Transport and control problems with boundary costs : regularity and summability of optimal and equilibrium densities

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À mon frère Ali…

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Résumé: Une première partie de cette thèse est dédiée à l'étude de la régularité de la densité de transport σ dans le problème de Monge entre deux mesures f^+ et f^- sur un domaine Ω. Tout d'abord, on étudie la question de la sommabilité L^p de cette densité de transport entre une mesure f^+ et sa projection sur le bord $(P_{\partial\Omega})_\# f^+$, qui ne découle pas en fait des résultats connus (dus à De Pascale - Evans - Pratelli - Santambrogio) sur la densité de transport entre deux densités L^p, comme dans notre cas la mesure cible est singulière. Par une méthode de symétrisation, dès que Ω est convexe ou satisfait une condition de boule uniforme extérieure, nous prouvons les estimations L^p (si $f^+ \in L^p$, alors $\sigma \in L^p$). En plus, nous analysons le cas où on paye des coûts supplémentaires $g^\pm$ sur le bord, en prouvant que la densité de transport σ est dans L^p dès que $f^\pm \in L^p$, Ω satisfait une condition de boule uniforme extérieure et, $g^\pm$ sont $\lambda^\pm-$Lipschitiziens avec $\lambda^\pm < 1$ et semi-concaves. Ensuite, on s'attaque à la régularité d'ordre supérieur ($W^{1,p}$, $C^{0,\alpha}$, BV $\cdots$) de la densité de transport σ entre deux densités régulières f^+ et f^-. Plus précisément, nous fournissons une famille de contre-exemples à la régularité supérieure: nous prouvons que la régularité $W^{1,p}$ des mesures source et cible, f^+ et f^-, n'implique pas que la densité de transport est $W^{1,p}$, de même pour la régularité BV, et même $f^\pm \in C^\infty$ n'implique pas que σ est dans $W^{1,p}$, pour p grand. Ensuite, nous étudions la sommabilité L^p de la densité de transport entre deux mesures f^+ et f^- concentrées sur le bord. Plus précisément, nous prouvons que si f^+ et f^- sont dans $L^p(\partial\Omega)$, alors la densité de transport σ entre eux est dans $L^p(\Omega)$ dès que Ω est uniformément convexe et $p \leq 2$; de plus, nous introduisons un contre-exemple montrant que ce résultat n'est plus vrai si $p > 2$. Cela fournit des résultats de régularité $W^{1,p}$ sur la solution u du problème de gradient minimal avec donnée au bord g dans des domaines uniformément convexes (si $g \in W^{1,p}(\partial\Omega) \Rightarrow u \in W^{1,p}(\Omega)$).

Dans une deuxième partie, nous étudions un problème de contrôle optimal motivé par un modèle de jeux à champ moyen. D'abord, nous montrons des résultats de différentiabilité et semi-concavité sur la fonction valeur associée au problème de contrôle (le résultat de semi-concavité est optimal en ce qui concerne les hypothèses sur la régularité en temps). Ensuite, nous démontrons que la densité des agents ρ_t, dans le modèle MFG considéré, est dans L^p dès que la densité initiale $\rho_0 \in L^p$. En plus, nous arrivons à prouver l'existence d'un équilibre pour le problème MFG considéré dans un cas où la dynamique n'est pas régulière.

Dernièrement, nous considérons le problème stationnaire associé au problème MFG. Nous montrons que la densité d'équilibre n'est rien d'autre que la densité de transport entre une densité source f et sa projection sur le bord en utilisant une métrique Riemannienne non-uniforme comme coût de transport. Cela nous permet de démontrer que la densité d'équilibre ρ est dans L^p dès que la densité source $f \in L^p$. Par conséquent, nous arrivons à prouver aussi l'existence d'un équilibre stationnaire dans un cas où la dynamique n'est pas régulière.

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Introduction

Gaspard Monge a proposé en 1781 un problème, *Mémoire sur la théorie des déblais et des remblais* [93], qui, dans ses divers développements, suscite toujours un intérêt profond au sein d'une vaste communauté dans divers domaines de mathématiques. Son idée était de considérer un tas de sable (le déblai), représenté par f^+, et un trou (le remblai), représenté par f^-, du même volume, et il voulait trouver comment déplacer les sables de la pile au trou en minimisant le travail effectué. Nous pouvons formaliser ce problème dans une terminologie moderne comme suit: les données sont deux densités, f^+ et f^-, définies sur une région Ω, qui doivent être considérées comme la hauteur de la pile et la profondeur du trou. Un moyen de déplacer la masse est une fonction $T : \Omega \mapsto \Omega$, et le fait que ce soit effectivement un moyen de déplacer les sables dans le trou peut être exprimé par la condition $T_\# f^+ = f^-$, ce qui signifie que $\int_{T^{-1}(A)} f^+(x)\,\mathrm{d}x = \int_A f^-(y)\,\mathrm{d}y$ pour tout ensemble Borelien $A \subset \Omega$. Puisque, selon la formulation de Monge, le coût du déplacement d'une masse unitaire du point x au point y est la distance Euclidienne $|x-y|$, on peut se rendre compte que le coût total du transport correspondant à l'application T est

$$\int_\Omega |x - T(x)|\,\mathrm{d}f^+. \tag{0.1}$$

Le problème du transport de masse consiste alors à trouver la fonction T (appelée *application de transport optimale*) qui minimise (0.1) parmi toutes les applications de transport. L'existence des applications optimales a été abordée par de nombreux auteurs [1], [32], [58], [101] et [110] (voir aussi [42] pour un résultat plus général, qui est valable pour des normes arbitraires $||x-y||$).

Bien que ce problème pourrait ne pas avoir aucune solution, son relaxation (qui est le problème de Kantorovich [73]) en a toujours, au moins, une. Le problème relaxé consiste à trouver une mesure de Borel λ sur $\Omega \times \Omega$ (applée *plan de transport optimal*) satisfaisant $(\Pi_x)_\# \lambda = f^+$ et $(\Pi_y)_\# \lambda = f^-$, où Π_x, $\Pi_y : \Omega \times \Omega \mapsto \Omega$ sont les projections sur le premier et le second facteur, respectivement, qui minimise la fonctionnelle

$$\int_{\Omega \times \Omega} |x - y|\,\mathrm{d}\lambda$$

parmi toutes les mesures Boreliennes λ sur $\Omega \times \Omega$ satisfaisant $(\Pi_x)_\# \lambda = f^+$ et $(\Pi_y)_\# \lambda = f^-$. En fait, sous l'hypothèse que f^+ est absolument continue par rapport à la mesure de Lebesgue $\mathcal{L}^d$, les problèmes de Monge et Kantorovich sont équivalents, au sens que toute application de transport T telle que $T_\# f^+ = f^-$ induit un plan de transport $\lambda = (Id, T)_\# f^+$ et que, parmi les plans optimaux λ, il en existe un qui a cette forme (au contraire, il n'y a pas d'unicité, et d'autres plans de transport optimaux pourraient être de formes différentes). Pour plus des détails sur la théorie du transport optimal, son histoire et les principaux résultats, nous nous référons aussi à [**103**] et [**112**].

Dans l'analyse du problème de transport optimal ci-dessus, un outil clé consiste en la dualité convexe. En effet, il est possible de prouver que la maximisation de la fonctionnelle suivante

$$\int_\Omega u \, \mathrm{d}(f^+ - f^-)$$

parmi toutes les fonctions 1-Lipschitziennes u sur Ω, est le dual du problème de Kantorovich: il peut être obtenu à partir de problème primal par une procédure d'échange inf-sup appropriée, sa valeur est égale au minimum du problème du Kantorovich, et, pour tout plan de transport λ et pour toute fonction $u \in \mathrm{Lip}_1(\Omega)$, on a

$$\int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\lambda \geq \int_{\Omega\times\Omega} (u(x)-u(y)) \, \mathrm{d}\lambda = \int_\Omega u(x) \, \mathrm{d}f^+(x) - \int_\Omega u(y) \, \mathrm{d}f^-(y) = \int_\Omega u \, \mathrm{d}(f^+ - f^-).$$

L'égalité des deux valeurs optimales implique que les solutions λ et u satisfont $u(x) - u(y) = |x-y|$ sur le support de λ (un segment $[x, y]$ maximal qui satisfait cette égalité sera nommé *un rayon de transport*), mais aussi que, à chaque fois que nous trouvons un plan de transport λ et une fonction $u \in \mathrm{Lip}_1$ satisfaisant $\int |x-y| \, \mathrm{d}\lambda = \int u \, \mathrm{d}(f^+ - f^-)$, elles sont toutes les deux optimales. Les maximiseurs dans le problème dual sont appelés *potentiels de Kantorovich*.

Dans une telle théorie, il est classique d'associer à un plan de transport optimal λ une mesure positive σ sur Ω, appelée *densité de transport*, qui représente la quantité de transport effectuée dans chaque région de Ω. Cette mesure σ est définie par

$$< \sigma, \varphi >= \int_{\Omega\times\Omega} \mathrm{d}\lambda(x,y) \int_0^1 \varphi(\omega_{x,y}(t)) |\omega'_{x,y}(t)| \, \mathrm{d}t \quad \forall \ \varphi \ \in \ C(\Omega)$$

où $\omega_{x,y}$ est une courbe paramétrant le segment reliant x à y. En d'autres termes, on a

$$\sigma(A) = \int_{\Omega\times\Omega} \mathcal{H}^1(A \cap [x,y]) \, \mathrm{d}\lambda(x,y) \quad \text{pour tout ensemble Borelien } A \subset \Omega$$

où $\mathcal{H}^1$ représente la mesure de Hausdorff 1-dimensionnelle. Cela signifie que $\sigma(A)$ représente "combien" le transport a lieu dans A, si les particules passent de leur origine x à leur destination y en ligne droite. Le rôle de cette mesure est très important: elle a été utilisée par exemple pour donner l'une des premières preuves d'existence d'une application de transport optimal T pour le problème de Monge [**58**], mais également en optimisation de forme [**14**].

Nous rappelons ici certaines propriétés de σ.

Proposition 0.1. *Supposons $f^+ \ll \mathcal{L}^d$. Dans ce cas, la densité de transport σ est unique (c.à.d. ne dépend pas du choix du plan de transport optimal λ) et $\sigma \ll \mathcal{L}^d$. En plus, si f^+ est dans $L^p(\Omega)$, avec $p < d/(d-1)$, alors σ est aussi dans $L^p(\Omega)$. Et, si f^+, f^- sont les deux dans $L^p(\Omega)$, alors σ appartient également à $L^p(\Omega)$.*

Ces propriétés sont bien connues dans la littérature, et nous nous référons à [48], [50], [51], [59] et [102]. La densité de transport σ apparaît également dans le problème de Beckmann suivant [9]

$$\min\left\{\int_\Omega |w|\,\mathrm{d}x:\ w\in\mathcal{M}^d(\Omega),\ \nabla\cdot w = f^+ - f^- \ \text{ dans }\ \bar\Omega\right\}, \tag{0.2}$$

où $\nabla\cdot w = f^+ - f^-$ dans $\bar\Omega$ est équivalent à dire que $\int\nabla\phi\cdot\mathrm{d}w + \int\phi\,\mathrm{d}(f^+ - f^-) = 0$ pour toute $\phi\in C^1(\bar\Omega)$. La relation entre ce problème et le problème de Kantorovich peut être considérée comme une conséquence de la dualité convexe. En effet, si l'on utilise la version duale de la contrainte de divergence, on peut obtenir un problème dual en interchangeant inf et sup:

$$\sup_u\left\{\int_\Omega u\,\mathrm{d}(f^+ - f^-) + \inf_w\left(\int_\Omega |w|\,\mathrm{d}x + \int_\Omega \nabla u\cdot\mathrm{d}w\right)\right\}$$

devient

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+ - f^-)\ :\ |\nabla u|\le 1\right\}.$$

Il suffit alors d'observer que la condition $|\nabla u|\le 1$ est équivalente à $u\in\mathrm{Lip}_1$ (en supposant que Ω est convexe) pour revenir au problème de Monge-Kantorovich.

En fait, il est possible de démontrer que le champ vectoriel w donné par $w = -\sigma\nabla u$, où u est un potentiel de Kantorovich, est une solution de problème de minimisation ci-dessus. Aussi, il est possible de prouver (voir, par exemple, [103, théorème 4.13]) que tous les minimiseurs de ce dernier problème sont de cette forme, et que donc le minimiseur est unique dès que $f^+ \ll \mathcal{L}^d$.

Les conditions d'optimalité primale-duale dans les problèmes ci-dessus peuvent également être écrites sous la forme d'une EDP: σ résout, avec le potentiel de Kantorovich u, le système de Monge-Kantorovich suivant

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f^+ - f^- & \text{dans }\ \Omega,\\ \sigma\nabla u\cdot\mathbf{n} = 0 & \text{sur }\ \partial\Omega,\\ |\nabla u|\le 1 & \text{dans }\ \Omega,\\ |\nabla u| = 1 & \sigma - \text{p.p.}\end{cases} \tag{0.3}$$

Dans le cadre de la congestion du trafic et du renforcement des membranes, voir [30], les auteurs utilisent une variante de ce problème, déjà présente dans [13] et [29], où le système de Monge-Kantorovich (0.3) est complété par une condition de Dirichlet au bord. La version la plus simple du système devient

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f^+ & \text{dans } \ \Omega, \\ u = 0 & \text{sur } \ \partial\Omega, \\ |\nabla u| \leq 1 & \text{dans } \ \Omega, \\ |\nabla u| = 1 & \sigma - \text{p.p.} \end{cases} \tag{0.4}$$

En termes de transport optimal, cela correspond au problème de transport vers le bord, c.à.d. on a une densité f^+ à l'intérieur de Ω et on la transporte vers le bord de manière optimale. Plus précisément, on veut étudier le problème suivant

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\lambda \,:\, \lambda\in\mathcal{M}^+(\Omega\times\Omega),\, (\Pi_x)_\#\lambda = f^+ \ \text{ et } \ (\Pi_y)_\#\lambda\subset\partial\Omega\right\}.$$

Puisque la mesure $(\Pi_y)_\#\lambda$ sur $\partial\Omega$ est complètement arbitraire, il est clair que le choix optimal est de la prendre égale à $P_\# f^+$, où

$$P(x) = \operatorname{argmin}\{|x-y|,\ y\in\partial\Omega\} \ \text{ pour tout } \ x\in\Omega.$$

Cela signifie que nous allons considérer le problème suivant

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\lambda,\ \lambda\in\Pi(f^+, P_\# f^+)\right\},$$

ce qui revient aussi à résoudre

$$\min\left\{\int_\Omega |w|\mathrm{d}x \,:\, w\in L^1(\Omega,\mathbb{R}^d),\ \nabla\cdot w = f^+ \ \text{ dans } \ \mathring{\Omega}\right\}. \tag{0.5}$$

Nous pouvons exprimer la contrainte dans un sens faible en testant contre les fonctions $u\in C^1_c(\Omega)$ (ou C^1 s'annulant sur $\partial\Omega$), et le dual de ce problème devient

$$\sup\left\{\int_\Omega u\,\mathrm{d}f^+ \ :\ u\in\mathrm{Lip}_1(\Omega),\, u=0 \ \text{ sur } \ \partial\Omega\right\}.$$

Dans le chapitre 3, nous serons principalement concernés par la régularité de la densité de transport σ, entre f^+ et $P_\# f^+$, en termes de la régularité de f^+. En fait, on pourrait se demander si la densité de transport σ est dans L^p ou pas, quand $f^+ \in L^p$. Notons qu'on ne pourrait pas utiliser la proposition 0.1 (pour $p \geq d/(d-1)$), puisque dans ce cas la mesure cible $P_\# f^+$ est concentrée sur le bord de Ω et est donc singulière. Cependant, nous verrons plus loin (au Chapitre 3) que le même résultat L^p sera également vrai, par une technique de symétrisation. Plus précisément, on a le résultat suivant:

Proposition 0.2. *La densité de transport σ entre f^+ et $P_\# f^+$ est dans $L^p(\Omega)$ dès que $f^+ \in L^p(\Omega)$ et sous l'hypothèse que Ω satisfait une condition de boule uniforme extérieure.*

La preuve (voir Chapitre 3) se base sur une technique de symétrisation; en fait, si Ω est un polyèdre, σ est égal à la restriction à Ω d'une densité de transport entre f^+ et une nouvelle densité f^- obtenue en symétrisant f^+ à travers les faces composant la frontière $\partial\Omega$. Un argument similaire peut être effectué pour les domaines avec des faces "rondes" (appelés *round polyhedra*) et, par un argument d'approximation, pour des domaines arbitraires satisfaisant une condition de boule uniforme extérieure.

La condition de la boule uniforme extérieure garantit que si $f^+ \in L^\infty(\Omega)$, alors $P_\# f^+$ a une densité bornée par rapport à la mesure de Hausdorff sur $\partial\Omega$. Donc, on pourrait se demander si cette dernière condition est la bonne hypothèse pour obtenir la sommabilité L^∞ de la densité de transport: si $f^+ \in L^\infty(\Omega)$ et $f^- \in L^\infty(\partial\Omega)$, est-il vrai que la densité de transport entre ces deux mesures est dans $L^\infty(\Omega)$?

Or, nous donnerons, au Chapitre 3, un exemple où $f^+ \in L^\infty(\Omega)$ et $f^- \in L^\infty(\partial\Omega)$ mais la densité de transport entre f^+ et f^- n'est pas dans $L^\infty(\Omega)$. En d'autres termes, si $\sigma(f^+, f^-)$ désigne la densité de transport entre f^+ et f^-, alors on a les assertions suivantes:

$$f^+ \in L^\infty(\Omega) \Rightarrow \sigma(f^+, P_\# f^+) \in L^\infty(\Omega),$$

$$f^+ \in L^\infty(\Omega),\ f^- \in L^\infty(\partial\Omega) \not\Rightarrow \sigma(f^+, f^-) \in L^\infty(\Omega).$$

Une généralisation du problème de transport vers le bord peut être obtenue quand on ajoute des coûts sur le bord [**90**]. En d'autres termes, nous voulons transporter une certaine quantité de matériel représentée par f^+, dans Ω, (f^+ encode la quantité de matériau et son emplacement) vers un trou avec une distribution donnée par f^-, également définie dans Ω. Le but est de transporter toute la masse de f^+ vers f^- ou bien vers le bord (c.à.d. exporter la masse de f^+ à l'extérieur). En faisant cela, nous payons le coût de transport donné par la distance Euclidienne $|x-y|$ et quand une unité de masse est sortie à travers un point $y \in \partial\Omega$, un coût supplémentaire donné par $g^-(y)$, la taxe d'exportation. Nous avons également la contrainte de remplir le trou complètement, c.à.d. que nous devons importer de la masse, si nécessaire, de l'extérieur de Ω en payant les frais de transport plus un coût supplémentaire $-g^+(x)$, la taxe d'importation, pour chaque unité de masse qui pénètre à travers un point $x \in \partial\Omega$. Nous avons la liberté de choisir d'exporter ou d'importer de la masse, à condition qu'on transporte toute la masse f^+ et qu'on couvre aussi toute la masse de f^-. L'objectif principal ici est de minimiser le coût total de cette opération, qui est donné par le coût de transport plus les taxes d'exportation/importation. Notons que dans ce problème de transport il en a deux masses sur le bord qui sont inconnues (qui encodent la masse exportée et la masse importée). Notons également que la condition d'équilibre de masse

$$\int_\Omega f^+(x)\,\mathrm{d}x = \int_\Omega f^-(y)\,\mathrm{d}y$$

n'est pas demandée car nous pouvons importer ou exporter de la masse à travers la frontière si nécessaire. En d'autres termes, nous considérons le problème suivant

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\mathrm{d}\lambda + \int_{\partial\Omega} g^-\mathrm{d}(\Pi_y)_\#\lambda - \int_{\partial\Omega} g^+\mathrm{d}(\Pi_x)_\#\lambda : ((\Pi_x)_\#\lambda)_{|\mathring{\Omega}} = f^+, ((\Pi_y)_\#\lambda)_{|\mathring{\Omega}} = f^-\right\}.$$

Nous pouvons démontrer que ce problème est équivalent au problème suivant

$$\min\left\{\int_\Omega |w|\,\mathrm{d}x + \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \,:\, w\in L^1(\Omega,\mathbb{R}^d),\, \chi^\pm\in\mathcal{M}^+(\partial\Omega),\, \nabla\cdot w = f+\chi\right\}.$$

D'autre part, on peut prouver que le dual de ce problème est

$$\sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-) \,:\, u\in \mathrm{Lip}_1(\Omega),\; g^+\le u\le g^- \;\text{ sur }\; \partial\Omega\right\}.$$

En plus, le système (0.4), dans ce cas, sera complété par la condition $g^+ \le u \le g^-$ sur $\partial\Omega$, c.à.d., (0.4) devient

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{dans }\ \Omega,\\ g^+\le u\le g^- & \text{sur }\ \partial\Omega,\\ |\nabla u|\le 1 & \text{dans }\ \Omega,\\ |\nabla u| = 1 & \sigma-\text{p.p.}\end{cases} \tag{0.6}$$

Dans le chapitre 4, nous serions intéressés à étudier la sommabilité L^p de la densité de transport σ associé à ce problème, qui ne découle pas, en fait, de la proposition 0.1, puisque à nouveau les mesures cibles ne sont pas dans L^p car elles ont des parties concentrées sur $\partial\Omega$. Rappelons-nous que dans le chapitre 3 nous prouvons que si $g^+ = g^- = 0$, alors la densité de transport σ est dans L^p à condition que $f\in L^p$ et Ω satisfait une condition de boule uniforme extérieure. Notre but, dans le chapitre 4, sera, donc, de prouver le même résultat L^p que dans le chapitre 3, mais cette fois pour des coûts plus généraux g^+ et g^-.

Pour ce faire, l'idée sera de décomposer un plan de transport optimal λ comme une somme de trois plans de transport λ_{ii}, λ_{ib} et λ_{bi}, où chacun de ces plans résout un problème de transport particulier. Plus précisément, si λ est un plan de transport optimal et si ν^+ représente une partie de f^+ qu'on va exporter et ν^- une partie de f^- pour laquelle on va importer une masse de l'extérieure, alors on pourra décomposer le plan λ en trois parties: λ_{ii} c'est le plan qui transporte $f^+ - \nu^+$ vers $f^- - \nu^-$, λ_{ib} transporte le reste de la masse de f^+, c.à.d. ν^+, vers le bord (c.à.d. on exporte la masse ν^+), λ_{bi} qui importe une masse de l'extérieure pour remplir la masse qui reste de f^-, c.à.d. ν^-. En fait, le plan λ_{ii} résout le problème suivant

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\lambda\,:\,\lambda\in\Pi(f^+-\nu^+,f^--\nu^-)\right\},$$

le plan λ_{ib} résout

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\lambda+\int_{\partial\Omega}g^-\,\mathrm{d}\chi^-\,:\,\lambda\in\Pi(\nu^+,\chi^-),\,\mathrm{spt}(\chi^-)\subset\partial\Omega\right\}$$

et λ_{bi} minimise

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\lambda-\int_{\partial\Omega}g^+\,\mathrm{d}\chi^+\,:\,\lambda\in\Pi(\chi^+,\nu^-),\,\mathrm{spt}(\chi^+)\subset\partial\Omega\right\}.$$

Ensuite, nous étudierons la sommabilité L^p des densités de transport σ_{ii}, σ_{ib} et σ_{bi} associées à ces plans de transport λ_{ii}, λ_{ib} et λ_{bi}, respectivement. De cette façon, nous obtenons la sommabilité de la densité de transport σ associée au plan de transport optimal λ. En fait, la densité σ_{ii} ne pose pas de problèmes car λ_{ii} est un plan de transport optimal entre deux densités L^p (qui sont $f^+ - \nu^+$ et $f^- - \nu^-$) et donc, $\sigma_{ii} \in L^p$. Par contre, on voit que ce n'est pas le cas pour σ_{ib} et σ_{bi}, puisque, dans ces deux cas, le plan de transport aura lieu entre une densité L^p et une mesure singulière concentrée sur le bord (qui n'est pas donc, dans L^p). L'étude de la sommabilité de σ_{bi} est assez similaire à celui de σ_{ib} et, donc, il suffit d'étudier la sommabilité de σ_{ib}.

En fait, on pourra voir facilement que le choix optimal pour exporter la masse ν^+ à l'extérieure est de la transporter vers $T_\#\nu^+$ où T est définie comme suit:

$$T(x)=\mathrm{argmin}\{|x-y|+g^-(y)\,:\,y\in\partial\Omega\}\quad\text{pour tout }\ x.$$

En particulier, λ_{ib} résout

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\lambda\,:\,\lambda\in\Pi(\nu^+,T_\#\nu^+)\right\}.$$

D'abord, nous supposons que notre domaine Ω est un *round polyhedron* de rayon r. Dans ce cas, sous l'hypothèse que $g^- \in C^2(\partial\Omega)$ avec $|\nabla g^-| < 1$, nous démontrons que le Jacobien J_t de l'application $T_t := (1-t)I + tT$ satisfait l'estimation suivante

$$J_t \geq C(1-t)$$

où C est une constante strictement positive qui dépend de d, r, diam(Ω), $||\nabla g^-||_\infty$ et de la borne supérieure de D^2g^-.

Cette estimation sur le Jacobien J_t nous sera suffisante pour obtenir la sommabilité L^p de la densité de transport σ_{ib}. Plus précisément, nous arriverons à démontrer l'estimation

$$||\sigma_{ib}||_{L^p(\Omega)} \leq C||\nu^+||_{L^p(\Omega)},$$

où C est une constante qui dépend seulement de d, r, diam(Ω), $||\nabla g^-||_\infty$ et de la borne supérieure de D^2g^-. Ensuite, nous généralisons ce résultat à chaque domaine ayant une boule uniforme extérieure.

Proposition 0.3. *La densité de transport σ entre ν^+ et $T_\#\nu^+$ est dans $L^p(\Omega)$ dès que $\nu^+ \in L^p(\Omega)$, sous l'hypothèse que Ω satisfait une condition de boule uniforme extérieure et le coût g^- est $\lambda-$Lip, avec $\lambda < 1$, et semi-concave.*

Dans le chapitre 5, nous nous intéresserons à l'étude de la régularité d'ordre supérieur ($W^{1,p}$, $C^{0,\alpha}$ et BV) pour la densité de transport σ entre deux densités régulières f^+ et f^-. Ce problème est ouvert et difficile. Le seul résultat connu est contenu dans [61], mais ne concerne que le cas de la dimension 2 et exige des hypothèses très restrictives sur f^+ et f^- (densités continues sur des supports convexes disjoints et bornées inférieurement sur leur supports). Dans leur papier, l'objectif principal est la preuve de la continuité de l'application optimale T (qui n'est pas unique, et le résultat concerne donc une application de transport privilégiée, celle qui est monotone sur chaque rayon de transport) et la continuité de la densité de transport σ n'est qu'un sous-produit de l'analyse developpée pour T. Récemment, une nouvelle stratégie de preuve a été proposée dans [84], basée sur les estimations de Ma-Trudinger-Wang [110], pour prouver la continuité de T. Hélas, le résultat n'est pas complet: voulant prouver que T est Lipschitz, [84] n'arrive qu'à démontrer que les valeurs propres de DT sont bornées mais, DT n'étant pas symétrique, cela ne permet pas de conclure, et un contre-exemple est même proposé. Pourtant, dans ce contre-exemple T est $C^{0,\alpha}$, ce qui laisse ouverte la question de la continuité de T. Or, dans [45], les auteurs arrivent à construire deux densités f^+ et f^- $\alpha-$Höldériennes tel que le transport optimal monotone T entre eux n'est pas $\alpha-$Höldérien, c.à.d., on a l'assertion suivante

$$f^\pm \in C^{0,\alpha} \nRightarrow T \in C^{0,\alpha}, \ \ \forall \ \alpha \in (0,1).$$

Mais, cela n'implique pas que la densité de transport entre f^+ et f^- n'est pas régulière puisque toute application de transport optimal T produit la même densité de transport σ, ce qui nous permet, alors, de choisir la plus régulière parmi eux pour étudier la régularité de σ. Par conséquent, la question de régularité $C^{0,\alpha}$ ou Lipschitz de la densité de transport σ reste ouverte.

D'autre part, dans [83], les auteurs prouvent la continuité de l'application de transport optimal monotone T sous l'hypothèse que f^+ et f^- soient deux densités positives, continues avec $\mathrm{spt}(f^+) \subset \mathrm{spt}(f^-)$ et l'un des ensembles $\{f^+ > f^-\}$, $\{f^- > f^+\}$ soit convexe (la densité de transport σ est également continue dans ce cas). D'autres résultats existent en ce qui concerne la régularité de la densité de transport dans certaines directions: dans [58], il a été prouvé que lorsque $f^{\pm}$ sont Lipschitz continues avec des supports disjoints (et avec des conditions techniques supplémentaires sur les supports), alors la densité de transport est localement Lipschitzienne "le long des rayons de transport". Dans [31], les auteurs ont généralisé le résultat au cas où les densités f^+ et f^- sont seulement dans L^p, sans aucunes conditions sur les supports; ils prouvent que si $f^{\pm} \in L^p(\Omega)$, alors pour presque tout $x \in \Omega$, la densité de transport $\sigma \in W^{1,p}_{loc}(R_x)$, où R_x est le rayon de transport passant par x. Or, comme on a l'assertion

$$f^{\pm} \in L^p \Rightarrow \sigma \in L^p,$$

on pourrait se demander si l'assertion suivante est correcte ou pas

$$f^{\pm} \in W^{1,p} \Rightarrow \sigma \in W^{1,p}\,?$$

Pour des problèmes de congestion de trafic, [39] a introduit une généralisation de la densité de transport, appelée intensité de trafic, qui donne lieu à des problèmes d'équilibre dans un jeu à potentiel représentant les choix des agents dans un domaine congestionné. Ensuite, [21] a montré l'équivalence du modèle choisi avec le modèle proposé par Beckmann dans [9]. Il s'agit de résoudre

$$\min\left\{\int_{\Omega} F(w(x))\,dx \ :\ w : \Omega \mapsto \mathbb{R}^d,\, \nabla \cdot w = f\right\},$$

où $F : \mathbb{R}^d \mapsto \mathbb{R}$ est une fonction convexe, superlinéaire, et telle que $F(w) \geq |w|$ (pour représenter le fait que le coût de transport augmente en présence de congestion). Le w optimal peut être identifié à l'aide d'une EDP elliptique dégénérée: en effet, on vérifie facilement que $\nabla F(w)$ doit être un gradient, et on trouve donc $\nabla \cdot (\nabla F^*(\nabla u)) = f$, où F^* est la transformé de Flenchel de F. Le problème de cette équation, de type $p-$Laplacien, est que $F^* = 0$ sur la boule unité, ce qui en fait une équation très dégénérée. La question de la régularité de $w = \nabla F^*(\nabla u)$ a été étudiée dans [21] (bornes L^∞ et H^1) et dans [44, 104] (continuité de w), sous des hypothèses de non-dégénerescence de F^* en dehors de la boule. Or, pour des modèles de trafic basés sur l'homogénéisation de modèles de réseaux [6], d'autres choix de F sont nécessaires, et cela donne $F^*(z) = \sum_{i=1}^{d}(|z_i| - 1)_+^p$. Le problème dans ce cas est extrêment plus difficile, du fait de la dégénerescence sur une zone non bornée. Brasco et ses collaborateurs ont travaillé en profondeur sur ce problème, en obtenant des résultats intéressants (mais durs) dans [18] et [20]. D'autres approches ont aussi été utilisées dans [47].

Pour étudier la régularité supérieure de la densité de transport σ, ou de façon équivalente la régularité du flot optimal w dans le cas où $F(w) = |w|$, une idée sera d'étudier le cas $F_\varepsilon(w) = |w| + \varepsilon|w|^2$ et de faire tendre ε vers 0. Malheureusement, les estimations H^1, par exemple, sur les flots optimaux w_ε ne passent pas à la limite et, on ne peut rien dire à propos de la régularité du flot optimal w.

En effet, nous donnerons, au Chapitre 5, une famille de contre-exemples aux régularités supérieures de la densité de transport σ. En particulier, nous démontrons que la régularité $W^{1,p}$ des densités f^+ et f^- n'implique pas que la densité de transport σ soit dans $W^{1,p}$ aussi, c.à.d., on a l'assertion suivante

$$f^\pm \in W^{1,p} \not\Rightarrow \sigma \in W^{1,p},\ \ \forall\ p > 1.$$

En plus, on a

$$f^\pm \in C^{0,\alpha} \not\Rightarrow \sigma \in C^{0,\alpha},\ \ \forall\ \alpha \in (0,1).$$

Proposition 0.4. *On a les assertions suivantes:*

$$f^\pm \in BV(\Omega) \quad \not\Rightarrow \quad \sigma \in BV(\Omega), \tag{0.7}$$

$$\forall\ p > 1,\, \varepsilon > 0,\ \ f^\pm \in W^{1,p}(\Omega) \quad \not\Rightarrow \quad \sigma \in W_{loc}^{1,\frac{2p+\varepsilon}{p+1}}(\Omega), \tag{0.8}$$

$$f^\pm \in C^1(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in H^1(\Omega), \tag{0.9}$$

$$f^\pm \in W^{1,\infty}(\Omega) \quad \not\Rightarrow \quad \sigma \in H^1_{loc}(\Omega), \tag{0.10}$$

$$\forall\ \alpha \in (0,1),\ \ f^\pm \in C^{1,\alpha}(\Omega) \quad \not\Rightarrow \quad \sigma \in W_{loc}^{1,2+\alpha}(\Omega), \tag{0.11}$$

$$f^\pm \in C^{2,1}(\Omega) \quad \not\Rightarrow \quad \sigma \in W_{loc}^{1,3}(\Omega), \tag{0.12}$$

$$f^\pm \in C^{\infty}(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in W^{1,3}(\Omega), \tag{0.13}$$

$$f^\pm \in C^{\infty}(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in W_{loc}^{1,5}(\Omega), \tag{0.14}$$

$$\forall\ \alpha \in (0,1),\, \varepsilon > 0,\ \ f^\pm \in C^{0,\alpha}(\Omega) \quad \not\Rightarrow \quad \sigma \in C_{loc}^{0,\frac{\alpha}{\alpha+2}+\varepsilon}(\Omega), \tag{0.15}$$

$$\forall\ \varepsilon > 0,\ \ f^\pm \in C^1(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{1}{3}+\varepsilon}(\Omega), \tag{0.16}$$

$$\forall\ \varepsilon > 0,\ \ f^\pm \in C^{0,1}(\Omega) \quad \not\Rightarrow \quad \sigma \in C_{loc}^{0,\frac{1}{3}+\varepsilon}(\Omega), \tag{0.17}$$

$$\forall\ \alpha \in (0,1),\, \varepsilon > 0,\ \ f^\pm \in C^{1,\alpha}(\Omega) \quad \not\Rightarrow \quad \sigma \in C_{loc}^{0,\frac{1+\alpha}{3+\alpha}+\varepsilon}(\Omega), \tag{0.18}$$

$$\forall \varepsilon > 0,\ \ f^\pm \in C^{2,1}(\Omega) \quad \not\Rightarrow \quad \sigma \in C_{loc}^{0,\frac{1}{2}+\varepsilon}(\Omega), \tag{0.19}$$

$$\forall \varepsilon > 0,\ \ f^\pm \in C^{\infty}(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{1}{2}+\varepsilon}(\Omega), \tag{0.20}$$

$$\forall \varepsilon > 0,\ \ f^\pm \in C^{\infty}(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in C_{loc}^{0,\frac{2}{3}+\varepsilon}(\Omega). \tag{0.21}$$

Les contre-exemples que nous allons construire seront inspirés par [**45, 84**]. Plus précisément, pour produire de tels contre-exemples, l'idée sera de fixer un réel $\gamma > 0$ et de considérer les rayons de transport $(l_a)_{a\in(0,1)}$ où chaque rayon l_a est défini comme suit

$$x_2 = \frac{a^\gamma}{2}(x_1 + a), \;\; x_1 \in [-a, 1].$$

En plus, la densité f^+ sera donnée tandis que la densité f^- sera à choisir de façon à ce que les rayons de transport entre f^+ et f^- seront exactement les segments $(l_a)_a$. Ce réel $\gamma > 0$ joue un grand rôle dans la construction des contre-exemples: en fait, on voudra choisir à chaque fois un γ convenable pour obtenir un contre-exemple à la régularité $W^{1,p}$ (pour un certain p), $C^{0,\alpha}$ (pour un certain α) ou BV. En particulier, pour faire un contre-exemple à la régularité $W^{1,p}$, pour un $p \to 1$, on devra choisir un $\gamma \to +\infty$. La régularité $W^{1,p}$ ($C^{0,\alpha}$ ou BV) de la fonction f^- dépend, éventuellement, du choix de paramètre γ. Donc, pour chaque $\gamma > 0$, on voudrait vérifier que f^- est $W^{1,p}$ et pourtant, σ ne l'est pas.

Dans le chapitre 6, nous nous intéressons à une nouvelle application des estimations L^p sur la densité de transport. Il s'agit d'étudier la régularité $W^{1,p}$ de la solution, en 2D, du problème du gradient minimal [**108, 64**]

$$\min\left\{\int_\Omega |\nabla u|\,\mathrm{d}x \;:\; u \in BV(\Omega),\, u_{|\partial\Omega} = g\right\}, \tag{0.22}$$

quand Ω est un domaine uniformément convexe, $u_{|\partial\Omega}$ désigne la trace de u et $g : \partial\Omega \mapsto \mathbb{R}$ est une fonction L^1 donnée. Tout d'abord, nous rappelerons la connection entre (0.22) et le problème de Beckmann (voir aussi [**65**])

$$\inf\left\{\int_\Omega |w|\,\mathrm{d}x \;:\; w \in L^1(\Omega, \mathbb{R}^2),\, \nabla\cdot w = 0 \;\text{ dans }\; \mathring{\Omega},\, w\cdot\mathbf{n} = f \;\text{ sur }\; \partial\Omega\right\}, \tag{0.23}$$

où f est la dérivée tangentielle de g (i.e., $f = \partial g/\partial\tau$, τ est la tangente au bord de Ω) et $\mathbf{n}$ est la normale extérieure au $\partial\Omega$. Plus précisément, il est possible de prouver que si u est une solution de (0.22), alors le champ $w = R_{\frac{\pi}{2}}\nabla u$ résout (0.23), où $R_{\frac{\pi}{2}}$ désigne une rotation avec angle $\frac{\pi}{2}$ autour de l'origine. Or, ce problème (0.23) est aussi équivalent au problème de Monge-Kantorovich entre deux mesures, f^+ et f^-, concentrées sur le bord:

$$\min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\lambda \;:\; \lambda \in \Pi(f^+, f^-)\right\}, \tag{0.24}$$

où $f = f^+ - f^-$. D'autre part, on rappelle que si λ est un minimiseur du (0.24), alors le champ de vecteur w_λ donné par

$$< w_\lambda, \xi >:= \int_{\Omega\times\Omega} \mathrm{d}\lambda(x,y) \int_0^1 \xi((1-t)x+ty)\cdot(y-x)\,\mathrm{d}t,\ \ \forall\ \xi \in C(\Omega,\mathbb{R}^2)$$

est un minimiseur de (0.23). En plus, tout minimiseur de (0.23) vient d'un minimiseur λ de (0.24). Pourtant, nous ne savons pas, par exemple, si ce minimiseur est unique ou pas, parce que les seuls résultats connus à propos de l'unicité de w_λ nécessite qu'au moins l'un des deux f^+ ou f^- soit dans $L^1(\Omega)$, ce qui n'est pas le cas ici comme f^+ et f^- sont concentrées sur le bord. Cependant, nous sommes en mesure de prouver que si Ω est strictement convexe, si f^+ ou f^- est non-atomique, et s'ils n'ont pas de masse commune, alors il existe un unique plan de transport optimal λ et donc, un unique minimiseur w pour (0.23).

Par conséquent, étudier la régularité $W^{1,p}$ de la solution u revient à étudier la sommabilité L^p du flot optimal w. Rappelons-nous que les seuls résultats connus à propos de la sommabilité L^p du flot optimal w demandent qu'au moins une mesure entre f^+ et f^- soit dans $L^p(\Omega)$ (voir Proposition 0.1). La sommabilité L^p du w dans le cas où on transporte une mesure f^+, concentrée sur le bord, vers une autre f^-, concentrée sur le bord aussi, n'est pas connue. En particulier, si $f^\pm \in L^p(\partial\Omega)$, est-il vrai que le flot optimal w entre f^+ et f^- est dans $L^p(\Omega,\mathbb{R}^2)$?

En fait, en utilisant un argument d'approximation par des mesures atomiques, nous montrons que si $f^\pm \in L^p(\partial\Omega)$, alors le flot minimal w est dans $L^p(\Omega,\mathbb{R}^2)$, à condition que Ω soit uniformément convexe et $p \le 2$.

PROPOSITION 0.5. *Si Ω est uniformément convexe, alors, pour une donnée au bord g dans $W^{1,p}(\partial\Omega)$, la solution u est dans $W^{1,p}(\Omega)$, pour tout $p \le 2$.*

D'autre part, par un contre-exemple, nous montrons que ce résultat ne reste plus valable si $p > 2$. Plus précisément, on a un contre-exemple où f^+ et f^- sont dans $L^\infty(\partial\Omega)$, mais le flot optimal n'est pas dans $L^p(\Omega,\mathbb{R}^2)$, pour tout $p > 2$.

PROPOSITION 0.6. *On a l'assertion suivante*

$$g \in \mathrm{Lip}(\partial\Omega) \nRightarrow u \in W^{1,p}(\Omega),\ \ \forall\ p > 2.$$

Par des estimations du même type, on voit que l'hypothèse classique $g \in C^{1,1}(\partial\Omega)$ donnera $u \in \mathrm{Lip}(\Omega)$. Plus généralement, nous prouvons que si $g \in C^{1,\alpha}(\partial\Omega)$, alors $u \in W^{1,\frac{2}{1-\alpha}}(\Omega)$.

À partir du Chapitre 7 nous passons à un sujet de recherche différent, en lien avec la théorie récente des MFG [79, 80, 81, 67, 68, 69, 28, 87, 7, 8, 33, 38, 63, 78, 11, 34]. On verra, pourtant, que plusieurs liens apparaissent avec les sujets développés dans les chapitres précédents. Comme point de départ, la théorie des MFG était très liée au contrôle optimal [37, 43]. Dans le chapitre 7, nous considérons un problème de sortie d'un domaine en temps minimal. Le temps terminal des trajectoires n'est pas fixe, mais c'est le premier auquel elles atteignent le bord de Ω. Plus précisément, pour chaque $x_0 \in \Omega$ et $t_0 \in \mathbb{R}^+$, on considère la trajectoire $\gamma^{t_0,x_0,u}$ solution de

$$\begin{cases} \gamma'(t) = k(t,\gamma(t))\, u(t), & t \geq t_0, \\ \gamma(t_0) = x_0, \end{cases}$$

où $u : [t_0,\infty) \mapsto \bar{B}(0,1)$ est une fonction mesurable et $k : \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}^+$ est une fonction donnée (appelée *dynamique*). De plus, on se donne une fonction $g : \partial\Omega \mapsto \mathbb{R}^+$ définie sur le bord. Le but est de minimiser ce coût

$$\tau^{t_0,x_0,u} + g(\gamma_\tau^{t_0,x_0,u})$$

parmi tous les controles u, où $\tau^{t_0,x_0,u}$ est le premier instant pour lequel la trajectoire $\gamma^{t_0,x_0,u}$ touche le bord en un point noté $\gamma_\tau^{t_0,x_0,u}$.

Premièrement, nous démontrons l'existence d'un contrôle optimal u sous certaines hypothèses sur k et g. D'autre part, si φ est la fonction valeur associée à notre problème de contrôle optimal, c.à.d.,

$$\varphi(t,x) = \inf_u \{\tau^{t,x,u} + g(\gamma_\tau^{t,x,u})\}, \;\; \forall\; (t,x) \in \mathbb{R}^+ \times \Omega,$$

alors φ est une solution de viscosité du problème

$$\begin{cases} -\,\partial_t\varphi(t,x) \,+\, k(t,x)|\nabla\varphi(t,x)| = 1, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ \varphi(t,x) = g(x), & (t,x) \in \mathbb{R}^+ \times \partial\Omega. \end{cases}$$

Nous démontrons aussi que cette fonction valeur φ est Lipschitzienne sur $\mathbb{R}^+ \times \Omega$. De plus, il est possible de prouver que $\partial_t\varphi \geq c-1$, pour un certain $c > 0$, ce qui est équivalent à une borne inférieure $|\nabla\varphi| \geq c > 0$.

D'autre part, nous analysons quelques conditions d'optimalité pour notre problème de contrôle. Notre objectif sera d'obtenir plus de régularité sur les trajectoires optimales. En particulier, nous prouvons que si u est un contrôle optimal, alors u est Lipschitzien, ce qui est équivalent à dire que la trajectoire optimale γ est $C^{1,1}$. Aussi, nous prouvons la suivante

Proposition 0.7. *Si* $\gamma : [t_0, t_0 + \tau_\gamma] \mapsto \Omega$ *est une trajectoire optimale pour* $(t_0, \gamma(t_0))$ *(où* $\tau_\gamma := \tau^{t_0,\gamma(t_0),u}$ *et* u *est le contrôle optimal correspondant), alors la fonction valeur* φ *est différentiable en* $(t, \gamma(t))$*, pour tout* $t \in (t_0, t_0 + \tau_\gamma)$.

Par conséquent, on aura l'égalité

$$\gamma'(t) = -k(t,\gamma(t))\,\frac{\nabla\varphi(t,\gamma(t))}{|\nabla\varphi(t,\gamma(t))|},\ \ \forall\ t \in (t_0, t_0 + \tau_\gamma).$$

D'autre part, nous allons raffiner le résultat de semi-concavité donné dans [37] en montrant qu'au lieu de supposer que la dynamique k est $C^{1,1}$ en (t, x), seule une borne inférieure sur $\partial_t k$ (tout en gardant l'hypothèse $C^{1,1}$ en x) est suffisante pour obtenir la semi-concavité de φ par rapport à x. Pour gérer la dépendance en temps, nous devrons renforcer aussi la régularité du bord. Plus précisément, on a

Proposition 0.8. *Si* $\partial_t k \geq -c$, $|\nabla^2_x k| \leq C$ *et* $\partial\Omega$ *est* $C^{1,1}$*, alors la fonction valeur* φ *est semi-concave par rapport à* x.

On rappelle que si la dynamique k ne dépend pas du temps, alors une condition d'une boule uniforme extérieure (au lieu d'une hypothèse $C^{1,1}$ sur $\partial\Omega$) est suffisante pour avoir la semi-concavité de la fonction valeur φ.

Dans le chapitre 8, nous étudierons un problème de jeux à champ moyen où on a une densité d'agents, représentée par ρ_0, dans un domaine Ω, et le but de chaque agent est de quitter le domaine Ω à travers son bord $\partial\Omega$ en temps minimal (ou de façon plus générale en minimisant un coût qui est supposé être donné par le temps nécessaire pour atteindre un point de sortie éventuel z plus un coût sur le bord $g(z)$ au point z). Afin de prendre en compte des phénomènes de congestion, nous supposons que la vitesse maximale de chaque agent est bornée par une dynamique k, c.à.d.,

$$|\gamma'(t)| \leq k(\rho_t, \gamma(t)),$$

où $\gamma(t)$ donne la position de l'agent à chaque instant t, ρ_t est l'évolution de la densité ρ_0 au temps t et $k : \mathcal{P}(\Omega) \times \Omega \mapsto \mathbb{R}^+$ est une fonction de congestion donnée. Nous donnerons une formulation Lagrangienne de ce probème. Il s'agit de décrire l'évolution des agents par une mesure η sur l'ensemble $\mathcal{C}(\mathbb{R}^+, \Omega)$ de trajectoires possibles sur Ω. En fait, pour chaque $x \in \Omega$, on considère le problème suivant

$$\inf\left\{\tau_\gamma + g(\gamma(\tau_\gamma))\ :\ \gamma(0) = x,\ |\gamma'(t)| \leq k((e_t)_\#\eta, \gamma(t))\ \text{ p.p. } t,\ \gamma(t) = \gamma(\tau_\gamma)\ \ \forall\ t > \tau_\gamma\right\},$$

où

$$\tau_\gamma := \inf\{s \geq 0\ :\ \gamma(s) \in \partial\Omega\}.$$

Une mesure η est appelée équilibre si son image par l'évaluation au temps $t = 0$ coïncide avec la distribution initiale ρ_0 et $\eta-$presque toute trajectoire γ minimise le coût $\tau_\gamma + g(\gamma(\tau_\gamma))$ parmi toutes les trajectoires admissibles qui démarrent de $\gamma(0)$. Nous montrerons l'existence d'un équilibre η, sous l'hypothèse que la dynamique k est continue en (ρ, x) et Lipschitzienne par rapport à x, en reformulant cette notion en termes d'un problème de point fixe (voir [87]).

Nous donnerons ensuite une caractérisation de l'équilibre, en montrant que la distribution d'agents ρ_t satisfait une équation de continuité dont le champ de vitesse dépend du gradient de la fonction valeur φ du problème de contrôle associé au problème de jeux à champ moyen considéré. Cette équation de continuité sur ρ, satisfaite au sens des distributions, sera couplée avec une équation de Hamilton–Jacobi sur φ, satisfaite au sens de viscosité. Plus précisément, nous montrerons que, sous des hypothèses convenables sur la dynamique k, $\rho : t \mapsto \rho_t$ et φ sont solutions du système suivant

$$
\begin{cases}
\partial_t \rho(t,x) - \nabla \cdot \left(\rho(t,x)\, k(\rho_t, x) \dfrac{\nabla \varphi(t,x)}{|\nabla \varphi(t,x)|} \right) = 0, & (t,x) \in (0,\infty) \times \Omega, \\
-\partial_t \varphi(t,x) + k(\rho_t, x) |\nabla \varphi(t,x)| = 1, & (t,x) \in \mathbb{R}^+ \times \Omega, \\
\rho(0,x) = \rho_0(x), & x \in \Omega, \\
\varphi(t,x) = g(x), & (t,x) \in \mathbb{R}^+ \times \partial\Omega.
\end{cases}
\tag{0.25}
$$

D'autre part, sous des hypothèses de régularité sur le domaine Ω et la dynamique k, on a ce qui suit

Proposition 0.9. *Si ρ_0 est une densité dans $L^p(\Omega)$, alors la restriction de ρ_t à $\mathring{\Omega}$ est aussi absolument continue et à densité dans $L^p(\Omega)$ pour tout $t \geq 0$, avec un contrôle de la norme L^p de la densité de ρ_t par celle de la densité de ρ_0, c.à.d., il existe une constante C tel que*

$$||\rho_t||_{L^p(\Omega)} \leq e^{Ct} ||\rho_0||_{L^p(\Omega)}, \;\; \forall \; t \in \mathbb{R}^+.$$

Ces estimations L^p seront très utiles pour démontrer l'existence d'un équilibre dans le cas où la dynamique k est définie comme suit

$$k(\rho, x) = c\left(\int_\Omega \chi(x-y) 1_{\mathring{\Omega}}(y) \, \mathrm{d}\rho(y) \right), \;\; \forall \; (\rho, x) \in \mathcal{P}(\Omega) \times \Omega.$$

La signification de cette dynamique est que chaque agent évalue une densité moyenne des agents autour de lui à travers le terme intégral, χ étant un noyau de convolution et l'indicatrice nous permet de ne pas prendre en compte les agents ayant déjà quitté le domaine, et qui restent sur $\partial\Omega$. Sa vitesse maximale dépend de cette évaluation de la densité à travers une fonction c, qui est supposée être décroissante.

