ON THE STEINER PROPERTY FOR PLANAR MINIMIZING CLUSTERS. THE ANISOTROPIC CASE.

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Abstract. In this paper we discuss the Steiner property for minimal clusters in the plane with an anisotropic double density. This means that we consider the classical isoperimetric problem for clusters, but volume and perimeter are defined by using two densities. In particular, the perimeter density may also depend on the direction of the normal vector. The classical "Steiner property" for the Euclidean case (which corresponds to both densities being equal to 1) says that minimal clusters are made by finitely many $C^{1,\gamma}$ arcs, meeting in finitely many "triple points". We can show that this property holds under very weak assumptions on the densities. In the parallel paper [15] we consider the isotropic case, i.e., when the perimeter density does not depend on the direction, which makes most of the construction much simpler. In particular, in the present case the three arcs at triple points do not necessarily meet with three angles of 120°, which is instead what happens in the isotropic case.

1. Introduction

In recent years, a lot of attention is being paid to isoperimetric problems in $\mathbb{R}^N$ depending on two densities. This means that we are given two l.s.c. functions $g : \mathbb{R}^N \to (0, +\infty)$ and $h : \mathbb{R}^N \times \mathbb{S}^{N-1} \to (0, +\infty)$, usually called *densities*, and the volume and perimeter of any set $E \subseteq \mathbb{R}^N$ of locally finite perimeter are defined by

$$|E| = \int_E g(x)\, dx\,, \qquad\qquad P(E) = \int_{\partial^* E} h(x, \nu_E(x))\, d\mathcal{H}^1(x)\,, \tag{1.1}$$

where, as usual, $\partial^* E$ is the reduced boundary of E and, for every $x \in \partial^* E$, $\nu_E(x)$ is the outer normal vector to E at x (see [3] for definitions and properties of sets of finite perimeter). There are several reasons why this problem is attracting a big interest, that we are not going to describe here, we limit ourselves to point out some basic bibliography, more information can be found there and in the references therein [36, 6, 5, 26, 4, 1, 10, 18, 2, 7, 12].

In this paper we consider the isoperimetric problem for clusters. In other words, we do not want to minimize the perimeter of a single set of given volume, but of a "cluster", that is, a group of sets with given volumes. This is not simply the "sum" of isoperimetric problems for single sets, because the common boundary is only counted once. A practical example of such a problem is given by soap bubbles, which behave more

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or less as minimal clusters with the Euclidean density. Of course a single bubble must be a ball; however, when there are two bubbles, the best situation is not given by two distinct balls, but by a cluster with the usual shape of two soap bubbles, which minimize the total perimeter by having a large common portion of the boundary. Also the problem of studying minimal clusters, in the Euclidean case, has been deeply investigated in the last decades, and completely solved for "double bubbles", i.e., when the cluster is made by two sets (see [11, 20, 35], and see also [40, 29, 30, 31] for the case of three or four sets with equal volumes). In the planar case $N = 2$ it is known that minimal clusters enjoy a strong regularity property. More precisely, the boundary of any minimal cluster is made by finitely many $C^{1,\gamma}$ arcs (and then, by standard regularity, they are actually C^∞), which meet in finitely many junction points. Each of these junction points is actually a triple point –that is, exactly three arcs meet– and the three arcs form three angles of 120°. This property, usually called *Steiner property*, is now widely known, two very good references are the classical paper [38] and the recent book [22].

Our goal is to extend the regularity of minimal clusters in the plane to the case when perimeter and volume are given by two densities, and we are able to do this in a wide generality. In the parallel paper [15] we consider the isotropic case, that is, when the density h only depends on the point but not on the direction of the normal vector, and in that case the whole construction is rather simple. In the more general anisotropic case, that we consider here, the underlying idea is still simple, but several technical points become much more complicated. Moreover, the "120° property", which is still true in the isotropic case, becomes false, at this regard see the discussion in Section 3.2.

To consider the isoperimetric problem for clusters, the first thing to do is to extend the definition (1.1) of volume and perimeter of a single set. For a given $m \geq 2$, a *m-cluster* is a collection $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ of m essentially disjoint sets of locally finite perimeter in $\mathbb{R}^2$, and its *volume* is the vector $|\mathcal{E}| = (|E_1|, |E_2|, \dots, |E_m|) \in (\mathbb{R}^+)^m$. We set, for brevity, $E_0 = \mathbb{R}^2 \setminus (\cup_{i=1}^m E_i)$ and $\partial^*\mathcal{E} = \cup_{i=1}^m \partial^* E_i$. The *perimeter* of a cluster $\mathcal{E}$ is then defined as

$$P(\mathcal{E}) = \frac{P(\cup_{i=1}^m E_i) + \sum_{i=1}^m P(E_i)}{2}\,. \tag{1.2}$$

It is very important to understand the meaning of this definition, which is discussed in detail in Section 1.1 below. Hence, a reader who sees this definition for the first time might want to read that section before going on with this introduction.

In order to present our main result, a few definitions are in order. We start with the *strict convexity* and the *uniform roundedness* of h in the second variable.

Definition 1.1 (Strict convexity and uniform roundedness in the second variable)**.** *Let $h : \mathbb{R}^2 \times \mathbb{S}^1 \to (0, +\infty)$ be given, and extend it to the whole $\mathbb{R}^2 \times \mathbb{R}^2$ by positive 1-homogeneity, i.e., set $h(x, \lambda\nu) = \lambda h(x, \nu)$ for every $x \in \mathbb{R}^2$, $\lambda \geq 0$ and $\nu \in \mathbb{S}^1$. We say* *that* h is strictly convex in the second variable *if for every* $x \in \mathbb{R}^2$ *the unit ball*

$$\mathcal{C}(x) = \Big\{\nu \in \mathbb{R}^2 :\ h(x,\nu) \leq 1\Big\} \tag{1.3}$$

is strictly convex. This is equivalent to ask that $h\Big(x, t\nu+(1-t)\mu\Big) < th(x,\nu)+(1-t)h(x,\mu)$ *for every* $x \in \mathbb{R}^2$ *and* $0 < t < 1$*, and for every two vectors* ν*,* $\mu \in \mathbb{R}^2$ *which are not positively parallel (i.e.,* $\nu = \lambda\mu$ *for some* $\lambda \geq 0$*). We say that* h is locally uniformly round in the second variable *if for every bounded* $D \subseteq \mathbb{R}^2$ *there exists a constant* $c > 0$ *such that the curvature of every ball* $\mathcal{C}(x)$ *with* $x \in D$ *is bounded from below by* c*. In other words, for every* $x \in D$*,* $\nu \in \mathbb{S}^1$*,* $w \in \mathbb{R}^2$ *with* $|w| \leq 1$ *and* $w \perp \nu$ *one has*

$$\frac{h(x,\nu+w)+h(x,\nu-w)}{2} \geq h(x,\nu) + c|w|^2\,. \tag{1.4}$$

It is readily observed that regularity may fail if h is not strictly convex and uniform round in the second variable. The basic idea is that the shortest path between two points close to each other can be far from the straight line if h is not strictly convex, and the uniform roundedness is necessary to quantify this closeness (this is related with the so-called "excess"). Moreover, the fact that junction points are necessarily triple points is related to the regularity of the unit ball, or in other words, to the fact that $h \in \mathrm{C}^1$. In fact, as we will discuss in detail in Section 3.1, quadruple points may occur for the L^1 density $h(x,\nu) = \max\{|\nu_1|, |\nu_2|\}$. This density is not C^1, but it is also not strictly convex nor uniformly round. However, a simple modification of this density (also described in Section 3.1) allows to obtain quadruple points for a density which is strictly convex and uniformly round but still not C^1. Instead, as soon as h is C^1 and strictly convex in the second variable, multiple points are necessarily triple points and the boundary of $\mathcal{E}$ is done by locally finitely many curves (this is the content of Section 2.3). Summarizing, to obtain a Steiner property for minimal clusters (that is, $\partial\mathcal{E}$ is done by regular arcs meeting in triple points, see Definition 1.3) one has to assume that h is strictly convex, C^1 and uniformly round in the second variable. Our main result, Theorem A, says that under these assumptions, and together with the same $\varepsilon - \varepsilon^\beta$ property and volume growth condition as in the isotropic case, it is still true that the Steiner property holds.

Definition 1.2 (η-growth condition and $\varepsilon - \varepsilon^\beta$ property for clusters)**.** *Given a power* $\eta \geq 1$*, an* η-growth condition *is said to hold if there exist two positive constants* $C_{\rm vol}$ *and* R_η *such that, for every* $x \in \mathbb{R}^2$ *and every* $r < R_\eta$*, the ball* $B(x,r)$ *has volume* $|B(x,r)| \leq C_{\rm vol} r^\eta$*. We say that the* local η-growth condition *holds if for any bounded domain* $D \subset\subset \mathbb{R}^2$ *there exist two constants* $C_{\rm vol}$ *and* R_η *such that the above property holds for balls* $B(x,r) \subseteq D$*.*

We say that a cluster $\mathcal{E}$ *satisfies the* $\varepsilon - \varepsilon^\beta$ *property for some* $0 < \beta \leq 1$ *if there exist three positive constants* R_β*,* $C_{\rm per}$ *and* $\bar\varepsilon$ *such that, for every vector* $\varepsilon \in \mathbb{R}^m$ *with* $|\varepsilon| \leq \bar\varepsilon$

and every $x \in \mathbb{R}^2$, there exists another cluster $\mathcal{F}$ such that

$$\mathcal{F}\Delta\mathcal{E} \subseteq \mathbb{R}^2 \setminus B(x, R_\beta)\,, \qquad |\mathcal{F}| = |\mathcal{E}| + \varepsilon\,, \qquad P(\mathcal{F}) \leq P(\mathcal{E}) + C_{\rm per}|\varepsilon|^\beta\,. \tag{1.5}$$

If this holds then, for each $t \leq \bar\varepsilon$, we call $C_{\rm per}[t]$ the smallest constant such that the above property is true for every $|\varepsilon| \leq t$. Clearly $t \mapsto C_{\rm per}[t]$ is an increasing function, and $C_{\rm per}[\bar\varepsilon] \leq C_{\rm per}$.

We underline that both the above assumptions are satisfied for a wide class of densities. In particular, the growth (or local growth) condition clearly holds with $\eta = 2$ whenever the density g is bounded (or locally bounded). Concerning the $\varepsilon - \varepsilon^\beta$ property, this is a crucial tool when dealing with isoperimetric problems. It is simple to observe that it is valid with $\beta = 1$ for every cluster of locally finite perimeter whenever the density h is regular enough (at least Lipschitz) in the first variable. It is also known that, if h is α-Hölder in the first variable, then every cluster of locally finite perimeter satisfies the $\varepsilon - \varepsilon^\beta$ property with

$$\beta = \frac{1}{2-\alpha}\,,$$

the proof can be found in [8] for the special case $g = h$ and in [33] for the general case. The case $\alpha = 0$ is particular, also because there is not a unique possible meaning of "0-Hölder function". More precisely, the $\varepsilon - \varepsilon^{1/2}$ property holds as soon as h is locally bounded. If h is continuous, instead, not only the $\varepsilon - \varepsilon^{1/2}$ property holds, but in addition $C_{\rm per}[t] \searrow 0$ if $t \searrow 0$. We can be even more precise: $C_{\rm per}[t] \lesssim \sqrt{\omega_h(t)}$, being ω_h the modulus of continuity of h in the first variable (see [34]). Notice that the required regularity for h here is in the first variable. In particular, ω_h is defined as $\omega_h(t) = \sup\{|h(x,\nu) - h(y,\nu)| : \nu \in \mathbb{S}^1,\ |y-x| \leq t\}$.

We can now give the formal definition of the Steiner property, already described above, and of the Dini property.

Definition 1.3 (Steiner property)**.** *A cluster $\mathcal{E}$ is said to satisfy the* Steiner property *if $\partial\mathcal{E}$ is a locally finite union of* C^1 *arcs, and each junction point is endpoint of exactly three different arcs, arriving with three different tangent vectors.*

Definition 1.4 (Dini property)**.** *We say that an increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$* satisfies the Dini property *if for every $C > 1$ one has*

$$\sum_{n\in\mathbb{N}} \varphi(C^{-n}) < +\infty\,,$$

which in particular implies $\lim_{t\searrow 0}\varphi(t) = 0$. We say that φ satisfies the 1/2-Dini property *if $\sqrt{\varphi}$ satisfies the Dini property. A uniformly continuous function f is said* Dini continuous *whenever*

$$\int_0^1 \frac{\omega_f(t)}{t}\,dt < +\infty\,,$$

where ω_f is the modulus of continuity of f. It is known that f is Dini continuous if and only if ω_f satisfies the Dini property. We say that f is 1/2-Dini continuous *if ω_f satisfies the 1/2-Dini property.*

We are now in position to state the main result of the present paper.

Theorem A (Steiner regularity for minimal clusters). *Let $g : \mathbb{R}^2 \to (0, +\infty)$ be a l.s.c. function, and let $h : \mathbb{R}^2 \times \mathbb{S}^1 \to (0, +\infty)$ be a continuous function, which is C^1, strictly convex and uniformly round in the second variable in the sense of Definition 1.1. Let $\mathcal{E}$ be a minimal cluster, and let us assume that for some η, β the local η-growth condition holds, as well as the $\varepsilon - \varepsilon^\beta$ property for $\mathcal{E}$. Assume also that h is locally 1/2-Dini continuous in the first variable, and that either*

(i) $\eta\beta > 1$, or

(ii) $\eta\beta = 1$ and the function $t \mapsto C_{\mathrm{per}}[t]$ satisfies the 1/2-Dini property.

Then $\mathcal{E}$ satisfies the Steiner property, and if $\eta\beta > 1$ and h is locally α-Hölder in the first variable then the arcs of $\partial\mathcal{E}$ are actually $\mathrm{C}^{1,\gamma}$ with $\gamma = \frac{1}{2}\min\{\eta\beta - 1, \alpha\}$.

It is to be observed that this result strongly generalizes the classical Euclidean case. In fact, we require that $\eta\beta \geq 1$, while in the Euclidean case one has $\eta = 2$ and $\beta = 1$. This result also extends the isotropic case considered in [15], but the proof there is considerably simpler.

Concerning the 1/2-Dini property, it is a standard assumption to get the C^1 regularity of the boundary, see for instance [37, 22]. Notice that one can always apply Theorem A if g is locally bounded and h is locally 1/4-Dini continuous in the first variable (i.e., $\sqrt[4]{\omega_h}$ satisfies the Dini property), since in this case $\eta = 2$ and $\beta = 1/2$, and the required continuity of h and C_{per} follows by the fact that $C_{\mathrm{per}} \lesssim \sqrt{\omega_h}$, already observed above.

Finally, we remark that under quite mild assumption (which broadly cover the Euclidean case) the boundedness of the minimal clusters is known, see for instance [8, 10, 32, 33]. Of course, whenever optimal clusters are bounded, the arcs given by Theorem A become finite and not just locally finite.

1.1. **Meaning of the definition of perimeter for clusters.** This short section is devoted to discuss in detail the definition (1.2) of the perimeter for clusters, which might be a bit obscure at first sight. We do this with the aid of the example in Figure 1, where a 3-cluster is shown. We start by observing that $\mathcal{H}^1$-almost every $x \in \partial^*\mathcal{E}$ belongs to either a single one of the boundaries $\partial^* E_i$, $1 \leq i \leq m$, or to two ones. The reduced boundary of $\cup_{i=1}^m E_i$ is done exactly by the points of $\partial^*\mathcal{E}$ which belong to a single one. Therefore, every point of $\partial^*\mathcal{E}$ is counted exactly twice in the expression (1.2), and this explains the reason of the factor 1/2. It is crucial to understand what happens with the direction of the normal vectors. More precisely, consider a point x which belongs to a single boundary

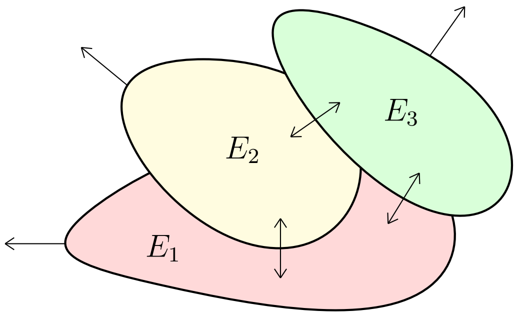

Figure 1. Example of a 3-cluster.

$\partial^* E_i$, and then also to the boundary of $\cup_{i=1}^m E_i$. In this case, the outer normal vector to E_i and to $\cup_{i=1}^m E_i$ at x is the same, call it $\nu(x)$. In the expression (1.2), then, the two times in which the point x is considered give both a contribution $h(x,\nu(x))$, multiplied by $1/2$, thus the infinitesimal perimeter of the cluster produced by the point x is simply $h(x,\nu(x))$. This is consistent with the fact that $\nu(x)$ is "the normal vector to $\mathcal{E}$ at x". Consider, instead, a point x which belongs to $\partial^* E_i$ and to $\partial^* E_j$ for two different indices $1 \leq i,\, j \leq m$. The two normal vectors at x to E_i and to E_j are then opposite, hence there is a vector $\nu(x)$ such that the contribution of x to the perimeter of the cluster is $\frac{1}{2}h(x,\nu(x)) + \frac{1}{2}h(x,-\nu(x))$. This is consistent with the fact that both $\nu(x)$ and $-\nu(x)$ are outer normal vectors at x, either to E_i or to E_j. This also explains why in Figure 1 at some points of $\partial^*\mathcal{E}$ an arrow of length 1 is attached, and at other points two opposite arrows of length $1/2$.

A couple of final comments are now in order. First of all, in the isotropic case, i.e., when h only depends on x, then the contribution of every point $x \in \partial^*\mathcal{E}$ is simply $h(x)$, hence (1.2) can be rewritten in the much simpler form

$$P(\mathcal{E}) = \int_{\partial^*\mathcal{E}} h(x)\, d\mathcal{H}^1(x)\,.$$

There is also an intermediate situation, namely, if the perimeter density is anisotropic but symmetric, that is, $h(x,\nu) = h(x,-\nu)$ for every $x \in \mathbb{R}^2$, $\nu \in \mathbb{S}^1$. Also in this case it is possible to express the perimeter of the cluster $\mathcal{E}$ in a much simpler way than (1.2), that is,

$$P(\mathcal{E}) = \int_{\partial^*\mathcal{E}} h(x,\nu(x))\, d\mathcal{H}^1(x)\,,$$

where the vector $\nu(x)$ is defined as above for every $x \in \partial^*\mathcal{E}$.

We remark that the anisotropic but symmetric case is only slightly more complicate to treat than the isotropic case, and most of the difficulties of the case considered in this paper are due to the asymmetry. Roughly speaking, the big issue in the non symmetric case is that the set E_0 behaves in a different way than the sets E_i with $1 \leq i \leq m$, while in the symmetric case there is locally no difference between the different sets.

We also underline that some authors suggest, for the perimeter of a cluster in the case with densities, to use the definition $P(\mathcal{E}) = \frac{1}{2}\sum_{i=0}^m P(E_i)$ in place of (1.2). This is of course a possible choice. However, in this case the contribution to the perimeter of the cluster given by any point $x \in \partial^*\mathcal{E}$ is always given by $\frac{1}{2}h(x,\nu(x)) + \frac{1}{2}h(x,-\nu(x))$, regardless whether or not x belongs to $\partial^* E_0 = \partial^*\Big(\cup_{i=1}^m E_i\Big)$. This is then completely equivalent to replacing h with the density $\tilde{h}$ given by $\tilde{h}(x,\nu) = (h(x,\nu)+h(x,-\nu))/2$. In other words, with this choice one only has to consider the symmetric case, and as said above this would require a considerably simpler proof for our main result.

2. Proof of the main result

The proof of the main result, Theorem A, is presented in this section. In turn, this is subdivided in five subsections. While the first one collects some standard definitions and technical tools, in the second one we present the basic geometric estimate from which the fact that junction points are necessarily triple points follows. This estimate, which is trivial in the isotropic case, follows by convexity via a suitable first order expansion in general. The third subsection is devoted to show that there are (locally) finitely many junction points, each of which where exactly three different sets meet, and in the fourth one we obtain the regularity. The actual proof of the theorem, presented in the last subsection, basically only consists in putting the different parts together.

Since we aim to prove Theorem A, from now on we assume that h is continuous and that the local η-growth condition holds for some $\eta \geq 1$. Moreover, we assume that $\mathcal{E}$ is a minimal cluster, for which the $\varepsilon - \varepsilon^\beta$ property holds, in such a way that either assumption (i) or (ii) of Theorem A holds.

2.1. Some definitions and technical tools.

Let us fix some notation, that will be used through the rest of the paper. Since we are interested in a local property, in the proof of Theorem A we will immediately start by fixing a big closed ball $D \subseteq \mathbb{R}^2$, and the whole construction will be performed there. Hence, all the following definitions will depend upon D, in particular we assume that $|B(x,r)| \leq C_{\rm vol} r^\eta$ for every ball $B(x,r)$ with $x \in D$ and $r \leq {\rm diam}(D)$.

Since h is continuous, we can call $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ its modulus of continuity in the first variable inside D, that is,

$$\omega(t) := \sup\left\{\Big|h(x,\nu) - h(y,\nu)\Big| :\ \nu \in \mathbb{S}^1,\ x,\ y \in D,\ |y-x| \leq t\right\}.$$

In particular, if h is locally α-Hölder in the first variable, then $\omega(r) \leq Cr^\alpha$ for a suitable constant C. Moreover, we will call $0 < h_{\rm min} \leq h_{\rm max}$ the maximum and the minimum of h in $D \times \mathbb{S}^1$.

Observe that, if assumption (ii) of Theorem A is satisfied, then in particular one can choose the constant $C_{\rm per}$ to be as small as desired, up to decrease the value of $\bar\varepsilon$. As a consequence, we assume that the constant $\bar\varepsilon$ of the $\varepsilon-\varepsilon^\beta$ property is so small that

$$C_{\rm per} < \min\Big\{C^1_{\rm per},\, C^2_{\rm per},\, C^3_{\rm per},\, C^4_{\rm per},\, C^5_{\rm per}\Big\} \qquad \text{if } \eta\beta = 1\,, \tag{2.1}$$

where all the constants $C^i_{\rm per}$ depend solely on h and on D, and are defined in (2.18), (2.26), (2.45), (2.56) and (2.59) respectively.

Lemma 2.1 (Isoperimetric inequality with exponent)**.** *For every set $E\subseteq D$ we have*

$$P(E) \geq \frac{h_{\min}}{C_{\rm vol}^{1/\eta}}\, |E|^{1/\eta}\,.$$

Proof. By standard approximation, it is enough to prove the inequality for a polygonal set $E\subseteq\mathbb{R}^2$. Under this assumption, a classical result from Gustin (see [19]) says that E can be covered with countably many balls $B_i = B(x_i,r_i)$ in such a way that

$$\mathcal{H}^1(\partial^* E) \geq 2\sqrt{2}\sum\nolimits_i r_i\,.$$

If $E\subseteq D$, we can clearly assume all the balls to be such that $x_i\in D$ and $r_i\leq {\rm diam}(D)$ for every i. Then, keeping in mind that $\eta\geq 1$, for any $E\subseteq D$ we deduce

$$\begin{aligned}P(E) &\geq h_{\min}\mathcal{H}^1(\partial^*E) \geq 2\sqrt{2}h_{\min}\sum\nolimits_i r_i \geq \frac{2\sqrt{2}h_{\min}}{C_{\rm vol}^{1/\eta}}\sum\nolimits_i |B(x_i,r_i)|^{1/\eta}\\ &\geq \frac{2\sqrt{2}h_{\min}}{C_{\rm vol}^{1/\eta}}\Big(\sum\nolimits_i |B(x_i,r_i)|\Big)^{1/\eta} \geq \frac{2\sqrt{2}h_{\min}}{C_{\rm vol}^{1/\eta}}\,|E|^{1/\eta}\,,\end{aligned}$$

so the thesis is concluded. ☐

We introduce now the (standard) notation of relative perimeter. Given a set $E\subseteq\mathbb{R}^2$ of locally finite perimeter, or a cluster $\mathcal{E}$, and given a Borel set $A\subseteq\mathbb{R}^2$, the *relative perimeter of E (or $\mathcal{E}$) inside A* is the measure of the boundary of E (or $\mathcal{E}$) within A, i.e.,

$$P(E;A) = \int_{A\cap\partial^*E} h(x,\nu_E(x))\,d\mathcal{H}^1(x)\,,\qquad P(\mathcal{E};A) = \frac{P(\cup_{i=1}^m E_i;A) + \sum_{i=1}^m P(E_i;A)}{2}\,,$$

compare with (1.2).

We conclude this short section by presenting (a very specific case of) a fundamental result due to Vol'pert, see [39] and also [3, Theorem 3.108]).

