coarea formula, the Γ-convergence of sets of finite perimeter or in the space of BV-functions is equivalent. Thus also in their setting, the densities agree. The density of the absolutely continuous part can be calculated via (3). This property eventually allows us to express φ via (3) since the density of the absolutely continuous part can be calculated via (3) and the density of the singular part agrees with the energy density in (2), see Proposition 2.6. Using convex duality (see [40]) and using (3) we show in Theorem 2.7 that φ is crystalline, and estimate an upper bound on the number of extreme points of the corresponding Wulff shape. We would like to stress that (3) is not only a useful tool in our proof but it can be used also for computational purposes as it is a finite and not an asymptotic cell formula.

The paper is organized as follows. In Section 2 we describe the mathematical setting and state the main theorems of our paper. In Section 3 we prove Proposition 2.6, the alternate representation formula for φ. In Section 4 we show that, in the case of finite range interactions, the density φ is always crystalline. We present some numerical simulations of our findings in the last chapter.

2. Setting of the problem and statement of the main result

2.1. **Notation.** We denote by $\mathcal{B}(\mathbb{R}^d)$ the collection of all Borel-Sets in $\mathbb{R}^d$. For every $A \subset \mathbb{R}^d$ we denote by $|A|$ its d-dimensional Lebesgue measure. Given $r > 0$, we denote by $(A)_r := \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) < r\}$ the r-neighbourhood of A. Given $\tau \in \mathbb{R}^d$, we set $A + \tau := \{x + \tau : x \in A\}$. The set $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$ is the set of all d-dimensional unit vectors. For $v, w \in \mathbb{R}^d$ we denote by $\langle v, w\rangle$ their scalar product. We denote by $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$ the standard orthonormal basis of $\mathbb{R}^d$. Given $C \subset \mathbb{R}^d$ convex, we denote by $\mathrm{extreme}(C)$ its extreme points. Given $\rho > 0$, we denote by $Q_\rho := [-\rho/2, \rho/2)^d$ the half open cube centred in 0 with side-length ρ. For $\nu \in \mathbb{S}^{d-1}$, we set $Q^\nu_\rho := R^\nu Q_\rho$, where R^ν is a rotation such that $R^\nu e_d = \nu$. Furthermore, given $x \in \mathbb{R}^d$ we set $Q^\nu_\rho(x) := x + Q^\nu_\rho$ (resp. $Q_\rho(x) = x + Q_\rho$). Given $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B_r(x)$ the open ball with radius $r > 0$ and center x. We denote by ω_d the volume of the unit ball in $\mathbb{R}^d$. Given $\nu \in \mathbb{S}^{d-1}$ we define

$$u_\nu(x) := \begin{cases} 1 & \text{if } \langle \nu, x\rangle \geq 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For $z \in \mathbb{R}$ we denote by $(z)^+ := \max\{z, 0\}$ the positive part of z.

2.2. **Discrete energies and homogenized surface energy density.** In this paragraph we define the discrete energies we want to consider and the main object of homogenized surface energy density.

Let $\mathcal{L} \subset \mathbb{R}^d$ satisfy the following two conditions:

(S1) (Discreteness) There exists $c > 0$ such that for all $x \in \mathcal{L}$ there holds

$$\mathrm{dist}(x, \mathcal{L} \setminus \{x\}) \geq c\,.$$

(S2) (Periodicity) There exists $T \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$ there holds

$$\mathcal{L} + Tz = \mathcal{L}\,.$$

Note that the two assumptions (S1) and (S2) include multi-lattices, such as the hexagonal closed packing in three dimensions, and bravais lattices, such as $\mathbb{Z}^d$, or the face centred cubic lattice in three dimensions.

We consider interaction coefficients $c_{i,j} : \mathcal{L} \times \mathcal{L} \to [0, +\infty)$ and the corresponding (localized) ferromagnetic spin energies of the form

$$E(u, A) := \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j}(u(i) - u(j))^+\,, \tag{5}$$

where $u \colon \mathcal{L} \to \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R}^d)$. If $A = \mathbb{R}^d$ we omit the dependence on the set and write $E(u) := E(u, \mathbb{R}^d)$. We want to remark that we are considering interactions on the directed graph instead of the undirected graph.

We introduce the following three hypothesis on the interaction coefficients $c_{i,j} \colon \mathcal{L} \times \mathcal{L} \to [0, +\infty)$ hold true:

(H1) (Periodicity) There holds

$$c_{i+Tz,j+Tz} = c_{i,j}$$

for all $i, j \in \mathcal{L}$, $z \in \mathbb{Z}^d$.

(H2) (Decay of Interactions) For all $i \in \mathcal{L}$ there holds

$$\sum_{j \in \mathcal{L}} c_{i,j}|i-j| < +\infty\,.$$

(H3) (Finite Range Interactions) There exists $R > 1$ such that

$$c_{i,j} = 0$$

for all $i, j \in \mathcal{L}$ such that $|i - j| \geq R$.

It is obvious, that hypothesis (H3) implies hypothesis (H2). Note that, if (H1) and (H2) are satisfied then

$$\max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j}|i-j| = \max_{i \in \mathcal{L}\cap Q_T} \sum_{j \in \mathcal{L}} c_{i,j}|i-j| < +\infty$$

and for all $R > 0$, there exists $C_R > 0$ such that $C_R \to 0$ as $R \to +\infty$ and

$$\max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq R}} c_{i,j}|i-j| \leq C_R\,.$$

Definition 2.1. Let $c_{i,j}$ satisfy (H1) and (H2). We then define the *homogenized surface energy density* $\varphi : \mathbb{R}^d \to [0, +\infty)$ as the convex positively homogeneous function of degree one such that for all $\nu \in \mathbb{S}^{d-1}$ we have

$$\varphi(\nu) := \lim_{\delta \to 0} \lim_{S \to +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q^\nu_S) : u \colon \mathcal{L} \to \{0,1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S} \right\} \tag{6}$$

with u_ν defined in (4).

Remark 2.2. The definition above can be interpreted as a passage from discrete to continuum description as follows. Given $\varepsilon > 0$, we consider the scaled energies

$$E_\varepsilon(u) := \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} \varepsilon^{d-1} c_{i,j} (u(\varepsilon i) - u(\varepsilon j))^+\,,$$

where $u : \varepsilon\mathcal{L} \to \{0,1\}$. Upon identifying u with its piecewise-constant interpolation, we can regard these energies to be defined on $L^1_{\text{loc}}(\mathbb{R}^d)$. We know that their Γ-limit is infinite outside the space $BV_{\text{loc}}(\mathbb{R}^d)$, where it has the form

$$E_0(u) := \int_{\partial^*\{u=1\}} \varphi(\nu)\,\mathrm{d}\mathcal{H}^{d-1}$$

with φ given by (6).[1] Here, $\partial^*\{u = 1\}$ denotes the reduced boundary of the set $\{u = 1\}$ and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$ (cf. [4], Chapters 2.8 and 3.5).

Remark 2.3. Testing with u_ν in (6), using (S1) and (H2), it is easy to see that $\varphi(\nu) \leq C$ for all $\nu \in \mathbb{S}^{d-1}$. Therefore, due to the convexity and the fact that it is a positively one homogeneous function of degree one, φ is continuous.

[1]Actually, the integral representation for the Γ-limit has only been shown for undirected graphs. However, a slight modification of the proof shows that it is still true for directed graphs.

2.3. **Statement of the main result.** In this section we state the main result.

Definition 2.4. Given $\varphi\colon \mathbb{R}^d \to [0,+\infty)$ convex, positively homogeneous of degree one, we define the Wulff set of φ by

$$W_\varphi := \{\zeta \in \mathbb{R}^d : \langle \zeta, \nu\rangle \leq \varphi(\nu) \text{ for all } \nu \in \mathbb{S}^{d-1}\}\,.$$

We say that φ is crystalline, if W_φ is a polytope.

Remark 2.5. From the definition of the Wulff set, it is clear that

$$\varphi(\nu) = \sup_{\zeta \in W_\varphi} \langle \nu, \zeta\rangle\,.$$

Furthermore, one can check, that if φ is crystalline, then the set $\{\varphi \leq 1\}$ is a polytope.

The next proposition shows that with our proof, we obtain a finite cell formula in order to calculate φ instead of the asymptotic one, given in (6). We think that this result in itself is interesting, since it allows for calculations on finite size systems in order to compute φ for general Ising-systems. This result is in spirit very close to [9, 13], where convex and positively 1-homogenous continuum energies are considered. In this case, the surface energy density and the energy with respect to the absolutely continuous part coincide.

Proposition 2.6. *Let $c_{i,j}\colon \mathcal{L}\times\mathcal{L} \to [0,+\infty)$ be interaction coefficients such that* (H1) *and* (H2) *hold true. Then*

$$\varphi(\nu) = \frac{1}{T^d} \inf\left\{E(u,Q_T) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot\rangle \text{ is } T\text{-periodic}\right\}\,. \tag{7}$$

Theorem 2.7. *Let $c_{i,j}\colon \mathcal{L}\times\mathcal{L} \to [0,+\infty)$ be interaction coefficients such that* (H1) *and* (H3) *hold true. Then, the homogenized surface energy density φ is crystalline. Denote by*

$$N := \#\{(i,j) \in \mathcal{L}\cap Q_T \times \mathcal{L} : c_{i,j} \neq 0\}\,.$$

Then,

$$\#\mathrm{extreme}(W_\varphi) \leq 2^N\,.$$

The next example shows that without assumption (H2) Theorem 2.7 fails to hold true.

Example 2.8. To construct the example we first observe that if $f\colon \mathbb{R}^d \to [0,+\infty)$ is crystalline, then $D^2 f$ is a Radon-measure with support contained in finitely many hyper-planes. To see this, note that if $f\colon \mathbb{R}^d \to [0,+\infty)$ is crystalline, then there exist $\{\xi_k\}_{k=1}^N \subset \mathbb{R}^d$ such that

$$f(\nu) = \max_{1\leq k\leq N} \langle \xi_k, \nu\rangle\,.$$

Here, we assume that $\{\xi_k\}_{k=1}^N$ is chosen minimal, i.e. $\xi_k \neq \lambda \xi_j$ for some $\lambda > 0$ and for some $j \neq k$. This assumption ensures that all the vectors ξ_k play an active role in the definition of f. Now, $Df \in BV_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ is given by

$$Df(\nu) = \sum_{k=1}^N \chi_{V_k}(\nu)\xi_k\,,\ \text{ where } V_k = \left\{\nu \in \mathbb{R}^d\colon f(\nu) = \langle \xi_k, \nu\rangle\right\}\,.$$

Then

$$D^2 f(\nu) = \sum_{1\leq k<j\leq N} (\xi_k - \xi_j)\otimes \nu_{kj}\mathcal{H}^{d-1}\lfloor_{\partial V_k \cap \partial V_j}\,,$$

where $\partial V_k \cap \partial V_j = \left\{\nu \in \mathbb{R}^d\colon f(\nu) = \langle \xi_k, \nu\rangle = \langle \xi_j, \nu\rangle\right\}$. and $\nu_{kj} \in \mathbb{S}^{d-1}$ denotes the normal pointing towards the set V_k.

Let now $\mathcal{L} = \mathbb{Z}^d$ and $c_{i,j} = c_{j-i} = c_{i-j}$ (in the following denoted by $\{c_\xi\}_{\xi\in\mathbb{Z}^d}$) be such that $c_\xi > 0$ for all $\xi \in \mathbb{Z}^d$ and

$$\sum_{\xi\in\mathbb{Z}^d} c_\xi|\xi| < +\infty\,.$$

It is then obvious that $c_{i,j}$ is 1-periodic, (H1) and (H2) hold true, but (H3) is violated. Therefore, due to Proposition 2.6, we have

$$\varphi(\nu) = \sum_{\xi\in\mathbb{Z}^d} c_\xi|\langle\xi,\nu\rangle|\,.$$

This is true, since the only admissible functions in the minimum problem given by Proposition 2.6 are $u_\nu(i) = \langle\nu, i\rangle + c$ for some $c \in \mathbb{R}$. We claim that

$$D\varphi(\nu) = \sum_{\xi\in\mathbb{Z}^d} \mathrm{sign}(\langle\xi,\nu\rangle)\, c_\xi\,\xi\,,$$

where sign: $\mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{sign}(t) = \begin{cases} 1 & t \geq 0; \\ -1 & t < 0. \end{cases}$$

Therefore

$$D^2\varphi = 2\sum_{\xi\in\mathbb{Z}^d} c_\xi\xi\otimes\frac{\xi}{|\xi|}\mathcal{H}^{d-1}\lfloor_{\{\nu:\ \langle\xi,\nu\rangle=0\}}\,. \tag{8}$$

This can be seen by approximation. Consider $\varphi_R : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\varphi_R(\nu) = \sum_{\substack{\xi\in\mathbb{Z}^d\\|\xi|\leq R}} c_\xi|\langle\xi,\nu\rangle|\,,\quad D\varphi_R(\nu) = \sum_{\substack{\xi\in\mathbb{Z}^d\\|\xi|\leq R}} \mathrm{sign}(\langle\xi,\nu\rangle)\, c_\xi\,\xi\,,$$

Then

$$D^2\varphi_R = 2\sum_{\substack{\xi\in\mathbb{Z}^d\\|\xi|\leq R}} c_\xi\xi\otimes\frac{\xi}{|\xi|}\mathcal{H}^{d-1}\lfloor_{\{\nu:\ \langle\xi,\nu\rangle=0\}}\,.$$

Now

$$|D^2\varphi_R|(B_r) \leq Cr^{d-1}\sum_{\xi\in\mathbb{Z}^d} c_\xi|\xi|\,,$$

so the total variation of $D^2\varphi_R$ is (locally) uniformly bounded with limiting measure $D^2\varphi$ and $D\varphi_R \to D\varphi$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, actually weakly in BV. Hence, (8) is shown. Now, since $c_\xi > 0$ for all $\xi \in \mathbb{Z}^d$ it is obvious that $D^2\varphi$ is not supported on finitely many hyper-planes. Thus φ cannot be crystalline.

Note that φ is differentiable in totally irrational directions.[2] A similar property is known to hold, in the continuous setting [5, 24], for homgenized surface tensions. We can state a result showing that this is still the case in the discrete setting, under assumptions (H1) and (H2).

Proposition 2.9. *Under the assumptions of Proposition 2.6, φ is differentiable in any totally irrational direction.*

[2] p is totally irrational if there is no $q \in \mathbb{Z}^d \setminus \{0\}$ such that $\langle q, p\rangle = 0$.

It is expected that it should be, "in general", not differentiable in the other directions, at least whenever the minimizers u in (7) are constant on an infinite set, however the proofs in [5, 24] rely on ellipticity properties of the problem and are less easy to transfer to the discrete case. The proof of Proposition 2.9, which mimicks the proof in [24], is postponed to Section 5, and relies on the dual representation (100) introduced later on.

3. Proof of Proposition 2.6

This section is devoted to the proof of Proposition 2.6. We assume throughout this section that assumptions (S1), (S2) and (H1), (H2) are satisfied. The proof consists in showing that φ can be characterized by several (equivalent) cell-formulas and therefore passing from (6) to (7).

First, we will state and prove some elementary properties of E that will be used throughout this section.

Lemma 3.1. *Let $A \in \mathcal{B}(\mathbb{R}^d)$.*

(i) *Let $|A| > 1$ and $\nu \in \mathbb{R}^d$. Then*
$$E(\langle \nu, \cdot \rangle, A) \leq C|\nu||(A)_c|\,.$$
(ii) *Let $u\colon \mathcal{L} \to \mathbb{R}$. For all $t \in \mathbb{R}, \lambda > 0$ there holds*
$$E(\lambda u + t, A) = \lambda E(u, A)$$
and $u \mapsto E(u, A)$ is convex. In particular,
$$E(u + v, A) \leq E(u, A) + E(v, A)$$
for all $u, v\colon \mathcal{L} \to \mathbb{R}$.

(iii) *Let $u\colon \mathcal{L} \to \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R}^d)$ be such that $A \subset B$. Then*
$$E(u, A) \leq E(u, B)\,.$$
(iv) *Let $u\colon \mathcal{L} \to \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R}^d)$ be such that $A \cap B = \emptyset$. Then*
$$E(u, A \cup B) = E(u, A) + E(u, B)\,.$$
(v) *There holds*
$$\#\{i \in \mathcal{L} \cap A\} \leq \frac{1}{c^d \omega_d}|(A)_c|\,,$$
where c is given by (S1).

(vi) *Let $u\colon \mathcal{L} \to \mathbb{R}$. Then, the function $\tau \mapsto E(u(\cdot - \tau), A + \tau)$ is T-periodic.*

Proof. We start by proving (ii)-(iv) in Step 1, then (v) and (vi) in Step 2 and Step 3 respectively, and finally (i) in Step 4.

Step 1.(Proof of (ii) - (iv)) All the three statements are a direct consequence of (5) and the fact that $c_{i,j} \geq 0$.