Pour ce faire, l'idée sera d'approcher la dynamique k par des dynamiques plus régulières k_ε, où k_ε est supposée être de la forme

$$k_\varepsilon(\rho,x) = c\bigg(\int_\Omega \chi(x-y)\psi^\varepsilon(y)\,\mathrm{d}\rho(y)\bigg),\ \ \forall\ (\rho,x)\in\mathcal{P}(\Omega)\times\Omega.$$

Ici ψ^ε est une fonction cut-off qui converge vers $1_{\mathring{\Omega}}$ quand $\varepsilon\to 0$. Si η^ε est un équilibre associé à la dynamique k_ε, alors $\eta^\varepsilon \rightharpoonup \eta$ où η sera un équilibre associé à la dynamique k. Cela découle du fait que les estimations L^p sur $\rho_t^\varepsilon := 1_{\mathring{\Omega}}\cdot(e_t)_\#\eta^\varepsilon$ sont uniformes en ε, ce qui permet de montrer la convergence uniforme de la fonction valeur φ_ε, associée au problème de contrôle avec la dynamique k_ε, à la fonction valeur φ, associée avec la dynamique k.

Dans le chapitre 9, nous étudions le problème de jeux à champ moyen stationnaire de (0.25) avec une source f. En d'autres termes, nous considérons d'abord le même problème qu'auparavant avec l'ajout du fait qu'à chaque instant t, une densité additionnelle f (independente de t) entre dans le jeu. Dans ce cas, le système (0.25) serait

$$\begin{cases}\partial_t\rho(t,x)-\nabla\cdot\left(\rho(t,x)\,k(\rho_t,x)\dfrac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|}\right)=f, & (t,x)\in(0,\infty)\times\Omega,\\ -\partial_t\varphi(t,x)+k(\rho_t,x)|\nabla\varphi(t,x)|=1, & (t,x)\in\mathbb{R}^+\times\Omega,\\ \rho(0,x)=\rho_0(x), & x\in\Omega,\\ \varphi(t,x)=0, & (t,x)\in\mathbb{R}^+\times\partial\Omega.\end{cases}\tag{0.26}$$

De ce système, on considère la version stationnaire, qui est la suivante

$$\begin{cases}-\nabla\cdot\left(\rho\,k(\rho,\cdot)\,\frac{\nabla\varphi}{|\nabla\varphi|}\right)=f & \text{dans}\ \ \Omega,\\ k(\rho,\cdot)\,|\nabla\varphi|=1 & \text{dans}\ \ \Omega,\\ \varphi=0 & \text{sur}\ \ \partial\Omega.\end{cases}\tag{0.27}$$

Tout d'abord, nous voulons étudier l'existence d'un équilibre pour le problème stationnaire associé à (0.27). En fixant $T>0$ suffisamment grand, nous posons $\rho^\eta := \int_0^T (e_t)_\#\eta\,\mathrm{d}t$, pour toute mesure η sur $C(\mathbb{R}^+,\Omega)$. Pour tout $x\in\Omega$, nous considérons le problème

$$\min\Big\{\tau_\gamma\ :\ \gamma(0)=x,\ |\gamma'(t)|\le k(\rho^\eta,\gamma(t))\ \ \text{p.p.}\ \ t\in(0,\tau_\gamma),\ \gamma(t)=\gamma(\tau_\gamma)\in\partial\Omega\ \ \forall\ t>\tau_\gamma\Big\}.$$

Encore une fois, η est un équilibre pour le problème stationnaire si son image par l'évaluation au temps $t = 0$ est égal à f et η est concentrée sur l'ensemble des courbes optimales, c.à.d. $\eta-$p.p. γ est une courbe optimale pour $\gamma(0)$. Dans le cas où la dynamique k est régulière, on peut démontrer, comme au Chapitre 8, l'existence d'un équilibre η en utilisant une méthode du point fixe.

D'autre part, on verra que le système (0.27) n'est rien d'autre que celui de Monge-Kantorovich pour le problème de transport optimal entre la densité f et la frontière en présence d'une métrique non-uniforme d_c, où $c = k^{-1}$. En d'autres termes, nous considérons le problème de transport

$$\min\left\{\int_\Omega d_c(x,y)\,\mathrm{d}\lambda \;:\; \lambda \in \mathcal{M}^+(\Omega\times\Omega),\, (\Pi_x)_\#\lambda = f,\, (\Pi_y)_\#\lambda \subset \partial\Omega\right\} \tag{0.28}$$

où

$$d_c(x,y) = \inf\left\{\int_0^1 c(\gamma(t))|\gamma'(t)|\,\mathrm{d}t \;:\; \gamma \in C^1([0,1],\Omega),\, \gamma(0) = x \;\text{ et }\; \gamma(1) = y\right\},\ \forall\, x,\, y \in \Omega.$$

Puisque la mesure $(\Pi_y)_\#\lambda$ sur $\partial\Omega$ est complètement arbitraire, alors il est clair que le choix optimal est de la prendre égale à $P_\# f$, où

$$P(x) = \mathrm{argmin}\,\{d_c(x,y),\ y \in \partial\Omega\} \quad \text{pour tout } \; x \in \Omega,$$

ce qui signifie que $\lambda := (Id, P)_\# f$ est l'unique plan de transport optimal pour (0.28), qui est également le même que

$$\min\left\{\int_{\Omega\times\Omega} d_c(x,y)\,\mathrm{d}\lambda \;:\; \lambda \in \Pi(f, P_\# f)\right\}. \tag{0.29}$$

Ce qui est aussi équivalent au problème suivant

$$\max\left\{\int_\Omega u\,\mathrm{d}f \;:\; |\nabla u| \le c,\ u = 0 \;\text{ sur }\; \partial\Omega\right\}. \tag{0.30}$$

Maintenant, nous voulons généraliser la notion de la densité de transport au cas où le coût de transport n'est pas la distance Euclidienne, mais c'est plutôt la distance géodésique avec un poids c. Dans ce cas, les rayons de transport seront des géodésiques (et pas forcément des segments). Nous posons alors

$$\sigma := \int_{\Omega\times\Omega} \mathcal{H}^1 \llcorner \gamma_{x,y}\,\mathrm{d}\lambda(x,y),$$

où $\gamma_{x,y}$ est une géodésique reliant x à y. De façon équivalente,

$$< \sigma, \phi >= \int_{\Omega\times\Omega} \mathrm{d}\lambda(x,y) \int_0^1 \phi(\gamma_{x,y}(t))\, |\gamma'_{x,y}(t)| \mathrm{d}t \quad \forall\ \phi\ \in\ C(\Omega).$$

D'autre part, le problème de Beckmann (0.2) devient

$$\min \left\{ \int_\Omega c\, \mathrm{d}|w| \ :\ w \in \mathcal{M}^d(\Omega),\ \nabla\cdot w = f \ \text{ dans }\ \mathring{\Omega} \right\}. \tag{0.31}$$

En plus, si on pose

$$< w, \xi >= \int_{\Omega\times\Omega} \mathrm{d}\lambda(x,y) \int_0^1 \xi(\gamma_{x,y}(t)) \cdot \gamma'_{x,y}(t)\, \mathrm{d}t, \quad \forall\ \xi\ \in\ C(\Omega, \mathbb{R}^d),$$

alors, la mesure vectorielle w résout (0.31). La version la plus compliquée du système (0.4) devient

$$\begin{cases} -\nabla\cdot\left(\sigma\,\frac{\nabla u}{|\nabla u|}\right) = f & \text{dans }\ \Omega, \\ u = 0 & \text{sur }\ \partial\Omega, \\ |\nabla u| \le c & \text{dans }\ \Omega, \\ |\nabla u| = c & \sigma - \text{p.p.} \end{cases} \tag{0.32}$$

La question que nous considérons maintenant est de savoir si la densité de transport σ, dans (0.32), de f à $P_{\#}f$ (ou de manière équivalente, le champ de vecteur optimal w dans (0.31)) est dans $L^p(\Omega)$ quand $f \in L^p(\Omega)$. Pour cette raison, nous démontrons que le Jacobien J_t de l'application $x \mapsto P_t(x)$, où $P_t(x)$ est le point de la géodésique entre x et $P(x)$ situé à une distance $(1-t)\, d_c(x, \partial\Omega)$ du bord, est borné inférieurement par une constante strictement positive C, multipliée par un facteur $(1-t)$, c.à.d. on a

$$J_t \ge C(1-t),$$

dès que Ω et c sont lisses. Grâce à cette borne, nous arriverons à démontrer l'estimation suivante

$$||\sigma||_{L^p(\Omega)} \le C||f||_{L^p(\Omega)},$$

où C est une constante qui dépend seulement de d, diam(Ω), $c_{\min}$, $c_{\max}$, $||\nabla c||_\infty$, $||D^2 c||_\infty$ et de la borne inférieure de la courbure de $\partial\Omega$. Donc, on a le résultat suivant

PROPOSITION 0.10. *La densité de transport σ entre f et $P_\# f$ est dans $L^p(\Omega)$ dès que $f \in L^p(\Omega)$ et, sous les hypothèses que Ω satisfait une condition de boule uniforme extérieure et que c soit $C^{1,1}$.*

Revenons au problème de jeux à champ moyen stationnaire, nous observons que la densité d'équilibre ρ n'est rien d'autre que la densité de transport σ entre f et $P_\# f$. Et donc, on a

$$||\rho||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)}.$$

Ces estimations L^p seront aussi très utiles pour généraliser le résultat d'existence d'un équilibre pour le problème stationnaire au cas où la dynamique k est moins régulière. Plus précisément, nous démontrons l'existence d'un équilibre stationnaire dans le cas où k est définie comme suit

$$k(\rho, x) = h\left(\int_\Omega \chi(x-y) 1_{\overset{\circ}{\Omega}}(y)\, \mathrm{d}\rho(y) \right), \ \forall\ (\rho, x) \in \mathcal{P}(\Omega) \times \Omega,$$

exactement comme ce qu'on a fait au Chapitre 8 dans le cas évolutif. La différence est que ici les hypothèses sont sur le terme source f et pas sur la donnée initiale ρ_0 qui n'a pas lieu d'être dans un problème stationnaire.

Le lecteur pourra remarquer que le traitement de la donnée au bord g, les estimations L^p, et la notion de la densité de transport sont finalement le point clé et le fil conducteur de la thèse, y compris dans la partie MFG.

CHAPTER 1

Preliminaries on the Monge-Kantorovich problem

Let Ω be a compact domain in $\mathbb{R}^d$ and f^+, f^- be two finite non-negative Borel measures on Ω with the same total mass; i.e. f^+, $f^- \in \mathcal{M}^+(\Omega)$ and $f^+(\Omega) = f^-(\Omega)$. The goal of the transport problem is to move f^+ onto f^-: this means, roughly speaking, that one needs a map specifying where to move the mass. Taking then any Borel map $T : \Omega \mapsto \Omega$, we try to understand what should mean to consider it as a way to transport the distribution of mass given by f^+: the idea is that one should move all the mass which is in a point x into the point $T(x)$. Yet, this simple "pointwise" point of view is not formally correct unless f^+ is the sum of countably many point masses, but it leads to the correct idea that the mass which will be in any Borel set $A \subset \Omega$ after the movement is $f^+(T^{-1}(A))$. Since we want to move f^+ on f^-, it should happen that this mass equals $f^-(A)$. Following this intuitive argument, a Borel map $T : \Omega \mapsto \Omega$ is said to be a *transport map* from f^+ to f^- if $T_\# f^+ = f^-$, where the push-forward is defined as

$$T_\# f^+ \,(A) := f^+(T^{-1}(A)) \quad \text{for all Borel set } A \subset \Omega.$$

Once we know what a transport map is, we are interested in finding a cheapest one; to explain what this means, we need to consider a continuous cost function $c(x,y)$ (typically, we take $c(x,y) = |x-y|$): its meaning is that the cost of moving a unit mass from the point $x \in \Omega$ to the point $y \in \Omega$ is $c(x,y)$. Recalling the meaning of the mass following the transport map T, it is natural to consider, as the cost of the transport T, the quantity

$$\int_\Omega c(x, T(x))\,\mathrm{d}f^+. \tag{1.1}$$

The task will be to find an *optimal transport map*, which is a transport map minimizing the quantity (1.1) among all the transport maps from f^+ to f^-. In other words, we consider the following problem, which is already introduced by Monge in [**93**],

$$\text{(MP)} \qquad \inf\left\{\int_\Omega c(x,T(x))\,\mathrm{d}f^+ \,:\, T_\# f^+ = f^-\right\}.$$

Yet, even if this problem is quite easy to state, it is not easy at all to solve it and in general, it may also have no solutions (to see that, just consider the case where $f^+ = \delta_x$, for some $x \in \Omega$, and f^- is any probability measure different than a Dirac mass; in this case no transport map can exist).

However, the first key ideas for studying the Monge problem are due to Kantorovich [73, 74] in the 1940's: he proposed to consider as admissible ways to move the mass all the measures γ defined in $\Omega \times \Omega$ admitting f^+ and f^- as marginals; each of these measures will be called *transport plan*. The meaning of this definition is to allow the splitting of masses; roughly speaking, consider the mass contained in a point x: according to Monge's formulation, it should be entirely moved to the point $T(x)$, while Kantorovich's idea is to distribute it in Ω more freely, provided that the final distribution of the points results to be the target measure f^-. More precisely, the Kantorovich problem is the following

$$\text{(KP)} \qquad \min\left\{\int_{\Omega\times\Omega} c(x,y)\,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+,f^-)\right\}$$

where

$$\Pi(f^+,f^-) = \left\{\gamma \in \mathcal{M}^+(\Omega\times\Omega) \,:\, (\Pi_x)_\#\gamma = f^+,\, (\Pi_y)_\#\gamma = f^-\right\}$$

and Π_x, Π_y are the two projections of $\Omega \times \Omega$ onto Ω. Let us note that the Kantorovich problem of finding an optimal transport plan is a generalization of the Monge one of finding an optimal transport map. Indeed, if T is a transport map from f^+ to f^-, then $\gamma_T := (Id,T)_\# f^+ \in \Pi(f^+,f^-)$ and, we have

$$\int_{\Omega\times\Omega} c(x,y)\,\mathrm{d}\gamma_T = \int_{\Omega\times\Omega} c(x,y)\,\mathrm{d}(Id,T)_\# f^+ = \int_{\Omega} c(x,T(x))\,\mathrm{d}f^+.$$

This generalization is extremely useful for many reasons; let us briefly discuss some of them. First of all, one can show that the Kantorovich problem is "much easier", since it is immediately seen to admit a solution. In fact, the set of all transport plans between f^+ and f^- belongs to the normed space $\mathcal{M}^+(\Omega\times\Omega)$, and in particular we will see that it is, for the weak convergence of measures, a compact subset of it; moreover, the cost in (KP) is a linear function of the transport plan. On the other hand, if $(T_n)_n$ is a minimizing sequence of transport maps, then, up to a subsequence, $T_n \rightharpoonup T$ weak* in L^∞; but, this is not sufficient to get that T is a transport map. Consequently, it is much easier to compare different plans than different maps. In addition, another big difference between (KP) and (MP) is about symmetry: for the Kantorovich problem, exchanging f^+ and f^- does not have any effect, and it is completely equivalent to transport f^+ on f^- or f^- on f^+ provided we replace $c(x,y)$ with $c(y,x)$; for the Monge problem, this is absolutely not true.

Proposition 1.1. *(KP) admits a solution.*

Proof. We need to show that the set $\Pi(f^+,f^-)$ is compact and that $\gamma \mapsto K(\gamma) := \int c\,\mathrm{d}\gamma$ is continuous and then, to apply Weierstrass's Theorem. The continuity of K follows immediately from the definition of the weak convergence of measures and the fact that the cost c is continuous. For the compactness, take a sequence $(\gamma_n)_n \subset \Pi(f^+,f^-)$. They are measures with the same total mass (which is $f^+(\Omega) = f^-(\Omega)$) and so, they are bounded in $\mathcal{M}^+(\Omega\times\Omega)$. Hence, there exists a subsequence $\gamma_{n_k} \rightharpoonup \gamma$ converging to a non-negative measure γ. We just need to check that $\gamma \in \Pi(f^+, f^-)$. This may be done by fixing $\phi \in C(\Omega)$ and using $\int \phi(x)\,\mathrm{d}\gamma_{n_k} = \int \phi\,\mathrm{d}f^+$ and passing to the limit, which gives $\int \phi(x)\,\mathrm{d}\gamma = \int \phi\,\mathrm{d}f^+$. This shows that $(\Pi_x)_\#\gamma = f^+$. The same may be done for Π_y. $\square$

On the other hand, the problem (KP) is a linear optimization under convex constraints, given by linear equalities and, so an important tool will be duality theory, which is typically used for convex problems. In fact, by an inf-sup exchange, we are able to find a formal dual problem (DP) for (KP). To do that, let us express the constraint $\gamma \in \Pi(f^+, f^-)$ in the following way: note that, if $\gamma \in \mathcal{M}^+(\Omega \times \Omega)$, then we have

$$\sup_{u^\pm \in C(\Omega)} \left\{ \int_\Omega u^+\,\mathrm{d}f^+ + \int_\Omega u^-\,\mathrm{d}f^- - \int_{\Omega\times\Omega} (u^+(x) + u^-(y))\,\mathrm{d}\gamma \right\} = \begin{cases} 0 & \text{if } \gamma \in \Pi(f^+, f^-) \\ +\infty & \text{else.} \end{cases}$$

Hence, we may look at the problem, we get

$$\min_{\gamma \in \mathcal{M}^+(\Omega\times\Omega)} \left\{ \int_{\Omega\times\Omega} c\,\mathrm{d}\gamma \;+ \sup_{u^\pm \in C(\Omega)} \left\{ \int_\Omega u^+\,\mathrm{d}f^+ + \int_\Omega u^-\,\mathrm{d}f^- - \int_{\Omega\times\Omega} (u^+(x) + u^-(y))\,\mathrm{d}\gamma \right\} \right\}$$

and consider interchanging sup and inf:

$$\sup_{u^\pm \in C(\Omega)} \left\{ \int_\Omega u^+\,\mathrm{d}f^+ + \int_\Omega u^-\,\mathrm{d}f^- \;+ \inf_{\gamma \in \mathcal{M}^+(\Omega\times\Omega)} \left\{ \int_{\Omega\times\Omega} (c(x,y) \,-\, (u^+(x) + u^-(y)))\,\mathrm{d}\gamma \right\} \right\}.$$

If we come back to the maximization over (u^+, u^-), one can rewrite the inf in γ as a constraint on u^+ and u^-:

$$\inf_{\gamma \in \mathcal{M}^+(\Omega\times\Omega)} \left\{ \int_{\Omega\times\Omega} (c(x,y) \,-\, (u^+(x) + u^-(y)))\,\mathrm{d}\gamma \right\} = \begin{cases} 0 & \text{if } u^+ \oplus u^- \le c \text{ on } \Omega\times\Omega \\ -\infty & \text{else,} \end{cases}$$

where $u^+ \oplus u^-$ denotes the function defined through $(u^+ \oplus u^-)(x,y) := u^+(x) + u^-(y)$. Finally, we get the following dual problem

$$\text{(DP)} \qquad \sup \left\{ \int_\Omega u^+\,\mathrm{d}f^+ + \int_\Omega u^-\,\mathrm{d}f^- \;:\; u^\pm \in C(\Omega),\, u^+ \oplus u^- \le c \right\}.$$

In fact, there was a great development in studying the duality relationship between problems (KP) and (DP): a main ingredient was the extension of the notion of superdifferential for concave functions as proposed by Rockafellar [98], leading to the notions of $c-$concavity and $c-$superdifferential (see [75, 99, 100]). For completeness, let us introduce an alternative proof (which is essentially taken from [103]) based on a simple convex analysis trick.

Proposition 1.2. *The duality formula* $\min$ *(KP)* $=\sup$ *(DP) holds.*

Proof. For every $p \in C(\Omega\times\Omega)$, set

$$H(p) := -\sup\left\{\int_\Omega u^+\,\mathrm{d}f^+ + \int_\Omega u^-\,\mathrm{d}f^- \,:\, u^\pm\in C(\Omega),\, u^+\oplus u^- \le c-p\right\}.$$

Then, it is not difficult to see that $H(p)\in\mathbb{R}\cup\{+\infty\}$, for all $p\in C(\Omega\times\Omega)$. This follows immediately from the fact that for a maximizing sequence $(u_n^+,u_n^-)_n$, we can always assume that these functions share the same modulus of continuity as $c-p$ (in fact, if we replace u_n^- by v_n^- where $v_n^-(y) := \min\{c(x,y)-p(x,y)-u_n^+(x) \,:\, x\in\Omega\}$, for every $y\in\Omega$, the constraints are preserved and the integrals increased) and that they are uniformly bounded (this may be done if we note that adding a constant to u_n^+ and subtracting it to u_n^- is always possible) and so, to apply Ascoli-Arzelà's Theorem. Moreover, we have that

• H is convex: take p_0 and p_1 with their optimal potentials (u_0^+,u_0^-) and (u_1^+,u_1^-). For $t\in[0,1]$, define $p_t := (1-t)p_0 + tp_1$, $u_t^+ := (1-t)u_0^+ + tu_1^+$ and $u_t^- := (1-t)u_0^- + tu_1^-$. Yet, the pair (u_t^+,u_t^-) is admissible in the max defining $-H(p_t)$ and so, we have

$$H(p_t) \le -\left(\int_\Omega u_t^+\,\mathrm{d}f^+ + \int_\Omega u_t^-\,\mathrm{d}f^-\right) = (1-t)H(p_0) + tH(p_1).$$

• H is l.s.c.: take $p_n\to p$ uniformly in $\Omega\times\Omega$ and extract a subsequence $(p_{n_k})_k$ realizing the $\liminf$ of $H(p_n)$. From uniform convergence, the sequence $(p_{n_k})_k$ is equicontinuous and bounded. Hence, the corresponding optimal potentials $(u_{n_k}^+,u_{n_k}^-)_k$ are also equicontinuous and bounded and so, we can assume $u_{n_k}^+\to u^+$ and $u_{n_k}^-\to u^-$ uniformly in Ω. As

$$u_{n_k}^+\oplus u_{n_k}^- \le c - p_{n_k},$$

then

$$u^+\oplus u^- \le c-p$$

and

$$H(p) \le -\left(\int_\Omega u^+\,\mathrm{d}f^+ + \int_\Omega u^-\,\mathrm{d}f^-\right) = \liminf_n H(p_n).$$

Now, let us compute $H^\star : \mathcal{M}(\Omega\times\Omega)\mapsto\mathbb{R}\cup\{+\infty\}$, the Legendre transform of H. For

$\gamma \in \mathcal{M}(\Omega \times \Omega)$, we have

$$H^\star(\gamma) = \sup\left\{ \int_{\Omega\times\Omega} p\,\mathrm{d}\gamma - H(p) \,:\, p \in C(\Omega\times\Omega)\right\}$$

$$= \sup\left\{ \int p\,\mathrm{d}\gamma + \int u^+ \mathrm{d}f^+ + \int u^- \mathrm{d}f^- : u^\pm \in C(\Omega),\, p \in C(\Omega\times\Omega),\, u^+ \oplus u^- \le c - p\right\}.$$

If $\gamma \notin \mathcal{M}^+(\Omega\times\Omega)$, i.e. there is a non-negative continuous function p_0 such that $\int p_0\,\mathrm{d}\gamma < 0$, one can take $u^\pm = 0$, $p_n = c - np_0$, and for $n \to +\infty$, we get $H^\star(\gamma) = +\infty$. On the contrary, if $\gamma \in \mathcal{M}^+(\Omega\times\Omega)$, we should choose the largest possible p, i.e., $p := c - (u^+ \oplus u^-)$. This gives

$$\begin{aligned} H^\star(\gamma) &= \int c\,\mathrm{d}\gamma \;+\; \sup\left\{\int u^+\,\mathrm{d}f^+ + \int u^-\,\mathrm{d}f^- - \int (u^+(x) + u^-(y))\,\mathrm{d}\gamma \;:\; u^\pm \in C(\Omega)\right\} \\ &= \begin{cases} \int c\,\mathrm{d}\gamma & \text{if } \gamma \in \Pi(f^+, f^-) \\ +\infty & \text{else.}\end{cases}\end{aligned}$$

Yet, we have already seen that H is convex and l.s.c., then $H^{\star\star} = H$. In particular, we have $H^{\star\star}(0) = H(0)$. Yet, $H(0) = -\sup\,(\mathrm{DP})$ and,

$$H^{\star\star}(0) := \sup\{< 0, \gamma > - H^\star(\gamma) \,:\, \gamma \in \mathcal{M}(\Omega\times\Omega)\} = -\min\,(\mathrm{KP}). \quad \square$$

Using this duality result (i.e., $\min\,(\mathrm{KP}) = \sup\,(\mathrm{DP})$), we are able to give the following stability result that we will need in the sequel:

Proposition 1.3. *Let γ_n be an optimal transport plan between f^+ and f_n^-, and assume that $f_n^- \rightharpoonup f^-$. Then, up to a subsequence, $\gamma_n \rightharpoonup \gamma$, where γ is an optimal transport plan between f^+ and f^-. Moreover, if all the plans γ_n are induced by transport maps T_n and γ is induced by a map T, then we have $T_n \to T$ in $L^2(f^+)$.*

Proof. Firstly, it is easy to see that there is a subsequence $\gamma_{n_k} \rightharpoonup \gamma$ with $\gamma \in \Pi(f^+, f^-)$. Let $(u^+_{n_k}, u^-_{n_k})$ be a corresponding maximizer for (DP) between f^+ and $f^-_{n_k}$. From Proposition 1.2, we have

$$\int_\Omega u^+_{n_k}\,\mathrm{d}f^+ + \int_\Omega u^-_{n_k}\,\mathrm{d}f^-_{n_k} = \int_{\Omega\times\Omega} c\,\mathrm{d}\gamma_{n_k} \to \int_{\Omega\times\Omega} c\,\mathrm{d}\gamma.$$

In addition, we can suppose that $u^\pm_{n_k}$ are equicontinuous and equibounded, and so there are two subsequences $u^+_{n_k} \to u^+$ and $u^-_{n_k} \to u^-$ with $u^+ \oplus u^- \le c$. Hence,

$$\int_\Omega u^+ \,\mathrm{d}f^+ + \int_\Omega u^- \,\mathrm{d}f^- = \int_{\Omega\times\Omega} c \,\mathrm{d}\gamma,$$

which implies that γ is an optimal transport plan between f^+ and f^-, and (u^+, u^-) is the corresponding maximizer for (DP). The last part of the statement, when plans are induced by maps, can be deduced by the weak convergence of the plans. Using $\gamma_n = (Id, T_n)_\# f^+$ and $\gamma_n \rightharpoonup \gamma := (Id, T)_\# f^+$ and testing the weak convergence against the test function $\phi(x, y) = \xi(x)\cdot y$ we obtain

$$\int_\Omega \xi(x)\cdot T_n(x)\,\mathrm{d}f^+(x) \to \int_\Omega \xi(x)\cdot T(x)\,\mathrm{d}f^+(x),$$

which means that we have the weak convergence $T_n \rightharpoonup T$ in $L^2(f^+)$. We can now test against $\phi(x, y) = |y|^2$ and obtain

$$\int_\Omega |T_n(x)|^2 \,\mathrm{d}f^+(x) \to \int_\Omega |T(x)|^2 \,\mathrm{d}f^+(x),$$

which proves the convergence of the L^2 norm. This gives strong convergence in $L^2(f^+)$. □

Concerning the existence of an optimal transport map for (MP): the first general existence result has been proved when the cost is $c(x, y) = |x - y|^2$: it was obtained independently in 1984 by Knott and Smith, [76], and in 1987 by Brenier, [22, 23]. After their first results, many generalizations ($c(x, y) = |x - y|^p$, $p > 1$) come out, see for example [10, 62, 97, 113]. Here, for the sake of generality, we provide a proof of existence of an optimal transport map when the cost is $c(x, y) = h(x-y)$, where h is a strictly convex function, which includes the quadratic and the power cases. In fact, the duality $\min\,(\mathrm{KP}) = \sup\,(\mathrm{DP})$ implies that optimal γ and (u^+, u^-) satisfy

$$u^+(x) + u^-(y) = h(x - y) \text{ on } \mathrm{spt}(\gamma).$$

We recall that the function u^+ shares the same modulus of continuity as the cost c. Hence, u^+ is Lipschitz continuous in this case. If $f^+ \ll \mathcal{L}^d$, then for $\gamma-$a.e. (x, y), we get

$$\nabla u^+(x) \in \partial h(x - y),$$

where ∂h denotes the subdifferential of h. As h is strictly convex, then this shows at the same time that every optimal transport plan is induced by a transport map and that this transport map is

$$x \mapsto T(x) := x - (\partial h)^{-1}(\nabla u^+(x)).$$

Since the potential u^+ does not depend on γ, then this map is uniquely determined and so, there is a unique optimal transport plan γ (which is in fact induced by the map T). In the quadratic case, one can easily see that there is a convex function u such that the optimal transport map T is the gradient of u, i.e., $T = \nabla u$.

On the other hand, it has been really hard to give some answer about the existence of an optimal transport map in the Euclidean case (i.e., when $c(x,y) = |x-y|$). The main difficulty of this problem is the fact that the cost $|x-y|$ is convex but not strictly convex. More precisely, due to the lack of strict convexity of the Euclidean cost, the uniqueness of the optimal transport plan is in general not true, except for particular situations, and moreover not all the optimal transport plans are actually transport maps. Therefore, there is the additional trouble of selecting a particular optimal transport plan, which comes from a map. The proof of existence of such a map has took a lot of time: in the work of Evans and Gangbo [58], it was considered the case when f^+ and f^- are two positive Lipschitz densities supported in disjoint sets. Afterwards Caffarelli, Feldmann and McCann [32] and Trudinger and Wang [110] independently extended the result to the case when f^+ and f^- are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$. Then, Ambrosio [1, 4] proved that it was sufficient the absolute continuity of the measure f^+, while f^- could be any measure; his proof is based essentially on the notion of $c-$ *cyclical monotonicity*. In order to understand this, we first need to analyse the support of the optimal γ. In fact, one can easily see that the support of any optimal transport plan γ for (KP) is $c-$cyclically monotone, i.e., for any $k \in \mathbb{N}$, any finite set of pairs $(x_1, y_1), ..., (x_k, y_k) \in \operatorname{spt}(\gamma)$ and any permutation σ, we have

$$\sum_{i=1}^{k} c(x_i, y_i) \leq \sum_{i=1}^{k} c(x_i, y_{\sigma(i)}).$$

This property is a generalization of the cyclical monotonicity introduced by Rockafellar in [98], and it was first considered by Knott and Smith in [77]; a detailled discussion can be found in [62]. In the Euclidean case: this implies that, for all (x,y), $(x',y') \in \operatorname{spt}(\gamma)$, we have

$$|x-y| + |x'-y'| \leq |x-y'| + |x'-y|.$$

This inequality has the intuitive meaning that if an optimal transport plan moves x to y and x' to y', then this must be more convenient than moving x to y' and x' to y. In particular, it implies that the segments $[x,y]$ and $[x',y']$ cannot intersect at an interior point for one of them, except they have the same direction. This is a well-known property in the mass transportation problem with Euclidean cost. To be more precise, we will introduce the notion of *transport rays*. First, let us note that, in the Euclidean case, if (u^+, u^-) is a maximizer of (DP), then one can suppose that u^+ is $1-$Lipschitz and $u^- = -u^+$. As a consequence of that, we get the following

$$\min\left\{\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma : \ \gamma \in \Pi(f^+, f^-)\right\} = \sup\left\{\int_{\Omega} u \,\mathrm{d}(f^+ - f^-) \ : \ u \in \operatorname{Lip}_1(\Omega)\right\}. \tag{1.2}$$

The equality of the two optimal values implies that optimal γ and u satisfy $u(x) - u(y) = |x-y|$ on the support of γ, but also that, whenever we find some admissible γ and u satisfying

$\int |x-y|\,\mathrm{d}\gamma = \int u\,\mathrm{d}(f^+ - f^-)$, they are both optimal. Let u be such a maximizer (which is called *Kantorovich potential*). We call transport ray any non-trivial (i.e., different from a singleton) segment $[x, y]$ such that $u(x) - u(y) = |x - y|$ that is maximal for the inclusion among segments of this form (this definition makes sense since u is affine on the whole segment $[x, y]$). This notion has been first introduced by Evans and Gangbo in [**58**], even if Monge himself had in mind something similar (see also [**1, 32, 50**]). Following this definition, we see that an optimal transport plan has to move the mass along the transport rays. Moreover, we call $\mathcal{S}$ the union of all nondegenerate transport rays, $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) be the set of lower (resp. upper) endpoints of nondegenerate transport rays (i.e., those where u is maximal (resp. minimal) on the transport ray, say the points x (resp. y) in the definition $u(x) - u(y) = |x - y|$). Finally, we denote by $\mathcal{D}$ the set of double points, i.e., those whose belong to several transport rays.

In fact, it is not difficult to prove (see, for instance, [**103**]) that the Kantorovich potential u is differentiable at any interior point z of a transport ray $[x, y]$ with $\nabla u(z) = e := (x-y)/|x-y|$. To see that, take $z' \in B(z, \varepsilon)$, $\varepsilon > 0$ is small enough, and let z'' be the projection of z' on the segment $[x, y]$. Then, there are a vector v orthogonal to e and a small t such that $z' = z'' + tv$ and so, one has

$$u(z') = u(z'' + tv) = u(z'' + tv) - u(z'') + e \cdot (z' - z) + u(z).$$

Yet, one can check easily that

$$|u(z'' + tv) - u(z'')| = o(|z' - z|).$$

As a consequence of that, two different transport rays can only meet at a boundary point for both of them, and in such a case, one can show that u must be not differentiable at such a point (this implies that $\mathcal{L}^d(\mathcal{D}) = 0$). Moreover, the transport rays have some regularity; they satisfy *Property N* for "negligibility". Let us introduce the notion of this property.

DEFINITION 1.4. *We say that Property N for "negligibility" holds, for a given Kantorovich potential u, if for every set $B \subset \Omega$ such that:*

- $B \subset \mathcal{S}$
- $B \cap r$ *is at most countable for every transport ray* r,

then $\mathcal{L}^d(B) = 0$.

Notice that this property is not always satisfied by any disjoint family of segments, and there is an exemple (by Alberti, Kirchheim, and Preiss, later improved by Ambrosio, Kirchheim and Pratelli; see [**3**]) where a disjoint family of segments contained in a cube is such that the collection of their middle points has a strictly positive measure. Yet, one can prove that the direction of the transport rays satisfies additional properties, which guarantee *Property N*. More precisely, we are able to show (see, for instance, [**103**]) that the gradient of a Kantorovich potential u is in fact countably Lipschitz. To see that, let us define

$$\mathcal{S}_\varepsilon = \left\{ x \in \mathcal{S} \ : \ \exists z \in \mathcal{S} \ \text{ with } \ u(x) - u(z) = |x - z| > \varepsilon \right\}, \quad \varepsilon > 0,$$

which is roughly speaking made of those points in the transport rays that are at least at a distance ε apart from the upper boundary point of the rays. It is clear that $\cup_{\varepsilon > 0} \mathcal{S}_\varepsilon = \mathcal{S} \setminus \mathcal{S}^-$. In addition, it is easy to check that, if $x \in \mathcal{S}_\varepsilon$, then

$$u(x) = \inf_{y \notin B(x,\varepsilon)} |x - y| + u(y).$$

Hence, the restriction of u to each set $\mathcal{S}_\varepsilon$ is semi-concave. Using this fact, we get the following (see [**103**]):

Proposition 1.5. *The property N for "negligibility" holds for a given Kantorovich potential u.*

Proof. Without loss of generality, suppose that ∇u is Lipschitz. Consider a set B in the definition of Property N (see Definition 1.4). Take $x \in B$. So, x belongs to some transport ray r. Yet, it is clear that this ray r intersects at least one hyperplane $x_i = q$, for some $i \in \{1, ..., d\}$ and $q \in \mathbb{Q}$, at exactly one point of its interior (we denote by $H_{i,q}$ such an hyperplane and by $B_{i,q}$ the set of all points x in B having this property). In this way, one has $B = \cup_{i,q} B_{i,q}$. Let $R_{i,q}$ be the set of all transport rays that meet the hyperplane $H_{i,q}$ at exactly one point of its interiors (we denote by $I_{i,q}$ the set of all the intersection points). Set

$$A_{i,q} = \left\{ (y,t) \in I_{i,q} \times \mathbb{R} \, : \, \exists \ r \in R_{i,q}, \, y \in r \ \text{ and } \ y + t\nabla u(y) \in r \setminus D \right\}.$$

Now, let us define the map $\xi_{i,q} : A_{i,q} \mapsto \mathbb{R}^d$ by setting, for $(y,t) \in A_{i,q}$, $\xi_{i,q}(y,t) = y + t\nabla u(y)$. The map $\xi_{i,q}$ is injective, since getting the same point as the image of (y,t) and of (y',t') would mean that two different transport rays cross at such point. $B_{i,q}$ is contained in the image of $\xi_{i,q}$ by construction, so that $\xi_{i,q}$ is a bijection between $B'_{i,q} := \xi^{-1}(B_{i,q})$ and $B_{i,q}$. The map $\xi_{i,q}$ is also Lipschitz, as a consequence of the Lipschitz behavior of ∇u. Note that $B'_{i,q}$ is a subset of $H_{i,q} \times \mathbb{R}$ containing at most countably many points on every line $\{y\} \times \mathbb{R}$. By Fubini's theorem, this implies $\mathcal{L}^d(B'_{i,q}) = 0$. Then we have also $\mathcal{L}^d(B_{i,q}) = \mathcal{L}^d(\xi_{i,q}(B'_{i,q})) \leq \mathrm{Lip}(\xi_{i,q})^d \mathcal{L}^d(B'_{i,q})$, which implies $\mathcal{L}^d(B_{i,q}) = 0$. $\square$

Finally, we are ready to find an optimal transport plan γ somehow better than the others (i.e., induced by a map). In fact, the idea was to consider the following

$$\min \left\{ \int_{\Omega \times \Omega} |x - y| \, + \, \varepsilon |x - y|^2 \, \mathrm{d}\gamma \, : \, \gamma \in \Pi(f^+, f^-) \right\}, \quad \varepsilon > 0. \tag{1.3}$$

If γ_ε is an optimal transport plan for (1.3), then it is not difficult to see that $\gamma_\varepsilon \rightharpoonup \gamma$ when $\varepsilon \to 0$, where γ is an optimal transport plan for (KP) with Euclidean cost. Moreover, by the $c-$cyclical monotonicity of $\mathrm{spt}(\gamma_\varepsilon)$ when $\varepsilon \to 0$, one can prove that, for (x, y), $(x', y') \in \mathrm{spt}(\gamma)$ such that x, x', y and y' are all points of a same transport ray r, we have

$$|x-y|^2 + |x'-y'|^2 \leq |x-y'|^2 + |x'-y|^2,$$

which is equivalent to say that

$$(x-x') \cdot (y-y') \geq 0\,. \tag{1.4}$$

Let us define an order relation on such a transport ray through $x \leq x' \Leftrightarrow u(x) \geq u(x')$. This implies that if $x \leq x'$, then $y \leq y'$. Now, let us define the interval I_x as the minimal interval I such that $\mathrm{spt}(\gamma) \cap (\{x\} \times r) \subset \{x\} \times I$. As the interiors of all these intervals are disjoint and ordered, then there can be at most a countable quantity of points x such that I_x is not a singleton. Using Proposition 1.5, we infer that the optimal transport plan γ will be induced by a map T as soon as $f^+ \ll \mathcal{L}^d$ (thanks to (1.4), this plan γ is monotone nondecreasing along each transport ray; it is so-called the *monotone optimal transport plan* and the map T, which corresponds to it, the *monotone optimal transport map*). So, we have the following

THEOREM 1.6. *If $f^+ \ll \mathcal{L}^d$, then (MP) reaches a minimum.*

CHAPTER 2

Transport density

2.1. Definitions

In the mass transportation problem with Euclidean cost (supposing also that the domain Ω is convex), it is classical to associate with any optimal transport plan γ for (KP) a non-negative measure σ_γ on Ω, called *transport density*, which represents the amount of transport taking place in each region of Ω. This measure σ_γ is defined by

$$(2.1) \qquad < \sigma_\gamma, \varphi >= \int_{\Omega\times\Omega} \mathrm{d}\gamma(x,y) \int_0^1 \varphi(\omega_{x,y}(t))|\omega'_{x,y}(t)|\,\mathrm{d}t \quad \text{for all } \varphi \in C(\Omega)$$

where $\omega_{x,y}$ is a curve parameterizing the straight line segment connecting x to y. Notice in particular that one can write

$$(2.2) \qquad \sigma_\gamma(A) = \int_{\Omega\times\Omega} \mathcal{H}^1(A \cap [x,y])\,\mathrm{d}\gamma(x,y) \quad \text{for every Borel subset } A \subset \Omega$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure. This means that $\sigma_\gamma(A)$ stands for "how much" the transport takes place in A, if particles move from their origin x to their destination y on straight lines.

This measure σ_γ had been already considered by Janfalk (see [71]). Moreover, in the work of Evans and Gangbo, it was one of the main tools to build an optimal transport map for the Monge's problem; it was the additional ingredient that they used to recover enough information to move correctly the mass inside each transport ray. More precisely, their construction used the approximation of the so-called $p-$Laplacian: they considered the solutions u_p of the problem

$$-\nabla\cdot(|\nabla u_p|^{p-2}\nabla u_p) = f^+ - f^-$$

and, passing to a limit for $p \to +\infty$, they prove then that there is some $u \in \mathrm{Lip}_1(\Omega)$ such that, up to a subsequence, u_p converges uniformly to u and $|\nabla u_p|^{p-2}\nabla u_p$ weakly$^\star-$converges in L^∞ to $\sigma\nabla u$, where σ is a non-negative bounded density. Then, in particular, one has

$$(2.3) \qquad -\nabla\cdot(\sigma\nabla u) = f^+ - f^-.$$

The function u is a Kantorovich potential, while σ is the transport density between f^+ and f^-. From the definition (2.1), we see that a transport density σ_γ does not depend uniquely on f^+ and f^-, but also on the choice of the optimal transport plan γ. However, the uniqueness of this measure is true under some assumptions on the data. In fact, we have the following result (see, for instance, [59, 103]).

Proposition 2.1. *Suppose $f^+ \ll \mathcal{L}^d$ or $f^- \ll \mathcal{L}^d$. Then the transport density is unique, that is, any optimal transport plan γ between f^+ and f^- defines the same transport density σ_γ.*

Proof. Suppose $f^+ \ll \mathcal{L}^d$ and let u be a Kantorovich potential for the transport between f^+ and f^-. Define the map $R : \Omega \times \Omega \mapsto \mathcal{R}$, valued in the set $\mathcal{R}$ of all transport rays, sending each pair (x, y) into the ray containing x (this is well-defined $\gamma-$a.e. since $f^+ \ll \mathcal{L}^d$ and $\mathcal{L}^d(\mathcal{D}) = 0$, where $\mathcal{D}$ is the set of double points). So, we can write $\gamma = \gamma^r \otimes \alpha$, where $\alpha = R_\# \gamma$ (notice that the plan γ^r will be optimal between its own marginals, for $\alpha-$a.e. $r \in \mathcal{R}$). Hence, we have $\sigma_\gamma = \sigma_{\gamma^r} \otimes \alpha$. It is clear that the measure α does not depend on γ, since it has been obtained as an image measure through a map only depending on x and hence, only on f^+. On the other hand, σ_{γ^r} is the 1D transport density associated with the optimal transport plan γ^r and so, one can see easily that it uniquely depends on the marginal measures of γ^r. But, $(\Pi_x)_\# \gamma^r$ (resp. $(\Pi_y)_\#(\gamma^r \llcorner (\Omega \times \mathcal{D}^c))$) must coincide with the disintegration of f^+ (resp. $f^- \llcorner \mathcal{D}^c$) according to the map R and then, it does not depend on γ. And, the measure $(\Pi_y)_\#(\gamma^r \llcorner (\Omega \times \mathcal{D}))$ can only be concentrated on the two endpoints of the transport ray r. Yet, an endpoint where u is maximal cannot contain any mass of the target measure unless the source one has an atom at the "beginning" of the transport ray. But, this is not the case for $\alpha-$a.e. ray $r \in \mathcal{R}$, as $f^+ \ll \mathcal{L}^d$ and Property N holds (see Proposition 1.5). Hence, $(\Pi_y)_\#(\gamma^r \llcorner (\Omega \times \mathcal{D}))$ is a single Dirac for $\alpha-$a.e. $r \in \mathcal{R}$, with mass equals to $1 - (\Pi_y)_\#(\gamma^r \llcorner (\Omega \times \mathcal{D}^c))$. Yet, this last quantity does not depend on γ but only on f^+ and f^-. The same result is true if f^- (in place of f^+) is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, since we recall that the transport plans between f^+ and f^- remain the same swapping f^+ and f^-. □

2.2. L^p summability

There are several papers, mainly by De Pascale and Pratelli, Evans and Feldmann and McCann, adressing absolute continuity (or more generally, L^p summability) of the transport density (see, for instance, [48, 50, 51]). In [50], the authors show estimates on the dimension of the transport density σ in terms of the dimensions of the measures f^+ and f^-, and they represent the connection between the study of the dimension and that of the summability of σ. Alternatively, in [102], the author gives a short proof for this summability result; the idea is based on displacement interpolation and on approximation by discrete measures. More precisely, we have the following

Proposition 2.2. *Suppose $f^+ \ll \mathcal{L}^d$ or $f^- \ll \mathcal{L}^d$. Then, the unique transport density σ between f^+ and f^- is also absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$.*

Proof. Let γ be an optimal transport plan between f^+ and f^-, and let f_t be the interpolation between these two measures, i.e., $f_t = (\Pi_t)_\# \gamma$ where $\Pi_t(x, y) := (1 - t)x + ty$ (note that $f_0 = f^+$ and $f_1 = f^-$). From (2.1), it is easy to see that

$$\sigma \leq C \int_0^1 f_t \, \mathrm{d}t. \tag{2.4}$$

Hence, to prove that σ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, it is sufficient to prove that almost every measure f_t is absolutely continuous with respect to $\mathcal{L}^d$. First, we suppose that f^- is finitely atomic (the points $(x_i)_{i=1,...,n}$ being its atoms). In this case, we will choose γ to be the monotone optimal transport plan between f^+ and f^- (let T be the corresponding monotone optimal transport map). For each $i \in \{1, ..., n\}$, set $\Omega_i := T^{-1}(\{x_i\})$ and let $\Omega_i(t)$ be the image of Ω_i through the map $x \mapsto (1-t)x + tT(x)$. It is easy to see that these sets $(\Omega_i(t))_{i=1,...,n}$ are disjoint, for every $t \in [0,1]$. So, for every Borel subset $A \subset \Omega$, we have

$$f_t(A) = \sum_{i=1}^n f_t(A \cap \Omega_i(t)) = \sum_{i=1}^n f^+\Big(\frac{A \cap \Omega_i(t) - tx_i}{1-t}\Big) = f^+\Big(\bigcup_{i=1}^n \frac{A \cap \Omega_i(t) - tx_i}{1-t}\Big).$$

Yet,

$$\mathcal{L}^d\Big(\bigcup_{i=1}^n \frac{A \cap \Omega_i(t) - tx_i}{1-t}\Big) \leq \frac{1}{(1-t)^d}\,\mathcal{L}^d(A).$$

Then, $f_t \ll \mathcal{L}^d$, for all $t < 1$. Now, take a sequence f_n^- of atomic measures converging to f^-. From Proposition 1.3, we know that the corresponding optimal transport plans γ_n between f^+ and f_n^- converge to an optimal transport plan γ between f^+ and f^-, and $f_{n,t} := (\Pi_t)_\# \gamma_n$ converge to the corresponding f_t. We conclude by observing that the absolute continuity estimates on $f_{n,t}$ may pass to the limit. □

Moreover, we are able to find estimates for the L^p summability of the transport density σ under stronger assumptions on the data. In fact, one can expect some results stronger than the last one, i.e. the L^1 result on σ (see Proposition 2.2), if the source (resp. target) measure f^+ (resp. f^-) belongs to L^p, for some $p > 1$. In [**50**], the authors gave some L^p estimates on the transport density σ via geometrical arguments. Yet, their result has been strengthened in [**102**], where the author proves the following

Proposition 2.3. *Suppose $f^+ \in L^p(\Omega)$. Then, if $p < d/(d-1)$, the unique transport density σ associated with the transport of f^+ onto f^- belongs to $L^p(\Omega)$ as well.*

Proof. Using Minkowski's inequality, (2.4) implies that

$$\| \sigma \|_{L^p(\Omega)} \leq C \int_0^1 \| f_t \|_{L^p(\Omega)} \,\mathrm{d}t.$$

First, consider the case where f^- is discrete: we know that f_t is absolutely continuous and that it coincides on each set $\Omega_i(t)$ with the density of a homothetic image of f^+ on Ω_i, the homothetic ratio being $(1-t)$. Hence, we have

$$\begin{aligned}\int_\Omega f_t(y)^p \,\mathrm{d}y &= \sum_{i=1}^n \int_{\Omega_i(t)} f_t(y)^p \,\mathrm{d}y = \sum_{i=1}^n \int_{\Omega_i(t)} \Big(\frac{f^+(\frac{y-tx_i}{1-t})}{(1-t)^d}\Big)^p \,\mathrm{d}y \\ &= (1-t)^{d(1-p)} \sum_{i=1}^n \int_{\Omega_i} f^+(x)^p \,\mathrm{d}x = (1-t)^{d(1-p)} \int_\Omega f^+(x)^p \,\mathrm{d}x.\end{aligned}$$

Consequently,

$$\| f_t \|_{L^p(\Omega)} = (1-t)^{-\frac{d}{p'}} \ \| f^+ \|_{L^p(\Omega)},$$

where $p' := p/(p-1)$. In addition, it is easy to see that this inequality, which is true in the discrete case, stays true at the limit as well (i.e., if f^- is not atomic) and then,

$$\| \sigma \|_{L^p(\Omega)} \leq C \int_0^1 \| f_t \|_{L^p(\Omega)} \,\mathrm{d}t \leq C\bigg(\int_0^1 (1-t)^{-\frac{d}{p'}} \,\mathrm{d}t \bigg) \| f^+ \|_{L^p(\Omega)} < +\infty. \quad \square$$

This result is actually sharp! One can give an example where $f^+ \in L^\infty$, but the singularity of the target measure f^- prevents σ from being $L^{\frac{d}{d-1}}$ (just consider the case where we send a bounded density f^+ to a Dirac mass).

