INFINITELY MANY SOLUTIONS FOR SCHRÖDINGER-NEWTON EQUATIONS

YEYAO HU, ALEKS JEVNIKAR, AND WEIHONG XIE

Abstract. We prove the existence of infinitely many non-radial positive solutions for the Schrödinger-Newton system

$$\begin{cases} \Delta u - V(|x|)u + \Psi u = 0, & x \in \mathbb{R}^3, \\ \Delta \Psi + \frac{1}{2}u^2 = 0, & x \in \mathbb{R}^3, \end{cases}$$

provided that $V(r)$ has the following behavior at infinity:

$$V(r) = V_0 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right) \qquad \text{as } r \to \infty,$$

where $\frac{1}{2} \leq m < 1$ and a, V_0, θ are some positive constants. In particular, for any s large we use a reduction method to construct $s-$bump solutions lying on a circle of radius $r \sim (s \log s)^{\frac{1}{1-m}}$.

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1. Introduction and statement of results

In this paper we consider the following Schrödinger-Newton system

$$\begin{cases} \Delta u - V(x)u + \Psi u = 0, & x \in \mathbb{R}^3, \\ \Delta \Psi + \frac{1}{2}u^2 = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here V is a given external potential and Ψ is the Newtonian gravitational potential. The latter model was proposed in [22], where the wave function u represents a stationary solution for a quantum system describing a nonlinear modification of the Schrödinger equation with a Newtonian gravitational potential representing the interaction of the particle with its own gravitational field.

Note that the second equation of (1.1) (see [5]) has a unique positive solution $\Psi_u \in D^{1,2}(\mathbb{R}^3)$ given by

$$\Psi_u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy. \tag{1.2}$$

Thus, the system (1.1) is equivalent to the following single nonlocal equation:

$$-\Delta u + V(x)u = \frac{1}{8\pi} \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad x \in \mathbb{R}^3. \tag{1.3}$$

Clearly, (u, Ψ_u) is a solution of the system (1.1) if and only if u is a solution of the equation (1.3). The problem (1.3) appears also in the study of standing waves of nonlinear Hartree equations, see [4, 7, 24]. Moreover, it can be seen as a special case of the Choquard equation, see [10, 12, 16].

Let us consider the system (1.1) with $V(x) \equiv 1$, that is

$$\begin{cases} \Delta u - u + \Psi u = 0, & x \in \mathbb{R}^3, \\ \Delta \Psi + \frac{1}{2} u^2 = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

The existence of a unique ground state solution to the latter problem has been known since [13, 14] via variational methods, see also [17] for a more recent result. Moreover, the nondegeneracy of the ground state was proven in [27]. We refer to Theorem 2.1 for the precise statements.

Furthermore, the case $V(x) \not\equiv 1$ has been treated in [27] in the semi-classical regime, that is the singularly perturbed Schrödinger-Newton problem

$$-\varepsilon^2 \Delta u + V(x) u = \frac{1}{8\pi\varepsilon^2} \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad x \in \mathbb{R}^3, \tag{1.5}$$

where $\varepsilon > 0$ is a parameter and $\inf_{\mathbb{R}^3} V > 0$. The authors prove the existence of positive $K-$bump solutions to (1.5) concentrating at local maximum (minimum) or nondegenerate critical points of V as $\varepsilon \to 0$. Moreover, it is also shown that there is a strong interacting between each pair of bumps, which stands in comparison with the analogous result for Schrödinger equations [8]. It turns out that such positive solutions concentrating at non-degenerate critical points of V are unique for ε small enough, as recently shown in [15] by using local Pohozaev identities.

On the other hand, for (1.5) with $V(x) \not\equiv 1$ and $\varepsilon = 1$ there are fewer results. In particular, at least to our knowledge, there are no multiplicity results available in the literature. The aim of this paper is to obtain infinitely many non-radial solutions to (1.1) with radial potential $V(r)$, that is,

$$\begin{cases} \Delta u - V(|x|) u + \Psi u = 0, & x \in \mathbb{R}^3, \\ \Delta \Psi + \frac{1}{2} u^2 = 0, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

or equivalently,

$$-\Delta u + V(|x|) u = \frac{1}{8\pi} \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad x \in \mathbb{R}^3. \tag{1.7}$$

To this end, we make the following assumption on the behavior at infinity of V:

(H) There exist some constants a, θ, $V_0 > 0$ and $\frac{1}{2} \leq m < 1$, such that $V(x) \geq V_0$ and

$$V(r) = V_0 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}} \right) \quad \text{as } r \to \infty. \tag{1.8}$$

The main result is the following.

Theorem 1.1. *If V satisfies (H), then problem* (1.6) *has infinitely many non-radial positive solutions.*

For any s large we will construct solutions with s bumps approaching the infinity. Observe that since we can assume without loss of generality $V_0 = 1$, the condition (H) yields

$$\lim_{r\to\infty} V(r) = 1.$$

Therefore, we can use the single-peaked ground state solution U of (1.4), see also Theorem 2.1, as an approximate solution of (1.6). A combination of single-peaked solutions will give then rise to the $s-$bump solutions we are looking for.

We center the bumps in

$$\xi_i = \left(r\cos\frac{2(i-1)\pi}{s}, r\sin\frac{2(i-1)\pi}{s}, 0 \right), \quad i = 1, 2, \dots, s, \tag{1.9}$$

and we thus let

$$U_r = \sum_{i=1}^{s} U_{\xi_i}(x),$$

where $U_{\xi_i}(x) = U(x - \xi_i)$. The bumps are lying on a circle for which we choose a radius

$$r \in \left[r_0(s\log s)^{\frac{1}{1-m}}, r_1(s\log s)^{\frac{1}{1-m}} \right] \tag{1.10}$$

for some $r_1 > r_0 > 0$ and m is given in (H).