Step 2. (Proof of (v)) Note that
$$\bigcup_{i \in \mathcal{L} \cap A} B_c(i) \subset (A)_c$$
and therefore, due to (S1),
$$c^d \omega_d \#\{i \in \mathcal{L} \cap A\} = \left| \bigcup_{i \in \mathcal{L} \cap A} B_c(i) \right| \leq |(A)_c|\,.$$

This is the claim.

Step 3.(Proof of (vi)) Let $u\colon \mathcal{L} \to \mathbb{R}$ and $z \in \mathbb{Z}^d$. Then, using (H1) and (S1),

$$\begin{aligned} E(u(\cdot - Tz), A + Tz) &= \sum_{i\in\mathcal{L}\cap(A+Tz)} \sum_{j\in\mathcal{L}} c_{i,j}(u(i - Tz) - u(j - Tz))^+ \\ &= \sum_{i\in\mathcal{L}\cap A} \sum_{j\in(\mathcal{L}+Tz)} c_{i+Tz,j+Tz}(u(i) - u(j))^+ \\ &= \sum_{i\in\mathcal{L}\cap A} \sum_{j\in\mathcal{L}} c_{i,j}(u(i) - u(j))^+ = E(u, A)\,. \end{aligned}$$

Step 4.(Proof of (i)) Let $S > 1$ and $\nu \in \mathbb{R}^d$, then, due to (v), the fact that $S > 1$, (S1), (S2),(H1), and (H2), we have

$$E(\langle\nu,\cdot\rangle, A) = \sum_{i\in\mathcal{L}\cap A} \sum_{j\in\mathcal{L}} c_{i,j}|\langle\nu, i - j\rangle| \leq |\nu|\#\{i \in \mathcal{L} \cap A\} \max_{i\in\mathcal{L}} \sum_{j\in\mathcal{L}} c_{i,j}|i - j| \leq C|\nu||(A)_c|\,.$$

□

Lemma 3.2. *Let $S > 0$ and $\nu \in \mathbb{S}^{n-1}$. Then*

$$\begin{aligned} &\inf\left\{E(u, Q^\nu_S) : u\colon \mathcal{L} \to \mathbb{R}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S}\right\} \\ =&\inf\left\{E(u, Q^\nu_S) : u\colon \mathcal{L} \to \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S}\right\}\,. \end{aligned}$$

Proof. We divide the proof into two steps. We first show that the left hand side is smaller or equal than the right hand side and then we show the right hand side is smaller or equal than the left hand side.

Step 1. (Proof of '$\leq$') This inequality is clear, since the infimum on the left hand side is taken over a larger class of functions.

Step 2. (Proof of '$\geq$') Let us take $u : \mathcal{L} \to \mathbb{R}$ such that $u = u_\nu$ on $\mathbb{R}^d \setminus Q^\nu_{(1-\delta)S}$. It suffices to construct $u_0 : \mathcal{L} \to \{0, 1\}$ such that $u_0 = u_\nu$ on $\mathbb{R}^d \setminus Q^\nu_{(1-\delta)S}$ and

$$E(u_0, Q^\nu_S) \leq E(u, Q^\nu_S)\,. \tag{9}$$

We prove this inequality by induction on $N = \#\text{codomain}(u)$. If $N = 2$, then we must have $\text{codomain}(u) = \{0, 1\}$ and there is nothing to prove. Suppose that $\#\text{codomain}(u) = N + 1$, with $N \geq 2$. Since

$$E((u \wedge 1) \vee 0, Q^\nu_S) \leq E(u, Q^\nu_S)\,,$$

we can suppose that $\text{codomain}(u) \subset [0, 1]$ We can write $\text{codomain}(u) = \{a_k : k = 0, \dots, N\}$ with $0 = a_0 < a_1 < \dots, a_N = 1$. Let

$$u = \sum_{k=1}^N a_k \chi_{E_k}\,, \text{ with } \bigcup_{k=0}^N E_k = \mathbb{R}^d\,,$$

$\{x : \langle\nu, x\rangle \geq 0\} \setminus Q^\nu_{(1-\delta)S} \subset E_N$, and $\{x : \langle\nu, x\rangle < 0\} \setminus Q^\nu_{(1-\delta)S} \subset E_0$ so that $u(i) = u_\nu(i)$ on $\mathcal{L} \setminus Q^\nu_{(1-\delta)S}$. We define

$$u' = \sum_{k=2}^N a_k \chi_{E_k}\,, \text{ and } u'' = \sum_{k=2}^N a_k \chi_{E_k} + a_2 \chi_{E_1}\,. \tag{10}$$

Note that both $u'(i) = u''(i) = u^\nu(i)$ on $\mathcal{L} \setminus Q^\nu_{(1-\delta)S}$. Furthermore, we have that

$$E(u', Q^\nu_T) = E(u, Q^\nu_T) + a_1 \left(\sum_{k=2}^N \sum_{i \in \mathcal{L} \cap E_k} \sum_{j \in \mathcal{L} \cap E_1} c_{i,j} - \sum_{i \in \mathcal{L} \cap E_0} \sum_{j \in \mathcal{L} \cap E_1} c_{i,j} \right)$$

and

$$E(u'', Q^\nu_T) = E(u, Q^\nu_T) + (a_1 - a_2) \left(\sum_{k=2}^N \sum_{i \in \mathcal{L} \cap E_k} \sum_{j \in \mathcal{L} \cap E_1} c_{i,j} - \sum_{i \in \mathcal{L} \cap E_0} \sum_{j \in \mathcal{L} \cap E_1} c_{i,j} \right) .$$

Hence, either $E(u', Q^\nu_S) \leq E(u, Q^\nu_S)$ or $E(u'', Q^\nu_S) \leq E(u, Q^\nu_S)$. Since, by (10), $u', u'' : \mathcal{L} \to \#\mathrm{codomain}(u) \setminus \{a_1\}$ inequality (9) is true by induction. This concludes the proof. □

Let $\phi\colon \mathbb{R}^d \to [0, +\infty]$ be defined by

$$\phi(\nu) = \lim_{\delta \to 0} \lim_{S \to +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S) : u\colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S} \right\} . \tag{11}$$

$\phi_{\mathrm{per}}\colon \mathbb{R}^d \to [0, +\infty]$ is defined by

$$\phi_{\mathrm{per}}(\nu) = \lim_{k \to +\infty} \frac{1}{(kT)^d} \inf \left\{ E(u, Q_{kT}) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } (kT)\text{-periodic} \right\} . \tag{12}$$

The following lemma uses a standard cutoff-argument. However, due to the infinite range of interactions, the finite scale arguments need to be adapted.

Lemma 3.3. *Let $\nu \in \mathbb{R}^d$. Then*

$$\phi_{\mathrm{per}}(\nu) = \phi(\nu) .$$

Proof. In order to prove the Lemma, we show first $\phi_{\mathrm{per}}(\nu) \leq \phi(\nu)$ and then the reverse inequality. In order to do so, we modify competitors of the respective cell formulas in order to obtain a competitor for the other formula.

Due to the one homogeneity of both functions, we may assume that $\nu \in \mathbb{S}^{d-1}$.

Step 1.(Proof of '$\leq$') Since, in the definition of $\phi_{\mathrm{per}}(\nu)$, (resp. $\phi(\nu)$), the limit exists[3] we can assume without loss of generality that $S = kT$ for some $k \in \mathbb{N}$ with k large. Let $\delta > 0$ and let $u^\delta_k\colon \mathcal{L} \to \mathbb{R}$ be such that $u^\delta_k(i) = \langle \nu, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)kT}$ and

$$E(u^\delta_k, Q_{kT}) = \inf \left\{ E(u, Q_{kT}) : u\colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)kT} \right\} . \tag{13}$$

We assume that

$$||u^\delta_k||_{L^\infty(Q_{(1+\delta)kT})} \leq 2kT . \tag{14}$$

If that were not true we perform the following construction with $\tilde{u}^\delta_k(i) = (u^\delta_k(i) \vee (-2kT)) \wedge (2kT)$. Note that still $\tilde{u}^\delta_k(i) = \langle \nu, i \rangle$ on $Q_{(1+\delta)kT}$ for δ small enough. We define $v^\delta_k\colon \mathcal{L} \to \mathbb{R}$ by setting

$$v^\delta_k(i_0 + kTz) = u^\delta_k(i_0) + \langle \nu, kTz \rangle \text{ if } i_0 \in Q_{kT}, z \in \mathbb{Z}^d . \tag{15}$$

We claim that

$$|v^\delta_k(i) - v^\delta_k(j)| \leq CkT + |i - j| . \tag{16}$$

[3]The existence of the limit of ϕ_{per} is a direct consequence of Lemma 3.4 whereas for ϕ it follows by classical subadditivity arguments.

To see this note for $i = i_0 + Tkz, j = j_0 + Tkz'$, $i_0, j_0 \in Q_{kT}, z, z' \in \mathbb{Z}^d$, due to (15) and (14), we have

$$
\begin{aligned}
|v_k^\delta(i) - v_k^\delta(j)| &= |u_k^\delta(i_0) - u_k^\delta(j_0) + \langle \nu, kT(z-z')\rangle| \leq |u_k^\delta(i_0)| + |u_k^\delta(j_0)| + |kT(z-z')| \\
&\leq CkT + |i-j| + |i_0| + |j_0| \leq CkT + |i-j| \,.
\end{aligned}
$$

Clearly, $v_k^\delta(\cdot) - \langle \nu, \cdot\rangle$ is (kT)-periodic. Let us check that

$$
v_k^\delta(i) = v_k^\delta(i) \text{ for } i \in Q_{(1+\delta)kT} \,. \tag{17}
$$

This holds true for $i \in Q_{kT}$. It remains to be checked for $i \in Q_{(1+\delta)kT} \setminus Q_{kT}$. Let $i \in Q_{(1+\delta)kT} \setminus Q_{kT}$, i.e. $i = kTz + i_0$, where $i_0 \in Q_{kT}$ and $||z||_\infty = 1$. Let $n \in \{1, \dots, d\}$ be such that $|i_n| = ||i||_\infty$. Furthermore, let us assume for now that $i_n \geq 0$ and therefore, $z_n = 1$. Since $i \in Q_{(1+\delta)Tk}$ we have $i_n < (1+\delta)Tk/2$. Hence

$$
(i_0)_n = i_n - kTz_n < (1+\delta)Tk/2 - kT = (-1+\delta)Tk/2 \,.
$$

Hence $i_0 \in \mathcal{L} \setminus Q_{(1-\delta)kT}$ (The case that $i_n < 0$ is done analogously - note that Q_{kT} is defined as the half-open cube centred in 0. Hence, we need to make this distinction.). Therefore, by (15) and the definition of u_k^δ in $\mathcal{L} \setminus Q_{(1-\delta)kT}$,

$$
v_k^\delta(i) = v_k^\delta(i_0 + kTz) = v_k^\delta(i_0) + \langle \nu, kTz\rangle = u_k^\delta(i_0) + \langle \nu, kTz\rangle = \langle \nu, i_0\rangle + \langle \nu, kTz\rangle = u_k^\delta(i) \,.
$$

Hence, (17) holds true. Additionally,

$$
\inf \{E(u, Q_{kT}) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot\rangle \text{ is } (kT)\text{-periodic}\} \leq E(v_k^\delta, Q_{kT}) \,. \tag{18}
$$

We are finished with Step 1 if we prove

$$
E(v_k^\delta, Q_{kT}) \leq E(u_k^\delta, Q_{kT}) + \frac{C_k}{\delta}(kT)^d \,, \tag{19}
$$

where $C_k \to 0$ as $k \to +\infty$. In fact, using (13), (18), (19), dividing by $(kT)^d$, and letting $k \to +\infty$, we obtain the claim. Let us prove (19). We have, using (17),

$$
\begin{aligned}
E(v_k^\delta, Q_{kT}) &= \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L}} c_{i,j}(v_k^\delta(i) - v_k^\delta(j))^+ \\
&= \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L} \cap Q_{(1+\delta)kT}} c_{i,j}(v_k^\delta(i) - v_k^\delta(j))^+ + \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)kT}} c_{i,j}(v_k^\delta(i) - v_k^\delta(j))^+ \\
&\leq E(u_k^\delta, Q_{kT}) + \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j}|v_k^\delta(i) - v_k^\delta(j)| \,.
\end{aligned}
$$

Hence, in order to show (19), it remains to prove

$$
\sum_{i \in Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j}|v_k^\delta(i) - v_k^\delta(j)| \leq \frac{C_k}{\delta}(kT)^d \,. \tag{20}
$$

Using (16), (H2), and Lemma 3.1(v), we have

$$\begin{aligned}\sum_{i\in\mathcal{L}\cap Q_{kT}}\sum_{\substack{j\in\mathcal{L}\\|i-j|\geq\delta kT/2}} c_{i,j}|v_k^\delta(i)-v_k^\delta(j)| &\leq \sum_{i\in\mathcal{L}\cap Q_{kT}}\sum_{\substack{j\in\mathcal{L}\\|i-j|\geq\delta kT/2}} c_{i,j}(CkT+|i-j|)\\ &\leq\left(\frac{C}{\delta}+1\right)\sum_{i\in\mathcal{L}\cap Q_{kT}}\sum_{\substack{j\in\mathcal{L}\\|i-j|\geq\delta kT/2}} c_{i,j}|i-j|\\ &\leq \frac{C}{\delta}\#(\mathcal{L}\cap Q_{kT})\max_{i\in\mathcal{L}}\sum_{\substack{j\in\mathcal{L}\\|i-j|\geq\delta kT/2}} c_{i,j}|i-j| \leq \frac{C_k}{\delta}(kT)^d\,,\end{aligned}$$

where $C_k\to 0$ as $k\to+\infty$. This yields (20) and therefore the claim of Step 1.

Step 2.(Proof of '$\geq$') Let $u_k\colon\mathcal{L}\to\mathbb{R}$ be such that $u_k(\cdot)-\langle\nu,\cdot\rangle$ is (kT)-periodic and

$$E(u_k,Q_{kT})=\inf\left\{E(u,Q_{kT}):u\colon\mathcal{L}\to\mathbb{R},u(\cdot)-\langle\nu,\cdot\rangle\text{ is }(kT)\text{-periodic}\right\}$$

Fix $S\in\mathbb{N}$ such that $S=mkT>>kT$ for some $m\in\mathbb{N},m>>1$. Since $u_k(\cdot)-\langle\nu,\cdot\rangle$ is (kT)-periodic, we have

$$E(u_k,Q_{kT}(x_0))=E(u_k,Q_{kT})\text{ for all }x_0\in kT\mathbb{Z}^d$$

and therefore

$$E(u_k,Q_S)=\frac{S^d}{(kT)^d}E(u_k,Q_{kT})\,. \tag{21}$$

There exists a constant $C_k>0$ (we omit the dependence on T) such that, due to the fact that $u_k(\cdot)-\langle\nu,\cdot\rangle$ is (kT)-periodic, there holds

$$\max_{i\in\mathcal{L}}|u_k(i)-\langle\nu,i\rangle|=\max_{i\in\mathcal{L}\cap Q_{kT}}|u_k(i)-\langle\nu,i\rangle|\leq C_k\,. \tag{22}$$

Let $\varphi_S\in C_c^\infty(\mathbb{R}^d)$ be a cut-off function such that

$$\varphi_S(x)=1\text{ for }x\in Q_{(1-3\delta)S},\varphi_S(x)=0\text{ for }x\in\mathbb{R}^d\setminus Q_{(1-2\delta)S},\text{ and }||\nabla\varphi_S||_\infty\leq C_k^{-1}\,.$$

Define $u_S\colon\mathcal{L}\to\mathbb{R}$ by

$$u_S(i)=\varphi_S(i)u_k(i)+(1-\varphi_S(i))\langle\nu,i\rangle\,.$$

Then, $u_S(i)=\langle\nu,i\rangle$ for $i\in\mathcal{L}\setminus Q_{(1-\delta)S}$ and therefore

$$\inf\left\{E(u,Q_S):u\colon\mathcal{L}\to\mathbb{R},u(i)=\langle\nu,i\rangle\text{ on }\mathcal{L}\setminus Q_{(1-\delta)S}\right\}\leq E(u_S,Q_S)\,. \tag{23}$$

For all $i,j\in\mathcal{L}$ there holds

$$u_S(i)-u_S(j)=\varphi_S(i)\left(u_k(i)-u_k(j)\right)+(1-\varphi_S(i))\langle\nu,i-j\rangle+(\varphi_S(i)-\varphi_S(j))(u_k(i)-\langle\nu,i\rangle)\,,$$

which, together with (22), implies for all $i,j\in\mathcal{L}$

$$\begin{aligned}(u_S(i)-u_S(j))^+&\leq(u_k(i)-u_k(j))^++|i-j|+C_k^{-1}|u_k(i)-\langle\nu,i\rangle||i-j|\\&\leq(u_k(i)-u_k(j))^++C|i-j|\,.\end{aligned} \tag{24}$$

and

$$(u_S(i)-u_S(j))^+=(u_k(i)-u_k(j))^+\text{ for all }i,j\in Q_{(1-3\delta)S}\,. \tag{25}$$

Using (21), (24), and (25), we obtain

$$
\begin{aligned}
E(u_S, Q_S) \leq & \sum_{i\in\mathcal{L}\cap Q_S}\sum_{j\in\mathcal{L}} c_{i,j}(u_k(i)-u_k(j))^+ + C\sum_{i\in\mathcal{L}\cap Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}\setminus Q_{(1-3\delta)S}} c_{i,j}|i-j| \\
& + C\sum_{i\in\mathcal{L}\cap Q_S\setminus Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}} c_{i,j}|i-j| \\
= & \frac{S^d}{(kT)^d}E(u_k, Q_{kT}) + C\sum_{i\in\mathcal{L}\cap Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}\setminus Q_{(1-3\delta)S}} c_{i,j}|i-j| \\
& + C\sum_{i\in\mathcal{L}\cap Q_S\setminus Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}} c_{i,j}|i-j|\,.
\end{aligned}
\tag{26}
$$

We show that

$$
\sum_{i\in\mathcal{L}\cap Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}\setminus Q_{(1-3\delta)S}} c_{i,j}|i-j| \leq C_S S^d\,, \tag{27}
$$

where $C_S \to 0$ as $S \to +\infty$. Furthermore, we show that

$$
\sum_{i\in\mathcal{L}\cap Q_S\setminus Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}} c_{i,j}|i-j| \leq C\delta S^d\,. \tag{28}
$$

Note that from (27) and (28) we obtain the claim of Step 2 by using (23), (26), dividing by S^d, letting first $S \to +\infty$, then $k \to +\infty$ and lastly $\delta \to 0$.