What happens if both f^+ and f^- are in L^p? It is reasonable, in this case, to expect that the transport density σ between f^+ and f^- would be more summable than just $L^{\frac{d}{d-1}-\varepsilon}$ when $p \geq d/(d-1)$, since the target measure is no more singular. In fact, we saw in the previous estimates that the measures f_t inherit some regularity (absolute continuity or L^p summability) from f^+ exactly as it happens for homotheties of ratio $1-t$. This regularity degenerates as $t \to 1$, but we saw two cases where this degeneracy produced no problem: for proving absolute continuity, where the separate absolute continuous behavior of almost all the f_t, was sufficient, and for L^p estimates, provided "the degeneracy stays integrable". However, if f^- is also regular, then we can give estimate on f_t for $t \to 0$ starting from f^+ and for $t \to 1$ starting from f^-. Yet, let us note that in the previous estimates, we didn't know a priori that f_t shared the same behavior of piecewise homotheties of f^+; we go it as a limit from discrete approximations and, when we pass to the limit, we do not know which optimal transport plan γ will be selected as a limit of the optimal plans γ_n. This was not important above, since any optimal γ induces the same transport density σ (thanks to Proposition 2.1). But here, we would like to glue together estimates on f_t for $t \to 0$ which have been obtained by approximating f^- and estimates on f_t for $t \to 1$ which come from the approximation of f^+. Should the two approximations converge to two different transport plans, we could not put together the two estimates and deduce anything on σ.

Hence, the main technical issue that we need to consider is proving that one particular optimal transport plan can be approximated in both directions. To do that, the idea is to consider, first, the optimal transport plan γ_ε for (KP), with cost $|x-y|^{1+\varepsilon}$ (where $\varepsilon > 0$), and to show that the same estimates, as in the proof of Proposition 2.3, are still true for $f_{\varepsilon,t} := (\pi_t)_{\#}\gamma_\varepsilon$. The advantage, now, is that the optimal transport plan γ_ε is unique, and then, we can get uniform L^p estimates on $f_{\varepsilon,t}$ (with no degeneracy when $t \to 0$ or $t \to 1$) by approximating f^+ or f^- (the only fact that must be checked is the disjointness of the sets $\Omega_i(t)$, which follows, in fact, from the $c-$cyclical monotonicity of $\mathrm{spt}(\gamma_\varepsilon)$). We note that this strategy is different from the one given in [**102**] where the author shows that the monotone optimal transport plan can be approximated in both directions. Anyway, we are able to prove that

$$\| f_{\varepsilon,t} \|_{L^p(\Omega)} \leq C \max\{\| f^+ \|_{L^p(\Omega)}, \| f^- \|_{L^p(\Omega)}\}, \quad \forall \ t \in [0,1],\, \varepsilon > 0. \tag{2.5}$$

Hence, we get the following

PROPOSITION 2.4. *Suppose both f^+ and f^- belong to $L^p(\Omega)$. Then, the unique transport density σ associated with the transport of f^+ onto f^- belongs to $L^p(\Omega)$ as well.*

PROOF. It is easy to see that the optimal transport plans γ_ε converge to an optimal transport plan γ for (KP) with Euclidean cost, and then, $f_{\varepsilon,t}$ converge to f_t. Yet, the estimate (2.5) may pass to the limit, when $\varepsilon \to 0$, giving that

$$\| f_t \|_{L^p(\Omega)} \leq C \max\{\| f^+ \|_{L^p(\Omega)}, \| f^- \|_{L^p(\Omega)}\}.$$

Consequently, the transport density σ between f^+ and f^- belongs to $L^p(\Omega)$ and, we have the following estimate

$$\| \sigma \|_{L^p(\Omega)} \leq C \max\{\| f^+ \|_{L^p(\Omega)}, \| f^- \|_{L^p(\Omega)}\}. \quad \square$$

2.3. From transport density to Beckmann's problem

The equation (2.3) used by Evans and Gangbo had been independently considered and deeply generalized, in the context of shape optimization problems, by Bouchitté, Buttazzo and Seppecher [**13, 14, 15**] and it had been studied also by Iri [**70**] and Strang [**109**]. In fact, some problems which appeared to have correlations with the mass transportation were the Beckmann problem [**9**] and, the evolution problem considered by Brenier [**24, 25, 26**]. It turns out that, in all these connections, a main role is played by the transport density, which appears in each of these problems with different meanings.

Back to Monge's original problem, i.e. when the cost is equal to the Euclidean distance, we can take advantage of one more equivalent formulation of (KP), which is particularly interesting, as far as it is expressed as a divergence-constrained optimization problem. This is a particular case of the so-called *continuous model of transportation*, first, proposed by Beckmann in [**9**], which is the following

$$\text{(BP)} \qquad \inf\left\{\int_\Omega |w|\,\mathrm{d}x \,:\, w \in L^1(\Omega,\mathbb{R}^d),\, \nabla\cdot w = f^+ - f^-\right\},$$

where the divergence condition is to be read in the weak sense, with no-flux boundary conditions, i.e., $-\int_\Omega \nabla\phi\cdot w\,\mathrm{d}x = \int_\Omega \phi\,\mathrm{d}(f^+ - f^-)$ for any $\phi \in C^1(\bar{\Omega})$.

We note that the minimization of the L^1 norm under divergence constraints also has applications in image processing, as in [**27, 82**], in particular because the L^1 norm (and not its strictly convex variants) induces sparsity.

Let us note that the direct method in Calculus of Variations, to prove existence of a minimizer for (BP), does not work here (this is due to the fact that the space L^1 is not reflexive) and so, (BP) is a priori not well-posed in the L^1 space. To avoid this difficulty, we will choose the natural setting for (BP), i.e., we will replace the space $L^1(\Omega,\mathbb{R}^d)$ with the space of vector measures $\mathcal{M}^d(\Omega)$. This means that we want to consider the following

$$\text{(BP)}\qquad \min\left\{|w|(\Omega)\,:\,w\in\mathcal{M}^d(\Omega),\,\nabla\cdot w=f^+-f^-\right\},$$

where $|w|(\Omega)$ denotes the total variation of the vector measure w. Now, it is easy to see that this second version of (BP) has always a solution (using the direct method in Calculus of Variations). However, a minimizer w is a priori a vector measure (we do not known whether it belongs or not to $L^1(\Omega,\mathbb{R}^d)$). So, the idea is to construct a minimizer for (BP) (which is essentially a vector measure) and to show that, under some assumptions on the data, it belongs to L^1. In this way, we get a solution to the original (BP) (i.e., the one that is formulated in the space L^1 instead of the space of vector measures). More precisely, we want to show that the minimal value of (BP) equals that of (KP), and a solution of (BP) can be built from a solution of (KP); the two problems are hence equivalent. One first observes that the indicator function of the set of admissible w can be written as

$$\sup\left\{\int_\Omega\nabla\phi\cdot\mathrm{d}w+\int_\Omega\phi\,\mathrm{d}(f^+-f^-)\,:\,\phi\in C^1(\Omega)\right\}=\begin{cases}0 & \text{if}\quad \nabla\cdot w=f^+-f^-,\\ +\infty & \text{otherwise.}\end{cases}$$

So, the (BP) can be stated in an unconstrained form as

$$\min\left\{|w|(\Omega)\,+\,\sup\left\{\int_\Omega\nabla\phi\cdot\mathrm{d}w+\int_\Omega\phi\,\mathrm{d}(f^+-f^-)\,:\,\phi\in C^1(\Omega)\right\}\,:\,w\in\mathcal{M}^d(\Omega)\right\}.$$

Notice that

$$\inf\left\{|w|(\Omega)+\int_\Omega\nabla\phi\cdot\mathrm{d}w\,:\,w\in\mathcal{M}^d(\Omega)\right\}=\begin{cases}0 & \text{if}\quad |\nabla\phi|\le 1\\ -\infty & \text{otherwise.}\end{cases}$$

Hence, via a formal inf-sup exchange, we get

$$\sup\left\{\int_\Omega\phi\,\mathrm{d}(f^+-f^-)\,:\,|\nabla\phi|\le 1\right\}.$$

The latter being exactly the dual formulation of Monge's problem (see (1.2)). In fact, we can see easily that $\sup(\mathrm{DP}) \leq \min(\mathrm{BP})$. Indeed, for all admissible ϕ in (DP) and w in (BP), we have

$$\int_\Omega \phi \,\mathrm{d}(f^+ - f^-) = \int_\Omega (-\nabla\phi)\cdot \mathrm{d}w \leq \int_\Omega 1 \,\mathrm{d}|w| = |w|(\Omega).$$

Now, let γ be an optimal transport plan between f^+ and f^-, and build a vector measure w_γ as follows

$$< w_\gamma, \xi >:= \int_{\Omega\times\Omega} \mathrm{d}\gamma(x,y) \int_0^1 \xi(w_{x,y}(t))\cdot w'_{x,y}(t)\,\mathrm{d}t, \quad \text{for every } \xi \in C(\Omega, \mathbb{R}^d), \tag{2.6}$$

where $w_{x,y}$ being a parameterization of the segment $[x,y]$. Recalling (2.1), it is not difficult to see that

$$w_\gamma = -\sigma_\gamma \nabla u,$$

where σ_γ is the transport density associated with the optimal transport plan γ and u is the Kantorovich potential between f^+ and f^-. On the other hand, it is easy to check that this measure w_γ satisfies the divergence constraint in (BP), since for every $\phi \in C^1(\Omega)$, we have

$$\begin{aligned} - < w_\gamma, \nabla\phi > &= -\int_{\Omega\times\Omega} \mathrm{d}\gamma(x,y) \int_0^1 \frac{d}{dt}\phi(w_{x,y}(t))\,\mathrm{d}t \\ &= \int_{\Omega\times\Omega} (\phi(x) - \phi(y))\,\mathrm{d}\gamma(x,y) \\ &= \int_\Omega \phi\,\mathrm{d}(f^+ - f^-). \end{aligned}$$

Hence,

$$\min(\mathrm{BP}) \leq |w_\gamma|(\Omega) \leq \sigma_\gamma(\Omega) = \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma = \min(\mathrm{KP}) = \sup(\mathrm{DP}).$$

Thus, the equality $\min(\mathrm{BP}) = \min(\mathrm{KP})$ holds and, the vector measure w_γ is in fact a solution to (BP). Yet, from Proposition 2.2, the transport density σ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, provided that $f^+ \ll \mathcal{L}^d$ or $f^- \ll \mathcal{L}^d$. Hence, the original (BP) reaches a minimum, as soon as f^+ or f^- is in $L^1(\Omega)$.

In addition, we note that the pair (σ, u) solves a particular PDE system, which is the so-called Monge-Kantorovich system,

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f^+ - f^- & \text{in } \Omega \\ \sigma\nabla u\cdot\mathbf{n} = 0 & \text{on } \partial\Omega \\ |\nabla u| \leq 1 & \text{in } \Omega \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{2.7}$$

Now, we want to show that in fact any minimizer for (BP) comes from an optimal transport plan for (KP). In Chapter 6, we will also need this result in the case where the transport cost c is given by a strictly convex norm $||\cdot||$. So, let us give the proof in this more general case. From Proposition 1.2, we have

$$\min\left\{\int_{\Omega\times\Omega}||x-y||\,\mathrm{d}\gamma\;:\;\gamma\in\Pi(f^+,f^-)\right\}=\sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\;:\;||\nabla u||_{\star,\infty}\le 1\right\},$$

since u is 1−Lip with respect to $||\cdot||$ is equivalent to $||\nabla u||_{\star,\infty}\le 1$, as soon as Ω is convex, where $||\cdot||_{\star,\infty}$ is defined as follows

$$||\nabla u||_{\star,\infty}=\sup_{x\in\Omega}||\nabla u(x)||_\star$$

and

$$||v||_\star:=\sup\left\{v\cdot\xi\;:\;||\xi||\le 1\right\},\text{ for every } v\in\mathbb{R}^d.$$

With a general norm, (BP) becomes

$$\min\left\{||w||(\Omega)\;:\;w\in\mathcal{M}^d(\Omega),\,\nabla\cdot w=f^+-f^-\right\},$$

where $||w||$ denotes the variation measure associated with the vector measure w, i.e., $||w||(E):=\sup_\pi\sum_{A\in\pi}||w(A)||$, for every measurable set $E\subset\Omega$, where the supremum is taken over all partitions π of E into a countable number of disjoint measurable subsets.

Note that, so far, we have not stated the equivalence between the problems (BP) and (KP) in the case of a general norm. Before showing that, let us define the transport density σ_γ, associated with some optimal transport plan γ, as follows

$$<\sigma_\gamma,\phi>=\int_{\Omega\times\Omega}\mathrm{d}\gamma(x,y)\int_0^1\phi((1-t)x+ty)||x-y||\,\mathrm{d}t,\quad\text{for all } \phi\in C(\Omega). \tag{2.8}$$

Define w_γ exactly as in (2.6). We see easily that $||w_\gamma||\le\sigma_\gamma$. Yet, w_γ is admissible in (BP) and $||w_\gamma||(\Omega)\le\sigma_\gamma(\Omega)=\min\,(\mathrm{KP})=\sup\,(\mathrm{DP})\le\min\,(\mathrm{BP})$, where the last inequality follows from the fact that, if $\nabla\cdot w=f^+-f^-$ and $||\nabla u||_{\star,\infty}\le 1$, then we have $\int_\Omega u\,\mathrm{d}(f^+-f^-)=-\int_\Omega\nabla u\cdot\mathrm{d}w\le||w||(\Omega)$. This yields that w_γ is an optimal flow for (BP).

Now, let us come back to the proof of the claim, that is the fact that any optimal flow for (BP) comes from an optimal transport plan for (KP). First of all, we will introduce some objects that generalize both σ_γ and w_γ.

Let $\mathcal{C}$ be the set of absolutely continuous curves $\alpha : [0,1] \mapsto \Omega$. We call *traffic plan* any non-negative measure Q on $\mathcal{C}$ with total mass equal to $f^+(\Omega) = f^-(\Omega)$, and

$$\int_{\mathcal{C}} L(\alpha)\,\mathrm{d}Q(\alpha) < +\infty,$$

where $L(\alpha)$ is the length of the curve α, i.e. $L(\alpha) = \int_0^1 ||\alpha'(t)||\,\mathrm{d}t$ (note that the length is measured according to the norm $||\cdot||$). We define the *traffic intensity* $i_Q \in \mathcal{M}^+(\Omega)$ as follows

$$\int_\Omega \phi\,\mathrm{d}i_Q = \int_{\mathcal{C}} \left(\int_0^1 \phi(\alpha(t))||\alpha'(t)||\,\mathrm{d}t \right) \mathrm{d}Q(\alpha) \quad \text{for all } \phi \in C(\Omega).$$

This definition (which is taken from [39] and adapted to the case of a general norm) is a generalization of the notion of transport density σ_γ. The interpretation is the following: for a subregion A, $i_Q(A)$ represents the total cumulated traffic in A induced by Q, i.e., for every path we compute "how long" it stays in A, and then we average on paths. We also associate a vector measure w_Q (called *traffic flow*) with any traffic plan Q via

$$\int_\Omega \xi \cdot \mathrm{d}w_Q = \int_{\mathcal{C}} \left(\int_0^1 \xi(\alpha(t)) \cdot \alpha'(t)\,\mathrm{d}t \right) \mathrm{d}Q(\alpha) \quad \text{for all } \xi \in C(\Omega, \mathbb{R}^d).$$

Taking a gradient field $\xi = \nabla\phi$ in the previous definition yields

$$\int_\Omega \nabla\phi \cdot \mathrm{d}w_Q = \int_{\mathcal{C}} \big(\phi(\alpha(1)) - \phi(\alpha(0))\big)\,\mathrm{d}Q(\alpha) = \int_\Omega \phi\,\mathrm{d}((e_1)_\# Q - (e_0)_\# Q),$$

where e_t is the evaluation map at time t, i.e. $e_t(\alpha) := \alpha(t)$, for all $\alpha \in \mathcal{C}$, $t \in [0,1]$. From now on, we will restrict our attention to admissible traffic plans Q, i.e. traffic plans such that $(e_0)_\# Q = f^+$ and $(e_1)_\# Q = f^-$, since, in this case, w_Q will be an admissible flow in (BP), i.e. one has

$$\nabla \cdot w_Q = f^+ - f^-.$$

Lemma 2.5. *Let w be a flow such that $\nabla \cdot w = f^+ - f^-$. Then, there is an admissible traffic plan Q such that $||w - w_Q||(\Omega) + i_Q(\Omega) = ||w||(\Omega)$.*

Proof. Following [103, Section 4.2.3], we suppose, first, that f^+, f^- and w are smooth with $f^+, f^- > 0$. Let $X(.,x)$ be the flow solution of

$$\begin{cases} \partial_t X(t,x) = \frac{w(X(t,x))}{(1-t)f^+(X(t,x)) \,+\, tf^-(X(t,x))}, \\ X(0,x) = x. \end{cases}$$

Then, we define the measure Q as follows:

$$\int_{\mathcal{C}} \psi(\alpha)\,\mathrm{d}Q(\alpha) := \int_\Omega \psi(X(.,x))\,\mathrm{d}f^+(x), \text{ for every } \psi \in C(\mathcal{C}).$$

By construction, it is not difficult to check that the flow map X satisfies the following

$$\frac{d}{dt}\bigg(((1-t)f^+(X(t,x)) + tf^-(X(t,x)))\det \nabla_x X(t,x)\bigg) = 0,$$

which implies that

$$f^+(x) = f_t(X(t,x))\det \nabla_x X(t,x),$$

where f_t is the standard interpolation between f^+ and f^-, i.e. $f_t = (1-t)f^+ + tf^-$. Hence, $f_t = (X(t,.))_\# f^+$, which guarantees, in particular, that $(e_0)_\# Q = f^+$ and $(e_1)_\# Q = f^-$. In addition, one can see easily that $w_Q = w$ and $i_Q = ||w||$.

For the general case: following also [103, Theorem 4.10], take a convolution of w (resp. of f^+ and f^-) with a Gaussian kernel η_ε and take care of the boundary behavior (for more details, see [103, Lemma 4.8]), we obtain smooth vector fields w_ε and positive smooth densities $f_\varepsilon^\pm$ with $\nabla \cdot w_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-$ such that $w_\varepsilon \rightharpoonup w$ (resp. $||w_\varepsilon|| \rightharpoonup ||w||$ because of standard properties of convolutions) and $f_\varepsilon^\pm \rightharpoonup f^\pm$. Let $(Q_\varepsilon)_\varepsilon$ be the sequence of traffic plans such that, for every $\varepsilon > 0$, $w_{Q_\varepsilon} = w_\varepsilon$ and $i_{Q_\varepsilon} = ||w_\varepsilon||$. The measures Q_ε were constructed so that $(e_0)_\# Q_\varepsilon = f_\varepsilon^+$ and $(e_1)_\# Q_\varepsilon = f_\varepsilon^-$, which implies, at the limit, that $Q_\varepsilon \rightharpoonup Q$ (since $\int_{\mathcal{C}} L(\alpha)\,\mathrm{d}Q_\varepsilon(\alpha) = ||w_\varepsilon||(\Omega) \leq C$ and so, Q_ε is tight) with $(e_0)_\# Q = f^+$ and $(e_1)_\# Q = f^-$. Moreover, it is not difficult to check that Proposition 4.7 in [103] is still true if we replace the Euclidean norm $|\cdot|$ by another one $||\cdot||$. In particular, we have

$$\begin{aligned} \int_\Omega \xi \cdot \mathrm{d}w &= \lim_\varepsilon \int_\Omega \xi \cdot \mathrm{d}w_\varepsilon = \lim_\varepsilon \int_{\mathcal{C}} \bigg(\int_0^1 \xi(\alpha(t)) \cdot \alpha'(t)\,\mathrm{d}t\bigg)\,\mathrm{d}Q_\varepsilon(\alpha) \\ &\geq \int_\Omega \xi \cdot \mathrm{d}w_Q \;+\; ||\xi||_{\star,\infty}\,(i_Q(\Omega) - ||w||(\Omega)), \end{aligned}$$

for all $\xi \in C(\Omega, \mathbb{R}^d)$. Hence, $||w - w_Q||(\Omega) + i_Q(\Omega) \leq ||w||(\Omega)$. Yet, the other inequality is always true since $||w_Q|| \leq i_Q$. Then, we get that $||w - w_Q||(\Omega) + i_Q(\Omega) = ||w||(\Omega)$. $\square$

Proposition 2.6. *Let w be an optimal flow for (BP), then there is an optimal transport plan γ for (KP) such that $w = w_\gamma$.*

Proof. From Lemma 2.5, there is an admissible traffic plan Q such that $||w - w_Q||(\Omega) + i_Q(\Omega) = ||w||(\Omega)$. The optimality of the flow w and the fact that $||w_Q|| \leq i_Q$ imply that $w_Q = w$ and $i_Q = ||w||$. Hence,

$$\begin{aligned} ||w||(\Omega) = i_Q(\Omega) = \int_{\mathcal{C}} L(\alpha)\,\mathrm{d}Q(\alpha) \geq \int_{\mathcal{C}} ||\alpha(0) - \alpha(1)||\mathrm{d}Q(\alpha) &= \int_{\Omega\times\Omega} ||x - y||\,\mathrm{d}(e_0, e_1)_\# Q \\ &\geq \min\,(\mathrm{KP}). \end{aligned}$$

Yet, the equalities $||w||(\Omega) = \min\,(\mathrm{BP}) = \min\,(\mathrm{KP})$ imply that the above inequalities are in fact equalities. This means that Q must be concentrated on segments (this is the point where the strict convexity of the norm $||\cdot||$ is needed). Also, the measure $\gamma = (e_0, e_1)_\# Q$, which belongs to $\Pi(f^+, f^-)$, must be optimal in (KP) and, we have $w = w_Q = w_\gamma$. $\square$

If $f^+ \ll \mathcal{L}^d$ or $f^- \ll \mathcal{L}^d$, we also obtain uniqueness of optimal flow w for (BP) in the Euclidean case, since an optimal γ induces the same w_γ (thanks to Proposition 2.1). This has not been investigated in the case of general norms. We will see in Chapter 6 that we use uniqueness of the optimal w, but in a case where the optimal γ is unique.

CHAPTER 3

Summability estimates via symmetrization techniques

In this chapter we consider the mass transportation problem in a compact domain Ω where a non-negative mass f^+ in the interior is sent to the boundary $\partial\Omega$. This problem appears, for instance, in some shape optimization issues. We prove summability estimates on the associated transport density σ, which is the transport density from a diffuse measure to a measure on the boundary $f^- = (P_{\partial\Omega})_\# f^+$ ($P_{\partial\Omega}$ being the projection on the boundary), hence singular. Via a symmetrization trick, as soon as Ω is convex or satisfies a uniform exterior ball condition, we prove L^p estimates (if $f^+ \in L^p$, then $\sigma \in L^p$). Finally, by a counter-example we prove that if $f^+ \in L^\infty(\Omega)$ and f^- has bounded density w.r.t. the surface measure on $\partial\Omega$, the transport density σ between f^+ and f^- is not necessarily in $L^\infty(\Omega)$, which means that the fact that $f^- = (P_{\partial\Omega})_\# f^+$ is crucial.

This chapter is taken from a joint article with F. Santambrogio, which will be published in *ESAIM: Control, Optimisation and Calculus of Variations,* [53].

3.1. About optimal transport with Dirichlet regions

In [13] & [29] a transport problem between measures with different mass is proposed, in the presence of a so-called *Dirichlet Region*. A Dirichlet region $\Sigma \subset \Omega$ is a closed set where transportation is free, and one can study the following problem

$$\min\left\{\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma,\ \gamma \in \Pi_\Sigma(f^+, f^-)\right\},$$

where

$$\Pi_\Sigma(f^+, f^-) := \left\{\gamma \in \mathcal{M}^+(\Omega\times\Omega)\ :\ ((\Pi_x)_\#\gamma)\llcorner(\Omega\setminus\Sigma) = f^+,\ ((\Pi_y)_\#\gamma)\llcorner(\Omega\setminus\Sigma) = f^-\right\}.$$

It is not difficult to see that this problem corresponds to a transport problem where it is possible to add arbitrary mass to $f^\pm$ on Σ, but the transport cost between points on Σ is set to 0. A simple variant, that we will develop in Chapter 4, concerns the case where the mass we add on Σ "pays" something, i.e. adding a cost $\int g^+(x)\,\mathrm{d}((\Pi_x)_\#\gamma)\llcorner\Sigma + \int g^-(y)\,\mathrm{d}((\Pi_y)_\#\gamma)\llcorner\Sigma$. This is what is done, for instance, in [46, 90] in the case $\Sigma = \partial\Omega$, where $g^\pm$ represent import/export costs.

Anyway, here we consider the easiest case, which is $f^- = 0$. In this case the transport plan γ can only transport mass from the density f^+ on Ω to Σ. Since its marginal $(\Pi_y)_\#\gamma$ on Σ is completely arbitrary, then it is clear that the optimal choice is to take it equal to $(P_\Sigma)_\# f^+$, where

$$P_\Sigma(x) = \operatorname{argmin}\{|x-y|,\ y \in \Sigma\}\ \text{ for all } x.$$

By this definition, P_Σ is a priori multivalued, but the argmin is a singleton on all the points where the function $x \mapsto d(x, \Sigma)$ is differentiable, which means a.e. (here as well, the assumption $f^+ \ll \mathcal{L}^d$ is crucial).

In this chapter we will concentrate on the case where Σ is a negligible (lower-dimensional) subset of Ω and, more precisely, for a "nice" domain Ω, we will consider $\Sigma = \partial\Omega$ (as in [**13, 30, 90**]). This means that we will consider the following problem

$$\min \left\{ \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma, \ \gamma \in \Pi(f^+, (P_{\partial\Omega})_\# f^+) \right\}.$$

This is also the same as

$$\min \left\{ \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma, \ (\Pi_x)_\# \gamma = f^+, \ \mathrm{spt}((\Pi_y)_\# \gamma) \subset \partial\Omega \right\}.$$

In the Beckmann's formulation, this also amounts to solve

$$\min \left\{ \int_\Omega |w| \, \mathrm{d}x : \ w \in L^1(\Omega, \mathbb{R}^d), \ \mathrm{spt}(\nabla \cdot w - f^+) \subset \partial\Omega \right\}. \tag{3.1}$$

If we write the condition $\mathrm{spt}(\nabla \cdot w - f^+) \subset \partial\Omega$ as $\nabla \cdot w = f^+$ inside $\mathring{\Omega}$, we can express this condition in a weak sense by testing functions $u \in C^1_c(\Omega)$ (or C^1 functions, vanishing on $\partial\Omega$), and the dual of this problem becomes (in Chapter 4, we will prove this duality result in a more general case, i.e., when we add boundary costs)

$$\sup \left\{ \int_\Omega u \, \mathrm{d}(f^+ - f^-) \ : \ u \in C^1(\Omega), \, |\nabla u| \le 1, \, u = 0 \ \text{on} \ \partial\Omega \right\}.$$

This relaxes on the set of Lip_1 functions vanishing on the boundary $\partial\Omega$. In this way, the Dirichlet region Σ really hosts a Dirichlet boundary condition!

REMARK 3.1. *We observe that in this framework the convexity of Ω is no longer needed to guarantee the equivalence between* (BP) *and* (KP)*. Indeed, for C^1 functions vanishing on $\partial\Omega$ we have* $\mathrm{Lip}(u) = \sup |\nabla u|$*, which is not true for Ω non convex, without the condition $u = 0$ on $\partial\Omega$. Equivalently, we can think that the transport rays $[x, T(x)]$ will never exit Ω, from the fact that the target measure is on $\partial\Omega$ and is arbitrary: in case of multiple intersections of the segment $[x, T(x)]$ with the boundary, then $P_{\partial\Omega}(x)$ would coincide with the first one.*

The pair (σ, u), where σ is the transport density between f^+ and its projection on the boundary $(P_{\partial\Omega})_\# f^+$, and u is the Kantorovich potential (which is in fact the distance function to the boundary $x \mapsto d(x, \partial\Omega)$), models (in a statical or dynamical framework) the configuration of stable or growing sandpiles, where u gives the pile shape and σ stands for sliding layer (see, for instance, [**35, 49, 52, 96**]). In the framework of both traffic congestion and membrane reinforcement, in [**30**] the authors also consider the easiest version of the Monge-Kantorovich system (2.7), which becomes

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = f^+ & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| \leq 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{3.2}$$

The question that we consider now is whether the transport density σ from f^+ to $(P_{\partial\Omega})_\# f^+$ (or equivalently, the optimal vector field w in (3.1)) is in $L^p(\Omega)$ when $f^+ \in L^p(\Omega)$. We cannot use Proposition 2.4, since in this case the target measure $(P_{\partial\Omega})_\# f^+$ is concentrated on the boundary of Ω and hence, is not L^p itself. However, from Propositions 2.2 & 2.3, we get that the transport density σ, between f^+ and its projection on the boundary, is in L^p as soon as $f^+ \in L^p$ and $p < d' := d/(d-1)$. Anyway, the summability question in the Dirichlet case is an interesting one, required in some estimates in [**30**], since it is non-trivial for $p \geq d'$ because f^- is singular. In this chapter, thanks to a symmetrization argument, we give positive answer under some geometric conditions on $\partial\Omega$. Note that [**48**] already contained a similar, but weaker, result: indeed, the methods used in [**48**] allows to get the L^p estimate we look for, for $p < \infty$, on a convex domain, since a boundary term in an integration by parts happens to have a sign. As far as results are concerned (since, anyway, the strategy is completely different), the novelty in the present work are the case $p = \infty$ and the case where Ω only satisfies an exterior ball condition, instead of being convex.

To prove that, the idea is the following: we will show, first, that the transport density σ between f^+ and $(P_{\partial\Omega})_\# f^+$ is in $L^p(\Omega)$ provided $f^+ \in L^p(\Omega)$ and Ω is a *polyhedron*. In this case, we prove that σ is equal to the restriction to Ω of the transport density from f^+ to a new density f^- obtained by symmetrizing f^+ across the faces composing the boundary $\partial\Omega$. A similar argument can be performed for domains with "round" faces (called *round polyhedra*) and, by an approximation argument, for arbitrary domains satisfying an exterior ball condition. The presentation, for completeness and pedagogical purposes, goes step-by-step from the convex case to the case of domains with an exterior ball condition, by aproximations, and is done for every p.

3.2. L^p estimates via symmetrization

In this section, we will first develop some tools, based on a symmetrization argument, to show that the transport density σ from f^+ to $(P_{\partial\Omega})_\# f^+$ is also the restriction of a transport density $\tilde{\sigma}$, which is associated with the transport from f^+ to another suitable density f^-, supported outside Ω. Then, we will apply this fact so as to produce the desired L^p estimates on σ.

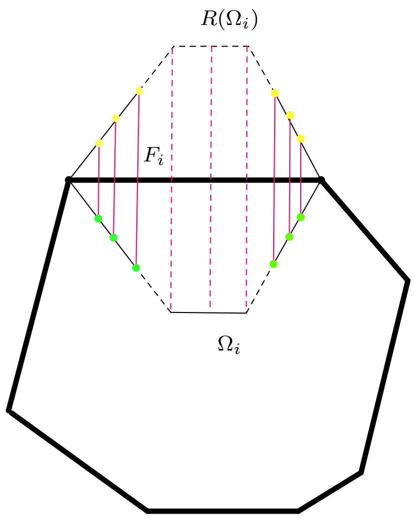

FIGURE 1

We will start by supposing that Ω is a convex polyhedron with n faces F_i $(i = 1, ..., n)$, and denote by Ω_i the set of points whose projection onto $\partial\Omega$ lies in F_i:

$$\Omega_i = \left\{ x \in \Omega \,:\, d(x, \partial\Omega) = d(x, F_i) \right\}.$$

We can write $\Omega = \bigcup_i \Omega_i$, and the union is almost disjoint (we have $\mathcal{L}^d(\Omega_i \cap \Omega_j) = 0$ for all $i \neq j$). Let R be the map obtained by reflecting with respect to the boundary each subdomain Ω_i. More precisely, for all $x \in \Omega_i \setminus \bigcup_{j \neq i} \Omega_j$, the point $R(x)$ is the reflexion of x with respect to F_i (see Figure 1). In this way R is well-defined for a.e. $x \in \Omega$.

Suppose that $f^+ \in L^p(\Omega)$ and set $f^- := R_\# f^+$. It is clear that f^- is an absolutely continuous measure, with density given by $f^-(Ry) = f^+(y)$ for all $y \in \Omega$. Let $\widetilde{\Omega}$ be any large compact convex set containing $\Omega \cup R(\Omega)$. We observe that $f^- \in L^p(\widetilde{\Omega})$ and $||f^-||_{L^p} = ||f^+||_{L^p}$.

We are now interested in the following fact concerning the corresponding transport density. We will denote by $\sigma(f^+, f^-)$ the transport density from f^+ to f^- (which is unique and belongs to L^1 as soon as $f^+ \ll \mathcal{L}^d$; thanks to Propositions 2.1 & 2.2).

PROPOSITION 3.2. *Suppose that Ω is a polyhedron. Take $f^+ \ll \mathcal{L}^d$ and define f^- as above through $f^- = R_\# f^+$. Then,*

$$(\sigma(f^+, f^-)) \llcorner \Omega = \sigma(f^+, (P_{\partial\Omega})_\# f^+).$$

Moreover, if $f^+ \in L^p(\Omega)$, then the transport density between f^+ and $(P_{\partial\Omega})_\# f^+$ is in $L^p(\Omega)$.

Proof. First, we will show that R is an optimal transport map from f^+ to f^-. Set,

$$u(x) = \begin{cases} \mathrm{d}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\mathrm{d}(x, \partial\Omega) & \text{else.} \end{cases}$$

From $|x - R(x)| = 2|x - P_{\partial\Omega}(x)|$, we have

$$\int_\Omega |x - R(x)| \, \mathrm{d}f^+(x) = 2 \int_\Omega |x - P_{\partial\Omega}(x)| \, \mathrm{d}f^+(x).$$

On the other hand, u is 1-Lip and

$$\begin{aligned} \int_\Omega u \, \mathrm{d}(f^+ - f^-) &= \int_\Omega |x - P_{\partial\Omega}(x)| \, f^+(x) \, \mathrm{d}x + \int_\Omega |R(x) - P_{\partial\Omega}(R(x))| \, f^+(x) \, \mathrm{d}x \\ &= 2 \int_\Omega |x - P_{\partial\Omega}(x)| \, \mathrm{d}f^+(x). \end{aligned}$$

Consequently, R is an optimal transport map between f^+ and f^-, and u is a Kantorovich potential. We observe that the segment $[x, R(x)]$ intersects $\partial\Omega$ at the point $P_{\partial\Omega}(x)$ and that we have

$$[x, R(x)] \cap \Omega = [x, P_{\partial\Omega}(x)].$$

But the map $x \mapsto P_{\partial\Omega}(x)$ is of course optimal in the transport from f^+ to $(P_{\partial\Omega})_\# f^+$. Hence, using (2.2), we immediately get

$$(\sigma(f^+, f^-)) \llcorner \Omega = \sigma(f^+, (P_{\partial\Omega})_\# f^+)$$

and we conclude by using Proposition 2.4. □

Now, we will give a more general construction, inspired from the previous one, which will allow to deal with the case of a domain with an exterior ball condition.

Suppose that the boundary of Ω is a union of a finite number of parts of sphere of radius

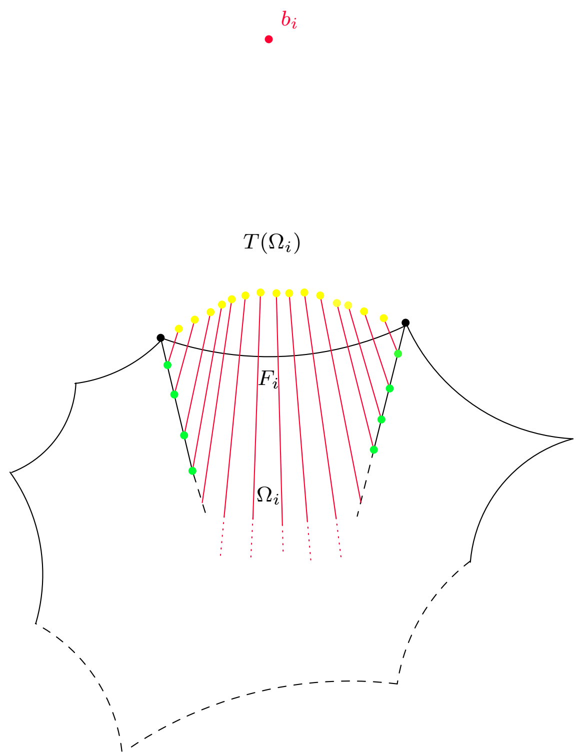

FIGURE 2

r. We will call the domains with this property *round polyhedra* (see Figure 2). Set again

$$\Omega_i := \left\{ x \in \ \Omega : \ P_{\partial\Omega}(x) \in F_i \right\}$$

where $F_i \subset \partial B(b_i, r)$ is the ith part in the boundary of Ω, contained in a sphere centered at b_i. More precisely, we suppose that $B := \bigcup_i B(b_i, r)$ disconnects $\mathbb{R}^d$ and that Ω is equal to the union of all the bounded connected components of $\mathbb{R}^d \setminus B$. We define

$$T(x) := b_i + \left(r - \frac{|x - b_i| - r}{L} \frac{r}{2} \right) \frac{x - b_i}{|x - b_i|} \quad \text{for all } \ x \in \Omega_i$$

where $L := \operatorname{diam}(\Omega)$ and b_i is the center of the sphere corresponding to F_i. Again we choose a large domain $\widetilde{\Omega}$ containing $\Omega \cup T(\Omega)$.

PROPOSITION 3.3. *Suppose that* $f^+ \in L^p(\Omega)$ *and set* $f^- := T_\# f^+$, *then* $f^- \in L^p(\widetilde{\Omega})$ *with* $||f^-||_{L^p} \leq C||f^+||_{L^p}$, *where the constant* C *only depends on* d, r *and* L.

PROOF. Compute the Jacobian of the map T: on Ω_i, we have

$$DT(x) = \frac{r}{2L}\left(-I + \frac{r+2L}{|x-b_i|}\big(I - e(x)\otimes e(x)\big)\right),$$

where $e(x) := (x-b_i)/|x-b_i|$. It is easy to see that $DT(x)$ is a symmetric matrix with one eigenvalue equals to $-\frac{r}{2L}$ and $d-1$ eigenvalues equal to

$$\lambda(x) := \frac{r}{2L}\frac{r+2L-|x-b_i|}{|x-b_i|} = \frac{r}{2L}\left(\frac{r+2L}{|x-b_i|}-1\right).$$

Using $|x-b_i| \le r+L$, we get

$$\lambda(x) \ge \frac{r}{2(r+L)}.$$

This provides, for $J := |\det(DT)|$, the lower bound

$$J(x) \ge \frac{r^d}{2^d(r+L)^{d-1}L}$$

which is, by the way, independent of i and of the number of spherical parts composing $\partial\Omega$. Moreover, from $f^-(T(x)) = f^+(x)/J(x)$, we get

$$\int |f^-(y)|^p\,dy = \int |f^-(y)|^{p-1}\,df^- = \int |f^-(T(x))|^{p-1}\,df^+ = \int \frac{f^+(x)^p}{J(x)^{p-1}}\,dx \le C\int f^+(x)^p\,dx,$$

where $C := (\inf J(x))^{1-p} > 0$. By raising to power $1/p$, this provides

$$||f^-||_{L^p} \le C(r,L,d)^{1/p-1}||f^+||_{L^p}$$

and the constant can be taken independent of p. In particular, the estimate is also valid for $p=\infty$. □

PROPOSITION 3.4. *Suppose that Ω is a round polyhedron. Take $f^+ \ll \mathcal{L}^d$ and define f^- as above through $f^- = T_\# f^+$. Then*

$$(\sigma(f^+,f^-))\llcorner\Omega = \sigma(f^+,(P_{\partial\Omega})_\# f^+).$$

Moreover, if $f^+ \in L^p(\Omega)$*, then the transport density between* f^+ *and* $(P_{\partial\Omega})_\# f^+$ *is in* $L^p(\Omega)$*.*

Proof. The proof will follow the same lines of Proposition 3.2. We will show again the optimality of T for the transport of f^+ to f^- by producing a Kantorovich potential. In this case, we set

$$u(x) = \min_{i=1,..,n} |x - b_i|.$$

The function u is of course 1-Lip and we have

$$\begin{aligned}
\int_\Omega u \,\mathrm{d}(f^+ - f^-) &= \int_\Omega u(x) f^+(x)\,\mathrm{d}x - \int_\Omega u(T(x)) f^+(x)\,\mathrm{d}x \\
&= \sum_{i=1}^n \int_{\Omega_i} u(x) f^+(x)\,\mathrm{d}x - \int_{\Omega_i} u(T(x)) f^+(x)\,\mathrm{d}x \\
&= \sum_{i=1}^n \int_{\Omega_i} (|x - b_i| - |T(x) - b_i|)\, f^+(x)\,\mathrm{d}x.
\end{aligned}$$

Yet, by definition of T, the points b_i, x and $T(x)$ are aligned (with $T(x) \in [x, b_i]$) and then, $|x - b_i| - |T(x) - b_i| = |x - T(x)|$. So, we get

$$\int_\Omega u \,\mathrm{d}(f^+ - f^-) = \int_\Omega |x - T(x)| f^+(x)\,\mathrm{d}x.$$

Consequently, T is an optimal transport map between f^+ and f^-, and u is the corresponding Kantorovich potential. Now, we observe in this case as well that the segment $[x, T(x)]$ intersects $\partial\Omega$ at the point $P_{\partial\Omega}(x)$ and that we have

$$[x, T(x)] \cap \Omega = [x, P_{\partial\Omega}(x)].$$

Hence, using (2.2) again, we immediately get

$$(\sigma(f^+, f^-)) \llcorner \Omega = \sigma(f^+, (P_{\partial\Omega})_\# f^+)$$

and we conclude by Proposition 2.4. □

Remark 3.5. *One can easily see that, both in Propositions 3.2 and 3.4, the restriction property of the transport density* σ *also holds for the optimal flow* w *of (3.1).*

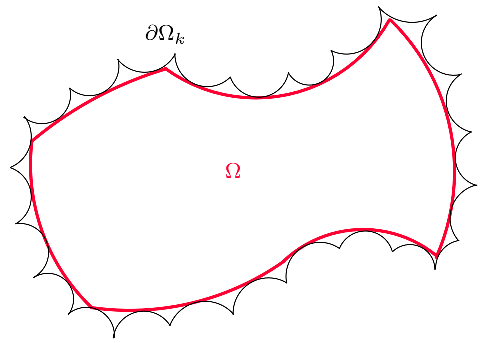

FIGURE 3

We will now generalize, via a limit procedure, the previous construction to arbitrary convex domains, or more generally domains satisfying a uniform ball condition. Before doing that, let us give a suitable definition for this last condition:

DEFINITION 3.6. *We say that a bounded domain $\Omega \subset \mathbb{R}^d$ satisfies an exterior ball condition of radius $r > 0$ if for every point $x \in \mathbb{R}^d \setminus \Omega$ and $x_0 \in \partial\Omega$ with $d(x, \Omega) = |x - x_0| > 0$ we have $d(y, \Omega) = r$ for $y := x_0 + r\frac{x-x_0}{|x-x_0|}$ (and hence, $x_0 = P_{\partial\Omega}(x)$ is also a projection of y onto $\partial\Omega$).*

This definition means that for every $x \in \partial\Omega$ there exists $y \in \mathbb{R}^d \setminus \Omega$ such that $|x - y| = r$ and $B(y, r) \cap \Omega = \emptyset$, where these balls of radius r can "roll" on the boundary. It could seem more restrictive than the usual definition which only requires the existence of a ball for every point of the boundary, but actually for compact sets they can be proven to be equivalent, up to reducing the radius r. However, for simplicity, we just choose the definition which best fits the use we will make of it. Now, we need an approximation lemma about sets satisfying an exterior ball condition. More precisely:

LEMMA 3.7. *For every bounded domain $\Omega \subset \mathbb{R}^d$ satisfying an exterior ball condition of radius $r > 0$, there exists a sequence of round polyhedra Ω_k such that*

- $\Omega \subset \Omega_k$,
- $\mathrm{diam}(\Omega_k) \le \mathrm{diam}(\Omega) + 2r$,
- $\partial\Omega_k$ *is made of parts of sphere of radius* r,
- $\partial\Omega_k \to \partial\Omega$ *in the Hausdorff sense, and* $P_{\partial\Omega_k}(x) \to P_{\partial\Omega}(x)$ *for a.e.* $x \in \Omega$.

PROOF. Set $A := \{x : d(x, \Omega) = r\}$ and let $A_k \subset A$ be a sequence of finite sets converging in the Hausdorff sense to A (to produce them, just take a countable dense subset of A, order its points, and put the first k points in A_k). Set $\Omega_k := \{x : d(x,\Omega) \le r \le d(x,A_k)\}$. For large k, the set Ω_k is a round polyhedron with boundary composed of parts of spheres of radius r centered at points of A_k. Indeed, it is clear that the points on $\partial\Omega_k$ are contained either in these spheres, or in A, but as soon as the Hausdorff distance between A_k and A is smaller than r, we have $d(x,A_k) < r$ for every $x \in A$, hence $A \cap \Omega_k = \emptyset$ and the points of A cannot be on the boundary of Ω_k. Moreover, one easily see that we have $\Omega \subset \Omega_k$ and that Ω_k is contained in a compact set. Up to a subsequence, we can suppose $\Omega_k \to \Omega'$ in the Hausdorff sense, with $\Omega \subset \Omega'$. Yet, passing to the limit in the definition of Ω_k, we get $\Omega' \subset \{x : d(x,\Omega) \le r \le d(x,A)\}$. This is enough to obtain $\Omega' = \Omega$: take a point x with $d(x,\Omega) \le r \le d(x,A)$ and suppose that it does not belong to Ω; let $x_0 \in \partial\Omega$ be such that $|x - x_0| = d(x,\Omega)$ and set $y := x_0 + r\frac{x-x_0}{|x-x_0|}$. From the definition 3.6, we have that $y \in A$, which is a contradiction, as $d(x,y) < r$.

Hence, we have $\Omega \subset \Omega_k$ and $\Omega_k \to \Omega$ in the Hausdorff sense. The last part of the statement (convergence of the boundaries and of the projections onto the boundaries) is a general consequence of these facts. □

We can now state the following

Proposition 3.8. *Suppose that $\Omega \subset \mathbb{R}^d$ is a compact domain satisfying a uniform exterior ball condition of radius $r > 0$. Then there exists a larger domain $\widetilde{\Omega}$ such that for every positive measure $f^+ \ll \mathcal{L}^d$, there exists $f^- \ll \mathcal{L}^d$, supported on $\widetilde{\Omega} \setminus \Omega$ with*

$$(\sigma(f^+, f^-)) \llcorner \Omega = \sigma(f^+, (P_{\partial\Omega})_\# f^+).$$

Moreover, for every $p \in [1, +\infty]$, we have

$$||f^-||_{L^p(\widetilde{\Omega})} \le C||f^+||_{L^p(\Omega)},$$

where C is a constant only depending on d, r and $L := \mathrm{diam}(\Omega)$.

Proof. It is enough to act by approximation. In the case where Ω is convex, we can write it as an intersection of half-spaces, and hence we can approximate Ω as the limit of a sequence of polyhedra Ω_k, while in the case where Ω satisfies a uniform exterior ball condition, we will write it as a limit of round polyhedra (see Figure 3) as we pointed out in Lemma 3.7. Then, we just build the reflection maps R_k (or T_k) as in Propositions 3.2 and 3.4, and we get a sequence of measures f_k^- supported on $\widetilde{\Omega} \setminus \Omega_k$ with $||f_k^-||_{L^p(\widetilde{\Omega})} \le C||f^+||_{L^p(\Omega_k)} = C||f^+||_{L^p(\Omega)}$. We also have

$$(\sigma(f^+, f_k^-)) \llcorner \Omega = \sigma(f^+, (P_{\partial\Omega_k})_\# f^+) \llcorner \Omega.$$

Then, it is enough to extract a converging subsequence from the sequence f_k^-, note that we have $(P_{\partial\Omega_k})_\# f^+ \rightharpoonup (P_{\partial\Omega})_\# f^+$, and use Proposition 1.3. □

As a consequence, we can now obtain

THEOREM 3.9. *Suppose that Ω satisfies a uniform exterior ball condition of radius $r > 0$. Then, the transport density σ between f^+ and $(P_{\partial\Omega})_\# f^+$ is in $L^p(\Omega)$ provided f^+ is in $L^p(\Omega)$, and*

$$||\sigma||_{L^p(\Omega)} \leq C||f^+||_{L^p(\Omega)},$$

where the constant C only depends on d, r and $L = \mathrm{diam}(\Omega)$.

PROOF. We just need to use Proposition 3.8, which guarantees that σ is the restriction to Ω of the transport density between two L^p measures. □

We finish this section by two remarks on the proof of the above result.

REMARK 3.10. *In this particular case where the transport has not a fixed target measure on $\partial\Omega$, the transport density σ linearly depends on f^+: in this case, L^p estimates could be obtained via interpolation (via the celebrated Marcinkiewicz interpolation theorem, [**86, 114**]) as soon as one has L^1 and L^∞ estimates. Since L^1 (and L^p for $p < d'$) are well-known, this means that it would be enough to write L^∞ estimates. Yet, we did not see any significant simplification in concentrating on L^∞ estimates instead of L^p, which is the reason why we decided not to evoke general interpolation theorems but we performed explicit estimates.*

REMARK 3.11. *Another observation concerns the fact that we proved Proposition 3.8 by approximation. Apart from the fact that we first developed the convex case (just for the sake of simplicity), the reader would have preferred a direct formulation, valid in the case of an arbitrary domain Ω with an exterior ball condition, instead of passing through round polyhedra. This would be possible, by defining a map $T(x) := P_{\partial\Omega}(x) + c(P_{\partial\Omega}(x) - x)$, for small $c > 0$. It can be proven, by studying the properties of the Jacobian of $P_{\partial\Omega}$, that T is injective and $|\det(DT)|$ is bounded from below as soon as c is small (depending on r and L), but we considered that the proof in the case of round polyhedra was easier.*

3.3. An L^∞ bound on f^- with respect to the surface measure on $\partial\Omega$ is not enough

In this section, we show that the L^∞ estimates for the transport density (again, note by Remark 3.10 that the case $p = \infty$ is the most interesting one) fail if we only assume summability (or boundedness) of the densities of f^+ w.r.t. the Lebesgue measure on Ω and of f^- w.r.t. the Hausdorff measure $\mathcal{H}^{d-1}$ on $\partial\Omega$. Indeed, when we consider a domain Ω with a uniform exterior ball condition and we take $f^+ \in L^\infty$, we can easily prove that $(P_{\partial\Omega})_\# f^+$ has a bounded density w.r.t. $\mathcal{H}^{d-1} \llcorner \partial\Omega$. One could wonder whether this is the correct assumption to prove, for instance, $\sigma \in L^\infty$, and the answer is negative.