Theorem 1.1 will be an immediate consequence of the following result.

Theorem 1.2. *If V satisfies (H), then there exists an integer $s_0 > 0$ such that for all $s > s_0$, the problem* (1.6) *admits a positive solution u_s satisfying*

$$u_s = U_{r_s} + w_s,$$

where $r_s \in \left[r_0(s\log s)^{\frac{1}{1-m}}, r_1(s\log s)^{\frac{1}{1-m}} \right]$, $w_s \in \mathcal{E}$, $\mathcal{E}$ *is defined in* (1.11) *and*

$$\int_{\mathbb{R}^3} (|Dw_s|^2 + w_s^2) \to 0 \qquad \text{as } s \to \infty.$$

The space $\mathcal{E}$ for the error term is given by

(1.11)

$$\mathcal{E} = \Big\{ u \in H^1(\mathbb{R}^3) : u \text{ is even in } x_2,\, x_3,$$
$$u(r\cos\theta, r\sin\theta, x_3) = u\left(r\cos\left(\theta + \frac{2\pi i}{s}\right), r\sin\left(\theta + \frac{2\pi i}{s}\right), x_3 \right),$$
$$i = 1, 2, \dots, s-1 \Big\}.$$

We will use a reduction argument to prove the above result. This is quite standard in the singularly perturbed problems alike (1.5) for $\varepsilon \to 0$. Here we exploit the loss of compactness of the domain and use the number s of the bumps of the solution u_s as the parameter to build up the spike solutions.

This strategy was firstly introduced by Wei and Yan in [31] and it was later successful used to study other elliptic problems, see for example [1, 2, 19, 20, 21, 25, 26, 28, 29, 30]. The method was originally designed in [31] to prove the existence of infinitely many solutions for the Schrödinger equation

$$-\Delta u + V(|x|)u = u^p, \quad u > 0, \tag{1.12}$$

provided V satisfies the above condition (H) with $m > 1$ instead of $\frac{1}{2} \leq m < 1$. The bump solutions are there modeled on the ground state of the Schrödinger equation (1.12) with $V(x) \equiv 1$. Compared to this result we have to face here new difficulties due to the non-local nature of the problem. A similar non-local problem, the Schrödinger-Possion system, was considered in [3, 9]. However, differently from our approach, the bump solutions are there modeled still on the ground state of the Schrödinger equation (1.12). As a matter of fact, in all [31] and [3, 9] the bumps are lying on a circle of radius

$$r_s \sim s \log s,$$

which is essential in the constructions of the spike solutions. On the contrary, in our case, to handle the non-local features and the fact that we build up the bump solutions directly on the ground state of the Schrödinger-Newton system (1.4), we end up constructing spike solutions lying on a circle of radius

$$r_s \sim (s \log s)^{\frac{1}{1-m}}.$$

See Remark 3.1 for further discussion in this respect.

The paper is organized as follows. In section 2 we set up the problem and estimate the energy of the approximate solution, in section 3 we give the proof of the main result.

2. An estimate for the energy of approximate solutions

In this section we present some preliminaries and give an estimate of the energy of the approximate solution. We first collect some key known results which will be used in the sequel.

Theorem 2.1. ([17], [27])
(Existence) *There exists a unique radial solution* (U, Ψ) *with*

$$U(x) \to 0, \quad \Psi(x) \to 0 \quad \text{as } |x| \to \infty,$$

of the following problem

$$\begin{cases} \Delta u - u + \Psi u = 0, & in\ \mathbb{R}^3, \\ \Delta\Psi + \frac{1}{2}u^2 = 0, & in\ \mathbb{R}^3, \\ u,\ \Psi > 0, \quad u(0) = \max_{x\in\mathbb{R}^3} u(x). \end{cases} \tag{2.1}$$

Moreover, U is strictly decreasing and

$$\lim_{|x|\to\infty} U(x)|x|e^{|x|} = \lambda_0, \quad \lim_{y\to\infty}\frac{U'(x)}{U(x)} = -1, \tag{2.2}$$

and

$$\lim_{|x|\to\infty} \Psi(x)|x| = \lambda_1 \tag{2.3}$$

for some constants $\lambda_0,\ \lambda_1 > 0$.

(Nondegeneracy) *Suppose that $\phi \in H^1(\mathbb{R}^3)$ satisfies the following eigenvalue problem:*

$$-\Delta\phi + \phi = \frac{1}{8\pi}\left(\int_{\mathbb{R}^3}\frac{U^2(y)}{|x-y|}dy\right)\phi + \frac{1}{4\pi}\left(\int_{\mathbb{R}^3}\frac{U(y)\phi(y)}{|x-y|}dy\right)U.$$

Then

$$\phi \in span\left\{\frac{\partial U}{\partial x_i}, i = 1,2,3\right\}.$$

We next introduce the functional-analytic framework. Define the inner product and norm on the workspace $H^1(\mathbb{R}^3)$ by

$$\langle u, v\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla v + V(|x|)uv) \quad \text{and} \quad \|u\| = (u,u)^{\frac{1}{2}},$$

respectively. Since V is bounded, the norm $\|\cdot\|$ is equivalent to the standard norm. By Hardy-Littlewood-Sobolev inequality, Hölder inequality and Sobolev inequality, for $u, v, w, \phi \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u(y)v(y)w(x)\phi(x)}{|x-y|} \le \|uv\|_{L^p(\mathbb{R}^3)}\|w\phi\|_{L^q(\mathbb{R}^3)} \le C\|u\|\|v\|\|w\|\|\phi\|, \tag{2.4}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{5}{3}$. In particular,

$$||\Psi_u||^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3}\Psi_u u^2 \le C\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \le C||u||^4. \tag{2.5}$$

Throughout this paper, C denotes various positive constants whose exact value is inessential.

