INTERACTION BETWEEN OSCILLATIONS AND SINGULAR PERTURBATIONS IN A ONE-DIMENSIONAL PHASE-FIELD MODEL

ANNIKA BACH, TERESA ESPOSITO, ROBERTA MARZIANI, AND CATERINA IDA ZEPPIERI

Abstract. We study the relative impact of fine-scale heterogeneities and singular perturbations in a one-dimensional phase-field model of Ambrosio-Tortorelli type. We show that the limit functional is always of Mumford-Shah type, with a surface term depending on the mutual converging rate of the oscillation and the perturbation parameter.

1. Introduction

In this note we study the asymptotic behaviour, via Γ-convergence, of one-dimensional integral functionals combining oscillations and singular perturbations occurring on two possibly different length-scales. The functionals we consider are of Ambrosio-Tortorelli type and, for $\varepsilon > 0$ and $u, v \in W^{1,2}(a,b)$, they are defined as

$$F_\varepsilon(u,v) = \int_a^b \left(v^2(u')^2 + \frac{(1-v)^2}{\varepsilon} + \varepsilon\varphi\Big(\frac{x}{\delta}\Big)(v')^2 \right) dx\,, \tag{1.1}$$

where $\varphi \in L^\infty(\mathbb{R})$ is a 1-periodic function. The scale-parameter $\delta = \delta(\varepsilon) > 0$ is infinitesimal as $\varepsilon \to 0$ and represents the characteristic length of some underlying heterogeneities. If

$$\alpha := \inf \varphi, \qquad \beta := \sup \varphi, \quad \text{with} \quad \alpha > 0$$

then, up to a multiplicative constant, F_ε is bounded both from below and from above by the Ambrosio-Tortorelli functional [1, 2]; that is, we have

$$\int_a^b v^2(u')^2\,dx + \int_a^b \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon\alpha(v')^2 \right) dx \le F_\varepsilon(u,v) \le \int_a^b v^2(u')^2\,dx + \int_a^b \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon\beta(v')^2 \right) dx.$$

Therefore, as in the Ambrosio-Tortorelli approximation, the parameter ε determines the length-scale of the diffuse approximation of the jump set of the limit variable. Indeed if $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(a,b) \times W^{1,2}(a,b)$ is a sequence along which F_ε is equi-bounded then, necessarily, $v_\varepsilon \to 1$ in $L^2(a,b)$, while the first term in (1.1) favours those configurations where v_ε is asymptotically negligible, in the regions where u'_ε blows-up. Then, as in the case of the Modica-Mortola functional [13, 14], v_ε makes a transition between 0 and 1 in a small layer of width proportional to ε. The cost of this transition is of order one and is bounded between the two constants $2\sqrt{\alpha}$ and $2\sqrt{\beta}$, the 2 appearing for symmetry reasons (cf. Remark 3.4). Moreover, the Γ-limit of F_ε (if it exists) shall satisfy

$$\int_a^b (u')^2\,dx + 2\sqrt{\alpha}\#S(u) \le \Gamma\text{-}\lim F_\varepsilon(u) \le \int_a^b (u')^2\,dx + 2\sqrt{\beta}\#S(u), \tag{1.2}$$

where $S(u)$ denotes the set of discontinuity points of u (and the limit variable v is omitted since it is equal to the constant function 1). The bounds in (1.2) then imply that the domain of the Γ-limit of F_ε is the space of piecewise-Sobolev functions $P\text{-}W^{1,2}(a,b)$. The latter coincides with the space of functions u which can be written as the sum of a Sobolev function $\tilde{u} \in W^{1,2}(a,b)$ and a piecewise-constant function $u^{\rm pc}$; thus $u' = \tilde{u}'$ and $S(u) = S(u^{\rm pc})$.

The main result of this note is Theorem 2.1 which establishes a Γ-convergence result for the functionals F_ε in every parameter regime; *i.e.,* for every $\ell \in [0,+\infty]$, where

$$\ell := \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)}.$$

Specifically, we show that the sequence (F_ε) Γ-converges, with respect to the $L^1(a,b)$-convergence, to a functional which is always of Mumford-Shah type; *i.e.,*

$$F^\ell(u) = \int_a^b (u')^2\,dx + \mathbf{m}^\ell \# S(u)\,, \quad u \in P\text{-}W^{1,2}(a,b), \tag{1.3}$$

with a (constant) surface energy-density $\mathbf{m}^\ell$ depending on the combined effect of the oscillations and the singular perturbation.

More precisely, we show that the lower bound in (1.2) is optimal when $\ell = 0$. That is, if $\varepsilon \ll \delta$, then $\mathbf{m}^0 = 2\sqrt{\alpha}$ and the Γ-limit of F_ε is given by the functional

$$F^0(u) = \int_a^b (u')^2\,dx + 2\sqrt{\alpha}\# S(u), \quad u \in P\text{-}W^{1,2}(a,b). \tag{1.4}$$

In this case a scale separation takes place. Indeed, formally, if in (1.1) we first take the Γ-limit in ε, and keep δ fixed, we obtain the inhomogeneous free-discontinuity functionals (see [11])

$$\int_a^b (u')^2\,dx + 2\sum_{(a,b)\cap S(u)} \sqrt{\varphi\Big(\frac{x}{\delta}\Big)}, \quad u \in P\text{-}W^{1,2}(a,b),$$

whose Γ-limit as $\delta \to 0$ is exactly given by (1.4) (see *e.g.*, [7, Section 9.3]).

For $\ell = +\infty$, which corresponds to the case $\delta \ll \varepsilon$, we also observe a scale separation. In fact being the oscillation parameter δ smaller than the approximation parameter ε, in this regime, the Γ-limit of F_ε is the same as that of the homogeneous functionals

$$\int_a^b v^2(u')^2\,dx + \int_a^b \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon\,\varphi_{\rm hom}(v')^2\right)dx,$$

where $\varphi_{\rm hom}$ is the harmonic mean of φ in $(0,1)$; *i.e.,*

$$\varphi_{\rm hom} := \left(\int_0^1 \frac{1}{\varphi(t)}\,dt\right)^{-1}.$$

Therefore, passing to the limit as $\varepsilon \to 0$ gives

$$F^\infty(u) = \int_a^b (u')^2\,dx + 2\sqrt{\varphi_{\rm hom}}\,\# S(u), \quad u \in P\text{-}W^{1,2}(a,b), \tag{1.5}$$

that is, $\mathbf{m}^\infty = 2\sqrt{\varphi_{\rm hom}}$. We notice that, in general, $\varphi_{\rm hom} \le \beta$.

Finally, in the case $\ell \in (0,+\infty)$ the parameters ε and δ, being of the same order, interact with one another producing a surface energy $\mathbf{m}^\ell$ which depends on their interplay according to the following formula

$$\mathbf{m}^\ell = \inf_{z\in[0,1)} \inf\left\{\int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x)(v')^2\big)\,dx \colon v \in W^{1,2}_{\rm loc}(\mathbb{R})\,,\ v(z/\ell) = 0\,,\ \lim_{t\to\pm\infty} v(t) = 1\right\}. \tag{1.6}$$

We notice that, in contrast to the typical optimal-profile problem for the Ambrosio-Tortorelli functional (cf. (3.6)) which determines both $\mathbf{m}^0$ and $\mathbf{m}^\infty$, the minimisation problem in (1.6) involves the (unscaled) Modica-Mortola term in F_ε on the whole real line, instead of $(0,+\infty)$. This is due to the presence of the inhomogeneity φ, which breaks the usual symmetry of the problem. Moreover, an additional optimisation on the parameter $z \in [0,1)$ is needed to determine the "starting point" of an optimal transition. This feature makes the present problem different from the corresponding one for the Modica-Mortola functional considered in [3, 4] (see also [7, Chapter 9]).

Eventually, we conclude the limit analysis of the functionals F_ε by proving that the surface energy density $\mathbf{m}^\ell$ is continuous with respect to the parameter ℓ; *i.e.,* we show that

$$\lim_{\ell\to 0^+}\mathbf{m}^\ell=\mathbf{m}^0 \quad\text{and}\quad \lim_{\ell\to+\infty}\mathbf{m}^\ell=\mathbf{m}^\infty.$$

We finally observe that the functionals in (1.1) have also a mechanical interpretation. Indeed they can be seen as a one-dimensional variational model for damage in heterogeneous materials, according to, *e.g.*, [10, 12, 15, 16]. We also notice that due to the presence of the two interacting scales ε and δ, a Γ-convergence analysis for the corresponding n-dimensional model makes it necessary to resort to a more abstract method of proof, as shown in [6]. This method relies, among other, on the Γ-convergence analysis for general (scale-dependent, non periodic) elliptic functionals recently developed in [5]. In particular, the result established in [5] shows that the Γ-limit of the n-dimensional counterpart of (1.1) is always of brittle type, this fact being a consequence of a volume-surface decoupling which takes place in the Γ-limit. On the other hand, the one-dimensional problem studied in this note can be solved directly, by hands, taking advantage of the simple form of the functionals F_ε and of the structure of the space of one-dimensional special functions of bounded variation, $SBV^2(a,b)$, which coincides with the space of piecewise-Sobolev functions $P\text{-}W^{1,2}(a,b)$. In particular, in the proof of the upper-bound inequality (in the three different scaling regimes), the structure of $P\text{-}W^{1,2}(a,b)$ allows us to treat the regular and singular part of the limit variable u separately, without resorting to the abstract decoupling result established in [5].

2. Setting of the problem and statement of the main result

In this section we define the phase-field functionals we are going to analyse and we state our main result.

Let $\varphi\in L^\infty(\mathbb{R})$ be a 1-periodic function and set

$$\alpha:=\inf\varphi,\quad \beta:=\sup\varphi; \tag{2.1}$$

we additionally assume that $\alpha>0$.

Let $\varepsilon>0$ and let $\delta_\varepsilon>$ be such that $\lim_{\varepsilon\to0}\delta_\varepsilon=0$. For $a,b\in\mathbb{R}$ with $a<b$ we consider the one-dimensional integral functionals $F_\varepsilon\colon L^1(a,b)\times L^1(a,b)\longrightarrow[0,+\infty]$ defined by

$$F_\varepsilon(u,v):=\begin{cases}\displaystyle\int_a^b\Big(v^2(u')^2+\frac{(1-v)^2}{\varepsilon}+\varepsilon\,\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v')^2\Big)\,dx & u,\,v\in W^{1,2}(a,b),\ 0\le v\le 1,\\ +\infty & \text{otherwise.}\end{cases} \tag{2.2}$$

We notice that thanks to (2.1) the functionals F_ε satisfy

$$AT_\varepsilon^\alpha(u,v)\le F_\varepsilon(u,v)\le AT_\varepsilon^\beta(u,v)\,, \tag{2.3}$$

where, for $\lambda>0$, AT_ε^λ is the one-dimensional Ambrosio-Tortorelli functional given by

$$AT_\varepsilon^\lambda(u,v):=\begin{cases}\displaystyle\int_a^b\Big(v^2(u')^2+\frac{(1-v)^2}{\varepsilon}+\varepsilon\lambda(v')^2\Big)\,dx & u,\,v\in W^{1,2}(a,b),\ 0\le v\le 1,\\ +\infty & \text{otherwise.}\end{cases} \tag{2.4}$$

For later use it is convenient to define the localised functionals

$$F_\varepsilon(u,v,I):=\begin{cases}\displaystyle\int_I\Big(v^2(u')^2+\frac{(1-v)^2}{\varepsilon}+\varepsilon\,\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v')^2\Big)\,dx & u,\,v\in W^{1,2}(a,b),\ 0\le v\le 1,\\ +\infty & \text{otherwise,}\end{cases} \tag{2.5}$$

where $I \subset (a,b)$ is any open interval. Analogously, we define a localised version of the Modica-Mortola term in F_ε by setting

$$G_\varepsilon(v,I) := \begin{cases} \displaystyle\int_I \Big(\frac{(1-v)^2}{\varepsilon} + \varepsilon\,\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v')^2\Big)\,dx & v \in W^{1,2}(a,b),\ 0 \le v \le 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

As for the Ambrosio-Tortorelli functional, the Γ-limit of F_ε will be defined on a space of discontinuous functions. Then, to describe the domain of the limit functional, we need to introduce the space $P\text{-}W^{1,2}(a,b)$. The latter denotes the space of *piecewise* $W^{1,2}(a,b)$-functions defined on the interval (a,b). That is, $u \in P\text{-}W^{1,2}(a,b)$ if and only if there exists a *finite* partition of (a,b), $a = t_0 < t_1 < \ldots < t_M = b$, such that $u \in W^{1,2}(t_i, t_{i+1})$, for every $i = 1, \ldots, M-1$. The discontinuity set of a function $u \in P\text{-}W^{1,2}$ is denoted by $S(u)$ and it coincides with the minimal of such sets of points.