We will construct an example of $f^\pm$, where f^+ has a bounded density w.r.t. $\mathcal{L}^d$ in Ω and f^- w.r.t. $\mathcal{H}^{d-1} \llcorner \partial\Omega$ (for instance Ω is a big square containing the support of f^+ and its boundary contains the support of f^-), but $\sigma \notin L^\infty$ (we will also investigate the summability of σ). Set

$$f^+ := \mathcal{L}^2 \llcorner A, \quad f^- := \mathcal{H}^1 \llcorner ([2, 3] \times \{0\})$$

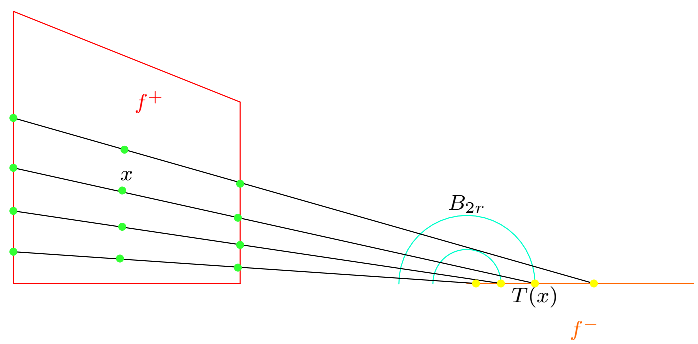

FIGURE 4

where A is a trapeze with vertices $(0,0)$, $(1,0)$, $(1,\frac{4}{5})$ and $(0,\frac{6}{5})$ (see Figure 4). For every $\varepsilon \in [0,1]$, let l_ε be the segment joining the two points $(0,w(\varepsilon))$ and $(2+\varepsilon,0)$, where

$$w(\varepsilon) := \frac{2(2+\varepsilon)\varepsilon}{3+2\varepsilon}.$$

First, it is easy to see that $f^+(\Delta_\varepsilon) = f^-(\Delta_\varepsilon)$, for every $\varepsilon \in [0,1]$, where Δ_ε is the triangle limited by $(0,0)$, $(2+\varepsilon,0)$ and $(0,w(\varepsilon))$. Then by [95] (or exactly as in Chapter 5), we can construct an optimal mapping T, which pushes f^+ to f^-, with $\{l_\varepsilon\}$ as its transport rays.

Let σ be the transport density between f^+ and f^-. For simplicity of notation, we denote the ball of center $(2,0)$ and radius r by B_r, and in this case we have

$$\sigma(B_{2r}) = \int \mathcal{H}^1(B_{2r} \cap [x,y])\,\mathrm{d}\gamma(x,y) \geq r\,\gamma\left(\{(x,y) : B_r \cap [x,y] \neq \emptyset\}\right),$$

where we used the fact that for every (x,y) s.t. $B_r \cap [x,y] \neq \emptyset$, we have $\mathcal{H}^1(B_{2r} \cap [x,y]) \geq r$. Note that there exists a value $\varepsilon_r \in (0,1)$ such that $\{x : B_r \cap [x,T(x)] \neq \emptyset\} = \Delta_{\varepsilon_r}$ and l_{ε_r} is tangent to the ball B_r. Hence,

$$\sigma(B_{2r}) \geq r f^+(\{x :\ B_r \cap [x,T(x)] \neq \emptyset\}) = r f^+(\Delta_{\varepsilon_r}) \simeq r\varepsilon_r.$$

If we denote by θ the angle between the two segments $[(0,0),(2+\varepsilon_r,0)]$ and l_{ε_r}, then we have

$$\sin(\theta) = \frac{r}{\varepsilon_r}.$$

Yet, $\sin(\theta) \simeq \varepsilon_r$ for r small enough. So, we get $\varepsilon_r \simeq r^{\frac{1}{2}}$. Thus, for r small enough

$$\sigma(B_{2r}) \geq c r^{\frac{3}{2}}, \tag{3.3}$$

which implies that σ cannot be bounded in a neighborhood of $(2,0)$, otherwise we would have

$$Cr^2 = ||\sigma||_{L^\infty(\Delta_1)} |B_{2r}| \geq \sigma(B_{2r}) \geq c r^{\frac{3}{2}}$$

which is a contradiction for small r. In addition, it is possible to see $\sigma \notin L^4(\Delta_1)$, otherwise, by Hölder inequality, we would get

$$\frac{\sigma(B_{2r})}{r^{3/2}} \leq \frac{|B_{2r}|^{3/4} \left(\int_{B_{2r}} \sigma^4\right)^{1/4}}{r^{3/2}} \to 0,$$

which is a contradiction with (3.3).

Actually, a finer analysis even proves $\sigma \in L^p(\Delta_1)$ if and only if $p < 3$. To prove this we need to use heavier computations. Fix ε_0 small enough and take $x \in \Delta_{\varepsilon_0}$: there exist $\varepsilon \in [0, \varepsilon_0]$ and $s \in [0,1]$ such that

$$x = (1-s)(2+\varepsilon, 0) + s(0, w(\varepsilon)).$$

Recalling (2.1), for all $\varphi \in C(\Delta_{\varepsilon_0})$, we have

$$< \sigma, \varphi > := \int_0^1 \int_{\Delta_{\varepsilon_0}} |x - T(x)|\, \varphi((1-t)x + tT(x))\, f^+(x)\, \mathrm{d}x\, \mathrm{d}t.$$

So, by a change of variable, we get, in the variable (ε, s),

$$\begin{aligned} \sigma(\varepsilon, s) &= \frac{\sqrt{(2+\varepsilon)^2 + w(\varepsilon)^2} \int_s^1 f^+((1-t)(2+\varepsilon), t w(\varepsilon)) |J(\varepsilon, t)|\, \mathrm{d}t}{|J(\varepsilon, s)|} \\ &= \frac{\sqrt{(2+\varepsilon)^2 + w(\varepsilon)^2} \int_{1-\frac{1}{2+\varepsilon}}^1 |J(\varepsilon, t)|\, \mathrm{d}t}{|J(\varepsilon, s)|} \simeq \frac{\int_{1-\frac{1}{2+\varepsilon}}^1 |J(\varepsilon, t)|\, \mathrm{d}t}{|J(\varepsilon, s)|}, \end{aligned}$$

where $|J| := |D_{(\varepsilon,s)}(x_1, x_2)|$. Using the fact that $|J(\varepsilon, s)| \simeq s + \varepsilon$, we get

$$\sigma(\varepsilon, s) \simeq \frac{1}{s+\varepsilon}$$

and

$$||\sigma||^p_{L^p(\Delta_1)} \simeq \int_0^{\varepsilon_0} \int_0^1 \frac{1}{(s+\varepsilon)^{p-1}} \,\mathrm{d}s \,\mathrm{d}\varepsilon \simeq \int_0^{\varepsilon_0} \frac{1}{\varepsilon^{p-2}} \,\mathrm{d}\varepsilon.$$

Notice that as $d = 2$ and $f^+ \in L^\infty(\Delta_1)$, by Proposition 2.3 we know that automatically $\sigma \in L^p(\Delta_1)$ for all $p < 2$. The fact that here we get $\sigma \in L^p(\Delta_1)$ for all $p < 3$ depends on the fact that we send a mass f^+ to a mass f^- which is distributed on a segment, and not to a Dirac mass. We are not in the worst possible case!

So, we have shown that if f^+ is a bounded density w.r.t. $\mathcal{L}^d$ and f^- is a bounded density w.r.t. $\mathcal{H}^{d-1} \llcorner \partial\Omega$ (and, f^- is different than the projection of f^+ onto $\partial\Omega$), then the transport density σ between f^+ and f^- is not, in general, in L^p, for all $p \geq 3$. In Chapter 6, we will study the summability of the transport density σ between two measures, f^+ and f^-, concentrated on the boundary.

CHAPTER 4

Summability estimates with boundary costs

In this chapter we analyze a mass transportation problem in a compact domain with the possibility to transport mass to/from the boundary, paying a cost given by the Euclidean distance plus an extra cost depending on the exit/entrance point. This problem appears in import/export model, as well as in some shape optimization problems. We study the L^p summability of the transport density σ which does not follow from Proposition 2.4, as the target measures are not absolutely continuous but they have some parts which are concentrated on the boundary. We also provide the relevant duality arguments to connect the corresponding Beckmann and Kantorovich problems to a formulation with Kantorovich potentials with Dirichlet boundary conditions.

This chapter is taken from my article [54], which will be published in *Journal of Convex Analysis.*

In [90], the authors introduce a variant of the classical Kantorovich problem (KP). They study a mass transportation problem between two masses f^+ and f^- (which do not have a priori the same total mass) with the possibility of transporting some mass to/from the boundary, paying the transport cost $c(x,y) = |x-y|$ plus an extra cost $g^-(y)$ for each unit of mass that comes out from a point $y \in \partial\Omega$ (the export taxes) or $-g^+(x)$ for each unit of mass that enters at the point $x \in \partial\Omega$ (the import taxes). This means that we can use $\partial\Omega$ as an infinite reserve/repository, we can take as much mass as we wish from the boundary, or send back as much mass as we want, provided that we pay the transportation cost plus the import/export taxes. In other words, given the set

$$\Pi b(f^+, f^-) := \left\{ \gamma \in \mathcal{M}^+(\Omega \times \Omega) \,:\, ((\Pi_x)_\# \gamma) \llcorner \mathring{\Omega} = f^+,\; ((\Pi_y)_\# \gamma) \llcorner \mathring{\Omega} = f^- \right\},$$

we minimize the quantity

$$\text{(KPb)} \qquad \min \left\{ \int_{\Omega \times \Omega} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega} g^- \,\mathrm{d}(\Pi_y)_\# \gamma - \int_{\partial\Omega} g^+ \,\mathrm{d}(\Pi_x)_\# \gamma \,:\, \gamma \in \Pi b(f^+, f^-) \right\}.$$

The equality $\min(\mathrm{KP}) = \min(\mathrm{BP})$ (see Section 2.3) implies that $\min(\mathrm{KPb}) = \min(\mathrm{BPb})$, where (BPb) is the following variant of (BP)

$$\min\left\{|w|(\Omega)+\int_{\partial\Omega} g^-\,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \,:\, w\in\mathcal{M}^d(\Omega),\, \chi\in\mathcal{M}(\partial\Omega),\, \nabla\cdot w = f+\chi\right\}.$$

On the other hand, the authors of [90] prove that the dual of (KPb) (or equivalently, (BPb)) is the following

$$\text{(DPb)}\qquad \sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\,:\, u\in \mathrm{Lip}_1(\Omega),\ g^+\le u\le g^- \ \text{ on }\ \partial\Omega\right\}.$$

We will give an alternative proof for this duality formula that we consider simpler than that in [90]. In fact, one can easily see that it follows immediately from a formal inf-sup exchange: we may look at the problem (KPb), we get

$$\min\left\{\sup\left\{\int_\Omega u\,\mathrm{d}(f+\chi)\,:\, u\in\mathrm{Lip}_1(\Omega)\right\}+\int_{\partial\Omega} g^-\,\chi^- - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \,:\, \chi^\pm\in\mathcal{M}^+(\partial\Omega)\right\}$$

and consider interchanging inf and sup:

$$=\sup\left\{\int_\Omega u\,\mathrm{d}f\,+\,\inf\left\{\int_{\partial\Omega}(u-g^+)\,\mathrm{d}\chi^+ + \int_{\partial\Omega}(g^- -u)\,\mathrm{d}\chi^- : \chi^\pm\in\mathcal{M}^+(\partial\Omega)\right\} : u\in\mathrm{Lip}_1(\Omega)\right\}.$$

Yet,

$$\inf\left\{\int_{\partial\Omega}(u-g^+)\mathrm{d}\chi^+ + \int_{\partial\Omega}(g^- -u)\mathrm{d}\chi^- : \chi^\pm\in\mathcal{M}^+(\partial\Omega)\right\} = \begin{cases} 0 & \text{if } g^+\le u\le g^- \text{ on } \partial\Omega\\ -\infty & \text{else.}\end{cases}$$

4.1. Monge-Kantorovich problems with boundary costs: existence, characterization and duality

In this section, we analyze the problem (KPb). Except for the duality proof, we will also decompose it into subproblems. One of this subproblems involves a transport plan γ_{ib} (with its transport density σ_{ib}), where i and b stand for interior and boundary (conversely, we also have a transport plan γ_{bi} with σ_{bi}). We will show that some questions, including summability of the transport density σ, reduce to the study of the summability of σ_{ib} and σ_{bi}. First of all, we suppose $g^\pm\in C(\partial\Omega)$ and we assume the following inequality

$$g^+(x)-g^-(y)\le |x-y| \text{ for all } x,\, y\in\partial\Omega. \tag{4.1}$$

Under this assumption, we have the following:

PROPOSITION 4.1. *(KPb) reaches a minimum.*

PROOF. Set

$$K(\gamma) := \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma \; + \int_{\partial\Omega} g^- \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+ \,\mathrm{d}(\Pi_x)_\#\gamma, \;\; \forall\; \gamma \in \mathcal{M}^+(\Omega\times\Omega).$$

Then, K is continuous with respect to the weak convergence of measures in $\Pi b(f^+, f^-)$. Indeed, if $(\gamma_n)_n$ is a sequence in $\Pi b(f^+, f^-)$ such that $\gamma_n \rightharpoonup \gamma$, then, for every n, there exists $\chi_n^\pm \in \mathcal{M}^+(\partial\Omega)$ such that

$$(\Pi_x)_\#\gamma_n = f^+ + \chi_n^+ \,, \;\; (\Pi_y)_\#\gamma_n = f^- + \chi_n^-$$

and

$$\chi_n^\pm \rightharpoonup \chi^\pm,$$

where $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. As $g^\pm \in C(\partial\Omega)$, then

$$K(\gamma_n) \to K(\gamma).$$

On the other hand, we observe that if $\gamma \in \Pi b(f^+, f^-)$ and $\tilde{\gamma} := \gamma \llcorner (\partial\Omega\times\partial\Omega)^c$, then $\tilde{\gamma}$ also belongs to $\Pi b(f^+, f^-)$. In addition, we have

$$\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega} g^- \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+ \,\mathrm{d}(\Pi_x)_\#\gamma$$

$$= \int_{\partial\Omega\times\partial\Omega} (|x-y| + g^-(y) - g^+(x)) \,\mathrm{d}\gamma + \int_{(\partial\Omega\times\partial\Omega)^c} |x-y| \,\mathrm{d}\gamma + \int_{\Omega^\circ\times\partial\Omega} g^-(y) \,\mathrm{d}\gamma - \int_{\partial\Omega\times\Omega^\circ} g^+(x) \,\mathrm{d}\gamma.$$

As

$$|x-y| + g^-(y) - g^+(x) \geq 0,$$

we get

$$\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma \; + \int_{\partial\Omega} g^- \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+ \,\mathrm{d}(\Pi_x)_\#\gamma$$

$$\geq \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\tilde{\gamma} \; + \int_{\partial\Omega} g^- \,\mathrm{d}(\Pi_y)_\#\tilde{\gamma} - \int_{\partial\Omega} g^+ \,\mathrm{d}(\Pi_x)_\#\tilde{\gamma}.$$

Now, let $(\gamma_n)_n \subset \Pi b(f^+, f^-)$ be a minimizing sequence. Then, we can suppose that

$$\gamma_n(\partial\Omega\times\partial\Omega) = 0.$$

In this case, we get

$$\begin{aligned}\gamma_n(\Omega\times\Omega) &\leq \gamma_n(\Omega^0\times\Omega)+\gamma_n(\Omega\times\Omega^0)\\ &= f^+(\Omega)+f^-(\Omega).\end{aligned}$$

Hence, there exist a subsequence $(\gamma_{n_k})_{n_k}$ and a plan $\gamma\in\Pi b(f^+,f^-)$ such that $\gamma_{n_k}\rightharpoonup\gamma$. But, the continuity of K implies that this plan γ is in fact a minimizer for (KPb). □

Let us note that the proof of the duality formula of (KPb), in [90], is based on the Fenchel-Rocafellar duality Theorem and it is decomposed into two steps: firstly, the authors suppose that the inequality in (4.1) is strict and secondly, they use an approximation argument to cover the other case. Now, we want to give an alternative proof for this duality formula, similar to the one introduced in Proposition 1.2, but, here, via a perturbation of the boundary costs $g^\pm$.

PROPOSITION 4.2. *Under the assumption (4.1), we have the following equality*

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega}g^-\,\mathrm{d}(\Pi_y)_\#\gamma-\int_{\partial\Omega}g^+\,\mathrm{d}(\Pi_x)_\#\gamma\,:\,\gamma\in\Pi b(f^+,f^-)\right\}\quad \textit{(KPb)}$$

$$=\sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\,:\,u\in\mathrm{Lip}_1(\Omega),\,g^+\leq u\leq g^-\ \textit{on}\ \partial\Omega\right\}\quad \textit{(DPb).}$$

Notice that if (4.1) is not satisfied, then both sides of this equality are $-\infty$*.*

PROOF. For every $(p^+,p^-)\in C(\partial\Omega)\times C(\partial\Omega)$, set

$$H(p^+,p^-):=-\sup\left\{\int_\Omega u\,\mathrm{d}(f^+-f^-)\,:\,u\in\mathrm{Lip}_1(\Omega),\,g^++p^+\leq u\leq g^--p^-\ \text{on}\ \partial\Omega\right\}.$$

First, it is easy to see that $H(p^+,p^-)\in\mathbb{R}\cup\{+\infty\}$. In addition, we claim that H is convex and l.s.c.

- For convexity: take $t\in(0,1)$ and (p_0^+,p_0^-), $(p_1^+,p_1^-)\in C(\partial\Omega)\times C(\partial\Omega)$, and let u_0, u_1 be their optimal potentials. Set

$$p_t^+:=(1-t)p_0^++tp_1^+\,,\ p_t^-:=(1-t)p_0^-+tp_1^-$$

and

$$u_t:=(1-t)u_0+tu_1.$$

As
$$g^+ + p_0^+ \le u_0 \le g^- - p_0^- \quad \text{and} \quad g^+ + p_1^+ \le u_1 \le g^- - p_1^- \text{ on } \partial\Omega,$$
then
$$g^+ + p_t^+ \le u_t \le g^- - p_t^- \text{ on } \partial\Omega.$$

In addition, u_t is 1-Lipschitz. Consequently, u_t is admissible in the max defining $-H(p_t^+, p_t^-)$ and then,
$$H(p_t^+, p_t^-) \le -\int_\Omega u_t \, \mathrm{d}(f^+ - f^-) = (1-t)H(p_0^+, p_0^-) + tH(p_1^+, p_1^-).$$

• For semi-continuity: take $p_n^+ \to p^+$ and $p_n^- \to p^-$ uniformly on $\partial\Omega$. Let $(p_{n_k}^+, p_{n_k}^-)_{n_k}$ be a subsequence realizing the lim inf of $H(p_n^+, p_n^-)$ (for simplicity of notation, we still denote it $(p_n^+, p_n^-)_n)$ and let $(u_n)_n$ be their corresponding optimal potentials. As u_n is a 1-Lipschitz function and $(p_n^+)_n$, $(p_n^-)_n$ are equibounded, then, by Ascoli-Arzelà Theorem, there exist a 1-Lipschitz function u and a subsequence $(u_{n_k})_{n_k}$ such that $u_{n_k} \to u$ uniformly in Ω. As
$$g^+ + p_{n_k}^+ \le u_{n_k} \le g^- - p_{n_k}^- \text{ on } \partial\Omega,$$
then
$$g^+ + p^+ \le u \le g^- - p^- \text{ on } \partial\Omega.$$

Consequently, the potential u is admissible in the max defining $-H(p^+, p^-)$ and, one has
$$H(p^+, p^-) \le -\int_\Omega u \, \mathrm{d}(f^+ - f^-) = \liminf_n H(p_n^+, p_n^-).$$

Hence, we get $H^{\star\star} = H$. In particular, $H^{\star\star}(0,0) = H(0,0)$. But by the definition of H, we have $H(0,0) = -\sup(\text{DPb})$. On the other hand, let us compute $H^{\star\star}(0,0)$. Take $\chi^\pm$ in $\mathcal{M}(\partial\Omega)$, then we have
$$H^\star(\chi^+, \chi^-) := \sup_{p^\pm \in C(\partial\Omega)} \left\{ \int_{\partial\Omega} p^+ \, \mathrm{d}\chi^+ + \int_{\partial\Omega} p^- \, \mathrm{d}\chi^- - H(p^+, p^-) \right\}$$
$$= \sup_{p^\pm \in C(\partial\Omega),\, u \in \mathrm{Lip}_1(\Omega)} \left\{ \int_{\partial\Omega} p^+ \, \mathrm{d}\chi^+ + \int_{\partial\Omega} p^- \, \mathrm{d}\chi^- + \int_\Omega u \, \mathrm{d}(f^+ - f^-) : g^+ + p^+ \le u \le g^- - p^- \text{ on } \partial\Omega \right\}.$$

If $\chi^+ \notin \mathcal{M}^+(\partial\Omega)$, i.e. there exists $p_0^+ \in C(\partial\Omega)$ such that $p_0^+ \ge 0$ and $\int_{\partial\Omega} p_0^+ \, \mathrm{d}\chi^+ < 0$, we may see that
$$H^\star(\chi^+, \chi^-) \ge -n \int_{\partial\Omega} p_0^+ \, \mathrm{d}\chi^+ + \int_{\partial\Omega} g^- \, \mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \, \mathrm{d}\chi^+ \underset{n \to +\infty}{\longrightarrow} +\infty$$

and similarly, if $\chi^- \notin \mathcal{M}^+(\partial\Omega)$. Now, suppose both $\chi^\pm \in \mathcal{M}^+(\partial\Omega)$. As $g^+ + p^+ \le u \le g^- - p^-$ on $\partial\Omega$, we should choose the largest possible p^+ and p^-, i.e. $p^+ = u - g^+$ and $p^- = g^- - u$ on $\partial\Omega$. Hence, we get

$$H^\star(\chi^+,\chi^-) = \sup\left\{ \int_\Omega u \,\mathrm{d}(f+\chi) \,:\, u \in \mathrm{Lip}_1(\Omega)\right\} + \int_{\partial\Omega} g^- \,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+.$$

By Proposition 1.2, we infer that

$$H^\star(\chi^+,\chi^-) = \min\left\{\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+ + \chi^+, f^- + \chi^-)\right\} + \int_{\partial\Omega} g^- \,\mathrm{d}\chi^- - \int_{\partial\Omega} g^+ \,\mathrm{d}\chi^+$$

$$= \min\left\{\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega} g^- \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g^+ \,\mathrm{d}(\Pi_x)_\#\gamma \,:\, \gamma \in \Pi(f^+ + \chi^+, f^- + \chi^-)\right\}.$$

Finally, we have

$$H^{\star\star}(0,0) = \sup\left\{ - H^\star(\chi^+,\chi^-) \,:\, \chi^+,\, \chi^- \in \mathcal{M}^+(\partial\Omega)\right\} = -\min\,(\mathrm{KPb}). \quad \square$$

Let γ be a minimizer for (KPb) and let us denote by χ^+ and χ^- the two non-negative measures concentrated on the boundary of Ω such that $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. Then, we may see easily that γ is also a minimizer for the following problem

$$\min\left\{\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma \,:\, \gamma \in \Pi(\mu^+,\mu^-)\right\}$$

where $\mu^\pm := f^\pm + \chi^\pm$. Moreover, if u is a maximizer for (DPb), then we have the following:

PROPOSITION 4.3. *The function u is also a Kantorovich potential between μ^+ and μ^-, i.e., u solves the following problem*

$$\sup\left\{\int_\Omega \varphi \,\mathrm{d}(\mu^+ - \mu^-) \,:\, \varphi \in \mathrm{Lip}_1(\Omega)\right\}.$$

PROOF. Let φ be a Kantorovich potential between μ^+ and μ^-. As u is 1-Lip and $g^+ \le u \le g^-$ on $\partial\Omega$, then we have

$$\int_\Omega u\,\mathrm{d}(f^+ - f^-) + \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ - \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- \le \int_\Omega u\,\mathrm{d}(\mu^+ - \mu^-) \le \int_\Omega \varphi\,\mathrm{d}(\mu^+ - \mu^-).$$

Yet, by Proposition 1.2, we have

$$\int_\Omega \varphi\,\mathrm{d}(\mu^+ - \mu^-) = \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma,$$

where γ is the fixed optimal transport plan for (KPb). Using Proposition 4.2, we infer that the above inequalities are in fact equalities and u is a Kantorovich potential between μ^+ and μ^-. $\square$

Now, suppose that Ω is convex and, set $w := -\sigma\nabla u$, where σ is the transport density associated with the optimal transport plan γ. Then, using Proposition 4.3, we infer that the vector measure w solves

$$\min\left\{|w|(\Omega)\,:\,w \in \mathcal{M}^d(\Omega),\,\nabla\cdot w = \mu^+ - \mu^-\right\}.$$

Yet, from the fact that $\min(\mathrm{BPb}) = \min(\mathrm{KPb})$, we can conclude that the vector measure w and the boundary measure χ solve together (BPb) ($\chi^\pm$ are the import/export measures). In addition, the pair (σ, u) solves the following system, which is a variant of the Monge-Kantorovich one:

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f^+ - f^- & \text{in } \Omega, \\ g^+ \le u \le g^- & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{4.2}$$

However, the same result will be true, even if Ω is not convex, by using the following:

PROPOSITION 4.4. *Suppose that*

$$|g^+(x) - g^-(y)| \le |x-y| \quad \textit{for all } x,\,y \in \partial\Omega,$$

i.e. $g^+ = g^- := g$ *where* g *is a 1-Lipschitz function on* $\partial\Omega$*. Then there exists a minimizer* $\gamma^\star$ *for (KPb) such that for all* $(x,y) \in \mathrm{spt}(\gamma^\star)$*, we have* $[x,y] \subset \Omega$*. In addition, if* g *is* λ*-Lipschitz with* $\lambda < 1$*, then for any minimizer* γ *of (KPb) and for all* $(x,y) \in \mathrm{spt}(\gamma)$*,* $[x,y] \subset \Omega$*.*

PROOF. First of all, set

$$E := \Big\{ (x,y) \in \Omega \times \Omega,\ [x,y] \subset \Omega \Big\}.$$

Let us define the map p^+ as follows

$$\begin{aligned} p^+ : \Omega \times \Omega &\mapsto \partial\Omega \times \Omega \\ (x,y) &\mapsto (x',y) \end{aligned}$$

where x' is the last point of intersection between the segment $[x,y]$ and the boundary if $(x,y) \notin E$ and $x' = x$ else.

Also set

$$\begin{aligned} p^- : \Omega \times \Omega &\mapsto \Omega \times \partial\Omega \\ (x,y) &\mapsto (x,y') \end{aligned}$$

where y' is the first point of intersection between the segment $[x,y]$ and the boundary if $(x,y) \notin E$ and $y' = y$ else.

Take a minimizer γ for (KPb) and, set

$$\gamma^\star := \gamma \llcorner E \,+\, (p^+)_\#(\gamma \llcorner E^c) \,+\, (p^-)_\#(\gamma \llcorner E^c).$$

It is easy to see that $\gamma^\star \in \Pi b(f^+, f^-)$. Moreover, $\gamma^\star$ is better than γ in (KPb), i.e., $K(\gamma^\star) \leq K(\gamma)$. Indeed, we have

$$\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma^\star + \int_{\partial\Omega} g \,\mathrm{d}(\Pi_y)_\#\gamma^\star - \int_{\partial\Omega} g \,\mathrm{d}(\Pi_x)_\#\gamma^\star$$

$$= \int_E |x-y| \,\mathrm{d}\gamma + \int_{E^c} (|x-y'| + |x'-y| + g(y') - g(x')) \,\mathrm{d}\gamma + \int_{\partial\Omega} g \,\mathrm{d}(\Pi_y)_\#\gamma - \int_{\partial\Omega} g \,\mathrm{d}(\Pi_x)_\#\gamma.$$

Yet,

$$|x-y'| + |x'-y| + g(y') - g(x') \leq |x-y'| + |x'-y| + |x'-y'| = |x-y|.$$

Consequently, $\gamma^\star$ is a minimizer for (KPb) and, for all $(x,y) \in \mathrm{spt}(\gamma^\star)$, we have $[x,y] \subset \Omega$. The second statement follows directly from the last inequality, which becomes strict. □

Using Proposition 4.4, even if Ω is not convex, we infer that there is a solution (σ, u) to the following system, provided that the boundary cost g is 1-Lip,

$$\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma-\text{a.e.} \end{cases} \tag{4.3}$$

We note that this system describe the growth of a sandpile on a bounded table, with a wall on the boundary of a height g, under the action of a vertical source here modeled by f (see [46]).

Now, we are interested to study the L^p summability of the transport density σ, which does not follow from Proposition 2.4, since in this case the source and target measures are not in L^p as they have some parts, $\chi^{\pm}$, which are concentrated on $\partial\Omega$. First of all, we note that to get a L^p summability on σ, it is natural to suppose that g is strictly better than 1-Lipschitz. Indeed, we can find a positive density $f \in L^p(\Omega)$ and a boundary measure $\chi \in \mathcal{M}^+(\partial\Omega)$ such that the transport density σ between f and χ is not in $L^p(\Omega)$ (see, for instance, Section 3.3). So, if u is the Kantorovich potential between f and χ (which is 1-Lipschitz), then (σ, u) solves (4.3) with $g = u$.

For this aim, we want to decompose the optimal transport plan γ as a sum of three transport plans γ_{ii}, γ_{ib} and γ_{bi}, where each of these plans solves a particular transport problem (i and b stand for interior and boundary). Next, we will study the L^p summability of the transport densities σ_{ii}, σ_{ib} and σ_{bi} associated with these transport plans γ_{ii}, γ_{ib} and γ_{bi}, respectively. In this way, we get the summability of the transport density σ associated with the optimal transport plan γ. Set,

$$\gamma_{ii} := \gamma \llcorner (\Omega^\circ \times \Omega^\circ),\ \gamma_{ib} := \gamma \llcorner (\Omega^\circ \times \partial\Omega),\ \gamma_{bi} := \gamma \llcorner (\partial\Omega \times \Omega^\circ),\ \gamma_{bb} := \gamma \llcorner (\partial\Omega \times \partial\Omega) = 0$$

and

$$\nu^+ := (\Pi_x)_\#\gamma_{ib}\,,\ \nu^- := (\Pi_y)_\#\gamma_{bi}\,.$$

Consider the three following problems:

$$\text{(P1)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+ - \nu^+, f^- - \nu^-)\right\}$$

$$\text{(P2)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega} g^-\,\mathrm{d}\chi^- \,:\, \gamma \in \Pi(\nu^+, \chi^-),\ \mathrm{spt}(\chi^-) \subset \partial\Omega\right\}$$

$$\text{(P3)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma - \int_{\partial\Omega} g^+\,\mathrm{d}\chi^+ \,:\, \gamma \in \Pi(\chi^+, \nu^-),\ \mathrm{spt}(\chi^+) \subset \partial\Omega\right\}.$$

Let γ_1 (resp. γ_2 and γ_3) be a solution of (P1) (resp. (P2) and (P3)). It is easy to see that γ_{ii}, γ_{ib} and γ_{bi} are admissible in (P1), (P2) and (P3), respectively. Then, one has

$$\int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma_1 \le \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma_{ii},$$

$$\int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma_2 + \int_{\partial\Omega} g^- \, \mathrm{d}(\Pi_y)_\# \gamma_2 \le \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma_{ib} + \int_{\partial\Omega} g^- \, \mathrm{d}(\Pi_y)_\# \gamma_{ib}$$

and

$$\int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma_3 - \int_{\partial\Omega} g^+ \, \mathrm{d}(\Pi_x)_\# \gamma_3 \le \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma_{bi} - \int_{\partial\Omega} g^+ \, \mathrm{d}(\Pi_x)_\# \gamma_{bi}.$$

Yet, we see easily that the transport plan $\tilde{\gamma} := \gamma_1 + \gamma_2 + \gamma_3$ belongs to $\Pi b(f^+, f^-)$ and, we have

$$\int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\tilde{\gamma} + \int_{\partial\Omega} g^- \, \mathrm{d}(\Pi_y)_\# \tilde{\gamma} - \int_{\partial\Omega} g^+ \, \mathrm{d}(\Pi_x)_\# \tilde{\gamma}$$

$$\le \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma + \int_{\partial\Omega} g^- \, \mathrm{d}(\Pi_y)_\# \gamma - \int_{\partial\Omega} g^+ \, \mathrm{d}(\Pi_x)_\# \gamma.$$

Then, $\tilde{\gamma}$ also solves the problem (KPb) and so, we obtain that γ_{ii}, γ_{ib} and γ_{bi} solve (P1), (P2) and (P3), respectively.

We want to characterize the optimal transport plan γ_{ib}. For this aim, let us define the multi-valued map T_{ib} as follows

$$T_{ib}(x) := \operatorname{argmin} \left\{ |x-y| + g^-(y), \, y \in \partial\Omega \right\}, \quad \text{for all } x \in \Omega.$$

We have the following

Lemma 4.5. *The multi-valued map T_{ib} has a Borel selector function.*

Proof. To prove that T_{ib} has a Borel selector function, it is enough to show that the graph of T_{ib} is closed (see, for instance, [**5, 40**]). Take a sequence $(x_n, y_n)_n$ in the graph of T_{ib} such that $(x_n, y_n) \to (x, y)$. As $y_n \in T_{ib}(x_n)$, then we have

$$|x_n - y_n| + g(y_n) \le |x_n - z| + g(z), \quad \text{for all } z \in \partial\Omega.$$

Passing to the limit, we get

$$|x-y| + g(y) \leq |x-z| + g(z), \quad \text{for all } z \in \partial\Omega$$

and then, $y \in T_{ib}(x)$. □

Then, the plan $\gamma_{T_{ib}} := (Id, T_{ib})_{\#}\nu^+$ is admissible in (P2). In addition, for any admissible $\gamma \neq \gamma_{T_{ib}}$ in (P2), we have

$$\int_{\Omega\times\partial\Omega} (|x-y| + g^-(y))\, \mathrm{d}\gamma_{T_{ib}} \; < \; \int_{\Omega\times\partial\Omega} (|x-y| + g^-(y))\, \mathrm{d}\gamma,$$

which implies that $\gamma_{ib} = \gamma_{T_{ib}} = (Id, T_{ib})_{\#}\nu^+$. Moreover, the transport plan γ_{ib} solves the following problem

$$\min\left\{\int_{\Omega\times\Omega} |x-y|\, \mathrm{d}\gamma \,:\, \gamma \in \Pi(\nu^+, (T_{ib})_{\#}\nu^+)\right\}.$$

In the same way, we get that the transport plan γ_{bi} is of the form $(T_{bi}, Id)_{\#}\nu^-$, where the map T_{bi} is defined as follows

$$T_{bi}(y) := \operatorname{argmin}\left\{|x-y| - g^+(x),\, x \in \partial\Omega\right\}, \quad \text{for all } y \in \Omega.$$

In particular, γ_{bi} also solves

$$\min\left\{\int_{\Omega\times\Omega} |x-y|\, \mathrm{d}\gamma \,:\, \gamma \in \Pi((T_{bi})_{\#}\nu^-, \nu^-)\right\}.$$

Now, let σ (resp. σ_{ii}, σ_{ib} and σ_{bi}) be the transport density associated with the optimal transport plan γ (resp. γ_{ii}, γ_{ib} and γ_{bi}), therefore $\sigma = \sigma_{ii} + \sigma_{ib} + \sigma_{bi}$. By Proposition 2.4, the transport density σ_{ii} belongs to $L^p(\Omega)$ as soon as $f^{\pm} \in L^p(\Omega)$. Hence, it is enough to study the summability of σ_{ib} (the case of σ_{bi} will be analogous), to get that of σ.

4.2. L^p summability of the transport density

In this section, we will study the L^p summability of the transport density σ_{ib}, under the assumption that Ω satisfies a uniform exterior ball condition and, by supposing that the boundary cost g is λ-Lip with $\lambda < 1$ and semi-concave. First, we will suppose that Ω has a very particular shape, i.e. its boundary is composed of parts of sphere of radius r (i.e., a *round polyhedron*), and then, by an approximation argument, we are able to generalize the result to any domain having a uniform exterior ball. Let us consider again the following transport problem

$$\text{(P)} \quad \min \left\{ \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega} g \,\mathrm{d}\chi \,:\, \gamma \in \Pi(f,\chi),\, \mathrm{spt}(\chi) \subset \partial\Omega \right\}.$$

Suppose that the boundary cost g is λ-Lip with $\lambda < 1$ and, set

$$T(x) := \operatorname{argmin}\{|x-y| + g(y),\, y \in \partial\Omega\} \quad \text{for all } x \in \Omega.$$

Then, we have the following:

Proposition 4.6. *$T(x)$ is a singleton Lebesgue-almost everywhere.*

Proof. Set

$$\phi(x) := \min\{|x-y| + g(y),\, y \in \partial\Omega\} \quad \text{for all } x \in \Omega.$$

It is clear that ϕ is 1-Lip, therefore it is differentiable Lebesgue-almost everywhere. Take $x_0 \in \Omega$ and suppose that there exist y_0 and $y_1 \in \partial\Omega$ such that

$$\phi(x_0) = |x_0 - y_0| + g(y_0) = |x_0 - y_1| + g(y_1).$$

As

$$\phi(x) - |x - y_0| \le g(y_0) \quad \text{for all } x \in \Omega,$$

then the function: $x \mapsto \phi(x) - |x - y_0|$ reaches a maximum at x_0 and so, $\nabla\phi(x_0) = \frac{x_0 - y_0}{|x_0 - y_0|}$. In the same way, we get $\nabla\phi(x_0) = \frac{x_0 - y_1}{|x_0 - y_1|}$. Hence, we have $\frac{x_0 - y_0}{|x_0 - y_0|} = \frac{x_0 - y_1}{|x_0 - y_1|}$, which is a contradiction as y_1 is in the half line with vertex x_0 and passing through y_0 (indeed in this case, one has $|g(y_0) - g(y_1)| = |y_0 - y_1|$). $\square$

Proposition 4.7. *If $x \in \Omega$ and $y \in T(x)$, then $(x,y) \cap \partial\Omega = \emptyset$.*

Proof. Suppose that this is not the case, i.e. there exist $x \in \Omega$, $y \in T(x)$ and some point $z \in (x,y) \cap \partial\Omega$. By definition of T, we have

$$|x-y| + g(y) \le |x-z| + g(z).$$

Then

$$|z-y| = |x-y| - |x-z| \le g(z) - g(y) \le \lambda|z-y|,$$

which is a contradiction. $\square$

Let Σ be the set of all the points $x \in \Omega$ where $T(x)$ is not a singleton (thanks to Proposition 4.6, one has $\mathcal{L}^d(\Sigma) = 0$). Then, we have the following:

PROPOSITION 4.8. *If* $x \in \Omega$, $y \in T(x)$ *and* $z \in (x, y)$, *then* $z \notin \Sigma$ *and* $T(z) = \{y\}$.

PROOF. For every $w \in \partial\Omega$ such that $w \notin T(x)$, we have

$$\begin{aligned} |z-y| + g(y) &= |x-y| - |x-z| + g(y) \\ &< |x-w| + g(w) - |x-z| \\ &\leq |z-w| + g(w). \end{aligned}$$

If $w \in T(x)$, we also have

$$\begin{aligned} |z-y| + g(y) &= |x-w| + g(w) - |x-z| \\ &< |z-w| + g(w), \end{aligned}$$

where the last strict inequality follows from Proposition 4.7. □

We recall that the plan $\gamma_T := (Id, T)_{\#}f$ is the unique minimizer for (P). In addition, γ_T solves the following problem

$$\min\left\{\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f, (T)_{\#}f)\right\}.$$

For simplicity of notation, we will denote this minimizer by γ instead of γ_T. Let σ be the transport density associated with the transport of f into $(T)_{\#}f$. By the definition of σ (see (2.1)), we have that, for all $\varphi \in C(\Omega)$,

$$\begin{aligned} <\sigma, \varphi> &= \int_{\Omega\times\Omega}\int_0^1 |x-y|\varphi((1-t)x+ty)\,\mathrm{d}t\,\mathrm{d}\gamma(x,y) \\ &= \int_\Omega\int_0^1 |x-T(x)|\varphi((1-t)x+tT(x))f(x)\,\mathrm{d}t\,\mathrm{d}x. \end{aligned}$$

Then

$$\sigma = \int_0^1 f_t\,\mathrm{d}t,$$

where

$$<f_t, \varphi> := \int_\Omega |x-T(x)|\varphi(T_t(x))f(x)\,\mathrm{d}x \quad \text{for all } \varphi \in C(\Omega)$$

and

$$T_t(x) := (1-t)x + tT(x) \quad \text{for all } x \in \Omega.$$

Notice that in the definition of f_t, differently from what done in Section 2.2, we need to keep the factor $|x - T(x)|$, which will be essential in the estimates. In addition, we have that $f_t \ll \mathcal{L}^d$ as soon as one has $f \ll \mathcal{L}^d$ (see Proposition 2.2).

Now, we will introduce the two following propositions, whose proofs, for simplicity of exposition, are postponed to Section 4.4.

PROPOSITION 4.9. *Suppose that Ω is a round polyhedron and g is in $C^2(\partial\Omega)$ with $|\nabla g| < 1$. Then, the closure $\bar{\Sigma}$ of the set Σ is negligible and T is a C^1 function on $\Omega \backslash \bar{\Sigma}$.*

We want to give an explicit formula of f_t in terms of f and T. Let φ be in $C(\Omega)$, then we have

$$\int_\Omega \varphi(y)\,\mathrm{d}f_t(y) = \int_\Omega \varphi(T_t(x))|x - T(x)|f(x)\,\mathrm{d}x.$$

Take a change of variable $y = T_t(x)$. By Proposition 4.8, we get easily that

$$x = \frac{y - tT(y)}{1-t} \quad \text{and} \quad |x - T(x)| = \frac{|y - T(y)|}{1-t}.$$

Consequently,

$$\int_\Omega \varphi(y) f_t(y)\,\mathrm{d}y = \int_{\Omega_t} \varphi(y)\,\frac{|y - T(y)|}{1-t} f\left(\frac{y - tT(y)}{1-t}\right)|J_t(y)|\,\mathrm{d}y,$$

where $\Omega_t := T_t(\Omega)$ and $J_t(y) := (\det(DT_t(x)))^{-1}$ for all $y = T_t(x) \in \Omega_t$. Yet, this implies that

$$f_t(y) = \frac{|y - T(y)|}{1-t} f\left(\frac{y - tT(y)}{1-t}\right)|J_t(y)|1_{\Omega_t}(y) \quad \text{for a.e. } y \in \Omega.$$

Notice that a point y belongs to Ω_t if and only if $|y - T(y)| \leq (1-t)l(y)$ where $l(y)$ is the length of the transport ray containing y, i.e.,

$$l(y) := \sup\left\{|x - T(x)| : \, x \in \Omega \cap \{T(y) + s(y - T(y)), \, s \geq 1\}, \, T(x) = T(y)\right\}.$$

The following proposition gives a lower bound on the Jacobian J_t, which is in fact sufficient to prove our L^p estimates on the transport density σ:

PROPOSITION 4.10. *Suppose that* Ω *is a round polyhedron and* $g \in C^2(\partial\Omega)$ *with* $|\nabla g| \leq \lambda < 1$*. Then, there exists a constant* $C := C(d,\, diam(\Omega),\, \lambda,\, r,\, M) > 0$*, where* $D^2 g \leq MI$*, such that, for a.e.* $x \in \Omega$*, we have the following estimate*

$$|\det(DT_t(x))| \geq C(1-t).$$

We are now ready to prove the L^p summability of the transport density σ. Then, we have the following result.

PROPOSITION 4.11. *Suppose that* Ω *is a round polyhedron and* $g \in C^2(\partial\Omega)$ *with* $|\nabla g| \leq \lambda < 1$*. Then, the transport density* σ *belongs to* $L^\infty(\Omega)$ *provided that* $f \in L^\infty(\Omega)$*.*

PROOF. By Proposition 4.10, we have

$$\begin{aligned}
\| \sigma \|_{L^\infty(\Omega)} &= \sup_{y \in \Omega} \left(\int_0^1 f_t(y)\, \mathrm{d}t \right) \\
&= \sup_{y \in \Omega} \left(\int_0^{1-\frac{|y-T(y)|}{l(y)}} \frac{|y-T(y)| f(\frac{y-tT(y)}{1-t})}{(1-t)|\det(DT_t(x))|}\, \mathrm{d}t \right) \\
&\leq C^{-1} \| f \|_{L^\infty(\Omega)} \sup_{y \in \Omega} \left(\int_0^{1-\frac{|y-T(y)|}{l(y)}} \frac{|y-T(y)|}{(1-t)^2}\, \mathrm{d}t \right).
\end{aligned}$$

Yet, it is easy to see that

$$\sup_{y \in \Omega} \int_0^{1-\frac{|y-T(y)|}{l(y)}} \frac{|y-T(y)|}{(1-t)^2}\, \mathrm{d}t \;\leq\; \mathrm{diam}(\Omega).$$

Then,

$$\| \sigma \|_{L^\infty(\Omega)} \leq C \,\| f \|_{L^\infty(\Omega)},$$

for some constant C depending only on d, $\mathrm{diam}(\Omega)$, λ, r and M, where M is any constant such that $D^2 g \leq MI$. $\square$

PROPOSITION 4.12. *Let* Ω *be a round polyhedron,* $g \in C^2(\partial\Omega)$ *with* $|\nabla g| < 1$ *and suppose* $f \in L^p(\Omega)$ *for some* $p \in [1, +\infty]$*. Then, the transport density* σ *also belongs to* $L^p(\Omega)$*.*

PROOF. We observe that as the transport is between f and $(T)_\# f$, then the transport density σ linearly depends on f: in this case, L^p estimates could be obtained via interpolation as soon as one has L^1 and L^∞ estimates (see, for instance, [86]). In order to get an L^1 estimate, it is enough to remember the implication $f \in L^1(\Omega) \Rightarrow \sigma \in L^1(\Omega)$ from Proposition 2.2, and that we have

$$||\sigma||_{L^1(\Omega)} \leq \mathrm{diam}(\Omega)\, ||f||_{L^1(\Omega)}.$$

In addition, the L^∞ estimates follow from Proposition 4.11. □

REMARK 4.13. *The same proof as in Proposition 4.11 could also be adapted to proving Proposition 4.12, but a suitable use of a Hölder inequality would be required (see Proposition 9.2).*

We will now generalize, via a limit procedure, the result of Proposition 4.12 to arbitrary domain having a uniform exterior ball (see Definition 3.6).

PROPOSITION 4.14. *Let Ω be a domain having a uniform exterior ball of radius r. Then, the transport density σ between f and $(T)_{\#}f$ belongs to $L^p(\Omega)$ provided that $f \in L^p(\Omega)$ and, the boundary cost g is $\lambda-$Lip with $\lambda < 1$ and semi-concave.*

PROOF. This proposition can be proven using the lemma 3.7. To do that, take a sequence of domains $(\Omega_k)_k$ such that: the boundary of each Ω_k is a union of parts of sphere of radius r, $\partial\Omega_k \to \partial\Omega$ in the Hausdorff sense and $\Omega \subset \Omega_k \subset \tilde{\Omega}$ for some large compact set $\tilde{\Omega}$. First of all, we suppose that $g \in C^2(\partial\Omega)$. Let γ_k be an optimal transport plan between f and $(T^k)_{\#}f$, i.e. the plan γ_k solves

$$\min\left\{\int_{\tilde{\Omega}\times\tilde{\Omega}} |x-y| \, \mathrm{d}\gamma \,:\, \gamma \in \Pi(f,(T^k)_{\#}f)\right\},$$

where $T^k(x) := \operatorname{argmin}\{|x-y| + g(y),\, y \in \partial\Omega_k\}$. Let σ_k be the transport density associated with the optimal transport plan γ_k. From Proposition 4.12, we have

$$\sigma_k \in L^p(\Omega_k)$$

and

$$\| \sigma_k \|_{L^p(\Omega_k)} \le C \| f \|_{L^p(\Omega)},$$

for some constant $C := C(d, \operatorname{diam}(\Omega), \lambda, r, M)$, where M is a constant such that $D^2 g \le MI$. Then, up to a subsequence, we can assume that $\sigma_k \rightharpoonup \sigma$ weakly in $L^p(\tilde{\Omega})$. Moreover, we have the following estimate

$$\| \sigma \|_{L^p(\Omega)} \le \liminf_k \| \sigma_k \|_{L^p(\Omega_k)} \le C \| f \|_{L^p(\Omega)} .$$

Now, it is sufficient to show that this σ is in fact the transport density associated with the transport of f into $(T)_{\#}f$. Firstly, we observe that for a given x, $(T_k(x))_k$ converges, up to a subsequence, to a point $y \in \partial\Omega$ such that $y \in \operatorname{argmin}\{|x-z| + g(z),\, z \in \partial\Omega\}$. Since this point is unique for a.e. x, we get (with no need to pass to a subsequence):

$$T^k(x) \to T(x)$$

and

$$(T^k)_{\#}f \rightharpoonup (T)_{\#}f \text{ in the sense of measures.}$$

By Proposition 1.3, we get that

$$\gamma_k \rightharpoonup \gamma,$$

where γ solves

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma\,:\,\gamma\in\Pi(f,(T)_{\#}f)\right\}.$$

Let σ_γ be the unique transport density between f and $(T)_{\#}f$. As $\gamma_k \rightharpoonup \gamma$, we find that $\sigma_k \rightharpoonup \sigma_\gamma$ (this follows immediately from (2.1)). Consequently, $\sigma_\gamma = \sigma \in L^p(\Omega)$ and we have the following estimate

$$\|\,\sigma_\gamma\,\|_{L^p(\Omega)} \leq C\,\|\,f\,\|_{L^p(\Omega)},$$

for some constant $C := C(d,\,\mathrm{diam}(\Omega),\,\lambda,\,r,\,M)$, where M is a constant such that $D^2 g \leq MI$.

Finally, the approximation of a semi-concave function g with smoother functions is also standard. Then, it is not difficult to check again that our result is still true for a semi-concave function g. □

4.3. A geometric lemma

In the particular case $g = 0$, we are able to prove Proposition 4.10, for arbitrary domain Ω having a uniform exterior ball, via a geometric argument which will not be available for the general case.