Theorem 2.2 (Vol'pert)**.** *Let $E\subseteq\mathbb{R}^2$ be a set of locally finite perimeter, and let $x\in\mathbb{R}^2$ be fixed. Then, for a.e. $r>0$, one has that*

$$\partial^*E\cap\partial B(x,r) = \partial^*\Big(E\cap\partial B(x,r)\Big)\,.$$

Notice that, for almost every $r > 0$, both sets in the above equality are done by finitely many points. In particular, $E \cap \partial B(x,r)$ is a subset of the circle $\partial B(x,r)$, and its boundary has to be considered in the 1-dimensional sense. More precisely, for almost every $r > 0$ the set $E \cap \partial B(x,r)$ essentially consists of a finite union of arcs of the circle, and the intersection of $\partial^* E$ with the circle is simply the union of the endpoints of all of them. Through the rest of the paper, we will often consider intersections of sets with balls. Even if this will not be repeated every time, we will always consider balls for which Vol'pert Theorem holds true.

2.2. **The 90° property.** This section is devoted to present a geometric estimate, which is the main reason why junction points are triple points. Let us consider for a moment the Euclidean perimeter, let A, O, B be three points in $\mathbb{R}^2$, and let us assume that $A\widehat{O}B$ is the greatest angle of the triangle AOB. A simple trigonometric computation ensures that the shortest connected set containing the three points is the union of the segments BO and OA if $A\widehat{O}B$ is greater than 120°, while otherwise it is the union of the three segments AP, BP and OP, being P the unique point of the triangle AOB such that the angles $O\widehat{P}B$, $B\widehat{P}A$ and $A\widehat{P}O$ are all 120°, see Figure 2. We can call this the "120° property". As a simple consequence, once one knows that the boundary of a minimal cluster is done by C^1 arcs, it follows that all the junction points must be points where the different arcs meet with angles of 120°, in particular they must be triple points (i.e., three arcs meet).

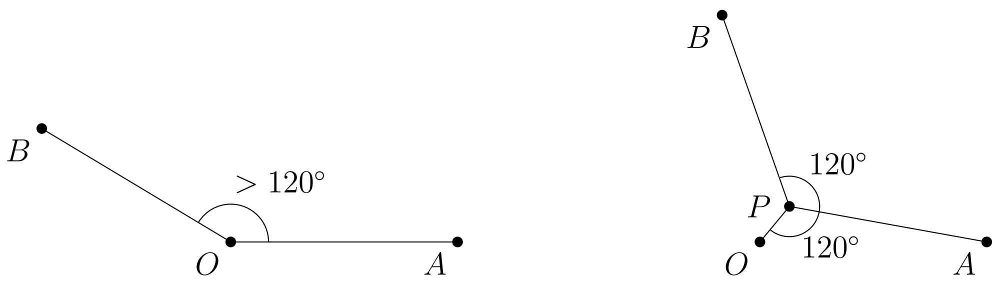

Figure 2. The "120° property". The shortest connected set containing three points A, O, B in two cases.

Let us now pass to consider a general (strictly convex and C^1) density for the perimeter, only depending on the direction, so that the unit ball $\mathcal{C} = \mathcal{C}(x)$ defined in (1.3) is the same for every $x \in \mathbb{R}^2$. As soon as $\mathcal{C}$ is not a Euclidean ball, the 120° property easily fails. Nevertheless, in order to understand whether or not the shortest connected set containing the three points A, O and B is the union of the segments BO and OA, there is still an interesting angle. Namely, the angle between the direction of OA and the tangent direction to $\mathcal{C}$ at the point in direction OB. It can be shown that this angle must be at least 90°, see Step III in the proof of Proposition 2.3. Roughly speaking, this is enough to rule out quadruple points, because they should correspond to four angles of exactly 90°, and in turn this is impossible by the *strict* convexity of the norm. The situation is not really so simple, but this is somehow the underlying idea.

Before giving the claim of the property, it is convenient to recall that we are considering perimeter densities which are not necessarily symmetric (the result below in the case of a symmetric density is much simpler to prove). As discussed in Section 1.1, a consequence of this is that we cannot speak of length of segments, but of length of *oriented* segments. Therefore, if we define $\mathfrak{h} : \mathbb{S}^1 \to \mathbb{R}^+$ as $\mathfrak{h}(\nu) = h(\hat{\nu})$, where for every ν the angle $\hat{\nu}$ is obtained by rotating ν of 90° clockwise, and again we extend $\mathfrak{h}$ to the whole $\mathbb{R}^2$ by positive 1-homogeneity, then for instance the perimeter inside the ball of the cluster depicted in Figure 3 is given by

$$\mathfrak{h}(OA) + \frac{\mathfrak{h}(OB) + \mathfrak{h}(BO)}{2} + \frac{\mathfrak{h}(OC) + \mathfrak{h}(CO)}{2} + \mathfrak{h}(DO) + \mathfrak{h}(OE) + \mathfrak{h}(FO)\,,$$

where for every P, $Q \in \mathbb{R}^2$ we denote by PQ the *oriented* segment pointing from P to Q. We are now in position to state and prove the "90° property".

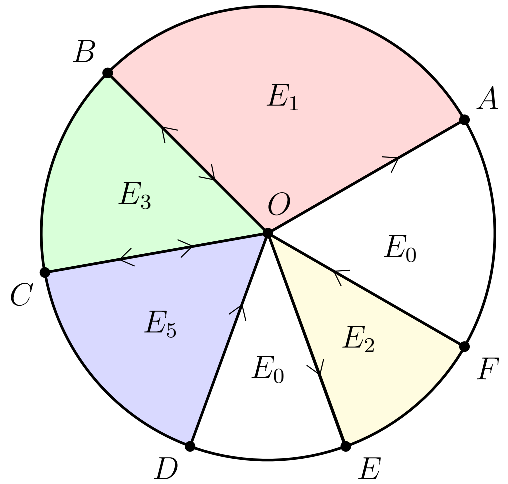

Figure 3. The intersection of a simple cluster with a ball and its perimeter.

Proposition 2.3 (The 90° property)**.** *Let* $\overline{h} : \mathbb{R}^2 \to \mathbb{R}^+$ *be a* C^1*, positively* 1*-homogeneous function, strictly positive except at* 0 *and with strictly convex unit ball, and let us denote by* $\overline{P}$ *the perimeter obtained by substituting* $h(x,\nu)$ *with* $\overline{h}(\nu)$ *in (1.1). There exists* $\delta > 0$ *such that the following is true. Let* $\mathcal{E}' \subseteq \mathbb{R}^2$ *be a cluster whose boundary, inside the unit ball* $B(0,1)$*, is done by a finite number of radii of the ball. If these radii are more than three, then there exists another cluster* $\mathcal{F} \subseteq \mathbb{R}^2$*, coinciding with* $\mathcal{E}'$ *outside the ball* $B(0,1)$*, such that*

$$\overline{P}(\mathcal{F}) \leq \overline{P}(\mathcal{E}') - \delta\,. \tag{2.2}$$

Proof. We will call for simplicity "slice" each of the sectors of the ball $B(0,1)$ having two consecutive radii of $\partial\mathcal{E}'$ in the boundary. We will say that a slice is "white" if it is contained in E_0, otherwise we will say that it is "coloured". As already done before, for every $\nu \in \mathbb{S}^1$ we call $\mathfrak{h}(\nu) = \overline{h}(\hat\nu)$, being $\hat\nu$ the angle obtained rotating ν of 90° clockwise, and we call $\tilde{\mathfrak{h}}$ the "symmetrized version" of $\mathfrak{h}$, that is $\tilde{\mathfrak{h}}(\nu) = (\mathfrak{h}(\nu) + \mathfrak{h}(-\nu))/2$. We let $K > 0$ be a number such that

$$\frac{1}{K} \le \overline{h}(\nu) \le K \qquad \forall \nu \in \mathbb{S}^1 . \tag{2.3}$$

We assume then that there are at least four slices, and we look for a cluster $\mathcal{F}$ satisfying (2.2). The proof is divided for clarity in a few steps.

<u>Step I.</u> *The minimal angle $\theta_{\rm min}$.*
First of all, we observe that the thesis is true if one of the angles is too small, that is, there exist $\theta_{\rm min} > 0$ and $\delta_1 > 0$ such that a cluster $\mathcal{F}$ satisfying (2.2) with δ_1 in place of δ can be found if one of the angles between the radii is less than $\theta_{\rm min}$. In fact, let P, Q be two consecutive points of $\partial^*\mathcal{E}' \cap \partial B(0,1)$, making with the origin a small angle θ, being Q slightly after P in the counterclockwise sense. Let us consider the three slices around the two radii OP and OQ. There are five possibilities: either the three slices are all coloured; or only the external slice having OP in the boundary is white; or the internal slice is white; or only the external slice having OQ in the boundary is white; or both the external ones are white. In the first three cases, we let $\mathcal{F}$ be the unique cluster such that

$$\mathcal{F} = \mathcal{E}' \text{ in } \mathbb{R}^2 \setminus B(0,1) , \qquad\qquad \partial\mathcal{F} = \Big(\partial\mathcal{E}' \setminus OQ\Big) \cup PQ .$$

Instead, in the fourth case we let $\mathcal{F}$ be the cluster so that

$$\mathcal{F} = \mathcal{E}' \text{ in } \mathbb{R}^2 \setminus B(0,1) , \qquad\qquad \partial\mathcal{F} = \Big(\partial\mathcal{E}' \setminus OP\Big) \cup PQ ,$$

and in the last case $\mathcal{F}$ is the cluster such that

$$\mathcal{F} = \mathcal{E}' \text{ in } \mathbb{R}^2 \setminus B(0,1) , \qquad\qquad \partial\mathcal{F} = \Big(\partial\mathcal{E}' \setminus (OQ \cup OP)\Big) \cup PQ .$$

Keeping in mind the definition of perimeter and of $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$, as well as (2.3), as soon as θ is small enough, only depending on K, in the first two cases we have

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \tilde{\mathfrak{h}}(PQ) - \tilde{\mathfrak{h}}(OQ) \le 2K \sin(\theta/2) - \frac{1}{K} \le -\frac{1}{2K} .$$

Analogously, in the fourth case we have

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \tilde{\mathfrak{h}}(PQ) - \tilde{\mathfrak{h}}(OP) \le 2K \sin(\theta/2) - \frac{1}{K} \le -\frac{1}{2K} .$$

In the fifth case, we have

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \mathfrak{h}(QP) - \mathfrak{h}(QO) - \mathfrak{h}(OP) \le 2K \sin(\theta/2) - \frac{2}{K} \le -\frac{1}{2K} .$$

Let us finally consider the third case. In this case, if the two external slices have "different colours", i.e. they belong to two different sets E_i, then

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \mathfrak{h}(PQ) - \mathfrak{h}(OQ) + \tilde{\mathfrak{h}}(PO) - \mathfrak{h}(PO)\,,$$

while if they have the same colour, then

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \mathfrak{h}(PQ) - \mathfrak{h}(OQ) - \mathfrak{h}(PO)\,.$$

We have then

$$\begin{aligned}\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') &\leq \mathfrak{h}(PQ) - \mathfrak{h}(OQ) + \tilde{\mathfrak{h}}(PO) - \mathfrak{h}(PO)\\ &= \mathfrak{h}(PQ) - \mathfrak{h}(OQ) + \frac{\mathfrak{h}(OP)}{2} - \frac{\mathfrak{h}(PO)}{2}\\ &\leq 2K\sin(\theta/2) - \mathfrak{h}(OQ) + \frac{\mathfrak{h}(OP)}{2} - \frac{1}{2K} \leq 2K\sin(\theta/2) - \frac{1}{2K} \leq -\frac{1}{3K}\,,\end{aligned}$$

where the second last inequality, namely $\mathfrak{h}(OP) \leq 2\mathfrak{h}(OQ)$, is true by continuity of $\overline{h}$ as soon as θ is small enough. Summarizing, the existence of $\theta_{\min}$ and δ_1 as claimed follows, and this step is concluded.

In the next steps we will show that, for any cluster $\mathcal{E}'$ as in the claim, with at least four slices, and with all the radii making angles larger than $\theta_{\min}$, there exists some cluster $\mathcal{F}$, coinciding with $\mathcal{E}'$ outside of the unit ball, such that $\overline{P}(\mathcal{F}) < \overline{P}(\mathcal{E}')$. In particular, since the points of $\partial^*\mathcal{E}'$ in $\partial B(0,1)$ are at most $2\pi/\theta_{\min}$, by continuity of $\overline{h}$ and compactness of $\mathbb{S}^1$ there must be a constant $\delta_2 > 0$, only depending on $\overline{h}$, such that $\overline{P}(\mathcal{F}) \leq \overline{P}(\mathcal{E}') - \delta_2$. Together with Step I, this will then clearly conclude the proof, with $\delta = \min\{\delta_1, \delta_2\}$.

Step II. Proof with more than a white slice.
We now show the thesis if there are at least two white slices. In fact, in this case, we can take two white slices, corresponding to two arcs with endpoints P, Q, and R, S respectively, in such a way that $S\hat{O}P < \pi$, and the angle $S\hat{O}P$ corresponds to a sector of circle which does not intersect E_0 (see Figure 4). We define then $\mathcal{F}$ by "joining" the two

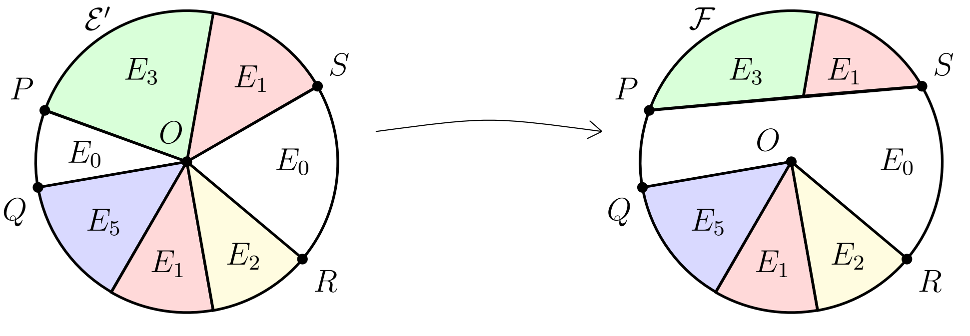

Figure 4. The situation in Step II.

white slices as in the Figure. Notice that $\partial\mathcal{F}$ is obtained by $\partial\mathcal{E}'$ removing the radii PO

and OS and adding the chord PS, and by correspondingly shortening the radii contained in the sector. As a consequence,

$$\overline{P}(\mathcal{F}) \leq \overline{P}(\mathcal{E}') - \mathfrak{h}(PO) - \mathfrak{h}(OS) + \mathfrak{h}(PS)\,, \tag{2.4}$$

where the inequality is strict if and only if the sector contains more than a single slice, as in the example depicted in the Figure. Since $\overline{h}$ is strictly convex in the sense of Definition 1.1, and PO and OS are not parallel because $S\hat{O}P < \pi$, we have

$$\mathfrak{h}(PS) = 2\mathfrak{h}\left(\frac{PO + OS}{2}\right) < \mathfrak{h}(PO) + \mathfrak{h}(OS)\,,$$

thus by (2.4) we obtain $\overline{P}(\mathcal{F}) < \overline{P}(\mathcal{E}')$ and this step is concluded.

<u>Step III.</u> *Proof with an angle $A\hat{O}C < \pi$ containing a single radius, between two coloured slices.*

The next step consists in proving the thesis if there are three consecutive radii, AO, BO and CO, in such a way that both the slices between them are coloured, and that $A\hat{O}C < \pi$. The situation is depicted in Figure 5, left, where the two slices are denoted by E_1 and

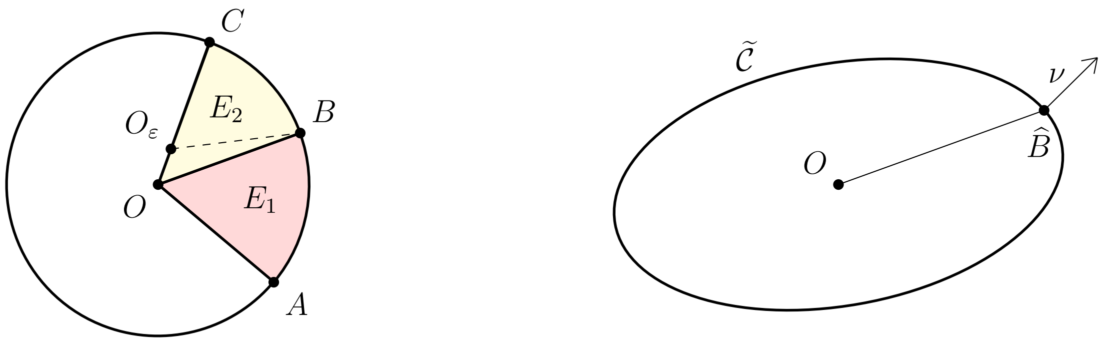

Figure 5. The situation in Step III.

E_2 just to fix the ideas. Without loss of generality we may assume that, as in the figure, the points are ordered from A to C in the counterclockwise sense. Outside of the sector AOC there can be other radii, not depicted in the figure. Let us denote by $\tilde{\mathcal{C}}$ the unit ball corresponding to $\tilde{\mathfrak{h}}$. Calling ν, as in Figure 5, right, the outer normal to $\tilde{\mathcal{C}}$ at $\hat{B} = B/\tilde{\mathfrak{h}}(OB)$, for any direction $\eta \in \mathbb{S}^1$ one has

$$\tilde{\mathfrak{h}}(O\hat{B} + \varepsilon\eta) = \tilde{\mathfrak{h}}\Big(O\hat{B} + \varepsilon(\eta\cdot\nu)\nu\Big) + o(\varepsilon) = 1 + \varepsilon\tilde{\mathfrak{h}}(OB)\,\frac{\eta\cdot\nu}{OB\cdot\nu} + o(\varepsilon) \tag{2.5}$$

for $|\varepsilon| \ll 1$. Set then, as in the figure, $O_\varepsilon = \varepsilon C$ for some small $\varepsilon > 0$, and consider the cluster $\mathcal{F}$ obtained from $\mathcal{E}'$ by substituting the radius OB with the segment $O_\varepsilon B$. Notice that the difference $\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}')$ is only determined by the different contribution of OB and $O_\varepsilon B$; indeed, the segment OO_ε contributes to $\overline{P}(\mathcal{F})$ exactly as to $\overline{P}(\mathcal{E}')$, because on one side of the segment nothing happens, while on the other side the set E_2 is replaced by E_1, and this does not make any difference since both are coloured. Therefore,

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \tilde{\mathfrak{h}}(O_\varepsilon B) - \tilde{\mathfrak{h}}(OB) = \tilde{\mathfrak{h}}(OB - \varepsilon OC) - \tilde{\mathfrak{h}}(OB)\,.$$

Keeping in mind the first order expansion (2.5), and observing that $OB\cdot\nu > 0$ by convexity of $\widetilde{\mathcal{C}}$, we derive that $\overline{P}(\mathcal{F}) < \overline{P}(\mathcal{E}')$ for $0 < \varepsilon \ll 1$ if $OC\cdot\nu > 0$. The step is then concluded in this case. Since we can perform the same argument with the segment OA in place of OC, the step is proved unless

$$OA\cdot\nu \leq 0\,, \qquad\qquad OC\cdot\nu \leq 0\,. \tag{2.6}$$

And in turn, we can observe that (2.6) is impossible. Indeed, the set $\{\eta \in \mathbb{R}^2 : \eta\cdot\nu \leq 0\}$ is a half-space. And since $A\widehat{O}C < \pi$, if this half-space contains both OA and OC then it must contain also OB, while as already observed $OB\cdot\nu > 0$.

<u>Step IV.</u> *Proof with an angle $B\widehat{O}D < \pi$ containing a single radius.*
The next step consists in proving the thesis if there are two consecutive slices making together an angle strictly less than π. Notice that this is exactly what we have done in Step III, except for the fact that we assumed there both slices to be coloured. In this step we have then only to consider the case when one of the two slices is white. We assume the radii to be BO, CO and DO, and without loss of generality we assume the points B, C and D to be ordered in the counterclockwise sense, and the slice between the radii BO and CO to be the white one, as in Figure 6.

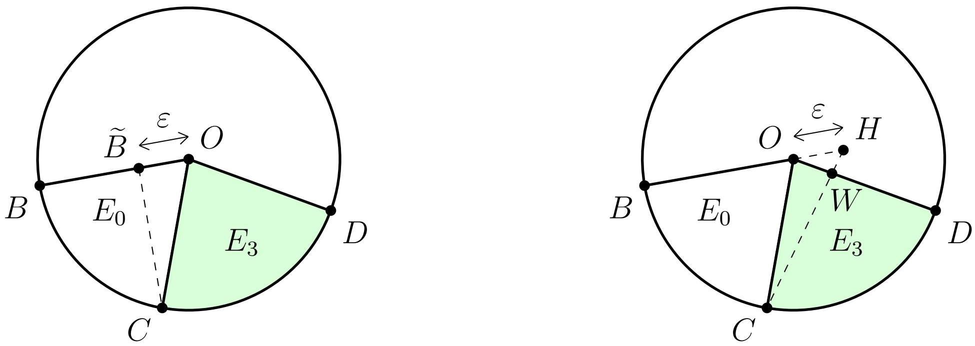

Figure 6. The situation in Step IV.

We define first a possible competitor $\mathcal{F}$ as in Figure 6, left. Namely, for a small, positive ε we define $\widetilde{B} = \varepsilon B$ and we let $\mathcal{F}$ be the cluster obtained by $\mathcal{E}'$ substituting the radius OC with the segment $\widetilde{B}C$. This time, the difference between $\overline{P}(\mathcal{E}')$ and $\overline{P}(\mathcal{F})$ is given not only by the different contribution of OC and $\widetilde{B}C$, but also by the fact that the small segment $O\widetilde{B}$ is between a white and a coloured slice in $\mathcal{E}'$, while it is between two coloured slices in $\mathcal{F}$. Actually, the segment $O\widetilde{B}$ is not even in $\partial^*\mathcal{F}$ if the coloured slice on the other side of OB has the same colour as the slice of the sector COD. Therefore,

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') \leq \mathfrak{h}(\widetilde{B}C) - \mathfrak{h}(OC) + \frac{\mathfrak{h}(O\widetilde{B}) - \mathfrak{h}(\widetilde{B}O)}{2}, \tag{2.7}$$

and the inequality is strict if and only if the coloured slice on the other side of OB and the slice of the sector COD have the same colour. There is a constant $\kappa \in \mathbb{R}$ such that

$$\mathfrak{h}(\widetilde{B}C) - \mathfrak{h}(OC) = \kappa\varepsilon + o(\varepsilon)$$

(the exact value of κ can be found as in (2.5), but in this step this is not important). Hence, from (2.7) we get

$$\overline{P}(\mathcal{F}) \leq \overline{P}(\mathcal{E}') + \varepsilon\left(\kappa + \frac{\mathfrak{h}(OB) - \mathfrak{h}(BO)}{2}\right) + o(\varepsilon),$$

so that the competitor $\mathcal{F}$ concludes the proof in this case unless

$$\kappa + \frac{\mathfrak{h}(OB) - \mathfrak{h}(BO)}{2} \geq 0. \tag{2.8}$$

Let us then assume that this last inequality holds true, and let us define a different competitor, as in Figure 6, right. More precisely, again for a small positive ε we define $H = -\varepsilon B$, and we let W be the point of intersection between the segments HC and OD. The cluster $\mathcal{F}$ is then obtained by substituting the radius OC with the segment WC. Arguing as before, and keeping in mind that the slice on the other side of OD is surely coloured by Step II, we have this time

$$\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') = \mathfrak{h}(WC) - \mathfrak{h}(OC) + \frac{\mathfrak{h}(OW) - \mathfrak{h}(WO)}{2}. \tag{2.9}$$

Notice that

$$\mathfrak{h}(WC) - \mathfrak{h}(OC) = \mathfrak{h}(HC) - \mathfrak{h}(OC) - \mathfrak{h}(HW) = -\kappa\varepsilon - \mathfrak{h}(HW) + o(\varepsilon),$$

so by (2.9) we get the thesis with some small $\varepsilon > 0$ if

$$\lim_{\varepsilon \searrow 0} \left(\kappa + \frac{\mathfrak{h}(HW)}{\varepsilon} + \frac{\mathfrak{h}(WO) - \mathfrak{h}(OW)}{2\varepsilon}\right) > 0,$$

which in turn, thanks to (2.8), is surely true if

$$\lim_{\varepsilon \searrow 0} \frac{2\mathfrak{h}(HW) + \mathfrak{h}(WO) - \mathfrak{h}(OW) + \mathfrak{h}(OH) - \mathfrak{h}(HO)}{\varepsilon} > 0. \tag{2.10}$$

Consider now the triangle WOH and observe that, by elementary geometric relations,

$$W\widehat{O}H = D\widehat{O}H = \pi - B\widehat{O}D, \quad H\widehat{W}O = C\widehat{W}D > C\widehat{O}D, \quad O\widehat{H}W = B\widehat{H}C > \frac{B\widehat{O}C}{2}.$$

Hence, the three angles of the triangle WOH depend on ε, but they are all greater than a strictly positive constant which does not depend on ε. Since $\overline{h}$ is strictly convex, there exists then a constant $\delta_3 > 0$ such that

$$\mathfrak{h}(HW) + \mathfrak{h}(WO) \geq (1+\delta_3)\mathfrak{h}(HO)\,, \qquad \mathfrak{h}(OH) + \mathfrak{h}(HW) \geq (1+\delta_3)\mathfrak{h}(OW)\,.$$

We deduce

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\frac{2\mathfrak{h}(HW)+\mathfrak{h}(WO)-\mathfrak{h}(OW)+\mathfrak{h}(OH)-\mathfrak{h}(HO)}{\varepsilon} &\geq \lim_{\varepsilon\searrow 0}\frac{\delta_3\Big(\mathfrak{h}(HO)+\mathfrak{h}(OW)\Big)}{\varepsilon}\\ &\geq \lim_{\varepsilon\searrow 0}\frac{\delta_3\mathfrak{h}(HO)}{\varepsilon} = \delta_3\mathfrak{h}(OB) > 0\,,\end{aligned}$$

so (2.10) is established and the proof follows also in this case.

