ON THE STEINER PROPERTY FOR PLANAR MINIMIZING CLUSTERS. THE ISOTROPIC CASE.

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Abstract. We consider the isoperimetric problem for clusters in the plane with a double density, that is, perimeter and volume depend on two weights. In this paper we consider the isotropic case, in the parallel paper [14] the anisotropic case is studied. Here we prove that, in a wide generality, minimal clusters enjoy the "Steiner property", which means that the boundaries are made by $C^{1,\gamma}$ regular arcs, meeting in finitely many triple points with the 120° property.

1. Introduction

In this paper we consider the isoperimetric problem with double density for planar clusters. This means that we are given two l.s.c. functions g, $h : \mathbb{R}^2 \to \mathbb{R}^+$ (the *densities*), and the volume and perimeter of any set $E \subseteq \mathbb{R}^2$ of locally finite perimeter are given by

$$|E| = \int_E g(x)\, dx\,, \qquad\qquad P(E) = \int_{\partial^* E} h(x)\, d\mathcal{H}^1(x)\,, \tag{1.1}$$

where $\partial^* E$ is the reduced boundary as usual (see [3] for definitions and properties of sets of finite perimeter). The *isoperimetric problem* consists then, as always, in the search of sets of given (weighted) volume which minimize the (weighted) perimeter. Of course, depending on g and h, different situations may occur. This generalization of the standard isoperimetric problem has gained a rapidly increasing interest in the last decades due to the work of several authors, see for instance [33, 7, 6, 23, 5, 1, 10, 17, 2, 8, 11] and the references therein.

The *isoperimetric problem for clusters*, instead, consists in minimizing the total perimeter of a union of sets with given volume. More precisely, for a given $m \in \mathbb{N}$, an *m-cluster* is a collection $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ of m essentially disjoint sets of locally finite perimeter. For brevity, we will denote $E_0 = \mathbb{R}^2 \setminus (\cup_{i=1}^m E_i)$. The *volume* of a cluster $\mathcal{E}$ is the vector $|\mathcal{E}| = (|E_1|, |E_2|, \dots, |E_m|) \in (\mathbb{R}^+)^m$, while its *perimeter* is

$$P(E) = \frac{P(\cup_{i=1}^m E_i) + \sum_{i=1}^m P(E_i)}{2} = \frac{\sum_{i=0}^m P(E_i)}{2} = \int_{\partial^*\mathcal{E}} h(x)\, d\mathcal{H}^1(x)\,,$$

where the "boundary" $\partial^*\mathcal{E}$ of a cluster is defined as the union of the boundaries, that is,

$$\partial^*\mathcal{E} = \cup_{i=1}^m \partial^* E_i\,.$$

A cluster which minimizes the perimeter among all those with fixed volume is usually called *minimal cluster*. The isoperimetric problem for clusters has been deeply studied in the last decades. In particular, for the Euclidean case, corresponding to $g \equiv h \equiv 1$, the problem is often referred to as the "double bubble problem" in the case $m = 2$, which was completely solved in [19, 32]. Also the "triple bubble" and the "quadruple bubble", corresponding to $m = 3$ and $m = 4$, have been studied, see [38] and [26, 27, 28] respectively.

The aim of this paper is to prove that minimal clusters enjoy the "Steiner property", that is, their boundaries are composed by finitely many $\mathrm{C}^{1,\gamma}$ arcs, which meet in finitely many triple points with the 120° property, see Definition 1.2 below. While this property is widely known for the Euclidean case, see for instance the classical paper [36] or the recent book [20], we generalize it to a much more general case. In particular, we will prove the validity of the Steiner property as soon as an $\varepsilon - \varepsilon^\beta$ property and a volume growth condition hold.

Definition 1.1 (η-growth condition and $\varepsilon - \varepsilon^\beta$ property for clusters)**.** *Given a power $\eta \geq 1$, an* η-growth condition *is said to hold if there exist two positive constants C_{vol} and R_η such that, for every $x \in \mathbb{R}^2$ and every $r < R_\eta$, the ball $B(x,r)$ has volume $|B(x,r)| \leq C_{\mathrm{vol}} r^\eta$. We say that the* local η-growth condition *holds if for any bounded domain $D \subset\subset \mathbb{R}^2$ there exist two constants C_{vol} and R_η such that the above property holds for balls $B(x,r) \subseteq D$.*

Moreover, we say that a cluster $\mathcal{E}$ satisfies the $\varepsilon - \varepsilon^\beta$ property for some $0 < \beta \leq 1$ if there exist three positive constants R_β, C_{per} and $\bar\varepsilon$ such that, for every vector $\varepsilon \in \mathbb{R}^m$ with $|\varepsilon| \leq \bar\varepsilon$ and every $x \in \mathbb{R}^2$, there exists another cluster $\mathcal{F}$ such that

$$\mathcal{F}\Delta\mathcal{E} \subseteq \mathbb{R}^2 \setminus B(x, R_\beta)\,, \qquad |\mathcal{F}| = |\mathcal{E}| + \varepsilon\,, \qquad P(\mathcal{F}) \leq P(\mathcal{E}) + C_{\mathrm{per}}|\varepsilon|^\beta\,. \tag{1.2}$$

In this case, for every $0 < t \leq \bar\varepsilon$ we call $C_{\mathrm{per}}[t]$ the smallest constant such that the above property is true whenever $|\varepsilon| \leq t$. The map $t \mapsto C_{\mathrm{per}}[t]$ is clearly increasing and $C_{\mathrm{per}}[\bar\varepsilon] \leq C_{\mathrm{per}}$.

We underline that both the above assumptions are satisfied for a wide class of densities. In particular, the growth (or local growth) condition clearly holds with $\eta = 2$ whenever the density g is bounded (or locally bounded). Concerning the $\varepsilon - \varepsilon^\beta$ property, this is a crucial tool when dealing with isoperimetric problems. It is simple to observe that it is valid with $\beta = 1$ for every cluster of locally finite perimeter whenever the density h is regular enough (at least Lipschitz). It is also known that, if h is α-Hölder, then every cluster of locally finite perimeter satisfies the $\varepsilon - \varepsilon^\beta$ property with

$$\beta = \frac{1}{2-\alpha}\,,$$

the proof can be found in [9] for the special case $g = h$ and in [30] for the general case. The case $\alpha = 0$ is particular, also because there is not a unique possible meaning of "0-Hölder function". More precisely, the $\varepsilon - \varepsilon^{1/2}$ property holds as soon as h is locally bounded. If h is continuous, instead, not only the $\varepsilon - \varepsilon^{1/2}$ property holds, but in addition $\lim_{t\to 0} C_{\rm per}[t] = 0$. We can be even more precise: the function $t \mapsto C_{\rm per}[t]$ can be bounded by $\sqrt{\omega_h}$, where ω_h is the modulus of continuity of h (see [31]).

We can now give the formal definitions of the Steiner property, already described above, and of the Dini property.

Definition 1.2 (Steiner property). *A cluster $\mathcal{E}$ is said to satisfy the* Steiner property *if $\partial^*\mathcal{E}$ is a locally finite union of C^1 arcs, and at each junction point exactly three arcs are meeting, having tangent vectors which form three angles of $\frac{2}{3}\pi$.*

Definition 1.3 (Dini property). *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\varphi(0) = 0$. We say that φ* satisfies the Dini property *if for every $C > 1$ one has*

$$\sum_{n\in\mathbb{N}} \varphi(C^{-n}) < +\infty\,,$$

and we say that φ satisfies the 1/2-Dini property *if $\sqrt{\varphi}$ satisfies the Dini property. A uniformly continuous function f is Dini continuous if and only if its modulus of continuity ω_f satisfies the Dini property. We say that f* is 1/2-Dini continuous *if ω_f satisfies the 1/2-Dini property.*

We are now in position to state the main result of the present paper.

Theorem 1.4 (Steiner regularity for minimal clusters). *Let h be locally 1/2-Dini continuous, let $\mathcal{E}$ be a minimal cluster, and let us assume that for some η, β the local η-growth condition holds, as well as the $\varepsilon - \varepsilon^\beta$ property for $\mathcal{E}$. Assume in addition that either*

(i) $\eta\beta > 1$, or

(ii) $\eta\beta = 1$ and the function $t \mapsto C_{\rm per}[t]$ satisfies the 1/2-Dini property.

Then $\mathcal{E}$ satisfies the Steiner property, and if $\eta\beta > 1$ and h is locally α-Hölder then the arcs of $\partial^*\mathcal{E}$ are actually $\mathrm{C}^{1,\gamma}$ with $\gamma = \frac{1}{2}\min\{\eta\beta - 1, \alpha\}$.

A few comments are now in order. First of all we observe that, in the classical Euclidean case, one has $\beta = 1$ and $\eta = 2$, hence the assumptions of our result cover an extremely more general case than the classical one.

Concerning the case $\eta\beta = 1$, in order to get the Steiner property of $\mathcal{E}$ we have added a Dini-type property on $C_{\rm per}$, while if $\eta\beta > 1$ no additional assumption was needed. In fact, the C^1 regularity of the boundary fails if $\eta\beta = 1$ without extra assumptions. On the bright side, Theorem 1.4 can always be applied if g is locally bounded and h is 1/4-Dini continuous (i.e., $\sqrt[4]{\omega_h}$ satisfies the Dini property). Indeed, in this case $\eta = 2$ and

$\beta = 1/2$, and the required continuity of h and $C_{\rm per}$ follows by the fact that $C_{\rm per} \lesssim \sqrt{\omega_h}$, already observed above. The fact that, in order to get C^1 regularity of the boundary, some 1/2-Dini property is needed, is standard, see for instance [35, 20].

It is to be remarked that the boundedness of an optimal cluster is false in general, but true under quite mild assumptions. The boundedness of isoperimetric sets, or optimal clusters, is a well studied question, also because it is deeply connected with the existence, see for instance [9, 10, 29, 30]. Of course, whenever optimal clusters are a priori known to be bounded, as soon as Theorem 1.4 applies then they are made by a finite (and not just locally finite) union of regular arcs.

We conclude this introduction by pointing out that, in the isoperimetric problem with double density, one can consider an anisotropic density for the perimeter, that is, the density h may also depend on the direction of the normal vector. In other words, h is defined on $\mathbb{R}^2 \times \mathbb{S}^1$, and the term $h(x)$ in the right definition in (1.1) is replaced by $h(x, \nu_E(x))$, where $\nu_E(x)$ is the unit normal vector to $\partial^* E$ at $x \in \partial^* E$. The anisotropic case is of course more complicate to treat, but it is a very important generalization, in particular it allows to cover the case of Riemannian manifolds, where the density h is naturally anisotropic, since it depends on the directional derivative of the Riemann tensor. We are able to study the Steiner property also in the general anisotropic case, to which the parallel paper [14] is devoted. An important peculiarity of the isotropic case is the 120° property, which is in general false in the anisotropic case.