Lemma 4.15. *Let $P_{\partial\Omega}$ be the projection on the boundary of Ω, i.e.,*

$$P_{\partial\Omega}(x) := \mathit{argmin}\,\{|x-y|,\,y\in\partial\Omega\}\ \ \text{for all}\ \ x.$$

Then, $P_{\partial\Omega}$ is the gradient of a convex function. In addition, if Ω has a uniform exterior ball of radius $r>0$, then for a.e. $x\in\Omega$, the positive symmetric matrix $DP_{\partial\Omega}(x)$ has $d-1$ eigenvalues larger than $\frac{r}{r+d(x,\partial\Omega)}$.

Proof. Set

$$u(x) := \sup\left\{x\cdot y - \frac{1}{2}|y|^2,\,y\in\partial\Omega\right\}.$$

As we can rewrite $u(x)$ as follows

$$u(x) = \sup\left\{-\frac{1}{2}|x-y|^2 + \frac{1}{2}|x|^2,\,y\in\partial\Omega\right\},$$

then the supremum is attained at $P_{\partial\Omega}(x)$ and $\nabla u(x) = P_{\partial\Omega}(x)$ for a.e. x. This implies that $P_{\partial\Omega}$ is the gradient of a convex function, which is in fact coherent with Brenier Theorem [22, 23] (see also Chapter 1).

Now, take $x_0 \in \Omega$ and let y_0 be the center of a ball $B(y_0, r)$ such that $B(y_0, r) \cap \Omega = \emptyset$ and $|y_0 - P_{\partial\Omega}(x_0)| = r$. Then x_0, $P_{\partial\Omega}(x_0)$ and y_0 are aligned. Indeed if not, we get $|x_0 - y_0| < |x_0 - P_{\partial\Omega}(x_0)| + r$, but $|x_0 - y_0| = |x_0 - z| + |z - y_0|$ for some $z \in [x_0, y_0] \cap \partial\Omega$, which is a contradiction as $|x_0 - P_{\partial\Omega}(x_0)| \leq |x_0 - z|$ and $r \leq |z - y_0|$. Moreover, we have

$$\begin{aligned} u(x) &= \sup\left\{\frac{1}{2}|x|^2 - \frac{1}{2}|x-y|^2,\ y \in \partial\Omega\right\} \\ &\geq \frac{1}{2}|x|^2 - \frac{1}{2}|x-y_x|^2, \text{ for some } y_x \in [x, y_0] \cap \partial\Omega \\ &\geq \frac{1}{2}|x|^2 - \frac{1}{2}(|x-y_0| - r)^2 \ := v(x). \end{aligned}$$

As $u(x_0) = v(x_0)$, then the function: $x \mapsto u(x) - v(x)$ has a minimum at x_0. Hence, we get that $D^2 u(x_0) \geq D^2 v(x_0)$ and the eigenvalues of $D^2 u(x_0)$ are bounded from below by those of $D^2 v(x_0)$. Yet, it is easy to show that

$$D^2 v(x_0) = \frac{r}{r + d(x_0, \partial\Omega)} \left(I - e(x_0) \otimes e(x_0)\right),$$

where $e(x_0) := \frac{x_0 - y_0}{|x_0 - y_0|}$.

Then, we conclude by observing that the eigenvalues of this matrix are 0 and $\frac{r}{r + d(x_0, \partial\Omega)}$ (with multiplicity $d-1$). $\square$

Set

$$P_t(x) := (1-t)x + tP_{\partial\Omega}(x).$$

By Lemma 4.15, we have

$$\det(DP_t(x)) \geq (1-t)\left(1 - t + t\frac{r}{r + d(x, \partial\Omega)}\right)^{d-1}.$$

Set $y := P_t(x)$. As $d(y, \partial\Omega) = (1-t)d(x, \partial\Omega)$, then the Jacobian at y satisfies the following estimate

$$J_t(y) := \frac{1}{\det(DP_t(x))} \leq \frac{1}{1-t}\left(\frac{r+d(x,\partial\Omega)}{r+(1-t)d(x,\partial\Omega)}\right)^{d-1}$$
$$= \frac{1}{(1-t)^d}\left(\frac{(1-t)r+d(y,\partial\Omega)}{r+d(y,\partial\Omega)}\right)^{d-1}.$$

Now, suppose $f \in L^\infty(\Omega)$ and let σ be the transport density between f and $(P_{\partial\Omega})_\# f$. Then, we have the following pointwise inequality

$$\sigma(y) \leq \| f \|_{L^\infty(\Omega)} \int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{d(y,\partial\Omega)}{(1-t)^{d+1}}\left(\frac{(1-t)r+d(y,\partial\Omega)}{r+d(y,\partial\Omega)}\right)^{d-1} \mathrm{d}t.$$

Hence,

$$\sigma(y) \leq C \| f \|_{L^\infty(\Omega)} \int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{d(y,\partial\Omega)}{(1-t)^{d+1}} \frac{(1-t)^{d-1}\, r^{d-1} + (d(y,\partial\Omega))^{d-1}}{(r+d(y,\partial\Omega))^{d-1}} \,\mathrm{d}t$$

$$\leq \frac{C\, d(y,\partial\Omega) \| f \|_\infty}{(r+d(y,\partial\Omega))^{d-1}}\left(\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^2}\,\mathrm{d}t + (d(y,\partial\Omega))^{d-1}\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^{d+1}}\,\mathrm{d}t\right).$$

Yet,

$$\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^2}\,\mathrm{d}t \;+\; (d(y,\partial\Omega))^{d-1}\int_0^{1-\frac{d(y,\partial\Omega)}{l(y)}} \frac{1}{(1-t)^{d+1}}\,\mathrm{d}t \leq \frac{C}{d(y,\partial\Omega)}.$$

Then,

$$\sigma(y) \leq \frac{C \| f \|_\infty}{(r+d(y,\partial\Omega))^{d-1}}.$$

This provides a very useful and pointwise estimate on σ. It shows that σ is bounded as soon as $r > 0$, or if we are far from the boundary $\partial\Omega$. By interpolation (see [**86**]), we also get that σ belongs to $L^p(\Omega)$ provided that $f \in L^p(\Omega)$. So as a particular case, we get the results of Section 4.2 in the case $g = 0$ whenever $r > 0$.

4.4. Technical proofs

In this section, we want to give the proofs of Propositions 4.9 & 4.10. First of all, suppose that Ω is a round polyhedron and set

$$\Omega_i := \{x = (x_1, x_2, ..., x_d) \in \Omega \,:\, T(x) \in F_i\},$$

where $F_i \subset \partial B(b_i, r)$ is the ith part in the boundary of Ω, contained in a sphere centered at b_i and with radius $r > 0$ (see Figure 2). Then, we have the following:

PROPOSITION 4.16. *For all $x \in \mathring{\Omega}$, there is no pair (i, j) with $i \neq j$ such that $T(x) \in F_i \cap F_j$.*

PROOF. Suppose that this is not the case at some point $x \in \mathring{\Omega}$, i.e. there exist two different faces F_i and F_j such that $T(x) \in F_i \cap F_j$. By Proposition 4.7, the segment $[x, T(x)]$ cannot intersect the boundary of Ω at an interior point. Then, taking into account the geometric form of Ω, we infer that there exist two tangent vectors v_i and v_j in $T(x)$ on F_i and F_j, respectively, such that

$$x - T(x) = \alpha_i v_i + \alpha_j v_j,$$

for some two positive constants α_i and α_j. Let γ_i and γ_j be two curves plotted on F_i and F_j, respectively, so that $\gamma_i'(0) = v_i$ and $\gamma_j'(0) = v_j$. For $t \geq 0$ small enough, we define the following functions

$$f_i(t) := |x - \gamma_i(t)| + g(\gamma_i(t))$$

and

$$f_j(t) := |x - \gamma_j(t)| + g(\gamma_j(t)).$$

By optimality of $T(x) = \gamma_i(0) = \gamma_j(0)$, we infer that f_i and f_j reach a minimum at $t = 0$ and so, $f_i'(0)$, $f_j'(0) \geq 0$. Hence,

$$-\frac{x - T(x)}{|x - T(x)|} \cdot {\gamma_i}'(0) + \nabla g(T(x)) \cdot {\gamma_i}'(0) \geq 0$$

and

$$-\frac{x - T(x)}{|x - T(x)|} \cdot {\gamma_j}'(0) + \nabla g(T(x)) \cdot {\gamma_j}'(0) \geq 0.$$

Now, if we multiply the first inequality by α_i, the second one by α_j and we take the sum, we get

$$-|x - T(x)| \, + \, \nabla g(T(x)) \cdot (x - T(x)) \geq 0$$

and

$$1 \leq |\nabla g(T(x))| \leq \lambda,$$

which is a contradiciton. $\square$

PROPOSITION 4.17. *For every $x \in \Omega_i \backslash \Sigma$, there is a neighborhood of x contained in Ω_i.*

PROOF. Suppose that this is not the case at some point x. Then, there exists a sequence $(x_n)_n$ such that $x_n \to x$ and $T(x_n) \in F_j$ for some $j \neq i$. Yet, up to a subsequence, we can assume that $T(x_n) \to y \in F_j$. By definition of T, we have

$$|x_n - T(x_n)| + g(T(x_n)) \leq |x_n - z| + g(z) \text{ for all } z \in \partial\Omega.$$

Passing to the limit, we get

$$|x-y|+g(y)\leq|x-z|+g(z) \text{ for all } z\in\partial\Omega,$$

which is in contradiction with Proposition 4.16. □

Consider Ω_1 (eventually it will be the same for the other Ω_i). Suppose that Proposition 4.9 is true and fix $x\in\Omega_1\backslash\bar{\Sigma}$. After a translation and rotation of axis, we can suppose that the tangent space at $T(x)$ on F_1 is contained in the plane $x_d=0$. Let $\varphi:U\mapsto\mathbb{R}$, where $U\subset\mathbb{R}^{d-1}$, be a parameterization of F_1, i.e. for any $z:=(z_1,...,z_d)\in F_1$, we have $\bar{z}:=(z_1,...,z_{d-1})\in U$ and $z_d=\varphi(\bar{z})$ (notice that an explicit formula of φ is not needed in the sequel). Set

$$\xi(\bar{z}):=\sqrt{|\bar{x}-\bar{z}|^2+(x_d-\varphi(\bar{z}))^2}+g(\bar{z},\varphi(\bar{z})) \text{ for all } \bar{z}\in U.$$

For any $i\in\{1,....,d-1\}$, one has

$$\frac{\partial\xi}{\partial z_i}(\bar{z})=\frac{(z_i-x_i)-(x_d-\varphi(\bar{z}))\frac{\partial\varphi}{\partial z_i}(\bar{z})}{\sqrt{|\bar{x}-\bar{z}|^2+(x_d-\varphi(\bar{z}))^2}}+\frac{\partial g}{\partial z_i}(\bar{z},\varphi(\bar{z}))+\frac{\partial g}{\partial z_d}(\bar{z},\varphi(\bar{z}))\frac{\partial\varphi}{\partial z_i}(\bar{z}).$$

Set $T(x):=(\bar{T}(x),\varphi(\bar{T}(x)))$, where $\bar{T}(x):=(T_1(x),...,T_{d-1}(x))$. Then, we have

$$\bar{T}(x)=\operatorname{argmin}\{\xi(\bar{z}),\ \bar{z}\in U\}.$$

By Proposition 4.16, $\bar{T}(x)\in\mathring{U}$. Hence,

$$\frac{\partial\xi}{\partial z_i}(\bar{T}(x))=0 \text{ for all } i\in\{1,....,d-1\}$$

or equivalently,

$$\frac{T_i(x)-x_i}{\tau(x)}+\frac{\partial g}{\partial z_i}(T(x))-\frac{(x_d-\varphi(\bar{T}(x)))}{\tau(x)}\frac{\partial\varphi}{\partial z_i}(\bar{T}(x))+\frac{\partial g}{\partial z_d}(T(x))\frac{\partial\varphi}{\partial z_i}(\bar{T}(x))=0, \tag{4.4}$$

for all $i\in\{1,....,d-1\}$, where $\tau(x):=|x-T(x)|$. Yet, by Proposition 4.17, the equality in (4.4) holds in a neighborhood of x. Then, differentiating (4.4) with respect to x_j and taking into account the fact that in this new system of coordinates we have

$$\frac{\partial \varphi}{\partial z_i}(\bar{T}(x)) = 0 \;\text{ for all }\; i \in \{1, ..., d-1\},$$

we get

$$\begin{aligned} &\frac{\partial T_i}{\partial x_j}(x) - \frac{(x_i - T_i(x))}{\tau(x)^2}\sum_{k=1}^{d-1}(x_k - T_k(x))\frac{\partial T_k}{\partial x_j}(x) + \tau(x)\sum_{k=1}^{d-1}\frac{\partial^2 g}{\partial z_i \partial z_k}(T(x))\frac{\partial T_k}{\partial x_j}(x) \\ (4.5) \quad &- \;(x_d - \varphi(\bar{T}(x)))\sum_{k=1}^{d-1}\frac{\partial^2 \varphi}{\partial z_i \partial z_k}(\bar{T}(x))\frac{\partial T_k}{\partial x_j}(x) + \tau(x)\frac{\partial g}{\partial z_d}(T(x))\sum_{k=1}^{d-1}\frac{\partial^2 \varphi}{\partial z_i \partial z_k}(\bar{T}(x))\frac{\partial T_k}{\partial x_j}(x) \\ &= \delta_{ij} \;-\; \frac{(x_i - T_i(x))(x_j - T_j(x))}{\tau(x)^2} \end{aligned}$$

for all $i,\, j \in \{1, ..., d-1\}$.

On the other hand, we have

$$DT_t(x) = (1-t)I + tDT(x) = \begin{pmatrix} 1-t+t\frac{\partial T_1}{\partial x_1} & t\frac{\partial T_1}{\partial x_2} & ... & t\frac{\partial T_1}{\partial x_d} \\ t\frac{\partial T_2}{\partial x_1} & 1-t+t\frac{\partial T_2}{\partial x_2} & ... & t\frac{\partial T_2}{\partial x_d} \\ ... & & & \\ 0 & ... & 0 & 1-t \end{pmatrix}.$$

Then,

$$\det(DT_t(x)) = (1-t)\det(A),$$

where $A := \Big((1-t)\delta_{ij} + t\frac{\partial T_i}{\partial x_j}(x)\Big)_{i,j=1,....,d-1}$.

Set

$$P := \left(\delta_{ij} - \frac{(x_i - T_i(x))(x_j - T_j(x))}{\tau(x)^2}\right)_{ij}$$

and

$$N := \left(\tau(x)\frac{\partial^2 g}{\partial z_i \partial z_j}(T(x)) - (x_d - \varphi(\bar{T}(x)))\frac{\partial^2 \varphi}{\partial z_i \partial z_j}(\bar{T}(x)) + \tau(x)\frac{\partial g}{\partial z_d}(T(x))\frac{\partial^2 \varphi}{\partial z_i \partial z_j}(\bar{T}(x))\right)_{ij}.$$

Suppose that $P+N$ is invertible for a.e. $x \in \Omega$ (see Proposition 4.18 below). Then, from (4.5), we observe that

$$\left(\frac{\partial T_i}{\partial x_j}(x)\right)_{i,j=1,\dots,d-1} = (P+N)^{-1}P.$$

Hence,

$$A = (1-t)I + t(P+N)^{-1}P = (P+N)^{-1}(P+(1-t)N)$$

and

$$\det(A) = \frac{\det(P+(1-t)N)}{\det(P+N)}.$$

We note that the matrix $P+N = \tau(x)\, D^2\xi(\bar{T}(x))$ and so, by optimality of $\bar{T}(x)$, it is nonnegative. On the other hand, as $D^2\varphi(\bar{T}(x)) = \frac{-1}{r}I$ and $D^2 g \le MI$, then

$$P+N \le C(d, \operatorname{diam}(\Omega), \lambda, r, M)I$$

and so,

$$\det(P+N) \le C(d, \operatorname{diam}(\Omega), \lambda, r, M). \tag{4.6}$$

From (4.4), we have

$$\frac{x_i - T_i(x)}{|x-T(x)|} = \frac{\partial g}{\partial z_i}(T(x)), \text{ for any } i \in \{1, ..., d-1\}.$$

Then,

$$|\bar{x} - \bar{T}(x)| \le \lambda |x - T(x)|.$$

Yet, this implies that

$$\begin{aligned} < Pz, z > &= |z|^2 - \left(\frac{\bar{x}-\bar{T}(x)}{|x-T(x)|}\cdot z\right)^2 \\ &\ge |z|^2 - \frac{|\bar{x}-\bar{T}(x)|^2}{|x-T(x)|^2}|z|^2 \\ &\ge (1-\lambda^2)|z|^2. \end{aligned}$$

Hence, $P \ge (1-\lambda^2)I$. Now, we are ready to give a lower bound for $\det(A)$. First, if $t \ge \frac{1}{2}$, then we have

$$P + (1-t)N = tP + (1-t)(P+N) \ge \frac{1}{2}P \ge \frac{1-\lambda^2}{2}I$$

and so,

$$\det(P + (1-t)N) \geq \left(\frac{1-\lambda^2}{2}\right)^{d-1}. \tag{4.7}$$

If $t < \frac{1}{2}$, then one has

$$P + (1-t)N \;\geq\; (1-t)(P+N) \;\geq\; \frac{1}{2}(P+N),$$

which implies that

$$\det(P + (1-t)N) \geq \frac{1}{2^{d-1}} \det(P+N). \tag{4.8}$$

Combining (4.6), (4.7) & (4.8), we infer that there is a constant $C > 0$ depending only on d, $\mathrm{diam}(\Omega)$, λ, r and M, for some constant M with $D^2 g \leq MI$, such that

$$\det(A) \geq C$$

or equivalently,

$$\det(DT_t(x)) \geq C(1-t).$$

Finally, we introduce the proof of the proposition 4.9.

PROOF. Set $h := (h_i)_{i=1,\dots,d-1}$, where for any i,

$$h_i(x,y) := \frac{y_i - x_i}{\sqrt{|\bar{x}-y|^2 + (x_d - \varphi(y))^2}} \;+\; \frac{\partial g}{\partial z_i}(y,\varphi(y)) \;-\; \frac{x_d - \varphi(y)}{\sqrt{|\bar{x}-y|^2 + (x_d - \varphi(y))^2}}\frac{\partial \varphi}{\partial z_i}(y)$$

$$+\; \frac{\partial g}{\partial z_d}(y,\varphi(y))\frac{\partial \varphi}{\partial z_i}(y)$$

for all $(x,y) \in \Omega_1 \times U$.

By Proposition 4.18 (see below), the matrix $(\frac{\partial h_i}{\partial y_j}(x,\bar{T}(x)))_{1\leq i,j\leq d-1}$ is invertible at a.e. x. Yet, we have $h(x,\bar{T}(x)) = 0$. Then, by the implicit function theorem, there exist an open neighborhood $V_1 \subset \Omega_1$ of x, a neighborhood $V_2 \subset U$ of $\bar{T}(x)$ and a function $q : V_1 \to V_2$ of class C^1 such that, for all $x' \in V_1$ and $y \in V_2$, we have

$$h(x',y) = 0 \Leftrightarrow y = q(x').$$

But, for all $x' \in V_1$, one has $h(x',\bar{T}(x')) = 0$. Hence, $\bar{T} = q$ and T is a C^1 function on V_1. Moreover, we can also suppose that $V_1 \subset \Omega_1 \backslash \Sigma$. Indeed, if this is not the case, then there exists a sequence $(x_n)_n$ such that $x_n \to x$ and, for all n, $x_n \in \Sigma$ (i.e., for all n, there exist $z_n, w_n \in \operatorname{argmin}\{|x_n - y| + g(y),\, y \in \partial\Omega\}$ such that $z_n \neq w_n$). As $x \notin \Sigma$, then $(z_n)_n$ and $(w_n)_n$ converge to $T(x)$. In particular, $\bar{z}_n, \bar{w}_n \in U$, $h(x_n, \bar{z}_n) = h(x_n, \bar{w}_n) = 0$ and so, $\bar{z}_n = \bar{w}_n = q(x_n)$, which is a contradiction. □

It remains to prove the following:

PROPOSITION 4.18. *The matrix $P + N$ is invertible at a.e. point $x \in \Omega$.*

PROOF. It is enough to prove that the matrix $P + N$ is not invertible only at a countable number of points on each transport ray, since in this case one can use Proposition 1.5, to infer that the set of all these points is in fact negligible. To do that, fix $x \in \Omega$ and set $y := T_t(x)$, where $t \in (0, 1]$. In fact, the matrix P is constant along the transport ray passing through x. Moreover, if N' plays the role of the matrix N for the point y, then we have $N' = (1 - t)N$. Hence,

$$P + N' = P + (1 - t)N > 0.$$

Consequently, $P + N$ is invertible at a point $x \in \Omega$ as soon as x is not a lower boundary point of some transport ray (i.e., if $x \notin \mathcal{S}^+$). □

Finally, we get that the closure $\bar{\Sigma}$ of the set of double points Σ is negligible. In addition, T is a C^1 function on $\Omega \backslash \bar{\Sigma}$.

CHAPTER 5

Lack of regularity of the transport density

In this chapter, we provide a family of counter-examples to the regularity of the transport density in the classical Monge-Kantorovich problem. In particular, we prove that the $W^{1,p}$ regularity of the source and target measures does not imply that the transport density σ is $W^{1,p}$, that the BV regularity of $f^\pm$ does not imply that σ is BV and that $f^\pm \in C^\infty$ does not imply that σ is $W^{1,p}$, for large p.

This chapter is taken from my article [55], which will be published in *Journal de Mathématiques Pures et Appliquées.*

The higher order regularity of the transport density σ is the object of a wide debate; the only positive known results are in $\mathbb{R}^2$: if $f^\pm$ are two positive densities, continuous and have compact, disjoint, convex support, then the "monotone optimal transport map" T is continuous except on a negligible set (the endpoints of transport rays) and the transport density σ is actually continuous everywhere [61]. Moreover, in [83], the authors prove the continuity of the same map T under the assumptions that $f^\pm$ are two positive densities, continuous with $\mathrm{spt}(f^+) \subset \mathrm{spt}(f^-)$ and one of the sets $\{f^+ > f^-\}$, $\{f^- > f^+\}$ is convex and, the transport density σ is also continuous in this case. Other results exist as far as the regularity in some directions is concerned: in [58], it has been proven that when $f^\pm$ are Lipschitz continuous with disjoint supports (and with some extra technical conditions on the supports), then the transport density is locally Lipschitz continuous "along transport rays". Also in [31], the authors have a more general result for the case of just summable $f^\pm$ without any extra conditions on supports; they prove that if $f^\pm \in L^p(\Omega)$, then for a.e. $x \in \Omega$, the restriction of the transport density σ to the transport ray passing through x is in $W^{1,p}_{loc}$. A *conjecture* of *Buttazzo* was the following: "if f^+ and f^- are smooth, then the transport density between them is Lipschitz". As one can see, the $W^{1,p}(\Omega)$ (resp. $C^{0,\alpha}(\Omega), BV(\Omega), ...$) regularity of the transport density σ is an interesting question, and the aim of this chapter is to give a (negative) answer to it!

In this chapter we focus on examples relating the regularity of the initial data $f^\pm$ with the regularity of the transport density σ. As a starting point, the following example shows that in general, the transport density σ is not more regular than the initial data: consider $\chi^+ := [0,1]^2$, $\chi^- := [2,3] \times [0,1]$ and set $f^+(x_1,x_2) := f_1(x_1)f_2(x_2)$, where we suppose that f^+ is concentrated on χ^+, and take $f^-(x_1,x_2) := f^+(x_1-2,x_2)$, for every $(x_1,x_2) \in \chi^-$. In this case, it is easy to compute the transport density σ between f^+ and f^-, so we get $\sigma(x_1,x_2) = (\int_0^{x_1} f_1(t)\,\mathrm{d}t)f_2(x_2)$ for every $x := (x_1,x_2) \in \chi^+$. Hence, the transport density σ has the same regularity as $f^\pm$ in the x_2-variable. Yet, we will give examples where the regularity of the transport density σ is worse than the regularity of the initial data $f^\pm$. In particular, we will prove among others the following statements:

$$f^{\pm} \in BV(\Omega) \quad \not\Rightarrow \quad \sigma \in BV(\Omega), \tag{5.1}$$

$$\text{for all} \quad p > 1, \qquad f^{\pm} \in W^{1,p}(\Omega) \quad \not\Rightarrow \quad \sigma \in W^{1,p}(\Omega), \tag{5.2}$$

$$f^{\pm} \in C^{\infty}(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in W^{1,3}(\Omega), \tag{5.3}$$

$$\text{for all} \quad \alpha \in (0,1), \qquad f^{\pm} \in C^{0,\alpha}(\Omega) \quad \not\Rightarrow \quad \sigma \in C^{0,\alpha}(\Omega), \tag{5.4}$$

$$\text{for all} \quad \varepsilon > 0, \qquad f^{\pm} \in C^{\infty}(\bar{\Omega}) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{1}{2}+\varepsilon}(\Omega). \tag{5.5}$$

5.1. Main Results

Inspired by [**45, 84**], we will construct a family of counter-examples by, first, choosing which lines will be transport rays. Set $\gamma > 0$ and consider the following transport rays:

$$l_a := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \, x_2 = \frac{a^{\gamma}}{2}\,(x_1 + a),\, x_1 \in (-a, 1) \right\}, \quad a \in [0,1]. \tag{5.6}$$

It is clear that the segments $(l_a)_a$ do not mutually intersect. The domain representing both source and target will be $\Delta \subset \mathbb{R}^2$ (see Figure 1), where

$$\Delta := \text{interior of the triangle with vertices } (-1,0),\ (1,0) \text{ and } (1,1). \tag{5.7}$$

The initial and final density will have the form

$$f^+(x_1,x_2) = 1,\ f^-(x_1,x_2) = 1 + \beta(\zeta''(x_1) + \eta''(x_2)) \quad \text{for all} \quad (x_1,x_2) \in \Delta, \tag{5.8}$$

where $\zeta(x_1) := -x_1^2(x_1-1)^2$ (the choice of ζ is made essentially in such a way that $\zeta(1) = \zeta'(1) = 0$), η is a C^2 function with $\eta(0) = \eta'(0) = 0$ and $\beta > 0$ is chosen so that f^- will be a non-negative density. Note that η is constructed in such a way that the following mass balance condition for the region in the domain below each l_a is satisfied:

$$\int_{\Delta_a} f^+ = \int_{\Delta_a} f^- \quad \text{for all} \ a \in [0,1], \tag{5.9}$$

where Δ_a is the subgraph of l_a in Δ, namely the triangle formed by $(-a,0)$, $(1,0)$ and $(1, \frac{a^{\gamma}}{2}(1+$

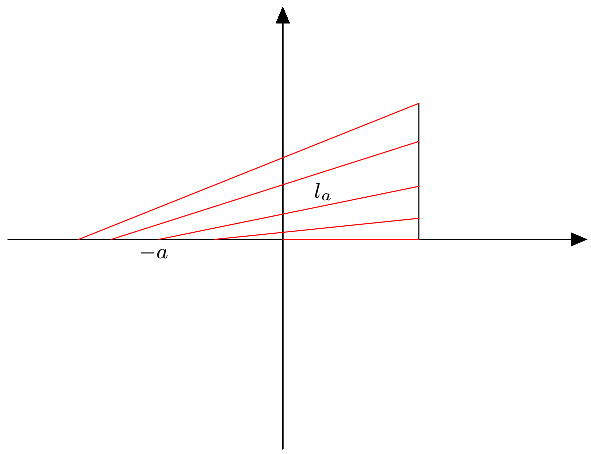

FIGURE 1

a)); or equivalently, (5.9) can be rewritten as

$$-\int_{\Delta_a} \zeta''(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\Delta_a} \eta''(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for all } a \in [0,1].$$

Yet, it is easy to see that

$$\begin{aligned}
-\int_{\Delta_a} \zeta''(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 &= -\int_0^{\frac{a^\gamma}{2}(1+a)} \int_{\frac{2}{a^\gamma}x_2 - a}^{1} \zeta''(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_0^{\frac{a^\gamma}{2}(1+a)} \zeta'\left(\frac{2}{a^\gamma}x_2 - a\right)\mathrm{d}x_2 \\
&= \frac{a^{\gamma+2}}{2}(1+a)^2
\end{aligned}$$

and as $\eta(0) = \eta'(0) = 0$, we have

$$\begin{aligned}
\int_{\Delta_a} \eta''(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 &= \int_{-a}^{1} \int_0^{\frac{a^\gamma}{2}(x_1+a)} \eta''(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \int_{-a}^{1} \eta'\left(\frac{a^\gamma}{2}(x_1+a)\right)\mathrm{d}x_1 \\
&= \frac{\eta(\frac{a^\gamma}{2}(1+a))}{\frac{a^\gamma}{2}}.
\end{aligned}$$

Then,

$$\eta(s) = s^2 a^2(s), \quad \forall\, s \in (0,1)$$

where $a(s)$ is the unique solution of

$$s = \frac{a^\gamma}{2}(1+a). \tag{5.10}$$

In fact, by the implicit function theorem, it is easy to see that there exists a C^∞ function h defined in a neighborhood of 0 such that

$$s = \frac{h(s)}{2^{\frac{1}{\gamma}}}(1+h(s))^{\frac{1}{\gamma}}.$$

Hence, $h(s^{\frac{1}{\gamma}})$ is a solution to (5.10) and $\eta(s) = s^2 h^2(s^{\frac{1}{\gamma}})$. After tedious computations, we can check that

$$\eta'(s) = 2s\, h^2(s^{\frac{1}{\gamma}}) \;+\; \frac{2}{\gamma} s^{\frac{1}{\gamma}+1}\; h(s^{\frac{1}{\gamma}})\, h'(s^{\frac{1}{\gamma}}),$$

$$\eta''(s) = 2\, h^2(s^{\frac{1}{\gamma}}) \;+\; \left(\frac{6}{\gamma} \;+\; \frac{2}{\gamma^2}\right) s^{\frac{1}{\gamma}}\; h(s^{\frac{1}{\gamma}})\, h'(s^{\frac{1}{\gamma}}) \;+\; \frac{2}{\gamma^2}\, s^{\frac{2}{\gamma}} (\, (h'(s^{\frac{1}{\gamma}}))^2 \;+\; h(s^{\frac{1}{\gamma}})\, h''(s^{\frac{1}{\gamma}})),$$

$$\begin{aligned}\eta'''(s) = \left(\frac{4}{\gamma} + \frac{6}{\gamma^2} + \frac{2}{\gamma^3}\right) s^{\frac{1}{\gamma}-1} h(s^{\frac{1}{\gamma}}) h'(s^{\frac{1}{\gamma}}) + \left(\frac{6}{\gamma^3} + \frac{6}{\gamma^2}\right) s^{\frac{2}{\gamma}-1}(\, (h'(s^{\frac{1}{\gamma}}))^2 + h(s^{\frac{1}{\gamma}}) h''(s^{\frac{1}{\gamma}}))\\ + \;\frac{6}{\gamma^3}\, s^{\frac{3}{\gamma}-1}\; h'(s^{\frac{1}{\gamma}})\, h''(s^{\frac{1}{\gamma}}) + \frac{2}{\gamma^3}\, s^{\frac{3}{\gamma}-1}\; h(s^{\frac{1}{\gamma}})\, h'''(s^{\frac{1}{\gamma}}),\end{aligned}$$

and

$$\begin{aligned}\eta''''(s) = &\left(\frac{14}{\gamma^4}+\frac{12}{\gamma^3}-\frac{2}{\gamma^2}\right) s^{\frac{2}{\gamma}-2}(h'(s^{\frac{1}{\gamma}}))^2 + \left(\frac{2}{\gamma^4}+\frac{4}{\gamma^3}-\frac{2}{\gamma^2}-\frac{4}{\gamma}\right) s^{\frac{1}{\gamma}-2} h(s^{\frac{1}{\gamma}}) h'(s^{\frac{1}{\gamma}}) + \frac{6}{\gamma^4} s^{\frac{4}{\gamma}-2}(h''(s^{\frac{1}{\gamma}}))^2\\ &+ \left(\frac{14}{\gamma^4} + \frac{12}{\gamma^3} - \frac{2}{\gamma^2}\right) s^{\frac{2}{\gamma}-2}\; h(s^{\frac{1}{\gamma}})\, h''(s^{\frac{1}{\gamma}}) \;+\; \left(\frac{36}{\gamma^4} + \frac{12}{\gamma^3}\right) s^{\frac{3}{\gamma}-2}\; h'(s^{\frac{1}{\gamma}})\, h''(s^{\frac{1}{\gamma}}) \;+\; \frac{8}{\gamma^4}\, s^{\frac{4}{\gamma}-2}\, h'(s^{\frac{1}{\gamma}})\, h'''(s^{\frac{1}{\gamma}})\\ &+ \left(\frac{12}{\gamma^4} \;+\; \frac{4}{\gamma^3}\right) s^{\frac{3}{\gamma}-2}\; h(s^{\frac{1}{\gamma}})\, h'''(s^{\frac{1}{\gamma}}) \;+\; \frac{2}{\gamma^4}\, s^{\frac{4}{\gamma}-2}\, h(s^{\frac{1}{\gamma}})\, h''''(s^{\frac{1}{\gamma}}).\end{aligned}$$

Hence,

$$\begin{cases} \eta(|.|) \in C^\infty[-1,1] & \text{if } \gamma = \frac{1}{2},\\ \eta \in C^\infty[0,1] \text{ and } \eta(|.|) \in C^{4,1}[-1,1] & \text{if } \gamma = 1,\\ \eta(|.|) \in C^{3,\frac{2}{\gamma}-1}[-1,1] & \text{if } 1 < \gamma < 2,\\ \eta \in C^3[0,1] \text{ and } \eta(|.|) \in C^{2,1}[-1,1] & \text{if } \gamma = 2,\\ \eta(|.|) \in C^{2,\frac{2}{\gamma}}[-1,1] \cap W^{3,\frac{\gamma}{\gamma-2}-\varepsilon}(-1,1) & \text{if } \gamma > 2,\, \varepsilon > 0. \end{cases} \tag{5.11}$$

Now, we will introduce the following key propositions, whose proofs, for simplicity of exposition, are postponed to Section 5.2.

PROPOSITION 5.1. *The transport density* σ *between* f^+ *and* f^- *is not in* $W^{1,p}(\Delta)$ *for all* p *satisfying*

$$p \geq \min\left\{\frac{\gamma}{\gamma-1}, \frac{\gamma+2}{\gamma}\right\}.$$

Then, we get the following

COROLLARY 5.2. *We have the following statements:*

$$\text{for all } \; p>1,\, \varepsilon>0, \quad f^{\pm} \in W^{1,p}(\Delta) \quad \not\Rightarrow \quad \sigma \in W^{1,\frac{2p+\varepsilon}{p+1}}(\Delta), \tag{5.12}$$

$$f^{\pm} \in C^1(\bar{\Delta}) \quad \not\Rightarrow \quad \sigma \in H^1(\Delta), \tag{5.13}$$

$$\text{for all } \; \alpha \in (0,1), \quad f^{\pm} \in C^{1,\alpha}(\Delta) \quad \not\Rightarrow \quad \sigma \in W^{1,2+\alpha}(\Delta), \tag{5.14}$$

$$f^{\pm} \in C^{\infty}(\bar{\Delta}) \quad \not\Rightarrow \quad \sigma \in W^{1,3}(\Delta). \tag{5.15}$$

PROOF. These statements follow immediately from (5.11) and the proposition 5.1. Indeed, for $\gamma > 2$: $\eta'' \in W^{1,\frac{\gamma}{\gamma-2}-\varepsilon}(0,1)$ (for any $\varepsilon > 0$) and the transport density σ is not in $W^{1,\frac{\gamma}{\gamma-1}}(\Delta)$, so (5.12) follows. To prove (5.13), take $\gamma = 2$ and then, in this case, we have that $\eta'' \in C^1[0,1]$ and $\sigma \notin H^1(\Delta)$. For $1 < \gamma < 2$: $\eta'' \in C^{1,\frac{2}{\gamma}-1}[0,1]$ and $\sigma \notin W^{1,\frac{\gamma+2}{\gamma}}(\Delta)$, so (5.14) follows. Finally, for $\gamma = 1$: $\eta \in C^{\infty}[0,1]$ and $\sigma \notin W^{1,3}(\Delta)$ and the statement (5.15) follows. □

PROPOSITION 5.3. *The transport density* σ *between* f^+ *and* f^- *is not in* $C^{0,\frac{1}{\gamma+1}+\varepsilon}(\Delta)$, *for every* $\varepsilon > 0$.

COROLLARY 5.4. *We have the following statements:*

$$\text{for all } \; \alpha \in (0,1),\, \varepsilon>0, \quad f^{\pm} \in C^{0,\alpha}(\Delta) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{\alpha}{\alpha+2}+\varepsilon}(\Delta), \tag{5.16}$$

$$\text{for all } \; \varepsilon>0, \quad f^{\pm} \in C^1(\bar{\Delta}) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{1}{3}+\varepsilon}(\Delta), \tag{5.17}$$

$$\text{for all } \; \alpha \in (0,1),\, \varepsilon>0, \quad f^{\pm} \in C^{1,\alpha}(\Delta) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{1+\alpha}{3+\alpha}+\varepsilon}(\Delta), \tag{5.18}$$

$$\text{for all } \; \varepsilon>0, \quad f^{\pm} \in C^{\infty}(\bar{\Delta}) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{1}{2}+\varepsilon}(\Delta). \tag{5.19}$$

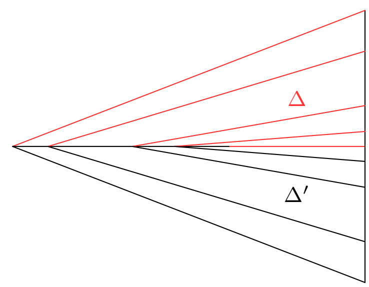

FIGURE 2

PROOF. These statements follow immediately from (5.11) and Proposition 5.3. Indeed, for $\gamma > 2:\ \eta'' \in C^{0,\frac{2}{\gamma}}[0,1]$, for $\gamma = 2:\ \eta'' \in C^1[0,1]$, for $1 < \gamma < 2:\ \eta'' \in C^{1,\frac{2}{\gamma}-1}[0,1]$ and, finally, for $\gamma = 1:\ \eta \in C^\infty[0,1]$. Yet, in all these cases, the transport density $\sigma \notin C^{0,\frac{1}{\gamma+1}+\varepsilon}(\Delta)$, for all $\varepsilon > 0$. □

To obtain counter-examples to interior regularity of the transport density, it suffices to reflect the domain across the x_1-axis. Let Δ' be the reflection of Δ with respect to the x_1-axis (see Figure 2) and set $\Omega := \Delta \cup \Delta'$. Extend the functions $f^\pm$ to Ω so that they are symmetric with respect to the x_1-axis. Let T be an optimal transport map from f^+ onto f^-, and let σ be the transport density between them, then it is easy to prove that the map S, which is equal to T on Δ and to the reflection of T with respect to the x_1-axis on Δ', is an optimal transport map between the extended densities and the transport density between them is equal to σ on Δ and to the reflection of σ, with respect to the x_1-axis, on Δ'. Using this fact and (5.11), we get the following statements:

$$\text{for all } p > 1,\ \varepsilon > 0,\ \ f^\pm \in W^{1,p}(\Omega) \quad \not\Rightarrow \quad \sigma \in W^{1,\frac{2p+\varepsilon}{p+1}}_{loc}(\Omega), \tag{5.20}$$

$$\text{for all } \alpha \in (0,1),\ \varepsilon > 0,\ \ f^\pm \in C^{0,\alpha}(\Omega) \quad \not\Rightarrow \quad \sigma \in C^{0,\frac{\alpha}{\alpha+2}+\varepsilon}_{loc}(\Omega), \tag{5.21}$$

$$\text{for all } \varepsilon > 0,\ \ f^\pm \in C^{0,1}(\Omega) \quad \not\Rightarrow \quad \sigma \in H^1_{loc}(\Omega) \cup C^{0,\frac{1}{3}+\varepsilon}_{loc}(\Omega), \tag{5.22}$$

$$\text{for all } \alpha \in (0,1),\ \varepsilon > 0,\ \ f^\pm \in C^{1,\alpha}(\Omega) \quad \not\Rightarrow \quad \sigma \in W^{1,2+\alpha}_{loc}(\Omega) \cup C^{0,\frac{1+\alpha}{3+\alpha}+\varepsilon}_{loc}(\Omega), \tag{5.23}$$

$$\text{for all } \varepsilon > 0,\ \ f^\pm \in C^{2,1}(\Omega) \quad \not\Rightarrow \quad \sigma \in W^{1,3}_{loc}(\Omega) \cup C^{0,\frac{1}{2}+\varepsilon}_{loc}(\Omega), \tag{5.24}$$

$$\text{for all } \varepsilon > 0,\ \ f^\pm \in C^\infty(\bar\Omega) \quad \not\Rightarrow \quad \sigma \in W^{1,5}_{loc}(\Omega) \cup C^{0,\frac{2}{3}+\varepsilon}_{loc}(\Omega). \tag{5.25}$$

5.2. Proof

In this section, we want to prove Propositions 5.1 & 5.3. Firstly, we will compute the transport density σ between f^+ and f^-. To do that, let us observe that the family $\{l_a,\, a\in(0,1)\}$, where l_a is defined as in (5.6), covers Δ so that for every $x:=(x_1,x_2)\in\Delta$, there exists a unique pair $(t,a)\in(0,1)^2$ such that $x\in l_a$ and $|x-(-a,0)|=tL(a)$, where $L(a)$ is the length of the segment l_a. In other words, we have

$$x \;=\; \left(-a+(1+a)t\,,\,(1+a)\frac{ta^\gamma}{2}\right).$$

Fix $(t,a)\in(0,1)^2$ and set,

$$\omega_\varepsilon := \left\{\left(-s+(1+s)\tau,\,(1+s)\frac{\tau s^\gamma}{2}\right),\,(\tau,s)\in(0,t)\times(a,a+\varepsilon)\right\}$$

where $\varepsilon>0$ is small enough. Recalling (2.7) and integrating $-\nabla\cdot(\sigma\nabla u)=f$ on ω_ε, we get

$$-\int_{\partial\omega_\varepsilon}\sigma\nabla u\cdot\mathbf{n}=\int_{\omega_\varepsilon}f. \tag{5.26}$$

Suppose that the family of segments $(l_a)_{a\in(0,1)}$ are, in fact, all the transport rays on which the optimal transport map, between f^+ and f^-, acts. In this case, we get that for every $x\in l_a$:

$$\nabla u(x)=\frac{(-a,0)-(1,(1+a)\frac{a^\gamma}{2})}{|(-a,0)-(1,(1+a)\frac{a^\gamma}{2})|}=\frac{-(1,\frac{a^\gamma}{2})}{\sqrt{1+(\frac{a^\gamma}{2})^2}},$$

which means that $\nabla u(x)\cdot\mathbf{n}=0$ if $\mathbf{n}$ is the unit orthogonal vector to l_a. Hence, (5.26) becomes

$$-\int_{s_\varepsilon}\sigma\nabla u\cdot\mathbf{n}=\int_{\omega_\varepsilon}f \tag{5.27}$$

where $s_\varepsilon:=\left\{(-s+(1+s)t,\,(1+s)\frac{ts^\gamma}{2}),\,s\in[a,a+\varepsilon]\right\}$. Yet, we have

$$\begin{aligned}\int_{\omega_\varepsilon} f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 &= \int_{\omega_\varepsilon} -\beta(\zeta''(x_1)+\eta''(x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &= \int_a^{a+\varepsilon}\int_0^t -\beta\Big(\zeta''(-s+(1+s)\tau)+\eta''\Big((1+s)\frac{\tau s^\gamma}{2}\Big)\Big)\,J(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s,\end{aligned}$$

where $J(\tau,s) := |\det(D_{(\tau,s)}(x_1,x_2))|$. But,

$$D_{(t,a)}(x_1,x_2) := \begin{pmatrix}\partial_t x_1 & \partial_a x_1\\ \partial_t x_2 & \partial_a x_2\end{pmatrix} = \begin{pmatrix}1+a & -1+t\\ \frac{(1+a)\,a^\gamma}{2} & (\gamma\,(1+a)+a)\,\frac{ta^{\gamma-1}}{2}\end{pmatrix}.$$

Then,

$$J(t,a) = (1+a)(\gamma\,(1+a)\,t+a)\frac{a^{\gamma-1}}{2}. \tag{5.28}$$

On the other hand,

$$-\nabla u\cdot\mathbf{n} = \frac{\partial_t x}{|\partial_t x|}\cdot R\,\frac{\partial_a x}{|\partial_a x|} = \frac{J(t,a)}{L(a)|\partial_a x|}$$

where $R := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is the rotation matrix. Hence,

$$-\int_{s_\varepsilon}\sigma\nabla u\cdot\mathbf{n} = \int_a^{a+\varepsilon}\sigma\Big(-s+(1+s)t,\ (1+s)\frac{ts^\gamma}{2}\Big)\frac{J(t,s)}{L(s)}\,\mathrm{d}s.$$

By (5.27), we infer that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_a^{a+\varepsilon}\sigma\Big(-s+(1+s)t,\ (1+s)\frac{ts^\gamma}{2}\Big)\frac{J(t,s)}{L(s)}\,\mathrm{d}s\\ &\qquad= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_a^{a+\varepsilon}\int_0^t -\beta\Big(\zeta''(-s+(1+s)\tau)+\eta''\Big((1+s)\frac{\tau s^\gamma}{2}\Big)\Big)\,J(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Finally, we get

$$\sigma(x) = \frac{L(a)\int_0^t -\beta(\zeta''(-a+(1+a)\tau)+\eta''((1+a)\frac{\tau a^\gamma}{2}))\,J(\tau,a)\,\mathrm{d}\tau}{J(t,a)}. \tag{5.29}$$

Now, we are ready to prove Proposition 5.3. Indeed, for every $\varepsilon > 0$, let $(t_\varepsilon, a_\varepsilon)$ be in $(0,1)^2$ such that

$$x_\varepsilon := (0,\varepsilon) = \Big(-a_\varepsilon + (1+a_\varepsilon)t_\varepsilon,\ (1+a_\varepsilon)\frac{t_\varepsilon a_\varepsilon^\gamma}{2}\Big).$$

As $\zeta''(0) < 0$ and $\eta''(0) = 0$, then, from (5.29), we can see easily that, close to the origin, we have

$$\sigma \approx \frac{\int_0^t J(\tau,a)\,\mathrm{d}\tau}{J(t,a)}$$

where the symbol $\approx$ stands for inequalities up to multiplicative constants depending on the data, i.e. on f, but not on x. Yet, by (5.28), one has

$$\int_0^t J(\tau,a)\,\mathrm{d}\tau = \frac{t}{2}\Big(J(t,a) + (1+a)\frac{a^\gamma}{2}\Big).$$

As $t_\varepsilon = a_\varepsilon/(1+a_\varepsilon)$ and $\varepsilon = a_\varepsilon^{\gamma+1}/2$, we infer that $t_\varepsilon \approx \varepsilon^{\frac{1}{\gamma+1}}$. Hence, we get that the value of the transport density σ at x_ε is

$$\sigma(x_\varepsilon) \approx \varepsilon^{\frac{1}{\gamma+1}}.$$

This completes the proof of the proposition 5.3. Next, to prove Proposition 5.1, we will only look at $\partial_{x_2}\sigma$ close to the origin and to do that, we want to compute, firstly, $\partial_t\sigma$ and $\partial_a\sigma$. So, differentiating (5.29) with respect to t and a respectively, we get

$$\partial_t\sigma(x) = L(a)\,f(x) - \frac{\partial_t J(t,a)}{J(t,a)}\,\sigma(x), \tag{5.30}$$

and

$$\partial_a\sigma(x) = \frac{L'(a)}{L(a)}\,\sigma(x) + R_1(t,a) + R_2(t,a), \tag{5.31}$$

where

$$R_1(t,a) := \frac{L(a)\int_0^t -\beta(\zeta''(-a+(1+a)\tau) + \eta''((1+a)\frac{\tau a^\gamma}{2}))\,\partial_a J(\tau,a)\,\mathrm{d}\tau}{J(t,a)} - \frac{\partial_a J(t,a)}{J(t,a)}\,\sigma(x)$$

and

$$R_2(t,a) := \frac{L(a)\int_0^t -\beta(-(1-\tau)\zeta'''(-a+(1+a)\tau) + \frac{\tau a^{\gamma-1}}{2}(\gamma(1+a)+a)\eta'''((1+a)\frac{\tau a^\gamma}{2}))J(\tau,a)\mathrm{d}\tau}{J(t,a)}.$$

Now, we claim that, for any $\gamma \geq \frac{1}{2}$,

$$\partial_a \sigma \approx \, t \, + \, \left(\frac{t}{t+a}\right)^2. \tag{5.32}$$

From Section 5.1, we have

$$|\eta'''(x_2)| \leq C\, x_2^{\frac{2}{\gamma}-1}, \text{ for all } \, x_2 \in (0,1). \tag{5.33}$$

Then,

$$\left|\frac{\tau a^{\gamma-1}}{2}(\gamma(1+a)+a)\,\eta'''\left((1+a)\frac{\tau a^\gamma}{2}\right)\right| \leq C\,\tau^{\frac{2}{\gamma}} a.$$

Now, as $\zeta'''(0) > 0$, we infer that

$$R_2(t,a) \approx t.$$

In addition, it is easy to see that

$$\frac{L'(a)}{L(a)}\,\sigma(x) = \frac{4 + a^{2\gamma} + \gamma(1+a)\,a^{2\gamma-1}}{(1+a)(4+a^{2\gamma})}\,\sigma(x) \approx t.$$

On the other hand,

$$R_1(t,a) \quad = \quad \frac{L(a)\int_0^t f(-a+(1+a)\tau, (1+a)\frac{\tau a^\gamma}{2})\,(J(t,a)\partial_a J(\tau,a) - J(\tau,a)\partial_a J(t,a))\,\mathrm{d}\tau}{J(t,a)^2}.$$

Yet, by (5.28), we have

$$\partial_a J(t,a) = \big(\gamma\,(\gamma-1)\,(1+a)^2\,t + (\gamma\,(1+a)\,(1+2t)+a)\,a\big)\,\frac{a^{\gamma-2}}{2}.$$

Then, it is not difficult to check that

$$J(t,a)\,\partial_a J(\tau,a) - J(\tau,a)\,\partial_a J(t,a) = \frac{\gamma}{4}(1+a)^2(t-\tau)a^{2\gamma-2}.$$

Using (5.28), we infer that

$$R_1(t,a) \approx \left(\frac{t}{t+a}\right)^2,$$

which completes the proof of (5.32). On the other hand, we want to prove the following

$$\partial_t \sigma \approx 1. \tag{5.34}$$

Fix $\varepsilon > 0$. As $\eta''(0) = 0$, then we can assume that, close to the origin, we have

$$|f(x) - 2\beta| < \varepsilon.$$

From (5.30), we get

$$\partial_t\sigma(x) \quad \geq \quad L(a)\left(2\beta - \varepsilon - \frac{\int_0^t (2\beta+\varepsilon)\,\partial_t J(t,a)\,J(\tau,a)\mathrm{d}\tau}{J(t,a)^2}\right).$$

Yet,

$$\partial_t J(t,a) = \frac{\gamma}{2}\,(1+a)^2\,a^{\gamma-1}$$

and then,

$$\frac{\int_0^t \partial_t J(t,a) J(\tau,a)\mathrm{d}\tau}{J(t,a)^2} = \frac{1}{2}\left(1 - \frac{a^2}{(\gamma\,(1+a)\,t+a)^2}\right).$$

Hence,

$$\partial_t\sigma(x) \geq \; \beta - \frac{3\varepsilon}{2} > 0,$$

for $\varepsilon > 0$ small enough. In the same way, one can also see that $\partial_t\sigma$ is bounded from above and then, (5.34) follows.