Step V. Conclusion.
We are now ready to conclude the thesis. By Step III and Step IV, the only case which is left open is when there are exactly four radii, say OA, OB, OC and OD, with the points A, B, C, D ordered in the counterclockwise sense, and $A\widehat{O}C = B\widehat{O}D = \pi$, as in Figure 7, left. Since by Step II there can be at most one white slice, we assume that the slices corresponding to the sectors AOB, BOC and COD are coloured. We are going

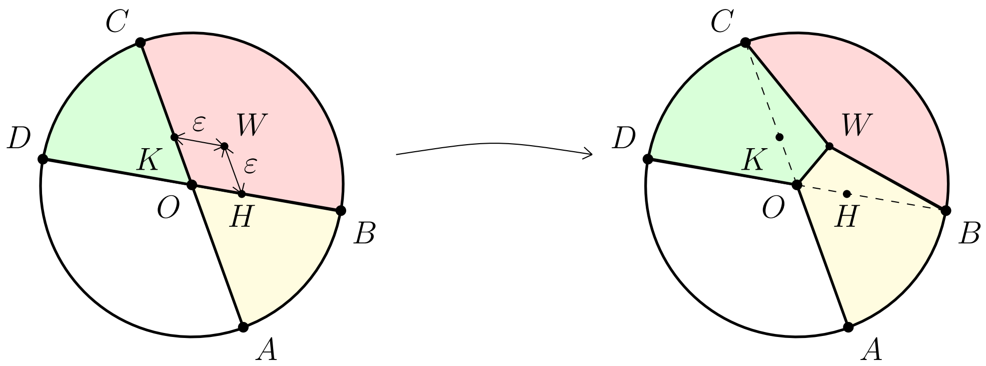

Figure 7. The situation in Step V.

to use only the fact that these slices are coloured, the fact whether or not so is also the slice DOA does not play any role. As in Step III, let us call ν the direction of the outer normal at $B/\tilde{\mathfrak{h}}(OB)$ to $\widetilde{\mathcal{C}} = \{\tilde{\mathfrak{h}} \leq 1\}$. Since both the sectors AOB and BOC are coloured, Step III already gives the proof unless (2.6) holds. As noticed in Step III, (2.6) is in fact impossible if $A\widehat{O}C < \pi$, and if $A\widehat{O}C = \pi$ it holds only if $OA \cdot \nu = OC \cdot \nu = 0$, which by the first order expansion (2.5) implies that

$$\tilde{\mathfrak{h}}(OB - \varepsilon OC) = \tilde{\mathfrak{h}}(OB) + o(\varepsilon)\,. \tag{2.11}$$

Repeating the same argument in the union of the sectors BOC and COD, which are also both coloured and correspond to the angle $B\widehat{O}D = \pi$, we get the thesis unless

$$\tilde{\mathfrak{h}}(OC - \varepsilon OB) = \tilde{\mathfrak{h}}(OC) + o(\varepsilon)\,. \tag{2.12}$$

To conclude, we have then only to find a suitable competitor under the assumption that (2.11) and (2.12) hold. In this final case, as in Figure 7, right, we call $H = \varepsilon B$, $K = \varepsilon C$ and $W = \varepsilon(B + C) = H + K$, and we define the cluster $\mathcal{F}$ substituting in $\partial\mathcal{E}'$ the radii OB and OC with the three segments OW, WB and WC. By construction, each of these segments in $\partial\mathcal{F}$ is between two coloured slices, thus by (2.11) and (2.12) we get

$$\begin{aligned}
\overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}') &= \tilde{\mathfrak{h}}(OW) + \tilde{\mathfrak{h}}(WB) + \tilde{\mathfrak{h}}(WC) - \tilde{\mathfrak{h}}(OB) - \tilde{\mathfrak{h}}(OC) \\
&= \tilde{\mathfrak{h}}(OW) + \tilde{\mathfrak{h}}(HB) + \tilde{\mathfrak{h}}(KC) - \tilde{\mathfrak{h}}(OB) - \tilde{\mathfrak{h}}(OC) + o(\varepsilon) \\
&= \varepsilon\Big(\tilde{\mathfrak{h}}(OB + OC) - \tilde{\mathfrak{h}}(OB) - \tilde{\mathfrak{h}}(OC)\Big) + o(\varepsilon)\,,
\end{aligned}$$

and by the strict convexity of $\overline{h}$ we deduce $\overline{P}(\mathcal{F}) < \overline{P}(\mathcal{E}')$ for some small, positive ε. The proof is then concluded. □

Remark 2.4. *It is important to observe that the constant $\delta = \delta(\overline{h})$ in the above proposition only depends on the norm $\overline{h}$. By continuity of h, we can then fix a constant $\delta > 0$, depending only on h and on D, such that $\delta \leq \delta(\overline{h})$ for every $\overline{h}$ of the form $\overline{h}(v) = h(x, v)$ for some $x \in D$. We will apply Proposition 2.3 with such a choice.*

We conclude this section by presenting a simple observation and an important consequence.

Lemma 2.5. *Let $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}^+$ be a convex and positively 1-homogeneous function, and for every path $\gamma : [0, 1] \to \mathbb{R}^2$ of finite length let us call* $\mathrm{len}(\gamma)$ *the "length of γ" defined by*

$$\mathrm{len}(\gamma) = \int_0^1 \mathfrak{h}(\gamma'(\sigma))\, d\sigma\,. \tag{2.13}$$

For any such path γ, then, one has

$$\mathrm{len}(\gamma) \geq \mathrm{len}(\tilde{\gamma})\,,$$

where $\tilde{\gamma} : [0, 1] \to \mathbb{R}^2$ is the affine path connecting $\gamma(0)$ with $\gamma(1)$.

Proof. This is a direct application of Jensen lemma,

$$\mathrm{len}(\gamma) = \int_0^1 \mathfrak{h}(\gamma'(\sigma))\, d\sigma \geq \mathfrak{h}\bigg(\int_0^1 \gamma'(\sigma)\, d\sigma\bigg) = \mathfrak{h}\Big(\gamma(1) - \gamma(0)\Big) = \mathrm{len}(\tilde{\gamma})\,.$$

□

Corollary 2.6. *Let* $\mathfrak{h}$ *and* len *be as in Lemma 2.5, and let* $\tau_1, \tau_2 : [0,1] \to \mathbb{R}^2$ *be two injective paths of finite length which have no intersection except the points* $P = \tau_1(0) = \tau_2(1)$ *and* $Q = \tau_1(1) = \tau_2(0)$*. Then, there exists an injective path* $\tau : [0,1] \to \mathbb{R}^2$ *with* $\tau(0) = P$ *and* $\tau(1) = Q$*, which is entirely contained in the (closed) region enclosed by* $\tau_1 \cup \tau_2$*, and such that, setting* $\hat\tau : [0,1] \to \mathbb{R}^2$ *as* $\hat\tau(t) = \tau(1-t)$*, one has*

$$\mathrm{len}(\tau) + \mathrm{len}(\hat\tau) \leq \mathrm{len}(\tau_1) + \mathrm{len}(\tau_2)\,. \tag{2.14}$$

Proof. By approximation, we can assume that the paths τ_1 and τ_2 are done by finitely many linear pieces, so that the region enclosed by $\tau_1 \cup \tau_2$ is a closed polygon $\mathcal{P}$. In addition, we can also assume that *every* couple of vertices of $\mathcal{P}$ (not only the couples of consecutive vertices) corresponds to a different direction, so in particular there are no three aligned vertices. We argue then by induction on the number N of sides of $\mathcal{P}$.

If $N = 3$, then necessarily one of the paths, say τ_1, is simply the segment PQ, and the other path is done by two linear pieces, say QB and BP. In this case, it is enough to call τ the segment between P and Q, and then (2.14) is obvious by Lemma 2.5, since

$$\mathrm{len}(\tau_1)+\mathrm{len}(\tau_2) = \mathrm{len}(PQ)+\mathrm{len}(QB)+\mathrm{len}(BP) \geq \mathrm{len}(PQ)+\mathrm{len}(QP) = \mathrm{len}(\tau)+\mathrm{len}(\hat\tau)\,.$$

Let us then assume that $N \geq 4$ and that the claim has been proven for all the polygons with strictly less sides than N. If there are two vertices B, D in τ_1 such that the open segment BD is contained in the interior of $\mathcal{P}$, then we can call $\tilde\tau_1$ the path obtained by τ_1 by substituting the whole part between B and D with the segment BD, and $\tilde\tau_2 = \tau_2$. The resulting polygon $\widetilde{\mathcal{P}}$ is contained in $\mathcal{P}$ and has strictly less than N vertices. By assumption, we find then a path τ as in the claim for the polygon $\widetilde{\mathcal{P}}$. The path τ is contained in $\widetilde{\mathcal{P}}$, hence in $\mathcal{P}$, and (2.14) holds true since, again by Lemma 2.5,

$$\mathrm{len}(\tau) + \mathrm{len}(\hat\tau) \leq \mathrm{len}(\tilde\tau_1) + \mathrm{len}(\tilde\tau_2) \leq \mathrm{len}(\tau_1) + \mathrm{len}(\tau_2)\,.$$

In the same way we argue if the two vertices B, D belong to τ_2.

Let us finally assume that there are no such vertices. A possible situation is depicted in Figure 8. We can take three consecutive vertices A, B, C in one of the paths, say τ_1, such that the angle at B is less than π (this is clearly possible since every polygon has at least three angles less than π, so at least one vertex of $\mathcal{P}$ different from P and Q corresponds to an angle less than π). Since the open segment AC cannot be contained in the interior of $\mathcal{P}$ by assumption, the triangle ABC contains other vertices of the polygon. In particular, there are two points $\widetilde{A}$ and $\widetilde{C}$ in AB and BC respectively, such that the open segment $\widetilde{A}\widetilde{C}$ is parallel to AC and intersects $\partial\mathcal{P}$ exactly at one point, say D. The point D is necessarily a vertex of the polygon, and the open segment BD is contained in the interior of $\mathcal{P}$, thus by assumption D must be contained in τ_2. Let us then call τ_1^1 the path obtained by taking the part of τ_1 between P and $\widetilde{A}$ and adding the segment $\widetilde{A}D$, and

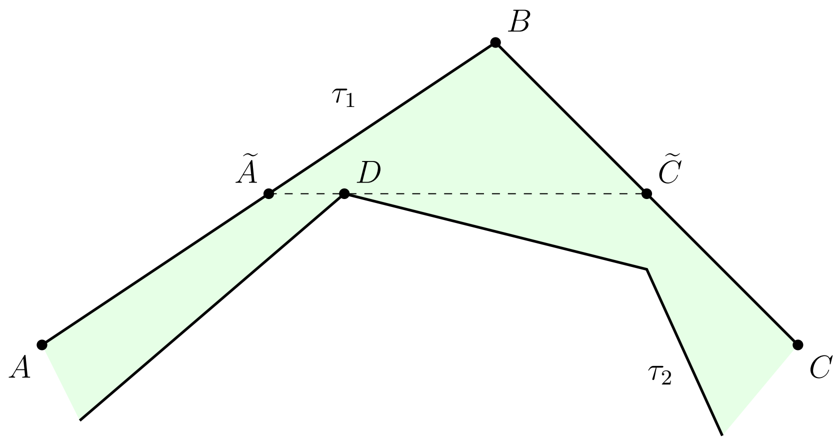

Figure 8. A possible situation in Corollary 2.6.

τ_1^2 the segment $D\widetilde{C}$ together with the part of τ_1 between $\widetilde{C}$ and Q. Moreover, subdivide τ_2 in the part τ_2^2 between Q and D, and the part τ_2^1 between D and P. The paths τ_1^1 and τ_2^1 enclose a polygon $\mathcal{P}^1$, while τ_1^2 and τ_2^2 enclose $\mathcal{P}^2$. Both polygons are contained in $\mathcal{P}$ and have strictly less sides than $\mathcal{P}$, thus by inductive assumption we obtain a path τ^1 in $\mathcal{P}^1$ between P and D and a path τ^2 in $\mathcal{P}^2$ beween D and Q, which satisfy the inequalities analogous to (2.14). The path τ obtained putting together τ^1 and τ^2 is then a path in $\mathcal{P}^1 \cup \mathcal{P}^2 \subseteq \mathcal{P}$, and satisfies (2.14) since applying once again Lemma 2.5 we have

$$\begin{aligned}\operatorname{len}(\tau) + \operatorname{len}(\hat{\tau}) &= \operatorname{len}(\tau^1) + \operatorname{len}(\widehat{\tau^1}) + \operatorname{len}(\tau^2) + \operatorname{len}(\widehat{\tau^2}) \\ &\leq \operatorname{len}(\tau_1^1) + \operatorname{len}(\tau_2^1) + \operatorname{len}(\tau_1^2) + \operatorname{len}(\tau_2^2) \leq \operatorname{len}(\tau_1) + \operatorname{len}(\tau_2)\,.\end{aligned}$$

□

2.3. **Finitely many triple points.** We now start our construction for proving Theorem A. Through this section and the following one, $\mathcal{E}$ is a fixed, minimal cluster, satisfying the assumptions of Theorem A, and D is a fixed, closed ball. The aim of this section is to show several preliminary properties of $\mathcal{E}$, eventually establishing that $\partial^*\mathcal{E}$ only admits (in D) finitely many junction points, and all of them are triple points. This will be obtained in Lemma 2.17.

We set $R_1 = \min\{R_\beta, R_\eta\}$. In the following, we will define several different values of R_i with $R_1 \geq R_2 \geq R_3 \cdots$. Each of these constants will only depend on $\mathcal{E}$, D, g and h.

Our first result is a simple observation following from the $\varepsilon - \varepsilon^\beta$ property and the η-growth condition, that one can use to build competitors. We will use it several times in the sequel.

Lemma 2.7 (Small ball competitor)**.** *Let $B(x,r) \subseteq D$ be a ball such that $|B(x,r)| < \bar{\varepsilon}/2$ and $r < R_1$, let $\mathcal{E}$, $\mathcal{E}'$ be any two clusters which coincide outside $B(x,r)$, and call $\varepsilon = |\mathcal{E}| - |\mathcal{E}'|$. There exists another cluster $\mathcal{E}''$ such that $|\mathcal{E}''| = |\mathcal{E}|$, $\mathcal{E}'' \cap B(x,r) = \mathcal{E}' \cap B(x,r)$*

and

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}|\varepsilon|^\beta \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol}r^\eta)^\beta\,. \tag{2.15}$$

This inequality is actually true also with $C_{\rm per}[|\varepsilon|]$ *in place of* $C_{\rm per}$.

Proof. We start by noticing that

$$|\varepsilon| \leq \sum_{i=1}^m |\varepsilon_i| \leq 2|B(x,r)| < \bar\varepsilon\,,$$

hence we can apply the $\varepsilon - \varepsilon^\beta$ property to $\mathcal{E}$ with constant ε and point x. Hence, there is another cluster $\mathcal{F}$ such that $\mathcal{F} = \mathcal{E}$ inside $B(x,R_\beta) \supseteq B(x,r)$, and moreover $|\mathcal{F}| = |\mathcal{E}| + \varepsilon$ and $P(\mathcal{F}) \leq P(\mathcal{E}) + C_{\rm per}[|\varepsilon|]|\varepsilon|^\beta$. We define then the cluster $\mathcal{E}''$ as the cluster which coincides with $\mathcal{E}'$ inside $B(x,r)$, and with $\mathcal{F}$ outside of $B(x,r)$. Its volume is then

$$\begin{aligned}|\mathcal{E}''| &= |\mathcal{E}' \cap B(x,r)| + |\mathcal{F} \setminus B(x,r)|\\ &= |\mathcal{E} \cap B(x,r)| + |\mathcal{E}'| - |\mathcal{E}| + |\mathcal{E} \setminus B(x,r)| + |\mathcal{F}| - |\mathcal{E}| = |\mathcal{E}|\,.\end{aligned}$$

Keeping in mind the growth condition, we have

$$|\varepsilon| \leq 2|B(x,r)| \leq 2C_{\rm vol}r^\eta\,.$$

As a consequence, the perimeter of $\mathcal{E}$ can be evaluated as

$$\begin{aligned}P(\mathcal{E}'') &= P(\mathcal{E}'; B(x,r)) + P\Big(\mathcal{F}; \mathbb{R}^2 \setminus B(x,r)\Big)\\ &= P(\mathcal{E}; B(x,r)) + P(\mathcal{E}') - P(\mathcal{E}) + P\Big(\mathcal{E}; \mathbb{R}^2 \setminus B(x,r)\Big) + P(\mathcal{F}) - P(\mathcal{E})\\ &\leq P(\mathcal{E}') + C_{\rm per}[|\varepsilon|]|\varepsilon|^\beta \leq P(\mathcal{E}') + C_{\rm per}[|\varepsilon|]\Big(2C_{\rm vol}r^\eta\Big)^\beta\,.\end{aligned}$$

Keeping in mind that $C_{\rm per}[|\varepsilon|] \leq C_{\rm per}[\bar\varepsilon] \leq C_{\rm per}$, the proof is then concluded. □

Lemma 2.8 (Perimeter in a ball is controlled by radius)**.** *There exists a constant* $R_2 \leq R_1$ *such that, for every* $B(x,r) \subseteq D$ *with* $r < R_2$*, one has*

$$\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) < 7\,\frac{h_{\max}}{h_{\min}}\,r\,. \tag{2.16}$$

Proof. We let $R_2 \leq R_1$ be so small that

$$C_{\rm vol}R_2^\eta < \frac{\bar\varepsilon}{2}\,, \qquad\qquad C_{\rm per}\Big(2C_{\rm vol}R_2^\eta\Big)^\beta < \frac{h_{\max}}{2}\,R_2\,. \tag{2.17}$$

Notice that the first inequality is true for every R_2 small enough. The same is true for the second one if $\eta\beta > 1$. Instead, if $\eta\beta = 1$, the second inequality is true regardless of the value of R_2 thanks to (2.1) as soon as we define

$$C^1_{\rm per} = \frac{h_{\max}}{2^{\beta+1}C_{\rm vol}^\beta}\,. \tag{2.18}$$

Take now $r < R_2$ and $x \in \mathbb{R}^2$ as in the claim. Let us define the cluster $\mathcal{E}'$ by setting $E_1' = E_1 \cup B(x,r)$ and $E_i' = E_i \setminus B(x,r)$ for every $2 \leq i \leq m$. Clearly

$$P(\mathcal{E}') \leq P(\mathcal{E}) - h_{\min}\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) + 2\pi r h_{\max} \,. \tag{2.19}$$

Let us call $\varepsilon \in \mathbb{R}^m$ the vector given by $\varepsilon_i = |E_i \cap B(x,r)|$ for every $2 \leq i \leq m$, and $\varepsilon_1 = -|B(x,r) \setminus E_1|$, so that $|\mathcal{E}| = |\mathcal{E}'| + \varepsilon$. Notice that $|B(x,r)| \leq C_{\rm vol} r^\eta < \bar\varepsilon/2$ by the first property in (2.17). Hence, we can apply Lemma 2.7 to get another cluster $\mathcal{E}''$ satisfying (2.15), so that

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol}r^\eta)^\beta < P(\mathcal{E}') + \frac{h_{\max}}{2}\, r$$

by the second property in (2.17), which is clearly valid with every $r < R_2$ in place of R_2. Putting this estimate together with (2.19), and recalling that $P(\mathcal{E}) \leq P(\mathcal{E}'')$ by minimality of $\mathcal{E}$ and since $|\mathcal{E}''| = |\mathcal{E}|$ by Lemma 2.7, we get

$$\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) \leq 2\pi r\,\frac{h_{\max}}{h_{\min}} + \frac{h_{\max}}{2h_{\min}}\, r < 7\,\frac{h_{\max}}{h_{\min}}\, r \,,$$

hence the proof is concluded. □

We can now show that there can be no "islands" in small balls, that is, if a set E_i intersects a small ball then it must also intersect its boundary. Notice that, at least for the moment, i cannot attain the value 0, hence it is still possible that there is a empty hole (or "lake") compactly contained inside a small ball. We will rule out this possibility later.

Lemma 2.9 (No-islands)**.** *There exist a constant $K > 0$, only depending on D, h and $\mathcal{E}$, and a constant $R_3 \leq R_2$, such that for every $r < R_3$, every finite perimeter set $G \subseteq B(x,r) \subseteq D$ and every $1 \leq i \leq m$, one has*

$$\mathcal{H}^1\Big(\partial^*(E_i \cap G)\Big) \leq K\mathcal{H}^1\Big(E_i \cap \partial^* G\Big)\,. \tag{2.20}$$

In particular, if for some $1 \leq i \leq m$ one has

$$|E_i \cap G| > 0\,, \tag{2.21}$$

then also

$$\mathcal{H}^1(E_i \cap \partial^* G) > 0\,. \tag{2.22}$$

Proof. Let $R_3 \leq R_2$ be a constant, to be precised later, take a set of finite perimeter G contained in a ball $B(x,r) \subseteq D$ with $r < R_3$, and fix $1 \leq i \leq m$. First of all, we notice that (2.20) is enough to conclude the thesis. In fact, if (2.21) holds true, then $\mathcal{H}^1(\partial^*(E_i \cap G)) > 0$, which by (2.20) gives (2.22).

Let us call $F = E_i \cap G$, and assume that $|F| > 0$, since otherwise (2.20) is emptily true. We claim that, provided K is large enough, if (2.20) is false then we can find a competitor, that is, a cluster $\mathcal{E}'$ such that

$$\mathcal{E}' = \mathcal{E} \text{ outside } F\,, \qquad\qquad P(\mathcal{E}') \leq P(\mathcal{E}) - \frac{h_{\min}^2}{3mh_{\max}}\,\mathcal{H}^1(\partial^* F)\,. \tag{2.23}$$

We can immediately observe that the existence of such a cluster is impossible if R_3 has been taken small enough, so that the thesis will follow by proving the claim. By Lemma 2.1, from (2.23), right, we deduce

$$P(\mathcal{E}') \leq P(\mathcal{E}) - \frac{h_{\min}^2}{3mh_{\max}^2}\,P(F) \leq P(\mathcal{E}) - \frac{h_{\min}^3}{3C_{\text{vol}}^{1/\eta} m h_{\max}^2}\,|F|^{1/\eta}\,. \tag{2.24}$$

Let us define $\varepsilon = |\mathcal{E}| - |\mathcal{E}'|$, so that by (2.23), left, $|\varepsilon| \leq 2|F|$. Applying Lemma 2.7, we get a cluster $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ and

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\text{per}}|\varepsilon|^\beta \leq P(\mathcal{E}') + C_{\text{per}}2^\beta|F|^\beta\,.$$

Putting this inequality together with (2.24), by the optimality of $\mathcal{E}$ we find

$$|F|^{\beta-1/\eta} \geq \frac{h_{\min}^3}{3\cdot 2^\beta C_{\text{per}} C_{\text{vol}}^{1/\eta} m h_{\max}^2}\,. \tag{2.25}$$

We can again distinguish the case $\eta\beta = 1$ and the case $\eta\beta > 1$. If $\eta\beta = 1$, then the above inequality is false thanks to (2.1) if we define

$$C_{\text{per}}^2 = \frac{h_{\min}^3}{3\cdot 2^\beta C_{\text{vol}}^{1/\eta} m h_{\max}^2}\,, \tag{2.26}$$

so we have already found the desired contradiction and the thesis follows simply by taking $R_3 = R_2$. Instead, if $\eta\beta > 1$, keeping in mind that

$$|F| \leq |G| \leq |B(x,r)| \leq C_{\text{vol}} r^\eta$$

by the growth condition, the estimate (2.25) is clearly false if r is small enough, so we can find some $R_3 \leq R_2$ such that the desired contradiction follows also in this case. Summarizing, the thesis follows if we prove the existence of a cluster $\mathcal{E}'$ satisfying (2.23) under the assumption that (2.20) does not hold and with K large enough.