We first prove (27). Note that, for S big enough, due to (H2) and Lemma 3.1(v), we have

$$
\begin{aligned}
\sum_{i\in\mathcal{L}\cap Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}\setminus Q_{(1-3\delta)S}} c_{i,j}|i-j| &\leq \sum_{i\in\mathcal{L}\cap Q_{(1-6\delta)S}}\sum_{\substack{j\in\mathcal{L}\\ |i-j|\geq\delta S}} c_{i,j}|i-j| \\
&\leq \#(\mathcal{L}\cap Q_S)\max_{i\in\mathcal{L}}\sum_{\substack{j\in\mathcal{L}\\ |i-j|\geq\delta S}} c_{i,j}|i-j| \\
&\leq C_S S^d\,,
\end{aligned}
$$

where $C_S \to 0$ as $S \to +\infty$. Next, we show (28). Using (H2), and Lemma 3.1(v), we obtain

$$
\begin{aligned}
\sum_{i\in\mathcal{L}\cap Q_S\setminus Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}} c_{i,j}|i-j| &\leq \sum_{i\in\mathcal{L}\cap Q_S\setminus Q_{(1-6\delta)S}}\sum_{j\in\mathcal{L}} c_{i,j}|i-j| \\
&\leq \#(\mathcal{L}\cap Q_S\setminus Q_{(1-6\delta)S})\max_{i\in\mathcal{L}}\sum_{j\in\mathcal{L}} c_{i,j}|i-j| \\
&\leq C\delta S^d\,.
\end{aligned}
$$

This is (28) and hence the claim of Step 2. □

The next Lemma shows that, using periodic boundary conditions, one can reduce from an asymptotic cell formula to a finite cell formula.

Lemma 3.4. *Let $\nu \in \mathbb{S}^1$. For all $k \in \mathbb{N}$ there holds*

$$
\begin{aligned}
&\frac{1}{(kT)^d}\inf\left\{E(u, Q_{kT}) : u\colon \mathcal{L}\to\mathbb{R}, u(\cdot)-\langle\nu,\cdot\rangle \textit{ is } (kT)\textit{-periodic}\right\} \\
&= \frac{1}{T^d}\inf\left\{E(u, Q_T) : u\colon \mathcal{L}\to\mathbb{R}, u(\cdot)-\langle\nu,\cdot\rangle \textit{ is } T\textit{-periodic}\right\}\,.
\end{aligned}
\tag{29}
$$

In particular

$$\phi_{\rm per}(\nu) = \frac{1}{T^d} \inf\left\{E(u,Q_T) : u\colon \mathcal{L}\to\mathbb{R}, u(\cdot)-\langle\nu,\cdot\rangle \text{ is } T\text{-periodic}\right\} . \tag{30}$$

Proof. We split the proof into two steps by first observing the (obvious) inequality that the right hand side in (29) is less than or equal to the left hand side. Then, we prove the converse inequality by using a superposition argument.

Step 1.(Proof of '$\leq$') Let $u\colon \mathcal{L}\to\mathbb{R}$ be such that $u(\cdot)-\langle\nu,\cdot\rangle$ is T-periodic and

$$E(u,Q_T) = \inf\left\{E(u,Q_T) : u\colon \mathcal{L}\to\mathbb{R}, u(\cdot)-\langle\nu,\cdot\rangle \text{ is } T\text{-periodic}\right\} . \tag{31}$$

Then, since $u(\cdot)-\langle\nu,\cdot\rangle$ is T-periodic, we have $E(u,Q_T(z)) = E(u,Q_T)$ for all $z\in\mathbb{Z}^d$. For $m\in\{0,\dots,k-1\}^d$ set

$$z_m^k = T\left(m - \frac{k-1}{2}\vec{1}\right) ,$$

where $\vec{1}_n = 1$ for all $n=1,\dots,d$. It is easy to check that $Q_T(z_m^k)\subset Q_{kT}$ for all $m\in\{0,\dots,k-1\}^d$. Hence, using (31), we obtain

$$\begin{aligned} E(u,Q_{kT}) \leq & \sum_{m\in\{0,\dots,k-1\}^d} E(u,Q_T(z_m^k)) = k^d E(u,Q_T) \\ & = \inf\left\{E(u,Q_T) : u\colon \mathcal{L}\to\mathbb{R}, u(\cdot)-\langle\nu,\cdot\rangle \text{ is } T\text{-periodic}\right\} . \end{aligned}$$

Noting that $u(\cdot)-\langle\nu,\cdot\rangle$ is (kT)-periodic we obtain the desired inequality.

Step 2.(Proof of '$\geq$') Let $u_k\colon\mathcal{L}\to\mathbb{R}$ be such that $u(\cdot)-\langle\nu,\cdot\rangle$ is (kT)-periodic. Define $u_T\colon\mathcal{L}\to\mathbb{R}$ by

$$u_T(i) = \frac{1}{k^d}\sum_{z\in\{0,\dots,k-1\}^d} u_k(i+Tz) .$$

Let us first check that $u_T(\cdot)-\langle\nu,\cdot\rangle$ is T-periodic. By definition, for $n\in\{1,\dots,d\}$ there holds

$$u_T(i+Te_n)-\nu(i+Te_n) = \frac{1}{k^d}\sum_{z\in\{0,\dots,k-1\}^d} u_k(i+Tz+Te_n) - \langle\nu, i+Te_n\rangle$$

We now split the sum to obtain

$$\begin{aligned} & u_T(i+Te_n)-\nu(i+Te_n) \\ = & \frac{1}{k^d}\sum_{\substack{z'\in\{0,\dots,k-1\}^d \\ z_n\in\{0,\dots,k-2\}}} u_k(i+Tz+Te_n) + \frac{1}{k^d}\sum_{\substack{z'\in\{0,\dots,k-1\}^d \\ z_n=k-1}} (u_k(i+Tz+Te_n) - k\langle\nu,Te_n\rangle) - \langle\nu,i\rangle . \end{aligned} \tag{32}$$

For the first term in the sum, shifting the indices, we obtain

$$\frac{1}{k^d}\sum_{\substack{z'\in\{0,\dots,k-1\}^d \\ z_n\in\{0,\dots,k-2\}}} u_k(i+Tz+Te_n) = \frac{1}{k^d}\sum_{\substack{z'\in\{0,\dots,k-1\}^d \\ z_n\in\{1,\dots,k-1\}}} u_k(i+Tz) . \tag{33}$$

For the second term in the sum, we obtain, due to the fact that $u(\cdot) - \langle \nu, \cdot \rangle$ is (kT)-periodic,

$$
\begin{aligned}
&\frac{1}{k^d} \sum_{\substack{z' \in \{0,\dots,k-1\}^d \\ z_n = k-1}} (u_k(i + Tz + Te_n) - k\langle \nu, Te_n \rangle) \\
=&\frac{1}{k^d} \sum_{\substack{z' \in \{0,\dots,k-1\}^d \\ z_n = k-1}} (u_k(i + T(z',0) + kTe_n) - \langle \nu, i + T(z',0) + kTe_n \rangle + \langle \nu, i + T(z',0) \rangle) \\
=&\frac{1}{k^d} \sum_{\substack{z' \in \{0,\dots,k-1\}^d \\ z_n = k-1}} u_k(i + T(z',0)) = \frac{1}{k^d} \sum_{\substack{z' \in \{0,\dots,k-1\}^d \\ z_n = 0}} u_k(i + Tz)\,.
\end{aligned}
$$

Using this together with (32) and (33), we get

$$
\begin{aligned}
u_T(i + Te_n) - \nu(i + Te_n) &= \frac{1}{k^d} \sum_{\substack{z' \in \{0,\dots,k-1\}^d \\ z_n \in \{1,\dots,k-1\}}} u_k(i + Tz) + \frac{1}{k^d} \sum_{\substack{z' \in \{0,\dots,k-1\}^d \\ z_n = 0}} u_k(i + Tz) - \langle \nu, i \rangle \\
&= \frac{1}{k^d} \sum_{z' \in \{0,\dots,k-1\}^d} u_k(i + Tz) - \langle \nu, i \rangle = u_T(i) - \langle \nu, i \rangle\,.
\end{aligned}
$$

This shows that in fact $u_T(\cdot) - \langle \nu, \cdot \rangle$ is T-periodic. Hence

$$
\inf \{E(u, Q_T) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic}\} \le E(u_T, Q_T) \tag{34}
$$

Now, using the convexity of E (cf. Lemma 3.1(ii)) , we obtain

$$
E(u_T, Q_T) \le \frac{1}{k^d} \sum_{z \in \{0,\dots,k-1\}^d} E(u_k(\cdot + Tz), Q_T) = \frac{1}{k^d} E(u_k, Q_{kT})\,.
$$

Note that $u_k\colon \mathcal{L} \to \mathbb{R}$ is such that $u_k(\cdot) - \langle \nu, \cdot \rangle$ is (kT)-periodic and arbitrary. This together with (34) yields the claim. Equation (30) follows from Step 1, Step 2, and the definition of ϕ_{per}, see (12). □

Let $\psi\colon \mathbb{R}^d \to [0, +\infty]$ be defined as the positively homogeneous function of degree one that for $\nu \in \mathbb{S}^{d-1}$ is defined by

$$
\psi(\nu) = \lim_{\delta \to 0} \lim_{S \to +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S^\nu) : u\colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S} \right\}. \tag{35}
$$

Lemma 3.5. $\psi\colon \mathbb{R}^d \to [0, +\infty]$ *satisfies the following properties:*

(i) *There exists $C > 0$ such that $\psi(\nu) \le C|\nu|$ for all $\nu \in \mathbb{R}^d$,*
(ii) *ψ is a continuous function.*

Proof. We divide the proof into two steps. We first prove (i) and then (ii). Throughout the proofs let $1 \ll S$.

Step 1.(Proof of (i)) Let $\nu \in \mathbb{S}^{d-1}$ it suffices to prove

$$
\psi(\nu) \le C\,.
$$

The general case then follows by one-homogeneity. In order to prove (i) we insert $u(i) = \langle \nu, i \rangle$ for all $i \in \mathcal{L}$ as a competitor in the cell formula. Using Lemma 3.1(i), we then have

$$
E(u, Q_S^\nu) = E(\langle \nu, \cdot \rangle, Q_S^\nu) \le C|\nu||(Q_S)_c| \le CS^d\,.
$$

Dividing by S^d and letting $S \to +\infty$ yields the claim.

Step 2.(Proof of (ii)) Due to the one-homogeneity, it suffices to consider the case where $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$. Let $\eta > 0$ and $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$ be such that $|\nu_1 - \nu_2| \leq \eta$. Our goal is to prove that there exists $C > 0$ independent of ν_1 and ν_2 such that

$$|\psi(\nu_1) - \psi(\nu_2)| \leq C\eta \,. \tag{36}$$

We only prove

$$\psi(\nu_1) - \psi(\nu_2) \leq C\eta \,, \tag{37}$$

since then (36) follows by exchanging ν_1 and ν_2 in (37). To this end let $\delta > 0$ small enough, $S > 0$ big enough, $u_1 \colon \mathcal{L} \to \mathbb{R}$ be such that $u_1(i) = \langle \nu_1, i \rangle$ on $\mathcal{L} \setminus Q^{\nu}_{(1-\delta)S}$ and

$$\frac{1}{S^d} E(u_1, Q^{\nu_1}_S) \leq \psi(\nu_1) + \eta \,. \tag{38}$$

We assume that

$$||u_1||_{L^\infty(Q^{\nu_1}_S)} \leq S \,. \tag{39}$$

If this were not the case, we consider

$$\tilde{u}_1(i) = \begin{cases} (u_1(i) \wedge S) \vee (-S) & i \in Q^{\nu_1}_{2S} \,, \\ u_1(i) & \text{otherwise.} \end{cases}$$

Note that for $i, j \in Q^{\nu_1}_{2S}$, due to truncation, $(\tilde{u}_1(i) - \tilde{u}_1(j))^+ \leq (u_1(i) - u_1(j))^+$, whereas in general there holds $|\tilde{u}_1(i) - \tilde{u}_1(j)| \leq CS + |i - j|$. From this, using Lemma 3.1(v) and (H2), we deduce

$$\begin{aligned} E(\tilde{u}_1, Q^{\nu_1}_S) &= \sum_{i \in \mathcal{L} \cap Q^{\nu_1}_S} \sum_{j \in \mathcal{L} \cap Q^{\nu_1}_{2S}} c_{i,j} (\tilde{u}_1(i) - \tilde{u}_1(j))^+ + \sum_{i \in \mathcal{L} \cap Q^{\nu_1}_S} \sum_{j \in \mathcal{L} \setminus Q^{\nu_1}_{2S}} c_{i,j} (\tilde{u}_1(i) - \tilde{u}_1(j))^+ \\ &\leq \sum_{i \in \mathcal{L} \cap Q^{\nu_1}_S} \sum_{j \in \mathcal{L} \cap Q^{\nu_1}_{2S}} c_{i,j} (u_1(i) - u_1(j))^+ + C \sum_{i \in \mathcal{L} \cap Q^{\nu_1}_S} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq S/2}} c_{i,j} |i - j| \\ &\leq E(u_1, Q^{\nu_1}_S) + C\#(\mathcal{L} \cap Q^{\nu_1}_S) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq S/2}} c_{i,j} |i - j| \leq E(u_1, Q^{\nu_1}_S) + C_S S^d \,, \end{aligned}$$

where $C_S \to 0$ as $S \to \infty$. In particular $C_S \leq \eta$ for S big enough. Hence, we can assume (39). There exists $C > 0$ such that for $\tilde{S} = (1 + C\eta)S$ there holds $Q^{\nu_2}_{(1-\delta)\tilde{S}} \supset Q^{\nu_1}_{(1+\delta)S}$. We now define $u_2 \colon \mathcal{L} \to \mathbb{R}$ by

$$u_2(i) = \langle \nu_2 - \nu_1, i \rangle + u_1(i) \,. \tag{40}$$

First, note that $u_2(i) = \langle \nu_2, i \rangle$ for all $i \in \mathcal{L} \setminus Q^{\nu_2}_{(1-\delta)\tilde{S}}$ and therefore

$$\inf \left\{ E(u, Q^{\nu_2}_{\tilde{S}}) : u \colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu_2, i \rangle \text{ on } \mathcal{L} \setminus Q^{\nu_2}_{(1-\delta)\tilde{S}} \right\} \leq E(u_2, Q^{\nu_2}_{\tilde{S}}) \,. \tag{41}$$

We claim that

$$E(u_2, Q^{\nu_2}_{\tilde{S}}) \leq E(u_1 Q^{\nu_1}_S) + \frac{C_S}{\delta} S^d + C\eta S^d + C\delta S^d \,, \tag{42}$$

where $C_S \to 0$ as $S \to +\infty$. We postpone the proof of (42) and show first how it implies (37). Dividing (42) by $\tilde{S}^d$, letting $\tilde{S}$ (therefore also S) tend to $+\infty$, $\delta \to 0$, and using (41) as well as (38), we get

$$\psi(\nu_2) \leq \psi(\nu_1) + C\eta \leq \psi(\nu_1) + C\eta \,.$$

This is (37). We now prove (42). Due to Lemma 3.1(ii), there holds

$$E(u_2, Q^{\nu_2}_{\tilde S}) \le E(u_1, Q^{\nu_2}_{\tilde S}) + E(\langle \nu_2 - \nu_1, \cdot\rangle, Q^{\nu_1}_{\tilde S})\,. \tag{43}$$