The associated energy functional to (1.7) is defined as follows

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)u^2) - \frac{1}{32\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(y)u^2(x)}{|x-y|}. \tag{2.6}$$

Obviously, $J(u) \in \mathcal{C}^2(H^1(\mathbb{R}^3),\mathbb{R})$.

Recall that $\xi_i = \left(r\cos\frac{2(i-1)\pi}{s}, r\sin\frac{2(i-1)\pi}{s}, 0\right)$. In what follows, we assume that

$$r \in I_s := \left[\left(\left(\frac{A_1}{64am\pi^2}\right)^{\frac{1}{1-m}} - \alpha\right)(s\log s)^{\frac{1}{1-m}}, \left(\left(\frac{A_1}{64am\pi^2}\right)^{\frac{1}{1-m}} + \alpha\right)(s\log s)^{\frac{1}{1-m}}\right], \tag{2.7}$$

where a and m is the constants in (H), $\alpha > 0$ is a small constant and

$$A_1 = \int_{\mathbb{R}^3} U^2. \tag{2.8}$$

To evaluate $J(U_r)$ we need to estimate the interacting term $\sum_{i=2}^s \int_{\mathbb{R}^3} \Psi_{U_{\xi_1}} U_{\xi_i}^2$. The following holds.

Lemma 2.2. *There is a small constant $\beta > 0$ such that*

$$\sum_{i=2}^{s} \int_{\mathbb{R}^3} \Psi_{U_{\xi_1}} U_{\xi_i}^2 = \frac{A_1^2}{32\pi^2}\frac{s\log s}{r} + O\left(\frac{1}{s^{\frac{2m}{1-m}+\beta}}\right). \tag{2.9}$$

where A_1 is introduced in (2.8).

Proof. A direct calculation gives that

$$\begin{aligned}
\sum_{i=2}^{s} \int_{\mathbb{R}^3} \Psi_{U_{\xi_1}} U_{\xi_i}^2 &= \frac{1}{32\pi}\sum_{i=2}^{s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U_{\xi_1}^2(y)U_{\xi_i}^2(x)}{|x-y|} \\
&= \frac{1}{32\pi}\sum_{i=2}^{s}\int_{\mathbb{R}^3} U^2(x)\int_{\mathbb{R}^3} U^2(y)\frac{1}{|x-y+(\xi_1-\xi_i)|} \\
&= \frac{1}{32\pi}\sum_{i=2}^{s}\frac{1}{|\xi_1-\xi_i|}\int_{\mathbb{R}^3} U^2(x)\int_{\mathbb{R}^3} U^2(y) + O\left(\sum_{i=2}^{s}\frac{1}{|\xi_1-\xi_i|^2}\right)
\end{aligned}$$

Observe that

$$\sum_{i=2}^{s}\frac{1}{|\xi_1-\xi_i|} = \frac{1}{2r}\sum_{i=1}^{s-1}\frac{1}{\sin\frac{i\pi}{s}}. \tag{2.10}$$

It is readily checked that

$$\int_{\frac{3}{2}}^{s-\frac{3}{2}} \frac{1}{\sin\frac{x\pi}{s}} \le \sum_{i=1}^{s-1}\frac{1}{\sin\frac{i\pi}{s}} \le \int_{\frac{1}{2}}^{s-\frac{1}{2}} \frac{1}{\sin\frac{x\pi}{s}}$$

and

$$\lim_{s\to\infty}\frac{1}{s\log s}\int_{\frac{3}{2}}^{s-\frac{3}{2}} \frac{1}{\sin\frac{x\pi}{s}} = \lim_{s\to\infty}\frac{1}{s\log s}\int_{\frac{1}{2}}^{s-\frac{1}{2}} \frac{1}{\sin\frac{x\pi}{s}} = \frac{2}{\pi}.$$

Thus, we have

$$\sum_{i=2}^{s}\frac{1}{|\xi_1-\xi_i|} = \frac{s\log s}{\pi r} + o_s(1). \tag{2.11}$$

Similarly,

$$O\left(\sum_{i=2}^{s}\frac{1}{|\xi_1-\xi_i|^2}\right)=O\left(\frac{1}{s^{\frac{2m}{1-m}+\beta}}\right).$$

Putting these facts together, (2.9) follows. $\square$

Next we give the energy estimate for the approximate solution U_r.

Lemma 2.3. *There is a small constant $\beta>0$ such that*

$$J(U_r)=s\left[\frac{A_2}{16\pi}+\frac{aA_1}{2}\frac{1}{r^m}-\frac{A_1^2}{128\pi^2}\frac{s\log s}{r}+O\left(\frac{1}{s^{\frac{2m}{1-m}+\beta}}\right)\right],$$

where A_1 is defined in (2.8) *and*

$$A_2=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{U^2(y)U^2(x)}{|x-y|}. \tag{2.12}$$

Proof.

$$\begin{aligned}J(U_r)&=\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla U_r|^2+U_r^2)+\frac{1}{2}\int_{\mathbb{R}^3}(V(|x|)-1)U_r^2-\frac{1}{32\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{U_r^2(y)U_r^2(x)}{|x-y|}\\&=I_1+I_2-I_3,\end{aligned}$$

where $I_1,\ I_2,\ I_3$ are defined by the last equality.