For every $0 \leq j \leq m$, $j \neq i$, we define $\Gamma_j := \partial^* F \cap \partial^* E_j$. Since

$$\partial^* G \cap E_i \subseteq \partial^* F \subseteq \partial^* E_i \cup \Big(\partial^* G \cap E_i\Big)\,,$$

and $\mathcal{H}^1$-a.e. point of $\partial^* E_i$ belongs also to $\partial^* E_j$ for exactly one $0 \leq j \leq m$, $j \neq i$, we have

$$\partial^* F = \Big(\partial^* G \cap E_i\Big) \cup \bigcup_{\substack{j\in\{0,1,\dots,m\}\\ j\neq i}} \Gamma_j \tag{2.27}$$

up to negligible sets, and the $m+1$ sets are essentially disjoint. Let us now consider the inequality

$$\mathcal{H}^1(\Gamma_0) \geq \left(1 - \frac{h_{\min}}{3h_{\max}}\right) \mathcal{H}^1(\partial^* F) \,. \tag{2.28}$$

The definition of the competitor $\mathcal{E}'$ will depend on whether or not this inequality holds true. First of all, we assume that the inequality holds. In this case, we let $\mathcal{E}'$ be the cluster such that $E_i' = E_i \setminus F$ and $E_j' = E_j$ for every $1 \leq j \leq m$, $j \neq i$, so that (2.23), left, is true. In order to compare $P(\mathcal{E})$ and $P(\mathcal{E}')$, we notice that

$$\partial^* \mathcal{E}' = \Big(\partial^* \mathcal{E} \setminus \Gamma_0\Big) \cup \Big(\partial^* G \cap E_i\Big) \,,$$

and also that, for each $1 \leq j \leq m$, $j \neq i$, the set Γ_j is contained both in $\partial^* \mathcal{E}$ and in $\partial^* \mathcal{E}'$, but its contribution to the perimeter is different. In fact, in $\partial^* \mathcal{E}$ the set Γ_j is a common boundary between two "coloured" sets, namely, E_i and E_j, while in $\partial^* \mathcal{E}'$ it is common boundary between a coloured and a white set, that is, E_j' and E_0'. Consequently, making use of (2.28), we can estimate

$$\begin{aligned} P(\mathcal{E}') &\leq P(\mathcal{E}) - h_{\min} \mathcal{H}^1(\Gamma_0) + h_{\max} \mathcal{H}^1(\partial^* F \setminus \Gamma_0) \\ &= P(\mathcal{E}) + h_{\max} \mathcal{H}^1(\partial^* F) - \Big(h_{\min} + h_{\max}\Big) \mathcal{H}^1(\Gamma_0) \leq P(\mathcal{E}) - \frac{h_{\min}}{3}\, \mathcal{H}^1(\partial^* F) \,, \end{aligned}$$

which is stronger than (2.23), right. We have then found the searched competitor if the estimate (2.28) holds.

Let us then finally assume that (2.28) is false. As a consequence, using (2.33) and the fact that (2.20) is false, we have

$$\sum_{\substack{j \in \{1, \dots, m\} \\ j \neq i}} \mathcal{H}^1(\Gamma_j) = \mathcal{H}^1(\partial^* F) - \mathcal{H}^1(\partial^* G \cap E_i) - \mathcal{H}^1(\Gamma_0) > \left(\frac{h_{\min}}{3h_{\max}} - \frac{1}{K}\right) \mathcal{H}^1(\partial^* F) \,.$$

Provided that K is large enough, we get then the existence of some $1 \leq \ell \leq m$, $\ell \neq i$ such that

$$\mathcal{H}^1(\Gamma_\ell) \geq \frac{h_{\min}}{3(m - 1/2) h_{\max}}\, \mathcal{H}^1(\partial^* F) \,. \tag{2.29}$$

We define this time $\mathcal{E}'$ the cluster such that $E_i' = E_i \setminus F$, $E_\ell' = E_\ell \cup F$, and $E_j' = E_j$ for every $1 \leq j \leq m$ different from i and ℓ. Notice that (2.23), left, is again true, and this time

$$\partial^* \mathcal{E}' = \Big(\partial^* \mathcal{E} \setminus \Gamma_\ell\Big) \cup \Big(\partial^* G \cap E_i\Big) \,.$$

Moreover, for every $1 \leq j \leq m$, $j \notin \{i, \ell\}$ the contribution of Γ_j to $P(\mathcal{E})$ and to $P(\mathcal{E}')$ is the same. As a consequence, we have

$$P(\mathcal{E}') \leq P(\mathcal{E}) - h_{\min} \mathcal{H}^1(\Gamma_\ell) + h_{\max} \mathcal{H}^1(\partial^* G \cap E_i) \leq P(\mathcal{E}) - \frac{h_{\min}^2}{3m h_{\max}} \mathcal{H}^1(\partial^* F) \,,$$

where the last inequality is true by (2.29), by the fact that (2.20) is false, and up to possibly increase the value of K. We have then proved (2.23), right, and the proof is completed. □

The same result is true also for the case $i = 0$. We can show it now in a simplified case, namely, for "holes" whose boundary entirely belongs to a same $\partial^* E_\ell$, with $1 \le \ell \le m$. Later on, we will show it in full generality.

Lemma 2.10 (No-lakes, part 1)**.** *Let $B(x,r) \subseteq D$ be a ball with $r < R_3$, and let $1 \le \ell \le m$. There is no set $F \subseteq B(x,r) \setminus E_\ell$, thus in particular no set $F \subseteq B(x,r) \cap E_0$, with $|F| > 0$ and $\partial^* F \subseteq \partial^* E_\ell$.*

Proof. We argue as in Lemma 2.9, but the situation is now much simpler. Assume the existence of x, r, ℓ and F as in the claim, and define $\mathcal{E}'$ the cluster such that $E'_\ell = E_\ell \cup F$, and $E'_j = E_j \setminus F$ for every $1 \le j \le m$, $j \ne \ell$. By assumption, $\partial^* \mathcal{E}' = \partial^* \mathcal{E} \setminus (\partial^* F \cup F)$, and in particular

$$P(\mathcal{E}') \le P(\mathcal{E}) - h_{\min} \mathcal{H}^1(\partial^* F)\,.$$

Since this inequality is stronger than (2.23), we conclude exactly as in Lemma 2.9. □

As a consequence of the last two results, we can observe a first regularity property for the sets E_i, $1 \le i \le m$, mild but useful.

Lemma 2.11. *For every ball $B(x,r) \subseteq D$ with $r < R_3$ and for every $1 \le i \le m$, the set $E_i \cap B(x,r)$ is an open set (taking the set of points with density 1 as representative). Moreover, for every connected component F of $E_i \cap B(x,r)$, there exists an injective curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ of finite length such that $\partial^* F = \partial F = \gamma(\mathbb{S}^1)$ up to $\mathcal{H}^1$-negligible subsets, and $\mathcal{H}^1\Big(\partial F \cap \partial B(x,r)\Big) > 0$.*

It is important to notice that, in the above result, the connectedness of F should be in principle meant in the measure theoretic sense, see the Appendix. However, an immediate consequence of the result itself is that the measure theoretical connected components are actually connected in the topological sense.

Proof of Lemma 2.11. We can assume that $|E_i \cap B(x,r)| > 0$, since otherwise the result is emptily true. Let then F be either the whole set $E_i \cap B(x,r)$ or one of its connected components. Since $F \subseteq E_i$, then (2.21) holds with $G = F$, hence also (2.22) is true, i.e., $\mathcal{H}^1(E_i \cap \partial^* F) > 0$. Observing that $\partial^* F \subseteq \partial^* E_i \cup \partial B(x,r)$ and $\mathcal{H}^1(E_i \cap \partial^* E_i) = 0$, we deduce that

$$0 < \mathcal{H}^1(E_i \cap \partial^* F) \le \mathcal{H}^1(\partial^* F \cap \partial B(x,r))\,.$$

Therefore, keeping in mind Theorem A.2, Lemma A.3 and Lemma A.5, all we have to do is to check that F is quasi-minimal and has no holes (in the sense of Definition A.4).

We start proving that F has no holes. By contradiction, assume the existence of $U \subseteq \mathbb{R}^2 \setminus F$ with $\mathcal{H}^2(U) > 0$ and such that $\partial^* F = \partial^* U \cup \partial^*(F \cup U)$, which implies that $U \subseteq B(x,r)$. Up to $\mathcal{H}^1$-negligible subsets, all points of $\partial^* U$ have density $1/2$ with respect to U, and also with respect to F since $\partial^* U \subseteq \partial^* F$. Since $U \cap F = \emptyset$, this implies that $\mathcal{H}^1$-a.e. point of $\partial^* U$ has density 1 with respect to $U \cup F$. And since $U \cup F \subseteq B(x,r)$, we deduce that $\partial^* U \cap \partial B(x,r)$ is $\mathcal{H}^1$-negligible, which recalling that $\partial^* U \subseteq \partial^* F \subseteq \partial^* E_i \cup \partial B(x,r)$ gives

$$\partial^* U \subseteq \partial^* E_i \qquad \mathcal{H}^1\text{-a.e.}. \tag{2.30}$$

Observe now that U does not intersect F, but it could still intersect E_i. Nevertheless,

$$\mathcal{H}^1\Big(\partial^*(E_i \cap U) \cap \partial^* U\Big) = 0\,. \tag{2.31}$$

Indeed, up to $\mathcal{H}^1$-negligible subsets, points of $\partial^*(E_i \cap U)$ have density $1/2$ with respect to $E_i \cap U$, and points of $\partial^* U \subseteq \partial^* F$ have density $1/2$ with respect to $F \subseteq E_i \setminus U$, so $\mathcal{H}^1$-a.e. point of $\partial^*(E_i \cap U) \cap \partial^* U$ has density 1 with respect to E_i, and by (2.30) this gives (2.31), which in particular implies that the set $V = U \setminus E_i$ satisfies $|V| > 0$. Summarizing, V is a subset of $B(x,r) \setminus E_i$, and by construction and (2.30) we have $\partial^* V \subseteq \partial^* U \cup \partial^* E_i \subseteq \partial^* E_i$. The existence of such a set V is excluded by Lemma 2.10, thus we have proved that F has no holes.

Hence, to conclude the proof, we only have to deal with the quasi-minimality. Since $F \subseteq B(x,r)$, it is enough to take a ball $B(z,\rho)$ intersecting F and with $\rho < R_3$. To prove the quasi-minimality we have to find a constant C_{qm}, only depending on D, h and $\mathcal{E}$, such that for every set H with $F \Delta H \subset\subset B(z,\rho)$ one has

$$\mathcal{H}^1\Big(\partial^* F \cap B(z,\rho)\Big) \leq C_{qm} \mathcal{H}^1\Big(\partial^* H \cap B(z,\rho)\Big)\,. \tag{2.32}$$

Let us call $G_1 = F \setminus H$ and $G_2 = H \setminus F$, and notice that $G_1,\, G_2 \subset\subset B(z,\rho)$. We clearly have

$$\partial^* F \cap B(z,\rho) \subseteq \partial^* G_1 \cup \partial^* G_2 \cup \Big(\partial^* H \cap B(z,\rho)\Big)\,. \tag{2.33}$$

Applying Lemma 2.9 to the set G_1 we get

$$\mathcal{H}^1(\partial^* G_1) = \mathcal{H}^1\Big(\partial^*(G_1 \cap E_i)\Big) \leq K \mathcal{H}^1\Big(E_i \cap \partial^* G_1\Big) \leq K \mathcal{H}^1\Big(\partial^* H \cap B(z,\rho)\Big) \tag{2.34}$$

where the first equality holds since $G_1 = G_1 \cap E_i$, and the last inequality is true since $\partial^* G_1 \subseteq \Big(\partial^* E_i \cup \partial^* H\Big) \cap B(z,\rho)$, and then $\mathcal{H}^1$-a.e. point of $\partial^* G_1 \cap E_i$ cannot be in $\partial^* E_i$, so it must be in $\partial^* H \cap B(z,\rho)$.

Let us now pass to consider G_2. Let us call $\widetilde{\mathcal{E}}$ the cluster such that $\widetilde{E}_i = E_i \cup G_2$ and $\widetilde{E}_j = E_j \setminus G_2$ for every $j \neq i$. As already observed while proving Lemma 2.10, we get a contradiction with the same argument of Lemma 2.9 if

$$P(\widetilde{\mathcal{E}}) \leq P(\mathcal{E}) - \frac{h_{\min}}{2}\,\mathcal{H}^1(\partial^* G_2)\,.$$

In fact, in the proof of Lemma 2.10 we were using the same inequality with $h_{\min}$ in place of $h_{\min}/2$, but both work since both inequalities are stronger than (2.23). Therefore, we know that

$$P(\widetilde{\mathcal{E}}) > P(\mathcal{E}) - \frac{h_{\min}}{2}\,\mathcal{H}^1(\partial^* G_2)\,. \tag{2.35}$$

Let us now observe that

$$\Big(\partial^* G_2 \cap \partial^* F\Big) \cap \partial^* \widetilde{\mathcal{E}} = \emptyset\,. \tag{2.36}$$

Indeed, $\mathcal{H}^1$-a.e. $y \in \partial^* G_2 \cap \partial^* F$ has density $1/2$ with respect to F. Moreover, it has density $1/2$ with respect to G_2, which does not intersect F, so density 1 with respect to $F \cup G_2 \subseteq \widetilde{E}_i$. Thus, $y \notin \partial^* \widetilde{\mathcal{E}}$ and (2.36) is established. Moreover, $\partial^* \widetilde{\mathcal{E}} \setminus \partial^* \mathcal{E} \subseteq \partial^* G_2$, which by (2.36) becomes

$$\partial^* \widetilde{\mathcal{E}} \setminus \partial^* \mathcal{E} \subseteq \partial^* G_2 \setminus \partial^* F\,.$$

By (2.35) we obtain the estimate

$$\begin{aligned}
\frac{h_{\min}}{2}\,\mathcal{H}^1(\partial^* G_2) > P(\mathcal{E}) - P(\widetilde{\mathcal{E}}) &\geq h_{\min}\mathcal{H}^1\Big(\partial^* G_2 \cap \partial^* F\Big) - h_{\max}\mathcal{H}^1\Big(\partial^* G_2 \setminus \partial^* F\Big)\\
&= h_{\min}\mathcal{H}^1\Big(\partial^* G_2\Big) - (h_{\min} + h_{\max})\mathcal{H}^1\Big(\partial^* G_2 \setminus \partial^* F\Big)\\
&\geq h_{\min}\mathcal{H}^1\Big(\partial^* G_2\Big) - (h_{\min} + h_{\max})\mathcal{H}^1\Big(\partial^* H \cap B(z,\rho)\Big)\,,
\end{aligned}$$

which can be rewritten as

$$\mathcal{H}^1(\partial^* G_2) < \frac{2(h_{\min} + h_{\max})}{h_{\min}}\,\mathcal{H}^1\Big(\partial^* H \cap B(z,\rho)\Big)\,.$$

Putting this inequality together with (2.34), we obtain (2.32) thanks to (2.33). □

Notice that, as a consequence of the above regularity result, for each ball $B(x,r) \subseteq D$ with $r < R_3$, and each $1 \leq i \leq m$, the boundary of $E_i \cap B(x,r)$ is done by a countable union of closed, injective curves. Since all these curves have to reach $\partial B(x,r)$, they are actually finitely many if Vol'pert Theorem holds for $B(x,r)$ and $\mathcal{H}^0(\partial^* \mathcal{E} \cap \partial B(x,r)) < +\infty$, which is true for almost each $r > 0$.

Observe now that, when two sets of the cluster have some common boundary, then two of these curves have some intersection. We show now that these intersections between different curves behave not too crazily.

Lemma 2.12. *There exists $R_4 \leq R_3$ such that the following holds. Let $B(x,r) \subseteq D$ be a ball with $r < R_4$, let γ_1, $\gamma_2 : \mathbb{S}^1 \to \overline{B(x,r)}$ be two curves as in Lemma 2.11, not necessarily different, and let τ_1, $\tau_2 : [0,1] \to B(x,r)$ be two injective subpaths of γ_1 and γ_2 such that*

$$\tau_1(0) = \tau_2(1)\,, \qquad\qquad \tau_1(1) = \tau_2(0)\,.$$

Then the paths τ_1 and τ_2 coincide, that is, $\tau_1((0,1)) = \tau_2((0,1))$. More in general, if

$$\tau_1\Big((0,1)\Big) \cap \tau_2\Big((0,1)\Big) = \emptyset\,, \qquad\qquad \tau_1(0) = \tau_2(1)\,,$$

then

$$\Big|\tau_1(0) - \tau_1(1)\Big| \leq C_1|\tau_1(1) - \tau_2(0)| \tag{2.37}$$

for some constant $C_1 > 1$ *depending only on* D, h *and* $\mathcal{E}$.

Before giving the proof of this result, we briefly explain its meaning, also with the aid of Figure 9. Let γ_1 and γ_2 be two curves as in Lemma 2.11, and let us select two subpaths,

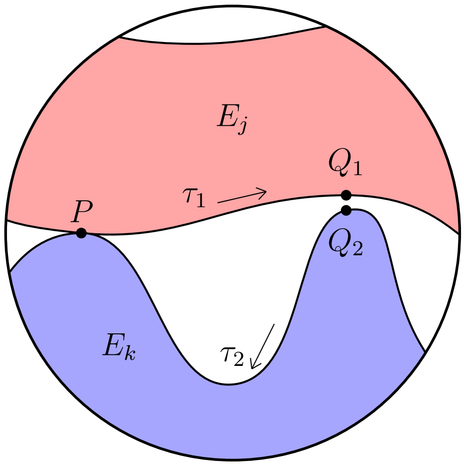

Figure 9. The curves τ_1 and τ_2 and the points P, Q_1 and Q_2 in Lemma 2.12.

τ_1 of γ_1 and τ_2 of γ_2, both entirely contained in the interior of the ball $B(x, r)$. The first part of the claim says that if the two paths have the same endpoints, then they have to coincide. In other words, if two curves γ_1 and γ_2 have two points in common, then they remain together between them. The second part of the claim considers a more general situation, namely, when τ_1 and τ_2 have disjoint interiors and one common endpoint, called P in the figure (in particular, τ_1 and τ_2 could be consecutive subpaths of a same curve $\gamma_1 = \gamma_2$). The other endpoints are called respectively Q_1 and Q_2. The inequality (2.37) then says that Q_1 and Q_2 cannot be too close, with respect to the distance between P and Q_1. We have then to exclude the situation depicted in the figure, where Q_1 and Q_2 are very close to each other.

Proof of Lemma 2.12. Let $R_4 \leq R_3$ be a small constant, which will be precised later. For simplicity of notation, as in Figure 9 we set

$$P = \tau_1(0) = \tau_2(1)\,, \qquad Q_1 = \tau_1(1)\,, \qquad Q_2 = \tau_2(0)\,, \qquad d = |Q_1 - Q_2|\,,$$

so that (2.37) can be rewritten as $|P - Q_1| \leq C_1 d$. We limit ourselves to show the second part of the thesis, that is, that (2.37) holds if τ_1 and τ_2 have disjoint interiors. Indeed, this implies that in the case of disjoint interiors it is impossible that $Q_1 = Q_2$. And as a consequence, if $Q_1 = Q_2$ then τ_1 and τ_2 have to coincide, because otherwise there are further subpaths of τ_1 and τ_2 with disjoint interiors and the same endpoints, which has been excluded.

We first assume that, as it happens in the figure,

$$\textit{the interior of the segment } Q_1Q_2 \textit{ does not intersect } \tau_1 \cup \tau_2\,, \tag{2.38}$$

at the end we will easily remove this assumption. Putting together τ_1, the segment Q_1Q_2, and τ_2, we obtain then an injective, closed path in $B(x,r)$, which encloses a closed region that we call G. Without loss of generality we assume that this path is percurred clockwise, as in the figure. We also call $1 \le j,\, k \le m$ the two indices such that the set enclosed by γ_1 (resp., γ_2) is contained in E_j (resp., E_k). Notice that j and k are not necessarily different, in particular if $\gamma_1 = \gamma_2$ then of course $j = k$. Observe that $\mathcal{H}^1$-a.e. point of $\tau_1 \cup \tau_2$ has density $1/2$ with respect to E_j or E_k, hence it cannot have density 1 with respect to any of the sets of the cluster. As a consequence, for every $1 \le i \le m$, regardless whether or not i coincides with one between j and k, applying (2.20) of Lemma 2.9 we get

$$\mathcal{H}^1(\partial^*(E_i\cap G)\cap(\tau_1\cup\tau_2)) \le \mathcal{H}^1(\partial^*(E_i\cap G)) \le K\mathcal{H}^1(E_i\cap\partial^* G) = K\mathcal{H}^1(E_i\cap Q_1Q_2)\,. \tag{2.39}$$

We can now easily reduce ourselves to the case when

$$\mathcal{H}^1\Big(\tau_1 \cap \partial^*(E_j \cap G)\Big) = 0\,, \qquad\qquad \mathcal{H}^1\Big(\tau_2 \cap \partial^*(E_k \cap G)\Big) = 0\,. \tag{2.40}$$

Indeed, the path τ_1 is part of the boundary of a connected component of $E_j \cap B(x,r)$, and then E_j is "on one side of τ_1". In other words, $\mathcal{H}^1$-a.e. point of τ_1 has density $1/2$ with respect to E_j and, up to $\mathcal{H}^1$-negligible sets, either all points of τ_1 have density $1/2$ with respect to $E_j \setminus G$, or they all have density $1/2$ with respect to $E_j \cap G$. The first case corresponds to the left assumption in (2.40), and it is the one depicted in Figure 9. As a consequence, if the left property of (2.40) fails, then by (2.39) with $i = j$ we have

$$|\tau_1(0) - \tau_1(1)| \le \mathcal{H}^1(\tau_1) \le \mathcal{H}^1\Big(\partial^*(E_j \cap G) \cap (\tau_1 \cup \tau_2)\Big) \le K|Q_1 - Q_2| = K|\tau_1(1) - \tau_2(0)|\,,$$

then (2.37) is already proved with $C_1 = K$. Similarly, if the right property of (2.40) is fails, then

$$|\tau_1(0) - \tau_1(1)| = |\tau_2(1) - \tau_1(1)| \le \mathcal{H}^1(\tau_2) + |\tau_1(1) - \tau_2(0)| \le (K+1)|\tau_1(1) - \tau_2(0)|\,,$$

hence we have again the validity of (2.37) with $C_1 = K + 1$. Therefore, from now on and without loss of generality we assume that (2.40) is true.

Putting together all the bounds (2.39) varying $1 \le i \le m$, we have

$$\mathcal{H}^1(\Gamma) \le Kd\,, \qquad \text{where} \qquad \Gamma = \bigcup\nolimits_{i=1}^m \partial^*(E_i \cap G) \cap (\tau_1 \cup \tau_2)\,. \tag{2.41}$$

This bound gives an important information. Namely, if $d \ll |Q_1 - P|$, then Γ is only a very small portion of $\tau_1 \cup \tau_2$. In other words, it is possible that some curves γ_h as in

Lemma 2.11 enter in G, at least if $Q_1 \neq Q_2$ (such curves are not shown in Figure 9 to keep the figure clear). Nevertheless, these curves cover only a small part of $\tau_1 \cup \tau_2$. As a consequence, the greatest part of $\tau_1 \cup \tau_2$ does not belong to Γ, and then by construction it is done by points which belong to $\partial^* E_0$.

We call now $\overline{h} : \mathbb{R}^2 \to \mathbb{R}^+$ the function given by $\overline{h}(v) = h(x, v)$ for every $v \in \mathbb{R}^2$, we set $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}$ as $\mathfrak{h}(\nu) = h(\hat{\nu})$ where $\hat{\nu}$ is the angle obtained by rotating ν of $90°$ clockwise, and we denote by len the length of curves given by (2.13). Applying then Corollary 2.6 to the paths $\tau_1 \cup Q_1Q_2$ and τ_2, which are two injective paths, disjoint except for the common endpoints (i.e., P and Q_2), we get an injective path $\tau : [0, 1] \to G$ connecting P and Q_2 and satisfying (2.14), which reads as

$$\operatorname{len}(\tau) + \operatorname{len}(\hat{\tau}) \leq \operatorname{len}(\tau_1 \cup Q_1Q_2) + \operatorname{len}(\tau_2) \leq \operatorname{len}(\tau_1) + \operatorname{len}(\tau_2) + h_{\max} d\,. \tag{2.42}$$

We call G_1 and G_2 the parts of G enclosed by $\tau_1 \cup Q_1Q_2 \cup \hat{\tau}$ and by $\tau \cup \tau_2$ respectively, and we are in position to define the competitor $\mathcal{E}'$. In fact, we set $E'_i = E_i \setminus G$ for every $i \notin \{j, k\}$. Moreover, if $j \neq k$ then we set $E'_j = E_j \cup G_1$ and $E'_k = E_k \cup G_2$, while if $j = k$ we set $E'_j = E_j \cup G$. Notice that, in this way, and keeping in mind (2.40), we remove from $\partial^*\mathcal{E}$ both τ_1 and τ_2, as well as the part of $\partial^*\mathcal{E}$ which was in the interior of G, if any. Conversely, we add a part of the segment Q_1Q_2 and, if $j \neq k$, the path τ (that could contain parts of τ_1 and τ_2, which would then be re-added to the boundary). In particular, notice that if $j \neq k$ then the path τ is added to $\partial^*\mathcal{E}'$, and it is common boundary between E'_j and E'_k. Instead, if $j = k$, then τ is simply not added to $\partial^*\mathcal{E}'$.