Now, due to Lemma 3.1(i) and the fact that $\tilde S \le 2S$, there holds

$$E(\langle \nu_2 - \nu_1, \cdot\rangle, Q^{\nu_1}_{\tilde S}) \le C|\nu_2 - \nu_1| S^d \le C\eta S^d\,. \tag{44}$$

Next, we prove

$$E(u_1, Q^{\nu_2}_{\tilde S}) \le E(u_1, Q^{\nu_1}_{S}) + C\delta S^d + \frac{C_S}{\delta} S^d\,, \tag{45}$$

where $C_S \to 0$ as $S \to +\infty$. We use Lemma 3.1(iv), to obtain

$$E(u_1, Q^{\nu_2}_{\tilde S}) = E(u_1, Q^{\nu_1}_{S}) + E(u_1, Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S})\,.$$

In order to prove (45) it suffices to prove

$$E(u_1, Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}) \le C\eta S^d + \frac{C_S}{\delta} S^d\,. \tag{46}$$

To see this we write

$$\begin{aligned} E(u_1, Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}) = & \sum_{i \in \mathcal{L} \cap Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}} \ \sum_{j \in \mathcal{L} \cap Q^{\nu_1}_{(1-\delta)S}} c_{i,j} (u_1(i) - u_1(j))^+ \\ & + \sum_{i \in \mathcal{L} \cap Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}} \ \sum_{j \in \mathcal{L} \setminus Q^{\nu_1}_{(1-\delta)S}} c_{i,j} (u_1(i) - u_1(j))^+\,. \end{aligned} \tag{47}$$

To estimate the first term, note that due to (39), we have $|u_1(i) - u_1(j)| \le CS + |i-j|$, and therefore, up to changing C, using (H2), and Lemma 3.1(iv), we get

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}} \ \sum_{j \in \mathcal{L} \cap Q^{\nu_1}_{(1-\delta)S}} c_{i,j} (u_1(i) - u_1(j))^+ &\le \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}} \ \sum_{\substack{j \in \mathcal{L} \\ |i-j| \ge \delta S/2}} c_{i,j} |i-j| \\ &\le \frac{C}{\delta} \#(\mathcal{L} \cap Q^{\nu_2}_{\tilde S_2}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \ge \delta S/2}} c_{i,j} |i-j| \le \frac{C_S}{\delta} S^d\,. \end{aligned} \tag{48}$$

To estimate the first term, we use the fact that $u_1(i) = \langle \nu_1, i\rangle$ on $\mathcal{L} \setminus Q^{\nu_1}_{(1-\delta)S}$, and Lemma 3.1 (i), to obtain

$$\sum_{i \in \mathcal{L} \cap Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}} \ \sum_{j \in \mathcal{L} \setminus Q^{\nu_1}_{(1-\delta)S}} c_{i,j} (u_1(i) - u_1(j))^+ \le E(\langle \nu_1, \cdot\rangle, Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S}) \le C|\nu_1| |(Q^{\nu_2}_{\tilde S} \setminus Q^{\nu_1}_{S})_c| \le C\eta S^d\,.$$

This together with (47) and (48) implies (46) which in turn, together with (43) and (44) implies (42) and therefore the conclusion of Step 2. □

Lemma 3.6. *$\phi\colon \mathbb{R}^d \to [0, +\infty]$ satisfies the following properties:*

(i) *There exists $C > 0$ such that $\phi(\nu) \le C|\nu|$ for all $\nu \in \mathbb{R}^d$,*
(ii) *ϕ is a positively homogeneous function of degree one,*
(iii) *ϕ is a continuous function.*

Proof. We divide the proof into two steps. We first prove (i) and then (ii). Throughout the proofs let $1 \ll S$.

Step 1.(Proof of (i) and (ii)) In order to prove (i) we insert $u(i) = \langle \nu, i\rangle$ for all $i \in \mathcal{L}$ as a competitor in the cell formula. Using Lemma 3.1(i), we then have

$$E(u, Q_S^\nu) = E(\langle \nu, \cdot\rangle, Q_S^\nu) \leq C|\nu||(Q_S)_c| \leq CS^d\,.$$

Dividing by S^d and letting $S \to +\infty$ yields the claim. (ii) follows by using Lemma (3.1)(ii) to obtain $E(\lambda u, Q_S) = \lambda E(u, Q_S)$ for all $\lambda > 0$ and by noting that, given $\nu \in \mathbb{R}^d$, if $u\colon \mathcal{L} \to \mathbb{R}$ satisfies $u(i) = \langle \nu, i\rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$, then $\lambda u(i) = \langle \lambda\nu, i\rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$. Employing this in (11) it is easy to see that ϕ is pos. one homogeneous function.

Step 2.(Proof of (iii)) In order to prove (ii), let $\nu_1, \nu_2 \in \mathbb{R}^d$. We prove that

$$|\phi(\nu_1) - \phi(\nu_2)| \leq C|\nu_1 - \nu_2|\,. \tag{49}$$

Here, we only prove

$$\phi(\nu_2) - \phi(\nu_1) \leq C|\nu_1 - \nu_2|\,, \tag{50}$$

since then (49) follows by exchanging ν_1 and ν_2. To this end let $u\colon \mathcal{L} \to \mathbb{R}$ be such that $u_1(i) = \langle \nu_1, i\rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$. We then define $u_2\colon \mathcal{L} \to \mathbb{R}$ by $u_2(i) = u_1(i) + \langle \nu_2 - \nu_1, i\rangle$. Clearly,

$$\inf\left\{E(u, Q_S) : u\colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i\rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\right\} \leq E(u_2, Q_S)\,. \tag{51}$$

Now, due to Lemma 3.1(i) and (ii),

$$\begin{aligned} E(u_2, Q_S) &= E(u_1 + \langle \nu_2 - \nu_1, \cdot\rangle, Q_S) \leq E(u_1, Q_S) + E(\langle \nu_2 - \nu_1, \cdot\rangle, Q_S) \\ &\leq E(u_1, Q_S) + C|\nu_1 - \nu_2||(Q_S)_c| \leq E(u_1, Q_S) + C|\nu_1 - \nu_2|S^d\,. \end{aligned}$$

Using (51), noting that u_1 is arbitrary, dividing by S^d, letting first S tend to $+\infty$, and then $\delta \to 0$, we obtain (50). □

The next Lemma shows that the asymptotic cell-formula describing the surface energy density is equal to the asymptotic cell-formula with affine boundary conditions.

Lemma 3.7. *Let $\nu \in \mathbb{R}^d$. Then*

$$\psi(\nu) = \varphi(\nu)\,.$$

Proof. Due the fact that both ψ and ϕ are positively homogeneous functions of degree one, it suffices to consider the case where $\nu \in \mathbb{S}^{d-1}$. Furthermore, since both functions are continuous, see Lemma 3.5(ii) and Remark 2.3, it suffices to prove the claim for $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$. For each such vector we can find $\nu_1, \ldots, \nu_{d-1} \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ such that the set $\{\nu_1, \ldots, \nu_{d-1}, \nu\}$ forms an orthonormal basis of $\mathbb{R}^d$. Then, there exists $\lambda \in \mathbb{N}$ such that

$$\lambda\nu_n = z_n \text{ for some } z_n \in \mathbb{Z}^d \text{ for all } n \in \{1, \ldots, d\}\,. \tag{52}$$

Step 1.(Proof of '$\leq$') Let $\{\nu_1, \ldots, \nu_{d-1}, \nu_d = \nu\} \subset \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ be an orthonormal basis as previously described and let $1 \ll S_1 \ll S_2$. We assume that $S_1 = \lambda T$, where λ satisfies (52) and T is given by (H1). Note that if λ satisfies (52), also $k\lambda$ satisfies (52) and therefore we can find a sequence $S_k = k\lambda T$ such that $S_k \to +\infty$ of the desired form. The existence of the limit in definition (6) of φ permits us to assume that S is of the specific form. Let $\delta > 0$ and $u_1\colon \mathcal{L} \to \{0,1\}$ be such that $u_1(i) = u_\nu(i)$ on $\mathcal{L} \setminus Q^\nu_{(1-\delta)S_1}$ and

$$E(u_1, Q^\nu_{S_1}) = \inf\left\{E(u, Q^\nu_{S_1}) : u\colon \mathcal{L} \to \{0,1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S_1}\right\}. \tag{53}$$

Due to the assumption on S_1 and Lemma 3.1(vi), we have

$$E(u_1(\cdot - z), Q^{\nu}_{S_1}(z)) = E(u_1, Q^{\nu}_{S_1}) \text{ for all } z = \lambda T \sum_{n=1}^{d} k_n \nu_n, k \in \mathcal{L}\,. \tag{54}$$

Set (omitting the dependence on S_1 and S_2)

$$\mathcal{Z} = \left\{ z = S_1 \sum_{n=1}^{d} k_n \nu_n : k \in \mathcal{L}, Q^{\nu}_{S_1}(z) \subset Q^{\nu}_{S_2 - 2R} \right\}.$$

We define $u_2 \colon \mathcal{L} \to \mathbb{R}$ by

$$u_2(i) = \begin{cases} S_1\left(u_1(i-z) - \frac{1}{2}\right) + \langle \nu, z\rangle & \text{if } z \in \mathcal{Z}, i \in Q^{\nu}_{S_1}(z)\,, \\ \langle \nu, i\rangle & \text{otherwise.} \end{cases}$$

By the definition of u_2, it is clear that

$$\inf\left\{ E(u, Q^{\nu}_{S_2}) : u \colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i\rangle \text{ on } \mathcal{L} \setminus Q^{\nu}_{(1-\delta)S_2} \right\} \leq E(u_2, Q^{\nu}_{S_2})\,. \tag{55}$$

It remains to show that

$$E(u_2, Q^{\nu}_{S_2}) \leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q^{\nu}_{S_1}) + C S_2^{d-1} S_1^2 + \frac{S_2^d}{\delta} C_{S_1}\,. \tag{56}$$

In fact, once we have shown (56), Step 1 follows from (55) and (53) by dividing with S_2^d and letting first tend S_2 to $+\infty$, then letting S_1 tend to $+\infty$, and lastly $\delta \to 0$. We are left to prove (56). In order to prove it we use Lemma 3.1(iv) to obtain

$$E(u_2, Q^{\nu}_{S_2}) = E(u_2, Q^{\nu}_{S_2 - 2S_1}) + E(u_2, Q^{\nu}_{S_2} \setminus Q^{\nu}_{S_2 - 2S_1}) \tag{57}$$

and we estimate the two terms on the right hand side separately. We claim that

$$E(u_2, Q^{\nu}_{S_2 - 2S_1}) \leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q^{\nu}_{S_1}) + \frac{S_2^d}{\delta} C_{S_1}\,. \tag{58}$$

Indeed, if $i \in Q^{\nu}_{S_2 - 2S_1}$, then there exists $z = S_1 \sum_{n=1}^{d} k_n \nu_n \in \mathcal{Z}$ such that

$$u_2(i) = S_1\left(u_1(i-z) - \frac{1}{2}\right) + \langle \nu, z\rangle \text{ for all } i \in Q^{\nu}_{(1+\delta)S_1}(z)\,. \tag{59}$$

Due to (40), this is clearly true for $i \in Q^{\nu}_{S_1}(z)$, while for $i \in Q^{\nu}_{(1+\delta)S_1}(z) \setminus Q^{\nu}_{S_1}(z)$ we have $i \in Q^{\nu}_{S_1}(z') \setminus Q^{\nu}_{(1-\delta)S_1}(z')$, for some $z' = S_1 \sum_{n=1}^{d} k'_n \nu_n$ with $||k - k'||_\infty = 1$. Then, due to the boundary conditions of u_1, we have

$$\begin{aligned} u_2(i) &= S_1\left(u_1(i-z') - \frac{1}{2}\right) + \langle \nu, z'\rangle = S_1\left(u_\nu(i-z') - \frac{1}{2}\right) + \langle \nu, z\rangle + \langle \nu, z'-z\rangle \\ &= S_1\left(u_\nu(i-z) - \frac{1}{2}\right) + \langle \nu, z\rangle = S_1\left(u_1(i-z) - \frac{1}{2}\right) + \langle \nu, z\rangle\,. \end{aligned}$$

Here, the third equality follows, from the fact that $||k - k'||_\infty = 1$ and therefore $\langle \nu, z'-z\rangle \in \{-S_1, 0, S_1\}$. To obtain the previous equality, we distinguish the following two cases:

$$\langle \nu, z'-z\rangle = \pm S_1 \implies u_\nu(i-z') - u_\nu(i-z) = \mp 1 \text{ and } \langle \nu, z'-z\rangle = 0 \implies u_\nu(i-z') = u_\nu(i-z)\,.$$

Now (59) together with (54) implies for $z \in \mathcal{Z}$

$$\begin{aligned} E(u_2, Q^\nu_{S_1}(z)) &= \sum_{i \in \mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L}\cap Q^\nu_{(1+\delta)S_1}(z)} c_{i,j}(u_2(i)-u_2(j))^+ \\ &\quad + \sum_{i \in \mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}(z)} c_{i,j}(u_2(i)-u_2(j))^+ \\ &\le E(u_1, Q^\nu_{S_1}(z)) + \sum_{i \in \mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}(z)} c_{i,j}|u_2(i)-u_2(j)| \\ &= S_1 E(u_1, Q^\nu_{S_1}) + \sum_{i \in \mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}(z)} c_{i,j}|u_2(i)-u_2(j)|\,. \end{aligned} \tag{60}$$

We estimate the second term on the right hand side of (60) to obtain (58). In fact, here we claim that

$$|u_2(i)-u_2(j)| \le \frac{C}{\delta}|i-j| \text{ for all } i \in Q^\nu_{S_1}(z), j \in \mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}(z) \tag{61}$$

for some $C>0$ independent of S_1, S_2 and δ. If this is true, then we get

$$\begin{aligned} \sum_{i \in Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}(z)} c_{i,j}|u_2(i)-u_2(j)| &\le \frac{C}{\delta} \sum_{i \in \mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{\substack{j\in\mathcal{L} \\ |i-j|\ge \delta S_1/2}} c_{i,j}|i-j| \\ &\le \#(\mathcal{L}\cap Q^\nu_{S_1}(z)) \max_{i\in\mathcal{L}} \sum_{\substack{j\in\mathcal{L} \\ |i-j|\ge \delta S_1/2}} c_{i,j}|i-j| \le \frac{C_{S_1}}{\delta} S_1^d\,. \end{aligned}$$

Hence, noting that for $z, z' \in \mathcal{Z}$ such that $z \neq z'$, we have $Q^\nu_{S_1}(z) \cap Q^\nu_{S_1}(z') = \emptyset$ and therefore $\#\mathcal{Z} \le S_2^d/S_1^d$, we get

$$\begin{aligned} E(u_2, Q^\nu_{S_2-2S_1}) \le \sum_{\substack{z\in\mathcal{Z} \\ Q^\nu_{S_1}(z)\subset Q^\nu_{S_2-2S_1}}} E(u_2, Q^\nu_{S_1}(S)) &\le \#\mathcal{Z}(S_1 E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_1^d) \\ &\le \frac{S_2^d}{S_1^{d-1}} E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_2^d\,. \end{aligned}$$

This is (58). It remains to prove (61). Note for $i \in Q^\nu_{S_1}(z)$, $j \in \mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}$ we have $|i-j| \ge \delta S_1/2$ as already used above. We claim that

$$|u_2(i)-u_2(j)| \le C(S_1 + |i-j|) \tag{62}$$

and therefore (61) holds true. Let us prove (62). There are three cases to consider:

(a) $i = i_0 + z$, $j = j_0 + z'$ $i_0 \in Q^\nu_{S_1}(z), j_0 \in Q^\nu_{S_1}(z')$, $z, z' \in \mathcal{Z}$,
(b) $i = i_0 + z$, $i_0 \in Q^\nu_{S_1}(z)$, $z \in \mathcal{Z}$ $j_0 \notin Q^\nu_{S_1}(z')$ for any $z' \in \mathcal{Z}$,
(c) $i \notin Q^\nu_{S_1}(z)$ for any $z \in \mathcal{Z}$ and $j \notin Q^\nu_{S_1}(z')$ for any $z' \in \mathcal{Z}$.

Case (a): Note that in the case where $i = i_0 + z$, $j = j_0 + z'$ for some $i_0, j_0 \in Q^\nu_{S_1}$ and for some $z, z' \in \mathcal{Z}$, we have

$$|u_2(i)-u_2(j)| \le |\langle \nu, z - z'\rangle| + CS_1 \le |\langle \nu, z + i_0 - z' - j_0\rangle| + |i_0 - j_0| + CS_1 \le |i-j| + CS_1$$

and therefore (62) holds true.

Case (b): Note that in the case where $i = i_0 + z$, $i_0 \in Q^\nu$, $z \in \mathcal{Z}$ and $j \notin Q^\nu_{S_1}(z)$ for any $z \in \mathcal{Z}$, we have

$$|u_2(i) - u_2(j)| \leq CS_1 + |\langle \nu, z - j\rangle| \leq CS_1 + |\langle \nu, i - j\rangle| + |i_0| \leq CS_1 + |i - j| \,.$$

Also here (62) holds true.