2. Proof of the main result

The proof of the main result, Theorem 1.4, is presented in this section. In turn, this is subdivided in four subsections. The first one collects some basic definitions and technical tools, in the second one we show that there are finitely many junction points, each of which where exactly three different sets meet, and in the third one we obtain the regularity. The actual proof of the theorem, presented in the last subsection, basically only consists in putting the different parts together.

Since we aim to prove Theorem 1.4, from now on we assume that h is locally 1/2-Dini continuous and that the local η-growth condition holds for some $\eta \geq 1$.

2.1. Some definitions and technical tools.

Let us fix some notation, that will be used through the rest of the paper. Since we are interested in a local property, in the proof of Theorem 1.4 we will immediately start by fixing a big closed ball $D \subseteq \mathbb{R}^2$, and the whole construction will be performed there. Hence, all the following definitions will depend upon D, in particular we assume that $|B(x,r)| \leq C_{\rm vol} r^\eta$ for every ball $B(x,r) \subseteq D$.

Since h is locally 1/2-Dini continuous, hence in particular continuous, we can call $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ its modulus of continuity inside D. In particular, if h is locally α-Hölder, then $\omega(r) \le Cr^\alpha$ for a suitable constant C. Moreover, we will call $0 < h_{\min} \le h_{\max}$ the maximum and the minimum of h in D. Keep in mind that, if $\eta\beta = 1$, then the function $t \mapsto C_{\rm per}[t]$ is assumed to satisfy the 1/2-Dini property, so in particular it is infinitesimal. As a consequence, we can fix $\bar\varepsilon$ so small that $C_{\rm per}$ is as small as desired, in particular we will use several times that

$$C_{\rm per} = C_{\rm per}[\bar\varepsilon] \ll \frac{h_{\min}^2}{C_{\rm vol}^\beta\, h_{\max}}\,. \tag{2.1}$$

We can easily observe now a simple estimate between volume and perimeter of any set E.

Lemma 2.1 (Isoperimetric inequality with exponent)**.** *For every set $E \subseteq D$ we have*

$$P(E) \ge \frac{h_{\min}}{C_{\rm vol}^{1/\eta}}\,|E|^{1/\eta}\,.$$

Proof. By approximation, we can limit ourselves to consider the case of a planar, polygonal set E. A classical result from Gustin (see [18]) says that such a set can be covered with countably many balls $B_i = B(x_i, r_i)$ in such a way that

$$\mathcal{H}^1(\partial^* E) \ge 2\sqrt{2}\sum\nolimits_i r_i\,.$$

Keeping in mind that $\eta \ge 1$, for any $E \subseteq D$ we deduce

$$\begin{aligned} P(E) &\ge h_{\min}\mathcal{H}^1(\partial^* E) \ge 2\sqrt{2}h_{\min}\sum\nolimits_i r_i \ge \frac{2\sqrt{2}h_{\min}}{C_{\rm vol}^{1/\eta}}\sum\nolimits_i |B(x_i,r_i)|^{1/\eta}\\ &\ge \frac{2\sqrt{2}h_{\min}}{C_{\rm vol}^{1/\eta}}\Big(\sum\nolimits_i |B(x_i,r_i)|\Big)^{1/\eta} \ge \frac{2\sqrt{2}h_{\min}}{C_{\rm vol}^{1/\eta}}\,|E|^{1/\eta}\,, \end{aligned}$$

so the proof is concluded. □

The following is a simple geometric fact. This is specific for the isotropic case. The analogous property in the anisotropic case is weaker and much more complicate to obtain.

Lemma 2.2 (The 120° net property)**.** *There exists a continuous and strictly increasing function $L : [0, 2\pi/3] \to [1, 2]$ such that, if x, y, z are three points in $\mathbb{R}^2$ such that $|y-x| = |z-x|$, and $\theta = z\widehat{x}y \in [0, 2\pi/3]$, then there exists a connected set Γ contained in the triangle xyz, containing x, y and z and such that $\mathcal{H}^1(\Gamma) = L(\theta)|y-x|$.*

Proof. There exists a unique point w, sometimes called Fermat point, contained in the triangle xyz, such that the three angles $y\widehat{w}z$, $z\widehat{w}x$ and $x\widehat{w}y$ are all equal to $2\pi/3$. In particular, $w = z = y$ in the case $\theta = 0$, while $w = x$ if $\theta = 2\pi/3$. The set Γ is then simply the union of the three segments xw, yw and zw. Calling $L(\theta)$ the length of Γ divided by $|y-x|$ (which of course does not depend on $|y-x|$ by rescaling), it is trivial to express $L(\theta)$ as an explicit trigonometric function of θ. The fact that this is a strictly increasing function of θ, with $L(0) = 1$ and $L(2\pi/3) = 2$, follows then by an elementary calculation. □

We introduce now the (standard) notation of relative perimeter. Given a set $E \subseteq \mathbb{R}^2$ of locally finite perimeter, or a cluster $\mathcal{E}$, and given a Borel set $A \subseteq \mathbb{R}^2$, the *relative perimeter of E (or $\mathcal{E}$) inside A* is the measure of the boundary of E (or $\mathcal{E}$) within A, i.e.,

$$P(E; A) = \int_{A\cap\partial^* E} h(x)\, d\mathcal{H}^1(x)\,, \qquad\qquad P(\mathcal{E}; A) = \int_{A\cap\partial^* \mathcal{E}} h(x)\, d\mathcal{H}^1(x)\,.$$

We conclude this short section by presenting (a very specific case of) a fundamental result due to Vol'pert, see [37] and also [3, Theorem 3.108].

Theorem 2.3 (Vol'pert)**.** *Let $E \subseteq \mathbb{R}^2$ be a set of locally finite perimeter, and let $x \in \mathbb{R}^2$ be fixed. Then, for a.e. $r > 0$, one has that*

$$\partial^* E \cap \partial B(x, r) = \partial^*\big(E \cap \partial B(x, r)\big)\,.$$

Notice that, for almost every $r > 0$, both sets in the above equality are done by finitely many points. In particular, $E \cap \partial B(x, r)$ is a subset of the circle $\partial B(x, r)$, and its boundary has to be considered in the 1-dimensional sense. More precisely, for almost every $r > 0$ the set $E \cap \partial B(x, r)$ essentially consists of a finite union of arcs of the circle, and the boundary is simply the union of the endpoints of all of them. Through the rest of the paper, we will often consider intersections of sets with balls. Even if this will not be written every time, we will always consider balls for which Vol'pert Theorem holds true.

2.2. **Finitely many triple points.** We now start our construction for proving Theorem 1.4. Through this section and the following one, $\mathcal{E}$ is a fixed, minimal cluster, satisfying the assumptions of Theorem 1.4, and D is a fixed, closed ball. The aim of this section is to show several preliminary properties of $\mathcal{E}$, eventually establishing that $\partial^*\mathcal{E}$ only admits (in D) finitely many "3-color points", see Definition 2.10 and Proposition 2.12. In the sequel we will derive that all the junction points (i.e., the points where at least 3 of the C^1 arcs of $\partial\mathcal{E}$ meet) are actually "3-color points", and in particular triple points (i.e., points where exactly 3 arcs meet).

We set $R_1 = \min\{R_\beta, R_\eta\}$. In the following, we will define several different values of R_i with $R_1 \geq R_2 \geq R_3 \cdots$. Each of these constants will only depend on $\mathcal{E}$, D, g and h (the dependence on $\mathcal{E}$, g and h is actually only through the constants $h_{\min}$ and $h_{\max}$ and on the values of the constants $C_{\rm per}$, $C_{\rm vol}$, $\bar\varepsilon$ and η of Definition 1.1).

Lemma 2.4 (Small ball competitor)**.** *Let $B(x, r) \subseteq D$ be a ball with $|B(x, r)| < \bar\varepsilon/2$ and $r < R_1$, and let $\mathcal{E}'$ be a cluster which coincides with $\mathcal{E}$ outside $B(x, r)$. There exists another cluster $\mathcal{E}''$ such that $|\mathcal{E}''| = |\mathcal{E}|$, $\mathcal{E}'' \cap B(x, r) = \mathcal{E}' \cap B(x, r)$ and, calling $\varepsilon = |\mathcal{E}| - |\mathcal{E}'|$,*

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}|\varepsilon|^\beta \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol} r^\eta)^\beta\,. \tag{2.2}$$

Proof. Since

$$|\varepsilon| \leq \sum_{i=1}^m |\varepsilon_i| \leq 2|B(x,r)| < \bar\varepsilon\,,$$

we can apply the $\varepsilon-\varepsilon^\beta$ property to $\mathcal{E}$ with constant ε and point x. Hence, there is another cluster $\mathcal{F}$ such that $\mathcal{F}=\mathcal{E}$ inside $B(x,R_\beta)\supseteq B(x,r)$, and moreover $|\mathcal{F}|=|\mathcal{E}|+\varepsilon$ and $P(\mathcal{F})\leq P(\mathcal{E})+C_{\rm per}|\varepsilon|^\beta$. We define then the cluster $\mathcal{E}''$ as the cluster which coincides with $\mathcal{E}'$ inside $B(x,r)$, and with $\mathcal{F}$ outside of $B(x,r)$. Its volume is

$$\begin{aligned}|\mathcal{E}''| &= |\mathcal{E}'\cap B(x,r)| + |\mathcal{F}\setminus B(x,r)|\\ &= |\mathcal{E}\cap B(x,r)| + |\mathcal{E}'| - |\mathcal{E}| + |\mathcal{E}\setminus B(x,r)| + |\mathcal{F}| - |\mathcal{E}| = |\mathcal{E}|\,.\end{aligned}$$

Keeping in mind the growth condition, we have $|\varepsilon|\leq 2|B(x,r)|\leq 2C_{\rm vol}r^\eta$. As a consequence, the perimeter of $\mathcal{E}''$ can be evaluated as

$$\begin{aligned}P(\mathcal{E}'') &= P(\mathcal{E}';B(x,r)) + P\big(\mathcal{F};\mathbb{R}^2\setminus B(x,r)\big)\\ &= P(\mathcal{E};B(x,r)) + P(\mathcal{E}') - P(\mathcal{E}) + P\big(\mathcal{E};\mathbb{R}^2\setminus B(x,r)\big) + P(\mathcal{F}) - P(\mathcal{E})\\ &\leq P(\mathcal{E}') + C_{\rm per}|\varepsilon|^\beta \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol}r^\eta)^\beta\,.\end{aligned}$$

The proof is then concluded. □

Lemma 2.5 (Length in a ball is controlled by radius)**.** *There exists a constant $R_2\leq R_1$ such that, for every $B(x,r)\subseteq D$ with $r<R_2$, one has*

$$\mathcal{H}^1(\partial^*\mathcal{E}\cap B(x,r)) < \frac{13}{2}\,r\,. \tag{2.3}$$

Proof. We let $R_2\leq R_1$ be so small that

$$C_{\rm vol}R_2^\eta < \frac{\bar\varepsilon}{2}\,, \qquad \omega(2R_2) < \frac{h_{\min}}{40}\,, \qquad C_{\rm per}\big(2C_{\rm vol}R_2^\eta\big)^\beta < h_{\min}\frac{R_2}{20}\,. \tag{2.4}$$

Notice that the first two inequalities are true for every R_2 small enough. The same is true for the third one if $\eta\beta>1$, while if $\eta\beta=1$ the last inequality is true, regardless of R_2, since $C_{\rm per}$ is very small by (2.1).