Let $PC(a,b)$ denote the space of *piecewise constant* functions on (a,b); then it is easy to check that

$$P\text{-}W^{1,2}(a,b) = W^{1,2}(a,b) + PC(a,b), \tag{2.7}$$

that is, $u \in P\text{-}W^{1,2}(a,b)$ if and only if

$$u = \tilde{u} + u^{\mathrm{pc}}, \tag{2.8}$$

with $\tilde{u} \in W^{1,2}(a,b)$ and $u^{\mathrm{pc}} \in PC(a,b)$. We also notice that the sum in (2.7) is not a direct sum since the constant functions belong to $W^{1,2}(a,b) \cap PC(a,b)$, therefore the decomposition in (2.8) is uniquely determined up to an additive constant.

Thanks to (2.8), for $u \in P\text{-}W^{1,2}(a,b)$ we have

$$u' = \tilde{u}' \quad \text{and} \quad S(u) = S(u^{\mathrm{pc}}).$$

Set

$$\ell := \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta_\varepsilon} \in [0,+\infty].$$

The following Γ-convergence theorem is the main result of this paper.

Theorem 2.1. *The sequence of functionals (F_ε) defined in (2.2) $\Gamma(L^1 \times L^1)$-converges to the functional $F^\ell \colon L^1(a,b) \times L^1(a,b) \longrightarrow [0,+\infty]$ defined as*

$$F^\ell(u,v) := \begin{cases} \displaystyle\int_a^b (u')^2\,dx + \mathbf{m}^\ell \# S(u) & u \in P\text{-}W^{1,2}(a,b),\ v = 1 \text{ a.e. in } (a,b)\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{2.9}$$

Moreover, the constant $\mathbf{m}^\ell > 0$ is defined as follows:

(1) if $\ell = 0$ and φ is upper semicontinuous then

$$\mathbf{m}^0 := 2\sqrt{\alpha}; \tag{2.10}$$

(2) if $\ell \in (0,+\infty)$ then

$$\mathbf{m}^\ell := \inf_{z\in[0,1)} \mathbf{m}^\ell_z\,, \tag{2.11}$$

with

$$\mathbf{m}^\ell_z := \inf\Big\{ \int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x + z)(v')^2\big)\,dx \colon v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}),\ 0 \le v \le 1,\ v(0) = 0,\ v(\pm\infty) = 1 \Big\}, \tag{2.12}$$

where $v(\pm\infty) := \lim_{x\to\pm\infty} v(x)$;

(3) if $\ell = +\infty$ then

$$\mathbf{m}^\infty := 2\left(\int_0^1 \frac{1}{\varphi(t)}\,dt\right)^{-1/2}. \tag{2.13}$$

Eventually, the constant $\mathbf{m}^\ell$ satisfies

$$\lim_{\ell\to 0^+}\mathbf{m}^\ell = \mathbf{m}^0 \quad \textit{and} \quad \lim_{\ell\to+\infty}\mathbf{m}^\ell = \mathbf{m}^\infty\,, \tag{2.14}$$

provided φ is upper semicontinuous.

3. Preliminary results

In this section we state and prove some preliminary results which will be used in what follows. We start recalling the convergence result for the 1-dimensional Ambrosio-Tortorelli functionals defined in (2.4) (see, *e.g.*, [8, Theorem 3.15])

Theorem 3.1. *For any $\lambda > 0$ the functionals AT^λ_ε defined in (2.4) $\Gamma(L^1\times L^1)$-converge as $\varepsilon\to 0$ to the functional*

$$MS^\lambda(u,v) := \begin{cases} \displaystyle\int_a^b (u')^2\,dx + 2\sqrt{\lambda}\#S(u) & u\in P\text{-}W^{1,2}(a,b),\ v=1 \ \textit{a.e. in } (a,b)\,,\\ +\infty & \textit{otherwise}\,. \end{cases}$$

The next proposition establishes a compactness result for sequences with equi-bounded energy and a lower bound for the first term in F_ε, which is independent of the parameter regime.

Proposition 3.2. *Let F_ε be as in (2.2) and let $(u_\varepsilon, v_\varepsilon)\subset W^{1,2}(a,b)\times W^{1,2}(a,b)$ be such that*

$$u_\varepsilon \to u \ \textit{ in } \ L^1(a,b) \quad \textit{and} \quad \sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty.$$

Then, there holds

(1) $v_\varepsilon \to 1$ in $L^2(a,b)$, $u\in P\text{-}W^{1,2}(a,b)$ and

$$\liminf_{\varepsilon\to 0}\int_a^b v_\varepsilon^2 (u_\varepsilon')^2\,dx \geq \int_a^b (u')^2\,dx\,; \tag{3.1}$$

(2) If $S(u) = \{t_1,\dots,t_N\}$ and $I_1,\dots,I_N$ are pairwise disjoint open subintervals in (a,b) such that $t_i\in I_i$, for every $i=1,\dots,N$, then there exist $s^1_\varepsilon,\dots,s^N_\varepsilon$ with $(s^i_\varepsilon)\subset I_i$ for every $\varepsilon>0$, such that

$$s^i_\varepsilon \to t_i \quad \textit{and} \quad v_\varepsilon(s^i_\varepsilon)\to 0 \quad \textit{as } \varepsilon\to 0, \tag{3.2}$$

for every $i=1,\dots,N$.

Proof. Thanks to (2.3), the proof readily follows from the corresponding one for the Ambrosio-Tortorelli functional (see, *e.g.*, [8, Theorem 3.15]). □

Remark 3.3. Let $(u_\varepsilon, v_\varepsilon)$ be as in Proposition 3.2 and $I_1,\dots,I_N$, s^i_ε as in Proposition 3.2 (2). Since (1) implies that, up to subsequences, $v_\varepsilon\to 1$ a.e. in (a,b), we can find $r^i,\tilde r^i\in I_i$ with $r^i < s^i_\varepsilon < \tilde r^i$ such that

$$\lim_{\varepsilon\to 0} v_\varepsilon(r^i) = \lim_{\varepsilon\to 0} v_\varepsilon(\tilde r^i) = 1. \tag{3.3}$$

In particular, since v_ε is continuous, we can apply the Intermediate Value Theorem to deduce that, for any $\eta\in(0,1/2)$ fixed, there exist $\tilde s^i_\varepsilon, r^i_\varepsilon, \tilde r^i_\varepsilon \in I_i$ (depending also on η) with $r^i_\varepsilon < \tilde s^i_\varepsilon < \tilde r^i_\varepsilon$ such that

$$v_\varepsilon(\tilde s^i_\varepsilon) = \eta\,,\ v_\varepsilon(r^i_\varepsilon) = v_\varepsilon(\tilde r^i_\varepsilon) = 1-\eta \quad \text{and} \quad v_\varepsilon \leq 1-\eta \ \text{ in } \ [r^i_\varepsilon, \tilde r^i_\varepsilon]\,. \tag{3.4}$$

Set $M := \sup_\varepsilon F_\varepsilon(u_\varepsilon, v_\varepsilon)$; since by assumption $M < +\infty$, from (3.4) we infer

$$M \geq \int_{r^i_\varepsilon}^{\tilde r^i_\varepsilon} \frac{(1-v_\varepsilon)^2}{\varepsilon}\,dx \geq \frac{\eta^2}{\varepsilon}(\tilde r^i_\varepsilon - r^i_\varepsilon) \quad \text{and} \quad M \geq \alpha \int_{\tilde s^i_\varepsilon}^{\tilde r^i_\varepsilon} \varepsilon (v'_\varepsilon)^2\,dx \geq \frac{\varepsilon\,\alpha(1-2\eta)^2}{\tilde r^i_\varepsilon - \tilde s^i_\varepsilon}\,,$$

where the last estimate follows from Jensen's Inequality. Therefore, for every $\varepsilon > 0$ we get

$$\frac{\alpha(1-2\eta)^2}{M} \leq \frac{\tilde r^i_\varepsilon - \tilde s^i_\varepsilon}{\varepsilon} < \frac{\tilde r^i_\varepsilon - r^i_\varepsilon}{\varepsilon} \leq \frac{M}{\eta^2} \tag{3.5}$$

and similarly for $\frac{\tilde s^i_\varepsilon - r^i_\varepsilon}{\varepsilon}$.

3.1. **The optimal-profile problem.** In this subsection we study the minimisation problem defining the constant $\mathbf{m}^\ell$ in (2.11). The latter represents the minimal cost of a two-sided transition from the value 0 to the value 1, on the real line, in terms of the unscaled Modica-Mortola term in F_ε. We thus refer to the corresponding minimisation problem as the optimal-profile problem. The analysis of $\mathbf{m}^\ell$ will be useful both to prove the Γ-convergence result in the regime $\delta_\varepsilon \sim \varepsilon$ and to establish (2.14) in Theorem 2.1.

We start recalling some properties of the corresponding optimal-profile problem for the Ambrosio-Tortorelli functionals AT^λ_ε defined in (2.4).

Remark 3.4. Let $\lambda > 0$; arguing as in, *e.g.*, [7, Chapter 6] it is immediate to check that

$$\begin{aligned} \sqrt{\lambda} &= \min\left\{ \int_0^{+\infty} \big((1-v)^2 + \lambda\,(v')^2\big)\,dx \colon v \in W^{1,2}_{\rm loc}(0,+\infty),\ 0 \leq v \leq 1,\ v(0)=0,\ v(+\infty)=1 \right\} \\ &= \inf_{T>0} \min\left\{ \int_0^{T} \big((1-v)^2 + \lambda\,(v')^2\big)\,dx \colon v \in W^{1,2}(0,T),\ 0 \leq v \leq 1,\ v(0)=0,\ v(T)=1 \right\}. \end{aligned} \tag{3.6}$$

Let $\mathbf{m}^\ell_z$ be as in (2.12); from (3.6) using a reflection argument and choosing either $\lambda = \alpha$ or $\lambda = \beta$, in view of (2.1) we get

$$2\sqrt{\alpha} \leq \mathbf{m}^\ell_z \leq 2\sqrt{\beta}\,, \tag{3.7}$$

for every $\ell \in (0,+\infty)$ and every $z \in [0,1)$.