Case (c): In this case $u_2(i) = \langle \nu, i\rangle$ and $u_2(j) = \langle \nu, j\rangle$ and therefore (62) holds true.

Next, we prove

$$E(u_2, Q^\nu_{S_2} \setminus Q^\nu_{S_2-2S_1}) \leq CS_2^{d-1}S_1^2 \,. \tag{63}$$

Now, using (62), (H2), and Lemma 3.1(v), we get

$$\begin{aligned} E(u_2, Q^\nu_{S_2} \setminus Q^\nu_{S_2-2S_1}) &\leq \sum_{i \in \mathcal{L} \cap Q^\nu_{S_2} \setminus Q^\nu_{S_2-2S_1}} \sum_{j \in \mathcal{L}} c_{i,j}|u_2(i) - u_2(j)| \\ &\leq CS_1 \# \left(\mathcal{L} \cap Q^\nu_{S_2} \setminus Q^\nu_{S_2-2S_1}\right) \left(1 + \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j}|i - j|\right) \leq CS_2^{d-1}S_1^2 \,. \end{aligned}$$

This yields (63). Now (57), (58), and (63) give (56) and therefore the conclusion of Step 1.

Step 2.(Proof of '$\geq$') Here, we proceed in two sub-steps. First we extend a competitor for ψ (for fixed S_1) periodically and then we perform a cut-off construction. Let $\{\nu_1, \ldots, \nu_{d-1}, \nu\}$ be an orthonormal basis as described at the beginning of the proof and let $1 \ll S_1 \ll S_2$. As in Step 1, we assume that $S_1 = \lambda T$, where λ satisfies (52) and T is given by (H1). Furthermore, we assume that $S_2 = kS_1$ for some $k \in \mathbb{N}$. Let $\delta > 0$ and $u_1 \colon \mathcal{L} \to \mathbb{R}$ be such that $u_1(i) = \langle \nu, i\rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S_1}$ and

$$E(u_1, Q^\nu_{S_1}) = \inf\left\{E(u, Q^\nu_{S_1}) : u\colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i\rangle \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S_1}\right\} \tag{64}$$

Due to the assumptions on S_1 and Lemma 3.1(vi), we have

$$E(u_1(\cdot - z), Q^\nu_{S_1}(z)) = E(u_1, Q^\nu_{S_1}) \text{ for all } z = \lambda T \sum_{n=1}^{d} k_n \nu_n, k \in \mathcal{L} \,. \tag{65}$$

Set as in Step 1 (omitting the dependence on S_1)

$$\mathcal{Z}' = \left\{ z = S_1 \sum_{n=1}^{d-1} k_n \nu_n : k \in \mathbb{Z}^{d-1} \right\} \,.$$

Step 2.1.(Periodic extension) We assume that

$$||u_1||_{L^\infty(Q^\nu_{S_1})} \leq S_1/2 \,, \tag{66}$$

since otherwise we consider $\hat{u}_1(i) = (u_1(i) \wedge (S_1/2)) \vee (-S_1/2)$. Let $\tilde{u}_1 \colon \mathcal{L} \to \mathbb{R}$ be defined by

$$\tilde{u}_1(i) = \begin{cases} S_1^{-1}u_1(i - z) + \frac{1}{2} & i \in Q^\nu_{S_1}(z), z \in \mathcal{Z}' \,, \\ u_\nu(i) & \text{otherwise.} \end{cases}$$

Note that, due to (66), we have

$$||\tilde{u}_1||_{L^\infty(Q^\nu_{S_2})} \leq 1 \,. \tag{67}$$

We prove that

$$E(\tilde{u}_1, Q^\nu_{S_2}) \leq \frac{S_2^{d-1}}{S_1^d} E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_2^{d-1} + C\delta S_2^{d-1} \tag{68}$$

for some $C_{S_1} \to 0$ as $S_1 \to +\infty$. In order to see this, first observe that for all $z \in \mathcal{Z}'$ there holds

$$|\tilde{u}_1(i) - \tilde{u}_1(j)| \leq S_1^{-1} |u_1(i-z) - u_1(j-z)| \text{ for all } i, j \in Q^\nu_{(1+\delta)S_1}(z)\,. \tag{69}$$

Clearly this holds true if $i, j \in Q^\nu_{S_1}(z)$. On the other hand, if $i \in Q^\nu_{(1+\delta)S_1}(z) \cap Q^\nu_{S_1}(z')$, then $i - z' \in \mathcal{L} \setminus Q^\nu_{(1-\delta)S_1}$. We therefore have $u_1(i-z) = \langle \nu, i-z\rangle = \langle \nu, i\rangle$ and with that

$$\tilde{u}_1(i) = S_1^{-1} u_1(i-z') + \frac{1}{2} = S_1^{-1}\langle \nu, i\rangle + \frac{1}{2} = S_1^{-1} u_1(i-z) + \frac{1}{2}\,.$$

Hence, it also holds true for $i, j \in (Q^\nu_{(1+\delta)S_1}(z) \cap Q^\nu_{S_1}(z')) \cup Q^\nu_{S_1}(z)$. If $i \in Q^\nu_{(1+\delta)S}(z) \setminus Q^\nu_S(z')$ for all $z' \in \mathcal{Z}'$. On the other hand if $i \notin Q^\nu_{S_1}(z')$ for all $z' \in \mathcal{Z}'$ it suffices to note that $u_\nu(i) = S_1^{-1}(u_1(i) \wedge (S_1/2)) \vee (-S_1/2)$. Hence, also in this case (69) holds true. We first show

$$E(\tilde{u}_1, Q^\nu_{S_1}(z)) \leq S_1^{-1} E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_2^{d-1}$$

for some $C_{S_1} \to 0$ as $S_1 \to +\infty$. Due to (65), (69), and Lemma 3.1(v) we have

$$\begin{aligned}
E(\tilde{u}_1, Q^\nu_{S_1}(z)) &= \sum_{i \in \mathcal{L} \cap Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L}} c_{i,j}(\tilde{u}_1(i) - \tilde{u}_1(j))^+ \leq S_1^{-1} \sum_{i \in \mathcal{L} \cap Q^\nu_{S_1}(z)} \sum_{j \in \mathcal{L} \cap Q^\nu_{(1+\delta)S_1}} c_{i,j}(u_1(i) - u_1(j))^+ \\
&\quad + \sum_{i \in \mathcal{L} \cap Q^\nu_{S_1}(z)} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S_1/2}} c_{i,j} |u_1(i) - u_1(j)| \\
&\leq S_1^{-1} E(u_1(\cdot - z), Q^\nu_{S_1}(z)) + \frac{C}{\delta S_1} \#(\mathcal{L} \cap Q^\nu_{S_1}(z)) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S_1/2}} c_{i,j} |i-j| \\
&\leq S_1^{-1} E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_1^{d-1}
\end{aligned}$$

for some $C_{S_1} \to 0$ as $S_1 \to +\infty$. Now note that $\#\mathcal{Z}' \leq S_2^{d-1}/S_1^{d-1}$ and therefore, due to Lemma 3.1(iv) there holds

$$E(\tilde{u}_1, \bigcup_{z \in \mathcal{Z}'} Q^\nu_{S_1}(z)) \leq \frac{S_2^{d-1}}{S_1^d} E(\tilde{u}_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_2^{d-1} \tag{70}$$

for some $C_{S_1} \to 0$ as $S_1 \to +\infty$. Next, we show

$$E(\tilde{u}_1, Q^\nu_{S_2} \setminus \bigcup_{z \in \mathcal{Z}'} Q^\nu_{S_1}(z)) \leq C\delta S_2^{d-1} + \frac{C_{S_1}}{\delta} S_2^{d-1}\,. \tag{71}$$

To this end, we introduce

$$H^\nu_{a,b} = \{x \in \mathbb{R}^d \colon b \leq \langle x, \nu\rangle < a\}\,.$$

Note that, by Lemma 3.1(iv), there holds

$$\begin{aligned}
E(\tilde{u}_1, Q^\nu_{S_2} \setminus \bigcup_{z \in \mathcal{Z}'} Q^\nu_{S_1}(z)) &= E(\tilde{u}_1, H^\nu_{(1+\delta)S_1/2, S_1/2}) + E(\tilde{u}_1, H^\nu_{S_2/2, (1+\delta)S_1/2}) \\
&\quad + E(\tilde{u}_1, H^\nu_{-S_1/2, (1+\delta)S_1/2}) + E(\tilde{u}_1, H^\nu_{-(1+\delta)S_1/2, -S_2/2})\,.
\end{aligned} \tag{72}$$

We only estimate the first to terms of (72), the other two being analogous. We first show

$$E(\tilde u_1, H^\nu_{(1+\delta)S_1/2,S_1/2}) \le C\delta S_2^{d-1} + \frac{C_{S_1}}{\delta} S_2^{d-1} \tag{73}$$

with $C_{S_1} \to 0$ as $S_1 \to +\infty$. Now, due to (H2), (67), (69), and Lemma 3.1(v), there holds

$$\begin{aligned}
E(\tilde u_1, H^\nu_{(1+\delta)S_1/2,S_1/2}) = & \sum_{i\in\mathcal{L}\cap H^\nu_{(1+\delta)S_1/2,S_1/2}\cap Q^\nu_{S_2}} \; \sum_{j\in\mathcal{L}\cap H^\nu_{(1+\delta)S_1/2,(1-\delta)S_1/2}\cap Q^\nu_{(1+\delta)S_2}} c_{i,j}(\tilde u_1(i) - \tilde u_1(j))^+ \\
& + \sum_{i\in\mathcal{L}\cap H^\nu_{(1+\delta)S_1/2,S_1/2}\cap Q^\nu_{S_2}} \; \sum_{j\in\mathcal{L}\setminus H^\nu_{(1+\delta)S_1/2,(1-\delta)S_1/2}\cap Q^\nu_{(1+\delta)S_2}} c_{i,j}(\tilde u_1(i) - \tilde u_1(j))^+ \\
\le & \; S_1^{-1} \#(\mathcal{L}\cap H^\nu_{(1+\delta)S_1/2,S_1/2}) \max_{i\in\mathcal{L}} \sum_{j\in\mathcal{L}} c_{i,j}|i-j| \\
& + \frac{1}{\delta S_1} \#(\mathcal{L}\cap H^\nu_{(1+\delta)S_1/2,S_1/2}) \max_{i\in\mathcal{L}} \sum_{\substack{j\in\mathcal{L}\\ |i-j|\ge\delta S_1/2}} c_{i,j}|i-j| \\
\le & \; C\delta S_2^{d-1} + \frac{C_{S_1}}{\delta} S_2^{d-1}\,.
\end{aligned}$$

This is (73). We now show that the second term in (72) is estimated by

$$E(\tilde u_1, H^\nu_{S_2/2,(1+\delta)S_1/2}) \le C_{S_1} S_2^{d-1} \tag{74}$$

where $C_{S_1} \to 0$ as $S_1 \to +\infty$. Note that $\tilde u_1(i) = u_\nu(i) = 1$ for all $i \in \mathcal{L}$ such that $\langle \nu, i\rangle \ge S_1/2$. Therefore, due to (H1), Lemma 3.1(v), (67), for S_1 big enough, there holds

$$\begin{aligned}
E(\tilde u_1, H^\nu_{S_2/2,(1+\delta)S_1/2}) \le & \; 2 \sum_{i\in H^\nu_{S_2/2,(1+\delta)S_1/2}\cap Q^\nu_{S_2}} \; \sum_{j\in\mathcal{L}\cap\{\langle\nu,i\rangle\le S_1/2\}} c_{i,j} \\
\le & \; C \max_{i_0,j_0\in\mathcal{L}\cap Q_T} \sum_{z\in T\mathbb{Z}^d\cap H^\nu_{S_2,(1+2/3\delta)S_1/2}\cap Q^\nu_{S_2}} \; \sum_{\substack{z'\in T\mathbb{Z}^d\\ \langle\nu,z'\rangle\le(1+1/3\delta)S_1/2}} c_{i_0+z,j_0+z'} \\
\le & \; C \max_{i_0,j_0\in\mathcal{L}\cap Q_T} \sum_{\substack{\zeta\in T\mathbb{Z}^d\\ |\zeta|\ge 1/6\delta S_1}} \; \sum_{\substack{z\in T\mathbb{Z}^d\cap H^\nu_{S_2,(1+2/3\delta)S_1/2}\cap Q^\nu_{S_2}\\ \langle\zeta+z,\nu\rangle\le(1+1/3\delta)S_1/2}} c_{i_0,j_0+\zeta} \\
\le & \; C \max_{i_0,j_0\in\mathcal{L}\cap Q_T} \sum_{\substack{\zeta\in T\mathbb{Z}^d\\ |\zeta|\ge 1/6\delta S_1}} c_{i_0,j_0+\zeta}|\zeta| S_2^{d-1} \\
\le & \; C \max_{i\in\mathcal{L}} \sum_{\substack{j\in\mathcal{L}\\ |i-j|\ge 1/12\delta S_1}} c_{i,j}|j-i| S_2^{d-1} \le C_{S_1} S_2^{d-1}\,.
\end{aligned}$$

This is (74). Now (73), (74) imply (71). (71) together with (70) implies (68) and with that the claim.

Step 2.2.(Cut-off construction) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\varphi(x) = 1 \text{ on } Q^\nu_{(1-2\delta)S_2}\,, \varphi(x) = 0 \text{ on } \mathbb{R}^d \setminus Q^\nu_{(1-\delta)S_2}\,, \text{ and } ||\nabla\varphi||_\infty \le C\delta^{-1}S_2^{-1}\,.$$

Let

$$u_2(i) = \varphi(i)\tilde u_1(i) + (1-\varphi(i))u_\nu(i)\,.$$

Clearly $u_2(i) = u_\nu(i)$ for $i \in \mathcal{L} \setminus Q^\nu_{(1-\delta)S_2}$ and therefore, due to Lemma 3.2, there holds

$$\inf\{E(u, Q^\nu_{S_2}) : u\colon \mathcal{L} \to \{0,1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q^\nu_{(1-\delta)S_2}\} \le E(u_2, Q^\nu_{S_2})\,. \tag{75}$$

We prove that

$$E(u_2, Q^\nu_{S_2}) \le E(\tilde{u}_1, Q^\nu_{S_2}) + C\delta S_2^{d-1} + \frac{C_{S_2}}{\delta} S_2^{d-1}\,, \tag{76}$$

where $C_{S_2} \to 0$ as $S_2 \to +\infty$. Note that, using (64),(68), and (75), this concludes Step 2 by dividing with S_2^{d-1} letting first $S_2 \to +\infty$, $S_1 \to +\infty$, and then $\delta \to 0$. It remains to prove (76). Clearly $u_2(i) = u_\nu(i)$ on $\mathcal{L} \setminus Q^\nu_{(1-\delta)S_2}$, and using (67), it is easy to see that

$$(u_2(i) - u_2(j))^+ \le (\tilde{u}_1(i) - \tilde{u}_1(j))^+ + \frac{C}{\delta S_2}|i-j||\tilde{u}_1(j) - u_\nu(j)| + |1 - \varphi(i)||u_\nu(i) - u_\nu(j)|\,.$$

Therefore,

$$\begin{aligned} E(u_2, Q^\nu_{S_2}) \le E(\tilde{u}_1, Q^\nu_{S_2}) &+ \frac{C}{\delta S_2} \sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{j \in \mathcal{L}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)| \\ &+ \sum_{i \in \mathcal{L} \cap Q^\nu_{S_2} \setminus Q^\nu_{(1-2\delta)S_2}} \sum_{j \in \mathcal{L}} c_{i,j}|u_\nu(i) - u_\nu(j)|\,. \end{aligned} \tag{77}$$

We show that

$$\sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{j \in \mathcal{L}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)| \le \frac{C_{S_1}}{\delta} S_2^d + \frac{C}{\delta} S_1 S_2^{d-1}\,. \tag{78}$$

To see this, we split the sum in two terms by writing

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{j \in \mathcal{L}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)| \le &\sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \le S_1}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)| \\ &+ \sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \ge S_1}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)|\,. \end{aligned}$$

Now note that $\tilde{u}_1(j) \neq u_\nu(j)$ only if $j \in H^\nu_{S_1/2, -S_1/2}$ therefore, using Lemma 3.1(v), (H2), and (67), we can estimate

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \le S_1}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)| &\le 2 \sum_{i \in \mathcal{L} \cap H^\nu_{3S_1/2, -3S_1/2} \cap Q^\nu_{S_2}} \sum_{j \in \mathcal{L}} c_{i,j}|i-j| \\ &\le 2\#(\mathcal{L} \cap H^\nu_{3S_1/2, -3S_1/2} \cap Q^\nu_{S_2}) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j}|i-j| \le C S_1 S_2^{d-1}\,. \end{aligned}$$

On the other hand, using again Lemma 3.1(v), (H2), and (67), we obtain

$$\sum_{i \in \mathcal{L} \cap Q^\nu_{S_2}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \ge S_1}} c_{i,j}|i-j||\tilde{u}_1(j) - u_\nu(j)| \le 2\#(\mathcal{L} \cap Q^\nu_{S_2}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \ge S_1}} c_{i,j}|i-j| \le C S_2^d C_{S_1}\,.$$

The previous two inequalities yield (78). Lastly we show

$$\sum_{i \in \mathcal{L} \cap Q^\nu_{S_2} \setminus Q^\nu_{(1-2\delta)S_2}} \sum_{j \in \mathcal{L}} c_{i,j}|u_\nu(i) - u_\nu(j)| \le C\delta S_2^{d-1},. \tag{79}$$

Using (S1) and (H2), we have

$$
\begin{aligned}
&\sum_{i\in\mathcal{L}\cap Q^\nu_{S_2}\setminus Q^\nu_{(1-2\delta)S_2}} \sum_{j\in\mathcal{L}} c_{i,j}|u_\nu(i)-u_\nu(j)| \\
&\le C \max_{i_0,j_0\in\mathcal{L}\cap Q_T} \sum_{z\in T\mathbb{Z}^d\cap Q^\nu_{(1+\delta)S_2}\setminus Q^\nu_{(1-3\delta)}} \sum_{z'\in T\mathbb{Z}^d} c_{i_0+z,j_0+z'}|u_\nu(i_0+z)-u_\nu(j_0+z')| \\
&\le C \max_{i_0,j_0\in\mathcal{L}\cap Q_T} \sum_{\zeta\in T\mathbb{Z}^d} \sum_{\substack{z\in T\mathbb{Z}^d\cap Q^\nu_{(1+\delta)S_2}\setminus Q^\nu_{(1-3\delta)} \\ -|\zeta|-\sqrt{d}T\le\langle z,\nu\rangle\le|\zeta|+\sqrt{d}T}} c_{i_0,j_0+\zeta} \\
&\le C\delta S_2^{d-1} \max_{i_0,j_0\in\mathcal{L}\cap Q_T} \sum_{\zeta\in T\mathbb{Z}^d} c_{i_0,j_0+\zeta}(|\zeta|+\sqrt{d}T) \\
&\le C\delta S_2^{d-1} \max_{i\in\mathcal{L}} \sum_{j\in\mathcal{L}} c_{i,j}(|i-j|+1) \le C\delta S_2^{d-1}\,.
\end{aligned}
$$

This implies (79). Now (77)-(79) imply (76) and with that the conclusion of Step 2.2. □

In the next Lemma we show that, assuming affine boundary conditions, the calculation of the asymptotic cell formula with respect to the coordinate cube and the calculation of the asymptotic cell formula with respect to the rotated cube are equivalent.