Let now $r<R_2$ and $x\in\mathbb{R}^2$ be as in the claim, and call $\tilde h_{\min}=\min\{h(x),\,x\in\overline{B(x,r)}\}$ and $\tilde h_{\max}=\max\{h(x),\,x\in\overline{B(x,r)}\}$. Let $\mathcal{E}'$ be the cluster defined by $E_1'=E_1\cup B(x,r)$ and $E_i'=E_i\setminus B(x,r)$ for every $2\leq i\leq m$. Clearly

$$P(\mathcal{E}') \leq P(\mathcal{E}) - P(\mathcal{E};B(x,r)) + 2\pi r\tilde h_{\max}\,. \tag{2.5}$$

Let us call $\varepsilon\in\mathbb{R}^m$ the vector given by $\varepsilon_i=|E_i\cap B(x,r)|$ for every $2\leq i\leq m$, and $\varepsilon_1=-|B(x,r)\setminus E_1|$, so that $|\mathcal{E}|=|\mathcal{E}'|+\varepsilon$. Notice that $|B(x,r)|\leq C_{\rm vol}r^\eta<\bar\varepsilon/2$ by the first property in (2.4). Hence, we can apply Lemma 2.4 to get another cluster $\mathcal{E}''$ satisfying (2.2), so that

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol}r^\eta)^\beta < P(\mathcal{E}') + h_{\min}\,\frac{r}{20}$$

by the third property in (2.4), which is clearly valid with every $r < R_2$ in place of R_2. Putting this estimate together with (2.5), and recalling that $P(\mathcal{E}) \le P(\mathcal{E}'')$ by minimality of $\mathcal{E}$ and since $|\mathcal{E}''| = |\mathcal{E}|$ by Lemma 2.4, we get

$$\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) \le \frac{P(\mathcal{E}; B(x,r))}{\tilde h_{\min}} \le 2\pi r \,\frac{\tilde h_{\max}}{\tilde h_{\min}} + \frac{h_{\min}}{\tilde h_{\min}}\,\frac{r}{20} \le 2\pi r\,\frac{\tilde h_{\max}}{\tilde h_{\min}} + \frac{r}{20} < \frac{13}{2}\, r\,,$$

where the last inequality follows from the second property in (2.4) since

$$\tilde h_{\max} \le \tilde h_{\min} + \omega(2r) \le \tilde h_{\min} + \frac{h_{\min}}{40} \le \frac{41}{40}\,\tilde h_{\min}\,.$$

□

Lemma 2.6 (At most 3 intersection points)**.** *There exist $R_3 \le R_2$ and $C_2 > 1$ such that*

$$\forall\, r \le R_3,\ \forall\, B(x,r) \subseteq D,\ \exists\ \frac{r}{C_2} < \rho < r: \quad \#\Big(\partial^*\mathcal{E} \cap \partial B(x,\rho)\Big) \le 3\,. \tag{2.6}$$

Proof. First of all, we show that (2.6) is true with 6 in place of 3 by choosing $R_3 = R_2$ and $C_2 = 14$ directly by Lemma 2.5. Indeed, suppose that this is false for some $B(x,r)$ as in the claim. Then, for every $r/14 < \rho < r$ we have $\#\big(\partial^*\mathcal{E} \cap \partial B(x,\rho)\big) \ge 7$. As a consequence, by coarea formula

$$\begin{aligned}\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) &\ge \mathcal{H}^1\big(\partial^*\mathcal{E} \cap (B(x,r) \setminus B(x,r/14))\big) \ge \int_{r/14}^{r} \mathcal{H}^0\big(\partial^*\mathcal{E} \cap \partial B(x,\rho)\big)\, d\rho \\ &\ge 7\,\frac{13}{14}\, r = \frac{13}{2}\, r\,,\end{aligned}$$

in contradiction with (2.3).

To conclude the proof is then enough to show that if (2.6) holds with some $k \ge 4$ in place of 3 and with two constants $C_{2,k}$ and $R_{3,k}$, then it also holds with $k-1$ in place of 3 and with two suitable constants $C_{2,k-1} \ge C_2$ and $R_{3,k-1} \le R_{3,k}$.

Let then $C_{2,k-1} \ge C_{2,k}$ and $R_{3,k-1} \le R_{3,k}$ be two constants to be specified later, and let $B(x,r) \subseteq D$ with $r \le R_{3,k-1}$. By assumption, there exists some $r/C_{2,k} < \tilde r < r$ for which $\partial^*\mathcal{E} \cap \partial B(x,\tilde r)$ contains at most k points. If the points are strictly less than k we are done, whatever the choice of $C_{2,k-1} \ge C_{2,k}$ and $R_{3,k-1} \le R_{3,k}$ is. Assume then that the points are exactly k, say $x_1, x_2, \dots, x_k$. We have to find some $r/C_{2,k-1} < \rho < r$ for which

$$\#\Big(\partial^*\mathcal{E} \cap \partial B(x,\rho)\Big) \le k-1\,.$$

Assume then by contradiction that for every such ρ (so in particular for every $r/C_{2,k-1} < \rho < \tilde r$) the opposite inequality holds, then by coarea formula again we deduce

$$\begin{aligned}\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,\tilde r)) &\ge \mathcal{H}^1\Big(\partial^*\mathcal{E} \cap \big(B(x,\tilde r) \setminus B(x, r/C_{2,k-1})\big)\Big) \ge k\left(\tilde r - \frac{r}{C_{2,k-1}}\right) \\ &\ge k\tilde r\left(1 - \frac{C_{2,k}}{C_{2,k-1}}\right).\end{aligned} \tag{2.7}$$

Up to renumbering, we can assume that the two points among x_1, x_2, $\dots$ x_k at minimal distance are x_1 and x_2, so in particular

$$x_1\widehat{x}x_2 \leq \frac{2\pi}{k} < \frac{2\pi}{3}\,.$$

Let us define the set $\Sigma \subseteq B(x,\tilde r)$ as the union of the $k-2$ segments $x_i x$ for $i\geq 3$ together with the set Γ given by Lemma 2.2 by setting $y=x_1$ and $z=x_2$. Recall that the set Γ is contained in the triangle xx_1x_2 by Lemma 2.2, hence it does not intersect the segments $x_i x$ with $i\geq 3$. As a consequence, the union of Σ with $\partial^*\mathcal{E}\setminus B(x,\tilde r)$ is the boundary of a uniquely defined cluster $\mathcal{E}'$ which coincides with $\mathcal{E}$ outside of $B(x,\tilde r)$. Notice that by construction, Lemma 2.2 and (2.7) one has

$$\begin{aligned}P(\mathcal{E}')-P(\mathcal{E}) &\leq \tilde h_{\max}\mathcal{H}^1(\Sigma)-\tilde h_{\min}\mathcal{H}^1(\partial^*\mathcal{E}\cap B(x,\tilde r))\\ &\leq \tilde h_{\max}\tilde r\left[k-2+L\left(\frac{2\pi}{k}\right)-\frac{\tilde h_{\min}}{\tilde h_{\max}}k\left(1-\frac{C_{2,k}}{C_{2,k-1}}\right)\right],\end{aligned}$$

having set again $\tilde h_{\min}=\min\{h(x),\,x\in\overline{B(x,r)}\}$ and $\tilde h_{\max}=\max\{h(x),\,x\in\overline{B(x,r)}\}$. Since $\tilde r<r\leq R_{3,k-1}$ and

$$\frac{\tilde h_{\min}}{\tilde h_{\max}}\geq 1-\frac{\omega(2r)}{h_{\min}}\geq 1-\frac{\omega(2R_{3,k-1})}{h_{\min}}\,,$$

then, up to choose $R_{3,k-1}$ small enough and $C_{2,k-1}$ big enough, this yields

$$P(\mathcal{E}')-P(\mathcal{E})\leq -c\tilde r\tilde h_{\max}\leq -c\tilde r h_{\min} \tag{2.8}$$

with $2c=2-L(2\pi/k)$. We apply again Lemma 2.4 with $\varepsilon=|\mathcal{E}|-|\mathcal{E}'|$ to get a cluster $\mathcal{E}''$ with $|\mathcal{E}''|=|\mathcal{E}|$ and such that

$$P(\mathcal{E}'')-P(\mathcal{E}')\leq C_{\rm per}(2C_{\rm vol}r^\eta)^\beta\,.$$

By the minimality of $\mathcal{E}$, putting this inequality together with (2.8) we obtain

$$C_{\rm per}(2C_{\rm vol}r^\eta)^\beta\geq c\tilde r h_{\min}\geq\frac{crh_{\min}}{C_{2,k}}\,,$$

hence

$$r^{\eta\beta-1}\geq\frac{c\,h_{\min}}{C_{2,k}C_{\rm per}2^\beta C_{\rm vol}^\beta}\,.$$

If $\eta\beta>1$, this gives a lower bound to r, hence we have the searched contradiction, up to possibly decrease $R_{3,k-1}$. Instead, if $\eta\beta=1$, the searched contradiction follows since $C_{\rm per}$ is very small by (2.1). □

Lemma 2.7 (No-islands)**.** *There exists $R_4\leq R_3$ such that for every $r\leq R_4$ and every $B(x,r)\subseteq D$, if for some $0\leq i\leq m$ one has*

$$|E_i\cap B(x,r)|>0\,, \tag{2.9}$$

then also

$$\mathcal{H}^1(E_i \cap \partial B(x,r)) > 0\,. \tag{2.10}$$

Notice that, in this lemma, i can also attain the value 0. Recall that E_0 has been defined as $\mathbb{R}^2 \setminus (\cup_{i=1}^m E_i)$.