Note that the property (2.2) of U indicates that

$$\int_{\mathbb{R}^3}U_{\xi_1}U_{\xi_i}=O\left(e^{-(1-\delta)|\xi_1-\xi_i|}\right)$$

for $i\neq 1$ and for any small $\delta>0$. We further have

$$\sum_{i=2}^{s}e^{-(1-\delta)|\xi_1-\xi_i|}\le Ce^{-(1-\delta)\frac{2\pi r}{s}}=O\left(e^{-(1-\delta)s^{\frac{m}{1-m}}(\log s)^{\frac{1}{1-m}}}\right)=o\left(e^{-s^{\frac{m}{1-m}}}\right). \tag{2.13}$$

By (2.1), (2.5) and (2.13), we have

$$\begin{aligned}I_1&=\frac{s}{2}\int_{\mathbb{R}^3}(|\nabla U_{\xi_1}|^2+U_{\xi_1}^2)+\frac{s}{2}\sum_{i=2}^{s}\int_{\mathbb{R}^3}(\nabla U_{\xi_1}\nabla U_{\xi_i}+U_{\xi_1}U_{\xi_i})\\&=\frac{s}{16\pi}A_2+\frac{s}{2}\int_{\mathbb{R}^3}\Psi_{U_{\xi_1}}U_{\xi_1}\left(\sum_{i=2}^{s}U_{\xi_i}\right)\\&=\frac{s}{16\pi}A_2+sO\left(\sum_{i=2}^{s}e^{-(1-\delta)|\xi_1-\xi_i|}\right)\\&=\frac{s}{16\pi}A_2+o\left(se^{-s^{\frac{m}{1-m}}}\right).\end{aligned} \tag{2.14}$$

Following [31, Proposition A.2], a elementary calculation shows that for any $t>0$,

$$\frac{1}{|x-\xi_1|^t}=\frac{1}{|\xi_1|^t}\left(1+O\left(\frac{|x|}{|\xi_1|}\right)\right),\ x\in B_{\frac{|\xi_1|}{2}}(0)$$

and then

$$\begin{aligned}\int_{\mathbb{R}^3}(V(|x|)-1)U_{\xi_1}^2 &= \int_{B_{\frac{r}{2}}(0)}(V(|x-\xi_1|)-1)U^2+O\left(e^{-(1-\eta)r}\right)\\ &= \int_{B_{\frac{r}{2}}(0)}\left[\frac{a}{|x-\xi_1|^m}+O\left(\frac{1}{|x-\xi_1|^{m+\theta}}\right)\right]U^2+O\left(e^{-(1-\delta)r}\right)\\ &= \frac{aA_1}{r^m}+O\left(\frac{1}{r^{m+\theta}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}I_2 &= \frac{s}{2}\int_{\mathbb{R}^3}(V(|x|)-1)U_{\xi_1}^2+\frac{s}{2}\int_{\mathbb{R}^3}(V(|x|)-1)U_{\xi_1}\left(\sum_{i=2}^{s}U_{\xi_i}\right)\\ &= \frac{aA_1s}{2r^m}++sO\left(\sum_{i=2}^{s}e^{-(1-\delta)|\xi_1-\xi_i|}\right)\\ &= \frac{aA_1s}{2r^m}+o\left(se^{-s^{\frac{m}{1-m}}}\right).\end{aligned} \tag{2.15}$$

Using (2.9) and (2.13), we compute

$$\begin{aligned}I_3 &= \frac{1}{4}\int_{\mathbb{R}^3}\Psi_{U_r}\left(sU_{\xi_i}^2+\sum_{i\neq j}U_{\xi_i}U_{\xi_j}\right)=\frac{s}{4}\int_{\mathbb{R}^3}\Psi_{U_{\xi_1}}U_r^2+o\left(e^{-s^{\frac{m}{1-m}}}\right)\\ &= \frac{s}{4}\int_{\mathbb{R}^3}\Psi_{U_{\xi_1}}\left(\sum_{i=1}^{s}U_{\xi_i}^2+\sum_{i\neq j}U_{\xi_i}U_{\xi_j}\right)+o\left(e^{-s^{\frac{m}{1-m}}}\right)\\ &= \frac{s}{4}\int_{\mathbb{R}^3}\Psi_{U_{\xi_1}}U_{\xi_1}^2+\frac{s}{4}\sum_{i=2}^{s}\int_{\mathbb{R}^3}\Psi_{U_{\xi_1}}U_{\xi_i}^2+o\left(e^{-s^{\frac{m}{1-m}}}\right)\\ &= \frac{s}{32\pi}A_2+\frac{A_1^2}{128\pi^2}\frac{s^2\log s}{r}+O\left(\frac{1}{s^{\frac{3m-1}{1-m}+\beta}}\right).\end{aligned} \tag{2.16}$$

We conclude from (2.14)–(2.16) that

$$J(U_r)=s\left[\frac{A_2}{16\pi}+\frac{aA_1}{2}\frac{1}{r^m}-\frac{A_1^2}{128\pi^2}\frac{s\log s}{r}+O\left(\frac{1}{s^{\frac{2m}{1-m}+\beta}}\right)\right],$$

as desired. □

3. Proof of the main results

In this section, we use the reduction method to reduce the construction of spike solutions for (1.7) to finding critical points of a certain function $\mathcal{F}(r)$, $r\in I_s$.

Recall that $\xi_1=(r,0,0)$ and we assume that

$$r\in I_s.$$

Define

$$\mathcal{E}_s = \left\{ v \in \mathcal{E} : \int_{\mathbb{R}^3} \left(T\left[U_{\xi_i}^2\right] Z_i v + 2T\left[U_{\xi_i} Z_i\right] U_{\xi_i} v \right) = 0 \right\}, \tag{3.1}$$

where

$$T[uv](x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u(y)v(y)}{|x-y|} dy \tag{3.2}$$

and

$$Z_i = \frac{\partial U_{\xi_i}}{\partial r}, \quad i = 1, 2, \ldots, s$$

for simplicity. Define an auxiliary function

$$F(w) = J(U_r + w) \quad w \in \mathcal{E}_s.$$

We want to find the critical points of F. To this end, we expand $F(w)$ in the following form:

$$F(w) = F(0) + E(w) + \frac{1}{2} L(w) + N(w), \quad w \in \mathcal{E}_s, \tag{3.3}$$

where

$$E(w) = \int_{\mathbb{R}^3} (V(|x|) - 1) U_r w + \frac{1}{2} \int_{\mathbb{R}^3} \left(\sum_{i=1}^{s} T\left[U_{\xi_i}^2\right] U_{\xi_i} - T\left[U_r^2\right] U_r \right) w; \tag{3.4}$$

$$L(w) = \int_{\mathbb{R}^3} (|\nabla w|^2 + V(|x|) w^2) - \frac{1}{2} \int_{\mathbb{R}^3} \left(T\left[U_r^2\right] w^2 + 2T[U_r w] U_r w \right) \tag{3.5}$$

and

$$N(w) = -\frac{1}{2} \int_{\mathbb{R}^3} T\left[w^2\right] w U_r - \frac{1}{8} \int_{\mathbb{R}^3} T\left[w^2\right] w^2. \tag{3.6}$$