The following lemma shows that the cost of an optimal profile depends continuously on the value attained by the competitors at zero.

Lemma 3.5. *For $\ell \in (0,+\infty)$, $z \in [0,1)$, and $t \in [0,1)$ let*

$$\mathbf{m}^\ell_z(t) := \inf\left\{ \int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x + z)(v')^2\big)\,dx \colon v \in W^{1,2}_{\rm loc}(\mathbb{R}),\ 0 \leq v \leq 1,\ v(0)=t\,, v(\pm\infty)=1 \right\}, \tag{3.8}$$

and set

$$\mathbf{m}^\ell(t) := \inf_{z\in[0,1)} \mathbf{m}^\ell_z(t)\,, \tag{3.9}$$

so that in particular $\mathbf{m}^\ell(0) = \mathbf{m}^\ell$, with $\mathbf{m}^\ell$ as in (2.11). Then $\lim_{t\to 0}\mathbf{m}^\ell(t) = \mathbf{m}^\ell$.

Proof. Let $\ell \in (0,+\infty)$ be fixed; let $z \in [0,1)$ be arbitrary and let $v \in W^{1,2}_{\rm loc}(\mathbb{R})$ be admissible for the infimum problem defining $\mathbf{m}^\ell_z$ in (2.12); *i.e.,* in particular, $v(0) = 0$. For any $t \in [0,1)$ the function

$$v_t := \min\{v + t, 1\}$$

is admissible for the infimum problem defining $\mathbf{m}^\ell_z(t)$ and satisfies

$$\int_{\mathbb{R}} \big((1-v_t)^2 + \varphi(\ell x + z)(v'_t)^2\big)\,dx \leq \int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x + z)(v')^2\big)\,dx\,.$$

Passing to the infimum in v and z we obtain both

$$\mathbf{m}^\ell(t) \le \mathbf{m}^\ell \quad \text{for every } t \in [0,1) \quad \text{and} \quad \limsup_{t\to 0} \mathbf{m}^\ell(t) \le \mathbf{m}^\ell. \tag{3.10}$$

Thus, to conclude it remains to show that

$$\liminf_{t\to 0} \mathbf{m}^\ell(t) \ge \mathbf{m}^\ell. \tag{3.11}$$

To this end, we fix $\eta \in (0,1/2)$ and for any $t \in (0,1/4)$ we choose $z_{\eta,t} \in [0,1)$ and $v_{\eta,t} \in W^{1,2}_{\text{loc}}(\mathbb{R})$ with $0 \le v_{\eta,t} \le 1$, $v_{\eta,t}(0) = t$, $v_{\eta,t}(\pm\infty) = 1$ such that

$$\int_{\mathbb{R}} \big((1-v_{\eta,t})^2 + \varphi(\ell x + z_{\eta,t})(v'_{\eta,t})^2\big)\, dx \le \mathbf{m}^\ell(t) + \eta\,. \tag{3.12}$$

Since $v_{\eta,t}(\pm\infty) = 1$ and $v_{\varepsilon,t}$ is continuous, we can apply the Intermediate Value Theorem to find $T^1_{\eta,t}, T^2_{\eta,t}$ with $T^1_{\eta,t} < 0 < T^2_{\eta,t}$ such that

$$v_{\eta,t}(T^1_{\eta,t}) = v_{\eta_t}(T^2_{\eta,t}) = 1-\eta \quad \text{and} \quad v_{\eta,t} \le 1-\eta \ \text{ on } \ [T^1_{\eta,t}, T^2_{\eta,t}]\,. \tag{3.13}$$

Notice that the second condition in (3.13) together with (3.12) implies that

$$\mathbf{m}^\ell(t) + \eta \ge \int_{T^1_{\eta,t}}^{T^2_{\eta,t}} (1-v_{\eta,t})^2\, dx \ge \eta^2 (T^2_{\eta,t} - T^1_{\eta,t})\,.$$

Thus, combining (3.7) and (3.10) yields

$$(T^2_{\eta,t} - T^1_{\eta,t}) \le \frac{2\sqrt{\beta} + \eta}{\eta^2} \quad \text{uniformly in } t\,. \tag{3.14}$$

Next we define

$$w_{\eta,t}(x) := \begin{cases} 1 - (\eta+t)(x - T^1_{\eta,t} + 1) & \text{if } T^1_{\eta,t} - 1 \le x < T^1_{\eta,t}\,, \\ \max\{0, v_{\eta,t}(x) - t\} & \text{if } T^1_{\eta,t} \le x \le T^2_{\eta,t}\,, \\ 1 + (\eta+t)(x - T^2_{\eta,t} - 1) & \text{if } T^2_{\eta,t} \le x < T^2_{\eta,t} + 1\,, \\ 1 & \text{otherwise in } \mathbb{R}\,, \end{cases}$$

(see Figure 1).

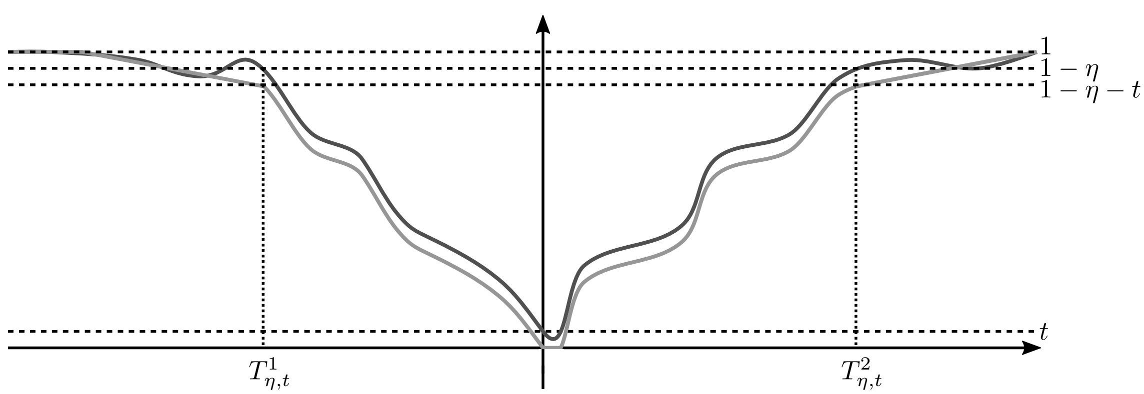

Figure 1. The function $v_{\eta,t}$ (in dark grey) and the modification $w_{\eta,t}$ (in light grey).

The first condition in (3.13) ensures that $w_{\eta,t} \in W^{1,2}_{\rm loc}(\mathbb{R})$. Moreover, we have $w_{\eta,t}(0) = v_{\eta,t}(0) - t = 0$ and $v_{\eta,t}(\pm\infty) = 1$. In particular, $w_{\eta,t}$ is admissible for $\mathbf{m}^\ell_z$ for any $z \in [0,1)$, so that

$$\mathbf{m}^\ell \leq \int_{\mathbb{R}} \big((1-w_{\eta,t})^2 + \varphi(\ell x + z_{\eta,t})(w'_{\eta,t})^2\big)\,dx\,. \tag{3.15}$$

Further, since the map $s \mapsto (1-s)^2$ is decreasing on $(-\infty,1)$ we get

$$\begin{aligned}\int_{T^1_{\eta,t}}^{T^2_{\eta,t}} &\big((1-w_{\eta,t})^2 + \varphi(\ell x + z_{\eta,t})(w'_{\eta,t})^2\big)\,dx \leq \int_{T^1_{\eta,t}}^{T^2_{\eta,t}} \big((1-v_{\eta,t}-t)^2 + \varphi(\ell x + z_{\eta,t})(v'_{\eta,t})^2\big)\,dx \\ &\leq (1+\eta)\int_{T^1_{\eta,t}}^{T^2_{\eta,t}} \big((1-v_{\eta,t})^2 + \varphi(\ell x + z_{\eta,t})(v'_{\eta,t})^2\big)\,dx + \Big(1+\frac{1}{\eta}\Big)t^2(T^2_{\eta,t} - T^1_{\eta,t})\,,\end{aligned} \tag{3.16}$$

where the second inequality follows by expanding the square $(1-v_{\eta,t}-t)^2$ and applying Young's Inequality to the term $2\sqrt{\eta}(1-v_{\eta,t})\frac{t}{\sqrt{\eta}}$. Eventually, by definition of $w_{\eta,t}$, from (2.1) we infer

$$\begin{aligned}&\int_{\mathbb{R}\setminus[T^1_{\eta,t},T^2_{\eta,t}]} \big((1-w_{\eta,t})^2 + \varphi(\ell x + z_{\eta,t})(w'_{\eta,t})^2\big)\,dx \\ &\leq (\eta+t)^2\bigg(\int_{T^1_{\eta,t}-1}^{T^1_{\eta,t}} (x - T^1_{\eta,t}+1)^2\,dx + \int_{T^2_{\eta,t}}^{T^2_{\eta,t}+1} (x - T^2_{\eta,t}-1)^2\,dx + 2\beta\bigg) = 2(\eta+t)^2\Big(\frac{1}{3}+\beta\Big)\,.\end{aligned} \tag{3.17}$$

Thus, inserting (3.14) in (3.16) and combining (3.12) with (3.15)–(3.17) we deduce that

$$\mathbf{m}^\ell \leq (1+\eta)\big(\mathbf{m}^\ell(t)+\eta\big) + \Big(1+\frac{1}{\eta}\Big)t^2\frac{2\sqrt{\beta}+\eta}{\eta^2} + 2(\eta+t)^2\Big(\frac{1}{3}+\beta\Big)\,.$$

Passing in the above inequality first to the liminf in t and then to the limit as $\eta \to 0$ we finally obtain (3.11). □

Remark 3.6. We observe that for $\ell \in (0,+\infty)$, in general the strict inequality $\mathbf{m}^0 < \mathbf{m}^\ell$ holds. To prove it, assume φ is continuous with $0 < \alpha = \min\varphi < \max\varphi = \beta$. Then, the direct methods and a truncation argument provide us with a pair $(\bar z, \bar v) \in [0,1) \times W^{1,2}_{\rm loc}(\mathbb{R})$ with $0 \leq \bar v \leq 1$, $\bar v(0) = 0$, $\bar v(\pm\infty) = 1$, such that

$$\mathbf{m}^\ell = \int_{\mathbb{R}} (1-\bar v)^2 + \varphi(\ell x + \bar z)(\bar v')^2\,dx\,. \tag{3.18}$$

Therefore the Young Inequality yields

$$\begin{aligned}\mathbf{m}^\ell &= \int_{\mathbb{R}} (1-\bar v)^2 + \varphi(\ell x + \bar z)(\bar v')^2\,dx \\ &= \int_{\mathbb{R}} \big((1-\bar v)^2 + \alpha(\bar v')^2\big)\,dx + \int_{\mathbb{R}} (\varphi(\ell x+\bar z)-\alpha)(\bar v')^2\,dx \\ &\geq 2\sqrt{\alpha} + \int_{\mathbb{R}} (\varphi(\ell x+\bar z)-\alpha)(\bar v')^2\,dx\,,\end{aligned} \tag{3.19}$$

with equality if and only if $\bar v$ satisfies $\alpha\,\bar v' = 1-\bar v$; *i.e.*, $\bar v = 1-\exp(-|x|/\sqrt{\alpha})$. If this is the case, then $|\bar v'(x)| > 0$ for every $x \in \mathbb{R}\setminus\{0\}$, which will imply that the second term on the right-hand side of (3.19) is strictly positive by the assumptions on φ. Thus, the claim follows.