Recalling that ω is the modulus of continuity of h in the first variable inside D, we select R_4 so small that $\omega(R_4) < h_{\min}/6$, thus for every $y \in B(x, r)$ and $\nu \in \mathbb{S}^1$ we have

$$\frac{5}{6}\,\overline{h}(\nu) \leq \overline{h}(\nu) - \frac{h_{\min}}{6} < \overline{h}(\nu) - \omega(r) \leq h(y, \nu) \leq \overline{h}(\nu) + \omega(r) < \overline{h}(\nu) + \frac{h_{\min}}{6} \leq \frac{7}{6}\,\overline{h}(\nu)\,.$$

As a consequence, keeping in mind that $\tau_1 \cup Q_1Q_2 \cup \tau_2$ is a clockwise, closed path, that points of $\tau_1 \setminus \Gamma$ (resp., $\tau_2 \setminus \Gamma$) belong to $\partial^* E_j \cap \partial^* E_0$ (resp., $\partial^* E_k \cap \partial^* E_0$), and using (2.42) and (2.41) we get

$$\begin{aligned} P(\mathcal{E}') - P(\mathcal{E}) &\leq \frac{7}{6}\,\frac{\operatorname{len}(\tau) + \operatorname{len}(\hat{\tau})}{2} - \frac{5}{6}\Big(\operatorname{len}(\tau_1) + \operatorname{len}(\tau_2)\Big) + h_{\max}\Big(\mathcal{H}^1(\Gamma) + d\Big) \\ &\leq -\frac{1}{4}\Big(\operatorname{len}(\tau_1) + \operatorname{len}(\tau_2)\Big) + h_{\max}\bigg(\mathcal{H}^1(\Gamma) + \frac{19}{12}\,d\bigg) \\ &\leq -\frac{h_{\min}}{4}\,\mathcal{H}^1(\tau_1 \cup \tau_2) + h_{\max}(K + 2)\,d\,. \end{aligned} \tag{2.43}$$

Let us call again $\varepsilon = |\mathcal{E}| - |\mathcal{E}'|$, so that by construction, by the (Euclidean) isoperimetric inequality we have

$$|\varepsilon|^\beta \leq (2|G|)^\beta \leq \frac{1}{(2\pi)^\beta}\Big(\mathcal{H}^1(\tau_1 \cup \tau_2) + d\Big)^{2\beta} \leq \mathcal{H}^1(\tau_1 \cup \tau_2)^{2\beta} + d^{2\beta} \leq \mathcal{H}^1(\tau_1 \cup \tau_2)^{2\beta} + d\,.$$

Notice that, in the last inequality, we have used the fact that $\beta \geq 1/2$, which is true since $\eta\beta \geq 1$ and the η-growth condition implies that $\eta \leq 2$, and the fact that $d < 1$, which is obvious as soon as $R_4 < 1/2$. We can now apply Lemma 2.7 to find a cluster $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ and such that, also by (2.43),

$$\begin{aligned} P(\mathcal{E}'') &\leq P(\mathcal{E}') + C_{\rm per}|\varepsilon|^\beta \\ &\leq P(\mathcal{E}) - \frac{h_{\min}}{4}\,\mathcal{H}^1(\tau_1\cup\tau_2) + \Big(h_{\max}(K+2) + C_{\rm per}\Big)\,d + C_{\rm per}\mathcal{H}^1(\tau_1\cup\tau_2)^{2\beta}\,. \end{aligned}$$

The optimality of $\mathcal{E}$ implies then that

$$\frac{h_{\min}}{4}\,\mathcal{H}^1(\tau_1\cup\tau_2) \leq \Big(h_{\max}(K+2) + C_{\rm per}\Big)\,d + C_{\rm per}\mathcal{H}^1(\tau_1\cup\tau_2)^{2\beta}\,.$$

We want to deduce that

$$\frac{h_{\min}}{5}\,\mathcal{H}^1(\tau_1\cup\tau_2) \leq \Big(h_{\max}(K+2) + C_{\rm per}\Big)\,d\,, \tag{2.44}$$

which is true if

$$C_{\rm per}\mathcal{H}^1(\tau_1\cup\tau_2)^{2\beta} \leq \frac{h_{\min}}{20}\,\mathcal{H}^1(\tau_1\cup\tau_2)\,.$$

If $\beta > 1/2$, this bound clearly holds as soon as $\mathcal{H}^1(\tau_1\cup\tau_2)$ is small enough, and in turn, since $\tau_1\cup\tau_2 \subseteq \partial^*\mathcal{E}$, this is true by Lemma 2.8, up to possibly decrease R_4. Instead, if $\beta = 1/2$, and then necessarily $\eta = 2$ and $\eta\beta = 1$, the bound holds by (2.1) as soon as we define

$$C_{\rm per}^3 = \frac{h_{\min}}{20}\,. \tag{2.45}$$

We have then shown the validity of (2.44), which implies (2.37) since $|\tau_1(1) - \tau_2(0)| = d$ and $|\tau_1(0)-\tau_1(1)| = |P-Q_1| \leq \mathcal{H}^1(\tau_1)$. Summarizing, we have concluded the proof under the additional assumption (2.38).

We are then only left to consider the case when the interiors of τ_1 and τ_2 are disjoint and the open segment Q_1Q_2 intersects points of $\tau_1\cup\tau_2$. In this case, since $\tau_1([0,1])$ and $\tau_2([0,1])$ are closed, we can define $\widetilde{Q}_1$ and $\widetilde{Q}_2$ two points with minimal distance among the pairs in $\tau_1\cap Q_1Q_2$ and $\tau_2\cap Q_1Q_2$. We can then call $\tilde\tau_1$ (resp., $\tilde\tau_2$) the subpath of τ_1 (resp., τ_2) between P and $\widetilde{Q}_1$ (resp., between $\widetilde{Q}_2$ and P). Since by minimality the open segment $\widetilde{Q}_1\widetilde{Q}_2$ does not intersect $\tilde\tau_1\cup\tilde\tau_2 \subseteq \tau_1\cup\tau_2$, we know the validity of (2.37) for $\tilde\tau_1$ and $\tilde\tau_2$. Therefore, we easily deduce the validity also for τ_1 and τ_2, since using the minimality of $\widetilde{Q}_1$ and $\widetilde{Q}_2$ we get

$$\begin{aligned} \Big|\tau_1(0)-\tau_1(1)\Big| &= |P-Q_1| \leq |P-\widetilde{Q}_1| + |\widetilde{Q}_1 - Q_1| \leq C_1|\widetilde{Q}_1-\widetilde{Q}_2| + |\widetilde{Q}_1-Q_1| \\ &\leq C_1|\widetilde{Q}_1 - Q_2| + |\widetilde{Q}_1 - Q_1| \leq C_1|Q_1-Q_2| = C_1|\tau_1(1)-\tau_2(0)|\,. \end{aligned}$$

□

An immediate corollary of the above result is the following one, which generalizes a part of the claim of Lemma 2.11 also to E_0.

Corollary 2.13. *Let $B(x,r) \subseteq D$ be a ball with $r < R_4$ and $\mathcal{H}^0(\partial^*\mathcal{E} \cap \partial B(x,r)) \in \mathbb{N}$, and let F be a connected component of $E_0 \cap B(x,r)$. There exists an injective map $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ of finite length such that $\partial^* F = \partial F = \gamma(\mathbb{S}^1)$ up to $\mathcal{H}^1$-negligible subsets.*

Proof. Let us take the collection $\{\gamma_j\}$ of all the curves given by Lemma 2.11, which parametrize the boundaries of all the connected components of the sets $E_i \cap B(x,r)$ with $1 \le i \le m$. Since all these curves have to reach $\partial B(x,r)$, by assumption and also using Vol'pert Theorem 2.2 we deduce that they are finitely many. For each j, the set $\gamma_j \cap B(x,r)$ is a disjoint, finite union if injective subpaths $\gamma_j^h : (0,1) \to B(x,r)$, each of them having both endpoints in $\partial B(x,r)$. Hence, altogether there are only finitely many paths γ_j^h. Notice that $\partial^*\mathcal{E} \cap B(x,r)$ coincides with the union of these paths. Moreover, $\mathcal{H}^1$-a.e. point $z \in \partial^*\mathcal{E} \cap B(x,r)$ belongs to exactly two boundaries $\partial^* E_i \cap B(x,r)$ with $0 \le i \le m$. If both boundaries correspond to indices $i \ne 0$, then z belongs to exactly two of the paths γ_j^h, and $z \notin \partial^* E_0$. Conversely, if one of the two boundaries corresponds to $i = 0$, thus in particular $z \in \partial^* E_0$, then z belongs to exactly one of the paths γ_j^h. Therefore, $\partial^* E_0 \cap B(x,r)$ coincides $\mathcal{H}^1$-a.e. with the points belonging to exactly one of the paths γ_j^h.

Notice now that, by construction and thanks to Lemma 2.12, the intersection between any two of the paths γ_j^h is either empty, or a single point, or a common closed subpath. As a consequence, $\partial^* E_0 \cap B(x,r)$ coincides $\mathcal{H}^1$-a.e. with the union of finitely many injective curves $\tau_k : (0,1) \to B(x,r)$ of finite length, which are the parts of the paths γ_j^h which do not belong to any other of the paths. By construction, the curves τ_k are pairwise disjoint. As a consequence, every endpoint of each curve τ_k must be also endpoint of some of the other curves, and then the claim immediately follows. □

Notice that, by Lemma 2.11 and Corollary 2.13, we now know that the sets $\partial\mathcal{E}$ and $\partial^*\mathcal{E}$ coincide in D up to $\mathcal{H}^1$-negligible subsets. We can now show the existence of arbitrarily small circles around each point of D with at most three points of $\partial\mathcal{E}$.

Lemma 2.14 (At most three points)**.** *There exist $R_5 \le R_4$ and $C_2 > 2$, only depending on h, D and $\mathcal{E}$, such that for every ball $B(x,r) \subseteq D$ with $r < R_5$ there is $r/C_2 \le \rho \le r$ such that*

$$\#\Big(\partial\mathcal{E} \cap \partial B(x,\rho)\Big) \le 3\,. \tag{2.46}$$

Proof. As already done several times, we start by defining $\overline{h} : \mathbb{R}^2 \to \mathbb{R}^+$ as $\overline{h}(v) = h(x,v)$, and $\overline{P}$ the perimeter obtained by substituting h with $\overline{h}$ in (1.1). Moreover, we call again $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}^+$ the function such that $\mathfrak{h}(\nu) = \overline{h}(\hat\nu)$ where $\hat\nu$ is the angle obtained by rotating ν of $90°$ clockwise, and we denote by len the length of curves given by (2.13).

Let now M be the smallest integer strictly larger than $1 + 7\,\frac{h_{\max}}{h_{\min}}$. Let $R_5 \le R_4$ and $C_2 > 2$ be two constants to be specified later, and let us take a ball $B(x,r) \subseteq D$ with

$r < R_5$. By Lemma 2.8, there exists a radius $r/M \leq r_0 \leq r$ such that $\partial B(x, r_0) \cap \partial\mathcal{E}$ is made by at most M points. More precisely, r_0 is a Lebesgue point of the function $\rho \mapsto \mathcal{H}^0(\partial B(x,\rho) \cap \partial\mathcal{E})$, and the value of this function at $\rho = r_0$ is at most M.

By Lemma 2.11, for every $1 \leq i \leq m$ the boundary of each connected component of the set $E_i \cap B(x, r_0)$ is a closed curve of finite length intersecting $\partial B(x, r_0)$. Intersecting these curves with the interior of the ball, we have finitely many paths of finite length inside $B(x, r_0)$ with both endpoints on $\partial B(x, r_0)$. Let us denote by $\psi_j : (0,1) \to B(x, r_0)$ these paths. For brevity, and with a small abuse of notation, we will denote by ψ_j also the image of the path, that is, $\psi_j([0,1])$. We can observe that these paths are at most M. Indeed, every path has two endpoints in $\partial B(x, r_0) \cap \partial\mathcal{E}$, and the fact that ρ is a Lebesgue point of $\rho \mapsto \mathcal{H}^0(\partial\mathcal{E} \cap \partial B(x, \rho))$ implies that each point of $\partial B(x, r_0) \cap \partial\mathcal{E}$ can be endpoint of at most two paths. For every j, we call

$$\rho_j = \min\Big\{|\psi_j(t) - x|,\, 0 < t < 1\Big\} \in [0, r_0)\,.$$

Remember that different paths may have parts in common, actually $\mathcal{H}^1$-a.e. point of $\partial\mathcal{E} \setminus \partial E_0$ belongs to two different paths, as already observed in Corollary 2.13 (where we called the paths γ_j^h instead of ψ_j). However, by Lemma 2.12, the intersection between any two of the paths is either empty, or a common closed subpath, which might be a single point. Let then ψ_k and ψ_l be two paths such that $\psi_k \cap \psi_l \neq \emptyset$, let us call $\gamma : [0,1] \to \mathbb{R}^2$ a parametrization of $\psi_k \cap \psi_l$, either injective or constant, and set $\rho^{\pm}_{k,l} \in [0, r_0)$ as

$$\rho^-_{k,l} = |\gamma(0) - x|\,, \qquad\qquad \rho^+_{k,l} = |\gamma(1) - x|\,.$$

Notice that the orientation of γ is not univoquely determined, but the pair $\rho^{\pm}_{k,l}$ is well-defined. Let now $H > 2C_1 + 1$ be a large constant, only depending on D, h and $\mathcal{E}$ and to be specified later. Since the paths ψ_j are at most M, the constants ρ_j and $\rho^{\pm}_{k,l}$ are at most M^2. As a consequence, we can fix

$$\frac{r_0}{H^{2M^2+3}} < r_1 < \frac{r_0}{H} \tag{2.47}$$

such that for every j, and for every pair (k,l) such that $\psi_k \cap \psi_l \neq \emptyset$ we have

$$\rho_j \notin \left[\frac{r_1}{H}, Hr_1\right], \qquad\qquad \rho^{\pm}_{k,l} \notin \left[\frac{r_1}{H}, Hr_1\right]. \tag{2.48}$$

Roughly speaking, this means that nothing "special" happens for a while around the circle $\partial B(x, r_1)$. More precisely, every path which enters in the ball $B(x, Hr_1)$ has also to enter also in the much smaller ball $B(x, r_1/H)$, and the intersection between every two paths has to start/end either outside of the ball $B(x, Hr_1)$, or inside the small ball $B(x, r_1/H)$. Up to renumbering, we assume that $\psi_j \cap B(x, Hr_1) \neq \emptyset$ if and only if $1 \leq j \leq M^-$ for some $M^- \leq M$.

Fix a path ψ_j with $1 \leq j \leq M^-$. By (2.48), this implies that also $\psi_j \cap B(x, r_1/H) \neq \emptyset$. We can then set $0 < t_j^0 < t_j^1 < t_j^2 < t_j^3 < 1$ as

$$t_j^0 = \inf\Big\{t \in [0,1] : |\psi_j(t) - x| \leq r_1\Big\}, \qquad t_j^1 = \inf\Big\{t \in [0,1] : |\psi_j(t) - x| \leq \frac{r_1}{H}\Big\}$$
$$t_j^2 = \sup\Big\{t \in [0,1] : |\psi_j(t) - x| \leq \frac{r_1}{H}\Big\}, \qquad t_j^3 = \sup\Big\{t \in [0,1] : |\psi_j(t) - x| \leq r_1\Big\}.$$

We can easily notice that

$$|\psi_j(t) - x| < Hr_1 \quad \forall\, t \in [t_j^0, t_j^3]. \tag{2.49}$$

Indeed, assume that $|\psi_j(t) - x| \geq Hr_1$ for some $t_j^0 \leq t \leq t_j^3$. Then, apply Lemma 2.12, having set τ_1 and τ_2 as the restrictions of ψ_j to $[t, t_j^3]$ and $[t_j^0, t]$, so that (2.37) implies

$$\Big|\tau_1(0) - \tau_1(1)\Big| \leq C_1|\tau_1(1) - \tau_2(0)| = C_1|\psi_j(t_j^3) - \psi_j(t_j^0)| \leq 2C_1 r_1,$$

and this gives a contradiction since

$$\Big|\tau_1(0) - \tau_1(1)\Big| = |\psi_j(t) - \psi_j(t_j^3)| \geq (H-1)r_1$$

and $H > 2C_1 + 1$. We call $\psi_j^{\rm int}$ the "interior part" of ψ_j, that is, the restriction of ψ_j to $[t_j^0, t_j^3]$, and we call also $\psi_j^{\rm int,1}$ and $\psi_j^{\rm int,2}$ the restrictions of ψ_j to $[t_j^0, t_j^1]$ and to $[t_j^2, t_j^3]$ respectively, which are two disjoint subpaths of $\psi_j^{\rm int}$.

Now, keep in mind that $\mathcal{H}^1$-a.e. point of $\partial\mathcal{E}$ belongs to the boundary of exactly two of the sets E_i, $0 \leq i \leq m$. In particular, there exist an index $1 \leq i \leq m$ such that $\psi_j \subseteq \partial E_i$, and there exists another index $0 \leq i' \leq m$ with $i' \neq i$ such that $\psi_j(t_j^0) \in \partial E_{i'}$. We subdivide the indices $1, 2, \cdots, M^-$ into the two subsets $\mathcal{I}_1^1$ and $\mathcal{I}_2^1$ by saying that $j \in \mathcal{I}_1^1$ if $i' \geq 1$, and $j \in \mathcal{I}_2^1$ if $i' = 0$. Now, we claim that

$$\psi_j\Big([t_j^0, t_j^1]\Big) \subseteq \partial E_{i'}. \tag{2.50}$$

To prove this property, we first assume that $j \in \mathcal{I}_1^1$, that is, $1 \leq i' \leq m$. As a consequence, $\psi_j(t_j^0)$ is a point of some curve $\psi_{j'}$, and then $\psi_j \cap \psi_{j'}$ is a non-empty subpath of ψ_j, in particular $\psi_j \cap \psi_{j'}$ contains $\psi_j\Big([t_j^0, \bar t]\Big)$ for some maximal value of $t_j^0 \leq \bar t \leq 1$. By (2.48) we know that $|\psi_j(\bar t) - x| \notin [r_1/H, Hr_1]$, since $|\psi_j(\bar t) - x| \in \{\rho_{j,j'}^-, \rho_{j,j'}^+\}$. We have then either that $|\psi_j(\bar t) - x| < r_1/H$, and then $\bar t \geq t_j^1$ by definition of t_j^1, or $|\psi_j(\bar t) - x| > Hr_1$, and then $\bar t \geq t_j^3$ by (2.49). In both cases, $\bar t \geq t_j^1$, and this shows (2.50) if $i' \geq 1$. In addition, this also shows that the point $\psi_j(t_j^0)$ is a "special point" also for $\psi_{j'}$, namely, it coincides with either $\psi_{j'}(t_{j'}^0)$ or $\psi_{j'}(t_{j'}^3)$.

Suppose now that $j \in \mathcal{I}_2^1$, i.e., $i' = 0$, so that $\psi_j(t_j^0)$ does not belong to any ψ_k with $k \neq j$. In particular, the point $\psi_j(t_j^0)$ belongs to a connected component of $E_0 \cap B(x, r_0)$. As before, we have a maximal $\bar t \geq t_j^0$ such that $\psi_j\Big([t_j^0, \bar t]\Big) \subseteq \partial E_0$, and proving (2.50) in this case again reduces to showing that $\bar t \geq t_j^1$. By construction, either $\bar t = 1$, and then (2.50) is already proved, or $\bar t < 1$, and then $\psi_j(\bar t) \in \psi_{j'}$ for some j', and in particular $|\psi_j(\bar t) - x|$

equals either $\rho^-_{j,j'}$ or $\rho^+_{j,j'}$. Exactly as before, this implies that $|\psi_j(\bar t)-x|\notin[r_1/H,Hr_1]$, and this proves $\bar t\geq t^1_j$. The property (2.50) is then proved. Of course, in the very same way, we have that $\psi_j(t^3_j)\in\partial E_i\cap\partial E_{i''}$, and that

$$\psi_j\Big([t^2_j,t^3_j]\Big)\subseteq\partial E_{i''}\,. \tag{2.51}$$

Moreover, we subdivide the indices 1, 2, $\cdots$, M^- also into the two subsets $\mathcal{I}^2_1$ and $\mathcal{I}^2_2$, by saying that $j\in\mathcal{I}^2_1$ if $i''\geq 1$, and $j\in\mathcal{I}^2_2$ if $i''=0$.

We are now ready to define the competitor cluster $\mathcal{E}'$. In fact, for every j we call $\widehat\psi^{\rm int,1}_j$ the segment connecting $\psi_j(t^0_j)$ and x, and $\widehat\psi^{\rm int,2}_j$ the segment connecting x and $\psi_j(t^3_j)$. By construction, in particular keeping in mind (2.50) and (2.51), we obtain that the set

$$\partial\mathcal{E}\setminus\Big(\bigcup\nolimits_{j=1}^{M^-}\psi^{\rm int}_j\Big)\cup\Big(\bigcup\nolimits_{j=1}^{M^-}\widehat\psi^{\rm int,1}_j\cup\widehat\psi^{\rm int,2}_j\Big)$$

is the boundary of a uniquely determined cluster, that we call $\mathcal{E}'$. In particular, we can observe that the "colours" of the boundaries of $\mathcal{E}'$ coincide with those of $\mathcal{E}$ in $B(x,Hr_1)\setminus B(x,r_1/H)$. More precisely, fix any $1\leq j\leq M^-$ and call i, i' and i'' as before, so that $\psi^{\rm int,1}_j\subseteq\partial E_i\cap\partial E_{i'}$, and $\psi^{\rm int,2}_j\subseteq\partial E_i\cap\partial E_{i''}$. Then, also the path $\widehat\psi^{\rm int,1}_j$ is contained in $\partial E'_i\cap\partial E'_{i'}$, and $\widehat\psi^{\rm int,2}_j$ is contained in $\partial E'_i\cap\partial E'_{i''}$. And finally, by the definition (1.2) of the perimeter and by construction this implies that

$$\begin{aligned}\overline P(\mathcal{E}')-\overline P(\mathcal{E})\leq\sum_{j\in\mathcal{I}^1_1}\frac{{\rm len}(\widehat\psi^{\rm int,1}_j)-{\rm len}(\psi^{\rm int,1}_j)}{2}+\sum_{j\in\mathcal{I}^2_1}\frac{{\rm len}(\widehat\psi^{\rm int,2}_j)-{\rm len}(\psi^{\rm int,2}_j)}{2}\\ \sum_{j\in\mathcal{I}^1_2}\Big({\rm len}(\widehat\psi^{\rm int,1}_j)-{\rm len}(\psi^{\rm int,1}_j)\Big)+\sum_{j\in\mathcal{I}^2_2}\Big({\rm len}(\widehat\psi^{\rm int,2}_j)-{\rm len}(\psi^{\rm int,2}_j)\Big)\,.\end{aligned} \tag{2.52}$$

We can easily estimate the terms of the above inequality. Indeed, for each $1\leq j\leq M^-$, $\widehat\psi^{\rm int,1}_j$ is the segment between $\psi_j(t^0_j)$ and x, while $\psi^{\rm int,1}_j$ is a path between $\psi_j(t^0_j)$ and some point $\psi_j(t^1_j)$ having distance r_1/H from x, and then by Lemma 2.5 we have that

$${\rm len}(\widehat\psi^{\rm int,1}_j)\leq{\rm len}(\psi^{\rm int,1}_j)+\frac{h_{\max}}{H}\,r_1\,,\qquad {\rm len}(\widehat\psi^{\rm int,2}_j)\leq{\rm len}(\psi^{\rm int,2}_j)+\frac{h_{\max}}{H}\,r_1\,.$$

Inserting these estimates in (2.52), we obtain that

$$\overline P(\mathcal{E}')-\overline P(\mathcal{E})\leq\frac{2Mh_{\max}}{H}\,r_1\,. \tag{2.53}$$

We claim now that $\partial B(x,r_1)\cap\partial\mathcal{E}$ contains at most 3 points. This will prove (2.46) with $\rho=r_1$, and recalling (2.47) and the fact that $r/M\leq r_0\leq r$ this will conclude the thesis with $C_2=MH^{2M^2+3}$. Suppose by contradiction that the claim is false, that is, $\partial B(x,r_1)\cap\partial\mathcal{E}$ contains at least 4 points. Then, applying Proposition 2.3 to the cluster $\mathcal{E}'$ in the ball $B(x,r_1)$ with the distance $\overline h$, we obtain another cluster $\mathcal{F}$ which equals $\mathcal{E}'$

outside of $B(x, r_1)$ and such that $\overline{P}(\mathcal{F}) \leq \overline{P}(\mathcal{E}') - \delta r_1$, so that (2.53) gives

$$\overline{P}(\mathcal{F}) \leq \overline{P}(\mathcal{E}) - \frac{\delta}{2} r_1 \tag{2.54}$$

as soon as $H > 4Mh_{\max}/\delta$. Keep in mind that, as observed in Remark 2.4, the constant δ only depends on h and D, but not on x or r.