We next will focus on the terms in the r.h.s. of (3.3). By (2.5), we know that $E(w)$ is a bounded linear functional in $\mathcal{E}_s$. In the following we estimate the error $\|E\|_{\mathcal{L}(\mathcal{E}_s, \mathcal{E}_s)}$.

Lemma 3.1. *There exist an integer $s_0 > 0$ and a small $\beta > 0$, such that for any $s > s_0$,*

$$\|E\|_{\mathcal{L}(\mathcal{E}_s, \mathcal{E}_s)} \le C \left(\frac{1}{s} \right)^{\frac{2m-1}{2(1-m)} + \beta}. \tag{3.7}$$

Proof. For the first term of $E(w)$, similar arguments as in (2.15) show that

$$\begin{aligned} \int_{\mathbb{R}^3} (V(|x|) - 1) U_r w &= s \int_{\mathbb{R}^3} (V(|x - \xi_1|) - 1) U w(x - \xi_1) \\ &= O\left(\frac{s}{r^m} \right) \|w\|. \end{aligned} \tag{3.8}$$

On the other hand, by (2.13) we compute

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(\sum_{i=1}^{s} T\left[U_{\xi_i}^2\right]U_{\xi_i} - T\left[U_r^2\right]U_r\right)w \\
&= -s\int_{\mathbb{R}^3} T\left[U_{\xi_1}^2\right]\sum_{i=2}^{s} U_{\xi_i}w - \int_{\mathbb{R}^3} T\left[\sum_{i\neq j} U_{\xi_i}U_{\xi_j}\right]U_r w \\
&= -s\sum_{i=2}^{s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{U_{\xi_1}^2(x)U_{\xi_i}(y)w(y)}{|x-y|} + O\left(\sum_{i\neq j}^{s} e^{-(1-\delta)|\xi_i-\xi_j|}\right) \\
&= -s\sum_{i=2}^{s} B_i + o\left(e^{-s^{\frac{m}{1-m}}}\right),
\end{aligned}
\tag{3.9}
$$

where

$$
B_i = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{U_{\xi_1}^2(x)U_{\xi_i}(y)w(y)}{|x-y|}.
$$

To estimate B_i, we set

$$
\Omega_j = \left\{x=(x',x_3)\in\mathbb{R}^2\times\mathbb{R} : \left\langle \frac{x'}{|x'|}, \frac{x_3}{|x_3|}\right\rangle \geq \cos\frac{\pi}{s}\right\}, \quad j=1,2,\dots,s. \tag{3.10}
$$

Then we know that for any $x\in\Omega_j$,

$$
|x-\xi_i| \geq \frac{1}{2}|\xi_j-\xi_i|. \tag{3.11}
$$

Indeed, it is easy to verify that

$$
|x-\xi_j| \leq |x-\xi_i|,
$$

which insures (3.11) if $|x-\xi_j|\geq\frac{1}{2}|\xi_i-\xi_j|$. Otherwise,

$$
|x-\xi_i| \geq |\xi_i-\xi_j| - |x-\xi_j| \geq \frac{1}{2}|\xi_j-\xi_i|.
$$

It follows from (2.2) and (3.11) that for any $x\in\Omega_j$,

$$
U_{\xi_i}(x) \leq Ce^{-(1-\delta)|y-\xi_i|} \leq Ce^{-\frac{1-\delta}{2}|\xi_j-\xi_i|} \tag{3.12}
$$

for any small $\delta > 0$. By (2.11), (2.13) and (3.12), we find

(3.13)

$$\begin{aligned}
\sum_{i=2}^{s} B_i &= \sum_{i=2}^{s} \int_{\Omega_1} \int_{\mathbb{R}^3} \frac{U_{\xi_1}^2(x) U_{\xi_i}(y) w(y)}{|x-y|} dy dx + \sum_{i,j=2}^{s} \int_{\Omega_j} \int_{\mathbb{R}^3} \frac{U_{\xi_1}^2(x) U_{\xi_i}(y) w(y)}{|x-y|} dy dx \\
&= \sum_{i=2}^{s} \int_{\Omega_1} \int_{\Omega_i} \frac{U_{\xi_1}^2(x) U_{\xi_i}(y) w(y)}{|x-y|} dy dx + \sum_{\substack{i=2, j=1, \\ i \neq j}}^{s} \int_{\Omega_1} \int_{\Omega_j} \frac{U_{\xi_1}^2(x) U_{\xi_i}(y) w(y)}{|x-y|} dy dx \\
&\quad + o\left(e^{-s^{\frac{m}{1-m}}} \right) \\
&= \sum_{i=2}^{s} \int_{\Omega_1} \int_{\Omega_i} \frac{U_{\xi_1}^2(x) U_{\xi_i}(y) w(y)}{|x-y|} dy dx + o\left(e^{-s^{\frac{m}{1-m}}} \right) \\
&\leq C \sum_{i=2}^{s} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_1)} \int_{\Omega_i} e^{-(1-\delta)(2|x-\xi_1|+|y-\xi_i|)} \frac{w(y)}{|x-y|} dy dx + o\left(e^{-s^{\frac{m}{1-m}}} \right) \\
&\leq C \sum_{i=2}^{s} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_1) \cap \Omega_1} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_i) \cap \Omega_i} e^{-(1-\delta)(2|x-\xi_1|+|y-\xi_i|)} \frac{w(y)}{|x-y|} dy dx + o\left(e^{-s^{\frac{m}{1-m}}} \right) \\
&\leq C \sum_{i=2}^{s} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_1)} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_i)} e^{-(1-\delta)(2|x-\xi_1|+|y-\xi_i|)} \frac{w(y)}{|x-y|} dy dx + o\left(e^{-s^{\frac{m}{1-m}}} \right) \\
&\leq C \sum_{i=2}^{s} \frac{2}{|\xi_1-\xi_i|} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_1)} \int_{B_{\frac{|\xi_1-\xi_i|}{8}}(\xi_i)} e^{-(1-\delta)(2|x-\xi_1|+|y-\xi_i|)} w(y) dy dx + o\left(e^{-s^{\frac{m}{1-m}}} \right) \\
&\leq C \frac{s}{r^m} o_r(1) = o\left(\frac{s}{r^m} \right),
\end{aligned}$$