Remark 3.7. For later reference it is useful to observe that for every $\ell \in (0,+\infty)$ and $z \in [0,1)$ the constant $\mathbf{m}^\ell_z$ in (2.12) can be equivalently expressed in terms of a minimisation problem where the test functions are suitably shifted, instead of the integrand. Indeed, consider the shifted function

$v_z := v(\cdot - \frac{z}{\ell})$; if $v \in W^{1,2}_{\rm loc}(\mathbb{R})$ then v_t belongs to $W^{1,2}_{\rm loc}(\mathbb{R})$, moreover $v(0) = 0$, $v(\pm\infty) = 1$ if and only if $v_z(\frac{z}{\ell}) = 0$, $v_z(\pm\infty) = 1$. Therefore, since the change of variables $y = x + \frac{z}{\ell}$ gives

$$\int_{\mathbb{R}} \Big((1-v)^2 + \varphi(\ell x + z)(v')^2\Big)\,dx = \int_{\mathbb{R}} \Big((1-v_z)^2 + \varphi(\ell y)(v_z')^2\Big)\,dy\,,$$

passing to the infimum we get

$$\mathbf{m}^\ell_z = \inf\left\{ \int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x)(v')^2\big)\,dx\colon v \in W^{1,2}_{\rm loc}(\mathbb{R}),\ 0 \le v \le 1,\ v(\tfrac{z}{\ell}) = 0\,,\ v(\pm\infty) = 1 \right\}. \tag{3.20}$$

Finally, in the next proposition we prove an alternative formula for the surface density of the Γ-limit in the regime $\delta_\varepsilon \sim \varepsilon$ (cf. [5, Theorem 8.4]).

Proposition 3.8. *Let $\ell \in (0, +\infty)$ and set*

$$\widetilde{\mathbf{m}}^\ell := \inf\Big\{ \int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x)(v')^2\big)\,dx\colon v \in W^{1,2}_{\rm loc}(\mathbb{R})\,,\ 0 \le v \le 1\,,\ v(\pm\infty) = 1\,,$$
$$\exists\, u \in W^{1,2}_{\rm loc}(\mathbb{R}) \text{ with } u(-\infty) = 0,\ u(+\infty) = 1 \text{ and } v\,u' = 0 \text{ a.e. in } \mathbb{R}\Big\}.$$

Then $\widetilde{\mathbf{m}}^\ell = \mathbf{m}^\ell$, where $\mathbf{m}^\ell$ is as in (2.11).

Proof. We first prove that $\mathbf{m}^\ell \ge \widetilde{\mathbf{m}}^\ell$.

To this end, let $\ell \in (0, +\infty)$ be fixed and $\eta \in (0, 1/2)$ be arbitrary; using the expression of $\mathbf{m}^\ell_z$ in (3.20) we choose $z_\eta \in [0, 1)$ and $v_\eta \in W^{1,2}_{\rm loc}(\mathbb{R})$ such that $v_\eta(\frac{z_\eta}{\ell}) = 0$, $v_\eta(\pm\infty) = 1$ and

$$\int_{\mathbb{R}} \big((1-v_\eta)^2 + \varphi(\ell x)(v_\eta')^2\big)\,dx \le \mathbf{m}^\ell + \eta\,. \tag{3.21}$$

Similarly as in Lemma 3.5 we can find $T^1_\eta, T^2_\eta, S^1_\eta, S^2_\eta$ with $T^1_\eta < S^1_\eta < \frac{z_\eta}{\ell} < S^2_\eta < T^2_\eta$ satisfying the following conditions:

$$v_\eta(T^1_\eta) = v_\eta(T^2_\eta) = 1 - \eta \quad \text{and} \quad v_\eta \le 1 - \eta \ \text{ on } [T^1_\eta, T^2_\eta] \tag{3.22}$$

$$v_\eta(S^1_\eta) = v_\eta(S^2_\eta) = \eta^2 \quad \text{and} \quad v_\eta \le \eta^2 \ \text{ on } [S^1_\eta, S^2_\eta]\,. \tag{3.23}$$

We then define a pair $(u_\eta, v_\eta) \in W^{1,2}_{\rm loc}(\mathbb{R}) \times W^{1,2}_{\rm loc}(\mathbb{R})$ with $(u_\eta, v_\eta)(-\infty) = (0, 1)$ and $(u_\eta, v_\eta)(+\infty) = (1, 1)$ by setting

$$u_\eta(x) := \begin{cases} 0 & \text{if } x < S^1_\eta\,, \\[2mm] \dfrac{x - S^1_\eta}{S^2_\eta - S^1_\eta} & \text{if } S^1_\eta \le x \le S^2_\eta\,, \\[2mm] 1 & \text{if } x > S^2_\eta\,, \end{cases}$$

$$w_\eta(x) := \begin{cases} 1 - (\eta + \eta^2)(x - T^1_\eta + 1) & \text{if } T^1_\eta - 1 \le x < T^1_\eta\,, \\[2mm] \max\{0, v_\eta(x) - \eta^2\} & \text{if } T^1_\eta \le x \le T^2_\eta\,, \\[2mm] 1 + (\eta + \eta^2)(x - T^2_\eta - 1) & \text{if } T^2_\eta < x \le T^2_\eta + 1\,, \\[2mm] 1 & \text{otherwise in } \mathbb{R}\,. \end{cases}$$

Clearly, $u_\eta \in W^{1,2}_{\rm loc}(\mathbb{R})$, while (3.22) ensures that also $w_\eta \in W^{1,2}_{\rm loc}(\mathbb{R})$. Moreover, the second condition in (3.23) implies that $w_\eta \equiv 0$ on $[S^1_\eta, S^2_\eta]$, hence $w_\eta\, u_\eta' = 0$ a.e. in $\mathbb{R}$. In particular, w_η is admissible for $\widetilde{\mathbf{m}}^\ell$. Then it only remains to estimate its energy. This can be done arguing in a similar way as in Lemma 3.5. Namely, by repeating the computation in (3.16)–(3.17) now replacing t with η^2 leads to

$$\begin{aligned}\widetilde{\mathbf{m}}^\ell \leq \int_{\mathbb{R}} \big((1-w_\eta)^2 + \varphi(\ell x)(w_\eta')^2\big)\,dx &\leq (1+\eta)\int_{\mathbb{R}} \big((1-v_\eta)^2 + \varphi(\ell x)(v_\eta')^2\big)\,dx \\ &\quad + \Big(1+\frac{1}{\eta}\Big)\eta^4(T_\eta^2 - T_\eta^1) + 2(\eta+\eta^2)^2\Big(\frac{1}{3}+\beta\Big)\,.\end{aligned} \tag{3.24}$$

Moreover, as in (3.14), we deduce from (3.21) and (3.22) that $T_\eta^2 - T_\eta^1 \leq \frac{2\sqrt{\beta}+\eta}{\eta^2}$. Inserting the latter in (3.24) and appealing to (3.21) yields

$$\widetilde{\mathbf{m}}^\ell \leq (1+\eta)(\mathbf{m}^\ell + \eta) + 2(\eta+\eta^2)\Big(\sqrt{\beta} + \eta + \frac{1}{3} + \beta\Big)\,,$$

hence the desired inequality follows by the arbitrariness of $\eta > 0$.

We now show that $\mathbf{m}^\ell \leq \widetilde{\mathbf{m}}^\ell$.

Let v be admissible for $\widetilde{\mathbf{m}}^\ell$; then there exist $u \in W^{1,2}_{\rm loc}(\mathbb{R})$ with $u(-\infty) = 0$, $u(+\infty) = 1$, and $v\,u' = 0$ a.e. in $\mathbb{R}$. Since $u \in W^{1,2}_{\rm loc}(\mathbb{R})$, the boundary conditions at $\pm\infty$ imply that u' cannot be equal to zero a.e. in $\mathbb{R}$. Since at the same time $v\,u' = 0$ a.e. in $\mathbb{R}$, we can find $\bar z \in \mathbb{R}$ with $v(\bar z) = 0$. Set $z := \ell\bar z - \lfloor \ell \bar z\rfloor \in [0,1)$ and $v_z := v(\cdot + (\bar z - \frac{z}{\ell}))$. Then $v_z(\frac{z}{\ell}) = 0$ and $v(\pm\infty) = 1$, while the 1-periodicity of φ together with the fact that $\ell\bar z - z = \lfloor \ell\bar z\rfloor \in \mathbb{Z}$ implies that

$$\begin{aligned}\int_{\mathbb{R}} \big((1-v)^2 + \varphi(\ell x)(v')^2\big)\,dx &= \int_{\mathbb{R}} \Big((1-v_z)^2 + \varphi\Big(\ell\Big(x + \bar z - \frac{z}{\ell}\Big)\Big)(v_z')^2\Big)\,dx \\ &= \int_{\mathbb{R}} \big((1-v_z)^2 + \varphi(\ell x)(v_z')^2\big)\,dx\,.\end{aligned}$$

Thus we conclude by passing to the infimum in v. $\square$

Remark 3.9. The proof of Proposition 3.8 actually shows that

$$\begin{aligned}\mathbf{m}^\ell = \inf_{T>0}\inf\Big\{ &\int_{-T}^{T} \big((1-v)^2 + \varphi(\ell x)(v')^2\big)\,dx \colon v \in W^{1,2}(-T,T)\,,\ 0 \leq v \leq 1\,,\ v(\pm T) = 1\,, \\ &\exists\, u \in W^{1,2}(-T,T)\ \text{ with }\ u(-T) = 0,\, u(T) = 1,\ \text{ and }\ v\,u' = 0 \text{ a.e. in } (-T,T)\Big\}\,.\end{aligned}$$

4. Oscillations on a larger scale than the singular perturbation

In this section we analyse the case when the oscillation parameter δ_ε is much larger than the singular-perturbation parameter ε; *i.e.,* the case $\ell = 0$.

Throughout this section the function φ is additionally assumed to be upper semicontinuous.

Proposition 4.1. *Let $\ell = 0$ and assume that φ is upper semicontinuous; then the sequence (F_ε) defined in* (2.2) *Γ-converges to the functional $F^0 \colon L^1(a,b) \times L^1(a,b) \longrightarrow [0,+\infty]$ defined as*

$$F^0(u,v) := \begin{cases} \displaystyle\int_a^b (u')^2\,dx + \mathbf{m}^0 \# S(u) & u \in P\text{-}W^{1,2}(a,b)\,,\ v = 1 \textit{ a.e. in } (a,b)\,, \\ +\infty & \textit{otherwise}\,, \end{cases} \tag{4.1}$$

where $\mathbf{m}^0 := 2\sqrt{\alpha}$.

Proof. Thanks to (2.3), from Theorem 3.1 we immediately deduce that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,v) \geq \Gamma\text{-}\liminf_{\varepsilon\to 0} AT_\varepsilon^\alpha(u,v) = \int_a^b (u')^2\,dx + 2\sqrt{\alpha}\#S(u)\,,$$

which by definition of $\mathbf{m}^0$ gives the lower-bound inequality. It thus remains to establish the upper-bound inequality.