Proof of Lemma 2.7. Take a ball $B(x,r) \subseteq D$ with $r \leq R_3$ such that, for some $0 \leq i \leq m$, (2.9) holds while (2.10) does not. We have to prove that this is absurd provided that r is is smaller than some $R_4 \leq R_3$ that we are going to specify later.

Let us call $F = E_i \cap B(x,r)$. Since (2.10) is false then, up to $\mathcal{H}^1$-negligible subsets, $\partial^* F \subseteq B(x,r)$, and in particular

$$\partial^* F \subseteq \bigcup_{\substack{j \in \{0,\,1,\,\ldots\,,\,m\}\\ j \neq i}} \partial^* E_j\,,$$

so that for some $0 \leq \ell \leq m$, $\ell \neq i$ we have

$$\mathcal{H}^1(\partial^* F \cap \partial^* E_\ell) \geq \frac{1}{m}\,\mathcal{H}^1(\partial^* F) \geq \frac{1}{h_{\max} m}\,P(F)\,.$$

Let us then define the cluster $\mathcal{E}'$ as the cluster such that $E_i' = E_i \setminus F$, $E_j' = E_j$ for every $j \notin \{i,\,\ell\}$ and $E_\ell' = E_\ell \cup F$. By construction and by Lemma 2.1 we have

$$\begin{aligned} P(\mathcal{E}') &\leq P(\mathcal{E}) - h_{\min}\,\mathcal{H}^1(\partial^* F \cap \partial^* E_\ell) \leq P(\mathcal{E}) - \frac{h_{\min}}{h_{\max} m}\,P(F) \\ &\leq P(\mathcal{E}) - \frac{h_{\min}^2}{h_{\max} C_{\rm vol}^{1/\eta} m}\,|F|^{1/\eta}\,. \end{aligned} \tag{2.11}$$

Let us define $\varepsilon = |\mathcal{E}| - |\mathcal{E}'|$, so that $|\varepsilon| \leq 2|F|$, and the latter is strictly positive by (2.9). Applying again Lemma 2.4, we get a cluster $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ and

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}|\varepsilon|^\beta \leq P(\mathcal{E}') + C_{\rm per} 2^\beta |F|^\beta\,.$$

Putting this inequality together with (2.11), by the optimality of $\mathcal{E}$ we find

$$(C_{\rm vol} r^\eta)^{1/\eta - \beta} \leq |B(x,r)|^{1/\eta-\beta} \leq |F|^{1/\eta-\beta} \leq \frac{h_{\max}}{h_{\min}^2}\,C_{\rm vol}^{1/\eta} C_{\rm per} 2^\beta m\,.$$

As usual, we have to distinguish two cases in order to conclude. If $\eta > 1/\beta$, then the inequality implies that r cannot be too small, hence $r > R_4$ for some $R_4 \leq R_3$. If $\eta = 1/\beta$, then the inequality is false because $C_{\rm per}$ is very small by (2.1). □

Remark 2.8. *Notice that the claim of the above lemma can be trivially generalised as follows. If there is a set $G \subseteq B(x,r)$ with Lipschitz boundary such that $|E_i \cap G| > 0$, then also $\mathcal{H}^1(E_i \cap \partial G) > 0$. The proof remains exactly the same, one only has to substitute $B(x,r)$ with G everywhere.*

Corollary 2.9 (At most 3 colors)**.** *For every* $B(x,R_4)\subseteq D$ *one has*

$$\#\Big\{0\leq i\leq m:\,|E_i\cap B(x,R_4/C_2)|>0\Big\}\leq 3\,.$$

Proof. We apply Lemma 2.6 to the ball $B(x,R_4)$, finding some $R_4/C_2<\rho<R_4$ for which $\partial^*\mathcal{E}\cap\partial B(x,\rho)$ consists of at most three points. As a consequence, there are at most three different indices $0\leq i\leq m$ such that $E_i\cap\partial B(x,\rho)$ has positive $\mathcal{H}^1$-measure (keep in mind that Vol'pert Theorem 2.3 holds true for the ball $B(x,\rho)$). Since $\rho<R_4$, by Lemma 2.7 we obtain that $|E_i\cap B(x,\rho)|$ can be strictly positive for at most three different indices $0\leq i\leq m$, and since $\rho>R_4/C_2$ the proof is concluded. □

Definition 2.10 (3-color point)**.** *A point* $x\in\mathbb{R}^2$ *is said a* 3-color point *if, for every* $r>0$*, we have*

$$\#\big\{0\leq i\leq m:\,|E_i\cap B(x,r)|>0\big\}\geq 3\,. \tag{2.12}$$

Notice that, in view of Corollary 2.9, for every 3-color point $x\in D$ the sets E_i satisfying (2.12) are actually exactly 3 for every $r<\min\{R_4/C_2,\,\mathrm{dist}(x,\partial D)\}$. This also motivates the name.

We can now improve the result of Lemma 2.6 for balls centered at a 3-color point, namely, in (2.6) the possibly large constant C_2 can be replaced by any constant strictly larger than 1. We are going to prove the result with constant 25/24 just because this is the value that we will need later, but it is clear from the proof that nothing changes with any other constant strictly larger than 1.

Lemma 2.11. *There exists* $R_5\leq R_4$ *such that, for any 3-color point* $x\in D$ *and any* $r\leq\min\{R_5,\,\mathrm{dist}(x,\partial D)/C_2\}$*, there is* $24r/25<\bar\rho<r$ *such that* $\#\Big(\partial^*\mathcal{E}\cap\partial B(x,\bar\rho)\Big)=3$.

Proof. Let $R_5\leq R_4$ be a constant to be specified later, and let x and r be as in the claim. By Lemma 2.7, we know that

$$\#\Big(\partial^*\mathcal{E}\cap\partial B(x,s)\Big)\geq 3\qquad\qquad \forall\,0<s\leq\min\{R_4,\,\mathrm{dist}(x,\partial D)\}\,. \tag{2.13}$$

By Lemma 2.6, we find some $r<\rho<C_2r$ such that

$$\#\Big(\partial^*\mathcal{E}\cap\partial B(x,\rho)\Big)\leq 3\,,$$

and the number is in fact exactly 3 by (2.13). Notice that we have applied Lemma 2.6 with C_2r in place of r, and this is possible only if $C_2r\leq\min\{R_3,\,\mathrm{dist}(x,\partial D)\}$, which in turn is admissible up to choose $R_5\leq R_3/C_2$.

Let us now call

$$\mu=\mathcal{H}^1\Big(\Big\{0<s<\rho:\,\#\Big(\partial^*\mathcal{E}\cap\partial B(x,s)\Big)\geq 4\Big\}\Big)\,,$$

so that the thesis follows as soon as we show that, with the right choice of R_5, $\mu < r/25$. In view of (2.13), by coarea formula we can estimate

$$P(\mathcal{E}; B(x,\rho)) \geq \tilde{h}_{\min}\big(3(\rho-\mu)+4\mu\big) = \tilde{h}_{\min}(3\rho+\mu)\,,$$

having again defined $\tilde{h}_{\min} = \min\{h(x),\, x \in \overline{B(x,\rho)}\}$ and $\tilde{h}_{\max} = \max\{h(x),\, x \in \overline{B(x,\rho)}\}$. Let us now define $\mathcal{E}'$ as the cluster which coincides with $\mathcal{E}$ outside of $B(x,\rho)$ and such that $\partial^*\mathcal{E}' \cap B(x,\rho)$ is done by the three segments joining the three points of $\partial^*\mathcal{E} \cap \partial B(x,\rho)$ with x. Observe that

$$P(\mathcal{E}'; B(x,\rho)) \leq 3\tilde{h}_{\max}\rho \leq 3\tilde{h}_{\min}\rho + 3\omega(2\rho)\rho\,.$$

As usual, we apply Lemma 2.4 to get a competitor $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ and

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol}\rho^\eta)^\beta\,.$$

Putting together the last three estimates, since $P(\mathcal{E}) \leq P(\mathcal{E}'')$ and $h_{\min} \leq \tilde{h}_{\min}$ we find

$$\mu \leq \frac{\rho}{h_{\min}}\Big(3\omega(2\rho) + 2^\beta C_{\rm per}C_{\rm vol}^\beta \rho^{\eta\beta-1}\Big) \leq \frac{C_2 r}{h_{\min}}\Big(3\omega(2\rho) + 2^\beta C_{\rm per}C_{\rm vol}^\beta \rho^{\eta\beta-1}\Big)\,,$$

hence the thesis reduces to check that the term in parentheses in the above estimate can be taken smaller than $h_{\min}/(25C_2)$. Concerning $3\omega(2\rho) \leq 3\omega(2C_2R_5)$, this is arbitrarily small as soon as R_5 is small enough, so we can suppose that this is smaller $h_{\min}/(50C_2)$, and to conclude we need the second term in parenthesis to be smaller than $h_{\min}/(50C_2)$. This is clearly true for R_5 small enough if $\eta\beta > 1$, while in the case $\eta\beta = 1$ this is true regardless of R_5 thanks to (2.1). □

Proposition 2.12 (3-color points are a positive distance apart)**.** *There exists $R_6 \leq R_5$ such that any two 3-color points x, x' with $B(x,R_5) \subseteq D$ have distance at least R_6.*

Proof. Let us assume by contradiction the existence of two 3-color points x and x' as in the claim with $d := |x - x'| < R_6$, where $R_6 \leq R_5$ is a constant to be specified later. Since the proof is a bit involved, we divide it in few steps for the sake of clarity.