Yet,

$$\partial_{x_2}\sigma = \partial_t\sigma\,\partial_{x_2}t \,+\, \partial_a\sigma\,\partial_{x_2}a$$

and

$$D_{(x_1,x_2)}(t,a) := \begin{pmatrix} \partial_{x_1}t & \partial_{x_2}t \\ \partial_{x_1}a & \partial_{x_2}a \end{pmatrix} = \frac{1}{J(t,a)}\begin{pmatrix} (\gamma\,(1+a)+a)\,\frac{ta^{\gamma-1}}{2} & 1-t \\ -\frac{(1+a)\,a^{\gamma}}{2} & 1+a \end{pmatrix}.$$

Hence, by (5.32) & (5.34), we get

$$\partial_{x_2}\sigma \approx \frac{1}{J}$$

and

$$||\partial_{x_2}\sigma||^p_{L^p(\Delta)} \approx \int_\Delta \frac{1}{J(t,a)^p}\,\mathrm{d}x_1\,\mathrm{d}x_2 \approx \int_0^\delta\int_0^\delta \frac{1}{J(t,a)^{p-1}}\,\mathrm{d}t\,\mathrm{d}a,$$

where $\delta > 0$ is small enough. Recalling (5.28), we see that the Jacobian $J \approx a^{\gamma-1}(t+a)$ and so,

$$\begin{aligned}||\partial_{x_2}\sigma||^p_{L^p(\Delta)} &\approx \int_0^\delta\int_0^\delta \frac{1}{a^{(\gamma-1)(p-1)}(t+a)^{p-1}}\,\mathrm{d}t\,\mathrm{d}a \\ &\approx \int_0^\delta r\,\mathrm{d}r\int_0^{\frac{\pi}{2}} \frac{1}{r^{\gamma(p-1)}\sin(\theta)^{(\gamma-1)(p-1)}(\cos(\theta)+\sin(\theta))^{p-1}}\,\mathrm{d}\theta \\ &\approx \int_0^\delta \frac{1}{r^{\gamma(p-1)-1}}\,\mathrm{d}r\int_0^{\frac{\pi}{2}} \frac{1}{\sin(\theta)^{(\gamma-1)(p-1)}}\,\mathrm{d}\theta.\end{aligned}$$

Then, the proposition 5.1 is proved. But, it remains to prove that the rays $(l_a)_a$ are all the transport rays between f^+ and f^-. Firstly, we observe that, for every $x := (x_1, x_2) \in \Delta$, there exists a unique $a := a(x) \in (0,1)$ (note that $a \in C^1(\Delta)$) such that $x \in l_a$, i.e.,

(5.35) $$x_2 = \frac{a^\gamma}{2}(x_1 + a).$$

Differentiating the equality (5.35) with respect to the x_1 and x_2 variables, we get the following

(5.36) $$\partial_{x_1}a = \frac{-a}{\gamma(x_1+a)+a}$$

and

(5.37) $$\partial_{x_2}a = \frac{2}{a^{\gamma-1}(\gamma(x_1+a)+a)}.$$

Now, let v_a be the unit vector of l_a, i.e.

$$v_a := \frac{-(1, \frac{a^\gamma}{2})}{\sqrt{1+\frac{a^{2\gamma}}{4}}},$$

then, we have

$$\nabla \times v_a := -\partial_{x_1}\Bigg(\frac{\frac{a^\gamma}{2}}{\sqrt{1+\frac{a^{2\gamma}}{4}}}\Bigg) + \partial_{x_2}\Bigg(\frac{1}{\sqrt{1+\frac{a^{2\gamma}}{4}}}\Bigg) = \frac{\gamma\, a^{\gamma-1}}{2(1+\frac{a^{2\gamma}}{4})^{\frac{3}{2}}}\Big(\partial_{x_1} a + \frac{a^\gamma}{2}\,\partial_{x_2} a\Big).$$

Yet, by (5.36) and (5.37), we infer that v_a is an irrotational vector field, which implies that there is a 1-Lipschitz function u such that

$$\nabla u(x) = v_a, \quad \text{for all } \ x \in l_a.$$

Hence, we have

$$u(x) - u(y) = |x-y| \qquad \text{for all } \ x,\, y \in l_a. \tag{5.38}$$

Finally, we want to prove that this function u is in fact the Kantorovich potential between f^+ and f^-. To do that, let us consider the disintegration of $f^\pm$ with respect to the segments $(l_a)_{a\in(0,1)}$. More precisely, we define a map $R : \Delta \mapsto \{l_a,\, a \in (0,1)\}$, valued in the set of all segments l_a, $a \in (0,1)$, sending each point $x \in \Delta$ into the unique segment l_a containing x. As by construction of f^- we have $f^+(\Delta_a) = f^-(\Delta_a)$ for every $a \in (0,1)$, we infer that there is a non-negative measure ν, defined on the set $\{l_a,\, a \in (0,1)\}$, such that $\nu = R_\# f^+ = R_\# f^-$. In particular, we can write $f^\pm = f_a^\pm \otimes \nu$. Now, set $\gamma_a := f_a^+ \otimes f_a^-$ and $\gamma := \gamma_a \otimes \nu$. Then, it is clear that the plan γ belongs to $\Pi(f^+, f^-)$. Moreover, one has that $\gamma-$a.e. pair (x,y) in $\Omega\times\Omega$ is contained in $\text{spt}(\gamma_a)$ for some $a \in (0,1)$, then both x and y are in l_a. Yet, this is sufficient to conclude, since we get

$$\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma = \int_{\Omega\times\Omega} (u(x)-u(y))\,\mathrm{d}\gamma(x,y) = \int_\Omega u\,\mathrm{d}(f^+ - f^-),$$

which implies that γ is an optimal transport plan between f^+ and f^-, and u is the corresponding Kantorovich potential.

5.3. BV counter-example

In this section, we will prove the statement (5.1). This means that we want to construct two densities $f^\pm \in BV(\Omega)$ such that the transport density σ between them is not in $BV(\Omega)$. First of all, we can see easily that for any $\gamma > 0$, the densities $f^\pm$, which are constructed in Section 5.1, are in $BV(\Omega)$, but it will be also the same for the transport density σ between them. Indeed, to get a counter-example to the $W^{1,p}$ regularity of the transport density, for $p \to 1$, we need a $\gamma \to \infty$. Hence, to get a BV counter-example, we could collect an infinity of triangles (constructed as in Section 5.1) with a sequence of exponents $\gamma_n \to \infty$ (where γ_n is the exponent of the slopes of the transport rays in the n-th triangle, see (5.6)). Actually, if we play on other parameters, we just need to take $\gamma_n = \gamma > 1$. To do that, let us define Δ_n as follows:

$$\Delta_n := \text{triangle with vertices } (-l_n, 0),\ \Big(1, -\frac{l_n^\gamma}{2}(1+l_n)\Big) \text{ and } \Big(1, \frac{l_n^\gamma}{2}(1+l_n)\Big)$$

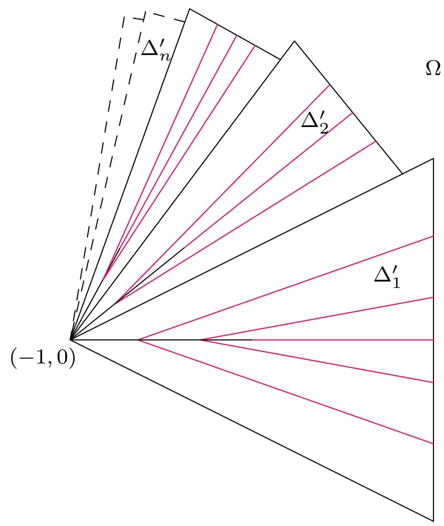

FIGURE 3

where $l_n := 1/n$. Set, $\Delta_1' := \Delta_1$ and, for all $n \geq 2$, define Δ_n' as a suitable roto-translation of Δ_n:

$$\Delta_n' := \{(x_1,x_2) \in \mathbb{R}^2 \,:\, (\cos(\theta_n)(x_1+l_1) + \sin(\theta_n)x_2 - l_n, -\sin(\theta_n)(x_1+l_1) + \cos(\theta_n)x_2) \in \Delta_n\},$$

where

$$\theta_n := \sum_{k=1}^{n-1} \alpha_k + \alpha_{k+1}$$

and

$$\sin(\alpha_k) := \frac{\frac{l_k^\gamma}{2}}{\sqrt{1+\left(\frac{l_k^\gamma}{2}\right)^2}}, \;\; \alpha_k \in \left(0, \frac{\pi}{2}\right).$$

Finally, set

$$\Omega := \bigcup_{n=1}^{\infty} \Delta_n'.$$

Fix $n \in \mathbb{N}^*$. Then, after a suitable roto-translation of axis, we can assume that $\Delta_n' = \Delta_n$. Set,

$$f^+(x_1,x_2) := 1$$

and

$$f^-(x_1,x_2) := f_n^-(x_1,x_2) := 1 \,+\, \beta(\zeta''(x_1) \,+\, \eta''(|x_2|)), \quad \text{for all } (x_1,x_2) \in \Delta_n',$$

where ζ and η are the same functions which are constructed in the section 5.1. Let us denote by σ the transport density between f^+ and f^-. Then, the restriction of σ to Δ'_n is the transport density σ_n between $f_n^+ := 1_{\Delta'_n}$ and f_n^-. Indeed, for all $n \in \mathbb{N}^*$, if T_n is an optimal transport map from f_n^+ onto f_n^- and if u_n is the corresponding Kantorovich potential such that $u_n(-1,0) = 0$, for all $n \in \mathbb{N}^*$, then it is not difficult to check that

$$T(x) := T_n(x), \text{ for a.e. } x \in \Delta'_n$$

is an optimal transport map from f^+ onto f^-, and the corresponding Kantorovich potential will be

$$u(x) := u_n(x), \text{ for all } x \in \Delta'_n.$$

By (2.1), we infer that the restriction of σ to Δ'_n is σ_n. Yet, by Section 5.2, we have already shown that

$$|\nabla \sigma_n| \approx \frac{1}{J_n},$$

where J_n is defined as in (5.28) on Δ_n. Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} ||\nabla \sigma_n||_{L^1(\Delta'_n)} &\approx \sum_{n=1}^{\infty} \int_0^{l_n} \int_0^{\delta} |\nabla \sigma_n(t,a)| J_n(t,a) \, \mathrm{d}t \, \mathrm{d}a \\ &\approx \sum_{n=1}^{\infty} l_n = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty, \end{aligned}$$

where $\delta > 0$ is small enough.

Hence, the transport density σ is not in $BV(\Omega)$. On the other hand, we will show that the target mass f^- is in $BV(\mathbb{R}^2)$. Using (5.33), it is easy to prove that

$$\sum_{n=1}^{\infty} ||\nabla f_n^-||_{L^1(\Delta'_n)} \le C \sum_{n=1}^{\infty} (l_n^{\gamma} + l_n^2) < +\infty.$$

In addition, for a fixed $n \in \mathbb{N}^*$ and after a suitable roto-translation of axis so that $\Delta'_n = \Delta_n$, we can assume that

$$f_n^-(x_1, x_2) = 1 + \beta(\zeta''(x_1) + \eta''(|x_2|))$$

and,

$$\begin{aligned} f_{n+1}^-(x_1, x_2) = 1 + \beta\Big(&\zeta''(\cos(\theta_{n+1})(x_1 + l_n) + \sin(\theta_{n+1})x_2 - l_{n+1}) \\ &+ \eta''(|-\sin(\theta_{n+1})(x_1 + l_n) + \cos(\theta_{n+1})x_2|)\Big) \end{aligned}$$

where

$$\theta_{n+1} := \alpha_n + \alpha_{n+1} \approx l_{n+1}^{\gamma}.$$

Hence, it is not difficult to check that

$$\left| f_{n+1}^{-}\left(x_1, \frac{l_n^{\gamma}}{2}(x_1+l_n)\right) - f_n^{-}\left(x_1, \frac{l_n^{\gamma}}{2}(x_1+l_n)\right)\right| \le C l_n^2.$$

Finally, we get

$$\sum_{n=1}^{\infty} \int_{\partial\Delta_n' \cap \partial\Delta_{n+1}'} |f_{n+1}^{-}(z) - f_n^{-}(z)| \mathrm{d}z \le C \sum_{n=1}^{\infty} l_n^2 < +\infty.$$

As f^- is bounded and $Per(\Omega) \approx \sum_n (l_n^{\gamma} + l_n^2) < +\infty$, we infer that the target mass $f^- \in BV(\mathbb{R}^2)$ and the statement (5.1) follows.

5.4. Counter-examples with compactly supported smooth densities on the whole plane

In this section, we want to show that is also possible to construct the target measure f^- so that it will be regular on $\mathbb{R}^2$. Firstly, let us observe that the function ζ (see Section 5.1) can be replaced by $\psi\zeta$, where ψ is a C^∞ function such that $\psi = 1$ on $[-1, 1-\varepsilon']$ and $\psi = 0$ on $[1-\varepsilon, 1]$, where $0 < \varepsilon < \varepsilon' < 1$. Let χ_1, χ_2 be two cut-off functions supported on $\Delta \cup R(\Delta)$, where R is the reflection map with respect to the x_1-axis, such that $\mathrm{spt}(\chi_2) \subset \{\chi_1 = 1\}$, $\Delta_{a_0} \cap \{x : x_1 \le 1-\varepsilon\} \subset \{\chi_1 = 1\}$ (where $a_0 \in (\varepsilon, 1)$ is such that $\mathrm{spt}(\chi_2) \subset \Delta_{a_0} \cup R(\Delta_{a_0})$), $\Delta_\varepsilon \cap \{x : x_1 \le 1-\varepsilon\} \subset \{\chi_2 = 1\}$ and χ_1, χ_2 are symmetric with respect to the x_1-axis. Set,

$$f^+ := \chi_1$$

and

$$f^- := \chi_1 + \beta\Bigg(\Big((\psi\zeta)''(x_1) + \eta''(|x_2|)\Big)\chi_2 + \varphi(x_1)c(a(x_1,|x_2|))\Bigg),$$

where φ is a non-negative C^∞ function such that $\mathrm{spt}(\varphi) \subset (1-\varepsilon', 1-\varepsilon)$ and c is to be determined in such a way that

$$\int_{\Delta_a} f^+ = \int_{\Delta_a} f^- \quad \text{for all } a \in (0,1),$$

which is equivalent to say that

$$-\int_{\Delta_a} ((\psi\zeta)''(x_1) + \eta''(x_2))\chi_2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\Delta_a} \varphi(x_1)c(a(x_1,x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad \text{for all } a \in (0,1).$$

Differentiating this equality with respect to a, we get

$$c(a) = \frac{-\int_{-a}^{1}(\gamma(x_1+a)+a)((\psi\zeta)''(x_1)+\eta''(\frac{a^\gamma}{2}(x_1+a)))\chi_2(x_1,\frac{a^\gamma}{2}(x_1+a))\,\mathrm{d}x_1}{\int_{-a}^{1}(\gamma(x_1+a)+a)\varphi(x_1)\,\mathrm{d}x_1}, \quad \forall\ a \in (0,1).$$

By (5.9), we have

$$-\int_{-a}^{1}(\gamma(x_1+a)+a)(\psi\zeta)''(x_1)\,\mathrm{d}x_1 = \int_{-a}^{1}(\gamma(x_1+a)+a)\eta''\Big(\frac{a^\gamma}{2}(x_1+a)\Big)\,\mathrm{d}x_1, \quad \text{for all } a \in (0,1).$$

Hence, for $a < \varepsilon$, we get

$$c(a) = \frac{\int_{-a}^{1}(\gamma(x_1+a)+a)\eta''(\frac{a^\gamma}{2}(x_1+a))(1-\chi_2(x_1,\frac{a^\gamma}{2}(x_1+a)))\,\mathrm{d}x_1}{\int_{-a}^{1}(\gamma(x_1+a)+a)\varphi(x_1)\,\mathrm{d}x_1}$$

and

$$c'(a) = \frac{1}{\int_{-a}^{1}(\gamma(x_1+a)+a)\varphi(x_1)\,\mathrm{d}x_1}\Bigg((\gamma+1)\int_{-a}^{1}\eta''\Big(\frac{a^\gamma}{2}(x_1+a)\Big)\Big(1-\chi_2\Big(x_1,\frac{a^\gamma}{2}(x_1+a)\Big)\Big)\,\mathrm{d}x_1$$

$$+\frac{a^{\gamma-1}}{2}\int_{-a}^{1}(\gamma(x_1+a)+a)^2\eta'''\Big(\frac{a^\gamma}{2}(x_1+a)\Big)\Big(1-\chi_2\Big(x_1,\frac{a^\gamma}{2}(x_1+a)\Big)\Big)\mathrm{d}x_1-(\gamma+1)\Big(\int_{-a}^{1}\varphi(x_1)\mathrm{d}x_1\Big)c(a)$$

$$-\frac{a^{\gamma-1}}{2}\int_{-a}^{1}(\gamma(x_1+a)+a)^2\,\eta''\Big(\frac{a^\gamma}{2}(x_1+a)\Big)\partial_{x_2}\chi_2\Big(x_1,\frac{a^\gamma}{2}(x_1+a)\Big)\,\mathrm{d}x_1\Bigg).$$

Using (5.33), we infer that

$$c'(a) \le Ca$$

and,

$$||\nabla(\varphi\, c(a))||^p_{L^p(\Delta)} \approx \int_{1-\varepsilon'}^{1-\varepsilon}\int_0^{\varepsilon}\Big(\varphi(x_1)^p\frac{|c'(a)|^p}{J(t,a)^p}+|\nabla\varphi(x_1)|^p|c(a)|^p\Big)\,\mathrm{d}x_2\,\mathrm{d}x_1 \approx \int_0^{\varepsilon}\frac{1}{a^{1-(\gamma-(\gamma-2)p)}}\,\mathrm{d}a.$$

Hence, for $\gamma > 2$,

$$f^- \in W^{1,\frac{\gamma}{\gamma-2}-\varepsilon}(\mathbb{R}^2), \text{ for all } \varepsilon > 0.$$

Similarly, we get that for $\gamma = \frac{1}{2}\colon\ f^- \in C^\infty(\mathbb{R}^2)$, for $\gamma = 1\colon\ f^- \in C^{2,1}(\mathbb{R}^2)$, for $1 < \gamma < 2\colon$ $f^- \in C^{1,\frac{2}{\gamma}-1}(\mathbb{R}^2)$ and, finally, for $\gamma = 2\colon\ f^- \in C^{0,1}(\mathbb{R}^2)$.

CHAPTER 6

Boundary-to-boundary transport densities, and applications to the BV least gradient problem in 2D

The least gradient problem (minimizing the BV norm with given boundary data) is known to be equivalent, in the plane, to the Beckmann minimal-flow problem with source and target measures located on the boundary of the domain. Motivated by this fact, we prove L^p summability results for the solution of the Beckmann problem in this setting, which improve upon previous results where the measures were themselves supposed to be L^p. This provides results about the $W^{1,p}$ regularity of the solution of the least gradient problem in uniformly convex domains.

This chapter is taken from a joint article with F. Santambrogio, [56].

6.1. Introduction

A classical problem in calculus of variations, which is of interest both with applications in image processing but also for its connection with minimal surfaces, is the so-called least gradient problem, considered for instance in [65, 94, 89, 108, 91, 12]. This is the problem of minimizing the total variation of the vector measure ∇u among all BV functions u defined on an open domain Ω with given boundary datum. If we consider

$$\inf\left\{|\nabla u|(\Omega)\,:\,u\in BV(\Omega),\,u_{|\partial\Omega}=g\right\}, \tag{6.1}$$

where $u_{|\partial\Omega}$ denotes the trace of u in the sense of BV functions and $|\nabla u|$ denotes the total variation measure of ∇u, this problem relaxes into

$$\min\left\{|\nabla u|(\Omega)+\int_{\partial\Omega}|u_{|\partial\Omega}-g|\mathrm{d}\mathcal{H}^{d-1}\,:\,u\in BV(\Omega)\right\}. \tag{6.2}$$

This can also be expressed in the following way: extend g into a BV function $\tilde{g}$ defined on a larger domain Ω', and then consider

$$\min\left\{|\nabla u|(\overline{\Omega})\,:\,u\in BV(\Omega'),\;u=\tilde{g}\;\text{ on }\;\Omega'\setminus\Omega\right\}. \tag{6.3}$$

The boundary datum g should be taken as a possible trace of BV functions, i.e. in $L^1(\partial\Omega)$, yet, the fact that the (a) solution u to (6.2) and (6.3) satisfies or not $u_{|\partial\Omega} = g$ could depend on g (and on the domain). In case we have $u_{|\partial\Omega} = g$, then u is also a solution of (6.1). In [64], the author proves existence of solutions to (6.1) for boundary data in $BV(\partial\Omega)$, while, in [106], the authors give an example of a function g such that (6.1) has no solution (g was chosen to be the characteristic function of a certain fat Cantor set, which does not lie in $BV(\partial\Omega)$).

In this chapter the least gradient problem will only be considered in the planar case $\Omega \subset \mathbb{R}^2$, and the boundary datum g will be at least in $BV(\partial\Omega)$ (something which makes perfectly sense, since $\partial\Omega$ is a closed curve, and we are just speaking about BV functions in 1D).

Following [65], we can see that there is a one-to-one correspondence between vector measures ∇u in (6.3) (considered as measures on $\overline{\Omega}$, so that we also include the part of the derivative of u which is on the boundary, i.e. the possible jump from $u_{|\partial\Omega}$ to g) and vector measures w satisfying, in $\overline{\Omega}$, $\nabla \cdot w = f$ where f is the measure obtained as the tangential derivative of $g \in BV(\partial\Omega)$; moreover, the mass of ∇u and of w are the same. Indeed, one just needs to take $w = R_{\frac{\pi}{2}}\nabla u$, where R_θ denotes a rotation with angle θ around the origin, and w solves the Beckmann problem

$$\inf\left\{ |w|(\overline{\Omega}) \,:\, w \in \mathcal{M}^2(\overline{\Omega}),\, \nabla\cdot w = f \right\}, \tag{6.4}$$

where we denote by $\mathcal{M}^2(\overline{\Omega})$ the space of finite vector measures on $\overline{\Omega}$ valued in $\mathbb{R}^2$. If we identify $f = \partial g/\partial \mathbf{t}$ ($\mathbf{t} := R_{-\frac{\pi}{2}}\mathbf{n}$ standing for the tangent vector to $\partial\Omega$) with its restriction to the boundary, we can also write the condition $\nabla \cdot w = f$ as $\nabla\cdot w = 0$ in Ω, $w \cdot \mathbf{n} = f$ on $\partial\Omega$ (when we write $\nabla \cdot w = f$ we mean $\int \nabla\varphi \cdot \mathrm{d}w = -\int \varphi \,\mathrm{d}f$ for every smooth test function φ, without imposing φ to have compact support, i.e. we also include boundary conditions).

The study of the least gradient problem can consequently be done by studying (6.4), and the question whether (6.1) has a solution becomes whether the solution to (6.4) gives mass to the boundary or not.

From Chapter 2, we have already seen that the Beckmann problem is strongly related to optimal transport theory, and is in some sense equivalent to the Monge-Kantorovich problem

$$\min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}\gamma :\ \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\, (\Pi_x)_{\#}\gamma = f^+ \ \text{ and } \ (\Pi_y)_{\#}\gamma = f^- \right\}, \tag{6.5}$$

where $f^\pm$ represent the positive and negative parts of f, i.e. two positive measures with the same mass. The scalar measure $\sigma = |w|$ obtained from an optimal w is the transport density (see (2.1)). Of course, L^p summability of σ is equivalent to $W^{1,p}$ regularity of the optimal u. From Propositions 2.3 & 2.4, we have the following L^p summability results on the transport density σ: in dimension d, for $p < d/(d-1)$, we have $w \in L^p$ as soon as at least one between f^+ or f^- is in L^p, while for general p (including $p = \infty$), this is proven when both f^+ and f^- belong to $L^p(\Omega)$. But in the case of interest for applications to the least gradient problem, the measures $f^\pm$ are singular (they are concentrated on the negligible set $\partial\Omega$). The only result obtained so far with measures concentrated on the boundary is the one that we presented in Chapter 3, where we considered the case where f^- is the projection of f^+ on $\partial\Omega$. On the other hand, this is far from the setting that we want to study now, since, first, in Chapter 3, only one of the two measures is on $\partial\Omega$ and, second, it is not an arbitrary measure but it is chosen to be the projection of the other.

We can say that, so far, the L^p summability of w in the case where both the measures f^+ and f^- are concentrated on the boundary is unknown. In particular, we do not know whether the optimal flow w belongs or not to $L^p(\Omega, \mathbb{R}^d)$ provided $f^\pm \in L^p(\partial\Omega)$.

The goal of the present chapter is exactly to investigate this kind of L^p summability results under suitable assumptions on the domain Ω, and then applying them to the $W^{1,p}$ regularity of the solution of (6.1).

This chapter is organized as follows. In Section 6.2, we adapt some well known facts concerning the usual Monge-Kantorovich problem to the precise setting of transport from the boundary to the boundary. In Section 6.3, we show positive results on the L^p summability of σ in the transport from a measure on the boundary to a measure on the boundary of a uniformly convex domain, in arbitrary dimension $d \geq 2$. In particular, we see that $f^\pm \in L^p(\partial\Omega) \Rightarrow \sigma \in L^p(\Omega)$ holds for $p \leq 2$ (to go beyond L^2 summability one needs extra regularity of the data). Section 6.4 gives indeed a counter-example where $f \in L^\infty$ but $\sigma \notin L^p$ for any $p > 2$. Section 6.5 summarizes the applications, in the case $d = 2$, of these results to the least gradient problem. Finally, Section 6.6 is devoted to the study of the anisotropic case.

Many of the results that we recover were already known thanks to different methods, but we believe that the connection with optimal transport and the technique we develop are interesting in themselves. Moreover, we believe that the following one is novel: if Ω is a uniformly convex domain in dimension 2, $p \leq 2$ and $g \in W^{1,p}(\partial\Omega)$, then the solution to (6.1) exists, is unique, and belongs to $W^{1,p}(\Omega)$ (this is our Theorem 6.11).

6.2. Monge-Kantorovich and Beckmann problems from boundary to boundary

We begin this section by recalling that an optimal transport map T for the Monge problem from f^+ onto f^- exists as soon as $f^+ \ll \mathcal{L}^d$ (see Theorem 1.6). Since we are interested in transport problems where f^+ is concentrared on the negligible set $\partial\Omega$, we will see later that the same is also true in other cases without absolutely continuous measures, and in particular in the case of interest for us.

On the other hand, we also recall that every solution of the Beckmann problem (6.4) is of the form $w = w_\gamma$ (see (2.6)) for an optimal transport plan γ (see Proposition 2.6). As in general Problem (6.5) can admit several different solutions, also (6.4) can have non-unique solutions. Yet, it is possible to prove that when either f^+ or f^- are absolutely continuous measures, then all different optimal transport plans γ induce the same vector measure w_γ; but, this is not the case now as both f^+ and f^- are concentrated on the boundary.

So, when the measures f^+ and f^- are concentrated on the boundary, we do not know yet, for instance, if an optimal transport map T between them exists or not, or if we have uniqueness of the transport density. Let us prove that any optimal γ in this case is induced by a transport map, which also implies uniqueness of γ and of σ_γ. We first start from the case $d = 2$, which is easier to deal with.

From now on we will suppose the condition that f^+ and f^- have no common mass, which means that there exist two disjoint sets A^+ and A^- contained in $\partial\Omega$ with $f^\pm$ concentrated on $A^\pm$ (beware that these sets are not necessarily the two supports of f^+ and f^-).

PROPOSITION 6.1. *Suppose that Ω is strictly convex, and $d = 2$. Then, if f^+ is atomless (i.e., $f^+(\{x\}) = 0$ for every $x \in \partial\Omega$) and f^+ and f^- have no common mass, there is a unique optimal transport plan γ for (6.5), between f^+ and f^-, and it is induced by a map T.*

PROOF. Let γ be an optimal transport plan between f^+ and f^-. Let $\mathcal{D}$ be the set of double points, that is those points whose belong to several transport rays. Take $x \in \mathcal{D}$ and let $r_x^\pm$ be two different transport rays starting from x. Let $\Delta_x \subset \Omega$ be the region delimited by r_x^+, r_x^- and $\partial\Omega$. As Ω is strictly convex, then we see easily that $|\Delta_x| > 0$ and the interior parts of all these sets Δ_x, $x \in \mathcal{D}$, are disjoint. This implies that the set $\mathcal{D}$ is at most countable and so $f^+(\mathcal{D}) = 0$ as f^+ is atomless. On the other hand, for every $x \in A^+ \setminus \mathcal{D}$ there is a unique transport ray r_x starting from x, and this ray r_x intersects A^- in - at most - one point, which will be denoted by $T(x)$. Hence, we get that $\gamma = (Id, T)_\# f^+$, which is equivalent to saying that γ is, in fact, induced by a map T. The uniqueness follows in the usual way: if two plans γ and γ' optimize (6.5), the same should be true for $(\gamma + \gamma')/2$. Yet, for this measure to be induced by a map, it is necessary to have $\gamma = \gamma'$. □

The higher-dimensional counterpart of the above result should replace the assumption that f^+ is atomless with the assumption that f^+ gives no mass to $(d-2)$-dimensional sets (i.e. sets of codimension 1 within the boundary). Yet, this seems more complicated to prove, and we will just stick to an easier result, in the case where f^+ is absolutely continuous w.r.t. to the $\mathcal{H}^{d-1}$ measure on $\partial\Omega$ (that we simply write $f^+ \in L^1(\partial\Omega)$).

PROPOSITION 6.2. *Suppose that Ω is strictly convex, and $d \geq 2$. Then, if $f^+ \in L^1(\partial\Omega)$ and f^+ and f^- have no common mass, there is a unique optimal transport plan γ for (6.5), and it is induced by a map T.*

PROOF. Let γ be an optimal transport plan between f^+ and f^-. According to the strategy above, it is enough to prove that for f^+-a.e. $x \in A^+$ there is at most a unique point $y \in A^-$ such that $(x, y) \in \operatorname{spt}\gamma$. We will parametrize $A^\pm$ via variables $s^\pm \in \mathbb{R}^{d-1}$. This is for sure possible since both $A^\pm$ do not fill the whole boundary $\partial\Omega$, and every proper subset of such a boundary is homeomorphic to a subset of $\mathbb{R}^{d-1}$, via an homeomorphism which can also be chosen to be locally bi-Lipschitz. Up to removing a negligible set, we can also assume that it is differentiable everywhere. Under this parameterization, we face a new transport problem in $\mathbb{R}^{d-1}$, with a new cost function $c(s^+, s^-) := |x(s^+) - y(s^-)|$, where $s^+ \mapsto x(s^+)$ and $s^- \mapsto y(s^-)$ are the above parameterization of A^+ and A^-.

Using standard arguments from optimal transport theory (see [103, Chapter 1]) one can see that the Kantorovich potentials in this new transport problem are locally Lipschitz continuous, and hence differentiable a.e. Thus it is enough to check that c satisfies the twist condition to prove that γ is necessarily induced by a map T, and that it is unique. Computing the gradient of c w.r.t. the variable s^+ one gets

$$\nabla_{s^+} c(s^+, s^-) = \frac{x(s^+) - y(s^-)}{|x(s^+) - y(s^-)|} Dx(s^+),$$

where $Dx(s^+)$ is the Jacobian matrix of the diffeomorphism x. We need to prove that this expression is injective in s^-. Having two different values of s^- (say, s_0^- and s_1^-) where these expressions coincide means, using that $s^+ \mapsto x(s^+)$ is a diffeomorphism, that the two unit vectors $x(s^+) - y(s_i^-)/|x(s^+) - y(s_i^-)|$ have the same projection onto the tangent space to $\partial\Omega$ at $x(s^+)$ (note that, from $A^+ \cap A^- = \emptyset$, we can assume $x(s^+) \neq y(s_i^-)$). Since they are unit vectors, and they both point to the interior of Ω, which is convex, then they should fully coincide. But this means that the direction connecting $x(s^+)$ to the points $y(s_i^-)$ is the same, and since all these points lie on the boundary of a strictly convex domain, we have $y(s_0^-) = y(s_1^-)$. $\square$

For the sake of the next section, we want stability results on the transport density. Suppose that f^+ and f^- are fixed, and that a unique optimal transport plan γ exists in the transportation from f^+ to f^-. In this case we will directly write σ instead of σ_γ, if no ambiguity arises. Given the optimal transport plan γ, let us define the measure f_t via

$$f_t = (\Pi_t)_\#(|x - y| \cdot \gamma) \tag{6.6}$$

where $\Pi_t(x, y) := (1 - t)x + ty$. From (2.1), the transport density σ may be easily written as

$$\sigma = \int_0^1 f_t \, \mathrm{d}t.$$

We also define a sort of partial transport density that will be useful in the sequel: given $\tau \leq 1$, set

$$\sigma^{(\tau)} = \int_0^\tau f_t \, \mathrm{d}t. \tag{6.7}$$

Note that $\sigma^{(\tau)}$ really depends on γ, i.e., differently from σ, it is not in general true that different optimal plans γ induce the same $\sigma^{(\tau)}$. On the other hand, we will only use this partial transport density in cases where the optimal γ is unique. In this case we can also obtain

PROPOSITION 6.3. *Suppose $f^+ \in \mathcal{M}^+(\Omega)$ is fixed and $f_n^- \rightharpoonup f^-$. Let γ_n be an optimal transport plan between f^+ and f_n^- and suppose that there is a unique optimal transport plan between f^+ and f^-. Fix $\tau \leq 1$ and define $\sigma_n^{(\tau)}$ according to (6.6) and (6.7) using γ_n, and $\sigma^{(\tau)}$ using γ. Then, we have $\sigma_n^{(\tau)} \rightharpoonup \sigma^{(\tau)}$.*

PROOF. This is a simple consequence of Proposition 1.3, of the continuity of the function $(x, y) \mapsto |x - y|$, and of the uniqueness of the optimal γ. □

6.3. L^p summability of boundary-to-boundary transport densities

In all that follows, Ω is a compact and uniformly convex domain in $\mathbb{R}^d$, f^+ and f^- are two non-negative Borel measures concentrated on the boundary, and at least one of them will belong to $L^1(\partial\Omega)$. Since we are only interested in the transport density between these two measures, we can always assume that they have no common mass, as the transport density only depends on the difference $f^+ - f^-$ and common mass can be subtracted to both of them. Then, by Propositions 6.1 and 6.2, there will exist one unique optimal transport plan between these two measures. We will make use of the transport density σ and of $\sigma^{(\tau)}$, defined in (6.7) and provide estimate on them. The main point is the following estimate.

PROPOSITION 6.4. *Suppose that the domain Ω is uniformly convex, with all its curvatures bounded from below by a constant $\kappa > 0$, take $p > 1$ and $f^+ \in L^p(\partial\Omega)$. Moreover, if $p > 2$ also suppose $\int f^+(x)^p \mathrm{d}(x, \mathrm{spt}(f^-))^{2-p} d\mathcal{H}^{d-1}(x) < +\infty$. Then, there exists a constant $C = C(\kappa, \mathrm{diam}(\Omega))$ such that we have*

$$\int_\Omega |\sigma^{(\tau)}|^p \mathrm{d}x \leq C \left(\int_0^\tau \frac{1}{(1-t)^{(d-1)(p-1)}} \mathrm{d}t \right) \int_{\partial\Omega} f^+(x)^p D(x)^{2-p} \mathrm{d}\mathcal{H}^{d-1}(x),$$

where $D(x) := |x - T(x)|$ is the distance between each point $x \in \partial\Omega$ and its image $T(x)$ in the optimal transport map which induces the optimal plan γ.

PROOF. Following the same strategy as in Chapter 2, we first assume that the target measure f^- is finitely atomic (the points $(x_j)_{j=1,\dots,m}$ being its atoms). Let T be the optimal transport map from f^+ onto f^-. For all $j \in \{1, \dots, m\}$, consider $T^{-1}(\{x_j\}) \subset \partial\Omega$, and partition it in finitely many smaller part, so that each can be represented by a single smooth chart parameterizing a part of $\partial\Omega$. We will call $(\chi_i)_{i=1,\dots,n}$ these parts. Let us call Ω_i the union of all transport rays starting from points in χ_i, all these rays pointing to a common point $x_{j(i)}$ (but we will write x_i for simplicity). Call $\Omega_i^{(\tau)}$ the set of points of the form $(1-t)x + tx_i$, with $x \in \chi_i$ and $t \leq \tau$. The sets Ω_i (and hence also $\Omega_i^{(\tau)}$) are essentially disjoint (the mutual intersections between them are Lebesgue-negligible). Set $\sigma_i^{(\tau)} := \sigma^{(\tau)} \llcorner \Omega_i$, for every $i \in \{1, ..., n\}$. Of course, $\sigma_i^{(\tau)}$ is concentrated on $\Omega_i^{(\tau)}$. In order to get L^p estimates on $\sigma^{(\tau)}$, we want to give an explicit formula of each $\sigma_i^{(\tau)}$. Fix $i \in \{1, ..., n\}$ and let α_i be a regular function such that, up to choosing a suitable system of coordinates, χ_i is contained in the graph of $s \mapsto \alpha_i(s)$, with $s \in \tilde{\chi}_i \subset \mathbb{R}^{d-1}$ (hence, the sets $\tilde{\chi}_i$ are the $(d-1)$-dimensional domains where the charts are defined). For every $y \in \Omega_i^{(\tau)}$, there are a unique point $x = (s, \alpha_i(s)) \in \chi_i$ and $t \in [0, \tau]$ such that

$$y := (y', y_d) = (1-t)x + tx_i = ((1-t)s + tx_i', (1-t)\alpha_i(s) + tx_{i,d}),$$

where we write $x_i := (x_i', x_{i,d})$ by separating the last (vertical) coordinate from the others. For all $\varphi \in C(\Omega_i)$, we get

$$\begin{aligned} \int_{\Omega_i} \varphi(y) \, \mathrm{d}\sigma_i^{(\tau)}(y) &= \int_{\chi_i} \int_0^\tau \varphi((1-t)x + tx_i)|x - x_i| \, \mathrm{d}t \, \mathrm{d}f^+(x) \\ &= \int_{\Omega_i^{(\tau)}} \varphi(y) \, \frac{|(s, \alpha_i(s)) - x_i| \, f^+(s, \alpha_i(s)) \sqrt{1 + |\nabla \alpha_i(s)|^2}}{J_i(t,s)} \, \mathrm{d}y, \end{aligned}$$

where $J_i(t,s) := |\det(D_{(s,t)}(y', y_d))|$.

Hence,

$$\sigma^{(\tau)}(y) = \frac{|(s, \alpha_i(s)) - x_i| \, f^+(s, \alpha_i(s)) \sqrt{1 + |\nabla \alpha_i(s)|^2}}{J_i(t,s)}, \quad \text{for a.e. } y \in \Omega_i^{(\tau)}. \tag{6.8}$$

Then, we have

$$\begin{aligned} ||\sigma^{(\tau)}||_{L^p(\Omega)}^p &= \sum_{i=1}^n \int_{\tilde{\chi}_i} \int_0^\tau \sigma^{(\tau)}((1-t)x + tx_i)^p J_i(t,s) \, \mathrm{d}t \, \mathrm{d}s \\ &= \sum_{i=1}^n \int_{\chi_i} \int_0^\tau \frac{|x - x_i|^p f^+(x)^p (1 + |\nabla \alpha_i(s)|^2)^{\frac{p-1}{2}}}{J_i(t,s)^{p-1}} \, \mathrm{d}t \, \mathrm{d}\mathcal{H}^{d-1}(x). \end{aligned}$$

Compute

$$D_{(s,t)}(y', y_d) = \begin{pmatrix} (1-t)\mathrm{I} & x_i' - s \\ (1-t)\nabla\alpha_i(s) & x_{i,d} - \alpha_i(s) \end{pmatrix},$$

where I is the $(d-1)\times(d-1)$ identity matrix. Up to considering sets χ_i which are very small, each one close to a point $x \in \partial\Omega$, and choosing a coordinate system where the vertical coordinate is parallel to the normal vector to $\partial\Omega$ at x, we can assume that $\nabla\alpha_i(s)$ is very small. At the limit, we can compute the above determinant as if it vanished, and thus we get $J_i(t,s) = (1-t)^{d-1}(x_{i,d} - \alpha(s))$ (as well as $1 + |\nabla\alpha_i(s)|^2 = 1$). This allows to write the change-of-variable coefficients in an intrinsic way, and thus obtain

$$||\sigma^{(\tau)}||^p_{L^p(\Omega)} = \sum_{i=1}^n \int_{\chi_i}\int_0^\tau \frac{|x-x_i|^p f^+(x)^p}{(1-t)^{(d-1)(p-1)}\big((x_i - x)\cdot \mathbf{n}(x)\big)^{p-1}}\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(x),$$

where $\mathbf{n}(x)$ is the inward normal vector to $\partial\Omega$ at x. Using the lower bound on the curvature of $\partial\Omega$ we have, for every pair of points x and x_i on $\partial\Omega$:

$$(x_i - x)\cdot\mathbf{n}(x) \geq c|x - x_i|^2,$$

for a constant $c = c(\operatorname{diam}\Omega, \kappa)$. Using then $x_i = T(x)$ for $x \in \chi_i$, this provides the desired formula

$$||\sigma^{(\tau)}||^p_{L^p(\Omega)} \leq C\int_{\partial\Omega}\int_0^\tau \frac{|x - T(x)|^{2-p} f^+(x)^p}{(1-t)^{(d-1)(p-1)}}\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(x).$$

This gives the desired result when f^- is atomic. If not, take a sequence $(f_n^-)_n$ of atomic measures converging to f^- and concentrated on $\operatorname{spt}(f^-)$. Call T_n the optimal maps from f^+ to f_n^- and $D_n(x) = |x - T_n(x)|$. By Proposition 6.3, the partial transport densities $\sigma_n^{(\tau)}$ converge to the corresponding partial transport density $\sigma^{(\tau)}$ and by Proposition 1.3 the optimal transport maps T_n also converge a.e. to the optimal transport map T inducing γ (up to extracting a subsequence, since $L^2(f^+)$ convergence implies a.e. convergence up to a subsequence). Moreover, we have $\int_{\partial\Omega} D_n(x)^{2-p} f^+(x)^p\,\mathrm{d}\mathcal{H}^{d-1}(x) \to \int_{\partial\Omega} D(x)^{2-p} f^+(x)^p\,\mathrm{d}\mathcal{H}^{d-1}(x)$ by dominated convergence, using either $p \leq 2$ and $f^+ \in L^p(\partial\Omega)$ or $\int_{\partial\Omega} \mathrm{d}(x, \operatorname{spt}(f^-))^{2-p} f^+(x)^p\,\mathrm{d}\mathcal{H}^{d-1}(x) < +\infty$, according to our assumptions. Using semicontinuity on the left hand side, we get

$$||\sigma^{(\tau)}||^p_{L^p(\Omega)} \leq \liminf_n ||\sigma_n^{(\tau)}||^p_{L^p(\Omega)} \leq C\left(\int_0^\tau \frac{1}{(1-t)^{(d-1)(p-1)}}\,\mathrm{d}t\right)\int_{\partial\Omega} D(x)^{2-p} f^+(x)^p\,\mathrm{d}\mathcal{H}^{d-1}(x)$$

and the result is proven in general. □

From the above estimate, we can deduce many integrability results.

PROPOSITION 6.5. *Suppose* $d \geq 2$ *and* $f^+ \in L^p(\partial\Omega)$ *with* $p < d/(d-1)$. *Then, the transport density* σ *between* f^+ *and any* $f^- \in \mathcal{M}^+(\partial\Omega)$ *is in* $L^p(\Omega)$.

PROOF. Note that our assumption on p implies $p < 2$. To prove this result it is enough to use Proposition 6.4 with $\tau = 1$, since in this case the integral $\int_0^1 \frac{1}{(1-t)^{(d-1)(p-1)}} \mathrm{d}t$ converges, and the term $D(x)^{2-p}$ is bounded. □

PROPOSITION 6.6. *Suppose that* $f^+, f^- \in L^p(\partial\Omega)$ *with* $p \leq 2$. *Then, the transport density* σ *between these two measures is in* $L^p(\Omega)$.

PROOF. In this case the integral in the estimate of $\sigma = \sigma^{(\tau)}$ with $\tau = 1$ diverges, so we need to adapt our strategy. Following again Chapter 2, we write

$$\sigma = \sigma^+ + \sigma^-,$$

where $\sigma^+ = \sigma^{(1/2)}$ and $\sigma^- = \sigma - \sigma^{(1/2)}$. In this case the L^p summability of f^+ guarantees that of σ^+ since $p \leq 2$ implies that $D(x)^{2-p}$ is bounded. Symmetrically, the L^p summability of f^- guarantees that of σ^-. Note that, thanks to Propositions 6.1 and 6.2, we do not face the same difficulties as in Chapter 2, where it was not obvious to glue together estimates on σ^+ obtained by approximating f^- and estimates on σ^- coming from the approximation of f^+. □

We will see in Section 6.4 that the same result is false for $p > 2$, and that in order to obtain higher integrability we need to assume much more on f^+ and f^-.

REMARK 6.7. *We do not discuss it here in details, but the summability result also works for Orlicz spaces with growth less than quadratic, i.e. we have, for every convex and superlinear function* $\Psi = \mathbb{R}^+ \mapsto \mathbb{R}^+$ *with* $\Psi(s) \leq C(s^2+1)$,

$$\int_\Omega \Psi(\sigma(x))\mathrm{d}x \leq C \int_{\partial\Omega} \Psi(|f(x)|)\mathrm{d}\mathcal{H}^{d-1}(x) + C.$$

This can be proven in similar ways with suitable manipulations on the function Ψ. *In particular, this implies that* $f \in L^1(\partial\Omega) \Rightarrow \sigma \in L^1(\Omega)$.

PROPOSITION 6.8. *Suppose that* $f^+, f^- \in C^{0,\alpha}(\partial\Omega)$ *for* $0 < \alpha \leq 1$. *Then, the transport density* σ *between these two measures is in* $L^p(\Omega)$ *for* $p = 2/(1-\alpha)$ *(with* $p = \infty$ *for* $\alpha = 1$*).*

PROOF. First, we check that we can apply Proposition 6.4, since in this case we need to use $p = 2/(1-\alpha) > 2$. Consider a point x with $f^+(x) > 0$, and take a point $y \in \mathrm{spt}(f^-)$ with $|x-y| = \mathrm{d}(x, \mathrm{spt}(f^-))$. Then we have $f^+(y) = 0$ (since f^+ and f^- have no mass in common) and $f^+(x) = |f^+(x) - f^+(y)| \leq C|x-y|^\alpha$. This provides $\mathrm{d}(x, \mathrm{spt}(f^-))^{2-p} f^+(x)^p \leq C\mathrm{d}(x, \mathrm{spt}(f^-))^{2-p+p\alpha}$. With our choice of p, this quantity is bounded since the exponent is non-negative (for $\alpha < 1$ the choice $p = 2/(1-\alpha)$ provides a zero exponent; for $\alpha = 1$ this exponent is equal to 2 for any p). This in particular guarantees $\int_{\partial\Omega} \mathrm{d}(x, \mathrm{spt}(f^-))^{2-p} f^+(x)^p \, \mathrm{d}\mathcal{H}^{d-1}(x) < +\infty$. Of course, the same can be performed on f^-. Then, the same strategy as in Proposition 6.6 shows

$$||\sigma||^p_{L^p(\Omega)} \leq C\bigg(\int_{\partial\Omega} |D(x)|^{2-p} f^+(x)^p \, \mathrm{d}\mathcal{H}^{d-1}(x) + \int_{\partial\Omega} |D^-(x)|^{2-p} f^-(x)^p \, \mathrm{d}\mathcal{H}^{d-1}(x) \bigg),$$

where $D^-(x) := |x - T^{-1}(x)|$ is defined as $D(x)$, but relatively to f^-. Using $D(x) = |x - T(x)| \geq \mathrm{d}(x, \mathrm{spt}(f^-))$, the quantity $D(x)^{2-p} f^+(x)^p \leq CD(x)^{2-p+p\alpha}$ is bounded. Since a similar argument can be performed on f^-, we obtain finiteness of the norm $||\sigma||_{L^p(\Omega)}$ (and for $\alpha = 1$ we obtain $\sigma \in L^\infty$ by passing to the limit $p \to \infty$). □

6.4. Counter-example to the $L^{2+\varepsilon}$ summability

In this section, we show that the L^p estimates for the transport density, in the case where $p > 2$, fail even if we assume $f^\pm \in L^\infty(\partial\Omega)$. More precisely, we will construct an example of $f^\pm$, where $f^\pm \in L^\infty(\partial\Omega)$, but the transport density σ between them does not belong to $L^{2+\varepsilon}(\Omega)$ for any $\varepsilon > 0$. For simplicity, this will be done in dimension $d = 2$.

Let Ω be a disk and let $(\chi_n^\pm)_n$ be a sequence of arcs in $\partial\Omega$ such that $\mathcal{H}^1(\chi_n^\pm) = \varepsilon_n$, for some sequence ε_n to be chosen later. We will put these arcs one after the other, so that they only have endpoints in common, and we assume that they are ordered in the following way: χ_{n-1}^+, χ_{n-1}^-, χ_n^-, χ_n^+, χ_{n+1}^+, χ_{n+1}^-, for all n (see Figure 1).