Let $u \in P\text{-}W^{1,2}(a,b)$; we construct a sequence $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(a,b) \times W^{1,2}(a,b)$ such that $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(a,b) \times L^1(a,b)$ and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \le \mathbf{m}^0 \# S(u)\,.$$

Since the construction of the recovery sequence $(u_\varepsilon, v_\varepsilon)$ will be performed locally, close to a discontinuity point of u, we can assume without loss of generality that $S(u) = \{t_0\}$, with $t_0 \in (a,b)$.

Let now $\tilde u \in W^{1,2}(a,b)$ and $u^{\rm pc} \in PC(a,b)$ be as in (2.8); without loss of generality, we choose $u^{\rm pc} = s\chi_{(a,t_0)}$, with $s \in \mathbb{R}$.

For $\eta > 0$ let $y_\eta \in (0,1)$ satisfy

$$\varphi(y_\eta) \le \alpha + \eta\,. \tag{4.2}$$

Applying (3.6) with $\lambda = \alpha$ we find $T_\eta > 0$ and $v_\eta \in W^{1,2}(0,T_\eta)$ such that $0 \le v_\eta \le 1$, $v_\eta(0) = 0$, $v_\eta(T_\eta) = 1$, and

$$\int_0^{T_\eta} \big((1-v_\eta)^2 + \alpha\,(v_\eta')^2\big)\,dx \le \sqrt{\alpha} + \eta\,. \tag{4.3}$$

Finally, set

$$t_0^\varepsilon := \left\lfloor \frac{t_0}{\delta_\varepsilon} \right\rfloor \delta_\varepsilon\,, \quad y_\eta^\varepsilon := \delta_\varepsilon y_\eta\,, \tag{4.4}$$

and let $\xi_\varepsilon > 0$ be such that $\xi_\varepsilon \ll \varepsilon$. Then, a recovery sequence for $F^0(u,1)$ is defined as $(u_\varepsilon, v_\varepsilon) = (\tilde u + \bar u_\varepsilon, v_\varepsilon)$ with $(\bar u_\varepsilon, v_\varepsilon) \subset W^{1,2}(a,b) \times W^{1,2}(a,b)$ given by

$$\bar u_\varepsilon(x) := \begin{cases} 0 & \text{if } x \le t_0^\varepsilon + y_\eta^\varepsilon + \dfrac{\xi_\varepsilon}{2}\,, \\[2ex] \dfrac{2s}{\xi_\varepsilon}\Big(x - \Big(t_0^\varepsilon + y_\eta^\varepsilon + \dfrac{\xi_\varepsilon}{2}\Big)\Big) & \text{if } t_0^\varepsilon + y_\eta^\varepsilon + \dfrac{\xi_\varepsilon}{2} < x < t_0^\varepsilon + y_\eta^\varepsilon + \xi_\varepsilon\,, \\[2ex] s & \text{if } x \ge t_0^\varepsilon + y_\eta^\varepsilon + \xi_\varepsilon\,. \end{cases}$$

and

$$v_\varepsilon(x) := \begin{cases} 0 & \text{if } |x - t_0^\varepsilon - y_\eta^\varepsilon| \le \xi_\varepsilon\,, \\[2ex] v_\eta\Big(\dfrac{|x - t_0^\varepsilon - y_\eta^\varepsilon| - \xi_\varepsilon}{\varepsilon}\Big) & \text{if } \xi_\varepsilon < |x - t_0^\varepsilon - y_\eta^\varepsilon| \le \xi_\varepsilon + \varepsilon T_\eta\,, \\[2ex] 1 & \text{if } |x - t_0^\varepsilon - y_\eta^\varepsilon| > \xi_\varepsilon + \varepsilon T_\eta\,, \end{cases}$$

(see Figure 2).

We notice that since $t_0^\varepsilon + y_\eta^\varepsilon \to t_0$, then by construction $u_\varepsilon := \tilde u + \bar u_\varepsilon \to u$ in $L^1(a,b)$, further, $v_\varepsilon \to 1$ in $L^1(a,b)$ and a.e. in (a,b). Therefore it remains to show that

$$\limsup_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, (a,b)) \le \int_a^b (u')^2\,dx + \mathbf{m}^0\,. \tag{4.5}$$

We start noticing that by construction

$$v_\varepsilon \bar u_\varepsilon' = 0 \ \text{ a.e. in } (a,b)$$

and therefore

$$\lim_{\varepsilon\to 0} \int_a^b v_\varepsilon^2 (u_\varepsilon')^2\,dx = \lim_{\varepsilon\to 0} \int_a^b v_\varepsilon^2 (\tilde u')^2\,dx = \int_a^b (\tilde u')^2\,dx\,, \tag{4.6}$$

the last equality following by the Dominated Convergence Theorem, since $\tilde u' \in L^2(a,b)$ and $0 \le v_\varepsilon \le 1$.

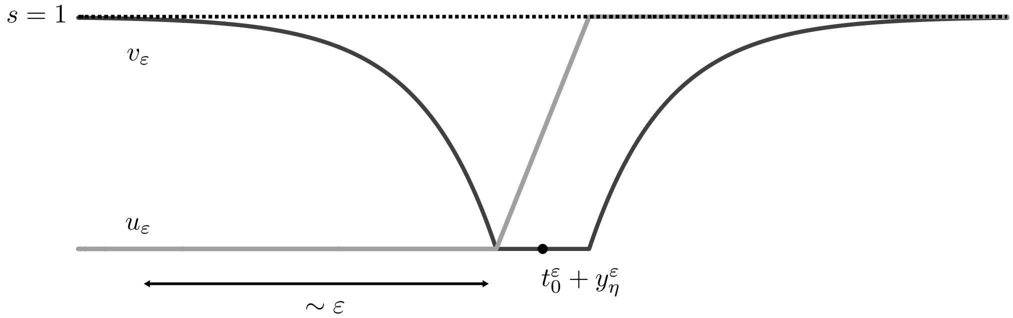

Figure 2. Recovery sequence in the case $s = 1$

Moreover, by definition of v_ε we also have

$$G_\varepsilon(v_\varepsilon,(a,b)) \leq 2\int_{t_0^\varepsilon+y_\eta^\varepsilon+\xi_\varepsilon}^{t_0^\varepsilon+y_\eta^\varepsilon+\xi_\varepsilon+\varepsilon T_\eta}\Big(\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\Big)\,dx+\frac{2\xi_\varepsilon}{\varepsilon}\,. \tag{4.7}$$

Since $\xi_\varepsilon \ll \varepsilon$, in (4.7) it only remains to estimate the integral on the right-hand side.

By a change of variables, recalling (4.4), and using the periodicity of φ we readily obtain

$$\int_{t_0^\varepsilon+y_\eta^\varepsilon+\xi_\varepsilon}^{t_0^\varepsilon+y_\eta^\varepsilon+\xi_\varepsilon+\varepsilon T_\eta}\Big(\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\Big)\,dx=\int_0^{T_\eta}\Big((1-v_\eta(x))^2+\varphi\Big(\frac{\varepsilon}{\delta_\varepsilon}x+y_\eta+\frac{\xi_\varepsilon}{\delta_\varepsilon}\Big)(v_\eta'(x))^2\Big)\,dx\,. \tag{4.8}$$

Since φ is upper semicontinuous and $\xi_\varepsilon \ll \varepsilon \ll \delta_\varepsilon$, applying the reverse Fatou Lemma we infer

$$\limsup_{\varepsilon\to 0}\int_0^{T_\eta}\varphi\Big(\frac{\varepsilon}{\delta_\varepsilon}x+y_\eta+\frac{\xi_\varepsilon}{\delta_\varepsilon}\Big)(v_\eta'(x))^2\,dx\leq\int_0^{T_\eta}\varphi(y_\eta)(v_\eta'(x))^2\,dx\,. \tag{4.9}$$

Therefore, gathering (4.7), (4.8), (4.9), and recalling the definition of y_η and v_η we get

$$\begin{aligned}\limsup_{\varepsilon\to 0}G_\varepsilon(v_\varepsilon,(a,b)) &\leq 2\int_0^{T_\eta}\big((1-v_\eta)^2+\varphi(y_\eta)(v_\eta')^2\big)\,dx\\ &\leq 2\int_0^{T_\eta}\big((1-v_\eta)^2+(\alpha+\eta)(v_\eta')^2\big)\,dx\\ &\leq 2\Big(1+\frac{\eta}{\alpha}\Big)(\sqrt{\alpha}+\eta)=\Big(1+\frac{\eta}{\alpha}\Big)(\mathbf{m}^0+2\eta)\,.\end{aligned} \tag{4.10}$$

Eventually, (4.5) follows by combining (4.6), (4.10) and letting $\eta \to 0$. □

Remark 4.2. We observe that in the proof of Proposition 4.1 the upper semicontinuity of φ is only needed to obtain the upper-bound inequality.

5. Oscillations on the same scale as the singular perturbation

In this section we analyse the case when the oscillation parameter δ_ε and the singular-perturbation parameter ε are of the same order; *i.e.,* the case $\ell \in (0,+\infty)$.

On account of Lemma 3.5 and Proposition 3.8 we prove the following result.

Proposition 5.1. *Let $\ell \in (0,+\infty)$; then the sequence (F_ε) defined in* (2.2) *Γ-converges to the functional $F^\ell \colon L^1(a,b)\times L^1(a,b) \longrightarrow [0,+\infty]$ defined as*

$$F^\ell(u,v) := \begin{cases} \displaystyle\int_a^b (u')^2\,dx + \mathbf{m}^\ell \# S(u) & u \in P\text{-}W^{1,2}(a,b)\,,\ v=1 \ \textit{a.e. in } (a,b)\,,\\ +\infty & \textit{otherwise,}\end{cases}$$

where $\mathbf{m}^\ell$ is as in (2.11).

Proof. We prove separately the lower-bound and the upper-bound inequalities.

Step 1: Lower-bound inequality.

Let $(u,v) \in L^1(a,b)\times L^1(a,b)$ be arbitrary and let $(u_\varepsilon,v_\varepsilon) \subset W^{1,2}(a,b)\times W^{1,2}(a,b)$ be such that

$$(u_\varepsilon,v_\varepsilon) \to (u,v) \ \text{ in } \ L^1(a,b)\times L^1(a,b) \quad \text{and} \quad \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,v_\varepsilon) < +\infty\,.$$

Then, up to subsequences (not relabelled) we can additionally assume that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon,v_\varepsilon) < +\infty$, therefore Proposition 3.2 immediately yields that $u \in P\text{-}W^{1,2}(a,b)$, $v=1$ a.e. in (a,b) and

$$\liminf_{\varepsilon\to 0}\int_a^b v_\varepsilon^2 (u_\varepsilon')^2\,dx \ge \int_a^b (u')^2\,dx\,. \tag{5.1}$$

Therefore, to prove the liminf inequality it suffices to show that

$$\liminf_{\varepsilon\to 0}\int_a^b \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon\,\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\right) dx \ge \mathbf{m}^\ell \# S(u)\,,$$

with $\mathbf{m}^\ell$ as in (2.11).