Step I. *A circle with three boundary points at around* $120°$.
We start by applying Lemma 2.11 to the point x and with $r = \frac{5}{4}d$, which is admissible as soon as $R_6 \leq \frac{4}{5}R_5$, finding a radius ρ with

$$\frac{6}{5}\,d = \frac{24}{25}\cdot\frac{5}{4}\,d < \rho < \frac{5}{4}\,d \tag{2.14}$$

such that $\partial^*\mathcal{E} \cap \partial B(x,\rho)$ contains exactly three points, say x_1, x_2, x_3. We want to show that these three points are close to be vertices of an equilateral triangle. More precisely, we will prove that

$$\theta = \min\Big\{x_1\widehat{x}x_2,\, x_2\widehat{x}x_3,\, x_3\widehat{x}x_1\Big\} > \Big(\frac{2}{3} - \frac{1}{15}\Big)\pi = \frac{3}{5}\,\pi\,. \tag{2.15}$$

Indeed, assuming just to fix the ideas that $\theta = x_1\widehat{x}x_2$, we define the cluster $\mathcal{E}'$ which coincides with $\mathcal{E}$ outside of $B(x,\rho)$ and such that $\partial^*\mathcal{E}'\cap B(x,\rho)$ is done by the segment xx_3 and the set Γ given by Lemma 2.2 with $y = x_1$ and $z = x_2$. By Lemma 2.4 we have a cluster $\mathcal{E}''$ with $|\mathcal{E}''| = |\mathcal{E}|$ such that

$$\begin{aligned} P(\mathcal{E}) &\leq P(\mathcal{E}'') \leq P(\mathcal{E}') + C_{\rm per}(2C_{\rm vol}\rho^\eta)^\beta \\ &\leq P(\mathcal{E}) + \rho\bigg(\tilde{h}_{\max}\big(1+L(\theta)\big) - 3\tilde{h}_{\min} + 2^\beta C_{\rm per}C_{\rm vol}^\beta\rho^{\eta\beta-1}\bigg)\,, \end{aligned} \tag{2.16}$$

where we have again defined $\tilde{h}_{\min} = \min\{h(x),\, x\in\overline{B(x,\rho)}\}$ and $\tilde{h}_{\max} = \max\{h(x),\, x\in\overline{B(x,\rho)}\}$, and where we have used the fact that

$$P(\mathcal{E};B(x,\rho)) \geq 3\tilde{h}_{\min}\rho\,,$$

which follows from Lemma 2.7 because x is a 3-color point and by coarea formula. Arguing as usual, a straightforward computation from (2.16) gives

$$2 - L(\theta) \leq \frac{1}{h_{\min}}\Big(3\omega(2\rho) + 2^\beta C_{\rm per}C_{\rm vol}^\beta\rho^{\eta\beta-1}\Big)\,.$$

As already done several times, both if $\eta\beta>1$ and if $\eta\beta=1$ we get that, provided R_6 is small enough, $L(\theta)$ is as close as 2 as we wish, in particular we can assume that (2.15) holds true.

For later use, we remark that from (2.16) and by definition of $\mathcal{E}'$ we have

$$\begin{aligned} P(\mathcal{E};B(x,\rho)) &\leq P(\mathcal{E}';B(x,\rho)) + C_{\rm per}(2C_{\rm vol}\rho^\eta)^\beta \leq 3\tilde{h}_{\max}\rho + C_{\rm per}(2C_{\rm vol}\rho^\eta)^\beta \\ &< \frac{19}{6}\,\tilde{h}_{\min}\rho\,, \end{aligned} \tag{2.17}$$

again up to possibly decrease the value of R_6.

<u>*Step II.*</u> *The estimate (2.20).*
In this step we define the "curved rectangles" Q_a and Q_b and we prove the estimate (2.20). The situation is depicted in Figure 1.

We start by observing that, since both x and x' are 3-color points, by Lemma 2.7,

$$P\big(\mathcal{E};B(x',\rho-d)\big) \geq 3\tilde{h}_{\min}(\rho-d)\,,\quad P\big(\mathcal{E};B(x,\rho-2(\rho-d))\big) \geq 3\tilde{h}_{\min}(2d-\rho)\,. \tag{2.18}$$

Among x_1, x_2 and x_3, let a and b be the two points having maximal distance from x', let ϕ be the angle defined by the property

$$(2d-\rho)\phi = 2\rho - 2d\,, \tag{2.19}$$

and let Q_a and Q_b be the curved rectangles defined by

$$\begin{aligned} Q_a &= \Big\{z\in B(x,\rho)\setminus B(x,2d-\rho):\,|z\widehat{x}a|<\phi\Big\}\,,\\ Q_b &= \Big\{z\in B(x,\rho)\setminus B(x,2d-\rho):\,|z\widehat{x}b|<\phi\Big\}\,. \end{aligned}$$

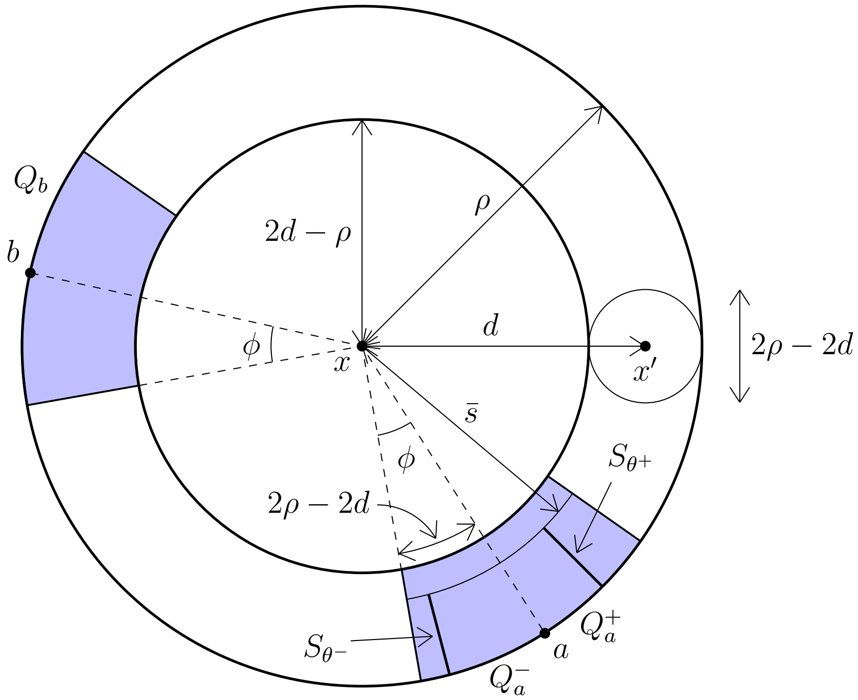

Figure 1. The situation in Proposition 2.12.

Notice that the angles $a\widehat{x}x'$ and $x'\widehat{x}b$ are both at least $3\pi/10$ by (2.15), while $\phi < 2/3 \approx 0.21\pi$ since $d > 4\rho/5$. As a consequence, a simple trigonometric computation ensures that, as in the figure, Q_a and Q_b are disjoint and they do not intersect the ball $B(x', \rho - d)$. The goal of this step is to show that

$$P(\mathcal{E}; Q_a) \geq \tilde{h}_{\min}(2\rho - 2d)\,, \qquad\qquad P(\mathcal{E}; Q_b) \geq \tilde{h}_{\min}(2\rho - 2d)\,. \tag{2.20}$$

Let us prove the estimate for Q_a, since there is no difference with the case of Q_b. If for almost every $2d - \rho < s < \rho$ one has $\#(\partial^*\mathcal{E} \cap \partial B(x, s) \cap Q_a) \geq 1$, the claim directly follows by integration.

Suppose then the existence of some $2d - \rho < \bar{s} < \rho$ such that the arc $\partial B(x, \bar{s}) \cap Q_a$ does not intersect $\partial^*\mathcal{E}$. Let us subdivide $Q_a = Q_a^+ \cup Q_a^-$, where $Q_a^\pm$ are the two parts in which Q_a is divided by the segment $\{z : z - x = \sigma(a - x),\ \frac{2d}{\rho} - 1 < \sigma < 1\}$. For every $0 < \theta < \phi$, let us now call S_θ the closed segment made by all the points $z \in Q_a^+$ such that $a\widehat{x}z = \theta$ and $\bar{s} \leq |z - x| \leq \rho$, and similarly for every $-\phi < \theta < 0$ we call S_θ the closed segment made by all the points $z \in Q_a^-$ such that $z\widehat{x}a = -\theta$ and $\bar{s} < |z - x| < \rho$. If $\#(\partial^*\mathcal{E} \cap S_\theta) \geq 1$ for almost every $0 < \theta < \phi$, or for almost every $-\phi < \theta < 0$, then again by integration and recalling (2.19) we obtain the searched estimate for $P(\mathcal{E}; Q_a)$.

We are then only left to consider the case when two angles $\theta^\pm$ with $-\phi < \theta^- < 0 < \theta^+ < \phi$ exist such that both the segments S_{θ^+} and S_{θ^-} have no intersection with $\partial^*\mathcal{E}$. In this case, putting together the two arcs $\partial B(x,\rho) \cap \bigcup_{\theta^- \leq \theta \leq \theta^+} S_\theta$ and $\partial B(x, \bar{s}) \cap \bigcup_{\theta^- \leq \theta \leq \theta^+} S_\theta$, and the two segments S_{θ^-} and S_{θ^+}, we obtain a Lipschitz, closed loop which intersects $\partial^*\mathcal{E}$ in a single point, namely, a. And in turn, this is impossible, because for any such loop the number of intersections with $\partial^*\mathcal{E}$ must be either empty, or done by at least two points. The validity of (2.20) is then established.

Step III. Estimate on $P(\mathcal{E}; B(x,\rho))$ from below.
In this last quick step we find an estimate of $P(\mathcal{E}; B(x,\rho))$ from below, which gives a contradiction with the estimate from above found in Step I, concluding the proof. Since the curved rectangles Q_a and Q_b and the balls $B(x, 2d-\rho)$ and $B(x', \rho-d)$ are pairwise disjoint, by (2.18), (2.20) and (2.14) we obtain

$$P(\mathcal{E}; B(x,\rho)) \geq \tilde{h}_{\min}(4\rho - d) \geq \frac{19}{6}\,\tilde{h}_{\min}\rho\,,$$

against (2.17). □

2.3. **Interface regularity.** This section is devoted to prove the following regularity result for the boundary of the optimal cluster $\mathcal{E}$.

Proposition 2.13 ($\mathrm{C}^{1,\gamma}$ regularity)**.** *There exists an increasing function $\xi:\mathbb{R}^+\to\mathbb{R}^+$ with $\lim_{r\to 0^+}\xi(r)=0$ such that the following property holds. Let $B(\bar{x},\bar{r})\subseteq D$ be a ball such that*

$$\#\Big\{0\leq i\leq m:\,|E_i\cap B(\bar{x},\bar{r})|>0\Big\}\leq 2\,,\qquad \#\left(\partial^*\mathcal{E}\cap\partial B(\bar{x},\bar{r})\right)<+\infty\,. \tag{2.21}$$

Then, $\partial^*\mathcal{E}\cap B(\bar{x},\bar{r})$ is a finite union of C^1 pairwise disjoint relatively closed curves such that, calling $\tau(x)\in\mathbb{P}^1$ the direction of the tangent vector at any $x\in\partial^*\mathcal{E}\cap B(\bar{x},\bar{r})$, one has

$$|\tau(y)-\tau(x)|\leq\xi(|y-x|) \tag{2.22}$$

for every x, $y\in\partial^*\mathcal{E}\cap B(\bar{x},\bar{r})$. Moreover, if $\eta\beta>1$ and h is locally α-Hölder continuous, then it is possible to take $\xi(t)=Kt^\gamma$ with some $K>0$ and

$$\gamma=\frac{1}{2}\,\min\{\eta\beta-1,\,\alpha\}\,, \tag{2.23}$$

so that in particular $\partial^*\mathcal{E}\cap B(\bar{x},\bar{r})$ is $\mathrm{C}^{1,\gamma}$.