Lemma 3.8. *Let $\nu\in\mathbb{R}^d$. Then*

$$\psi(\nu)=\phi(\nu)\,.$$

Proof. Before we start the proof, we would like to point out that the various steps of the proof are very similar to the steps of proof of Lemma 3.7. However, we decided to include them here for completeness.

First, note that fact that both ψ and ϕ are positively homogeneous functions of degree one (cf. (35) and Lemma 3.6(ii)) it suffices to consider the case where $\nu\in\mathbb{S}^{d-1}$. Thanks to Lemma 3.5(ii) and Lemma 3.6(iii) both functions are continuous. Thus it suffices to prove the claim for $\nu\in\mathbb{S}^{d-1}\cap\mathbb{Q}^d$. For each such vector we can find $\{\nu_1,\dots,\nu_{d-1}\}\in\mathbb{S}^{d-1}\cap\mathbb{Q}^d$ such that the set $\{\nu_1,\dots,\nu_{d-1},\nu_d=\nu\}$ forms an orthonormal basis of $\mathbb{R}^d$. For such an orthonormal basis, it is clear that there exists $\lambda\in\mathbb{N}$ such that

$$\lambda\nu_n=z_n \text{ for some } z_n\in\mathbb{Z}^d \text{ for all } n\in\{1,\dots,d\}\,. \tag{80}$$

Step 1.(Proof of '$\ge$') Let $\{\nu_1,\dots,\nu_{d-1},\nu_d=\nu\}\subset\mathbb{S}^{d-1}\cap\mathbb{Q}^d$ be the orthonormal basis described previously and let $1\ll S_1\ll S_2$. We assume that $S_1=\lambda T$, where λ satisfies (80) and T is given by (H1). Let $\delta>0$ and $u_1\colon\mathcal{L}\to\mathbb{R}$ be such that $u_1(i)=\langle\nu,i\rangle$ on $\mathcal{L}\setminus Q^\nu_{(1-\delta)S_1}$ and

$$E(u_1,Q^\nu_{S_1})=\inf\left\{E(u,Q^\nu_{S_1}):u\colon\mathcal{L}\to\mathbb{R},u(i)=\langle\nu,i\rangle \text{ on } \mathcal{L}\setminus Q^\nu_{(1-\delta)S_1}\right\}. \tag{81}$$

Due to the assumption on S_1 and Lemma 3.1(vi), we have that

$$E(u_1(\cdot-z),Q^\nu_{S_1}(z))=E(u_1,Q^\nu_{S_1}) \text{ for all } z=S_1\sum_{n=1}^d k_n\nu_n, k\in\mathcal{L}\,. \tag{82}$$

We can assume that

$$||u_1||_{L^\infty(Q^\nu_{S_1})}\le S_1\,, \tag{83}$$

since otherwise we consider $\hat{u}_1(i)=(u_1(i)\wedge S_1)\vee(-S_1)$.

We set

$$\mathcal{Z} = \left\{ z = S_1 \sum_{n=1}^{d} k_n \nu_n, k \in \mathcal{L}, Q^{\nu}_{S_1}(z) \subset Q_{(1-\delta)S_2} \right\}$$

and we define $u_2 \colon \mathcal{L} \to \mathbb{R}$ by

$$u_2(i) = \begin{cases} u_1(i-z) + \langle \nu, z \rangle & \text{if } z \in \mathcal{Z}, i \in Q^{\nu}_{S_1}(z)\,, \\ \langle \nu, i \rangle & \text{otherwise.} \end{cases} \tag{84}$$

By the definition of u_2 it is clear that

$$\inf \left\{ E(u, Q_{S_2}) : u \colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S_2} \right\} \leq E(u_2, Q_{S_2})\,. \tag{85}$$

We conclude the proof of Step 1 by showing that

$$E(u_2, Q_{S_2}) \leq \frac{S_2^d}{S_1^d} E(u_1, Q^{\nu}_{S_1}) + \frac{C_{S_1}}{\delta} S_2^d + C S_2^{d-1} S_1^2\,, \tag{86}$$

where $C_{S_1} \to 0$ as $S_1 \to \infty$. Once this is shown, Step 1 is proven. In fact, using it together with (81) and (85) and dividing by S_2^d and letting first S_2 tend to $+\infty$, then letting S_1 tend to $+\infty$, and finally letting $\delta \to 0$, we obtain the claim. We are left to prove (86). In order to do so, we employ Lemma 3.1(iv) to obtain

$$E(u_2, Q_{S_2}) = E\left(u_2, \bigcup_{z \in \mathcal{Z}} Q^{\nu}_{S_1}(z)\right) + E\left(u_2, Q_{S_2} \setminus \bigcup_{z \in \mathcal{Z}} Q^{\nu}_{S_1}(z)\right)\,. \tag{87}$$

We claim that

$$E\left(u_2, \bigcup_{z \in \mathcal{Z}} Q^{\nu}_{S_1}(z)\right) \leq \frac{S_2^d}{S_1^d} E(u_1, Q^{\nu}_{S_1}) + \frac{C_{S_1}}{\delta} S_2^d\,. \tag{88}$$

In order to see this, note that for all $z \in \mathcal{Z}$, we have

$$u_2(i) = u_1(i-z) + \langle \nu, z \rangle \text{ for all } i \in Q^{\nu}_{(1+\delta)S_1}(z)\,. \tag{89}$$

Consulting (84), this is clearly true for $i \in Q^{\nu}_{S_1}(z)$, while for $i \in Q^{\nu}_{(1+\delta)S_1}(z) \setminus Q^{\nu}_{S_1}(z)$ there are two cases to check:

(a) $i \in Q^{\nu}_{S_1}(z')$ for some $z' \in Z$,
(b) otherwise.

(a): If $i \in Q^{\nu}_{S_1}(z')$, then $i \in Q^{\nu}_{S_1}(z') \setminus Q^{\nu}_{(1-\delta)S_1}(z')$ for some $z' \in Z$. Then, due to the boundary conditions of u_1 we have

$$u_2(i) = u_1(i-z') + \langle \nu, z \rangle = \langle \nu, i-z' \rangle + \langle \nu, z' \rangle = \langle \nu, i \rangle = \langle \nu, i-z \rangle + \langle \nu, z \rangle = u_1(i-z) + \langle \nu, z \rangle$$

and (89) follows.

(b): If $i \notin Q^{\nu}_{S_1}(z')$ for any $z' \in Z$ then

$$u_2(i) = \langle \nu, i \rangle = \langle \nu, i-z \rangle + \langle \nu, z \rangle = u_1(i-z) + \langle \nu, z \rangle$$

and also here (89) follows. In order to proceed, note that

$$|u_2(i) - u_2(j)| \leq C(S_1 + |i-j|)\,. \tag{90}$$

To see this, we need to distinguish between three cases:

(a) $i \in Q^{\nu}_{S_1}(z), j \in Q^{\nu}_{S_1}(z')$ for $z, z' \in \mathcal{Z}$,
(b) $i \in Q^{\nu}_{S_1}(z)$ for $z \in \mathcal{Z}$,$j \notin Q^{\nu}_{S_1}(z')$ for any $z' \in \mathcal{Z}$,
(c) otherwise.

Case (a): In order to see this, note that, we have $|z-z'| = |z-i-z'+j+i-j| \le |i-j| - CS_1$. Therefore, using (83), we have for $i \in Q^\nu_{S_1}(z), j \in Q^\nu_{S_1}(z')$

$$|u_2(i)-u_2(j)| \le 2S_1 + |z-z'| \le CS_1 + |i-j| \,.$$

Case (b): Again, due to (83), we get

$$|u_2(i)-u_2(j)| = |u_1(i-z) + \langle \nu, z-j\rangle| \le S_1 + |i-z| + |j-i| \le CS_1 + |i-j| \,.$$

Case (c): This case is trivially true, due to the definition of u_2.

We are now in the position to prove (88). To this end, we prove

$$E(u_2, Q^\nu_{S_1}(z)) \le E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_1^d \,, \tag{91}$$

where $C_{S_1} \to 0$ as $S_1 \to +\infty$. Using this, Lemma 3.1(iv), and noting that for $z, z' \in \mathcal{Z}$ such that $z \neq z'$ we have $Q^\nu_{S_1}(z) \cap Q^\nu_{S_1}(z') = \emptyset$ and therefore $\#\mathcal{Z} \le S_2^d / S_1^d$, we get

$$E\left(u_2, \bigcup_{z\in\mathcal{Z}} Q^\nu_{S_1}(z)\right) = \sum_{z\in\mathcal{Z}} E\left(u_2, Q^\nu_{S_1}(z)\right) \le \#\mathcal{Z}\left(E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_1^d\right) \le \frac{S_2^d}{S_1^d} E(u_1, Q^\nu_{S_1}) + \frac{C_{S_1}}{\delta} S_2^d \,.$$

This is (88). Now, let us prove (91). Using (89) and (82), we have

$$\begin{aligned} E\left(u_2, Q^\nu_{S_1}(z)\right) &= \sum_{i\in\mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j\in\mathcal{L}\cap Q^\nu_{(1+\delta)S_1}} c_{i,j}(u_2(i)-u_2(j))^+ + \sum_{i\in\mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j\in\mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}} c_{i,j}(u_2(i)-u_2(j))^+ \\ &\le E(u_1, Q^\nu_{S_1}) + \sum_{i\in\mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j\in\mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}} c_{i,j}|u_2(i)-u_2(j)| \,. \end{aligned} \tag{92}$$

Now, we are in a position to estimate the second term on the right hand side of (92). Due to (90) and (H2), we have

$$\begin{aligned} \sum_{i\in\mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{j\in\mathcal{L}\setminus Q^\nu_{(1+\delta)S_1}} c_{i,j}|u_2(i)-u_2(j)| &\le C \sum_{i\in\mathcal{L}\cap Q^\nu_{S_1}(z)} \sum_{\substack{j\in\mathcal{L}\\ |i-j|\ge\delta S_1/2}} c_{i,j}(S_1+|i-j|) \\ &\le C\left(1+\frac{1}{\delta}\right)\#(\mathcal{L}\cap Q^\nu_{S_1}(z)) \max_{i\in\mathcal{L}} \sum_{\substack{j\in\mathcal{L}\\ |i-j|\ge\delta S_1/2}} c_{i,j}|i-j| \\ &\le \frac{C_{S_1}}{\delta} S_1^d \,. \end{aligned}$$

This together with (92) implies (91).

Next, we prove

$$E\left(u_2, Q_{S_2} \setminus \bigcup_{z\in\mathcal{Z}} Q^\nu_{S_1}(z)\right) \le CS_2^{d-1}S_1^2 \,. \tag{93}$$

To see this, we use (90), to obtain

$$\begin{aligned} E\left(u_2, Q_{S_2} \setminus \bigcup_{z\in\mathcal{Z}} Q^\nu_{S_1}(z)\right) &\le \sum_{i\in\mathcal{L}\cap Q_{S_2}\setminus\bigcup_{z\in\mathcal{Z}} Q^\nu_{S_1}(z)} \sum_{j\in\mathcal{L}} c_{i,j}|u_2(i)-u_2(j)| \\ &\le CS_1\#\left(\mathcal{L}\cap Q_{S_2}\setminus\bigcup_{z\in\mathcal{Z}} Q^\nu_{S_1}(z)\right) \max_{i\in\mathcal{L}} \sum_{j\in\mathcal{L}} c_{i,j}(|i-j|+1) \le CS_2^{d-1}S_1^2 \,. \end{aligned}$$

This is (93). Now (87),(88), and (93) imply (86).

Step 2.(Proof of '$\leq$') Let $1 \ll S_1 \ll S_2$. We assume that $S_1 = \lambda T$ for some $\lambda \in \mathbb{N}$ and T given by (H1). Let $\delta > 0$ and $u_1 \colon \mathcal{L} \to \mathbb{R}$ be such that $u_1(i) = \langle \nu, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S_1}$ and

$$E(u_1, Q_{S_1}) = \inf \left\{ E(u, Q_{S_1}) : u \colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S_1} \right\} . \tag{94}$$

We set

$$\mathcal{Z}' = \{ z = S_1 k : k \in \mathbb{Z}^d, Q_{S_1}(z) \subset Q^{\nu}_{(1-\delta)S_2} \}$$

and we define $u_2 \colon \mathcal{L} \to \mathbb{R}$ by

$$u_2(i) = \begin{cases} u_1(i-z) + \langle \nu, z \rangle & \text{if } z \in \mathcal{Z}', i \in Q_{S_1}(z) \, , \\ \langle \nu, i \rangle & \text{otherwise.} \end{cases}$$

It is clear that

$$\inf \left\{ E(u, Q^{\nu}_{S_2}) : u \colon \mathcal{L} \to \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q^{\nu}_{(1-\delta)S_2} \right\} \leq E(u_2, Q^{\nu}_{S_2}) \, . \tag{95}$$

Similar to proving (86) in Step 1, one can show that

$$E(u_2, Q^{\nu}_{S_2}) \leq \frac{S_2^d}{S_1^d} E(u_1, Q_{S_1}) + C S_2^{d-1} S_1^2 + \frac{C_{S_1}}{\delta} S_2^d \, .$$

Using (94) and (95), this implies the conclusion of Step 2. This can be seen by dividing by S_2^d and letting first S_2 tend to $+\infty$, then S_1 to $+\infty$, and lastly $\delta \to 0$. □

Proof of Proposition 2.6. Our goal is to prove

$$\varphi(\nu) = \frac{1}{T^d} \inf \left\{ E(u, Q_T) : u \colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic} \right\} \tag{96}$$

for all $\nu \in \mathbb{R}^d$. Due to Lemma 3.8, Lemma 3.7, and Lemma 3.3, we have

$$\varphi(\nu) = \psi(\nu) = \phi(\nu) = \phi_{\mathrm{per}}(\nu) \, . \tag{97}$$

Additionally, Lemma 3.4 ensures that

$$\phi_{\mathrm{per}}(\nu) = \frac{1}{T^d} \inf \left\{ E(u, Q_T) : u \colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic} \right\} .$$

This shows (96) and concludes the proof. □

4. Crystallinity of the homogenized surface energy density

This section is devoted to the proof of Theorem 2.7. We assume throughout this section that assumptions (S1), (S2) and (H1), (H3) are satisfied.

We define the set of edges $\mathcal{E}$ by

$$\mathcal{E} = \{ (i,j) \in (\mathcal{L} \cap Q_T) \times \mathcal{L} : c_{i,j} \neq 0 \} \text{ and } N = \# \mathcal{E} \, . \tag{98}$$

Proof of Theorem 2.7. We divide the proof into three steps. First, we derive a dual representation of φ. Then, using this representation, we show that φ is crystalline.