To this end we notice that if $S(u)=\emptyset$ then there is nothing to prove. Hence, we may assume that $S(u)=\{t_1,\dots,t_N\}$, with $N\ge 1$. Now, let $I_1,\dots,I_N$ be pairwise disjoint open intervals with $I_i \subset (a,b)$ and $t_i \in I_i$, for every $i=1,\dots,N$. We claim that

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon,I_i) \ge \mathbf{m}^\ell\,, \tag{5.2}$$

for every $i=1,\dots,N$, where G_ε is as in (2.6).

To prove the claim, we let $i\in\{1,\dots,N\}$ be arbitrary, and we invoke Proposition 3.2 (2) and Remark 3.3 to find $s^i_\varepsilon, r^i, \tilde r^i \in I_i$ with $r^i < s^i_\varepsilon < \tilde r^i$ satisfying

$$\lim_{\varepsilon\to 0} v_\varepsilon(s^i_\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} v_\varepsilon(r^i) = \lim_{\varepsilon\to 0} v_\varepsilon(\tilde r^i) = 1\,.$$

Set $z^i_\varepsilon := \frac{s^i_\varepsilon}{\delta_\varepsilon} - \Big\lfloor \frac{s^i_\varepsilon}{\delta_\varepsilon}\Big\rfloor \in [0,1)$; thanks to the 1-periodicity of φ, the change of variables $y = \frac{x-s^i_\varepsilon}{\ell\delta_\varepsilon}$ yields

$$\begin{aligned} G_\varepsilon(v_\varepsilon,I_i) &\ge \int_{r^i}^{\tilde r^i} \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\right) dx = \int_{\frac{r^i-s^i_\varepsilon}{\ell\delta_\varepsilon}}^{\frac{\tilde r^i-s^i_\varepsilon}{\ell\delta_\varepsilon}} \left(\frac{\ell\delta_\varepsilon}{\varepsilon}(1-w_\varepsilon)^2 + \frac{\varepsilon}{\ell\delta_\varepsilon}\varphi(\ell y + z^i_\varepsilon)(w_\varepsilon')^2\right) dy \\ &\ge \gamma_\varepsilon \int_{\frac{r^i-s^i_\varepsilon}{\ell\delta_\varepsilon}}^{\frac{\tilde r^i-s^i_\varepsilon}{\ell\delta_\varepsilon}} \left((1-w_\varepsilon)^2 + \varphi(\ell y + z^i_\varepsilon)(w_\varepsilon')^2\right) dy\,, \end{aligned} \tag{5.3}$$

where $w_\varepsilon(y) = v_\varepsilon(\ell\delta_\varepsilon y + s^i_\varepsilon)$ and

$$\gamma_\varepsilon := \min\left\{\frac{\ell\delta_\varepsilon}{\varepsilon}, \frac{\varepsilon}{\ell\delta_\varepsilon}\right\} \to 1 \quad \text{as} \ \ \varepsilon\to 0\,. \tag{5.4}$$

Since $v_\varepsilon(r^i), v_\varepsilon(\tilde r^i) \to 1$, using a linear interpolation as in the proof of Lemma 3.5 we can extend w_ε to a function $w_\varepsilon^i \in W^{1,2}_{\rm loc}(\mathbb{R})$ with $0 \le w_\varepsilon^i \le 1$ satisfying $w_\varepsilon^i(0) = v_\varepsilon(s_\varepsilon^i)$, $w_\varepsilon(\pm\infty) = 1$ and such that

$$\int_{\frac{r^i - s_\varepsilon^i}{\ell\delta_\varepsilon}}^{\frac{\tilde r^i - s_\varepsilon^i}{\ell\delta_\varepsilon}} \big((1-w_\varepsilon)^2 + \varphi(\ell y + z_\varepsilon^i)(w_\varepsilon')^2\big)\,dy = \int_{\mathbb{R}} \Big((1-w_\varepsilon^i)^2 + \varphi(\ell y + z_\varepsilon^i)\big((w_\varepsilon^i)'\big)^2\Big)\,dy + o_\varepsilon(1)\,, \tag{5.5}$$

as $\varepsilon \to 0$. Thus, since w_ε^i is admissible for $\mathbf{m}^\ell_{z_\varepsilon^i}(v_\varepsilon(s_\varepsilon^i)) \ge \mathbf{m}^\ell(v_\varepsilon(s_\varepsilon^i))$ and $v_\varepsilon(s_\varepsilon^i) \to 0$, gathering (5.3)–(5.5), passing to the liminf in ε and applying Lemma 3.5 yields

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, I_i) \ge \lim_{\varepsilon\to 0}\gamma_\varepsilon\,\liminf_{\varepsilon\to 0}\mathbf{m}^\ell(v_\varepsilon(s_\varepsilon^i)) = \mathbf{m}^\ell\,,$$

hence (5.2). Eventually, summing over i we get

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon,(a,b)) \ge \sum_{i=1}^N \liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, I_i) \ge \mathbf{m}^\ell N = \mathbf{m}^\ell \# S(u)\,,$$

which together with (5.1) gives the lower-bound inequality.

Step 2: Upper-bound inequality

As in the proof of Proposition 4.1 it suffices to construct a recovery sequence for $u = \tilde u + u^{\rm pc}$ with $\tilde u \in W^{1,2}(a,b)$ and $u^{\rm pc} = s\chi_{(a,t_0)}$, with $s\in\mathbb{R}$ and $t_0 \in (a,b)$. To this end, we fix $\eta > 0$ and according to Proposition 3.8 and Remark 3.9 we choose $T_\eta > 0$ and $(u_\eta, v_\eta) \in W^{1,2}(-T_\eta, T_\eta)\times W^{1,2}(-T_\eta,T_\eta)$ with $0 \le v_\eta \le 1$ satisfying $(u_\eta,v_\eta)(-T_\eta) = (0,1)$, $(u_\eta,v_\eta)(T_\eta) = (1,1)$, and $v_\eta\, u_\eta' = 0$ a.e. in $(-T_\eta, T_\eta)$ and

$$\int_{-T_\eta}^{T_\eta} \big((1-v_\eta)^2 + \varphi(\ell x)(v_\eta')^2\big)\,dx \le \mathbf{m}^\ell + \eta\,. \tag{5.6}$$

We extend (u_η, v_η) to $\mathbb{R}$ by setting $(u_\eta,v_\eta) := (\chi_{(0,+\infty)}, 1)$ in $\mathbb{R}\setminus(-T_\eta,T_\eta)$. Moreover, we set $t_0^\varepsilon := \lfloor \frac{t_0}{\delta_\varepsilon}\rfloor\delta_\varepsilon$ and define the pairs $(u_\varepsilon, v_\varepsilon) := (s\bar u_\varepsilon + \tilde u, v_\varepsilon)$ with $(\bar u_\varepsilon, v_\varepsilon)$ given by

$$\bar u_\varepsilon(x) := u_\eta\Big(\frac{x - t_0^\varepsilon}{\ell\delta_\varepsilon}\Big) \quad\text{and}\quad v_\varepsilon(x) := v_\eta\Big(\frac{x-t_0^\varepsilon}{\ell\delta_\varepsilon}\Big)\,.$$

By construction $u_\varepsilon \to u^{\rm pc} + \tilde u = u$ in $L^1(a,b)$, while $v_\varepsilon \to 1$ in $L^1(a,b)$ and a.e. in (a,b). It thus remains to estimate $F_\varepsilon(u_\varepsilon, v_\varepsilon)$. Since $v_\varepsilon u_\varepsilon' = 0$ a.e. in (a,b), as in (4.6) we deduce that

$$\lim_{\varepsilon\to 0}\int_a^b v_\varepsilon^2 (u_\varepsilon')^2\,dx = \lim_{\varepsilon\to 0}\int_a^b v_\varepsilon^2(\tilde u')^2\,dx = \int_a^b (\tilde u')^2\,dx\,. \tag{5.7}$$

Therefore, we are left to estimate $G_\varepsilon(v_\varepsilon,(a,b))$. By the choice of t_0^ε and the 1-periodicity of φ, a change of variables yields

$$\begin{aligned} G_\varepsilon(v_\varepsilon,(a,b)) &= \int_{t_0^\varepsilon - \ell\delta_\varepsilon T_\eta}^{t_0^\varepsilon + \ell\delta_\varepsilon T_\eta} \Big(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\Big)\,dx \\ &= \int_{-T_\eta}^{T_\eta}\Big(\frac{\ell\delta_\varepsilon}{\varepsilon}(1-v_\eta)^2 + \frac{\varepsilon}{\ell\delta_\varepsilon}\varphi(\ell x)(v_\eta')^2\Big)\,dx \le \widetilde\gamma_\varepsilon \int_{-T_\eta}^{T_\eta}\big((1-v_\eta)^2 + \varphi(\ell x)(v_\eta')^2\big)\,dx\,, \end{aligned} \tag{5.8}$$

where

$$\widetilde\gamma_\varepsilon := \max\Big\{\frac{\ell\delta_\varepsilon}{\varepsilon}, \frac{\varepsilon}{\ell\delta_\varepsilon}\Big\} \to 1 \quad \text{as}\ \ \varepsilon\to 0\,. \tag{5.9}$$

Using (5.6) and gathering (5.7)–(5.9) we readily obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \le \int_a^b (u')^2\,dx + \mathbf{m}^\ell + \eta\,,$$

hence the upper-bound inequality follows by the arbitrariness of $\eta > 0$. □

6. Oscillations on a smaller scale than the singular perturbation

In this section we analyse the case when the oscillation δ_ε parameter is much smaller than the singular-perturbation parameter ε; *i.e.,* the case $\ell = +\infty$.

Proposition 6.1. *Let $\ell = \infty$; then the sequence (F_ε) defined in (2.2) Γ-converges to the functional $F^\infty \colon L^1(a,b) \times L^1(a,b) \to [0,+\infty]$ defined as*

$$F^\infty(u,v) := \begin{cases} \displaystyle\int_a^b (u')^2\,dx + \mathbf{m}^\infty \# S(u) & u \in P\text{-}W^{1,2}(a,b),\ v = 1 \ \textit{a.e. in } (a,b)\,, \\ +\infty & \textit{otherwise}, \end{cases} \tag{6.1}$$

where

$$\mathbf{m}^\infty := 2\left(\int_0^1 \frac{1}{\varphi(t)}\,dt\right)^{-1/2}. \tag{6.2}$$

Proof. It is convenient to introduce the constant

$$\varphi_{\rm hom} := \left(\int_0^1 \frac{1}{\varphi(t)}\,dt\right)^{-1},$$

so that $\mathbf{m}^\infty = 2\sqrt{\varphi_{\rm hom}}$. We now divide the proof into two steps.

Step 1: Lower-bound inequality.

For any $(u,v) \in L^1(a,b) \times L^1(a,b)$ let $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(a,b) \times W^{1,2}(a,b)$ be such that

$$(u_\varepsilon, v_\varepsilon) \to (u,v) \ \text{ in } \ L^1(a,b) \times L^1(a,b) \quad \text{and} \quad \liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty\,.$$

Arguing as in Proposition 5.1 we assume without loss of generality that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$ and we apply Proposition 3.2 to deduce that $u \in P\text{-}W^{1,2}(a,b)$, $v = 1$ a.e. in (a,b) and

$$\liminf_{\varepsilon\to0} \int_a^b v_\varepsilon^2 (u_\varepsilon')^2\,dx \geq \int_a^b (u')^2\,dx\,.$$

We set $S(u) = \{t_1, \dots, t_N\}$ with $N \geq 1$ (if $S(u) = \emptyset$ there is nothing to prove) and we let $I_1, \dots, I_N$ be pairwise disjoint open intervals with $I_i \subset (a,b)$ and $t_i \in I_i$ for $i = 1, \dots, N$. Then if we show that

$$\liminf_{\varepsilon\to0} G_\varepsilon(v_\varepsilon, I_i) \geq \mathbf{m}^\infty \quad \text{for every } \ i = 1, \dots, N, \tag{6.3}$$

with $\mathbf{m}^\infty$ as in (6.2) we are done.