Lemma 2.14 (Almost alignment on a circle with two boundary points)**.** *There exist $R_7<R_6$ and a function $\xi_1:\mathbb{R}^+\to\mathbb{R}^+$ as in Proposition 2.13 and satisfying the Dini property such that the following holds. Let $B(\bar{x},\bar{r})$ be as in Proposition 2.13, and let $x\in\partial^*\mathcal{E}$ and $r<R_7$ be such that*

$$B(x,r)\subseteq B(\bar{x},\bar{r})\,,\qquad\qquad \partial B(x,r)\cap\partial^*\mathcal{E}=\{a,\,b\}$$

where a, b are two distinct points in $\partial^*\mathcal{E}$, such that $0 < a\widehat{x}b \le \pi$. Then, calling $d = \mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) - |b-a|$, we have

$$|a\widehat{x}b - \pi| \le \xi_1(r)\,, \qquad\qquad d \le \frac{r}{6}\,\xi_1(r)^2\,. \tag{2.24}$$

Proof. We define $\mathcal{E}'$ as the cluster which coincides with $\mathcal{E}$ outside of $B(x,r)$, and such that $\partial\mathcal{E}' \cap B(x,r)$ is done by the segment ab. We let $d = \mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,r)) - |b-a|$. Observe that, since $x \in \partial^*\mathcal{E}$, the fact that $r < R_6$ together with Lemma 2.7 ensures that for almost each $0 < s < r$ the set $\partial^*\mathcal{E} \cap \partial B(x,s)$ is non-empty, thus by Vol'pert Theorem it contains at least 2 points. As a consequence, by coarea formula, $\mathcal{H}^1(\partial\mathcal{E}' \cap B(x,r)) \ge 2r$, hence $d \ge 2r - |b-a|$. Setting $\tilde{h}_{\min}$ and $\tilde{h}_{\max}$ as usual, we have

$$P(\mathcal{E}') - P(\mathcal{E}) \le \tilde{h}_{\max}|b-a| - \tilde{h}_{\min}(|b-a| + d)\,.$$

As a consequence, minding that $\mathcal{E}\Delta\mathcal{E}' \subseteq B(x,r)$ and $|B(x,r)| \le C_{\rm vol} r^\eta$, by Lemma 2.4 –notice that in (2.2) one can clearly use $C_{\rm per}[|\varepsilon|]$ in place of $C_{\rm per}$– we readily obtain

$$d \le \frac{1}{h_{\min}}\Big(2r\omega(2r) + C_{\rm per}[2C_{\rm vol}r^\eta](2C_{\rm vol}r^\eta)^\beta\Big)\,.$$

Let us now set

$$\xi_1(r) = \bigg(\frac{6}{h_{\min}}\Big(2\omega(2r) + 2^\beta C_{\rm per}[2C_{\rm vol}r^\eta]C_{\rm vol}^\beta r^{\eta\beta-1}\Big)\bigg)^{1/2}\,,$$

so that the right estimate in (2.24) holds true. The left one then easily follows since

$$\frac{r}{6}\,\xi_1(r)^2 \ge d \ge 2r - |b-a| = 2r\Big(1 - \sin\big(a\widehat{x}b/2\big)\Big) \ge \frac{r}{6}\,(a\widehat{x}b - \pi)^2\,.$$

Hence, to conclude we only have to check the properties of ξ_1. The fact that ξ_1 is increasing is true by construction, and the fact that, as $r \searrow 0$, it goes to 0 is true since $\omega(r) \searrow 0$, and $r^{\eta\beta-1} \searrow 0$ if $\eta\beta > 1$, while $C_{\rm per}[2C_{\rm vol}r^\eta] \searrow 0$ if $\eta\beta = 1$ by assumption. In addition, since $r^\eta \le r$ because $\eta \ge 1$, then

$$\xi_1(r) \lesssim \sqrt{\omega(2r) + C_{\rm per}[2C_{\rm vol}r^\eta]r^{\eta\beta-1}} \lesssim \sqrt{\omega(2r)} + \sqrt{C_{\rm per}[2C_{\rm vol}r]r^{\eta\beta-1}}\,.$$

The Dini property of ξ_1 then readily follows. Indeed, the Dini property of $\sqrt{\omega(2r)}$ is true since by assumption h is locally 1/2-Dini continuous. Moreover, the Dini property of $\sqrt{C_{\rm per}[2C_{\rm vol}r]r^{\eta\beta-1}}$ is clear if $\eta\beta > 1$, while it comes from the 1/2-Dini property of $t \mapsto C_{\rm per}[t]$ if $\eta\beta = 1$. Finally, if $\eta\beta > 1$ and h is locally α-Hölder then we have

$$\xi_1(r) \lesssim \sqrt{\omega(2r) + r^{\eta\beta-1}} \lesssim \sqrt{r^\alpha + r^{\eta\beta-1}} \approx r^\gamma\,,$$

with γ given by (2.23). □

Lemma 2.15 (Almost alignment on every circle)**.** *There exists a function $\xi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ as in Lemma 2.14 such that the following holds. Let $B(\bar{x},\bar{r})$ be as in Proposition 2.13,* *and let* $x \in \partial^*\mathcal{E}$ *and* $r < R_7/(2C_2)$ *be such that* $B(x, 2C_2 r) \subseteq B(\bar{x}, \bar{r})$*. Then, there exists a direction* $\tau(x,r) \in \mathbb{P}^1$ *such that every* $y \in \partial^*\mathcal{E} \cap \partial B(x,r)$ *satisfies*

$$|\zeta(y-x) - \tau(x,r)| \leq \xi_2(r)\,, \tag{2.25}$$

where $\zeta(v) \in \mathbb{P}^1$ *is the direction of any vector* $v \in \mathbb{R}^2 \setminus \{0\}$*. Moreover, for every* $r' \in [r/2, r)$

$$|\tau(x,r) - \tau(x,r')| \leq 2C_2\xi_1(2C_2 r) + 2\xi_2(r)\,. \tag{2.26}$$

Proof. Since $r < R_7/(2C_2) < R_3/(2C_2)$, by Lemma 2.6 there exists some $2r < \rho < 2C_2 r$ such that $\partial B(x,\rho) \cap \partial^*\mathcal{E}$ contains at most three points. We claim that these points are actually 2. In fact, since $x \in \partial^*\mathcal{E}$ and $\rho < R_4$ then there must be at least two such points by Lemma 2.7. On the other hand, in view of (2.21) and again by Lemma 2.7, we obtain that $\#\big\{0 \leq i \leq m : \mathcal{H}^1(E_i \cap \partial B(x,\rho)) > 0\big\} \leq 2$, so that the number of points of $\partial B(x,\rho) \cap \partial^*\mathcal{E}$ must be even (keep in mind that, as always, Vol'pert Theorem holds for $B(x,\rho)$). The claim is then proved, and we can then call a and b these two points, and define $\tau(x,r) \in \mathbb{P}^1$ the direction of the segment ab. Notice that the vector $\tau(x,r)$ depends on x, on r, and on the choice of $2r < \rho < 2C_2 r$. The vector $\tau(x)$ of Proposition 2.13, instead, will only depend on x, as one can clearly deduce from (2.22).

Let now $y \in \partial^*\mathcal{E} \cap \partial B(x,r)$ be given, let us call $d_1 = |y-a|$ and $d_2 = |y-b|$ and assume, without loss of generality, that $d_1 \leq d_2$. For every $0 < s < d_2$, we call $G_s = B(y,s) \cap B(x,\rho)$. Since $y \in \partial^*\mathcal{E}$, by Lemma 2.7 and keeping in mind also Remark 2.8, we obtain that $\Gamma_s := \partial G_s \cap \partial^*\mathcal{E}$ contains at least two points. Since $s < d_2$, Γ_s cannot contain b, and it cannot contain a if $s < d_1$. Recalling that $\partial B(x,\rho) \cap \partial^*\mathcal{E} = \{a, b\}$, we deduce that $\Gamma_s \cap B(x,\rho)$ contains at least two points for $0 < s < d_1$, and at least one point for $d_1 < s < d_2$. By construction, this implies that

$$\mathcal{H}^1\big(\partial^*\mathcal{E} \cap B(x,\rho)\big) \geq d_1 + d_2\,.$$

As usual, we define the cluster $\mathcal{E}'$ coinciding with $\mathcal{E}$ outside of $B(x,\rho)$ and such that $\partial\mathcal{E}' \cap B(x,\rho)$ is given by the segment ab, so by Lemma 2.4 we readily obtain

$$h_{\min}\big(d_1 + d_2 - |a-b|\big) \leq 2\rho\omega(2\rho) + C_{\rm per}[2C_{\rm vol}\rho^\eta](2C_{\rm vol}\rho^\eta)^\beta\,.$$

Keeping in mind that $|y-x| = r$ while $|a-x| = |b-x| = \rho > 2r$, arguing as in Lemma 2.14 we find a function $\tilde{\xi}$ satisfying the Dini property and such that $|a\widehat{y}b - \pi| \leq \tilde{\xi}(r)$. In addition, $\tilde{\xi}(r) \lesssim r^\gamma$ if $\eta\beta > 1$ and h is locally α-Hölder, with γ given by (2.23). Moreover, Lemma 2.14 already gives that $|a\widehat{x}b - \pi| \leq \xi_1(\rho) \leq \xi_1(2C_2 r)$. Then, since $|y-x| = r > \rho/(2C_2)$, an immediate geometric argument provides a function ξ_2, satisfying the Dini property and with the same additional features as $\tilde{\xi}$, for which (2.25) is true.