Since $\mathcal{E}$ and $\mathcal{F}$ coincide outside $B(x, Hr_1)$, we can estimate

$$\begin{aligned}\mathcal{H}^1(\partial\mathcal{F} \cap B(x, Hr_1)) &\leq \frac{1}{h_{\min}} \overline{P}(\mathcal{F}; B(x, Hr_1)) \leq \frac{1}{h_{\min}} \overline{P}(\mathcal{E}; B(x, Hr_1)) \\ &\leq \frac{h_{\max}}{h_{\min}} \mathcal{H}^1(\partial\mathcal{E} \cap B(x, Hr_1))\,.\end{aligned} \tag{2.55}$$

Let now $R_5 \leq R_4$, only depending on h, D and $\mathcal{E}$, be a constant such that

$$\omega(R_5) < \frac{\delta h_{\min}^2}{28Hh_{\max}(h_{\min} + h_{\max})}\,.$$

As a consequence, by Lemma 2.8, (2.54) and (2.55) we have

$$\begin{aligned}P(\mathcal{F}) - P(\mathcal{E}) &\leq \overline{P}(\mathcal{F}) - \overline{P}(\mathcal{E}) + \omega(r)\Big(\mathcal{H}^1\big(\partial\mathcal{E} \cap B(x, Hr_1)\big) + \mathcal{H}^1\big(\partial\mathcal{F} \cap B(x, Hr_1)\big)\Big) \\ &\leq -\frac{\delta}{2} r_1 + 7H\omega(r)\left(1 + \frac{h_{\max}}{h_{\min}}\right) \frac{h_{\max}}{h_{\min}} r_1 \leq -\frac{\delta}{4} r_1\,.\end{aligned}$$

Applying then Lemma 2.7 we obtain a cluster $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ such that

$$P(\mathcal{E}'') \leq P(\mathcal{F}) + 2^\beta C_{\rm per} C_{\rm vol}^\beta (Hr_1)^{\eta\beta} \leq P(\mathcal{E}) + 2^\beta C_{\rm per} C_{\rm vol}^\beta H^{\eta\beta} r_1^{\eta\beta} - \frac{\delta}{4} r_1\,.$$

We can argue now as already done several times. Indeed, the contradiction $P(\mathcal{E}'') < P(\mathcal{E})$, which concludes the proof, follows up to possibly further decrease R_5 if $\eta\beta > 1$. Instead, if $\eta\beta = 1$, it follows by (2.1) as soon as we define

$$C_{\rm per}^4 = \frac{\delta}{2^{\beta+2} C_{\rm vol}^\beta H}\,. \tag{2.56}$$

□

Thanks to the above result, we can now show the "no-lakes" lemma in full generality.

Lemma 2.15 (No-lakes, general case)**.** *There is $R_6 \leq R_5$ such that, for every ball $B(x, r) \subseteq D$ with $r < R_6$, no connected component of E_0 can be compactly contained in $B(x, r/C_2)$.*

Proof. Let $R_6 \leq R_5$ be a constant to be specified later, let $B(x, r) \subseteq D$ and assume that $G \subset\subset B(x, r/C_2)$ is the closure of a connected component of E_0. We can reduce ourselves to the case that

$$\operatorname{diam}(G) \geq \frac{r}{2C_2}\,. \tag{2.57}$$

Indeed, otherwise let x' be any point internal to G, and let $r' = C_2\operatorname{diam}(G)$. Then $r' \leq r/2 \leq R_6$ and $G \subset\subset B(x', r'/C_2) \subseteq B(x', r') \subseteq B(x, r) \subseteq D$, thus we can replace $B(x, r)$ with $B(x', r')$ for which the analogous of (2.57) clearly holds. Hence, we assume without loss of generality that (2.57) holds true.

Applying Lemma 2.14, we find $r/C_2 \leq \rho \leq r$ such that $\partial\mathcal{E} \cap \partial B(x, \rho)$ has at most three points. Using Lemma 2.11 as already done in Corollary 2.13 and Lemma 2.14, we find then at most three paths $\psi_j : (0, 1) \to B(x, \rho)$, $j \in \{1, 2, 3\}$, of finite length and with $\psi_j(0)$, $\psi_j(1) \in \partial B(x, \rho)$, whose images contain the whole $\partial\mathcal{E} \cap B(x, \rho)$, thus in particular ∂G. Since any two of the paths ψ_j may intersect only in a connected subpath by Lemma 2.12, we deduce that the intersection of any ψ_j with ∂G is a connected path. As a consequence, ∂G is the union of at most three connected pieces, each contained in one of the ψ_j. Actually, these pieces have to be exactly three. In fact, it cannot be one since the paths ψ_j, being injective, may not contain loops, and they cannot be two because otherwise two paths ψ_j would have a non connected intersection.

Since ∂G intersects three different paths ψ_j, they have to be exactly three. As a consequence, also $\partial B(x, \rho) \cap \partial\mathcal{E}$ contains exactly three points, and each of them is endpoint of two different paths ψ_j. This shows that ∂E_0 does not intersect $\partial B(x, \rho)$. Moreover, each two of the three pahts ψ_1, ψ_2, ψ_3 have a non-empty intersection, which is a closed common subpath, with one endpoint in ∂G and the other endpoint in $\partial B(x, \rho)$. In particular, $E_0 \cap B(x, \rho)$ coincides with G.

Keep in mind that each ψ_j is part of the boundary of a connected component of $E_{\ell(j)} \cap B(x, \rho)$ for some $1 \leq \ell(j) \leq m$. Since every two of the paths have a non-negligible intersection, we deduce that $\ell(1)$, $\ell(2)$ and $\ell(3)$ are different indices. For simplicity of notation, and without loss of generality, we assume that $\ell(j) = j$ for each $j \in \{1, 2, 3\}$. Just to fix the ideas, we also call

$$A = \psi_1(0) = \psi_2(1)\,, \qquad B = \psi_3(0) = \psi_1(1)\,, \qquad C = \psi_2(0) = \psi_3(1)\,,$$

and let us call A', B', C' the other endpoints of the intersections $\psi_1 \cap \psi_2$, $\psi_1 \cap \psi_3$ and $\psi_2 \cap \psi_3$ respectively. The situation is depicted in Figure 10, left.

Notice now that $\operatorname{diam}(G) = |Q - P|$ for two points $P, Q \in \partial G$. Let us assume, just to fix the ideas, that $P \in \psi_1 \cap \partial G$, and that $Q \in (\psi_1 \cup \psi_3) \cap \partial G$. We can then apply Lemma 2.12 to the paths τ_1 and τ_2 given by the restriction of ψ_1 between P and B', and between A' and P respectively, so that (2.37) gives $|P - B'| \leq C_1|A' - B'|$. In the very same way, if $Q \in \psi_1$ we get $|Q - B'| \leq C_1|A' - B'|$, while if $Q \in \psi_3$ we get $|Q - B'| \leq C_1|B' - C'|$. As a consequence, also by (2.57) we deduce

$$\frac{r}{2C_2} \leq \operatorname{diam}(G) \leq C_1\Big(|A' - B'| + |B' - C'| + |C' - A'|\Big)\,. \tag{2.58}$$

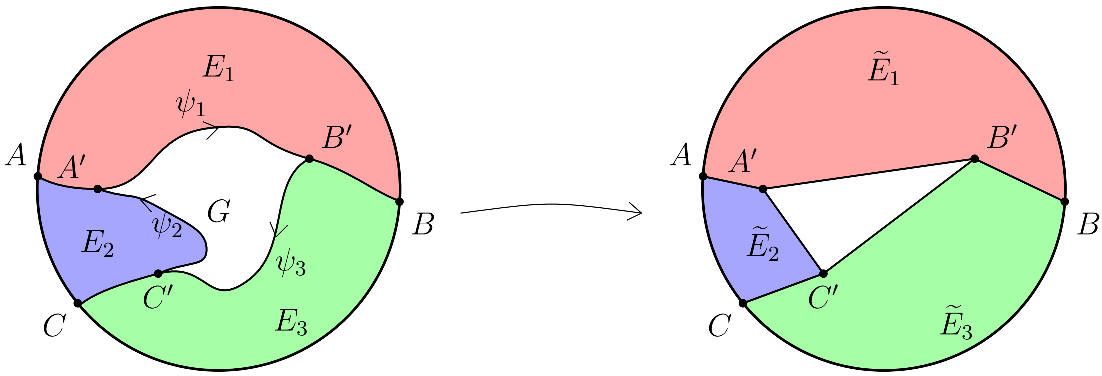

Figure 10. The paths ψ_1, ψ_2 and ψ_3 and the points A, B, C, A', B', C' in Lemma 2.15.

We set now once again $\overline{h}:\mathbb{R}^2\to\mathbb{R}^+$ as $\overline{h}(\nu)=h(x,\nu)$, $\overline{P}$ the perimeter obtained substituting h with $\overline{h}$ in (1.1), $\mathfrak{h}:\mathbb{R}^2\to\mathbb{R}^+$ as $\mathfrak{h}(\nu)=\overline{h}(\hat\nu)$, being $\hat\nu$ the angle obtained rotating $\nu\in\mathbb{R}^2$ of 90° clockwise, and $\mathrm{len}(\gamma)$ the lengt of any path γ as in (2.13). As in Figure 10, right, we define then a competitor $\widetilde{\mathcal{E}}$ by replacing the path ψ_1 with the union of the segments AA', $A'B'$ and $B'B$, the path ψ_2 with the segments CC', $C'A'$ and $A'A$, and the path ψ_3 with the segments BB', $B'C'$ and $C'C$. By Lemma 2.5 we have $\overline{P}(\widetilde{\mathcal{E}})\le\overline{P}(\mathcal{E})$.

Keeping in mind that h is strictly convex in the second variable (in the sense of Definition 1.1), and then the unit ball corresponding to $\mathfrak{h}$ is strictly convex, we have then a constant $\delta'>0$ such that

$$\mathrm{len}(A'C')\le\mathrm{len}(A'B')+\mathrm{len}(B'C')-8\delta' h_{\max}\mathrm{dist}(B',A'C')\,.$$

Putting this estimate together with the analogous ones for $C'B'$ and $B'A'$, we get

$$\begin{aligned}\mathrm{len}(A'C')+\mathrm{len}(C'B')+\mathrm{len}(B'A')&\le 2\Big(\mathrm{len}(A'B')+\mathrm{len}(B'C')+\mathrm{len}(C'A')\Big)\\&\qquad-4\delta' h_{\max}\Big(|A'-B'|+|B'-C'|+|C'-A'|\Big)\\&\le 2(1-2\delta')\Big(\mathrm{len}(A'B')+\mathrm{len}(B'C')+\mathrm{len}(C'A')\Big)\,.\end{aligned}$$

Up to exchange the letters, we have then that

$$\mathrm{len}(A'C')+\mathrm{len}(C'B')\le(1-2\delta')\Big(2\mathrm{len}(A'B')+\mathrm{len}(B'C')+\mathrm{len}(C'A')\Big)\,.$$

We are now in position to define a second competitor, $\mathcal{E}'$, simply adding the triangle $A'B'C'$ to $\widetilde{E}_1$, that is, we set $E'_j=\widetilde{E}_j$ for every $j\ne 1$, and $E'_1=\widetilde{E}_1\cup A'B'C'$. By definition, we have then

$$\begin{aligned}\overline{P}(\mathcal{E}') &= \overline{P}(\widetilde{\mathcal{E}}) - \operatorname{len}(A'B') + \frac{\operatorname{len}(A'C') - \operatorname{len}(C'A') + \operatorname{len}(C'B') - \operatorname{len}(B'C')}{2}\\ &\leq \overline{P}(\mathcal{E}) - \delta' \Big(2\operatorname{len}(A'B') + \operatorname{len}(B'C') + \operatorname{len}(C'A')\Big) \leq \overline{P}(\mathcal{E}) - \delta' \,\frac{h_{\min}}{2C_1C_2}\, r\,,\end{aligned}$$

where in the last inequality we have also used (2.58).

The conclusion is now standard. Since $\mathcal{E} = \mathcal{E}'$ outside $B(x,r)$, and Lemma 2.8 gives a bound of $\mathcal{H}^1(\partial\mathcal{E} \cap B(x,r))$ in terms of r, as soon as $R_6 \leq R_5$ is small enough we have $\omega(r)$ so small that the above inequality implies

$$P(\mathcal{E}') \leq P(\mathcal{E}) - \frac{\delta' h_{\min}}{3C_1C_2}\, r\,.$$

Then, Lemma 2.7 provides a further cluster $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ such that

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + 2^\beta C_{\rm per} C_{\rm vol}^\beta r^{\eta\beta} \leq P(\mathcal{E}) + 2^\beta C_{\rm per} C_{\rm vol}^\beta r^{\eta\beta} - \frac{\delta' h_{\min}}{3C_1C_2}\, r\,.$$

And finally, the last inequality gives the searched contradiction $P(\mathcal{E}'') < P(\mathcal{E})$ if $\eta\beta > 1$ up to possibly furher reduce R_6, while if $\eta\beta = 1$ the contradiction comes by (2.1) defining the constant

$$C_{\rm per}^5 = \frac{\delta' h_{\min}}{3\cdot 2^\beta C_1 C_2 C_{\rm vol}^\beta}\,, \tag{2.59}$$

which again only depends on h, A' and $\mathcal{E}$. The proof is then concluded. ☐

We can conclude this section by giving the definition of triple points and showing that there are only finitely many of them.

Definition 2.16 (Triple points)**.** *We say that* $x \in \mathbb{R}^2$ *is a* triple point *if*

$$\lim_{r\searrow 0}\ \#\Big\{0 \leq i \leq m:\ \big|E_i \cap B(x,r)\big| > 0\Big\} \geq 3\,.$$

Notice that, by Lemma 2.14, the no-islands Lemma 2.9 and the no-lakes Lemma 2.15, for every triple point x in the interior of D the above limit is necessarily 3.

Lemma 2.17 (Finitely many triple points)**.** *Let $B(x,r) \subseteq D$ be a ball with $r < R_6$ and being x a triple point. Then, there exists $r/C_2 \leq \rho \leq r$ such that $\partial\mathcal{E} \cap B(x,\rho)$ consists of three paths of finite length, connecting $\partial B(x,\rho)$ with x, disjoint except at x. As a consequence, there is no other triple point in $B(x,\rho)$, so in particular triple points in D are locally finitely many.*

Proof. First of all, we can apply Lemma 2.14 to find $r/C_2 \leq \rho \leq r$ such that Vol'pert Theorem 2.2 holds true for $B(x,\rho)$, ρ is a Lebesgue point of the function $s \mapsto \mathcal{H}^0(\partial\mathcal{E} \cap \partial B(x,s))$, and the value of this function at ρ is at most 3. Vol'pert Theorem implies that $\partial B(x,\rho)$ is the essentially disjoint union of either one, or two, or three open arcs belonging to different sets E_i. The no-islands Lemma 2.9 and the no-lakes Lemma 2.15 ensure that $|E_i \cap B(x,\rho)| > 0$ if and only if one of the above open arcs belong to E_i. Since x is a triple point, we deduce that $\partial\mathcal{E} \cap \partial B(x,\rho)$ consists of exactly three points, call them A, B, C for simplicity, that there are three distinct indices $0 \leq j_1, j_2, j_3 \leq m$ such that the arcs AB, BC, CA of $\partial B(x,\rho)$ belong to E_{j_1}, E_{j_2}, E_{j_3} respectively, and that $|E_i \cap B(x,\rho)| > 0$ if and only if $i \in \{j_1, j_2, j_3\}$.

Lemma 2.11 and Corollary 2.13 imply that each connected component of $E_i \cap B(x,\rho)$ has a boundary which is an injective, closed curve of finite length, and such a curve must reach $\partial B(x,\rho)$ by Lemma 2.9 and Lemma 2.15. By construction, each of the points A, B and C can be contained in at most two different curves among the boundaries of the connected components of $E_i \cap B(x,\rho)$. Therefore, for each $i \in \{j_1, j_2, j_3\}$ we have that $E_i \cap B(x,\rho)$ is made by a single connected component, and the boundary of such a connected component is the union of an arc of $\partial B(x,\rho)$ (in particular one between AB, BC and CA) and a path contained in the interior of $B(x,\rho)$. We call ψ_1, ψ_2 and ψ_3 these three arcs, and we set $\tau_1 = \psi_1 \cap \psi_2$, $\tau_2 = \psi_2 \cap \psi_3$ and $\tau_3 = \psi_3 \cap \psi_1$.

Keep in mind that $\mathcal{H}^1$-a.e. point of $\partial\mathcal{E}$ belongs to exactly two different boundaries ∂E_ℓ, with $0 \leq \ell \leq m$, hence in particular $\mathcal{H}^1$-a.e. point of $\partial\mathcal{E} \cap B(x,\rho)$ belongs to exactly two of the paths ψ_1, ψ_2 and ψ_3, that is, to one of the intersections τ_1, τ_2, τ_3. However, Lemma 2.12 implies that τ_1 is an injective closed path, which is a common subpath of ψ_1 and ψ_2, and the same is true for τ_2 and τ_3. In other words, ψ_1 is the essentially disjoint union of the connected paths τ_1 and τ_3, ψ_2 is the essentially disjoint union of τ_1 and τ_2, and ψ_3 is the essentially disjoint union of τ_2 and τ_3. The three paths τ_1, τ_2 and τ_3 meet then at some point $y \in B(x,\rho)$. Hence, we have proved that $\partial\mathcal{E} \cap B(x,\rho)$ is the union of the three paths τ_1, τ_2 and τ_3, and these three paths connect the points $A, B, C \in \partial B(x,\rho)$ with the internal point y, and they are disjoint except for the common point y. Consequently, every point of $B(x,\rho)$ different from y is *not* a triple point, and this ensures that $y = x$ and concludes the thesis. □

2.4. **Interface regularity.** This section is devoted to show the regularity of the boundary of the optimal cluster $\mathcal{E}$ away from the triple points, that is, where there are only two different sets. In this case, we show that $\partial\mathcal{E}$ is done by a union of regular curves. In particular, the goal of this section is to obtain the following result.

Proposition 2.18 ($\mathrm{C}^{1,\gamma}$ regularity). *There exists an increasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r\to 0^+} \xi(r) = 0$ with the following property. If $B(x,\bar r) \subseteq D$ is a ball with $\bar r < R_6$ and $\#\Big(\partial\mathcal{E} \cap \partial B(x,\bar r)\Big) = 2$, then $\partial\mathcal{E} \cap B(x,\bar r)$ is a C^1 curve of finite length having both endpoints in $\partial B(x,\bar r)$. Moreover, calling $\tau(y) \in \mathbb{P}^1$ the direction of the tangent vector at* *any $y \in \partial\mathcal{E} \cap B(x,\bar r)$, one has*

$$|\tau(y) - \tau(z)| \leq \xi(|y-z|) \tag{2.60}$$

for every $y,\, z \in \partial\mathcal{E} \cap B(x,\bar r)$. Finally, if $\eta\beta > 1$ and h is locally α-Hölder in the first variable, then it is possible to take $\xi(t) = Kt^\gamma$ with some $K > 0$ and

$$\gamma = \frac{1}{2}\min\{\eta\beta - 1,\, \alpha\}\,, \tag{2.61}$$

so that in particular $\partial\mathcal{E} \cap B(\bar x,\bar r)$ is $\mathrm{C}^{1,\gamma}$.

The first step is to show that in a small ball the boundary is close to a line.

Lemma 2.19 (Almost alignment in a circle)**.** *There exists a function $\xi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ as in Proposition 2.18 and satisfying the Dini property such that the following holds. Let $x \in \partial\mathcal{E}$ and $r < R_6$ be such that $B(x,r) \subseteq D$ and $\partial B(x,r) \cap \partial\mathcal{E}$ consists of two points, call them a and b. Then, for every $y \in \partial\mathcal{E} \cap B(x,r/2)$ one has $|y\hat{a}b| \leq \xi_1(r)$.*

Proof. We directly define $\mathcal{E}'$ as the cluster which coincides with $\mathcal{E}$ outside of $B(x,r)$ and such that $\partial\mathcal{E}' \cap B(x,r)$ is done by the segment ab. Keep in mind that, since $\partial\mathcal{E} \cap \partial B(x,r)$ is done by two points, Lemma 2.9 and Lemma 2.15 imply that $B(x,r)$ is the union of two connected regions, each one contained in a set E_i for some $0 \leq i \leq m$. By Lemma 2.11 and Corollary 2.13, we know that the common boundary between these regions, which coincides with the whole $\partial\mathcal{E} \cap B(x,r)$, is a path γ contained in $B(x,r)$ and connecting a to b. Since $y \in \partial\mathcal{E}$, in particular $y \in \gamma$. Let us call i_l and i_r the two indices in $\{0,\, 1,\, \dots\,,\, m\}$ so that the set E_{i_l} (resp., E_{i_r}) is on the left side of γ (resp., on the right side).