Step 1.(Dual representation) We define

$$\mathcal{C} = \Big\{ (\alpha_{i,j})_{i,j} : 0 \leq \alpha_{i,j} \leq c_{i,j}, \alpha_{i+Tz, j+Tz} = \alpha_{i,j} \text{ for all } z \in \mathcal{L}, \\ \sum_{j \in \mathcal{L}} (\alpha_{j,i} - \alpha_{i,j}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} \Big\} . \tag{99}$$

Our goal is to prove

$$\varphi(\nu) = \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j}\in\mathcal{C}} \left\langle \nu, \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}(i-j) \right\rangle . \tag{100}$$

Let $\nu \in \mathbb{R}^d$. Due to Proposition, 2.6 there holds

$$\begin{aligned}\varphi(\nu) = \phi_{\mathrm{per}}(\nu) &= \frac{1}{T^d} \inf \left\{ E(u, Q_T) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic} \right\} \\ &= \frac{1}{T^d} \inf \left\{ E(u + \langle \nu, \cdot\rangle, Q_T) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) \text{ is } T\text{-periodic} \right\} .\end{aligned}$$

It therefore suffices to prove that the function

$$\phi_{\mathrm{per}}(\nu) = \frac{1}{T^d} \inf \left\{ E(u + \langle \nu, \cdot\rangle, Q_T) : u\colon \mathcal{L} \to \mathbb{R}, u(\cdot) \text{ is } T\text{-periodic} \right\}$$

is crystalline. Note that, we can write

$$\phi_{\mathrm{per}}(\nu) = \frac{1}{T^d} \inf_{\substack{u\colon \mathcal{L}\to\mathbb{R} \\ u(\cdot)\, T\text{-per}}} \sup_{\substack{0\le\alpha_{i,j}\le c_{i,j} \\ \alpha_{i+Tz,j+Tz}=\alpha_{i,j}}} \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}(u(i) - u(j) + \langle \nu, i-j\rangle) . \tag{101}$$

Given $0 \le \alpha_{i,j} \le c_{i,j}$ such that $\alpha_{i+Tz,j+Tz} = \alpha_{i,j}$ for all $z \in \mathbb{Z}^d$, and $u\colon \mathcal{L} \to \mathbb{R}$ T-periodic, we have

$$\begin{aligned}\sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}(u(i) - u(j)) &= \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}u(i) - \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}u(j) \\ &= \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}u(i) - \sum_{j\in\mathcal{L}\cap Q_T} \sum_{z\in\mathbb{Z}^d} \sum_{i\in\mathcal{L}\cap Q_T} \alpha_{i,j+Tz}u(j+Tz) \\ &= \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}u(i) - \sum_{j\in\mathcal{L}\cap Q_T} \sum_{z\in\mathbb{Z}^d} \sum_{i\in\mathcal{L}\cap Q_T} \alpha_{i-Tz,j}u(j) \\ &= \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}u(i) - \sum_{j\in\mathcal{L}\cap Q_T} \sum_{i\in\mathcal{L}} \alpha_{i,j}u(j) \\ &= \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} (\alpha_{i,j} - \alpha_{j,i})u(i) .\end{aligned}$$

Note that, since in all steps the sum over i and, due to to (H3), the sum over j runs over a finite index set, the order of summation can be changed without changing the value of the various sums. This implies that, given $0 \le \alpha_{i,j} \le c_{i,j}$ such that $\alpha_{i+Tz,j+Tz} = \alpha_{i,j}$ for all $z \in \mathcal{L}$, we have

$$\inf_{\substack{u\colon \mathcal{L}\to\mathbb{R} \\ u(\cdot)\, T\text{-per}}} \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{ij}(u(i) - u(j)) = \begin{cases} 0 & \text{if } \sum_{j\in\mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} , \\ -\infty & \text{otherwise.} \end{cases} \tag{102}$$

Hence, using (97), (99), (101), and (102), we obtain (100).

Step 2.(Crystallinity) By Remark 2.5, we have

$$\varphi(\nu) = \frac{1}{T^d} \sup_{\zeta\in W_\varphi} \langle \nu, \zeta \rangle .$$

So that, by (100)

$$W_\varphi = \left\{ \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}(i-j) : (\alpha_{i,j})_{i,j} \in \mathcal{C} \right\} ,$$

with $\mathcal{C}$ given in (99). Recall N and $\mathcal{E}$ defined in (98). Define $L\colon \mathbb{R}^N \to \mathbb{R}^d$ by

$$L\left(\alpha_{i,j}\right)_{(i,j)\in\mathcal{E}} = \sum_{(i,j)\in\mathcal{E}} \alpha_{i,j}(i-j)\,. \tag{103}$$

We then find

$$W_\varphi = L\left([0,c_{i,j}]^N \cap V\right) , \tag{104}$$

where $V \subset \mathbb{R}^N$ is a linear subspace of co-dimension $T^d - 1$ given by

$$V = \left\{ \alpha_{i,j} \in \mathbb{R}^N : \sum_{j\in\mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} \right\} . \tag{105}$$

Hence, due to (104), W_φ is the image of the linear map L, given in (103), of a N-dimensional cube $[0, c_{i,\xi}]^N$ intersected with the linear subspace V, given in (105). The intersection of a cube with a linear subspace is a polytope, and thus also its image through a linear map. This proves that φ is crystalline.

Step 3.(Estimate on the number of vertices) Our goal is to prove that

$$\#\text{extreme}(W_\varphi) \leq 2^N , \tag{106}$$

where we recall N defined in (98). Let us note that, due to the Krein-Milman Theorem (cf. [19], Theorem 1.13) and (104), it is easy to see that there holds

$$\#\text{extreme}(W_\varphi) = \#\text{extreme}\left(L\left([0,c_{i,j}]^N \cap V\right)\right) \leq \#\text{extreme}\left([0,c_{i,j}]^N \cap V\right) .$$

In order to show (106), it remains to show

$$\#\text{extreme}\left([0,c_{i,j}]^N \cap V\right) \leq 2^N . \tag{107}$$

In order to obtain this estimate we construct a (non necessarily orthogonal) projection $P\colon \mathbb{R}^N \to \mathbb{R}^N$ such that

$$P([0,c_{i,j}]^N) = [0,c_{i,j}]^N \cap V . \tag{108}$$

By the Krein-Milman Theorem, it then follows that

$$\#\text{extreme}\left([0,c_{i,j}]^N \cap V\right) = \#\text{extreme}(P([0,c_{i,j}]^N)) \leq \#\text{extreme}([0,c_{i,j}]^N) = 2^N .$$

In order to construct P denote by $k = \dim(V)$ and let $\{v_1,\ldots,v_k\}$ be a basis of V. Add vectors $\{e_{i_1},\ldots,e_{i_{N-k}}\}$ from the standard orthonormal basis of $\mathbb{R}^N$ in order to form a basis of $\mathbb{R}^N$. For every $x \in \mathbb{R}^n$ we can write

$$x = x_v + x_c \text{ with } x_v = \sum_{j=1}^{k} \lambda_j v_j \text{ and } x_c = \sum_{j=1}^{N-k} \mu_j e_{i_j} ,$$

where $\lambda_k, \mu_k \in \mathbb{R}^N$ for all $j \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,N-k\}$ respectively. We define $P\colon \mathbb{R}^N \to \mathbb{R}^N$ by

$$Px = x_v\,.$$

It is easy to see that (108) holds true. In fact on the one hand, we have $P([0,c_{i,j}]^N) \supseteq P([0,c_{i,j}]^N \cap V) = [0,c_{i,j}]^N \cap V$. On the other hand given $x \in [0,c_{i,j}]^N$, we have

$$x = \sum_{j=1}^{N} \sigma_j e_j = \sum_{j=1}^{k} \lambda v_j + \sum_{j=1}^{N-k} \mu_j e_{i_j} = x_v + x_c\,.$$

Now, it is easy to see that $\mu_j = \sigma_{i_j}$ and therefore

$$x_v = \sum_{j=1}^{k} \lambda v_j = \sum_{\substack{j=1 \\ j\notin\{i_j : j\in\{1,\dots,N-k\}}}^{N} \sigma_j e_j\,.$$

This implies that $x_v \in [0,c_{i,j}]^N$ and clearly $x_v \in V$. This shows that $P([0,c_{i,j}]^N) \subseteq [0,c_{i,j}]^N \cap V$ and therefore (107). This concludes Step 3. □

5. Differentiability of the effective surface tension

In this Section, we prove Proposition 2.9 which states that φ is differentiable in totally irrational directions. It is a corollary of the two lemmas which we state and prove below.

Lemma 5.1. *Let $\nu \in \mathbb{S}^1$, let u be a minimizer in (7) and assume that for any $s \in \mathbb{R}$, the set $\{u = s\}$ is finite. Then φ is differentiable in ν.*

Proof. The expression (100) shows that φ is a convex, one-homogeneous function with subgradient at ν given by

$$\partial\varphi(\nu) = \left\{ \frac{1}{T^d} \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j\in\mathcal{L}} \alpha_{i,j}(i-j) \,:\, \alpha = (\alpha_{i,j})_{i,j} \in \mathcal{C} \text{ maximizer in (100)} \right\}$$

It is differentiable at ν if and only if the above set has exactly one element.

Let $\alpha, \alpha' \in \mathcal{C}$ be two maximizers in (100). Classical optimality conditions guarantee that for any i,j, if $u(i) \neq u(j)$, then:

$$\alpha_{i,j} = \alpha'_{i,j} = \begin{cases} c_{i,j} & \text{if } u(i)-u(j) > 0 \\ 0 & \text{if } u(i)-u(j) < 0. \end{cases} \tag{109}$$

Let us denote by $p, p' \in \partial\varphi$ the subgradients given by the dual variables, respectively, α and α', we claim that $p = p'$. One has:

$$p - p' = \frac{1}{T^d} \sum_{i\in\mathcal{L}\cap Q_T} \sum_{j:u(j)=u(i)} (\alpha_{i,j} - \alpha'_{i,j})(i-j). \tag{110}$$

Let $s \in \mathbb{R}$, $i_0 \in \mathcal{L} \cap Q_T$ with $u(i_0) = s$ and such that the finite set $J_s := \{j : u(j) = s\}$ has more than one element. For any i,j, let $\beta_{i,j} := \alpha_{i,j} - \alpha'_{i,j}$. Then

$$\begin{aligned}\sum_{i\in J_s} \sum_{j\in J_s} \beta_{i,j}(i-j) &= \sum_{z\in\mathbb{Z}^d} \sum_{i\in J_s\cap(Tz+Q_T)} \sum_{j\in J_s} \beta_{i,j}(i-j) \\ &= \sum_{z\in\mathbb{Z}^d} \sum_{i\in(J_s-Tz)\cap Q_T} \sum_{j\in J_s-Tz} \beta_{i,j}(i-j)\end{aligned}$$

where for the last line we have substituted (i,j) with $(i-Tz, j-Tz)$ and used that β is Q_T-periodic. In addition, we have that $u(i)=u(j)$ if and only if $u(i-Tz)=u(j-Tz)$ so that this can be rewritten:

$$\sum_{i\in J_s}\sum_{j\in J_s}\beta_{i,j}(i-j)=\sum_{z\in\mathbb{Z}^d}\sum_{i\in(J_s-Tz)\cap Q_T}\sum_{j:u(j)=u(i)}\beta_{i,j}(i-j)$$

By assumption, the sets $(J_s-Tz)\cap Q_T$, $z\in\mathbb{Z}^d$ are all disjoint. Otherwise, there would be i,z with $s=u(i-Tz)=u(i)+T\langle\nu,z\rangle=s$, yielding in particular that $\langle\nu,z\rangle=0$, and one would deduce that $i-kTz\in J_s$ for all $k\in\mathbb{Z}$, a contradiction since we assumed J_s was finite. As a consequence, showing that (110) vanishes is equivalent to showing that

$$\sum_{i\in J_s}\sum_{j\in J_s}\beta_{i,j}(i-j)=0 \tag{111}$$

for any $s\in\mathbb{R}$ (such that J_s is not empty and contains more than one point). Obviously, the expression in (111) is also

$$\sum_{i\in J_s}\sum_{j\in J_s}(\beta_{i,j}-\beta_{j,i})i$$

Thanks to the definition (99) of $\mathcal{C}$, one has for any i that $\sum_j \beta_{i,j}-\beta_{j,i}=0$, so that:

$$\sum_{i\in J_s}\sum_{j\in J_s}(\beta_{i,j}-\beta_{j,i})i=\sum_{i\in J_s}\sum_{j\notin J_s}(\beta_{j,i}-\beta_{i,j})i=0$$

thanks to (109). Hence, (111) holds and we deduce $p=p'$, which shows the lemma. □

Lemma 5.2. *Let $\nu\in\mathbb{S}^1$ be totally irrational and let u be a minimizer in* (7)*. Then for any $s\in\mathbb{R}$, the set $\{u=s\}$ is finite.*

Proof. Recalling the notation in the previous proof, let $s\in\mathbb{R}$ and consider the set $J_s:=\{u=s\}$. For $z\in\mathbb{Z}$, let $J_s^z=J_s\cap(Q_T+Tz)-Tz\subset Q_T$. For $i\in J_s^z$, $u(i)=s+T\langle z,\nu\rangle$. Since ν is totally irrational, we deduce that $J_s^z\cap J_s^{z'}=\emptyset$ for any $z\neq z'$, showing that all sets J_s^z but a finite number must be empty. Hence J_s is finite. □

6. Numerical illustration

6.1. **A simplified framework.** In this section, we address, as an illustrative experiment, the following issue. We consider a basic $2D$ cartesian graph $\{(i,j):0\le i\le M-1, 0\le j\le N-1\}$, representing for instance the pixels of an image, and we want to approximate on this discrete grid the two-dimensional total variation $\int_\Omega|Du|$, $u\in BV(\Omega)$. Here it is assumed that $\Omega\subset\mathbb{R}^2$ is a rectangle and that $\{0,\dots,M-1\}\times\{0,\dots,N-1\}$ is a discretization of Ω at a length scale $\sim 1/N\sim 1/M$.

There are of course many ways to do this, but we propose here to consider a family of discrete "graph" total variations, defined for a $(u_{i,j})_{i,j}\in\mathbb{R}^{M\times N}$ by:

$$\begin{aligned} J(u)=\sum_{i,j} c^+_{i+\frac12,j}(u_{i+1,j}-u_{i,j})^+ + c^-_{i+\frac12,j}(u_{i,j}-u_{i+1,j})^+ \\ + c^+_{i,j+\frac12}(u_{i,j+1}-u_{i,j})^+ + c^-_{i,j+\frac12}(u_{i,j}-u_{i,j+1})^+ \end{aligned} \tag{112}$$

and which involves only nearest-neighbour interactions in horizontal and vertical directions.

We assume in addition that the weight $c^\pm$ are T-periodic for some $T \in \mathbb{N}$, $T > 0$, that is, $C^\pm_{a+kT,b+lT} = c^\pm_{a,b}$ for any $(k,l) \in \mathbb{Z}^2$, $(a,b) = (i+\frac{1}{2}, j)$ or $(i, j+\frac{1}{2})$, as long as the points fall inside the grid.

For $T = 1$, $c^\pm_{a,b} \equiv 1$, it is standard that (112) approximates, in the continuum limit, the anisotropic total variation $\int_\Omega |\partial_1 u| + |\partial_2 u|$, which, if used for instance as a regularizer for image denoising or reconstruction, may produce undesired artefacts (although hardly visible on standard applications, see Figure 8).

A standard way to mitigate this issue (besides, of course, resorting to numerical analysis based on finite differences or elements in order to define more refined discretizations), is to add to (112) diagonal interactions, with appropriate weights, in order to improve the isotropy of the limit (see for instance [10]), with the drawback of complexifying the graph and the optimization. We show here that a similar effect can be attained by homogenization. To illustrate this, let us first consider the simplest situation, for $T = 2$.

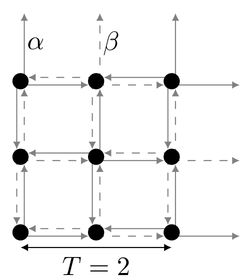

FIGURE 3. The alternating 2-periodic coefficients yielding the smallest anisotropy

In that case, one can explicitly build coefficients $c^\pm_{a,b}$, taking two values α, β (see Figure 3), which will yield the homogenized surface tension

$$\varphi(\nu) = (\sqrt{2}-1)\left(|\nu_1| + |\nu_2| + \frac{|\nu_1+\nu_2|}{\sqrt{2}} + \frac{|\nu_1-\nu_2|}{\sqrt{2}}\right) \tag{113}$$

whose 1-level set (or *Frank diagram*) is shown in Figure 4. Observe that this is the same

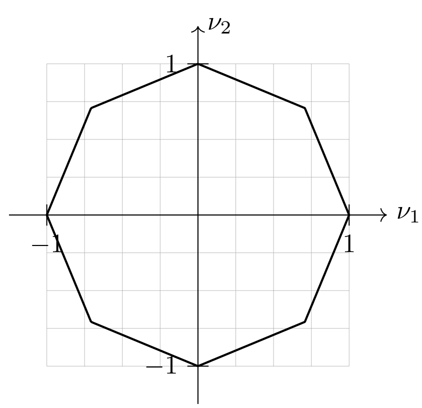

FIGURE 4. The Frank diagram $\{\nu : \varphi(\nu) \le 1\}$ given by (113)

anisotropy which would be obtained by using constant coefficients and adding interactions along the edges $((i,j),(i+1,j+1))$ and $((i,j),(i+1,j-1))$.