To conclude we only have to establish (2.26). Keep in mind that, by Lemma 2.14,

$$\mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,\rho)) - 2\rho \leq \mathcal{H}^1(\partial^*\mathcal{E} \cap B(x,\rho)) - |b-a| \leq \frac{\rho}{6}\,\xi_1(\rho)^2\,,$$

and this implies that

$$\mathcal{H}^1(\partial^*\mathcal{E}\cap B(x,r)) \leq 2r + \frac{\rho}{6}\,\xi_1(\rho)^2\,. \tag{2.27}$$

Let now $r'\in[r/2,r)$, and let $z\in\partial B(x,r')\cap\partial^*\mathcal{E}$. Let us call y the point of $\partial B(x,r)\cap\partial^*\mathcal{E}$ which is closest to z. Then, calling for brevity $\theta=\zeta(z-x)-\zeta(y-x)$, since $r'\geq r/2$ it is

$$\mathcal{H}^1(\partial^*\mathcal{E}\cap B(x,r)) - 2r \geq r\left(\sqrt{1+\sin^2\theta}-1\right) \geq \frac{\theta^2}{6}\,r\,.$$

By (2.27) and (2.25), we have then

$$|\zeta(z-x)-\tau(x,r)| \leq \sqrt{2C_2}\xi_1(\rho)+\xi_2(r) \leq \sqrt{2C_2}\xi_1(2C_2r)+\xi_2(r)\,.$$

To conclude it is then enough to apply (2.25) with r' in place of r and z in place of y, finally finding

$$|\tau(x,r')-\tau(x,r)| \leq |\zeta(z-x)-\tau(x,r')|+|\zeta(z-x)-\tau(x,r)| \leq \sqrt{2C_2}\xi_1(2C_2r)+\xi_2(r)+\xi_2(r')\,,$$

which is stronger than (2.26). □

Corollary 2.16. *Let $B(\bar x,\bar r)$ be as in Proposition 2.13 and let $x,\,y\in\partial^*\mathcal{E}$ be such that $r:=|y-x|<R_7/(2C_2)$ and $B(x,2C_2r)\cup B(y,2C_2r)\subseteq B(\bar x,\bar r)$. Then,*

$$|\tau(x,r)-\tau(y,r)| \leq 2\xi_2(r)\,.$$

Proof. It is possible to apply Lemma 2.15 both to x and y. Then, (2.25) gives that the direction $\zeta(y-x)$ of the vector $y-x$ differs at most $\xi_2(r)$ from both $\tau(x,r)$ and $\tau(y,r)$. The thesis is then obvious. □

We are now in position to prove Proposition 2.13.

Proof (of Proposition 2.13). We let ξ_1 and ξ_2 be the functions defined in Lemmas 2.14 and 2.15. For every $x\in\partial^*\mathcal{E}\cap B(\bar x,\bar r)$, it is possible to apply Lemma 2.15 for every r small enough. For every such r, taking in account (2.26), by obvious induction we get that for every $n\in\mathbb{N}$ and every $r'\in[r/2^n,r/2^{n-1})$ one has

$$|\tau(x,r)-\tau(x,r')| \leq 2C_2\sum_{j=0}^{n-1}\xi_1(2C_2r/2^j)+2\sum_{j=0}^{n-1}\xi_2(r/2^j)\,.$$

Let us then define

$$\xi_3(r) = 2C_2\sum_{j=0}^{+\infty}\xi_1(2C_2r/2^j)+2\sum_{j=0}^{+\infty}\xi_2(r/2^j)\,.$$

Notice that the series converges since the functions ξ_1 and ξ_2 have the Dini property. Moreover, if $\eta\beta>1$ and h is locally α-Hölder, then both ξ_1 and ξ_2 are bounded by a multiplicative constant (only depending on $h_{\min}$, β, η, ω, $C_{\rm per}$ and $C_{\rm vol}$) times r^γ, with γ given by (2.23). Hence, not only the series converges, but also $\xi_3(r)\leq Kr^\gamma$ with a constant K only depending on the data.

As a consequence, we obtain that $\tau(x,r')$ converges to a direction $\tau(x) \in \mathbb{P}^1$ for $r' \searrow 0$, and that $|\tau(x,r) - \tau(x)| \leq \xi_3(r)$. For every x, $y \in B(\bar{x},\bar{r})$ as in Corollary 2.16, then, we deduce that, calling $r = |y-x|$, one has

$$|\tau(x) - \tau(y)| \leq 2\xi_3(r) + 2\xi_2(r)\,.$$

We can finally set $\xi(r) = 2\xi_3(r) + 2\xi_2(r)$. Summarizing, we have shown that for *every* $x \in \partial^*\mathcal{E} \cap B(\bar{x},\bar{r})$ the normal vector to $\partial^*\mathcal{E}$ at x exists, and is orthogonal to $\tau(x)$. The above estimate, also keeping in mind (2.21), ensures then that $\partial^*\mathcal{E}$ is a finite union of C^1 curves. □

Corollary 2.17 (Single C^1 curve)**.** *Let $B(\bar{x},\bar{r}) \subseteq D$ be a ball as in Proposition 2.13, with the additional assumption that $\bar{r} < R_4$ and that $\#\big(\partial^*\mathcal{E} \cap \partial B(\bar{x},\bar{r})\big) = 2$. Then, $\partial\mathcal{E} \cap B(\bar{x},\bar{r})$ is a C^1 relatively closed curve, having both endpoints on $\partial B(\bar{x},\bar{r})$.*

Proof. Proposition 2.13 already tells us that $\partial\mathcal{E} \cap B(\bar{x},\bar{r})$ is a finite union of pairwise disjoint relatively closed C^1 curves. Every such curve cannot have an endpoint inside the ball $B(\bar{x},\bar{r})$, hence it is either a closed loop or a curve with both endpoints in $\partial B(\bar{x},\bar{r})$. On the other hand, a closed loop can be excluded since $\bar{r} < R_4$ thanks to Lemma 2.7 and Remark 2.8. Consequently, every curve has two endpoints in $\partial B(\bar{x},\bar{r})$ and, since there are only two points in $\partial^*\mathcal{E} \cap \partial B(\bar{x},\bar{r})$, we deduce that the curve is unique. □

2.4. **Conclusion.** In this short section we can now give the proof of Theorem 1.4, which basically consists in putting together the technical results of the preceding sections.

Proof of Theorem 1.4. Let $\mathcal{E} \subseteq \mathbb{R}^2$ be a minimal cluster, and let us fix two large, closed balls $D^- \subset\subset D \subseteq \mathbb{R}^2$.

Let $x \in D^- \cap \partial\mathcal{E}$ be any 3-color point in the boundary of $\mathcal{E}$ (there are finitely many of these points by Proposition 2.12). Then, by Lemma 2.11 there is some radius $r(x) < R_6$ such that the ball $B(x, r(x))$ is compactly contained in D, and its boundary contains exactly three points in $\partial^*\mathcal{E}$.

Let instead $x \in D^- \cap \partial\mathcal{E}$ be any point in the boundary of $\mathcal{E}$ which is not a 3-color point. Then, by definition and by Lemma 2.6 there is some radius $r(x) < R_6$ such that the ball $B(x, r(x))$ is compactly contained in D, has non-negligible intersection with at most 2 sets E_i with $0 \leq i \leq m$, and its boundary contains exactly two points in $\partial^*\mathcal{E}$ (in principle there could be at most three such points, but as already noticed in the proof of Lemma 2.15 they are necessarily 2).

By compactness, we can cover D^- with finitely many balls $B_j = B(x_j, r_j)$, having radii $r_j < R_6$ and with the following property. For every j, either x_j is a 3-color point and $\partial B_j \cap \partial^*\mathcal{E}$ is done by three points, or x_j is not a 3-color point, the ball B_j has non-negligible intersection with at most two different sets E_i, $0 \leq i \leq m$, and $\partial B_j \cap \partial^*\mathcal{E}$ is done by two points.

In the second case, by Proposition 2.13 and Corollary 2.17 we know that $\partial^*\mathcal{E} \cap B_j$ is done by a C^1 curve whose tangent vector τ satisfies the uniform estimate (2.22).

Let us then consider a ball B_j centered at a 3-color point, and let a be one of the three points of $\partial B_j \cap \partial^*\mathcal{E}$. The point a is not a 3-color point, by Proposition 2.12. Hence, a small ball centered in a has non-negligible intersection with only two different sets E_i, so using again Lemma 2.6 and Corollary 2.17 we obtain that $\partial^*\mathcal{E}$ is a uniformly C^1 curve near a. The same of course holds near b and c, the other two points of $\partial B_j \cap \partial^*\mathcal{E}$.

Therefore, there are three maximal (with respect to the inclusion) uniformly C^1 curves in $B_j \cap \partial^*\mathcal{E}$, having one endpoint respectively in a, b, c. Since, as just observed, $\partial^*\mathcal{E}$ is a C^1 curve around each point which is not a 3-color point, by maximality the second endpoint of each of the three curves must be a 3-color point inside B_j (keep in mind that the curves have finite length since $\mathcal{E}$ is a minimal cluster). This means that the three curves meet at x_j, which is the only 3-color point in B_j. Keeping in mind the uniform C^1 property of the curves, given by (2.22), we deduce that the three curves arrive with a well-defined tangent vector at x_j. In other words, $\partial^*\mathcal{E} \cap B_j$ contains three C^1 curves starting at a, b and c and meeting at x_j arriving with three tangent vectors. By Lemma 2.7 and Remark 2.8, $\partial^*\mathcal{E} \cap B_j$ cannot have other points except these three curves. Finally, the fact that the tangent vectors at x_j form three angles of $\frac{2}{3}\pi$ is an immediate consequence of Lemma 2.2.

The fact that, if $\eta\beta > 1$ and h is locally α-Hölder, then the arcs are not only C^1 but also $\mathrm{C}^{1,\gamma}$ is already given by Proposition 2.13. The proof is then concluded. □

3. Examples

3.1. Grushin plane. An interesting example arising from sub-Riemannian geometry is the so-called Grushin plane, corresponding to $\mathbb{R}^2$ endowed with densities

$$h(x,\nu) = \sqrt{\nu_1^2 + |x_1|^{2\alpha}\nu_2^2}, \qquad g \equiv 1, \quad x = (x_1, x_2) \in \mathbb{R}^2, \ \nu \in \mathbb{S}^1, \tag{3.1}$$

for $\alpha \geq 0$. In particular, for $\alpha = 1$ this is a 2-dimensional quotient of the Heisenberg group, setting of the celebrated Pansu's conjecture [24, 25].

In [22], the authors characterize isoperimetric sets in this framework. We also refer to [13] for a multidimensional generalization of the isoperimetric problem, and to [12, 16] for a first approach to clustering problems in the Grushin plane. Note that in these references the Grushin perimeter is defined in a more general way via De Giorgi's definition allowing for non-Euclidean rectifiable sets. In this paper, we do not need to work at this level of generality since a suitable (non-smooth) change of coordinates (see [22, Proposition 2.3]) reduces the problem to the study of the densities

$$h \equiv 1, \qquad g(x) = |(1+\alpha)x_1|^{-\frac{\alpha}{1+\alpha}}, \quad x \in \mathbb{R}^2, \tag{3.2}$$

for sets with locally finite Euclidean perimeter. Existence of minimal clusters for the densities in (3.2) is proved in the forthcoming paper [15].