since $w(y) = o_r(1)$ when $|y| \geq \frac{r}{2}$. We conclude from (3.8) and (3.9) that

$$|E(w)| \leq C \frac{s}{r^m} \|w\| \leq C \left(\frac{1}{s} \right)^{\frac{2m-1}{2(1-m)}+\beta} \|w\|$$

because of $\frac{1}{2} < m < 1$. We complete the proof. □

With the help of (2.4) we easily derive that

$$\int_{\mathbb{R}^3} (\nabla \psi_1 \nabla \psi_2 + V(|x|) \psi_1 \psi_2) - \frac{1}{2} \int_{\mathbb{R}^3} \left(T\left[U_r^2\right] \psi_1 \psi_2 + 2T[U_r \psi_1] U_r \psi_2 \right), \quad \psi_1, \psi_2 \in \mathcal{E}_s$$

is a bounded bilinear functional $\mathcal{E}_s$. So there is a bounded linear operator $L : \mathcal{E}_s \to \mathcal{E}_s$ such that

$$\langle L\psi_1, \psi_2 \rangle = \int_{\mathbb{R}^3} (\nabla \psi_1 \nabla \psi_2 + V(|x|) \psi_1 \psi_2) - \frac{1}{2} \int_{\mathbb{R}^3} \left(T\left[U_r^2\right] \psi_1 \psi_2 + 2T[U_r \psi_1] U_r \psi_2 \right), \ \psi_1, \psi_2 \in \mathcal{E}_s.$$

We now consider the invertibility of the operator L in $\mathcal{E}_s$.

Lemma 3.2. *There exist a constant $\zeta > 0$, independent of s, and an integer $s_0 > 0$ such that for any $s > s_0$,*

$$\|Lv\| \geq \zeta \|v\|, \quad v \in \mathcal{E}_s.$$

Proof. Suppose on the contrary that there exist $s \to \infty$, $r_s \in I_s$, and $v_s \in \mathcal{E}_s$ with

$$\|v_s\| = \sqrt{s} \quad \text{and} \quad \|Lv_s\| = o(\|v_s\|). \tag{3.14}$$

we first claim that if $\psi \in \mathcal{E}$, then $T[U\psi]$ satisfies

$$T[U\psi] \text{ is even in } x_k,\ k = 2, 3 \tag{3.15}$$

and

$$T[U\psi](r\cos\theta, r\sin\theta, x_3) = T[U\psi]\left(r\cos\left(\theta + \frac{2\pi i}{s}\right), r\sin\left(\theta + \frac{2\pi i}{s}\right), x_3\right). \tag{3.16}$$

Indeed, let $h = (h_{ij}) \in SO(3)$ and $h = \text{diag}(1, -1, -1)$. Then, obviously, $U \in \mathcal{E}$, which implies $U(x) = U(hx)$, so is ψ. Therefore,

$$T[U\psi](hx) = \int_{\mathbb{R}^3} \frac{U(y)\psi(y)}{|hx - y|} dy = \int_{\mathbb{R}^3} \frac{U(h^{-1}y)\psi(h^{-1}y)}{|x - h^{-1}y|} dy = \int_{\mathbb{R}^3} \frac{U(y)\psi(y)}{|x - y|} dy = T[U\psi](x),$$

which is equivalent to (3.15). A similar argument leads to (3.16).

In view of the definition of (3.10), we deduce from (3.14) and symmetry that

$$\begin{aligned} \frac{1}{k}\langle Lv_s, u\rangle &= \int_{\Omega_1} (\nabla v_s \nabla u + V(|x|)v_s u) - \int_{\Omega_1} \left(T\left[U_r^2\right] v_s u + 2T[U_r u]U_r v_s\right) \\ &= o\left(\frac{\|u\|}{\sqrt{k}}\right), \ \forall u \in \mathcal{E}_s \end{aligned} \tag{3.17}$$

and

$$\int_{\Omega_1} (|\nabla v_s|^2 + V(|x|)v_s^2) = 1. \tag{3.18}$$

Since

$$Cs^{\frac{m}{1-m}} \leq |\xi_2 - \xi_1| \leq |\xi_i - \xi_1|, \quad i \neq 1.$$

We have that $B_R(\xi_1) \subset \Omega_1$ for any $R > 0$. Let $\tilde{v}_s(x) = v_s(x + \xi_1)$. Then by (3.18),

$$\int_{B_R(0)} (|\nabla \tilde{v}_s|^2 + V(|x|)\tilde{v}_s^2) \leq 1.$$

Therefore, we may assume that there exists a $v \in H^1(\mathbb{R}^3)$, up to a subsequence, such that as $s \to \infty$,

$$\tilde{v}_s \to v, \quad \text{weakly in } H^1_{loc}(\mathbb{R}^3) \text{ and strogly in } L^2_{loc}(\mathbb{R}^3).$$