In order to obtain (113), one needs to tune α,β so that a vertical edge and a diagonal edge, in the most favorable position, have the same length (with a $\sqrt{2}$ factor for the diagonal, whose intersection with the periodicity cell is of course longer). This is ensured if $\alpha+\beta = 4\alpha/\sqrt{2}$, that is, $\beta = (2\sqrt{2}-1)\alpha$. We find that choosing

$$\begin{cases} \alpha = \frac{1}{4\sqrt{2}} \approx 0.1768 \\ \beta = (2\sqrt{2}-1)\alpha \approx 0.3232 \end{cases} \tag{114}$$

yields (113), as an effective homogenized anisotropy.

For larger periodicity cells, it seems difficult to do a similar analysis, first of all, because one should not expect the optimal minimizers, in most directions (if not all), to be given by straight lines, but rather by periodic perturbations of straigth lines. We propose an optimization process in order to compute the optimal weights $c^{\pm}_{a,b}$.

6.2. **The optimization method.** The effective surface tension is obtained by solving the cell problem:

$$\begin{aligned} \phi(\nu) = \min_u \Big\{ \sum_{(i,j)\in Y} & c^+_{i+\frac12,j}(u_{i+1,j}-u_{i,j})^+ + c^-_{i+\frac12,j}(u_{i,j}-u_{i+1,j})^+ \\ & + c^+_{i,j+\frac12}(u_{i,j+1}-u_{i,j})^+ + c^-_{i,j+\frac12}(u_{i,j}-u_{i,j+1})^+ \ : \\ & u_{i,j} - \nu\cdot\begin{pmatrix} i \\ j\end{pmatrix} \ Y\text{-periodic} \Big\} \end{aligned} \tag{115}$$

where $Y = \mathbb{Z}^2 \cap ([0,T)\times[0,T))$ is the periodicity cell. This is easily solved, for instance by a saddle-point algorithm [25] which aims at finding a solution to:

$$\begin{aligned} \phi(\nu) = \min_{vY\text{-periodic}} \max_{0\le w^{\pm}_{\bullet}\le 1} \sum_{(i,j)\in Y} & (w^+_{i+\frac12,j}c^+_{i+\frac12,j} - w^-_{i+\frac12,j}c^-_{i+\frac12,j})(v_{i+1,j}-v_{i,j}+\nu_1) \\ & + (w^-_{i,j+\frac12}c^-_{i,j+\frac12} - w^+_{i,j+\frac12}c^+_{i,j+\frac12})(v_{i,j+1}-v_{i,j}+\nu_2), \end{aligned}$$

where we have replaced the variable u with the periodic vector $v_{i,j} = u_{i,j} - \nu\cdot(i,j)^T$. For technical reasons, we need to "regularize" slightly this problem in order to make it differentiable with respect to the coefficients $\mathbf{c} = (c^{\pm}_{\bullet})$. This is done by introducing $\varepsilon > 0$ a (very) small parameter and adding to the previous objective the penalization

$$-\frac{\varepsilon}{2}\sum_{(i,j)\in Y}(w^+_{i+\frac12,j})^2 + (w^-_{i+\frac12,j})^2 + (w^-_{i,j+\frac12})^2 + (w^+_{i,j+\frac12})^2 + \frac{\varepsilon}{2}\sum_{(i,j)\in Y} v^2_{i,j}$$

which makes the problem strongly convex/concave and the solutions w,v unique. We call $\phi_\varepsilon(\nu)[\mathbf{c}]$ the corresponding value. The advantage of this regularization is that one can easily show that $\mathbf{c}\mapsto\phi_\varepsilon(\nu)[\mathbf{c}]$ is locally $C^{1,1}$, with a gradient given by:

$$\begin{aligned} \lim_{t\to 0}\frac{\phi_\varepsilon(\nu)[\mathbf{c}+t\mathbf{d}] - \phi_\varepsilon(\nu)[\mathbf{c}]}{t} = \sum_{(i,j)\in Y} & (w^+_{i+\frac12,j}d^+_{i+\frac12,j} - w^-_{i+\frac12,j}d^-_{i+\frac12,j})(v_{i+1,j}-v_{i,j}+\nu_1) \\ & + (w^-_{i,j+\frac12}d^-_{i,j+\frac12} - w^+_{i,j+\frac12}d^+_{i,j+\frac12})(v_{i,j+1}-v_{i,j}+\nu_2) \end{aligned}$$

where (w,v) solves the saddle-point problem which defines $\phi_\varepsilon(\nu)[\mathbf{c}]$.

Then, to find coefficients which ensure that ϕ is as "isotropic" as possible, one fixes a finite set of directions $(\nu_1, \dots, \nu_k)$ (typically, $(\cos(2\ell\pi/k), \sin(2\ell\pi/k))$ for $\ell = 1, \dots, k$), and use a first order gradient descent algorithm to optimize:

$$\mathcal{L}(\mathbf{c}) = \sum_{\ell=1}^{k} (\phi_\varepsilon(\nu_\ell)[\mathbf{c}] - 1)^2$$

The problem is easily solved for $Y = \{0,1\} \times \{0,1\}$, $k = 8$ and $(\nu_\ell)_{\ell=1}^{8}$ given as above. For larger periodicity cells and more directions, it easily gets trapped in local minima and we use a random initialization in order to be able to find satisfactory solutions. We then test the result by computing the un-regularized surface tension ϕ with the resulting coefficients $\mathbf{c}$. We show some results in the next section. Of course, taking a large value of ε will make the problem easier to solve, but the learned coefficients will not allow to reconstruct a satisfactory surface tension: we need to choose ε small, an order of magnitude below the error which we expect on the anisotropy of ϕ.

6.3. **Numerical results.** We show the outcome of the optimization, in the periodicity cell $Y = \{0, \dots T-1\} \times \{0, \dots, T-1\}$ for $T = 2, 4, 6, 8$. We plot first the set $\{\varphi \le 1\}$ or Frank diagram for the effective surface tensions. Figure 5 shows the diagram obtained, for $T = 2, 4, 8$.

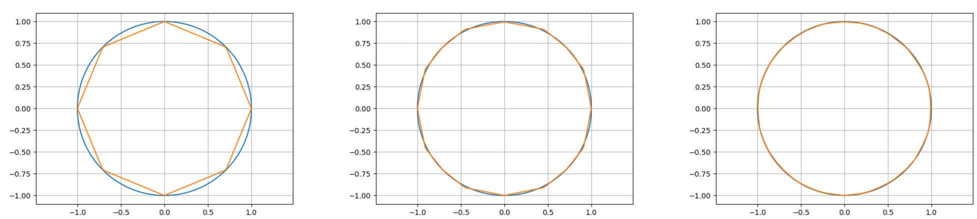

FIGURE 5. Frank diagrams of the effective anisotropies for $T = 2, 4, 8$.

For $T = 2$, the optimization yields the same anisotropy as our theoretical proposition (compare with Fig. 4). However, except when initialized with the values in (114), the algorithm usually outputs different values with the same effective anisotropy, see Fig. 6 (the values in (114) are in some sense better, as for instance a vertical edge will always have the same effective energy with these values, while with the comptuted values displayed in Fig. 6, it will need to pass through the edges in the second column of the cell in order to get the minimal energy).

For $T = 4$, one sees that the behaviour is almost isotropic, while for $T = 8$, the relative error with the perfect unit disk is about 1%. We illustrate this on an "inpainting" example, which consists in finding the minimal line in a given direction. We consider as an example the direction $(\cos 3\pi/8, \sin 3\pi/8)$, which is irrational, so that there cannot be a fully periodic solution. The figure 7 displays several minimal half-planes in this orientation. Observe that for this orientation, the results for $T = 4$ or 6 look nicer than the result obtained for $T = 8$.

We also show a denoising example based on the "ROF" method (which consists simply in minimizing the total variation (defined by the surface tension φ) of an image with a quadratic penalization of the distance to a noisy data, in order to produce a denoised version, see [41]) with the anisotropic tension $\varphi(\nu) = |\nu_1| + |\nu_2|$ ("$T = 1$") and the optimized homogenized surface tension for $T = 4$. The original image is degraded with a Gaussian noise with 10% standard deviation (with respect to the range of the values). Here, the difference between the

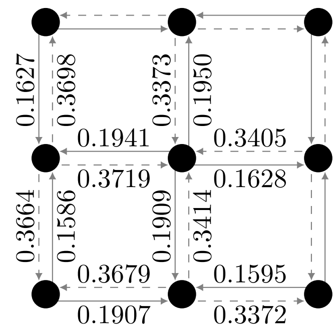

FIGURE 6. An example of optimized 2-periodic coefficients yielding the same anisotropy as the choice (114)

FIGURE 7. A minimal half-plane in the orientation $(\cos 3\pi/8, \sin 3\pi/8)$. Top left, boundary datum, the region where the perimeter is minimized is in gray. Top, middle: $\varphi(\nu) = |\nu_1| + |\nu_2|$. Top, right: optimal effective φ for $T = 2$. Bottom: for $T = 4$, 6, 8.

two regularizers is hardly perceptible (since the data term strongly influences the position of the discontinuities), yet a close-up (bottom row) allows to see a slight difference, for instance on the cheek where the $T = 1$ anisotropy produces block structures.

ACKNOWLEDGEMENTS

This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 -390685587, Mathematics Münster: Dynamics–Geometry–Structure.

FIGURE 8. "ROF" denoising example. Left: noisty image. Middle, denoised with $\varphi(\nu) = |\nu_1| + |\nu_2|$. Right: with the effective tension computed for $T = 4$.

REFERENCES

[1] R. ALICANDRO, A. BRAIDES, M. CICALESE. Phase and antiphase boundaries in binary discrete systems: a variational viewpoint. *Netw. Heterog. Media* **1** (2006), 85–107.

[2] R. ALICANDRO, M. CICALESE, M. RUF. Domain formation in magnetic polymer composites: an approach via stochastic homogenization. *Arch. Ration. Mech. Anal.* **218** (2015), 945–984.

[3] R. ALICANDRO, M.S. GELLI. Local and nonlocal continuum limits of Ising-type energies for spin systems. *SIAM J. Math. Anal.* **48** (2016), 895–931.

[4] L. AMBROSIO, N. FUSCO, D. PALLARA. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[5] F. AUER, V. BANGERT Differentiability of the stable norm in codimension one *Amer. J. Math.* **128.1** (2006), 215–238

[6] A. BACH, A. BRAIDES, M. CICALESE. Continuum Limits of Multibody Systems with Bulk and Surface Long-Range Interactions. *SIAM J. Math. Anal.* **52.4** (2020), 3600–3665.

[7] A. BLAKE, A. ZISSERMAN. Visual Reconstruction. MIT Press Series in Artificial Intelligence. MIT Press, Cambridge, (1987).

[8] X. BLANC, M. LEWIN. *The crystallization conjecture: a review.* EMS Surv. Math. Sci. **2** (2015), 225–306.

[9] G. BOUCHITTÉ, G. DAL MASO. Integral representation and relaxation of convex local functionals on $BV(\Omega)$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **20** (1993), 483–533.

[10] BOYKOV AND V. KOLMOGOROV. Computing geodesics and minimal surfaces via graph cuts *Proc. IEEE Int. Conf. on Computer Vision* **1** (2003), 26–33.

[11] A. BRAIDES. Γ-*convergence for Beginners.* Oxford Univ. Press, Oxford 2002.

[12] A. BRAIDES, M. CICALESE. Interfaces, modulated phases and textures in lattice systems. *Arch. Ration. Mech. Anal.* **223** (2017), 977–1017.

[13] A. BRAIDES, V. CHIADO PIAT. *A derivation formula for convex integral functionals defined on* $BV(\Omega)$. *J. Convex Anal* **2.1-2** (1995), 69–85.

[14] A. BRAIDES, A. DEFRANCESCHI. *Homogenization of multiple integrals.* Oxford Univ. Press, Oxford 1998.

[15] A. BRAIDES, L. KREUTZ. Optimal bounds for periodic mixtures of nearest-neighbour ferromagnetic interactions. *Rend. Lincei Mat. Appl* **28** (2017), 103–117.

[16] A. BRAIDES, L. KREUTZ. Design of lattice surface energies. *Calc. Var. Partial Differential Equations* **57**:97 (2018).

[17] A. Braides, A. Piatnitski. Homogenization of surface and length energies for spin systems. *J. Funct. Anal.* **264** (2013), 1296–1328.
[18] A. Braides, A.J. Lew, M. Ortiz. Effective cohesive behavior of layers of interatomic planes *Arch. Ration. Mech. Anal.* **180** (2006), 151–182.
[19] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Springer Science & Business Media, 2010.
[20] L. A. Caffarelli, R. de la Llave. Planelike minimizers in periodic media. *Comm. Pure Appl. Math.* **54** (2001), 1403–1441.
[21] L. A. Caffarelli, R. de la Llave. Interfaces of ground states in Ising models with periodic coefficients. *J. Stat. Phys.* **118** (2005), 687–719.
[22] A. Chambolle. Finite-differences discretizations of the Mumford-Shah functional. *M2AN Math. Model. Numer. Anal.* **33** (1999), 261–288.
[23] A. Chambolle, A. Giacomini, L.Lussardi. Continuous limits of discrete perimeters. *M2AN Math. Model. Numer. Anal.* **44** (2010), 207–230.
[24] A. Chambolle, M. Goldman, M. Novaga. Plane-like minimizers and differentiability of the stable norm. *J. Geom. Anal.* **24.3** (2014), 1447–1489.
[25] A. Chambolle, T. Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *J. Math. Imaging Vis.* **40** (2011), 120–145.
[26] A. Chambolle, G. Thouroude. Homogenization of interfacial energies and construction of plane-like minimizers in periodic media through a cell problem. *Netw. Heterog. Media* **4** (2009) 127–152.
[27] M. Cicalese, G.P. Leonardi. Maximal fluctuations on periodic lattices: an approach via quantitative Wulff inequalities. *Commun. Math. Phys.* **24** (2019), 1–14.
[28] G. Dal Maso. *An introduction to Γ-convergence.* Birkhäuser, Boston · Basel · Berlin 1993.
[29] S. Daneri, E. Runa. *Exact periodic stripes for minimizers of a local/nonlocal interaction functional in general dimension.* Arch. Ration. Mech. Anal. **231.1** (2019) 519–589.
[30] A. Figalli, Y.R. Zhang. *Strong stability for the Wulff inequality with a crystalline norm.* Comm. Pure Appl. Math. (2020).
[31] M. Friedrich, L. Kreutz. *Crystallization in the hexagonal lattice for ionic dimers.* Math. Models Methods Appl. Sci. **29** (2019), 1853–1900.
[32] M. Friedrich, L. Kreutz. *Finite crystallization and Wulff shape emergence for ionic compounds in the square lattice.* Nonlinearity **33** (2020), 1240–1296.
[33] M. Friedrich, B. Schmidt. *An analysis of crystal cleavage in the passage from atomistic models to continuum theory.* Arch. Ration. Mech. Anal. **217.1** (2015), 263–308.
[34] A. Giuliani, J. L. Lebowitz, E. H. Lieb. Checkerboards, stripes, and corner energies in spin models with competing interactions. *Phys. Rev. B* 84:064205, (2011).
[35] A. Giuliani, E. H. Lieb, R. Seiringer. Formation of stripes and slabs near the ferromagnetic transition. *Comm. Math. Phys.* **331** (2014), 333–350.
[36] A. Giuliani, R. Seiringer. Periodic striped ground states in Ising models with competing interactions. *Comm. Math. Phys.* **347** (2016), 983–1007.
[37] E. Mainini, B. Schmidt. Maximal Fluctuations Around the Wulff Shape for Edge-Isoperimetric Sets in $\mathbb{Z}^d$: A Sharp Scaling Law. *Comm. Math. Phys.* **380.2** (2020), 947–971.
[38] E. Presutti. *Scaling limits in statistical mechanics and microstructures in continuum mechanics.* Springer, Berlin, (2009).
[39] H.E, Stanley. *Introduction to Phase Transitions and Critical Phenomena.* Oxford Univ. Press, Oxford (1971).
[40] R.T. Rockafellar. *Convex analysis.* Vol. 36. Princeton university press, (1970).
[41] L. Rudin and S. J. Osher and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D* **60** (1992), 259–268.

(Antonin Chambolle) CEREMADE, CNRS, Université Paris-Dauphine, PSL, France

E-mail address: antonin.chambolle@ceremade.dauphine.fr

(Leonard Kreutz) WWU Münster, Germany

E-mail address: lkreutz@uni-muenster.de