We fix $\eta > 0$ and $i \in \{1, \dots, N\}$. By Proposition 3.2 and Remark 3.3 we can find $\tilde{s}^i_\varepsilon, r^i_\varepsilon, \tilde{r}^i_\varepsilon \in I_i$ with $r^i_\varepsilon < \tilde{s}^i_\varepsilon < \tilde{r}^i_\varepsilon$ such that

$$v_\varepsilon(\tilde{s}^i_\varepsilon) = \eta\,,\ v_\varepsilon(r^i_\varepsilon) = v_\varepsilon(\tilde{r}^i_\varepsilon) = 1-\eta \quad \text{and} \quad v_\varepsilon \leq 1-\eta \ \text{ in } \ [r^i_\varepsilon, \tilde{r}^i_\varepsilon]\,. \tag{6.4}$$

Then (3.5) implies that

$$\frac{\tilde{r}^i_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon} \in \left[\frac{\alpha(1-2\eta)^2}{M}, \frac{M}{\eta^2}\right] \quad \text{for every } \ \varepsilon > 0\,, \tag{6.5}$$

where $M := \sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$. Thanks to (6.4) the function $\tilde{v}_\varepsilon : \mathbb{R} \to [0,1]$ given by

$$\tilde{v}_\varepsilon(x) := \begin{cases} \eta & \text{if } \ x < \tilde{s}^i_\varepsilon\,, \\ v_\varepsilon(x) & \text{if } \ \tilde{s}^i_\varepsilon \leq x \leq \tilde{r}^i_\varepsilon\,, \\ (1-\eta) + \eta\dfrac{x - r^i_\varepsilon}{\varepsilon} & \text{if } \ \tilde{r}^i_\varepsilon < x \leq \tilde{r}^i_\varepsilon + \varepsilon\,, \\ 1 & \text{otherwise in } \mathbb{R} \end{cases} \tag{6.6}$$

belongs to $W^{1,2}_{\rm loc}(\mathbb{R})$. Moreover, set $t^i_\varepsilon := \Big\lfloor \frac{\tilde s^i_\varepsilon}{\delta_\varepsilon}\Big\rfloor \delta_\varepsilon \in (\tilde s^i_\varepsilon - \delta_\varepsilon, \tilde s^i_\varepsilon]$; using (2.1), by the definition of $\tilde v_\varepsilon$ we have

$$G_\varepsilon(v_\varepsilon, (\tilde s^i_\varepsilon, \tilde r^i_\varepsilon)) \geq G_\varepsilon(\tilde v_\varepsilon, (t^i_\varepsilon, \tilde r^i_\varepsilon + \varepsilon)) - \Big(\frac{1}{3} + \beta\Big)\eta^2 - (1-\eta)^2 \frac{\delta_\varepsilon}{\varepsilon}\,. \tag{6.7}$$

Eventually, by setting $w_\varepsilon(x) := \tilde v_\varepsilon(\varepsilon x + t^i_\varepsilon)$ and $T_\eta := \frac{M}{\eta^2} + 2$, the periodicity of φ, (6.5) and a change of variables yield

$$\begin{aligned} G_\varepsilon(\tilde v_\varepsilon, (t^i_\varepsilon, \tilde r^i_\varepsilon + \varepsilon)) &= \int_0^{T_\eta} \Big((1-w_\varepsilon)^2 + \varphi\Big(\frac{\varepsilon x}{\delta_\varepsilon}\Big)(w_\varepsilon')^2\Big)\,dx \\ &\geq \inf\left\{\int_0^{T_\eta}\Big((1-w)^2 + \varphi\Big(\frac{x}{\delta_\varepsilon/\varepsilon}\Big)(w')^2\Big)\,dx \colon w \in W^{1,2}(0,T_\eta)\,,\ w(0)=\eta\,,\ w(T_\eta)=1\right\}. \end{aligned} \tag{6.8}$$

Now, since $\delta_\varepsilon/\varepsilon \to 0$, by classical homogenisation (see *e.g.*, [7, Theorem 3.1]) we get

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \inf\left\{\int_0^{T_\eta}\Big((1-w)^2 + \varphi\Big(\frac{x}{\delta_\varepsilon/\varepsilon}\Big)(w')^2\Big)\,dx \colon w \in W^{1,2}(0,T_\eta)\,,\ w(0)=\eta\,,\ w(T_\eta)=1\right\} \\ &\quad= \min\left\{\int_0^{T_\eta}\big((1-w)^2 + \varphi_{\rm hom}(w')^2\big)\,dx \colon w \in W^{1,2}(0,T_\eta)\,,\ w(0)=\eta\,,\ w(T_\eta)=1\right\}. \end{aligned} \tag{6.9}$$

For any $w \in W^{1,2}(0,T_\eta)$ satisfying $w(0)=\eta$ and $w(T_\eta)=1$, an application of the Modica-Mortola trick together with a change of variables yields

$$\int_0^{T_\eta}\big((1-w)^2 + \varphi_{\rm hom}(w')^2\big)\,dx \geq 2\sqrt{\varphi_{\rm hom}}\int_0^{T_\eta}(1-w)|w'|\,dx = 2\sqrt{\varphi_{\rm hom}}\int_\eta^1 (1-s) = (1-\eta)^2\sqrt{\varphi_{\rm hom}}\,,$$

hence

$$\min\left\{\int_0^{T_\eta}\big((1-w)^2 + \varphi_{\rm hom}(w')^2\big)\,dx \colon w \in W^{1,2}(0,T_\eta)\,,\ w(0)=\eta\,,\ w(T_\eta)=1\right\} \geq (1-\eta)^2\sqrt{\varphi_{\rm hom}}\,. \tag{6.10}$$

Finally, gathering together (6.7)–(6.10) we obtain

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, (\tilde s^i_\varepsilon, \tilde r^i_\varepsilon)) \geq (1-\eta)^2\sqrt{\varphi_{\rm hom}} - \eta^2\Big(\frac{1}{3}+\beta\Big)\,. \tag{6.11}$$

Analogously it can be shown that

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, (r^i_\varepsilon, \tilde s^i_\varepsilon)) \geq (1-\eta)^2\sqrt{\varphi_{\rm hom}} - \eta^2\Big(\frac{1}{3}+\beta\Big)\,. \tag{6.12}$$

Hence, from (6.11) and (6.12) we deduce that

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon, I_i) \geq (1-\eta)^2\mathbf{m}^\infty - \eta^2\Big(\frac{1}{3}+\beta\Big)\,.$$

Eventually, by letting $\eta \to 0$ we obtain (6.3) and therefore the lower bound.

Step 2: Upper-bound inequality.

Let $u \in P\text{-}W^{1,2}(a,b)$ be fixed; As in the proof of Proposition 4.1 we assume without loss of generality that $S(u) = \{t_0\}$ for some $t_0 \in (a,b)$ and $u = \tilde u + u^{\rm pc}$ as in (2.8) with $\tilde u \in W^{1,2}(a,b)$ and $u^{\rm pc} = s\chi_{(a,t_0)}$ for some $s \in \mathbb{R}$.

We fix $\eta > 0$; applying (3.6) with $\lambda = \varphi_{\rm hom}$ we find $T_\eta > 0$ and $v_\eta \in W^{1,2}(0,T_\eta)$ satisfying $0 \le v_\eta \le 1$, $v_\eta(0) = 0$, $v_\eta(T_\eta) = 1$, and

$$\int_0^{T_\eta} (1-v_\eta)^2 + \varphi_{\rm hom}(v_\eta')^2\,dx \le \sqrt{\varphi_{\rm hom}} + \eta\,. \tag{6.13}$$

By invoking the classical homogenization theorem (see *e.g.*, [9, Theorem 14.5]), for any $\sigma \searrow 0$ we find a sequence $(w_\sigma) \subset W^{1,2}(0,T_\eta)$ such that $w_\sigma \to v_\eta$ in $L^2(0,T_\eta)$ as $\sigma \to 0$, $w_\sigma(0) = 0$, $w_\sigma(T_\eta) = 1$ and

$$\lim_{\sigma\to 0}\int_0^{T_\eta} \Big((1-w_\sigma)^2 + \varphi\Big(\frac{x}{\sigma}\Big)(w_\sigma')^2\Big)\,dx = \int_0^{T_\eta}\Big((1-v_\eta)^2 + \varphi_{\rm hom}(v_\eta')^2\Big)\,dx\,. \tag{6.14}$$

Now we let t_0^ε be as in (4.4), $\sigma_\varepsilon := \delta_\varepsilon/\varepsilon$ and $w_\varepsilon := w_{\sigma_\varepsilon}$ and define the pair $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(a,b)\times W^{1,2}(a,b)$ by setting $u_\varepsilon := \tilde u + \bar u_\varepsilon$ with

$$\bar u_\varepsilon(x) := \begin{cases} 0 & \text{if } \ x \le t_0^\varepsilon + \dfrac{\delta_\varepsilon}{2}\,, \\[2mm] \dfrac{2s}{\delta_\varepsilon}\Big(x - \Big(t_0^\varepsilon + \dfrac{\delta_\varepsilon}{2}\Big)\Big) & \text{if } \ t_0^\varepsilon + \dfrac{\delta_\varepsilon}{2} < x < t_0^\varepsilon + \delta_\varepsilon\,, \\[2mm] s & \text{if } \ x \ge t_0^\varepsilon + \delta_\varepsilon\,, \end{cases}$$

and

$$v_\varepsilon(x) := \begin{cases} 0 & \text{if } \ |x - t_0^\varepsilon| \le \delta_\varepsilon\,, \\[2mm] w_\varepsilon\left(\dfrac{|x-t_0^\varepsilon| - \delta_\varepsilon}{\varepsilon}\right) & \text{if } \ \delta_\varepsilon < |x-t_0^\varepsilon| < \delta_\varepsilon + \varepsilon\, T_\eta\,, \\[2mm] 1 & \text{if } \ \delta_\varepsilon + \varepsilon\,T_\eta \le |x - t_0^\varepsilon|\,, \end{cases}$$

(see Figure 3).

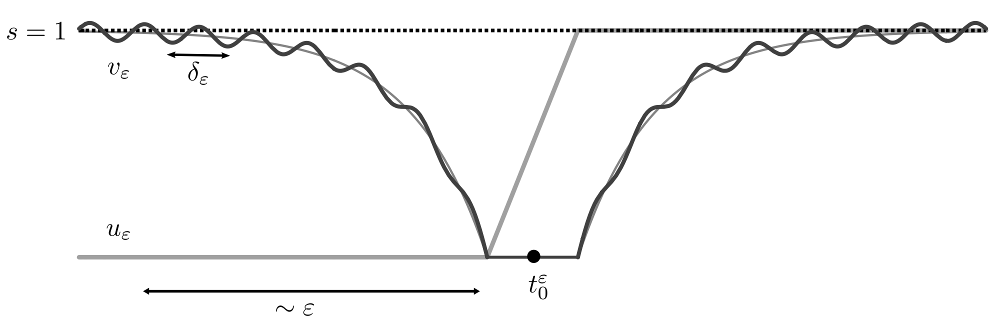

Figure 3. Recovery sequence in the case $s = 1$; v_ε in dark grey is obtained by superposing oscillations on the rescaled optimal profile.