Once again, for every $\nu \in \mathbb{S}^1$ we call $\hat\nu$ the angle obtained rotating ν of 90° clockwise. This time, we set $\mathfrak{h}(\nu) = h(x,\hat\nu)$ if $i_r = 0$, $\mathfrak{h}(\nu) = h(x,-\hat\nu)$ if $i_l = 0$, and $\mathfrak{h}(\nu) = (h(x,\hat\nu)+h(x,-\hat\nu))/2$ if both the indices i_l and i_r are different from 0. We use then (2.13) to define the length of curves with this choice of $\mathfrak{h}$. Also by Lemma 2.8 and Lemma 2.5, we have

$$\begin{aligned} P(\mathcal{E}') - P(\mathcal{E}) &\leq \mathrm{len}(ab) + \omega(r)|b-a| - \mathrm{len}(\gamma) + \omega(r)\mathcal{H}^1(\gamma) \\ &\leq \mathrm{len}(ab) - \mathrm{len}(ay) - \mathrm{len}(yb) + \left(2 + 7\,\frac{h_{\max}}{h_{\min}}\right) r\omega(r)\,. \end{aligned} \tag{2.62}$$

Let us now write for brevity $\theta = y\hat{a}b$. We claim that

$$\mathrm{len}(ay) + \mathrm{len}(yb) - \mathrm{len}(ab) \geq c' r \sin^2\theta\,, \tag{2.63}$$

for a constant c' which only depends on h and D. To show this estimate, we call $y_\perp$ the projection of y on ab. First of all, we can reduce ourselves to the "symmetric" case when $y_\perp$ is the middle point of ab. Indeed, assume that $y_\perp$ is not the middle point of ab and let $a'b'$ the shortest segment containing ab with middle point equal to $y_\perp$. If (2.63) holds true in the symmetric case, then in particular it holds true with a', b' and $\theta' = \widehat{ya'b'}$ in place of a, b and θ. Moreover, since $r/2 \leq |y-a|, |y-b| \leq 3r/2$, then $\sin\theta' \geq \sin\theta/3$. Since the triangular inequality implies

$$\operatorname{len}(ay) + \operatorname{len}(yb) - \operatorname{len}(ab) \geq \operatorname{len}(a'y) + \operatorname{len}(yb') - \operatorname{len}(a'b')\,,$$

we have then the validity of (2.63) in the general case, up to divide c' by 9. We are then left to show (2.63) in the symmetric case. Let us call

$$\nu = \frac{y_\perp - a}{|y_\perp - a|}\,, \qquad\qquad w = \frac{y - y_\perp}{4|y_\perp - a|}\,,$$

so that by construction $|\nu| = 1$ and $|w| \leq 1$. Keep in mind that $\mathfrak{h}$ is uniform round in the second variable (see Definition 1.1), and then (1.4) and the convexity give

$$\begin{aligned}\operatorname{len}(ay) + \operatorname{len}(yb) - \operatorname{len}(ab) &= |y_\perp - a|\Big(\mathfrak{h}(\nu+4w) + \mathfrak{h}(\nu-4w) - 2\mathfrak{h}(\nu)\Big)\\ &\geq \frac{r}{2}\Big(\mathfrak{h}(\nu+w) + \mathfrak{h}(\nu-w) - 2\mathfrak{h}(\nu)\Big) \geq rc|w|^2 \geq \frac{c}{16}\,r\sin\theta^2\,,\end{aligned}$$

and this proves (2.63). Inserting this inequality in (2.62), we obtain

$$P(\mathcal{E}') - P(\mathcal{E}) \leq -c'r\sin^2\theta + \left(2 + 7\,\frac{h_{\max}}{h_{\min}}\right) r\omega(r)\,.$$

Observe that $\Big||\mathcal{E}'| - |\mathcal{E}|\Big| \leq 2|B(x,r)| \leq 2C_{\rm vol}r^\eta$, and write for brevity $\widetilde{C_{\rm per}} = C_{\rm per}[2C_{\rm vol}r^\eta]$. Appling then once again Lemma 2.7, using the constant $\widetilde{C_{\rm per}}$ in place of $C_{\rm per}$ in (2.15), the optimality of $\mathcal{E}$ implies that

$$c'r\sin^2\theta \leq \left(2 + 7\,\frac{h_{\max}}{h_{\min}}\right) r\omega(r) + \widetilde{C_{\rm per}}(2C_{\rm vol}r^\eta)^\beta\,,$$

which implies

$$\theta \leq \xi_1(r) := \frac{\pi}{2}\left[\frac{1}{c'}\left(\left(2 + 7\,\frac{h_{\max}}{h_{\min}}\right)\omega(r) + \widetilde{C_{\rm per}}2^\beta C_{\rm vol}^\beta r^{\eta\beta-1}\right)\right]^{1/2}.$$

To conclude the proof, we have then to check that ξ_1 satisifies all the requirements. Keeping in mind that $\widetilde{C_{\rm per}} = C_{\rm per}[2C_{\rm vol}r^\eta]$ is an increasing function of r which goes to 0 when $r \searrow 0$, the fact that ξ_1 is an increasing function and that $\lim_{r\searrow 0}\xi_1(r) = 0$ is true by construction. If $\eta\beta > 1$ and h is α-Hölder in the first variable, then we obtain

$$\xi_1(r) \lesssim \sqrt{r^\alpha + r^{\eta\beta-1}} \approx r^\gamma\,,$$

with γ given by (2.61). Finally, up to multiplicative constants we have that

$$\xi_1(r) \leq \sqrt{\omega(r) + r^{\eta\beta-1}C_{\rm per}[2C_{\rm vol}r^\eta]} \leq \sqrt{\omega(r)} + \sqrt{r^{\eta\beta-1}C_{\rm per}[2C_{\rm vol}r]}\,,$$

where the last inequality comes because $\eta \geq 1$. The Dini property of ξ_1 then follows, since $r \mapsto \omega(r)$ satisfies the 1/2-Dini property, and the same is true for $r \mapsto C_{\rm per}[r]$ if $\eta\beta = 1$. □

Corollary 2.20. *For every $x \in D \cap \partial\mathcal{E}$ and every $r < \min\{\operatorname{dist}(x, \partial D), R_6\}/2C_2$ with the property that $\#\Big\{0 \le i \le m : |B(x,r) \cap E_i| > 0\Big\} = 2$, there is a direction $\tau = \tau(x,r) \in \mathbb{P}^1$ so that each point $y \in \partial\mathcal{E} \cap \partial B(x,r)$ satisfies*

$$|\zeta(y-x) - \tau(x,r)| \le 24C_2\xi_1(2C_2r) \tag{2.64}$$

where, for each $v \in \mathbb{R}^2 \setminus \{0\}$, we denote by $\zeta(v) \in \mathbb{P}^1$ the direction of v. In particular, if $w, z \in \partial\mathcal{E}$ are two points such that, calling $d = |w - z|$, both $\tau(w,d)$ and $\tau(z,d)$ are defined, then

$$|\tau(w,d) - \tau(z,d)| \le 48C_2\xi_1(2C_2d)\,. \tag{2.65}$$

In addition, for each $r' \in [r/2, r]$ one has

$$|\tau(x,r) - \tau(x,r')| \le 48C_2\xi_1(2C_2r)\,. \tag{2.66}$$

Proof. Since by assumption the ball $B(x, 2C_2r)$ is contained in D and its radius is less than $R_6 \le R_5$, we apply Lemma 2.14 and find some $2r < \rho < 2C_2r$ such that $\partial\mathcal{E}\cap\partial B(x,\rho)$ has at most three points. These points cannot be three since by assumption $B(x,\rho) \subseteq B(x, 2C_2r)$ intersects only two regions E_i, $0 \le i \le m$, and they cannot be less than 2 because $x \in \partial\mathcal{E}$ and by the no-islands Lemma 2.9 and the no-lakes Lemma 2.15. Therefore, $\partial\mathcal{E} \cap \partial B(x,\rho)$ has necessarily exactly two points, and we call them a and b for simplicity. The vector $\tau(x,r) \in \mathbb{P}^1$ can be then simply defined as the direction $\zeta(b-a)$ of the segment ab. Notice that this direction is not uniquely determined by x and r, since it also depends on the particular choice of ρ.

To check the properties of τ, let us take $y \in \partial B(x,r') \cap \partial\mathcal{E}$ for some $r/2 \le r' \le r$. Just to fix the ideas, let us call w_2 the second coordinate of any point $w \in \mathbb{R}^2$, and let us assume that the segment ab is horizontal, with $a_2 = b_2 = 0$. Since both y and x belong to $\partial\mathcal{E} \cap B(x,\rho/2)$, Lemma 2.19 gives $|x\hat{a}b| < \xi_1(\rho)$ and $|y\hat{a}b| < \xi_1(\rho)$, which implies

$$|x_2| = |x-a||\sin x\hat{a}b| \le \rho\sin\xi_1(\rho)\,, \qquad |y_2| = |y-a||\sin y\hat{a}b| \le 2\rho\sin\xi_1(\rho)\,,$$

and then

$$|\sin\zeta(y-x)| = \frac{|y_2 - x_2|}{|y-x|} \le \frac{3\rho\sin\xi_1(\rho)}{r'} \le 12C_2\sin\xi_1(\rho) \le 12C_2\sin\xi_1(2C_2r)\,.$$

Summarizing, since for every $0 \le \theta \le \pi/2$ we have $\theta \ge \sin\theta \ge 2\theta/\pi \ge \theta/2$, and since $\tau(x,r)$ is the horizontal direction, we have proved that

$$|\zeta(y-x) - \tau(x,r)| \le 24C_2\xi_1(2C_2r) \qquad \forall\, y \in \partial\mathcal{E} \cap \Big(\overline{B(x,r)} \setminus B(x,r/2)\Big)\,. \tag{2.67}$$

The particular case in which $y \in \partial B(x,r)$ is (2.64). Let now $r/2 \le r' \le r$, and take a point $y \in \partial\mathcal{E} \cap \partial B(x,r')$. Since we can apply (2.67) to x and y both with r and with r', we deduce

$$|\tau(x,r) - \tau(x,r')| \le |\zeta(y-x) - \tau(x,r)| + |\zeta(y-x) - \tau(x,r')| \le 48C_2\xi_1(2C_2r)\,,$$

which is (2.66). Finally, let $z,\, w \in \partial\mathcal{E}\cap\partial B(x,r)$ be such that, calling $d = |w-z|$, one has $2C_2 r < \min\Big\{\mathrm{dist}(z,\partial D),\, \mathrm{dist}(w,\partial D), R_6\Big\}$, so that both $\tau(z,d)$ and $\tau(w,d)$ are defined. Then, we can apply (2.67) with $r = d$, $x = z$, $y = w$, and also with $r = d$, $x = w$, $y = z$. This gives then (2.65). □

We are now in position to prove Proposition 2.18.

Proof (of Proposition 2.18). Let x and $\bar r$ be as in the claim of the proposition. The fact that $\partial\mathcal{E}\cap B(x,\bar r)$ is an injective curve of finite length with both endpoints in $\partial B(x,\bar r)$ has already been observed in Lemma 2.19. By Corollary 2.20, a direction $\tau(y,r)$ is defined for every $y \in \partial\mathcal{E}\cap B(x,\bar r)$ and for every $r < \min\{\mathrm{dist}(y,\partial D), R_6\}/2C_2$. Moreover, (2.66) holds true as soon as $r/2 \le r' \le r$. An obvious induction gives then, for every $n \in \mathbb{N}$ and every $r/2^n \le r' \le r$,

$$|\tau(y,r) - \tau(y,r')| \le 48C_2 \sum_{i=0}^{n-1} \xi_1(2C_2 r/2^i)\,. \tag{2.68}$$

We define then

$$\xi(r) = 144C_2 \sum_{i=0}^{+\infty} \xi_1(2C_2 r/2^i)\,.$$

Notice that the series converges because ξ_1 satisfies the Dini property. In particular, if $\eta\beta > 1$ and h is locally α-Hölder in the first variable, then $\xi_1(r) = K_1 r^\gamma$, with γ given by (2.61) and K_1 being a constant depending on $\mathcal{E}$, D, g and h. Thus, also $\xi(r) = K r^\gamma$ by definition.

By (2.68) we obtain that $\tau(y,r)$ converges to a direction when $r \searrow 0$, and we call $\tau(y) \in \mathbb{P}^1$ this limit direction. By construction, $|\tau(y,r) - \tau(y)| \le \xi(r)/3$. Therefore, for every point $z \in \partial\mathcal{E}\cap\partial B(y,r)$, recalling (2.64) we have then $|\zeta(z-y) - \tau(y)| \le \xi(r)/2$, so that $\tau(y)$ is the tangent vector at y of the curve $\partial\mathcal{E}\cap B(x,\bar r)$. Finally, (2.60) comes by (2.65). □

2.5. **Conclusion.** In this short last section we can now give the proof of Theorem A, which basically consists in putting together the technical results of the preceding sections.

Proof of Theorem A. Let $\mathcal{E} \subseteq \mathbb{R}^2$ be a minimal cluster, and let us fix two large, closed balls $D^- \subset\subset D \subseteq \mathbb{R}^2$. Let x be any point in $D^- \cap \partial\mathcal{E}$. If x is not a triple point, by Lemma 2.14 there is a small constant $r(x) < R_6$ such that $\partial\mathcal{E}\cap\partial B(x, r(x))$ consists of two points (the points cannot be three if $r(x)$ is small enough, as already noticed). By Proposition 2.18 we have then that $\partial\mathcal{E}\cap\partial B(x,r(x))$ is a C^1 curve, whose tangent vector satisfies the uniform estimate (2.60).

Suppose instead that x is a triple point. Then, again by Lemma 2.14, there is a small constant $r(x) < R_6$ such that $\partial\mathcal{E}\cap\partial B(x,r(x))$ consist of three points, call them a, b and c. Lemma 2.17 already gives that $\partial\mathcal{E}\cap B(x,r(x))$ is done by three paths of finite length, connecting a, b and c to x, and disjoint except for the common endpoint x. Let $z \neq x$ be any point of one of these paths. Since z is not a triple point, by the above argument we know that $\partial\mathcal{E}$ is a C^1 curve in a neighborhood of z, and the tangent vector satisfies the uniform estimate (2.60). As a consequence, the three paths are three C^1 paths, and they meet at x with three well-defined tangent vectors.

Summarizing, each point $x \in \partial\mathcal{E} \cap D^-$ is center of a ball, in the interior of which $\partial\mathcal{E}$ is given either by a single C^1 curve, or by three C^1 curves meeting with three tangent vectors in the center. By compactness, we can cover $\partial\mathcal{E} \cap D^-$ with finitely many such balls, and then the Steiner property of $\mathcal{E}$ follows. The $\mathrm{C}^{1,\gamma}$ regularity in the case that h is locally α-Hölder in the first variable and $\eta\beta > 1$ is given by Proposition 2.18. □

3. Final comments

This last section is devoted to present a couple of final comments about our result. The first observation is about the role of the C^1 property to obtain that multiple points are necessarily triple points, and the second one is about the directions of the arcs at triple points.

3.1. **The importance of the C^1 property to obtain triple points.** Our main result, Theorem A, concerns the Steiner property for minimal clusters, that is, the boundary of a minimal cluster is made by finitely many C^1 arcs which meet each other in triple points. We have shown that this property is true as soon as, together with the "correct" growth conditions and $\varepsilon - \varepsilon^\beta$ property, h is strictly convex, uniformly round and C^1 in the second variable. As already discussed in the Introduction, the importance of the strict convexity and uniform roundedness to obtain a Steiner propery is very simple to understand. Concerning the C^1 regularity of h, it is also clear that this is necessary to get local C^1 regularity of minimal clusters. This has nothing particular to do with the fact that we deal with clusters, the very same happens even in the much simpler case of isoperimetric sets. For instance, if h does not depend on the first variable, then isoperimetric sets are translations and homotheties of the unit ball of h, so they are exactly as regular as h is. Less obvious is the role played by the C^1 property of h in order to get triple points. This section is devoted to show by means of an example that quadruple points may occur for a density which is strictly convex and uniformly round but not C^1. In fact, as shown in [25], quadruple points are the worst than can happen, that is, a minimizing cluster for a generic norm in $\mathbb{R}^2$ may not have multiple points where more than four arcs meet. We start with the following weaker example, depicted in Figure 11, left.

Example 3.1. *Let us consider the L^1 norm in $\mathbb{R}^2$, that is, $\|\nu\| = \max\{|\nu_1|, |\nu_2|\}$ for every vector $\nu \in \mathbb{R}^2$. We want to show that an isoperimetric cluster may have a quadruple* *point in a region where $h(x,\nu) = \|\nu\|$, with the Euclidean volume $g \equiv 1$. Indeed, assume that $h(x,\nu) = \|\nu\|$ for every x inside the square $[-1,1] \times [-1,1]$, and that a minimal cluster $\mathcal{E}$ is so that the four sides of the square belong to E_1, E_2, E_3 and E_4 (this can be achieved by a suitable choice of the density outside the square). Moreover, assume that inside the square the volumes of E_1, E_2, E_3 and E_4 have to be 1 each. It is then easy to see that the perimeter of the cluster inside the square is at least 4, and it equals 4 if and only if the boundary of the cluster inside the square is done by the two diagonals. Since for this configuration the volumes of the four sets inside the square are all equal to 1, this must be the minimal cluster. The origin is then a quadruple point.*

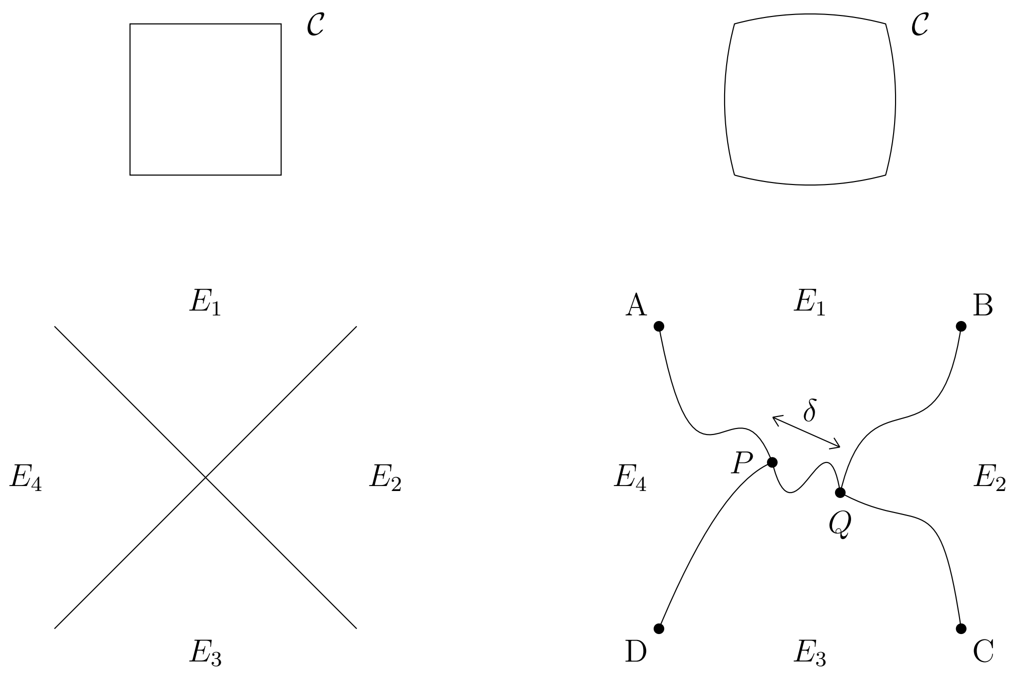

Figure 11. Left: the unit ball $\mathcal{C}$ and an isoperimetric cluster for Example 3.1. Right: the unit ball $\mathcal{C}$ and an (impossible) isoperimetric cluster without quadruple points for Example 3.2.

We can now slightly modify the above example, so that the density h becomes strictly convex and uniformly round, and still a quadruple point occurs.

Example 3.2. *We define this time a density h inside the square $\mathcal{Q} = [-1,1] \times [-1,1]$ as in Figure 11, right. It is very close to the L^1 density of Example 3.1, but the four sides of the unit ball are now substituted by four arcs, with strictly positive but very small curvature, and with the same endpoints, i.e. $(\pm 1, \pm 1)$. Notice that h is not C^1, but it is strictly convex and uniformly round. As in the example above, with a suitable choice of h outside of the square and with $g \equiv 1$ we can obtain a minimal cluster $\mathcal{E}$ so that the* *four sides of the square* $\mathcal{Q}$ *belong to the sets* E_1, E_2, E_3 *and* E_4, *and that these four sets have volume* 1 *each inside the square. We claim that then the minimal cluster has again a quadruple point at the origin. If this is false, then there is either a quadruple point at a point different from the origin, or (at least) two triple points. We can then call* P *and* Q *two multiple points, as in the figure, and assume that they do not coincide both with the origin, in particular they are either both triple points and distinct, or they could coincide and be a single quadruple point, but not in the origin. Notice that, since the density is very close to the one of Example 3.1, and in that case the boundary of the minimal cluster was done by the two digonals* AC *and* BD, *then both the points* P *and* Q *must be very close to the origin, so in particular* $d, \delta \ll 1$ *where we call* $\delta = |P-Q|$ *the Euclidean distance between the two points and* $d = \max\{|P-O|, |Q-O|\}$ *the Euclidean distance between the origin and the furthest of the two points, that we assume to fix the ideas to be* P. *By Lemma 2.5, the perimeter of the cluster inside the square is*

$$P(\mathcal{E};\mathcal{Q}) \geq \mathrm{len}(AP) + \mathrm{len}(DP) + \mathrm{len}(PQ) + \mathrm{len}(BQ) + \mathrm{len}(CQ)\,, \tag{3.1}$$

where as usual we denote by len *the length of a curve, or a segment, with repect to* h. *Notice that in this case there is no need to consider oriented segments, since the density is symmetric, and there is also no need to consider a clockwise rotation of* $90°$ *as through the rest of the paper, because the density remains the same after a rotation of* $90°$. *Notice that there is a constant* $C > 0$, *depending on* h *such that*

$$\mathrm{len}(AP) + \mathrm{len}(BP) + \mathrm{len}(CP) + \mathrm{len}(DP) \geq 4 + Cd\,.$$

In fact, the best constant C *for which the above inequality is true depends continuously on the curvature of the four arcs of* $\mathcal{C}$. *Since we have* $C = 1$ *for the case of Example 3.1, which corresponds to zero curvature, we can assume* $C > 1/2$ *up to have chosen a sufficiently low curvature. Similarly, we have*

$$\mathrm{len}(BQ) + \mathrm{len}(CQ) \geq \mathrm{len}(BP) + \mathrm{len}(CP) - C'\delta\,.$$

This estimate is again easily seen to hold with $C' = 2$ *for the density of Example 3.1, so with some* $C' < 3$ *in the present case. Inserting the last two estimates in (3.1), and keeping in mind that the optimal cluster has most perimeter* 4 *in* $\mathcal{Q}$ *(because we can use the cluster with boundary in* $\mathcal{Q}$ *given by the two diagonals as competitor), we get*

$$d < 6\delta\,, \tag{3.2}$$

that is, the points P *and* Q *cannot be much closer to each other than to the origin. There exists a third constant* $c > 0$ *such that*

$$\mathrm{len}(AP) + \mathrm{len}(DP) \geq \mathrm{len}(AO) + \mathrm{len}(DO) - c|P-O| = 2 - cd\,.$$

In fact, for the density of Example 3.1 this is true with $c = 0$*, because the unit ball for* h *in that case is a square. In our case, provided that the curvature of the arcs of* $\mathcal{C}$ *is suficiently small, we can assume* c *as small as desired. Similarly,*

$$\mathrm{len}(BQ) + \mathrm{len}(CQ) \geq 2 - c|Q - O| \geq 2 - cd\,.$$

Finally,

$$\mathrm{len}(PQ) \geq \frac{\sqrt{2}}{2}\,|P - Q| = \frac{\sqrt{2}}{2}\,\delta\,.$$

Inserting the last three estimates in (3.1), we get

$$cd \geq \frac{\sqrt{2}}{4}\,\delta\,,$$

and since as observed c *can be taken arbitrarily small this is gives a contradiction to (3.2). We have then proved that also for this modified density* h*, which is not* C^1 *but which is strictly convex and uniformly round, a minimal cluster can have a quadruple point.*

3.2. **The directions of the arcs at triple points.** This section is devoted to discuss which can be admissible directions for the tangents of $\partial^*\mathcal{E}$ at some triple point. First of all, we can immediately write down the first order minimality property.

Lemma 3.3. *Let* $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}^+$ *be a positively* 1*-homogeneous fuction, strictly positive and* C^1 *except at* 0 *and with strictly convex unit ball. Let moreover* A, B, C, O *be four distinct points such that* A, B *and* C *are nonaligned. Then,* O *uniquely minimizes the function* $L : \mathbb{R}^2 \to \mathbb{R}^+$ *given by*

$$L(P) := \mathfrak{h}(PA) + \mathfrak{h}(PB) + \mathfrak{h}(PC)$$

if and only if

$$\nabla\mathfrak{h}(OA) + \nabla\mathfrak{h}(OB) + \nabla\mathfrak{h}(OC) = 0\,. \tag{3.3}$$

Proof. The function $P \mapsto L(P)$ is strictly convex because so is the unit ball of $\mathfrak{h}$ and because A, B, C are nonaligned. Moreover, this function is C^1 except at A, B and C. Therefore, O uniquely minimizes L if and only if $\nabla L(O) = 0$. In particular,

$$\nabla L(O) = -\Big(\nabla\mathfrak{h}(OA) + \nabla\mathfrak{h}(OB) + \nabla\mathfrak{h}(OC)\Big)\,,$$

hence we have concluded. □

Notice that the above geometrical property characterizes the possible directions corresponding to triple points. Let us be more precise. Suppose for a moment, just for simplicity, that $\mathfrak{h}$ is symmetric, so that the length of curves is defined (otherwise one has to speak about *oriented* curves, as already done in Section 2). Let then A, B and C be three points in $\mathbb{R}^2$. By means of Lemma 2.5, it is very simple to notice that the shortest connected set containing A, B and C is always given by the three segments joining A, B

and C with some point O, which might coincide with one between A, B and C. Of course, this point O minimizes the function $L(P) = \mathfrak{h}(PA) + \mathfrak{h}(PB) + \mathfrak{h}(PC)$.

Let us consider this function in the general case when $\mathfrak{h}$ does not need to be symmetric. The existence of a point O minimizing L is obvious, and by the strict convexity of the unit ball we can observe that such a point is uniquely determined if the points are not aligned. In addition, if the points are not aligned and O is not one of them, the three directions OA, OB and OC necessarily satisfy the relation (3.3). It is interesting to observe "how many" admissible triples there are, and this is explained by the lemma below.

Lemma 3.4. *Let $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}^+$ be as in Lemma 3.3. Then, there exists at least a triple $\{A, B, C\}$ in $\partial\mathcal{C}$ which satisfies the property (3.3). In particular, for every $A \in \partial\mathcal{C}$, there is exactly a pair $\{B, C\}$ such that the triple $\{A, B, C\}$ satisfies (3.3) if $\mathfrak{h}$ is symmetric, while if $\mathfrak{h}$ is not symmetric it is also possible that there is no such pair, or several ones.*

Proof. We have already noticed that for every three non-aligned points A, B, C there exists a unique point O minimizing the function L defined above, and the three directions OA, OB and OC satisfy the property (3.3) if $O \notin \{A, B, C\}$. To prove the first part of the statement we want then to find three such points. Notice that the requirement that the points A, B, C belong to $\partial\mathcal{C}$ is just to fix their length, but since $\nabla\mathfrak{h}$ is 0-homogeneous this can be achieved for free just dividing the length of the segments OA, OB and OC by $\mathfrak{h}(OA)$, $\mathfrak{h}(OB)$ and $\mathfrak{h}(OC)$ recpectively.

We start by taking three non-aligned points $A_0, B, C \in \mathbb{R}^2$. If the corresponding minimizing point O is not one of them, we are already done. Otherwise, we can assume that $O = A_0$. We will obtain the first part of the statement by finding a point A_t such that A_t, B and C are still non-aligned, and the point O_t minimizing the function $L_t(P) = \mathfrak{h}(PA_t) + \mathfrak{h}(PB) + \mathfrak{h}(PC)$ is not in $\{A_t, B, C\}$.