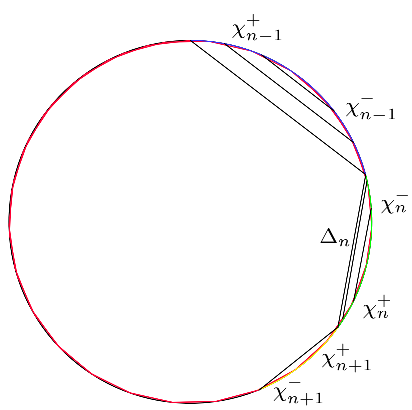

FIGURE 1

Set $f^\pm = \mathbb{1}_{\chi^\pm}$, where $\chi^\pm = \cup_n \chi_n^\pm$, and let T be the optimal transport map from f^+ to f^-. Correspondingly, let σ be the transport density. We see easily that the restriction of T to χ_n^+ is the optimal transport map T_n between f_n^+ and f_n^-, where $f_n^\pm$ is the restriction of $f^\pm$ to $\chi_n^\pm$. Moreover, if we denote by Δ_n the union of all transport rays from f_n^+ onto f_n^-, then the restriction of the transport density σ to Δ_n is the transport density σ_n between f_n^+ and f_n^-. We want to compute this density σ_n. Let $s \mapsto \alpha_n(s)$ be a parameterization of $\chi_n^+ \cup \chi_n^-$ where $s = 0$ corresponds to the boundary point between the two arcs. It is clear that $T_n(s, \alpha_n(s)) = (-s, \alpha_n(s))$, for all $s \in [0, \varepsilon_n]$. Then, for every $y \in \Delta_n$, there is a unique $(t,s) \in [0,1] \times [0, \varepsilon_n]$ such that

$$y := (y_1, y_2) = (1-t)(s, \alpha_n(s)) + t(-s, \alpha_n(s)) = ((1-2t)s, \alpha_n(s)).$$

Hence, for every $\varphi \in C(\Delta_n)$, we have

$$\begin{aligned}
\int_{\Delta_n} \varphi(y)\, \sigma_n(y)\, \mathrm{d}y &= \int_{\chi_n^+} \int_0^1 \varphi((1-t)x + tT_n(x))|x - T_n(x)| f_n^+(x)\, \mathrm{d}t\, \mathrm{d}x \\
&= \int_0^{\varepsilon_n} \int_0^1 \varphi((1-2t)s, \alpha_n(s))\, 2s \sqrt{1 + \alpha_n'(s)^2}\, \mathrm{d}t\, \mathrm{d}s \\
&= \int_{\Delta_n} \varphi(y)\, \frac{2s\sqrt{1+\alpha_n'(s)^2}}{J_n(t,s)}\, \mathrm{d}y,
\end{aligned}$$

where $J_n(t,s) := |\det(D_{(t,s)}(y_1, y_2))|$ on Δ_n.

Then, we get

$$\sigma_n(y) = \frac{2s\sqrt{1+\alpha_n'(s)^2}}{J_n(t,s)}, \quad \text{for a.e. } y \in \Delta_n.$$

Consequently, we obtain

$$\begin{aligned}
||\sigma||_{L^p(\Omega)}^p &= \sum_{n=1}^{\infty} \int_0^{\varepsilon_n} \int_0^1 \sigma_n((1-2t)s, \alpha_n(s))^p J_n(t,s)\, \mathrm{d}t\, \mathrm{d}s \\
&\approx \sum_{n=1}^{\infty} \int_0^{\varepsilon_n} \int_0^1 \frac{s^p}{J_n(t,s)^{p-1}}\, \mathrm{d}t\, \mathrm{d}s.
\end{aligned}$$

Computing

$$D_{(t,s)}(y_1, y_2) = \begin{pmatrix} -2s & 1-2t \\ 0 & \alpha_n'(s) \end{pmatrix},$$

we get

$$J_n(t,s) = 2s\, \alpha_n'(s) \approx s^2.$$

Finally, we have

$$||\sigma||_{L^p(\Omega)}^p \approx \sum_{n=1}^{\infty} \varepsilon_n^{3-p}.$$

This immediately shows that with this construction we cannot have $\sigma \in L^3$. Moreover, it is enough to choose a sequence ε_n satisfying

$$\sum_{n=1}^{\infty} \varepsilon_n < +\infty, \quad \sum_{n=1}^{\infty} \varepsilon_n^{\beta} = +\infty$$

for all $\beta < 1$, to prove $\sigma \notin L^p(\Omega)$ for all $p > 2$. Take for instance $\varepsilon_n = \frac{1}{n(\log(1+n))^2}$.

6.5. Applications to the BV least gradient problem

We collect in this section some corollaries of the results of the previous sections, which give interesting proofs for some properties of the BV least gradient problem in dimension $d = 2$. We need to restrict to $d = 2$ because only in this framework rotated gradients have prescribed divergence.

In all the cases, we will suppose $g \in BV(\partial\Omega)$. Note that this assumption is required to apply the classical theory of optimal transport to $f = \partial_{\mathbf{t}} g$; this requires to transport a measure onto another. If g was only in $L^1(\partial\Omega)$, then f would be the (one-dimensional) derivative of an L^1 function, i.e. an element of the dual of Lipschitz functions (since $W^{-1,1} = (W^{1,\infty})'$). It is not surprising that a Monge-Kantorovich theory is also possible in this case (because formula (1.2) characterizes the transport cost as the dual norm to the Lipschitz norm), see [**16**], but no estimates are possible.

Proposition 6.9. *If $\Omega \subset \mathbb{R}^2$ is strictly convex and $g \in BV(\partial\Omega)$, then Problem* (6.1) *has a solution (i.e. Problems* (6.2) *and* (6.3) *have a solution whose trace is g).*

Proof. We have already discussed the fact that we just need to exclude that the solution of (6.2) or (6.3) has a part of its distributional derivative on the boundary. After the rotation, this means that its trace agrees with g if and only if $\sigma(\partial\Omega) = 0$. Yet, in strictly convex domains, the transport density does not give mass to the boundary, because of the representation formula (2.2). □

Proposition 6.10. *If $\Omega \subset \mathbb{R}^2$ is strictly convex, and $g \in (BV \cap C^0)(\partial\Omega)$, then Problem* (6.1) *has a unique solution.*

Proof. Using again the rotation trick, we just need to prove uniqueness of the transport density. The condition $g \in C^0$ implies that its tangential derivative has no atoms, and we can apply Proposition 6.1. □

The following result is probably the main contribution of this chapter to the understanding of the least gradient problem, as we are not aware of similar results already existing in the literature.

THEOREM 6.11. *If $\Omega \subset \mathbb{R}^2$ is uniformly convex, and $g \in W^{1,p}(\partial\Omega)$ with $p \leq 2$, then the unique solution of Problem (6.1) belongs to $W^{1,p}(\Omega)$.*

PROOF. Setting $f = \partial_{\mathbf{t}} g$ and using f^+ and f^- as its positive and negative parts, the condition $g \in W^{1,p}(\partial\Omega)$ implies $f^\pm \in L^p(\partial\Omega)$. Hence, Proposition 6.6 implies $\sigma \in L^p(\Omega)$, and then $\nabla u \in L^p(\Omega, \mathbb{R}^2)$. □

PROPOSITION 6.12. *Even if $\Omega \subset \mathbb{R}^2$ is a disk, for every $p > 2$ there exists $g \in \mathrm{Lip}(\partial\Omega)$ such that the unique u solution of Problem (6.1) is not in $W^{1,p}(\Omega)$.*

PROOF. It is enough to take g as the antiderivative of the function $f = f^+ - f^-$ of the counter-example of Section 6.4. □

PROPOSITION 6.13. *If $\Omega \subset \mathbb{R}^2$ is uniformly convex, and $g \in C^{1,\alpha}(\partial\Omega)$ with $\alpha < 1$, then the unique solution of Problem (6.1) belongs to $W^{1,p}(\Omega)$ for $p = 2/(1-\alpha)$.*

PROOF. This is a consequence of Proposition 6.8. □

REMARK 6.14. *Note that the above $W^{1,p}$ regularity also implies Hölder bounds, since in dimension $d = 2$ we have $W^{1,p} \subset C^{0,1-2/p}$. In particular, using $p = 2/(1-\alpha)$, we get $g \in C^{1,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\alpha}(\Omega)$. Yet, this bound is not optimal, as it is known (see, for instance, [108]) that we have $g \in C^{1,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,(\alpha+1)/2}(\Omega)$. It is interesting to note that one would obtain exactly the desired $C^{0,(\alpha+1)/2}$ behavior if it was possible to use the Sobolev injection of $W^{1,p}$ corresponding to dimension 1 instead of dimension 2. This seems reasonable, using the fact that level lines of u are transport rays in the transport problem from f^+ to f^-, hence are line segments, but it is not easy to justify and goes beyond the scopes of this chapter.*

PROPOSITION 6.15. *If $\Omega \subset \mathbb{R}^2$ is uniformly convex, and $g \in C^{1,1}(\partial\Omega)$, then the unique solution of Problem (6.1) is Lipschitz continuous.*

PROOF. This is also a consequence of Proposition 6.8. □

REMARK 6.16. *Note that the above Lipschitz result is optimal, and perfectly coherent with the theory involving the bounded slope condition (see, for instance [107, 41]), since $C^{1,1}$ functions on the boundary of uniformly convex domains satisfy the bounded slope condition (see [66]).*

We finish this section with the following last remark.

REMARK 6.17. *We observe that we have not used Proposition 6.5 in this Section. Indeed, in the framework of the least gradient problem assuming assumptions on f^+ (i.e. on the positive part of the tangential derivative of the boundary datum) are not natural at all. Proposition 6.5 has been inserted in Section 6.3 just because it was an easy consequence of Proposition 6.4. Also consider that a simple result which could have been proven in Section 6.3 was the implication $f^\pm \in L^p(\partial\Omega) \Rightarrow \sigma \in L^p(\Omega)$ for arbitrary p (including $p > 2$) under the assumption $\mathrm{spt}(f^+) \cap \mathrm{spt}(f^-) = \emptyset$, but we did not considered it because this assumption, in terms of g, is very innatural: it would mean that g has some flat regions separating those with positive and negative derivatives.*

6.6. Anisotropic least gradient problem

This section is devoted to the anisotropic least gradient problem [64, 88]. We will show briefly that all our analysis could be done by replacing the Euclidean norm with another strictly convex one. Let φ be a given norm in $\mathbb{R}^2$. So, we consider the following problem

$$\inf\left\{\int_\Omega \varphi(\nabla u)\,:\, u \in BV(\Omega),\, u_{|\partial\Omega} = g\right\}, \tag{6.9}$$

where $\int_\Omega \varphi(\nabla u) = \int_\Omega \varphi(\nu^u(x))\,\mathrm{d}|\nabla u|$ (ν^u is the Radon-Nikodym derivative $\nu^u = \frac{d\nabla u}{d|\nabla u|}$) and $g \in BV(\partial\Omega)$ is a given boundary datum. On the other hand, let $||\cdot||$ be the *rotation-norm* of φ, i.e., $||\xi|| := \varphi(R_{-\frac{\pi}{2}}\xi)$, for every $\xi \in \mathbb{R}^2$. Recalling Section 6.1, we can see that the problem (6.9) is equivalent to the following one

$$\inf\left\{||w||(\overline{\Omega})\,:\, w \in \mathcal{M}^2(\overline{\Omega}),\, \nabla\cdot w = f\right\}, \tag{6.10}$$

where we denote by $||w||$ the variation measure associated with the vector measure w, and $f = \partial g/\partial \mathbf{t}$. More precisely, if u is a solution for (6.9), then $w = R_{\frac{\pi}{2}}\nabla u$ solves (6.10) and, we have $||w||(\overline{\Omega}) = \int_\Omega \varphi(\nabla u)$.

From Section 2.3, we have already seen that the problem (6.10) is also equivalent to the Monge-Kantorovich problem, with $c(x, y) = ||x - y||$ as a transport cost,

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}} ||x-y||\,\mathrm{d}\gamma :\ \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\, (\Pi_x)_\#\gamma = f^+ \ \text{ and }\ (\Pi_y)_\#\gamma = f^-\right\}, \tag{6.11}$$

where $f^\pm$ represent the positive and negative parts of f. In addition, we showed in Proposition 2.6 that if the norm φ (or equivalently, $||\cdot||$) is strictly convex, then any optimal flow w of (6.10) comes from an optimal transport plan γ for (6.11), i.e., $w = w_\gamma$ (see (2.6)).

Moreover, as soon as f^+ is atomless, f^+ and f^- have no common mass, and Ω is strictly convex, one can prove, exactly as in Proposition 6.1, that there is a unique optimal transport plan γ for (6.11), between f^+ and f^-, and it is induced by a map T. This also gives the uniqueness of the solution for (6.9).

Hence, it suffices to give L^p estimates on the transport density $\mathbf{a} := ||w||$, where w is the unique minimizer of (6.10), to get $W^{1,p}$ estimates on the unique solution u of (6.9). Let T be the optimal transport map between f^+ and f^-. Recalling (2.8), the transport density $\mathbf{a}$ may be easily written as

$$\mathbf{a} = \int_0^1 f_t \,\mathrm{d}t,$$

where

$$f_t = (T_t)_{\#}(||x - T(x)|| \cdot f^+)$$

and

$$T_t(x) := (1-t)x + tT(x).$$

Again, for every $\tau \leq 1$, we define

$$\mathbf{a}^{(\tau)} = \int_0^\tau f_t \,\mathrm{d}t.$$

Following the same lines of the proof of Proposition 6.4, one can prove, using the fact that $|| \cdot || \approx | \cdot |$, the following estimate

$$\begin{aligned}\int_\Omega |\mathbf{a}^{(\tau)}|^p \mathrm{d}x \;&\leq\; C\left(\int_0^\tau \frac{1}{(1-t)^{(p-1)}}\,\mathrm{d}t\right)\int_{\partial\Omega} \frac{||x-T(x)||^p}{|x-T(x)|^{2(p-1)}} f^+(x)^p \,\mathrm{d}\mathcal{H}^{d-1}(x)\\ &\approx\; C\left(\int_0^\tau \frac{1}{(1-t)^{(p-1)}}\,\mathrm{d}t\right)\int_{\partial\Omega} |x-T(x)|^{2-p} f^+(x)^p \,\mathrm{d}\mathcal{H}^{d-1}(x).\end{aligned}$$

Recalling also Proposition 6.6, we conclude that the transport density $\mathbf{a}$ is in $L^p(\Omega)$ as soon as $f^\pm \in L^p(\partial\Omega)$ with $p \leq 2$ and Ω is uniformly convex. Moreover, if $f^+, f^- \in C^{0,\alpha}(\partial\Omega)$ for $0 < \alpha \leq 1$, then the transport density $\mathbf{a}$ between these two measures is in $L^p(\Omega)$ for $p = 2/(1-\alpha)$ (with $p = \infty$ for $\alpha = 1$).

In terms of $W^{1,p}$ regularity for the solution u of (6.9), we infer that u is in $W^{1,p}(\Omega)$ as soon as $g \in W^{1,p}(\partial\Omega)$ with $p \leq 2$ and Ω is uniformly convex. In addition, if $g \in C^{1,\alpha}(\partial\Omega)$, then u is in $W^{1,\frac{2}{1-\alpha}}(\Omega)$, for $0 < \alpha \leq 1$.

CHAPTER 7

Exit-time optimal control problems

The control problems considered in this chapter will be called "exit-time" because the terminal time of the trajectories is not fixed, but it is the first one at which they reach a given target set. A typical example is the minimum time problem, where one wants to steer a point to the target in minimal time under some constraints on the dynamic. It is interesting to observe that the distance function can be regarded as the value function of a particular minimum-time problem, and so the properties of the distance function may serve as a guideline for the analysis of the general case. We first study the existence of optimal controls; then we introduce the value function, show that it solves a Hamilton-Jacobi equation, give a result about its Lipschitz continuity, analyze the optimality conditions and the properties of optimal trajectories and prove that the value function is differentiable along optimal trajectories, except possibly their endpoints. Finally, we prove the semi-concavity of the value function, with respect to x, under much weaker assumptions, on the smoothness of the dynamic with respect to time, than those in [**37**]. *Some statements in this chapter will be similar to those in* [**37**], *even if they are stated, here, in the non-autonomous case. But anyway, the differentiability property and the sharp semi-concavity of the value function seem to be novel.*

The original results in this chapter will be included in a joint paper with G. Mazanti, in preparation, [57].

7.1. Definition, existence and first properties

We begin by giving the definition of the control systems we are interested in.

Definition 7.1. *The control system, we are interested in, consists of a pair (k, U), where $U \subset \mathbb{R}^d$ is a compact set and $k : \mathbb{R}^+ \times \mathbb{R}^d \mapsto \mathbb{R}^+$ is a continuous function. The set U is called the control set, while k is called the dynamic of the system. The state equation associated with the system is*

$$\begin{cases} \gamma'(t) = k(t, \gamma(t))\, u(t), & \text{for a.e. } t \geq t_0, \\ \gamma(t_0) = x, \end{cases} \tag{7.1}$$

where $t_0 \in \mathbb{R}^+$, $x \in \mathbb{R}^d$ and $u : [t_0, \infty) \mapsto U$ is a measurable function (which is called a control). We denote the solution of (7.1) by $\gamma^{t_0,x,u}$ and we call it the trajectory of the system corresponding to the initial condition $\gamma(t_0) = x$ and to the control u.

Of course, this is only a very particular example of control system, but it is the simplest one which allows to study interesting exit time problems. In order to assume that the maximal speed of the trajectory $\gamma^{t_0,x,u}$ is bounded by the dynamic k, we take the control set $U = \bar{B}(0, 1)$.

Moreover, we list some basic assumptions on the dynamic k:

$$0 < k_{\min} := \inf k \quad \leq \quad k_{\max} := \sup k < +\infty, \tag{7.2}$$

$$|k(t,x_1) - k(t,x_2)| \quad \leq \quad L_x\,|x_1 - x_2|, \quad \text{for all } \; x_1,\, x_2 \in \mathbb{R}^d \; \text{ and } \; t \in \mathbb{R}^+. \tag{7.3}$$

Notice that the assumption (7.3) ensures the existence of a unique global solution to the state equation (7.1) for any choice of t_0, x and u. Let Ω be a bounded domain in $\mathbb{R}^d$: for a given trajectory $\gamma = \gamma^{t_0,x,u}$ of the system, we set

$$\tau^{t_0,x,u} = \inf\{\tau \geq 0 \,:\, \gamma^{t_0,x,u}(t_0 + \tau) \in \partial\Omega\},$$

with the convention $\tau^{t_0,x,u} = +\infty$ if $\gamma^{t_0,x,u}(t_0+\tau) \notin \partial\Omega$ for all $\tau \in [0,\infty)$. This means that we use $\partial\Omega$ as a target. We call $\tau^{t_0,x,u}$ the *exit-time* of the trajectory. If $\tau^{t_0,x,u} < +\infty$, we set for simplicity

$$\gamma_\tau^{t_0,x,u} := \gamma^{t_0,x,u}(t_0 + \tau^{t_0,x,u})$$

to denote the point where the trajectory reaches the target $\partial\Omega$. As $k_{\min} > 0$, one can see easily that, for all $x \in \Omega$ and at each time $t_0 \in \mathbb{R}^+$, there is always some control u such that $\tau^{t_0,x,u} < +\infty$.

An optimal control problem consists of choosing the control strategy u in the state equation (7.1) in order to minimize a given functional. Let $g : \partial\Omega \mapsto \mathbb{R}^+$ be a given continuous function. At each time $t_0 \in \mathbb{R}^+$ and for any initial point $x \in \Omega$, we minimize the following quantity

$$\tau^{t_0,x,u} \;+\; g(\gamma_\tau^{t_0,x,u}) \tag{7.4}$$

among all controls u. A control u and the corresponding trajectory $\gamma^{t_0,x,u}$ are called *optimal* for the point x at time t_0 if u minimizes (7.4). Now, suppose that

$$|g(x) - g(y)| \leq \lambda |x - y|, \;\; \text{for all } \; x,\, y \in \partial\Omega \tag{7.5}$$

with $\lambda < \frac{1}{k_{\max}}$. We note that this assumption is similar to the one in the problem studied in Chapter 4. Then, under the assumptions (7.2), (7.3) and (7.5), we have the following existence result.

PROPOSITION 7.2. *For each time $t_0 \in \mathbb{R}^+$ and for any initial point $x \in \Omega$, there exists an optimal control u for (7.4).*

PROOF. Let $(u_n)_n$ be a minimizing sequence. We set for simplicity $\tau_n := \tau^{t_0,x,u_n}$, $\gamma_n := \gamma^{t_0,x,u_n}$ and $z_n := \gamma^{t_0,x,u_n}_{\tau_n}$. As g is continuous on $\partial\Omega$, then $(\tau_n)_n$ is bounded and, up to a subsequence, τ_n converges to some $\bar{\tau}$. For each $n \in \mathbb{N}$, we can assume that $u_n(t) = 0$ for all $t > t_0 + \tau_n$. In this way, we get that γ_n converges uniformly to some $\gamma \in \mathrm{Lip}([t_0, \infty), \Omega)$ with $|\gamma'| \leq k_{\max}$. In addition, for a.e. $t \in (t_0, +\infty)$, we have

$$|\gamma_n'(t)| \leq k(t, \gamma_n(t)).$$

Letting $n \to +\infty$, we infer that $|\gamma'(t)| \leq k(t, \gamma(t))$, for a.e. $t \in (t_0, +\infty)$, which is equivalent to say that there is some control u such that γ is the associated trajectory to u with initial point x, at time t_0. On the other hand, we have

$$z_n \to \gamma(t_0 + \bar{\tau}),$$

which implies that $\gamma(t_0 + \bar{\tau}) \in \partial\Omega$ and $\tau := \tau^{t_0,x,u} \leq \bar{\tau}$. Yet, it is not possible to have $\tau < \bar{\tau}$. Indeed, one has

$$\lim_n \tau_n + g(z_n) = \bar{\tau} + g(\gamma(t_0 + \bar{\tau})) \leq \tau + g(\gamma(t_0 + \tau)),$$

which is a contradiction, as g is $\lambda-$Lipschitz with $\lambda < 1/k_{\max}$. Thus, we have $\tau = \bar{\tau}$ and this completes the proof that γ is an optimal trajectory and u is the associated optimal control. $\square$

REMARK 7.3. *The condition that g is $\lambda-$Lip with $\lambda < \frac{1}{k_{\max}}$ is crucial for this result. Without this condition, one should replace $\tau + g(\gamma(\tau))$ with $\inf\{t + g(\gamma(t)) : \gamma(t) \in \partial\Omega\}$.*

The value function of the problem is defined by

$$\varphi(t_0, x) = \min\{\tau^{t_0,x,u} + g(\gamma^{t_0,x,u}_\tau) : u \text{ is a control}\}, \quad t_0 \in \mathbb{R}^+, x \in \Omega. \tag{7.6}$$

The first important fact is that the value function φ satisfies the so-called *dynamic programming principle*:

LEMMA 7.4. *For any $t_0 \in \mathbb{R}^+$, $x \in \Omega$ and $u : [t_0, \infty) \mapsto \bar{B}(0,1)$ a control, we have*

$$\varphi(t_0, x) \leq t - t_0 + \varphi(t, \gamma^{t_0,x,u}(t)), \quad \textit{for all } t \in [t_0, t_0 + \tau^{t_0,x,u}],$$

with equality if u is optimal.

PROOF. Let u be a control for x, at time t_0, and fix $t \in [t_0, t_0 + \tau^{t_0,x,u}]$. Set $y = \gamma^{t_0,x,u}(t)$ and let v be an optimal control for y at time t, i.e., we have

$$\varphi(t, \gamma^{t_0,x,u}(t)) = \tau^{t,y,v} + g(\gamma_\tau^{t,y,v}).$$

Let us now define the control w as follows

$$w(s) := \begin{cases} u(s) & \text{if } \ s \in [t_0, t) \\ v(s) & \text{if } \ s \geq t. \end{cases}$$

Therefore, one has

$$\gamma^{t_0,x,w}(s) = \begin{cases} \gamma^{t_0,x,u}(s) & \text{if } \ s \in [t_0, t) \\ \gamma^{t,y,v}(s) & \text{if } \ s \geq t \end{cases}$$

and,

$$\varphi(t_0, x) \leq \tau^{t_0,x,w} + g(\gamma_\tau^{t_0,x,w}) = t - t_0 + \tau^{t,y,v} + g(\gamma_\tau^{t,y,v}).$$

Moreover, if the control u is optimal, then we get

$$\varphi(t, \gamma^{t_0,x,u}(t)) \leq \tau^{t,y,u} + g(\gamma_\tau^{t,y,u}) = \tau^{t_0,x,u} + t_0 - t + g(\gamma_\tau^{t_0,x,u}) = \varphi(t_0, x) + t_0 - t. \qquad \square$$

Now, we want to show that the value function φ is a viscosity solution of a suitable partial differential equation. In fact, we have the following classical result that we prove for completeness.

PROPOSITION 7.5. *The value function φ is a viscosity solution of the following Hamilton-Jacobi equation*

$$-\partial_t \varphi(t,x) + k(t,x)\,|\nabla_x \varphi(t,x)| - 1 = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathring{\Omega}. \tag{7.7}$$

Moreover, one has $\varphi(t,x) = g(x)$ for every $(t,x) \in \mathbb{R}^+ \times \partial\Omega$.

PROOF. First, we prove that φ is a viscosity subsolution of (7.7). Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathring{\Omega}$ and ϕ be a C^1 function such that $\phi(t_0,x_0) = \varphi(t_0,x_0)$ and $\phi(t,x) \geq \varphi(t,x)$ for (t,x) in a neighborhood of (t_0,x_0). Fix $v \in \mathbb{R}^d$ with $|v| = 1$. Let us consider the trajectory γ_v starting from x_0, at time t_0, and corresponding to the constant control v. Hence, by Lemma 7.4, one has, for $h > 0$ small enough, that

$$\phi(t_0,x_0) = \varphi(t_0,x_0) \leq h + \varphi(t_0 + h, \gamma_v(t_0+h)) \leq h + \phi(t_0+h, \gamma_v(t_0+h))$$

and so,

$$0 \leq h + h\,\partial_t \phi(t_0,x_0) + \int_{t_0}^{t_0+h} \gamma_v'(t)\,\mathrm{d}t \cdot \nabla_x \phi(t_0,x_0) + o\left(h + \left|\int_{t_0}^{t_0+h} \gamma_v'(t)\,\mathrm{d}t\right|\right)$$

$$\leq h + h\,\partial_t\phi(t_0,x_0) + (\nabla_x\phi(t_0,x_0)\cdot v)\int_{t_0}^{t_0+h} k(t,\gamma_v(t))\,\mathrm{d}t + o(h).$$

Thus, dividing by h and letting it to 0, one concludes that

$$1 + \partial_t\phi(t_0,x_0) + k(t_0,x_0)(\nabla_x\phi(t_0,x_0)\cdot v) \geq 0$$

and, since v is arbitrary, we then obtain

$$-\partial_t\phi(t_0,x_0) + k(t_0,x_0)\,|\nabla_x\phi(t_0,x_0)| \leq 1.$$

Let us now prove that φ is a viscosity supersolution of (7.7). Let $(t_0,x_0)\in\mathbb{R}^+\times\mathring{\Omega}$ and ϕ be a C^1 function such that $\phi(t_0,x_0)=\varphi(t_0,x_0)$ and $\phi(t,x)\leq\varphi(t,x)$ for (t,x) in a neighborhood of (t_0,x_0). Let γ be an optimal trajectory for x_0 at time t_0. Using Lemma 7.4, for $h>0$ small enough, we have

$$\phi(t_0,x_0)=\varphi(t_0,x_0)=h+\varphi(t_0+h,\gamma(t_0+h))\geq h+\phi(t_0+h,\gamma(t_0+h)),$$

which implies that,

$$\begin{aligned}0 &\geq h + h\,\partial_t\phi(t_0,x_0) + \int_{t_0}^{t_0+h}\gamma'(t)\,\mathrm{d}t\cdot\nabla_x\phi(t_0,x_0) + o\left(h+\left|\int_{t_0}^{t_0+h}\gamma'(t)\,\mathrm{d}t\right|\right)\\ &\geq h + h\,\partial_t\phi(t_0,x_0) - |\nabla_x\phi(t_0,x_0)|\int_{t_0}^{t_0+h}k(t,\gamma(t))\,\mathrm{d}t + o(h).\end{aligned}$$

Thus, again dividing by h and letting it to 0, one concludes that

$$-\partial_t\phi(t_0,x_0) + k(t_0,x_0)\,|\nabla_x\phi(t_0,x_0)| \geq 1,$$

proving that φ is a viscosity supersolution of (7.7) on $\mathbb{R}^+\times\mathring{\Omega}$. Finally, the fact that $\varphi(t,x)=g(x)$, for all $(t,x)\in\mathbb{R}^+\times\partial\Omega$, follows immediately by using the assumption (7.5). ☐

We denote by $D^+\varphi$ the superdifferential of φ, i.e., for every $(t,x)\in\mathbb{R}^+\times\Omega$,

$$D^+\varphi(t,x) := \left\{(h,p)\in\mathbb{R}\times\mathbb{R}^d \;:\; \limsup_{(s,y)\to(t,x)}\frac{\varphi(s,y)-\varphi(t,x)-(h,p)\cdot(s-t,y-x)}{|(s-t,y-x)|}\leq 0\right\}.$$

If we consider points along optimal trajectories different from the endpoints, we can prove that the elements of $D^+\varphi$ satisfy also (7.7). So, the following result holds.

PROPOSITION 7.6. *Let $\gamma:[t_0,t_0+\tau_0]\mapsto\Omega$ be an optimal trajectory for x_0, at time t_0, where $\tau_0=\tau^{t_0,x_0,u}$; u being the associated optimal control. Then, for any $t\in(t_0,t_0+\tau_0)$, we have*

$$-p_t + k(t,\gamma(t))\,|p_x| - 1 = 0, \quad \textit{for all}\;\; (p_t,p_x)\in D^+\varphi(t,\gamma(t)).$$

PROOF. Let us take $t \in (t_0, t_0 + \tau_0)$. We already know from the previous proof that

$$-p_t + k(t, \gamma(t))\, |p_x| - 1 \leq 0, \qquad \text{for all } \ (p_t, p_x) \in D^+\varphi(t, \gamma(t)).$$

So, it suffices to prove that the converse inequality also holds. First, we observe that, by the dynamic programming principle,

$$\varphi(t, \gamma(t)) = \varphi(t - h, \gamma(t - h)) - h, \quad 0 \leq h \leq t - t_0.$$

On the other hand, if $(p_t, p_x) \in D^+\varphi(t, \gamma(t))$, we have

$$\varphi(t - h, \gamma(t - h)) - \varphi(t, \gamma(t)) \leq -p_t h - p_x \cdot (\gamma(t) - \gamma(t - h)) + o(h).$$

Therefore, we find that

$$\begin{aligned}
0 &\leq -p_t h - p_x \cdot (\gamma(t) - \gamma(t - h)) - h + o(h) \\
&= -p_t h - \int_{t-h}^{t} k(s, \gamma(s))\ p_x \cdot u(s)\, \mathrm{d}s - h + o(h) \\
&\leq -p_t h + \int_{t-h}^{t} \sup_{s \in [t-h,t]} k(s, \gamma(s))\, |p_x|\, \mathrm{d}s - h + o(h) \\
&\leq h(-p_t + \sup_{s \in [t-h,t]} k(s, \gamma(s))\, |p_x| - 1) + o(h),
\end{aligned}$$

which yields the conclusion. □

Let us define now the *minimum time function* as follows:

$$T(t, x) = \inf\{\tau^{t,x,u} \ : \ u \text{ is a control}\}, \quad (t, x) \in \mathbb{R}^+ \times \Omega.$$

An optimal control u in $T(t, x)$ will be called *time-optimal control.* Notice that we have the following

LEMMA 7.7. *For every $(t, x) \in \mathbb{R}^+ \times \Omega$, one has $T(t, x) \leq k_{\min}^{-1}\, d(x, \partial\Omega)$.*

PROOF. Let γ be a geodesic (which is just a segment) from x to the boundary $\partial\Omega$ with $|\gamma'| = 1$. Set $\tilde{\gamma}(s) = \gamma(k_{\min}(s - t))$, for all $s \in [t, t + k_{\min}^{-1}\, d(x, \partial\Omega)]$. It is clear that $\tilde{\gamma}$ is an admissible trajectory, which means that there is a control u such that $\tilde{\gamma} = \gamma^{t,x,u}$. Hence, by definition of T, we infer that $T(t, x) \leq \tau^{t,x,u} = k_{\min}^{-1}\, d(x, \partial\Omega)$. □

In addition, one can also show that the exit-time of an optimal control for (7.4) can be estimated by the minimum time function T. More precisely, we have the following

PROPOSITION 7.8. *For every* $(t,x) \in \mathbb{R}^+ \times \Omega$*, if* u *is an optimal control for (7.4), then one has*

$$\tau^{t,x,u} \leq \frac{1+\lambda k_{\max}}{1-\lambda k_{\max}}\, T(t,x).$$

PROOF. Let u be an optimal control for (7.4) and v be a minimal-time control for (t,x). Then, we have

$$\tau^{t,x,u} + g(\gamma_\tau^{t,x,u}) \leq \tau^{t,x,v} + g(\gamma_\tau^{t,x,v})$$

and so,

$$\tau^{t,x,u} \leq \tau^{t,x,v} + \lambda|\gamma_\tau^{t,x,v} - \gamma_\tau^{t,x,u}| \leq \tau^{t,x,v} + \lambda k_{\max}(\tau^{t,x,v} + \tau^{t,x,u}).$$

Consequently, we get

$$\tau^{t,x,u} \leq \frac{1+\lambda k_{\max}}{1-\lambda k_{\max}}\, \tau^{t,x,v} = \frac{1+\lambda k_{\max}}{1-\lambda k_{\max}}\, T(t,x). \quad \square$$

Now, we want to give a result about the Lipschitz continuity of the value function φ. To do that, we will, first, introduce the following estimates on the trajectories, which will be repeatedly used in our analyis.

PROPOSITION 7.9. *For every* $x_0,\, x_1 \in \Omega$ *and* $t_0,\, t \in \mathbb{R}^+$*, there exists* $c := c(t-t_0) > 0$ *such that*

$$|\gamma^{t_0,x_0,u}(t) - \gamma^{t_0,x_1,u}(t)| \leq c|x_0 - x_1|$$

for all controls $u : [t_0,\infty) \mapsto \bar{B}(0,1)$.

PROOF. We have, by (7.3), that

$$\begin{aligned} |\gamma^{t_0,x_0,u}(t) - \gamma^{t_0,x_1,u}(t)| &= \left| x_0 - x_1 + \int_{t_0}^{t} (k(s,\gamma^{t_0,x_0,u}(s)) - k(s,\gamma^{t_0,x_1,u}(s)))\, u(s)\, \mathrm{d}s \right| \\ &\leq |x_0 - x_1| + L_x \int_{t_0}^{t} |\gamma^{t_0,x_0,u}(s) - \gamma^{t_0,x_1,u}(s)|\, \mathrm{d}s. \end{aligned}$$

Using Gronwall's inequality, we get

$$|\gamma^{t_0,x_0,u}(t) - \gamma^{t_0,x_1,u}(t)| \leq c\,|x_0 - x_1|. \quad \square$$

Now, we are ready to prove the following

PROPOSITION 7.10. *Let our system satisfy properties (7.2), (7.3) & (7.5). Then, the value function φ is Lipschitz continuous in $\mathbb{R}^+ \times \Omega$.*

PROOF. Take $x_0, x_1 \in \Omega$ and $t \in \mathbb{R}^+$. Suppose, without loss of generality, that $\varphi(t, x_0) \leq \varphi(t, x_1)$. Let u be an optimal control for x_0 at time t. First, assume that $\tau_0 := \tau^{t,x_0,u} \leq \tau^{t,x_1,u}$. Set $y_0 := \gamma^{t,x_0,u}(t+\tau_0)$ and $y_1 := \gamma^{t,x_1,u}(t+\tau_0)$. Then, by the dynamic programming principle (see Lemma 7.4), we have

$$\varphi(t, x_1) - \varphi(t, x_0) \quad \leq \quad \varphi(t+\tau_0, y_1) - g(y_0).$$

Now, let v be a time-optimal control for y_1, at time $t+\tau_0$. Then, using Lemma 7.7, we get

$$\begin{aligned}
\varphi(t, x_1) - \varphi(t, x_0) \quad &\leq \quad T(t+\tau_0, y_1) \,+\, g(\gamma_\tau^{t+\tau_0, y_1, v}) \,-\, g(y_0) \\
&\leq \quad k_{\min}^{-1}\, d(y_1, \partial\Omega) \,+\, \lambda |\gamma_\tau^{t+\tau_0, y_1, v} - y_0| \\
&\leq \quad k_{\min}^{-1}\, |y_1 - y_0| \,+\, \lambda |\gamma_\tau^{t+\tau_0, y_1, v} - y_0|.
\end{aligned}$$

Since

$$\begin{aligned}
|\gamma_\tau^{t+\tau_0, y_1, v} - y_0| \quad &\leq \quad |\gamma_\tau^{t+\tau_0, y_1, v} - y_1| \,+\, |y_1 - y_0| \\
&\leq \quad k_{\max}\, T(t+\tau_0, y_1) \,+\, |y_0 - y_1| \\
&\leq \quad \left(1 + \frac{k_{\max}}{k_{\min}}\right)|y_0 - y_1|,
\end{aligned}$$

we conclude, from Proposition 7.9, that

$$|\varphi(t, x_1) - \varphi(t, x_0)| \leq c|y_0 - y_1| \leq c|x_0 - x_1|.$$

If $\tau^{t,x_1,u} < \tau_0$: we have

$$\begin{aligned}
\varphi(t, x_1) - \varphi(t, x_0) \quad &\leq \quad \tau^{t,x_1,u} + g(\gamma_\tau^{t,x_1,u}) - \tau^{t,x_0,u} - g(y_0) \\
&\leq \quad \tau^{t,x_1,u} - \tau_0 + \lambda\, |\gamma_\tau^{t,x_1,u} - y_0|.
\end{aligned}$$

Set $y^\star = \gamma^{t,x_0,u}(t+\tau^{t,x_1,u})$. Then, we have

$$\begin{aligned}
|\gamma_\tau^{t,x_1,u} - y_0| \quad &\leq \quad |\gamma_\tau^{t,x_1,u} - y^\star| + |y^\star - y_0| \\
&\leq \quad |\gamma_\tau^{t,x_1,u} - y^\star| + k_{\max}(\tau_0 - \tau^{t,x_1,u}).
\end{aligned}$$

Hence, again by Proposition 7.9, we get

$$\begin{aligned}\varphi(t,x_1)-\varphi(t,x_0) &\leq (1-\lambda k_{\max})(\tau^{t,x_1,u}-\tau_0)+\lambda\,|\gamma_\tau^{t,x_1,u}-y^\star| \\ &\leq c|x_0-x_1|.\end{aligned}$$

Now, take $t_0 \in \mathbb{R}^+$, $x_0 \in \Omega$ and let u be an optimal control for x_0 at time t_0. Then, using Lemma 7.4, one has, for $\delta > 0$ small enough, that

$$\begin{aligned}&|\varphi(t_0+\delta,x_0)-\varphi(t_0,x_0)| \\ &\leq |\varphi(t_0+\delta,x_0)-\varphi(t_0+\delta,\gamma^{t_0,x_0,u}(t_0+\delta))|+|\varphi(t_0+\delta,\gamma^{t_0,x_0,u}(t_0+\delta))-\varphi(t_0,x_0)| \\ &\leq c|\gamma^{t_0,x_0,u}(t_0+\delta)-x_0|+\delta \\ &\leq (1+c\,k_{\max})\delta. \quad \square\end{aligned}$$

Under the assumptions (7.2), (7.3) and (7.5), we have the following

Proposition 7.11. *There exists $c > 0$ depending only on $k_{\min}$, $k_{\max}$, diam(Ω), λ and the Lipschitz constant L_x of the dynamic k with respect to x, such that, for every $x \in \Omega$ and t_0, $t_1 \in \mathbb{R}^+$ with $t_0 \neq t_1$,*

$$\frac{\varphi(t_1,x)-\varphi(t_0,x)}{t_1-t_0} \geq c-1. \tag{7.8}$$

Proof. Suppose, without loss of generality, that $t_0 < t_1$. Let γ_1 be an optimal trajectory for x, at time t_1, and u_1 be the associated optimal control. Define $\phi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ as a function satisfying

$$\begin{cases}\phi'(t)=\dfrac{k(t,\gamma_1(\phi(t)))}{k(\phi(t),\gamma_1(\phi(t)))} \\ \phi(t_0)=t_1.\end{cases} \tag{7.9}$$

Set $\gamma_0(t) = \gamma_1(\phi(t))$, for all $t \geq t_0$. By construction of ϕ, it is clear that there is a control u_0 such that $\gamma_0 = \gamma^{t_0,x,u_0}$ (more precisely, $u_0(t) = u_1(\phi(t))$, $\forall\ t \geq t_0$). Moreover, we have $\tau_0 := \tau^{t_0,x,u_0} = \phi^{-1}(t_1+\tau_1)-t_0$, where $\tau_1 := \tau^{t_1,x,u_1}$. So, $\phi(t_0+\tau_0) = t_1+\tau_1$ and $\phi(t_0+\tau_0)+g(\gamma_0(t_0+\tau_0)) = t_1+\varphi(t_1,x)$. On the other hand, from (7.9), it is easy to see that, for all t, $\bar{t} \geq t_0$, one has

$$\int_{\phi(t)}^{\phi(\bar{t})} k(s,\gamma_1(s))\,\mathrm{d}s = \int_t^{\bar{t}} k(s,\gamma_1(\phi(s)))\,\mathrm{d}s.$$

Now, set

$$G(\theta) = \int_{\theta}^{\phi(\bar{t})} k(s, \gamma_1(s))\, \mathrm{d}s, \ \ \forall\ \theta \in \mathbb{R}^+.$$

Then, using that G is bi-Lipschitz, we have

$$\begin{aligned}
|\phi(t) - t| &= \left| G^{-1}\Big(\int_t^{\bar{t}} k(s, \gamma_1(\phi(s)))\, \mathrm{d}s \Big) - G^{-1}\Big(\int_t^{\phi(\bar{t})} k(s, \gamma_1(s))\, \mathrm{d}s \Big) \right| \\
&\leq C \left| \int_t^{\bar{t}} k(s, \gamma_1(\phi(s)))\, \mathrm{d}s - \int_t^{\phi(\bar{t})} k(s, \gamma_1(s))\, \mathrm{d}s \right| \\
&\leq C \Big(|\phi(\bar{t}) - \bar{t}| + \left| \int_t^{\bar{t}} |k(s, \gamma_1(\phi(s))) - k(s, \gamma_1(s))|\, \mathrm{d}s \right| \Big) \\
&\leq C|\phi(\bar{t}) - \bar{t}| + C \left| \int_t^{\bar{t}} |\phi(s) - s|\, \mathrm{d}s \right|.
\end{aligned}$$

Using the fact that $\phi(t_0) = t_1 > t_0$, we infer that $\phi(t) > t$ for every $t \geq t_0$. Now, set $\bar{t} = t_0 + \tau_0$. Then, by Gronwall's inequality, we get

$$\phi(t) - t \leq C e^{C|t_0 + \tau_0 - t|} (\phi(t_0 + \tau_0) - (t_0 + \tau_0)).$$

Setting $t = t_0$, one has

$$c(t_1 - t_0) \leq \phi(t_0 + \tau_0) - (t_0 + \tau_0),$$

where $c > 0$ only depends on $k_{\min}$, $k_{\max}$, $\mathrm{diam}(\Omega)$, λ and the Lipschitz constant L_x of the dynamic k with respect to x. ☐

The proposition 7.11 yields a lower bound on the time derivative of the value function φ, which can be used to obtain information on the gradient of φ thanks to the Hamilton–Jacobi equation (7.7).

COROLLARY 7.12. *There exists $c > 0$ (which only depends on $k_{\min}$, $k_{\max}$, diam(Ω), λ and the Lipschitz constant L_x of the dynamic k with respect to x) such that $\partial_t \varphi(t, x) \geq c - 1$ and $|\nabla_x \varphi(t, x)| \geq c$, for all $(t, x) \in \mathbb{R}^+ \times \Omega$ where φ is differentiable.*

7.2. Optimality conditions and Pontryagin Maximum Principle

Now, we analyze some optimality conditions for our control problem. Our aim is to obtain results analogous to those in [37]; the statements and the methods of proof require suitable adaptations due to the fact that the dynamic, here, also depends on time. From now on, we assume properties (7.2), (7.3), (7.5) and the following additional regularity requirements:

$$\partial\Omega \text{ is of class } C^{1,1}, \tag{7.10}$$

$$\nabla_x k \text{ is continuous on } \mathbb{R}^+ \times \Omega, \tag{7.11}$$

$$g \in C^1(\partial\Omega). \tag{7.12}$$

We begin by proving a version of Pontryagin's maximum principle. We start with two preliminary results.

LEMMA 7.13. *Let $z \in \partial\Omega$ and let $\mathbf{n}$ be the outer normal to $\partial\Omega$ at z. Let $u^\star \in \bar{B}(0,1)$ be such that $u^\star \cdot \mathbf{n} > 0$ and consider, for $t \in \mathbb{R}^+$ and $x \in \Omega$, the exit time $\tau^{t,x,u^\star}$ under the constant control $u^\star$. Then there exists a neighborhood of z, which we denote by V, such that*

$$\tau^{t,x,u^\star} = -\frac{\mathbf{n}\cdot(x-z)}{k(t,z)\,\mathbf{n}\cdot u^\star} + o(|x-z|), \quad \forall\ x \in V.$$

PROOF. Let us consider the signed distance function $d^\pm$ from $\partial\Omega$, defined as

$$d^\pm(x) := \begin{cases} +d(x,\partial\Omega) & \text{if } x \in \Omega, \\ -d(x,\partial\Omega) & \text{else.} \end{cases}$$

Our smoothness assumption (7.10) on $\partial\Omega$ implies that $d^\pm$ is differentiable near z and $\nabla d^\pm(z) = -\mathbf{n}$. For every $t \in \mathbb{R}^+$, we define

$$F^t(x,s) = d^\pm(\gamma^{t,x,u^\star}(s)), \quad \text{for all } x \in \Omega,\ s \geq t.$$

Then F^t is differentiable for (x,s) near (z,t) and satisfies

$$F^t(z,t) = 0, \ \ \frac{\partial F^t}{\partial s}(z,t) = -k(t,z)\,\mathbf{n}\cdot u^\star \neq 0.$$

By the implicit function theorem, we can find a neighborhood V of z, a number $\delta > 0$ and a function $\mathbf{s} : V \mapsto (t-\delta, t+\delta)$ such that for all $x \in V$ and $s \in (t-\delta, t+\delta)$,

$$d^\pm(\gamma^{t,x,u^\star}(s)) = 0 \Leftrightarrow s = \mathbf{s}(x). \tag{7.13}$$

From (7.13) we see easily that $\mathbf{s}(x) - t$ coincides with $\tau^{t,x,u^\star}$, for all $x \in V$. Moreover, we have

$$\nabla \mathbf{s}(z) = -\frac{\nabla_x F^t(z,t)}{\frac{\partial F^t}{\partial s}(z,t)} = -\frac{\mathbf{n}}{k(t,z)\,\mathbf{n}\cdot u^\star},$$

which implies

$$\mathbf{s}(x) - t = \nabla \mathbf{s}(z)\cdot(x-z) + o(|x-z|) = -\frac{\mathbf{n}\cdot(x-z)}{k(t,z)\,\mathbf{n}\cdot u^\star} + o(|x-z|). \quad \square$$

LEMMA 7.14. *Given $z \in \partial\Omega$, let $\mathbf{n}$ be the outer normal to $\partial\Omega$ at z. Then, for every $t \in \mathbb{R}^+$, there exists a unique $\mu > 0$ such that $k(t,z)\,|\nabla g(z) - \mu\,\mathbf{n}| - 1 = 0$.*

PROOF. Since g is λ−Lip with $\lambda < \frac{1}{k_{\max}}$, we have $k(t,z)\,|\nabla g(z)| - 1 < 0$. Moreover, we see that $k(t,z)\,|\nabla g(z) - \mu\,\mathbf{n}| - 1 \to +\infty$ as $\mu \to +\infty$. This implies that there exists $\mu > 0$ such that $k(t,z)\,|\nabla g(z) - \mu\,\mathbf{n}| - 1 = 0$. Now we prove the uniqueness of μ. Arguing by contradiction we suppose that there exist $0 < \mu_1 < \mu_2$ such that $0 = k(t,z)\,|\nabla g(z) - \mu_1\,\mathbf{n}| - 1 = k(t,z)\,|\nabla g(z) - \mu_2\,\mathbf{n}| - 1$. We have

$$k(t,z)\,|\nabla g(z) - \mu_2\,\mathbf{n}| - 1 \quad \geq \quad k(t,z)\,\frac{\nabla g(z) - \mu_1\,\mathbf{n}}{|\nabla g(z) - \mu_1\,\mathbf{n}|}\cdot(\nabla g(z) - \mu_2\,\mathbf{n}) - 1$$

$$= (\mu_1 - \mu_2)\,k(t,z)\,\frac{\nabla g(z) - \mu_1\,\mathbf{n}}{|\nabla g(z) - \mu_1\,\mathbf{n}|}\cdot\mathbf{n}. \tag{7.14}$$

Yet,

$$-\mu_1\,k(t,z)\,\frac{\nabla g(z) - \mu_1\,\mathbf{n}}{|\nabla g(z) - \mu_1\,\mathbf{n}|}\cdot\mathbf{n} = 1 - \,k(t,z)\,\frac{\nabla g(z) - \mu_1\,\mathbf{n}}{|\nabla g(z) - \mu_1\,\mathbf{n}|}\cdot\nabla g(z) \; > \; 0.$$

Therefore,

$$\frac{\nabla g(z) - \mu_1\,\mathbf{n}}{|\nabla g(z) - \mu_1\,\mathbf{n}|}\cdot\mathbf{n} < 0, \tag{7.15}$$

So, using (7.14), (7.15), and $\mu_1 - \mu_2 < 0$, we get

$$k(t,z)\,|\nabla g(z) - \mu_2\,\mathbf{n}| - 1 > 0,$$

which is a contradiction. $\square$

We are now ready to state the Pontryagin Maximum Principle for this control problem.