Proposition 3.1. *Any minimal cluster $\mathcal{E}$ relative to the densities in* (3.2) *satisfies the Steiner property and the arcs of $\partial^*\mathcal{E}$ are $C^{1,\gamma}$ with $\gamma = 1/(2(\alpha+1))$.*

Proof. We show that the η-growth condition holds with $\eta = (\alpha+2)/(\alpha+1)$ and that any m-cluster $\mathcal{E}$ satisfies the $\varepsilon - \varepsilon^\beta$ property with $\beta = 1$. The conclusion follows then by Theorem 1.4 (note that h is the Euclidean density, hence regular).

We begin with the η-growth condition. For $x \in \mathbb{R}^2$ and $r > 0$, let us set $Q(x,r) = [x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r]$. In the following $C_\alpha > 0$ will be a constant only depending on α. Since $t \mapsto |t|^{-\frac{\alpha}{1+\alpha}}$ is decreasing for $t \in \mathbb{R}^+$, then

$$|Q(x,r)| \le |Q(0,r)| = 4(1+\alpha)^{-\frac{\alpha}{1+\alpha}}\, r \int_0^r x_1^{-\frac{\alpha}{1+\alpha}}\, dx_1 = C_\alpha\, r^{\frac{\alpha+2}{\alpha+1}}\,.$$

The η-growth condition then holds with $C_{\rm vol} = C_\alpha$ and $R_\eta = 1$.

We now pass to the $\varepsilon - \varepsilon^\beta$ property for m-clusters, starting from the case $m = 1$. Let E be a set of locally finite perimeter and finite Lebesgue measure. We fix $a, b \in \partial^* E$ such that $d = \min\{|a-b|, |a_1|, |b_1|\} > 0$ and we let $B_1 = B(a, d/4)$ and $B_2 = B(b, d/4)$. Note that on $B_1 \cup B_2$ we have $1/K < g < K$ for a constant $K > 0$ only depending on the choice of these two balls, and set $R_\beta = d/8$. By construction, for every $x \in \mathbb{R}^2$ the ball $B(x, R_\beta)$ can intersect at most one among B_1 and B_2, so to get (1.2) we can apply the standard Euclidean result in a ball among B_1 and B_2 not intersecting $B(x, R_\beta)$. To pass from the case $m = 1$ to the case $m > 1$, we can argue similarly as in the proof of [20, Theorem 2.9.14]. More precisely, for any $1 \le i \le m$, we can easily find finitely many indices $i_0, i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, m\}$ such that $i_0 = i$, $i_k = 0$, and for every $0 \le j < k$ there is a point $a_j \in \partial^* E_{i_j} \cap \partial^* E_{i_{j+1}}$ not lying on the x_2-axis. Since the necessary points to fix are at most $m(m+1)/2$, we can apply the Euclidean result finitely many times obtaining (1.2) for the special case when the vector $\varepsilon \in \mathbb{R}^m$ has a single non-zero coordinate. And from this we obviously conclude also for a generic vector. □

Remark 3.2. *The isoperimetric set for the densities in* (3.2) *has a $C^{1,\frac{1}{\alpha+1}}$ regular boundary, as follows by* [22]. *In particular, the regularity established in Proposition 3.1 is not sharp, at least for $m = 1$. Moreover, in* [15] *we prove that minimal clusters in this framework exist and are bounded so that they are made by a* finite *union of $C^{1,\frac{1}{2(\alpha+1)}}$ regular arcs.*

3.2. **Gaussian plane.** The Gaussian plane is $\mathbb{R}^2$ with densities

$$h(x) = g(x) = \frac{1}{2\pi}\, e^{-\frac{|x|^2}{2}}, \quad x \in \mathbb{R}^2 .$$

The isoperimetric problem with these densities, which is very important also for its connections with Probability, is deeply studied since the pioneering works [34, 4]. Recently, the characterization of optimal double bubbles in this framework has been given in [21], where the problem is studied in the more general n-dimensional Gaussian space. A simple application of our main result is the following.

Proposition 3.3. *Any minimal cluster $\mathcal{E}$ relative to the Gaussian densities satisfies the Steiner property and the arcs of $\partial^*\mathcal{E}$ are C^∞.*

Proof. We first apply Theorem 1.4 to prove that the Steiner property holds with $C^{1,\frac{1}{2}}$ regularity. Indeed, the η-growth condition is easily verified with $\eta = 2$ and the $\varepsilon - \varepsilon^\beta$ property for clusters holds with $\beta = 1$ thanks to [30, Theorem A].

To conclude the proof it is enough to observe that $h \equiv g$ is a smooth function on $\mathbb{R}^2$ and then the C^∞ regularity of the arcs follows by a standard variational argument. □

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References

[1] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo, M. R. Posteraro, Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^\alpha$. J. Math. Anal. Appl. **451** (2017), no. 1, 280–318.

[2] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo, M. R. Posteraro, The isoperimetric problem for a class of non-radial weights and applications, J. Differential Equations **267** (2019), no. 12, 6831–6871.

[3] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press (2000).

[4] C. Borell, The Brunn-Minkowski inequality in Gauss spaces. Invent. Math., **30** (1975), 207–216.

[5] W. Boyer, B. Brown, G. Chambers, A. Loving & S. Tammen, Isoperimetric regions in $\mathbb{R}^n$ with density r^p, Anal. Geom. Metr. Spaces **4** (2016), no. 1, 236–265.

[6] X. Cabré, X. Ros-Oton & J. Serra, Euclidean balls solve some isoperimetric problems with nonradial weights, C. R. Math. Acad. Sci. Paris **350** (2012), no. 21–22, 945–947.

[7] A. Cañete, M. Miranda & D. Vittone, Some isoperimetric problems in planes with density, J. Geom. Anal. **20** (2010), no. 2, 243–290.

[8] E. Cinti, F. Glaudo, A. Pratelli, J. Serra & X. Ros-Oton, Sharp quantitative stability for isoperimtric inequalities with homogeneous weights, preprint (2020).

[9] E. Cinti & A. Pratelli, The $\varepsilon - \varepsilon^\beta$ property, the boundedness of isoperimetric sets in $\mathbb{R}^N$ with density, and some applications, J. Reine Angew. Math. (Crelle) **728** (2017), 65–103.

[10] G. De Philippis, G. Franzina & A. Pratelli, Existence of Isoperimetric sets with densities "converging from below" on $\mathbb{R}^N$, J. Geom. Anal. **27** (2017), no. 2, 1086–1105.

[11] G. Franzina & A. Pratelli, Non-existence of isoperimetric sets in the Euclidean space with vanishing densities, preprint (2020).

[12] V. Franceschi, A minimal partition problem with trace constraint in the Grushin plane, Calc. Var. Partial Differential Equations **56** (2017), no. 4, 56–104.

[13] V. Franceschi & R. Monti, Isoperimetric problem in H-type groups and Grushin spaces, Rev. Mat. Iberoam. **32** (2016), no. 4, 1227–1258.

[14] V. Franceschi, A. Pratelli & G. Stefani, On the Steiner property for planar minimizing clusters. The anisotropic case, preprint (2020).

[15] V. Franceschi, A. Pratelli & G. Stefani, On the existence of planar minimizing clusters, preprint (2020).

[16] V. Franceschi & G. Stefani, Symmetric double bubbles in the Grushin plane, ESAIM Control Optim. Calc. Var., to appear (2019).

[17] I. McGillivray, An isoperimetric inequality in the plane with a log-convex density, Ric. Mat. **67** (2018), no. 2, 817–874.

[18] W. Gustin, Boxing inequalities, J. Math. Mech. **9** (1960), 229–239.

[19] M. Hutchings, F. Morgan, M. Ritoré & A. Ros, Proof of the double bubble conjecture, Ann. of Math. **155** (2002), no. 2, 459–489.

[20] F. Maggi, Sets of finite perimeter and geometric variational problems, Cambridge Studies in Advanced Mathematics **135**, Cambridge University Press, 2012.

[21] E. Milman & J. Neeman, The Gaussian Double-Bubble Conjecture, preprint (2018).

[22] R. Monti & D. Morbidelli, Isoperimetric inequality in the Grushin plane, J. Geom. Anal. **14** (2004), no. 2, 355–368.

[23] F. Morgan & A. Pratelli, Existence of isoperimetric regions in $\mathbb{R}^n$ with density, Ann. Global Anal. Geom. **43** (2013), no. 4, 331–365.

[24] P. Pansu, An isoperimetric inequality on the Heisenberg group, Conference on differential geometry on homogeneous spaces (Turin, 1983). Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 159–174.

[25] P. Pansu, Une inégalité isopérimétrique sur le groupe de Heisenberg, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 2, 127–130.

[26] E. Paolini & A. Tamagnini, Minimal clusters of four planar regions with the same area. ESAIM Control Optim. Calc. Var. **24** (2018), no. 3, 1303–1331.

[27] E. Paolini & A. Tamagnini, Minimal cluster computation for four planar regions with the same area, Geometric Flow **3** (2018), no. 1, 90–96.

[28] E. Paolini & V.M. Tortorelli, The quadruple planar bubble enclosing equal areas is symmetric, to appear on Calc. Var. PDE (2020).

[29] A. Pratelli & G. Saracco, On the isoperimetric problem with double density, Nonlinear Analysis **177**, Part B (2018), 733–752.

[30] A. Pratelli & G. Saracco, The $\varepsilon - \varepsilon^\beta$ property for an isoperimetric problem with double density, and the regularity of isoperimetric sets, to appear on Adv. Nonlinear Stud. (2020).

[31] A. Pratelli & V. Scattaglia, in preparation (2019).

[32] B.W. Reichardt, Proof of the double bubble conjecture in $\mathbb{R}^n$, J. Geom. Anal. **18** (2008), no. 1, 172–191.

[33] C. Rosales, A. Cañete, V. Bayle & F. Morgan, On the isoperimetric problem in Euclidean space with density, Calc. Var. Partial Differential Equations **31** (2008), no. 1, 27–46.

[34] V. N. Sudakov & B. S. Cirel'son, Extremal properties of half-spaces for spherically invariant measures, Zap. NauVolcn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 41:14–24, 165, 1974. Problems in the theory of probability distributions, II.

[35] I. Tamanini, Regularity results for almost-minimal oriented hypersurfaces in $\mathbb{R}^N$, Quaderni del Dipartimento di Matematica dell'Università del Salento (1984).

[36] J. Taylor, The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces, Ann. of Math. **103** (1976), no. 3, 489–539.

[37] A.I. Vol'pert, Spaces BV and quasilinear equations, Math. USSR Sb. **17** (1967), 225–267.

[38] W. Wichiramala, Proof of the planar triple bubble conjecture, J. Reine Angew. Math. **567** (2004), 1–49.

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