We claim that $(u_\varepsilon, v_\varepsilon)$ is a recovery sequence for $F^\infty(u,1)$. In fact, by construction $u_\varepsilon := \tilde u + \bar u_\varepsilon \to u$ in $L^1(a,b)$, $v_\varepsilon \to 1$ in $L^1(a,b)$ and a.e. in (a,b). Moreover, observing that

$$v_\varepsilon \bar u_\varepsilon' = 0 \text{ a.e. in } (a,b)$$

we get

$$\lim_{\varepsilon\to 0}\int_a^b v_\varepsilon^2(u_\varepsilon')^2\,dx = \lim_{\varepsilon\to 0}\int_a^b v_\varepsilon^2(\tilde u')^2\,dx = \int_a^b (\tilde u')^2\,dx = \int_a^b (u')^2\,dx\,. \tag{6.15}$$

On the other hand, by a change of variables and the periodicity of φ we deduce that

$$\begin{aligned}G_\varepsilon(v_\varepsilon,(a,b)) &\le 2\int_{t_0^\varepsilon+\delta_\varepsilon}^{t_0^\varepsilon+\delta_\varepsilon+\varepsilon T_\eta}\Big(\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\Big)\,dx+\frac{2\delta_\varepsilon}{\varepsilon}\\ &=2\int_0^{T_\eta}\left((1-w_\varepsilon)^2+\varphi\left(\frac{x}{\sigma_\varepsilon}\right)(w_\varepsilon')^2\right)\,dx+\frac{2\delta_\varepsilon}{\varepsilon}\,.\end{aligned}$$

The latter together with (6.14) and (6.13) yield

$$\limsup_{\varepsilon\to 0}G_\varepsilon(v_\varepsilon,(a,b))\le 2\sqrt{\varphi_{\mathrm{hom}}}+2\eta=\mathbf{m}^\infty+2\eta\,. \tag{6.16}$$

Finally, gathering together (6.15) and (6.16) we obtain

$$\limsup_{\varepsilon\to 0}\int_a^b\Big(v_\varepsilon^2(u_\varepsilon')^2+\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varphi\Big(\frac{x}{\delta_\varepsilon}\Big)(v_\varepsilon')^2\Big)\,dx\le\int_a^b(u')^2\,dx+\mathbf{m}^\infty+2\eta\,.$$

Thus, upon replacing v_ε by $0\vee(v_\varepsilon\wedge 1)$ we conclude by the arbitrariness of $\eta>0$. $\square$

7. Limit analysis of $\mathbf{m}^\ell$

We conclude this note by analysing the convergence of the constant $\mathbf{m}^\ell$ as $\ell\to 0^+$ and $\ell\to+\infty$. Namely, we prove (2.14), thus concluding the proof of Theorem 2.1.

Proposition 7.1. *Let $\ell\in(0,+\infty)$ and $\mathbf{m}^\ell$ be as in* (2.11). *Let moreover $\mathbf{m}^0$ and $\mathbf{m}^\infty$ be as in* (2.10) *and* (2.13), *respectively. Then*

$$\lim_{\ell\to+\infty}\mathbf{m}^\ell=\mathbf{m}^\infty\,. \tag{7.1}$$

If φ is upper semicontinuous, it also holds

$$\lim_{\ell\to 0^+}\mathbf{m}^\ell=\mathbf{m}^0\,. \tag{7.2}$$

Proof. The proof of (7.1) and (7.2) uses arguments which are similar to those employed in the proof of Proposition 6.1 and Proposition 4.1, respectively. For this reason, we only sketch this proof.

Step 1: Proof of (7.1).

We first show that

$$\liminf_{\ell\to+\infty}\mathbf{m}^\ell\ge\mathbf{m}^\infty\,. \tag{7.3}$$

To this end, we fix $\eta>0$; using a similar argument as in the proof of Lemma 3.5 we can find $T_\eta>0$ and for every $\ell\in(0,+\infty)$ a real number $z_{\eta,\ell}\in[0,1)$ and $v_{\eta,\ell}\in W^{1,2}(-T_\eta,T_\eta)$ such that $0\le v_{\eta,\ell}\le 1$, $v_{\eta,\ell}(\frac{z_{\eta,\ell}}{\ell})=0$, $v_\eta(\pm T_\eta)=1$ and

$$\int_{-T_\eta}^{T_\eta}\left((1-v_{\eta,\ell})^2+\varphi(\ell x)(v_{\eta,\ell}')^2\right)\,dx\le\mathbf{m}^\ell+\eta\,. \tag{7.4}$$

Note that T_η can be chosen independently of ℓ. We now define $\tilde v_{\eta,\ell}\in W^{1,2}(0,T_\eta)$ by setting

$$\tilde v_{\eta,\ell}:=\begin{cases}v_{\eta,\ell} & \text{if } \frac{z_{\eta,\ell}}{\ell}\le x\le T_\eta\,,\\ 0 & \text{if } 0\le x<\frac{z_{\eta,\ell}}{\ell}\,.\end{cases}$$

Since $z_{\eta,\ell}\in[0,1)$, we readily obtain

$$\begin{aligned}\int_{\frac{z_{\eta,\ell}}{\ell}}^{T_\eta}\big((1-v_{\eta,\ell})^2+\varphi(\ell x)(v_{\eta,\ell}')^2\big)\,dx &\geq \int_0^{T_\eta}\big((1-\tilde v_{\eta,\ell})^2+\varphi(\ell x)(\tilde v_{\eta,\ell}')^2\big)\,dx-\frac{1}{\ell}\\ &\geq \inf\bigg\{\int_0^{T_\eta}\big((1-v)^2+\varphi(\ell x)(v')^2\big)\,dx\colon v\in W^{1,2}(0,T_\eta),\ v(0)=0,\ v(T_\eta)=1\bigg\}-\frac{1}{\ell}\,.\end{aligned}$$

Thus, arguing as in the proof of Proposition 6.1, applying the classical homogenisation result together with the Modica-Mortola trick we deduce that

$$\liminf_{\ell\to+\infty}\int_{\frac{z_{\eta,\ell}}{\ell}}^{T_\eta}\big((1-v_{\eta,\ell})^2+\varphi(\ell x)(v_{\eta,\ell}')^2\big)\,dx\geq\frac{\mathbf{m}^\infty}{2}\,.$$

Since an analogous argument holds on $(-T_\eta,\frac{z_{\eta,\ell}}{\ell})$, in view of (7.4) we get

$$\liminf_{\ell\to+\infty}\mathbf{m}^\ell\geq\mathbf{m}^\infty-\eta\,,$$

from which we deduce (7.3) by letting $\eta\to0$.

Then, it remains to prove that

$$\limsup_{\ell\to+\infty}\mathbf{m}^\ell\leq\mathbf{m}^\infty\,.$$

We fix $\eta>0$; arguing as in the proof of Proposition 6.1 Step 2 we use (3.6) together with the classical homogenisation result with boundary conditions to find $T_\eta>0$ and a sequence $(v_{\eta,\ell})_\ell\subset W^{1,2}(0,T_\eta)$ satisfying $v_{\eta,\ell}(0)=0$, $v_{\eta,\ell}(T_\eta)=1$ and

$$\lim_{\ell\to+\infty}\int_0^{T_\eta}\big((1-v_{\eta,\ell})^2+\varphi(\ell x)(v_{\eta,\ell}')^2\big)\,dx\leq\frac{\mathbf{m}^\infty}{2}+\eta\,. \tag{7.5}$$

Upon truncation we can additionally assume that $0\leq v_{\eta,\ell}\leq1$. Since $v_{\eta,\ell}(0)=0$, the reflected function $\tilde v_{\eta,\ell}$ defined by setting $\tilde v_{\eta,\ell}(x):=v_{\eta,\ell}(|x|)$ belongs to $W^{1,2}(-T_\eta,T_\eta)$. Moreover, upon extending $\tilde v_{\eta,\ell}$ by 1 it is admissible for $\mathbf{m}_0^\ell$. Thus, (7.5) implies that

$$\limsup_{\ell\to+\infty}\mathbf{m}^\ell\leq\limsup_{\ell\to+\infty}\mathbf{m}_0^\ell\leq\lim_{\ell\to+\infty}2\int_0^{T_\eta}\big((1-v_{\eta,\ell})^2+\varphi(\ell x)(v_{\eta,\ell}')^2\big)\,dx\leq\mathbf{m}^\infty+2\eta\,,$$

which together with (7.3) gives (7.1) by the arbitrariness of $\eta>0$.

Step 2: Proof of (7.2).

By definition of $\mathbf{m}^0$, from (3.7) we immediately deduce that $\liminf_{\ell\to0}\mathbf{m}^\ell\geq\mathbf{m}^0$. To prove the opposite inequality, we fix $\eta>0$ and choose $y_\eta\in(0,1)$ such that $\sqrt{\varphi(y_\eta)}\leq\sqrt{\alpha}+\eta$. Moreover, we set

$$v_\eta(x):=1-\exp\bigg(-\frac{|x|}{\sqrt{\varphi(y_\eta)}}\bigg)\,.$$

Then $v_\eta\in W^{1,2}_{\rm loc}(\mathbb{R})$, $0\leq v_\eta\leq1$ and v_η satisfies $v_\eta(0)=0$, $v_\eta(\pm\infty)=1$ and

$$\int_{\mathbb{R}}\big((1-v_\eta)^2+\varphi(y_\eta)(v_\eta')^2\big)\,dx=2\sqrt{\varphi(y_\eta)}\leq\mathbf{m}^0+2\eta\,. \tag{7.6}$$

Since v_η is admissible for $\mathbf{m}_{y_\eta}^\ell\geq\mathbf{m}^\ell$, by the reverse Fatou Lemma and the upper semicontinuity of φ from (7.6) we deduce that

$$\begin{aligned}\limsup_{\ell\to0^+}\mathbf{m}^\ell\leq\limsup_{\ell\to0^+}\mathbf{m}_{y_\eta}^\ell&\leq\limsup_{\ell\to0^+}\int_{\mathbb{R}}\big((1-v_\eta)^2+\varphi(\ell x+y_\eta)(v_\eta')^2\big)\,dx\\ &\leq\int_{\mathbb{R}}\Big((1-v_\eta)^2+\limsup_{\ell\to0^+}\varphi(\ell x+y_\eta)(v_\eta')^2\Big)\,dx\leq\mathbf{m}^0+2\eta\,.\end{aligned}$$

By the arbitrariness of $\eta > 0$ this concludes the proof. □

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(A. Bach) Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, Italy
E-mail address: annika.bach@uniroma1.it

(T. Esposito)
E-mail address: teres.esposito@gmail.com

(R. Marziani) Angewandte Mathematik, WWU Münster, Germany
E-mail address: roberta.marziani@uni-muenster.de

(C. I. Zeppieri) Angewandte Mathematik, WWU Münster, Germany
E-mail address: caterina.zeppieri@uni-muenster.de