Up to change the names of the points, we assume that $\mathfrak{h}(BC) \leq \mathfrak{h}(CB)$. Then, we let A_1 be the point such that C is the middle point of the segment A_1B. Moreover, for any $0 < t < 1$ we write $A_t = tA_1 + (1-t)A_0$. For every $0 \leq t < 1$ the points A_t, B and C are not aligned, hence L_t is minimized by a unique point, that we call O_t. If, for some $0 < t < 1$, the point O_t is not one between A_t, B and C then we are done. If this does not happen, then by uniqueness and continuity we derive $O_t = A_t$ for every $0 < t < 1$. Again by continuity, the point A_1 is then a minimizer (not necessarily the unique one) of the function L_1. And in turn, this gives a contradiction because

$$L_1(A_1) = 3\mathfrak{h}(CB) > \mathfrak{h}(CB) + \mathfrak{h}(BC) = L_1(C) \,.$$

The first part of the claim is then proved.

Let us now pass to the second part. As already observed in Section 2.2, see in particular (2.5), for every $P \in \partial\mathcal{C}$ we have

$$\nabla\mathfrak{h}(OP) = \frac{\nu_P}{OP \cdot \nu_P}, \tag{3.4}$$

where ν_P denotes the outer unit normal vector to P at $\partial\mathcal{C}$. Fix now a direction $\eta \in \mathbb{S}^1$, and consider the line passing through O in direction η. As in Figure 12, left, we call R^- and R^+ the two intersections of this line with $\partial\mathcal{C}$, being $OR^- \cdot \eta < 0 < OR^+ \cdot \eta$, and Q^+, Q^- the two points of $\partial\mathcal{C}$ which respectively maximize and minimize the signed distance with the line. A simple geometric observation, coming from the strict convexity and regularity of $\partial\mathcal{C}$, shows the following. The function $\partial\mathcal{C} \ni P \mapsto \nabla\mathfrak{h}(OP) \cdot \eta = \partial\mathfrak{h}/\partial\eta(OP)$ is 0 in $P = Q^+$, then it continuously strictly increases when P is moved between Q^+ and R^+, reaching its maximum at $P = R^+$, then it continuously strictly decreases when P goes from R^+ to Q^-, and it is 0 again in $P = Q^-$. Similarly, the function decreases for P between Q^- and R^-, where the minimum is reached, and then it increases up to $P = Q^+$.

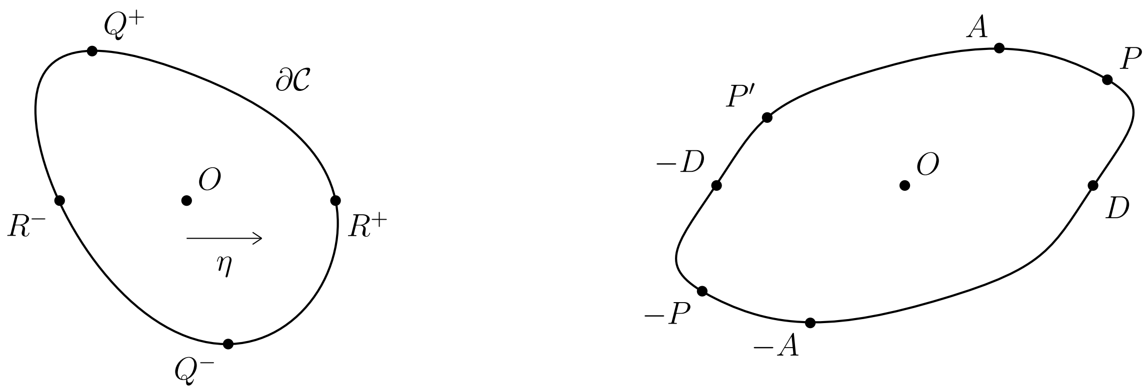

Figure 12. Left: the function $P \mapsto \nabla(OP) \cdot \eta$. Right: situation when $\mathfrak{h}$ is symmetric.

Let us now fix a point $A \in \partial\mathcal{C}$, and let us start by considering the symmetric case. We have to show that there exists a unique pair $\{B, C\}$ such that the triple $\{A, B, C\}$ satisfies (3.3). Up to a rotation, we assume that A is the point of $\partial\mathcal{C}$ with biggest second coordinate, so that $\nu_A = (0, 1)$. The situation is depicted in Figure 12, right. Keep in mind that, as observed above, $\partial\mathfrak{h}/\partial x(OP)$ is positive for points P in the "right part" of $\partial\mathcal{C}$, i.e., in the clockwise arc from A to $-A$, and it is negative for points in the "left part" of $\partial\mathcal{C}$. Since $\partial\mathfrak{h}/\partial x(OA) = 0$, this implies that a triple $\{A, B, C\}$ satisfying (3.3) must necessarily have one between B and C in the "right part" of $\partial\mathcal{C}$, and the other one in the left part. Let us now take a point P in the right part of $\partial\mathcal{C}$, and let us ask ourselves whether or not a suitable triple may exist with $B = P$. This happens if and only if there is some $C \in \partial\mathcal{C}$ such that, calling η the direction of the vector OA,

$$\frac{\partial\mathfrak{h}}{\partial x}(OC) = -\frac{\partial\mathfrak{h}}{\partial x}(OP), \qquad \frac{\partial\mathfrak{h}}{\partial\eta}(OC) + \frac{\partial\mathfrak{h}}{\partial\eta}(OP) = -\frac{\partial\mathfrak{h}}{\partial\eta}(OA), \tag{3.5}$$

The monotonicity of $\partial\mathfrak{h}/\partial x$ observed above, together with the symmetry of $\mathcal{C}$, ensures that there are exactly two points satisfying the first equality. One of them is $-P$, for which the second equality is surely false because $\partial\mathfrak{h}/\partial\eta(OC) + \partial\mathfrak{h}/\partial\eta(OP) = 0$, and the other one is some point P'. As shown in the figure, P is above $D = \partial\mathcal{C} \cap \{(x, 0) : x > 0\}$ if and only if P' is above $-D$. Summarizing, a suitable pair can only exist with $B = P$ and $C = P'$ for some P in the right part of $\partial\mathcal{C}$, and in particular we have only to take care of the second equality in (3.5) since the first one is true by construction. Notice that, if P continuously ranges from A to A' in the right part of $\partial\mathcal{C}$, then P' continuously ranges from A to A' in the left part, and there is a one-to-one correspondence between P and P'. Again recalling the observation above about the monotonicity of $\partial\mathfrak{h}/\partial\eta$, we have that the quantity $\partial\mathfrak{h}/\partial\eta(OP)$ is strictly decreasing when P moves from A to A', and the same happens for $\partial\mathfrak{h}/\partial\eta(OP')$. As a consequence, there can be at most a single point P such that the right equality in (3.5) holds with $C = P'$. And finally, the existence of such a point P is ensured by the continuity, since for $P = A$

$$\frac{\partial\mathfrak{h}}{\partial\eta}(OP') + \frac{\partial\mathfrak{h}}{\partial\eta}(OP) = 2\,\frac{\partial\mathfrak{h}}{\partial\eta}(OA) > -\frac{\partial\mathfrak{h}}{\partial\eta}(OA)$$

and for $P = -A$

$$\frac{\partial\mathfrak{h}}{\partial\eta}(OP') + \frac{\partial\mathfrak{h}}{\partial\eta}(OP) = -2\,\frac{\partial\mathfrak{h}}{\partial\eta}(OA) < -\frac{\partial\mathfrak{h}}{\partial\eta}(OA)\,.$$

Now, let us remove the assumption that $\mathfrak{h}$ is symmetric, and let us present an example in which no pair $\{B, C\}$ exists such that $\{A, B, C\}$ satisfies (3.3), and another example in which more than a single pair exists.

The first example, depicted in Figure 13, left, is very simple, it is enough to take as $\mathcal{C}$ a disk with radius 1 centered at the point $(0, -1/2)$, and call $A = (0, 1/2)$ and $A' = (0, -3/2)$. By (3.4), for every possible choice of $B, C \in \partial\mathcal{C}$ we have

$$\frac{\partial\mathfrak{h}}{\partial y}(OA) + \frac{\partial\mathfrak{h}}{\partial y}(OB) + \frac{\partial\mathfrak{h}}{\partial y}(OC) \geq \frac{\partial\mathfrak{h}}{\partial y}(OA) + 2\,\frac{\partial\mathfrak{h}}{\partial y}(OA') = 2 - \frac{4}{3} > 0\,,$$

and then $\{A, B, C\}$ cannot be an admissible triple.

Let us now present an example with more than a single admissible triple containing a given point A. As shown in Figure 13, right, we define $A = (0, 1)$ and we let $\partial\mathcal{C}$ coincide with the circle $\partial B(0, 1)$ for a short while around A. We let also $B = (\sin\theta, \cos\theta)$ for a small $\theta > 0$. We have then, by construction and recalling (3.4),

$$\nabla\mathfrak{h}(OA) + \nabla\mathfrak{h}(OB) = (0, 1) + (\sin\theta, \cos\theta)\,,$$

and then a point $C \in \partial\mathcal{C}$ completes a suitable triple together with A and B if and only $\nabla\mathfrak{h}(OC) = (-\sin\theta, -1 - \cos\theta)$. Again by (3.4), this is equivalent to say that the tangent line to $\partial\mathcal{C}$ at C is the line τ whose direction is orthogonal to the vector $(\sin\theta, 1 + \cos\theta)$, and whose signed distance from the origin is $-\big|(\sin\theta, 1 + \cos\theta)\big|^{-1}$.

Figure 13. Left: an example with no admissible triple containing A. Right: an example with multiple admissible triples containing A.

We can then fix a point $C \in \tau$, with first coordinate slightly negative. If $C \in \partial\mathcal{C}$ and τ is the tangent line to $\partial\mathcal{C}$ at C then $\{A,\, B,\, C\}$ is an admissible triple. Let us then call τ' (resp., B', C') the line obtained from τ (resp., the point obtained from B, C) with a symmetry with respect to the vertical axis $\{x = 0\}$. Therefore, as soon as $\partial\mathcal{C}$ contains the small arc of circle around A, and the points C and C' with tangent lines τ and τ' (and this is obviously possible with a non-symmetric unit ball $\mathcal{C}$), then both $\{A,\, B,\, C\}$ and $\{A,\, B',\, C'\}$ are admissible triples, so the uniqueness does not hold. $\square$

Let us briefly describe an explicit example of a unit ball and of a corresponding admissible triple.

Example 3.5. *For $p > 1$, let us consider the norm $\mathfrak{h}$ corresponding to the unit ball $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : |x|^p + |y|^p \leq 1\}$. Observe that this norm is symmetric. Let us now consider $A = (0, 1) \in \partial\mathcal{C}$. Then, a boring but elementary calculation ensures that the unique pair $\{B,\, C\} \in \partial\mathcal{C}$ such that $\{A,\, B,\, C\}$ satisfies (3.3) is given by the two points for which $A\hat{O}B = 2\pi - A\hat{O}C = \alpha$, being $\tan\alpha = -(2^p - 1)^{1/p}$. Notice that of course, for $p = 2$, this reduces to the well-known $120°$ rule.*

To conclude, we can "translate" the property (3.3) to triple points of optimal clusters. As already noticed several times, the study of the perimeter coincides with the study of the minimal length of curves, except that we have to rotate the normal vectors so to obtain the normal ones. And moreover, depending on the colours of the regions, the rotated function might have to be simmetrized. Precisely, we can prove the following result.

Proposition 3.6. *Let h satisfy the assumptions of Theorem A, and let O be a triple point of an optimal cluster $\mathcal{E}$. Call θ_1, θ_2, $\theta_3 \in \mathbb{S}^1$, ordered in clockwise sense, the directions of the three arcs of $\partial\mathcal{E}$ meeting at O. For every $\nu \in \mathbb{S}^1$, let us call $\hat{\nu}$ the direction obtained by rotating ν of $90°$ in the clockwise sense, and let $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}$ be given by $\mathfrak{h}(\nu) = h(O, \hat{\nu})$.*

Then, the minimality property $\Theta = 0$ holds, where $\Theta \in \mathbb{R}^2$ is defined as follows. If the three regions meeting at O are all coloured, then

$$\Theta = \frac{\nabla\mathfrak{h}(\theta_1) - \nabla\mathfrak{h}(-\theta_1)}{2} + \frac{\nabla\mathfrak{h}(\theta_2) - \nabla\mathfrak{h}(-\theta_2)}{2} + \frac{\nabla\mathfrak{h}(\theta_3) - \nabla\mathfrak{h}(-\theta_3)}{2}\,. \tag{3.6}$$

If the region between θ_1 and θ_2 is white, then

$$\Theta = \nabla\mathfrak{h}(\theta_1) - \nabla\mathfrak{h}(-\theta_2) + \frac{\nabla\mathfrak{h}(\theta_3) - \nabla\mathfrak{h}(-\theta_3)}{2}\,. \tag{3.7}$$

Notice that, if h is symmetric, then both the above definitions simply reduce to

$$\Theta = \nabla\mathfrak{h}(\theta_1) + \nabla\mathfrak{h}(\theta_2) + \nabla\mathfrak{h}(\theta_3)\,.$$

Proof. Let us call A, B, C the points in $\partial B(O,1)$ in the directions θ_1, θ_2 and θ_3. For every point $D \in B(O,1)$, let us define $L(D)$ as

$$L(D) = \frac{\mathfrak{h}(DA) + \mathfrak{h}(AD)}{2} + \frac{\mathfrak{h}(DB) + \mathfrak{h}(BD)}{2} + \frac{\mathfrak{h}(DC) + \mathfrak{h}(CD)}{2}$$

if the three regions meeting at O are all coloured, while

$$L(D) = \mathfrak{h}(DA) + \mathfrak{h}(BD) + \frac{\mathfrak{h}(DC) + \mathfrak{h}(CD)}{2}$$

if the regions between θ_1 and θ_2 is white. Notice that, thanks to the definitions (3.6) and (3.7), if $|D| = \varepsilon \ll 1$ then

$$L(D) = L(O) - \Theta \cdot D + o(\varepsilon)\,.$$

As a consequence, if $\Theta \neq 0$ there are a constant $c > 0$ and a point $D \in B(0,1)$ such that

$$L(D) = L(O) - c\,. \tag{3.8}$$

Let now $r \ll 1$ be a small constant. Keep in mind that O is a triple point, and the arcs meeting at O correspond to the directions θ_1, θ_2, θ_3. Therefore, as soon as r is small enough, $\partial\mathcal{E} \cap \partial B(O,r)$ consists of three points A', B' and C', and the directions of OA', OB' and OC' are arbitrarily close to θ_1, θ_2 and θ_3. Let us also define P' as the perimeter obtained by using (1.1) with h' in place of h, where h' is defined as $h'(x,\nu) = h(x,\nu)$ if $x \notin B(O,r)$, and $h'(x,\nu) = h(O,\nu)$ if $x \in B(O,r)$. In addition, for every point $Q \in B(O,r)$ we call $\mathcal{E}'_Q$ the cluster which equals $\mathcal{E}$ outside the ball $B(O,r)$, and such that $\partial\mathcal{E}' \cap B(O,r)$ is given by the three segments QA, QB and QC. In particular, we call $\mathcal{E}' = \mathcal{E}'_{rD}$. By (3.8) and rescaling, and also using Lemma 2.5, we can estimate

$$P'(\mathcal{E}') = P'(\mathcal{E}'_O) - cr \leq P'(\mathcal{E}) - cr\,. \tag{3.9}$$

We can then conclude by finding a contradiction with the same argument used several times in Section 2. Namely, if $r \ll 1$ we have that

$$\Big|P\Big(\mathcal{E}; B(O,r)\Big) - P'\Big(\mathcal{E}; B(O,r)\Big)\Big| \ll r\,, \qquad \Big|P\Big(\mathcal{E}'; B(O,r)\Big) - P'\Big(\mathcal{E}'; B(O,r)\Big)\Big| \ll r\,,$$

hence for r small enough (3.9) gives

$$P(\mathcal{E}') \leq P(\mathcal{E}) - \frac{c}{2}\, r\,.$$

And finally, this estimate together with Lemma 2.7 allows to find a competitor $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ and $P(\mathcal{E}'') < P(\mathcal{E})$, which is the searched contradiction. □

Appendix A. Some properties about quasi-minimal sets and porous sets

In this short appendix, we present some known results concerning quasi-minimal and porous sets, and their boundaries. We will not need to deal with densities, so we will only use the standard Euclidean volume $|\cdot|_{\rm Eucl}$ and perimeter $P_{\rm Eucl}$. First of all, we recall a couple of important standard definitions, see for instance [37, 9, 21] (we actually deal only with the case of subsets of $\mathbb{R}^N$, while [21] considers more general metric spaces with doubling measures).

Definition A.1 (Quasi-minimal sets, porous sets)**.** *Let $F \subseteq \mathbb{R}^N$ be a Borel set with locally finite perimeter. We say that F* is quasi-minimal *if there exists a constant C_{qm} such that, for every ball $B(x,r) \subseteq \mathbb{R}^N$ and every set $H \subseteq \mathbb{R}^N$ with $H\Delta F \subset\subset B(x,r)$, one has*

$$P_{\rm Eucl}\Big(F; B(x,r)\Big) \leq C_{qm} P_{\rm Eucl}\Big(H; B(x,r)\Big)\,.$$

We say that F is porous *if there exists $\delta > 0$ such that, for every $x \in \partial F$ (the topological boundary) and every small ball $B(x,r)$, there exist a ball $B(y,\delta r) \subseteq B(x,r) \cap F$ and a ball $B(z,\delta r) \subseteq B(x,r) \setminus F$.*

The following result is well-known, see for instance [9, Theorem 1.8] or [21, Theorem 5.2].

Theorem A.2. *Every quasi-minimal set $F \subseteq \mathbb{R}^N$ is porous.*

The convenience of the notion of porosity is mainly given by the following standard fact, that we prove just for completeness.

Lemma A.3. *Let $F \subseteq \mathbb{R}^N$ be a porous set. Then the set $F^{(1)}$ of the points of density 1 of F is open. Moreover, the reduced boundary $\partial^* F$ and the topological boundary ∂F coincide up to $\mathcal{H}^{N-1}$-negligible subsets.*

Proof. The inclusion $\partial^* F \subseteq \partial F$ is always satisfied. Let now x be any point in ∂F. By the definition of porosity, the density of F at x is between δ^N and $1-\delta^N$, so that $x \notin F^{(0)} \cup F^{(1)}$. Since $F^{(0)} \cup F^{(1)}$ fill the whole $\mathbb{R}^N \setminus \partial^* F$ up to zero $\mathcal{H}^{N-1}$-measure, we deduce that $\mathcal{H}^{N-1}(\partial F \setminus \partial^* F) = 0$. In particular, we have observed that a point of $F^{(1)}$ cannot belong to ∂F, hence it must be either in the interior of F, or in the interior of $\mathbb{R}^N \setminus F$. Since the latter possibility is excluded by the positive density, we deduce that $F^{(1)}$ is open. □

We conclude with a 2-dimensional property of porous sets without holes, that we formally define below. Also this property is not new, but we give a simple proof for completeness.

Definition A.4 (Holes). *Let $F \subseteq \mathbb{R}^2$ be a Borel set with locally finite perimeter. We say that F* has a hole U *if there exists a bounded set $U \subseteq \mathbb{R}^2 \setminus F$ with $\mathcal{H}^2(U) > 0$ and such that, up to $\mathcal{H}^1$-negligible sets,*

$$\partial^* F = \partial^* U \cup \partial^*(F \cup U)\,.$$

Lemma A.5. *Let $F \subseteq \mathbb{R}^2$ be an open, porous set of finite (Euclidean) perimeter, connected (in the measure theoretical sense) and without holes. Then, ∂F is a closed curve. More precisely, there exists an injective curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ of finite length such that $\partial F = \gamma(\mathbb{S}^1)$.*

Proof. We start by recalling that a set F is said *connected in the measure theoretical sense* if, whenever one writes $F = F' \cup F''$ with two essentially disjoint sets F', F'' so that, up to $\mathcal{H}^1$-negligible subsets, $\partial^* F = \partial^* F' \cup \partial^* F''$, it must be $\min\{\mathcal{H}^2(F'), \mathcal{H}^2(F'')\} = 0$. Notice that, as an immediate consequence of this lemma, we will obtain that F is actually connected also in the topological sense.

Since F has finite perimeter, by the compactness results in BV we have a sequence of smooth sets F_j such that

$$|F_j \Delta F|_{\rm Eucl} \to 0\,, \qquad\qquad P_{\rm Eucl}(F_j) \to P_{\rm Eucl}(F)\,. \tag{A.1}$$

<u>Step I.</u> *Reduction to the case of connected sets F_j.*

First of all, we want to reduce ourselves to the case when the sets F_j are connected. Since F_j is regular, we can write it as $F_j^1 \cup F_j^2$, where F_j^1 is the connected component with biggest area, and F_j^2 is the union of all the other connected components. Observe that, by the isoperimetric inequality,

$$P_{\rm Eucl}(F_j) \geq \frac{2\sqrt{\pi}}{\sqrt{|F_j^1|_{\rm Eucl}}}\,|F_j|_{\rm Eucl}\,,$$

and since $|F_j|_{\rm Eucl} \to |F|_{\rm Eucl}$ and $P_{\rm Eucl}(F_j) \to P_{\rm Eucl}(F)$ we deduce that $|F_j^1|_{\rm Eucl}$ is bounded away from 0. Up to a subsequence, we can assume that the characteristic functions of F_j^1 and F_j^2 weakly converge in BV to the characteristic functions of two sets, that we call F^1 and F^2. Notice that $F^1 \cap F^2 = \emptyset$ and $F^1 \cup F^2 = F$. By the lower semicontinuity of the perimeter under weak BV convergence and (A.1), we have

$$\begin{aligned}P_{\rm Eucl}(F) &\leq P_{\rm Eucl}(F^1) + P_{\rm Eucl}(F^2) \leq \liminf P_{\rm Eucl}(F_j^1) + \liminf P_{\rm Eucl}(F_j^2)\\ &\leq \liminf \Big(P_{\rm Eucl}(F_j^1) + P_{\rm Eucl}(F_j^2)\Big) = \liminf P_{\rm Eucl}(F_j) = P_{\rm Eucl}(F)\,,\end{aligned}$$

and since F is connected in the measure theoretical sense the first inequality is strict unless one of the two sets is negligible. Since the strict inequality is impossible and F^1 is not negligible by construction, we deduce that $|F^2|_{\rm Eucl} = \lim |F^2_j|_{\rm Eucl} = 0$, and as a byproduct the above chain of inequalities implies also that $P_{\rm Eucl}(F^2_j) \to 0$. Therefore, (A.1) still holds true replacing the sets F_j with the connected sets F^1_j, and this concludes the step.

Step II. Reduction to the case of sets F_j with ∂F_j connected.
We want now to reduce ourselves to the case when the sets F_j have connected boundaries (hence, they have no holes). Since F_j is a smooth, connected set, it is possible to write it as $F_j = G_j \setminus U_j$, where G_j has smooth, connected boundary, and $U_j \subset\subset G_j$. In particular, $P_{\rm Eucl}(F_j) = P_{\rm Eucl}(G_j) + P_{\rm Eucl}(U_j)$. Up to a subsequence, we can assume that the characteristic functions of G_j and of U_j weakly converge in BV to the characteristic functions of two sets, that we call G and U. Notice that $U \subseteq G$ and that $F = G \setminus U$, hence by lower semicontinuity of the perimeter we have

$$\begin{aligned}P_{\rm Eucl}(F) &= \lim P_{\rm Eucl}(F_j) = \lim \Big(P_{\rm Eucl}(G_j) + P_{\rm Eucl}(U_j)\Big)\\ &\geq \liminf P_{\rm Eucl}(G_j) + \liminf P_{\rm Eucl}(U_j) \geq P_{\rm Eucl}(G) + P_{\rm Eucl}(U)\\ &= P_{\rm Eucl}(F \cup U) + P_{\rm Eucl}(U)\,.\end{aligned}$$

Since F has no holes, we deduce that U is negligible, and that $P_{\rm Eucl}(U_j) \to 0$. As a consequence, we can replace the sets F_j with the sets G_j and (A.1) still holds true, which concludes also this step.

Step III. Conclusion.
By steps I and II, we have a sequence of sets F_j satisfying (A.1) and having a closed, regular curve as boundary. There are then smooth functions $\gamma_j : \mathbb{S}^1 \to \mathbb{R}^2$, injective and with $|\gamma_j'|$ constant (thus constantly equal to $P_{\rm Eucl}(F_j)/2\pi$). Up to subsequences, the functions γ_j uniformly converge to a Lipschitz function $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$. By construction $\partial F \subseteq \gamma(\mathbb{S}^1)$, thus

$$P_{\rm Eucl}(F) = \mathcal{H}^1(\partial F) \leq \mathcal{H}^1(\gamma) \leq \liminf \mathcal{H}^1(\gamma_j) = \liminf P_{\rm Eucl}(F_j) = P_{\rm Eucl}(F)\,.$$

We deduce that $\partial F = \gamma(\mathbb{S}^1)$, and the curve γ is injective since F is connected. □

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