From $v_s \in \mathcal{E}_s$ one derives that

$$\begin{aligned} \int_{\mathbb{R}^3} T\left[U_{\xi_1} Z_1\right] U_{\xi_1} v_s &= \int_{\mathbb{R}^3} T\left[U \frac{\partial U}{\partial y_1}\right](x - \xi_1) U_{\xi_1}(x) v_s(x) \\ &= \int_{\mathbb{R}^3} T\left[U \frac{\partial U}{\partial y_1}\right] U \tilde{v}_s. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^3} T\left[U_{\xi_1}^2\right] Z_1 v_s = \int_{\mathbb{R}^3} T\left[U^2\right] \frac{\partial U}{\partial x_1} \tilde{v}_s.$$

Moreover, $\tilde{v}_s$ is even in $x_k,\ k=2,3$. Hence, v is also even in $x_k,\ k=2,3$ and satisfies

$$\int_{\mathbb{R}^3} \left(T\left[U^2\right] \frac{\partial U}{\partial x_1} + 2T\left[U \frac{\partial U}{\partial y_1}\right] U\right) v = 0. \tag{3.19}$$

We shall show that v solves

$$-\Delta v + v = T\left[U^2\right] v + 2T\left[Uv\right] U. \tag{3.20}$$

Indeed, define the space

$$\bar{\mathcal{E}} = \left\{\phi \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(T\left[U^2\right] \frac{\partial U}{\partial x_1} + 2T\left[U \frac{\partial U}{\partial y_1}\right] U\right) \phi = 0\right\}.$$

For any $R > 0$, let

$$\phi \in \mathcal{C}_0^\infty(B_R(0)) \cap \bar{\mathcal{E}} \text{ and be even in } x_k,\ k = 2,3$$

and denote $\phi_s(x) := \phi(x+\xi_1)$. Then $\phi_s \in \mathcal{C}_0^\infty(B_R(0))$. We may identify $\phi_s \in \mathcal{E}_s$ by redefining the values outside Ω_1 with the symmetry. Following the arguments in [31], by (3.17) and similar arguments in (3.13), we obtain

$$\int_{\mathbb{R}^3} \left(\nabla v \nabla \phi + v\phi - T\left[U^2\right] v\phi - 2T[Uv]U\phi\right) = 0. \tag{3.21}$$

Moreover, since v is even in $x_k,\ k=2,3$, with the help of (3.15), it is easily shown that (3.21) holds for any function $\phi \in \mathcal{C}_0^\infty(B_R(0))$, ϕ odd in $x_k,\ k=2,3$. Therefore, for any $\phi \in \mathcal{C}_0^\infty(B_R(0)) \cap \bar{\mathcal{E}}$, one gets (3.21). Furthermore,

$$\int_{\mathbb{R}^3} \left(\nabla v \nabla \phi + v\phi - T\left[U^2\right] v\phi - 2T[Uv]U\phi\right) = 0, \quad \forall \phi \in \bar{\mathcal{E}} \tag{3.22}$$

due to the density of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$.

On the other hand, (3.22) is true for $\phi = \frac{\partial U}{\partial x_1}$. Putting these facts together, we obtain (3.22) for any $\phi \in H^1(\mathbb{R}^3)$. Namely, (3.20) holds.

Using the evenness in $x_k,\ k=2,3$ of v again and Theorem 2.1-(2), we derive that $v = c\frac{\partial U}{\partial x_1}$. By (3.19), we further have

$$v = 0.$$

Hence,

$$\int_{B_R(\xi_1)} v_s^2 = o_s(1), \quad \forall R > 0. \tag{3.23}$$

It is indicated by (3.12) that

$$U_r(x) \leq C e^{-(1-\delta)|x-\xi_1|}, \quad x \in \Omega_1. \tag{3.24}$$

Inserting $u = v_s$ in (3.17), we conclude from (2.4), (3.23) and (3.24) that

$$\begin{aligned} o(1) &= \int_{\Omega_1} (|\nabla v_s|^2 + V(|x|)v_s^2) - \int_{\Omega_1} \left(T\left[U_r^2\right] v_s^2 + 2T[U_r v_s]U_r v_s\right) \\ &\geq \int_{\Omega_1} (|\nabla v_s|^2 + V(|x|)v_s^2) + o(1) - Ce^{-(1-\delta)R} \int_{\Omega_1} v_s^2 \\ &\geq \frac{1}{2} \int_{\Omega_1} (|\nabla v_s|^2 + V(|x|)v_s^2) + o(1) \\ &= \frac{1}{2} + o(1), \end{aligned}$$

which is impossible for large s and large R. So we complete the proof. □

We now give the existence of the critical point for F.

Lemma 3.3. *There is an integer $s_0 > 0$ such that for each $s \geq s_0$, there exists a $\mathcal{C}^1$ map*

$$I_s \to \mathcal{E},\ r \mapsto w(r)$$

with $w(r) \in \mathcal{E}_s$ and

$$\langle \frac{\partial F(w)}{\partial w}, \phi \rangle = 0, \quad \forall \phi \in \mathcal{E}_s.$$

Moreover, there is a small constant $\beta > 0$ such that

$$\|w\| \leq C \left(\frac{1}{s}\right)^{\frac{2m-1}{2(1-m)}+\beta}. \tag{3.25}$$

Proof. It follows from Riesz Theorem that there is a $e_s \in \mathcal{E}_s$ satisfying

$$E(w) = (e_s, w) \text{ and } \|e_s\| = \|E\|_{\mathcal{L}(\mathcal{E}_s, \mathcal{E}_s)}.$$

Consequently, w, a critical point of F in $\mathcal{E}_s$, solves

$$e_s + \mathcal{L}w + N'(w) = 0. \tag{3.26}$$

It follows from Lemma 3.2 that $\mathcal{L}$ is invertible. Furthermore, we can write (3.26) as follows

$$w = P(w) := -\mathcal{L}^{-1}\left(e_s + N'(w)\right).$$

Let

$$M := \left\{ \varphi \in \mathcal{E}_s : \|\varphi\| \leq C \left(\frac{1}{s}\right)^{\frac{2m-1}{2(1-m)}+\beta} \right\}.$$

We claim that P is a contraction map from M to M. Indeed, in view of (3.6), we readily get

$$\|N'(\varphi)\| \leq C\|\varphi\|^2 \text{ and } \|N''(\varphi)\| \leq C\|\varphi\|.$$

We derive from Lemmas 3.1–3.2 that

$$\|P(\varphi)\| \leq C\|e_s\| + C\|\varphi\|^2 \leq C \left(\frac{1}{s}\right)^{\frac{2m-1}{2(1-m)}+\beta}.$$

Furthermore,

$$\|P(\varphi_1) - P(\varphi_2)\| \le C\|N'(\varphi_1) - N'(\varphi_2)\| \le C\left(\frac{1}{s}\right)^{\frac{2m-1}{2(1-m)}+\beta} \|\varphi_1 - \varphi_2\| \\ \le \frac{1}{2}\|\varphi_1 - \varphi_2\|$$

for s large. The proof is complete. □

Define

$$\mathcal{F}(r) = E(U_r + w), \quad \forall r \in I_s,$$

where I_s is defined in (2.7). It is easily checked that for s sufficiently large, if r is a critical point of $\mathcal{F}$, then $U_r + w$ is a solution of (1.7), see for example [23] or [8, 11].

Proof of Theorem 1.2. We conclude from (3.3) and Lemmas 2.3, 3.1 and 3.3 that

$$\begin{aligned} \mathcal{F}(r) &= E(U_r) + E(w) + \frac{1}{2}\langle Lw, w\rangle + N(w) \\ &= E(U_r) + O(\|E\|_{\mathcal{L}(\mathcal{E}_s,\mathcal{E}_s)}) \\ &= s\left[\frac{A_2}{16\pi} + \frac{aA_1}{2}\frac{1}{r^m} - \frac{A_1^2}{128\pi^2}\frac{s\log s}{r} + O\left(\frac{1}{s^{\frac{m}{1-m}+\beta}}\right)\right] \end{aligned} \tag{3.27}$$

Consider the following maximization problem

$$\max_{r\in I_r} \mathcal{F}(r). \tag{3.28}$$

It is easy to check that the function

$$g(r) = \frac{aA_1}{2}\frac{1}{r^m} - \frac{A_1^2}{128\pi^2}\frac{s\log s}{r} \tag{3.29}$$

has a maximum point

$$r_s = \left(\left(\frac{A_1}{64am\pi^2}\right)^{\frac{1}{1-m}} + o(1)\right)(s\log s)^{\frac{1}{1-m}}. \tag{3.30}$$

It follows from the expression of $\mathcal{F}(r)$ that the maximizer r_s is an interior point of I_r. Thus, system (1.6) admits a positive solution $u_s = U_{r_s} + w_s$ with the required properties. □

We conclude this section with the following remark about the assumption (H) on the potential V.

Remark 3.1. *Observe that when $0 < m < \frac{1}{2}$, we deduce from Lemmas 2.3, 3.1 and 3.3 that*

$$\begin{aligned}\mathcal{F}(r) &= E(U_r) + E(w) + \frac{1}{2}\langle Lw, w\rangle + N(w) \\ &= E(U_r) + O(\|E\|_{\mathcal{L}(\mathcal{E}_s,\mathcal{E}_s)}) \\ &= s\left[\frac{A_2}{16\pi} + \frac{aA_1}{2}\frac{1}{r^m} - \frac{A_1^2}{128\pi^2}\frac{s\log s}{r}\right] + c\frac{s^2}{r^{2m}}.\end{aligned}$$

If we consider the function

$$\bar{g}(r) = -\frac{A_1^2}{128\pi^2}\frac{s\log s}{r} + c\frac{s}{r^{2m}},$$

a direct calculation shows that $\bar{g}$ has a maximum point

$$\bar{r}_s = C(\log s)^{\frac{1}{m}}. \tag{3.31}$$

Note that

$$\lim_{r\to\infty}\frac{1/r^m}{s/r^{2m}} = \lim_{r\to\infty}\frac{r^m}{s} = 0$$

and one has to center the spikes on a circle of radius

$$r \in [(C-\alpha)(\log s)^{\frac{1}{m}}, (C+\alpha)(\log s)^{\frac{1}{m}}].$$

However, the distance between two spikes is

$$|\xi_1 - \xi_2| = 2r\sin\frac{\pi}{s} \sim \frac{2\pi r}{s} \to 0, \ \text{as } s \to \infty. \tag{3.32}$$

Therefore, it seems challenging to construct $s-$bump solutions for (1.6) *in this situation. A similar phenomenon happens for $m \geq 1$.*

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Yeyao Hu, School of Mathematics and Statistics, The Central South University, Changsha, Hunan 410083, P. R. China
Email address: huyeyao@csu.edu.cn

Aleks Jevnikar, Department of Mathematics, Computer Science and Physics, University of Udine, Via delle Scienze 206, 33100 Udine, Italy
Email address: aleks.jevnikar@uniud.it

Weihong Xie, School of Mathematics and Statistics, The Central South University, Changsha, Hunan 410083, P. R. China
Email address: xieweihong0218@163.com