THEOREM 7.15. *Let properties (7.2), (7.3), (7.5), (7.10), (7.11) and (7.12) hold, let $(t_0, x_0) \in \mathbb{R}^+ \times \Omega$ and let $\bar{u}$ be an optimal control for x_0, at time t_0. Set for simplicity*

$$\gamma := \gamma^{t_0,x_0,\bar{u}}, \qquad \tau_0 := \tau^{t_0,x_0,\bar{u}}, \qquad z := \gamma_\tau^{t_0,x_0,\bar{u}},$$

and denote by $\mathbf{n}$ the outer normal to $\partial\Omega$ at z. Let $\mu > 0$ be such that $k(t_0+\tau_0, z)\,|\nabla g(z) - \mu\,\mathbf{n}| - 1 = 0$ (μ is uniquely determined by the previous lemma). Let $p : [t_0, t_0+\tau_0] \mapsto \mathbb{R}^d$ be the solution to the system

$$\begin{cases} p'(t) = -\nabla_x k(t, \gamma(t))\,\bar{u}(t) \cdot p(t), \\ p(t_0+\tau_0) = \nabla g(z) - \mu\,\mathbf{n}. \end{cases} \tag{7.16}$$

Then, for a.e. $t \in [t_0, t_0+\tau_0]$,

$$-p(t) \cdot \bar{u}(t) = \max_{u \in \bar{B}(0,1)} -p(t) \cdot u.$$

PROOF. Let $t \in (t_0, t_0+\tau_0)$ be a Lebesgue point for the function $k(\,.\,, \gamma(.))\,\bar{u}(.)$, i.e., a value such that

$$\lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t+h} |k(s, \gamma(s))\,\bar{u}(s) - k(t, \gamma(t))\,\bar{u}(t)|\,\mathrm{d}s = 0. \tag{7.17}$$

Let us fix $u \in \bar{B}(0,1)$. For $\varepsilon > 0$ small, we define

$$u_\varepsilon(s) = \begin{cases} u, \quad s \in [t-\varepsilon, t], \\ \bar{u}(s), \quad s \in [t_0, t_0+\tau_0] \backslash [t-\varepsilon, t], \\ u^\star, \quad s > t_0 + \tau_0, \end{cases}$$

where $u^\star = -\frac{\nabla g(z) - \mu\,\mathbf{n}}{|\nabla g(z) - \mu\,\mathbf{n}|}$. We set $\gamma_\varepsilon = \gamma^{t_0,x_0,u_\varepsilon}$ and $\tau_\varepsilon = \tau^{t_0,x_0,u_\varepsilon}$. From Lemma 7.13, we see that τ_ε is finite if ε is small enough (if $\tau_\varepsilon > \tau_0$, then $\tau_\varepsilon \le \tau_0 + \tau^{t_0+\tau_0, \gamma_\varepsilon(t_0+\tau_0), u^\star}$; yet, $\gamma_\varepsilon(t_0+\tau_0) \to z$ when $\varepsilon \to 0$). Since $\bar{u}$ is an optimal control for x_0 at time t_0, we have

$$0 \le \tau_\varepsilon + g(\gamma_\varepsilon(t_0+\tau_\varepsilon)) - \varphi(t_0, x_0) = \tau_\varepsilon - \tau_0 + g(\gamma_\varepsilon(t_0+\tau_\varepsilon)) - g(z). \tag{7.18}$$

Taking into account (7.17), we find

$$\begin{aligned} \gamma_\varepsilon(t) - \gamma(t) &= \int_{t-\varepsilon}^{t} (k(s, \gamma_\varepsilon(s))\,u - k(s, \gamma(s))\,\bar{u}(s))\,\mathrm{d}s \\ &= \varepsilon\,k(t, \gamma(t))\,(u - \bar{u}(t)) + o(\varepsilon). \end{aligned}$$

Therefore,

$$\gamma_\varepsilon(s) = \gamma(s) + \varepsilon v(s) + o(\varepsilon), \quad s \in [t, t_0+\tau_0], \tag{7.19}$$

where $v(.)$ is the solution to the linearized system

$$\begin{cases} v'(s) = \bar{u}(s)\,\nabla_x k(s,\gamma(s))\cdot v(s) \\ v(t) = k(t,\gamma(t))\,(u-\bar{u}(t)). \end{cases}$$

Let us observe that

$$\frac{d}{ds}\,p(s)\cdot v(s) = 0. \tag{7.20}$$

It is now convenient to treat separately the cases $\tau_\varepsilon < \tau_0$ and $\tau_\varepsilon \geq \tau_0$.

• If $\tau_\varepsilon < \tau_0$:

We first observe that, by (7.19), the point $\gamma(t_0+\tau_\varepsilon)$ has distance of order $O(\varepsilon)$ from $\gamma_\varepsilon(t_0+\tau_\varepsilon) \in \partial\Omega$, and so $d_{\partial\Omega}(\gamma(t_0+\tau_\varepsilon)) = O(\varepsilon)$. Since the optimal control $\bar{u}$ steers $\gamma(t_0+\tau_\varepsilon)$ to the target in a time $\tau_0-\tau_\varepsilon$, we obtain, from Lemma 7.7 & Proposition 7.8, that

$$\tau_0 - \tau_\varepsilon = O(\varepsilon). \tag{7.21}$$

Moreover, we have

$$g(\gamma_\varepsilon(t_0+\tau_\varepsilon)) - g(z) = \nabla g(z)\cdot(\gamma_\varepsilon(t_0+\tau_\varepsilon)-z) + o(|\gamma_\varepsilon(t_0+\tau_\varepsilon)-z|).$$

By (7.2), (7.19) and (7.21) we have, for any $s\in[t_0+\tau_\varepsilon, t_0+\tau_0]$,

$$\begin{aligned} |\gamma_\varepsilon(s)-z| &\leq \int_s^{t_0+\tau_0} |k(r,\gamma_\varepsilon(r))\,u_\varepsilon(r)|\,\mathrm{d}r + |\gamma_\varepsilon(t_0+\tau_0)-z| \\ &\leq k_{\max}(t_0+\tau_0-s) + |\varepsilon v(t_0+\tau_0)+o(\varepsilon)| = O(\varepsilon). \end{aligned}$$

Therefore,

$$g(\gamma_\varepsilon(t_0+\tau_\varepsilon)) - g(z)$$

$$= \nabla g(z)\cdot(\gamma_\varepsilon(t_0+\tau_\varepsilon)-z) + o(\varepsilon)$$

$$= \nabla g(z)\cdot\Big(-\int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} k(s,\gamma_\varepsilon(s))\,u_\varepsilon(s)\,\mathrm{d}s + \varepsilon v(t_0+\tau_0)\Big) + o(\varepsilon)$$

$$= \nabla g(z)\cdot\Big(-\int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} k(t_0+\tau_0,z)\bar{u}(s)\mathrm{d}s - \int_{t_0+\tau_\varepsilon}^{t_0+\tau_0}(k(s,\gamma_\varepsilon(s))-k(t_0+\tau_0,z))\bar{u}(s)\mathrm{d}s + \varepsilon v(t_0+\tau_0)\Big) + o(\varepsilon)$$

$$\leq \nabla g(z) \cdot \Big(- \int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} k(t_0+\tau_0, z)\bar{u}(s)\mathrm{d}s + \varepsilon v(t_0+\tau_0) \Big)$$

$$+ \ \lambda(\tau_0 - \tau_\varepsilon) \sup_{s \in [t_0+\tau_\varepsilon, t_0+\tau_0]} (k(s, \gamma_\varepsilon(s)) - k(t_0+\tau_0, z)) + o(\varepsilon)$$

$$= \nabla g(z) \cdot \Big(- \int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} k(t_0+\tau_0, z)\, \bar{u}(s)\, \mathrm{d}s + \varepsilon v(t_0+\tau_0) \Big) + o(\varepsilon). \tag{7.22}$$

By the smoothness of $\partial\Omega$ we have, since $\gamma_\varepsilon(t_0+\tau_\varepsilon) \in \partial\Omega$, that

$$\mathbf{n} \cdot (z - \gamma_\varepsilon(t_0+\tau_\varepsilon)) = O(|z - \gamma_\varepsilon(t_0+\tau_\varepsilon)|^2) = o(\varepsilon).$$

Recalling also that

$$k(t_0+\tau_0, z)\, |\nabla g(z) - \mu\, \mathbf{n}| - 1 = 0 \quad \text{and} \quad |\bar{u}| \leq 1,$$

we obtain

$$\tau_\varepsilon - \tau_0 - \nabla g(z) \cdot \int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} k(t_0+\tau_0, z)\, \bar{u}(s)\, \mathrm{d}s$$

$$\leq \ -\mu\, \mathbf{n} \cdot \int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} k(t_0+\tau_0, z)\, \bar{u}(s)\, \mathrm{d}s$$

$$= -\mu\, \mathbf{n} \cdot (z - \gamma(t_0+\tau_\varepsilon)) - \mu\, \mathbf{n} \cdot \int_{t_0+\tau_\varepsilon}^{t_0+\tau_0} (k(t_0+\tau_0, z) - k(s, \gamma(s)))\, \bar{u}(s)\, \mathrm{d}s$$

$$= -\mu\, \mathbf{n} \cdot (z - \gamma(t_0+\tau_\varepsilon)) + o(\varepsilon) = -\mu\, \mathbf{n} \cdot (\gamma_\varepsilon(t_0+\tau_\varepsilon) - \gamma(t_0+\tau_\varepsilon)) - \mu\, \mathbf{n} \cdot (z - \gamma_\varepsilon(t_0+\tau_\varepsilon)) + o(\varepsilon)$$

$$= -\varepsilon\, \mu\, \mathbf{n} \cdot v(t_0+\tau_\varepsilon) + o(\varepsilon) = -\varepsilon\, \mu\, \mathbf{n} \cdot v(t_0+\tau_0) + o(\varepsilon).$$

Thus (7.18) implies, using also (7.22), that

$$0 \leq \varepsilon(\nabla g(z) - \mu\, \mathbf{n}) \cdot v(t_0+\tau_0) + o(\varepsilon).$$

Dividing by ε and letting $\varepsilon \to 0^+$ we obtain,

$$0 \leq (\nabla g(z) - \mu\, \mathbf{n}) \cdot v(t_0+\tau_0) = k(t, \gamma(t))\ p(t) \cdot (u - \bar{u}(t)),$$

where last inequality comes from (7.20). Then,

$$p(t) \cdot \bar{u}(t) \leq p(t) \cdot u.$$

• Now, suppose that $\tau_\varepsilon \geq \tau_0$:

By (7.19), $\gamma_\varepsilon(t_0+\tau_0) - z = \varepsilon v(t_0+\tau_0) + o(\varepsilon)$; thus, for ε small enough, we can apply Lemma 7.13 to obtain

$$\tau_\varepsilon - \tau_0 = \tau^{t_0+\tau_0,\gamma_\varepsilon(t_0+\tau_0),u^\star} = -\frac{(\gamma_\varepsilon(t_0+\tau_0)-z)\cdot\mathbf{n}}{k(t_0+\tau_0,z)\,u^\star\cdot\mathbf{n}} + o(\varepsilon) = -\frac{\varepsilon\, v(t_0+\tau_0)\cdot\mathbf{n}}{k(t_0+\tau_0,z)\,u^\star\cdot\mathbf{n}} + o(\varepsilon). \tag{7.23}$$

This implies in particular that

$$\tau_\varepsilon - \tau_0 = O(\varepsilon)$$

and

$$|\gamma_\varepsilon(s) - z| \leq |\gamma_\varepsilon(s) - \gamma_\varepsilon(t_0+\tau_0)| + |\gamma_\varepsilon(t_0+\tau_0) - z| = O(\varepsilon), \qquad \forall\ s \in [t_0+\tau_0, t_0+\tau_\varepsilon].$$

Thus we have

$$\begin{aligned} \gamma_\varepsilon(t_0+\tau_\varepsilon) - z &= \int_{t_0+\tau_0}^{t_0+\tau_\varepsilon} k(s,\gamma_\varepsilon(s))\,u^\star\,\mathrm{d}s + \gamma_\varepsilon(t_0+\tau_0) - z \\ &= (\tau_\varepsilon - \tau_0)\,k(t_0+\tau_0,z)\,u^\star + \varepsilon v(t_0+\tau_0) + o(\varepsilon). \end{aligned}$$

We get, from (7.18),

$$0 \leq (\tau_\varepsilon - \tau_0)\,(1 + k(t_0+\tau_0,z)\,\nabla g(z)\cdot u^\star) + \varepsilon\nabla g(z)\cdot v(t_0+\tau_0) + o(\varepsilon).$$

By definition of $u^\star$, we have

$$1 + k(t_0+\tau_0,z)\,\nabla g(z)\cdot u^\star = \mu\,k(t_0+\tau_0,z)\,\mathbf{n}\cdot u^\star.$$

So, using (7.23),

$$(\tau_\varepsilon - \tau_0)(1 + k(t_0+\tau_0,z)\,\nabla g(z)\cdot u^\star) = \mu(\tau_\varepsilon-\tau_0)k(t_0+\tau_0,z)\,u^\star\cdot\mathbf{n} = -\varepsilon\,\mu\,v(t_0+\tau_0)\cdot\mathbf{n} + o(\varepsilon).$$

Thus,

$$0 \leq \varepsilon(\nabla g(z) - \mu\,\mathbf{n})\cdot v(t_0+\tau_0) + o(\varepsilon).$$

Now we can divide by ε and let $\varepsilon \to 0^+$ to obtain the conclusion as in the first step. Hence,

$$p(t)\cdot\bar{u}(t) \leq p(t)\cdot u, \quad \text{for all } \ u \in \bar{B}(0,1). \quad \square$$

LEMMA 7.16. *Let* t_0, x_0, τ_0, γ, $\bar{u}$, μ *and* p *be as in the statement of Theorem 7.15. Then* $p(t) \neq 0$, *for every* $t \in [t_0, t_0 + \tau_0]$.

PROOF. From (7.16), one obtains that, for every $t \in [t_0, t_0 + \tau_0]$, $|p'(t)| \leq M|p(t)|$. Hence, we get

$$|p(t)| \leq |p(s)|\, e^{M|t-s|}$$

for every $s, t \in [t_0, t_0 + \tau_0]$. Hence, if there exists $s \in [t_0, t_0 + \tau_0]$ such that $p(s) = 0$, one concludes that $p(t) = 0$ for every $t \in [t_0, t_0 + \tau_0]$, which is a contradiction as $p(t_0 + \tau_0) = \nabla g(\gamma(t_0 + \tau_0)) - \mu\, \mathbf{n} \neq 0$, thanks to Lemma 7.14. □

Consequently, we get the following

COROLLARY 7.17. *The optimal control* $\bar{u}$ *is Lipschitz continuous on* $[t_0, t_0 + \tau_0]$. *And, the associated optimal trajectory* γ *from* x_0, *at time* t_0, *belongs to* $C^{1,1}([t_0, t_0 + \tau_0], \Omega)$, *as soon as* k *is also Lipschitz with respect to* t.

PROOF. Since $p(t) \neq 0$ for every $t \in [t_0, t_0 + \tau_0]$, it follows immediately from the theorem 7.15 that $\bar{u}(t) = -\frac{p(t)}{|p(t)|}$. Hence,

$$\begin{aligned} \bar{u}'(t) &= -\frac{|p(t)|\, p'(t) - \frac{p(t)\cdot p'(t)}{|p(t)|} p(t)}{|p(t)|^2} \\ &= -\nabla_x k(t, \gamma(t)) + \bar{u}(t) \cdot \nabla_x k(t, \gamma(t))\, \bar{u}(t). \end{aligned}$$

Hence, $|\bar{u}'(t)| \leq C$, for some constant C depending only on the Lipschitz constant L_x (w.r.t. x) of the dynamic k. □

Suppose that

$$|\nabla_x k(t, x_0) - \nabla_x k(t, x_1)| \leq L_{xx}\, |x_0 - x_1|, \quad \text{for all } x_0,\, x_1 \in \Omega,\, t \in \mathbb{R}^+. \tag{7.24}$$

Then, this also provides that $(\gamma, \bar{u})$ is the unique solution of the following system

$$\begin{cases} \gamma'(t) = k(t, \gamma(t))\, u(t), \\ u'(t) = -\nabla_x k(t, \gamma(t)) + u(t) \cdot \nabla_x k(t, \gamma(t))\, u(t), \\ \gamma(t_0) = x_0, \\ u_{|t=t_0} = \bar{u}(t_0). \end{cases} \tag{7.25}$$

Given an optimal trajectory γ for x_0 at time t_0, we will say that p is a dual arc associated with γ if it satisfies the properties of Theorem 7.15, that is, if it solves problem (7.16).

We now prove that the dual arc p of the theorem 7.15 is included in the superdifferential, with respect to x, $\nabla^+\varphi$ of the value function φ, where for every $t \in \mathbb{R}^+$, $x \in \Omega$,

$$\nabla^+\varphi(t,x) := \left\{p \in \mathbb{R}^d \,:\, \limsup_{h\to 0} \frac{\varphi(t,x+h)-\varphi(t,x)-p\cdot h}{|h|} \leq 0\right\}.$$

As a simple consequence of the definitions of $D^+\varphi$ and $\nabla^+\varphi$, we have the following inclusion $\Pi_x(D^+\varphi(t,x)) \subset \nabla^+\varphi(t,x)$. Moreover, the reverse inclusion holds as soon as φ is semi-concave (see, for instance, [**37**, Lemma 3.3.16]).

PROPOSITION 7.18. *Under the assumptions of Theorem 7.15, the arc p solution of system (7.16) satisfies*

$$p(t) \in \nabla^+\varphi(t,\gamma(t)), \quad \textit{for all } \; t \in [t_0, t_0+\tau_0).$$

PROOF. Fix $t \in [t_0, t_0+\tau_0)$. We will show that for all $h \in \mathbb{R}^d$ with $|h| = 1$,

$$\varphi(t,\gamma(t)+\varepsilon h) \leq \varphi(t,\gamma(t)) + \varepsilon p(t)\cdot h + o(\varepsilon)$$

with $o(\varepsilon)$ independent of h. Let us define a control strategy by setting

$$\widehat{u}(s) = \begin{cases} \bar{u}(s), & s \in [t, t_0+\tau_0], \\ u^\star, & s > t_0+\tau_0, \end{cases}$$

where $u^\star = -\frac{\nabla g(z)-\mu\,\mathbf{n}}{|\nabla g(z)-\mu\,\mathbf{n}|}$. Now, let us consider the trajectory $\gamma_\varepsilon := \gamma^{t,\gamma(t)+\varepsilon h,\widehat{u}}$, associated with $\widehat{u}$, starting from $\gamma(t)+\varepsilon h$, at time t. Set

$$\tau_\varepsilon := \tau^{t,\gamma(t)+\varepsilon h,\widehat{u}}.$$

Then we have, for all $s \in [t, t_0+\tau_0]$,

$$\gamma_\varepsilon(s) = \gamma(s) + \varepsilon v(s) + o(\varepsilon),$$

where v is the solution of the linearized problem

$$\begin{cases} v'(s) = \bar{u}(s)\, \nabla_x k(s,\gamma(s))\cdot v(s), & s \in [t, t_0+\tau_0] \\ v(t) = h, \end{cases}$$

and $o(\varepsilon)$ does not depend on h. We observe that

$$\varphi(t,\gamma(t)+\varepsilon h)-\varphi(t,\gamma(t))\le\tau_\varepsilon-(t_0+\tau_0-t)+g(\gamma_\varepsilon(t+\tau_\varepsilon))-g(z).$$

The terms on the right-hand side can be estimated as the corresponding terms in formula (7.18). Following the computations of the proof of Theorem 7.15, we obtain

$$\varphi(t,\gamma(t)+\varepsilon h)-\varphi(t,\gamma(t))\le\varepsilon\,(\nabla g(z)-\mu\,\mathbf{n})\cdot v(t_0+\tau_0)+o(\varepsilon)=\varepsilon\,p(t)\cdot h+o(\varepsilon).\quad\square$$

On the other hand, the subdifferential satisfies an analogous property, with a reversed time direction, as the next result shows. We denote by $\nabla^-\varphi$ the subdifferential of φ with respect to x, i.e., for every $(t,x)\in\mathbb{R}^+\times\Omega$,

$$\nabla^-\varphi(t,x):=\left\{p\in\mathbb{R}^d\;:\;\liminf_{y\to x}\frac{\varphi(t,y)-\varphi(t,x)-p\cdot(y-x)}{|y-x|}\ge 0\right\}.$$

PROPOSITION 7.19. *Let u be an optimal control for $(t_0,x_0)\in\mathbb{R}^+\times\Omega$ and γ be its associated optimal trajectory. Let $p:[t_0,t_0+\tau_0]\mapsto\mathbb{R}^d$ be any solution of the adjoint equation*

$$p'(t)=-\nabla_x k(t,\gamma(t))\,u(t)\cdot p(t),\quad t\in[t_0,t_0+\tau_0],\tag{7.26}$$

where $\tau_0=\tau^{t_0,x_0,u}$. Suppose that $p(t_0)\in\nabla^-\varphi(t_0,x_0)$, then $p(t)\in\nabla^-\varphi(t,\gamma(t))$, for all $t\in[t_0,t_0+\tau_0)$.

PROOF. Fix $t\in(t_0,t_0+\tau_0)$ and let h be any unit vector in $\mathbb{R}^d$. Let $v:[t_0,t]\mapsto\mathbb{R}^d$ be the solution of

$$\begin{cases}v'(s)=u(s)\,\nabla_x k(s,\gamma(s))\cdot v(s), & s\in[t_0,t]\\ v(t)=h.\end{cases}$$

For any $\varepsilon>0$, let us set $\gamma_\varepsilon=\gamma^{t_0,x_0+\varepsilon v(t_0),u}$. Then, it is not difficult to see that

$$\gamma_\varepsilon(s)=\gamma(s)+\varepsilon v(s)+o(\varepsilon),\quad s\in[t_0,t].$$

By the dynamic programming principle (see Lemma 7.4), the assumption that $p(t_0)$ belongs to $\nabla^-\varphi(t_0,x_0)$ and the fact that $p\cdot v$ is constant, we get

$$\begin{aligned}\varphi(t,\gamma_\varepsilon(t)) &\geq t_0 - t + \varphi(t_0, x_0 + \varepsilon v(t_0)) \\ &\geq t_0 - t + \varphi(t_0,x_0) + \varepsilon\, p(t_0)\cdot v(t_0) + o(\varepsilon) \\ &= \varphi(t,\gamma(t)) + \varepsilon\, p(t)\cdot h + o(\varepsilon).\end{aligned}$$

On the other hand, by the Lipschitz continuity of φ (see Proposition 7.10), we have

$$\varphi(t,\gamma_\varepsilon(t)) = \varphi(t,\gamma(t)+\varepsilon v(t)) + o(\varepsilon) = \varphi(t,\gamma(t)+\varepsilon h) + o(\varepsilon),$$

and so we conclude that

$$\varphi(t,\gamma(t)+\varepsilon h) \geq \varphi(t,\gamma(t)) + \varepsilon\, p(t)\cdot h + o(\varepsilon).$$

By the arbitrariness of h, we deduce that $p(t) \in \nabla^-\varphi(t,\gamma(t))$. □

7.3. Differentiability of the value function

In this section, we prove that the value function φ is differentiable along optimal trajectories, except possibly their endpoints. This kind of results is classical (see [**37**]) and one can even obtain more (for instance, in [**36**] smoothness of the value function in a neighborhood of optimal trajectories is proven under some suitable conditions). Yet, most of the literature is concerned with the autonomous case (see in particular [**37**]), which makes the question non-trivial in the non-autonomous one.

First of all, let us introduce the following result concerning the semi-concavity of the value function φ. The proof is omitted since in Section 7.4 we will prove a much finer result. Before that, we assume the following

$$k \in C^{1,1}(\mathbb{R}^+ \times \Omega), \tag{7.27}$$

$$g \text{ is semi-concave on } \partial\Omega. \tag{7.28}$$

Proposition 7.20. *Under the assumptions (7.2), (7.5), (7.10), (7.27) and (7.28), the value function φ is semi-concave on $\mathbb{R}^+ \times \Omega$.*

Proof. We refer, for instance, to [**37**, Theorem 8.2.7]. □

COROLLARY 7.21. *Let γ be an optimal trajectory for x_0, at time t_0, and u be the associated optimal control. If φ is differentiable at (t_0, x_0), then φ is differentiable at $(t, \gamma(t))$, for all $t \in [t_0, t_0 + \tau_0)$, where $\tau_0 = \tau^{t_0,x_0,u}$.*

PROOF. Fix $t \in [t_0, t_0 + \tau_0)$. If φ is differentiable at (t_0, x_0), then the subdifferential $\nabla^-\varphi(t_0, x_0)$ is a singleton, say $\nabla^-\varphi(t_0, x_0) = \{p_0\}$. Now, let p be a solution of (7.26) with initial condition $p(t_0) = p_0$. By Proposition 7.19, $p(t) \in \nabla^-\varphi(t, \gamma(t))$, which implies, in particular, that $\nabla^-\varphi(t, \gamma(t)) \neq \emptyset$. On the other hand, as φ is semi-concave (see Proposition 7.20), then $D^+\varphi(t, \gamma(t)) \neq \emptyset$ and so, $\nabla^+\varphi(t, \gamma(t)) \neq \emptyset$. Hence, φ is differentiable with respect to x at $(t, \gamma(t))$. Now, take $(p_t, p_x) \in D^+\varphi(t, \gamma(t))$. Then, $p_x = \nabla\varphi(t, \gamma(t))$. Yet, by Proposition 7.6, we have

$$-p_t + k(t, \gamma(t))\,|p_x| = 1, \tag{7.29}$$

which implies that p_t is uniquely determined by $\nabla\varphi(t, \gamma(t))$. Consequently, $D^+\varphi(t, \gamma(t))$ is a singleton and so, φ is differentiable at $(t, \gamma(t))$ (thanks again to the semi-concavity of the value function φ). Notice that we are not able to extend this result to the final time $t_0 + \tau_0$ as (7.29) does not hold a priori at the endpoint of an optimal trajectory. □

Now, let us introduce the following lemma, which shows that the uniform limit of optimal trajectories is an optimal trajectory.

LEMMA 7.22. *Let $(t_n, x_n)_n$ be a sequence in $\mathbb{R}^+ \times \Omega$ such that $t_n \to t$ and $x_n \to x$. For each n, let γ_n be an optimal trajectory for x_n, at time t_n, and u_n be the associated optimal control. Then, $\gamma_n \to \gamma$ and $u_n \to u$ uniformly, where γ is in fact an optimal trajectory for x, at time t, and u is its associated optimal control.*

PROOF. The fact that $\gamma_n \to \gamma$ and $u_n \to u$ follows immediately using Corollary 7.17. Yet, for every n, we have

$$\gamma_n'(s) = k(s, \gamma_n(s))\,u_n(s), \quad \text{for a.e. } s \geq t_n.$$

Passing to the limit as $n \to +\infty$, we get

$$\gamma'(s) = k(s, \gamma(s))\,u(s), \quad \text{for a.e. } s \geq t. \tag{7.30}$$

Moreover, $\gamma_n(t_n) = x_n$ implies, at the limit, that $\gamma(t) = x$. Hence, γ is an admissible trajectory for x, at time t, and u is its associated control. Now, set $\tau_n = \tau^{t_n,x_n,u_n}$ and $z_n = \gamma_n(t_n + \tau_n) \in \partial\Omega$. From Lemma 7.7 & Proposition 7.8, we have that $\tau_n \to \bar{\tau}$. In addition, $z_n \to \gamma(t + \bar{\tau}) \in \partial\Omega$. Hence, $\tau := \tau^{t,x,u} \leq \bar{\tau}$ and,

$$\varphi(t_n, x_n) = \tau_n + g(z_n) \to \bar{\tau} + g(\gamma(t + \bar{\tau})).$$

Yet, by the continuity of the value function φ (see, for instance, Proposition 7.10), we infer that

$$\varphi(t, x) = \bar{\tau} + g(\gamma(t + \bar{\tau})) \leq \tau + g(\gamma(t + \tau))$$

and then, thanks to (7.5), $\bar{\tau} = \tau$. $\square$

On the other hand, we have the following result about the uniqueness of optimal control at any interior point of an optimal trajectory.

PROPOSITION 7.23. *Let γ be an optimal trajectory from x_0 at time t_0, and set $\tau_0 = \tau^{t_0,x_0,u}$, where u is the associated optimal control. Then, under the assumption (7.24), for every $t \in (t_0, t_0 + \tau_0)$, u is the unique optimal control for $\gamma(t)$, at time t.*

PROOF. Fix $t \in (t_0, t_0 + \tau_0)$ and let v be an optimal control for $x := \gamma(t)$, at time t. Set

$$\tilde{u}(s) = \begin{cases} u(s), & \text{if } s < t, \\ v(s), & \text{if } s \geq t. \end{cases}$$

Then, it is clear that $\tilde{u}$ is an optimal control for x_0, at time t_0. Indeed, we have $\varphi(t_0, x_0) \leq \tau^{t_0,x_0,\tilde{u}} + g(\gamma_\tau^{t_0,x_0,\tilde{u}}) = t - t_0 + \varphi(t, x)$ and so, thanks to Lemma 7.4, the control $\tilde{u}$ is in fact optimal. Hence, by Corollary 7.17, $\tilde{u}$ is continuous, which proves that $u(t) = v(t) := q$. The fact that $u(s) = v(s)$, for all $s \geq t$, follows from the uniqueness of solution to the system (7.25) with initial conditions $\gamma(t) = x$ and $u(t) = q$. $\square$

We recall that a vector $(h, p) \in \mathbb{R}^+ \times \mathbb{R}^d$ is called a *reachable gradient* of φ at $(t, x) \in \mathbb{R}^+ \times \mathring{\Omega}$ if there is a sequence $\{(t_k, x_k)\}_k$ such that φ is differentiable at (t_k, x_k) for each $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} (t_k, x_k) = (t, x), \qquad \lim_{k\to\infty} D\varphi(t_k, x_k) = (h, p).$$

The set of all reachable gradients of φ at (t, x) is denoted by $D^\star\varphi(t, x)$. It is easily seen that $D^\star\varphi(t, x)$ is a compact set: it is closed by definition and it is bounded since φ is Lipschitz. From Rademacher's Theorem it follows that $D^\star\varphi(t, x) \neq \emptyset$ for every $(t, x) \in \mathbb{R}^+ \times \Omega$. We have the following

PROPOSITION 7.24. *Let t_0, x_0, γ, u, τ_0 and p be as in the proposition 7.19. Fix $t_1 \in (t_0, t_0 + \tau_0)$ and set $x_1 := \gamma(t_1)$. Suppose that $p(t_1) \in \Pi_x(D^\star\varphi(t_1, x_1))$, then $p(t) \in \Pi_x(D^\star\varphi(t, \gamma(t)))$, for all $t \in [t_1, t_0 + \tau_0]$.*

PROOF. If $p(t_1) \in \Pi_x(D^\star\varphi(t_1, x_1))$, then there is a sequence $(t_{1,n}, x_{1,n}) \in \mathbb{R}^+ \times \Omega$ such that $t_{1,n} \to t_1$, $x_{1,n} \to x_1$ and φ is differentiable at $(t_{1,n}, x_{1,n})$ with $\nabla\varphi(t_{1,n}, x_{1,n}) \to p(t_1)$. As φ is differentiable at $(t_{1,n}, x_{1,n})$, then, by Corollary 7.21, φ is differentiable at $(t, \gamma_n(t))$, for all $t \in [t_{1,n}, t_{1,n} + \tau_{1,n})$, where γ_n is an optimal trajectory for $x_{1,n}$, at time $t_{1,n}$, and $\tau_{1,n} = \tau^{t_{1,n},x_{1,n},u_n}$; u_n being the associated optimal control with γ_n. Let p_n be the solution of (7.26), associated with γ_n, with initial condition $p_n(t_{1,n}) = \nabla\varphi(t_{1,n}, x_{1,n})$, i.e., p_n is the solution of the following system

$$\begin{cases} p_n'(t) = -\nabla_x k(t,\gamma_n(t))\, u_n(t)\cdot p_n(t), & t\in[t_{1,n}, t_{1,n}+\tau_{1,n}] \\ p_n(t_{1,n}) = \nabla\varphi(t_{1,n}, x_{1,n}). \end{cases}$$

By Proposition 7.19, we have $p_n(t) = \nabla\varphi(t,\gamma_n(t))$ for all $t\in[t_{1,n}, t_{1,n}+\tau_{1,n})$. Yet, it is clear, from Lemma 7.22 & Proposition 7.23, that $u_n \to u$ and $\gamma_n\to\gamma$ uniformly, where u is the unique optimal control for x_1, at time t_1, and γ is its associated optimal trajectory. So, we also have $p_n \to p$ uniformly. Now, fix $t\in[t_1, t_0+\tau_0]$ and let $(t_n)_n$ be any sequence such that $t_n\in(t_{1,n}, t_{1,n}+\tau_{1,n})$, for all n, and $t_n\to t$. As $p_n(t_n) = \nabla\varphi(t_n,\gamma_n(t_n))$, we get that $p(t) = \lim_n \nabla\varphi(t_n,\gamma_n(t_n))$, which means that $p(t)\in\Pi_x(D^\star\varphi(t,\gamma(t)))$. □

PROPOSITION 7.25. *Given $(t_0,x_0)\in\mathbb{R}^+\times\Omega$, let $\gamma:[t_0,t_0+\tau_0]\mapsto\Omega$ be an optimal trajectory for x_0, at time t_0, where $\tau_0=\tau^{t_0,x_0,u}$; u being the associated optimal control. Then, φ is differentiable at all points $(t,\gamma(t))$, with $t\in(t_0,t_0+\tau_0)$.*

PROOF. Let us argue by contradiction and suppose that $D^+\varphi(t,\gamma(t))$ is not a singleton for some $t\in(t_0,t_0+\tau_0)$. Then, $D^\star\varphi(t,\gamma(t))$ contains at least two elements, say $(p_{t,0},p_{x,0})$ and $(p_{t,1},p_{x,1})$. Yet, from Proposition 7.6, we see that different elements of $D^\star\varphi(t,\gamma(t))$ have different space components, i.e. $p_{x,0}\neq p_{x,1}$. For any $\theta\in(0,1)$, we have $(1-\theta)(p_{t,0},p_{x,0})+\theta(p_{t,1},p_{x,1})\in D^+\varphi(t,\gamma(t))$ and so, recalling again Proposition 7.6, one has the following

$$-p_{t,0}+k(t,\gamma(t))\,|p_{x,0}|-1=0$$

$$-p_{t,1}+k(t,\gamma(t))\,|p_{x,1}|-1=0$$

and

$$-(1-\theta)p_{t,0}-\theta p_{t,1}+k(t,\gamma(t))\,|(1-\theta)p_{x,0}+\theta p_{x,1}|-1=0.$$

Hence, we have

$$|(1-\theta)p_{x,0}+\theta p_{x,1}| = (1-\theta)|p_{x,0}|+\theta|p_{x,1}|,$$

which implies that $p_{x,1}=\alpha\,p_{x,0}$, for some $\alpha>0$, $\alpha\neq1$. Now, let p_0 and p_1 be the solutions of (7.26), associated to the optimal (γ,u), with initial conditions $p_0(t)=p_{x,0}$ and $p_1(t)=p_{x,1}$, respectively. By the uniqueness of solution to (7.26), we see that $p_1=\alpha\,p_0$. In particular, we have $p_1(t_0+\tau_0)=\alpha\,p_0(t_0+\tau_0)$. Yet, by Proposition 7.24, we know that both $p_0(t_0+\tau_0)$ and $p_1(t_0+\tau_0)$ belong to $\Pi_x(D^\star\varphi(t_0+\tau_0,\gamma(t_0+\tau_0)))$. As $\varphi(t,x)=g(x)$ at every $(t,x)\in\mathbb{R}^+\times\partial\Omega$, then φ is differentiable with respect to t on $\mathbb{R}^+\times\partial\Omega$ and, $\partial_t\varphi=0$. This implies that $\Pi_t(D^\star\varphi(t_0+\tau_0,\gamma(t_0+\tau_0)))=\{0\}$. Hence, we obtain, using Proposition 7.6, that if $q_0,q_1\in\Pi_x(D^\star\varphi(t_0+\tau_0,\gamma(t_0+\tau_0)))$, then $|q_0|=|q_1|$. This implies that $|p_0(t_0+\tau_0)| = |p_1(t_0+\tau_0)| = \alpha|p_0(t_0+\tau_0)|$, which is a contradiction as $\alpha\neq1$. Hence, φ is differentiable at $(t,\gamma(t))$, for all $t\in(t_0,t_0+\tau_0)$. □

As a consequence of Theorem 7.15, Propositions 7.18 & 7.25, one can characterize an optimal control u in terms of the normalized gradient, with respect to x, of the value function φ. So, we finish this section by the following

COROLLARY 7.26. *Let $(t_0, x_0) \in \mathbb{R}^+ \times \Omega$ and $\gamma = \gamma^{t_0,x_0,u}$ be an optimal trajectory for x_0, at time t_0, where u is the associated optimal control. Then, for all $t \in (t_0, t_0 + \tau_0)$, where $\tau_0 := \tau^{t_0,x_0,u}$, one has*

$$\gamma'(t) = -k(t, \gamma(t)) \frac{\nabla_x \varphi(t, \gamma(t))}{|\nabla_x \varphi(t, \gamma(t))|}.$$

7.4. Sharp semi-concavity

In this section, we investigate the hypotheses under which the value function φ of our exit-time optimal control problem is semi-concave with respect to x. Actually, we will refine the semi-concavity result given in [37] by showing that, instead of assuming (7.27), only a lower bound on $\partial_t k$ (and eventually, we assume that k is $C^{1,1}$ with respect to x) is sufficient to get the semi-concavity of φ with respect to x. On the other hand, we note that this property is related not only to the regularity of the data, but also to the smoothness of the target $\partial\Omega$. We begin by the following

PROPOSITION 7.27. *For every x_0, $x_1 \in \Omega$ and $t_0 \in \mathbb{R}^+$, we have, under the assumptions (7.3) & (7.24), the following estimate*

$$\left|\gamma^{t_0,x_0,u}(t) \,+\, \gamma^{t_0,x_1,u}(t) \,-\, 2\,\gamma^{t_0,\frac{x_0+x_1}{2},u}(t)\right| \leq c|x_0 - x_1|^2, \quad \textit{for all } \; t \in [t_0, \infty),$$

where $c := c(t - t_0) > 0$ and $u : [t_0, \infty) \mapsto \bar{B}(0, 1)$ is any control strategy.

PROOF. First of all, we have

$$\left|\gamma^{t_0,x_0,u}(t) \,+\, \gamma^{t_0,x_1,u}(t) \,-\, 2\,\gamma^{t_0,\frac{x_0+x_1}{2},u}(t)\right|$$

$$= \left| \int_{t_0}^t k(s, \gamma^{t_0,x_0,u}(s))\, u(s)\, \mathrm{d}s \,+\, \int_{t_0}^t k(s, \gamma^{t_0,x_1,u}(s))\, u(s)\, \mathrm{d}s - 2 \int_{t_0}^t k(s, \gamma^{t_0,\frac{x_0+x_1}{2},u}(s))\, u(s)\, \mathrm{d}s \right|.$$

Then,

$$\left|\gamma^{t_0,x_0,u}(t) \,+\, \gamma^{t_0,x_1,u}(t) \,-\, 2\,\gamma^{t_0,\frac{x_0+x_1}{2},u}(t)\right|$$

$$\leq \int_{t_0}^{t} \left| k(s, \gamma^{t_0,x_0,u}(s)) + k(s, \gamma^{t_0,x_1,u}(s)) - 2k(s, \gamma^{t_0,\frac{x_0+x_1}{2},u}(s)) \right| \mathrm{d}s$$

$$\leq \int_{t_0}^{t} \left| k(s, \gamma^{t_0,x_0,u}(s)) + k(s, \gamma^{t_0,x_1,u}(s)) - 2k\left(s, \frac{\gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s)}{2}\right) \right| \mathrm{d}s$$

$$+ \quad 2\int_{t_0}^{t} \left| k\left(s, \frac{\gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s)}{2}\right) - k(s, \gamma^{t_0,\frac{x_0+x_1}{2},u}(s)) \right| \mathrm{d}s.$$

Yet, by (7.3), one has

$$2\int_{t_0}^{t} \left| k\left(s, \frac{\gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s)}{2}\right) - k(s, \gamma^{t_0,\frac{x_0+x_1}{2},u}(s)) \right| \mathrm{d}s$$

$$\leq L_x \int_{t_0}^{t} \left| \gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s) - 2\,\gamma^{t_0,\frac{x_0+x_1}{2},u}(s) \right| \mathrm{d}s.$$

On the other hand, as k is $C^{1,1}$ with respect to x, we can estimate the first term as follows

$$\left| k(s, \gamma^{t_0,x_0,u}(s)) + k(s, \gamma^{t_0,x_1,u}(s)) - 2k\left(s, \frac{\gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s)}{2}\right) \right|$$

$$\leq C|\gamma^{t_0,x_0,u}(s) - \gamma^{t_0,x_1,u}(s)|^2.$$

Hence, by Proposition 7.9, we obtain

$$\int_{t_0}^{t} \left| k(s, \gamma^{t_0,x_0,u}(s)) + k(s, \gamma^{t_0,x_1,u}(s)) - 2k\left(s, \frac{\gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s)}{2}\right) \right| \mathrm{d}s \leq C_1\,|x_1 - x_0|^2$$

and then,

$$\left| \gamma^{t_0,x_0,u}(t) + \gamma^{t_0,x_1,u}(t) - 2\,\gamma^{t_0,\frac{x_0+x_1}{2},u}(t) \right|$$

$$\leq C_1\,|x_1 - x_0|^2 + C \int_{t_0}^{t} \left| \gamma^{t_0,x_0,u}(s) + \gamma^{t_0,x_1,u}(s) - 2\gamma^{t_0,\frac{x_0+x_1}{2},u}(s) \right| \mathrm{d}s.$$

We conclude by using the Gronwall's inequality. □

We recall that the property (7.10) (or more generally, a uniform exterior ball condition on Ω) implies the semi-concavity of the distance function $d(\cdot, \overline{\mathbb{R}^d \backslash \Omega})$ in $\bar{\Omega}$. Actually, the semi-concavity of this distance function is needed in the proof of the semi-concavity of the value function φ. To prove this semi-concavity result on φ, we need to assume that (7.2), (7.3), (7.5), (7.10), (7.11), (7.12), (7.24) and (7.28) are satisfied. In addition, we suppose, instead of (7.27), that we only have a lower bound on the derivative of the dynamic k with respect to t, i.e.,

$$\partial_t k \geq -c. \tag{7.31}$$

Then, we have the following

Proposition 7.28. *The value function φ is semi-concave w.r.t. x, and the semi-concavity constant depends only on λ, $k_{\min}$, $k_{\max}$, κ, L_x, L_{xx}, M and $||\,[\partial_t k]^-\,||_\infty$, where κ is a bound on the curvatures of $\partial\Omega$, L_x is the Lipschitz constant of k with respect to x, L_{xx} is the Lipschitz constant of $\nabla_x k$ with respect to x and M is a constant such that $D^2 g \leq MI$.*

Proof. First of all, we consider arbitrary x, $x-h$, $x+h \in \Omega$ and we suppose, for simplicity of exposition, that $t_0 = 0$. We want to construct suitable trajectories steering the points x, $x-h$, $x+h$ to the target $\partial\Omega$. More precisely, let us take a control u which is optimal for x, at time 0, and consider the trajectories $\gamma^{0,x,u}$, $\gamma^{0,x-h,u}$, $\gamma^{0,x+h,u}$ starting from the points x, $x-h$, $x+h$, at time 0, associated with the same control u. We treat separately two different cases depending on which of these trajectories reaches the target first.

- $\tau_0 := \tau^{0,x,u} \leq \min\{\tau^{0,x-h,u}, \tau^{0,x+h,u}\}$:

Since u is optimal for x, at time 0, we have by the dynamic programming principle (see Lemma 7.4) that

$$\varphi(0,x-h) + \varphi(0,x+h) - 2\,\varphi(0,x) \;\leq\; \varphi(\tau_0,x^-) + \varphi(\tau_0,x^+) - 2g(\gamma_\tau^{0,x,u}),$$

where, for simplicity, we set

$$x^+ := \gamma^{0,x+h,u}(\tau_0) \quad \text{and} \quad x^- := \gamma^{0,x-h,u}(\tau_0).$$

Let us now take u^+, u^- two optimal controls for x^+ and x^-, at time τ_0, respectively. Let us set for simplicity $y^\pm = \gamma_\tau^{\tau_0,x^\pm,u^\pm}$ to denote the terminal points of the corresponding trajectories. Hence, we get

$$\varphi(\tau_0,x^-) + \varphi(\tau_0,x^+) - 2g(\gamma_\tau^{0,x,u}) = \tau^- + g(y^-) + \tau^+ + g(y^+) - 2g(\gamma_\tau^{0,x,u}),$$

where $\tau^{\pm} := \tau^{\tau_0,x^{\pm},u^{\pm}}$. Yet, by Lemma 7.7 & Proposition 7.8, we have

$$\tau^{\pm} \leq c\, d(x^{\pm}, \overline{\mathbb{R}^d \backslash \Omega}).$$

As the distance function $d(\,.\,,\overline{\mathbb{R}^d \backslash \Omega})$ is semi-concave, and taking into account that $\gamma_\tau^{0,x,u} \in \partial\Omega$, we obtain that

$$d(x^+, \overline{\mathbb{R}^d \backslash \Omega}) \;+\; d(x^-, \overline{\mathbb{R}^d \backslash \Omega})$$

$$= d(x^+, \overline{\mathbb{R}^d \backslash \Omega}) + d(x^-, \overline{\mathbb{R}^d \backslash \Omega}) - 2d\bigg(\frac{x^+ + x^-}{2}, \overline{\mathbb{R}^d \backslash \Omega}\bigg) + 2\bigg(d\bigg(\frac{x^+ + x^-}{2}, \overline{\mathbb{R}^d \backslash \Omega}\bigg) - d(\gamma_\tau^{0,x,u}, \overline{\mathbb{R}^d \backslash \Omega})\bigg)$$

$$\leq c|x^+ - x^-|^2 \;+\; |x^+ + x^- - 2\gamma_\tau^{0,x,u}| \leq\; ch^2,$$

where the last inequality follows from Propositions 7.9 & 7.27. On the other hand, from the assumptions on g, we have

$$g(y^-) + g(y^+) - 2\,g(\gamma_\tau^{0,x,u}) \;\;=\;\; g(y^-) + g(y^+) - 2g\bigg(\frac{y^+ + y^-}{2}\bigg) + 2\,\bigg(g\bigg(\frac{y^+ + y^-}{2}\bigg) - g(\gamma_\tau^{0,x,u})\bigg)$$

$$\leq c|y^+ - y^-|^2 + \lambda|y^+ + y^- - 2\,\gamma_\tau^{0,x,u}|. \tag{7.32}$$

Yet,

$$|y^+ - y^-| \leq |y^+ - x^+| + |x^+ - x^-| + |x^- - y^-| \leq |y^+ - x^+| + |x^- - y^-| + c|h|.$$

In addition, we have

$$|y^{\pm} - x^{\pm}| = \bigg|\int_{\tau_0}^{\tau_0+\tau^{\pm}} k(s, \gamma^{\tau_0,x^{\pm},u^{\pm}}(s))\, u^{\pm}(s)\,\mathrm{d}s\bigg| \leq k_{\max}\,\tau^{\pm} \leq ch^2,$$

which implies that

$$|y^+ - y^-| \leq c|h|.$$

For the second term in (7.32), we have

$$|y^+ + y^- - 2\,\gamma_\tau^{0,x,u}| \le |y^+ - x^+| + |x^+ + x^- - 2\,\gamma_\tau^{0,x,u}| + |x^- - y^-| \le ch^2.$$

Consequently,

$$\varphi(0,x-h) \,+\, \varphi(0,x+h) \,-\, 2\,\varphi(0,x) \le ch^2.$$

- $\tau_0 := \tau^{0,x-h,u} \le \min\{\tau^{0,x,u}, \tau^{0,x+h,u}\}$:

The analysis of this case will conclude the proof since the remaining case which is $\tau^{0,x+h,u} \le \min\{\tau^{0,x,u}, \tau^{0,x-h,u}\}$ is entirely symmetric. By the dynaminc programming principle (see Lemma 7.4), we have

$$\varphi(0,x-h) \,+\, \varphi(0,x+h) \,-\, 2\,\varphi(0,x) \quad \le \quad \varphi(\tau_0,x_2) - 2\,\varphi(\tau_0,x_1) + g(x_0),$$

where, for simplicity, we set

$$x_0 = \gamma^{0,x-h,u}(\tau_0), \qquad x_1 = \gamma^{0,x,u}(\tau_0), \qquad x_2 = \gamma^{0,x+h,u}(\tau_0).$$

We recall that u is also an optimal control starting from x_1, at time τ_0, with $\tau_1 := \tau^{\tau_0,x_1,u} = \tau^{0,x,u} - \tau_0$. As $x_0 \in \partial\Omega$, then, by Lemma 7.7, Propositions 7.8 & 7.9, we get that

$$\tau_1 \le c\, d(x_1, \overline{\mathbb{R}^d \backslash \Omega}) \le c|x_1 - x_0| \le c|h|.$$

We will use the control $u^\star(t) := u(\frac{t+\tau_0}{2})$ for the point x_2, at time τ_0. We have again two cases which require a separate analysis:

- $\tau_1 < \frac{\tau^{\tau_0,x_2,u^\star}}{2}$:

By the dynamic programming principle (see Lemma 7.4), we have

$$\varphi(\tau_0,x_2) - 2\,\varphi(\tau_0,x_1) + g(x_0) \le \varphi(\tau_0+2\tau_1, \gamma^{\tau_0,x_2,u^\star}(\tau_0+2\,\tau_1)) \,+\, g(x_0) \,-\, 2g(\gamma_\tau^{\tau_0,x_1,u}).$$

Let $z_1 = \gamma_\tau^{\tau_0,x_1,u} \in \partial\Omega$ and $z_2 = \gamma^{\tau_0,x_2,u^\star}(\tau_0+2\,\tau_1)$. Let v be an optimal control for z_2, at time $\tau_0+2\tau_1$, then, by Lemma 7.7 & Proposition 7.8, we have

$$\varphi(\tau_0 + 2\tau_1, z_2) \;+\; g(x_0) \;-\; 2g(z_1)$$

$$= \tau^{\tau_0+2\tau_1,z_2,v} + g(\gamma_\tau^{\tau_0+2\tau_1,z_2,v}) \;+\; g(x_0) \;-\; 2g(z_1)$$

$$\le c\, d(z_2, \overline{\mathbb{R}^d \backslash \Omega}) + g(\gamma_\tau^{\tau_0+2\tau_1,z_2,v}) \;+\; g(x_0) \;-\; 2g(z_1)$$

$$= c\, d(z_2, \overline{\mathbb{R}^d \backslash \Omega}) + 2\left(g\left(\frac{x_0 + \gamma_\tau^{\tau_0+2\tau_1,z_2,v}}{2} \right) - g(z_1) \right) + g(\gamma_\tau^{\tau_0+2\tau_1,z_2,v}) + g(x_0) - 2g\left(\frac{x_0 + \gamma_\tau^{\tau_0+2\tau_1,z_2,v}}{2} \right).$$

From (7.5) & (7.28), we infer that

$$\varphi(\tau_0 + 2\tau_1, z_2) + g(x_0) - 2g(z_1) \le c\, d(z_2, \overline{\mathbb{R}^d \backslash \Omega}) + c\, |\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - x_0|^2 + \lambda\, |x_0 + \gamma_\tau^{\tau_0+2\tau_1,z_2,v} - 2z_1|.$$

Yet, using Proposition 7.9, we have

$$|\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - x_0| \le |\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - z_2| + |z_2 - x_2| + |x_2 - x_0| \le |\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - z_2| + |z_2 - x_2| + c|h|.$$

In addition, by Lemma 7.7 & Proposition 7.8, one has

$$|\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - z_2| = \left| \int_{\tau_0+2\tau_1}^{\tau_0+2\tau_1+\tau_v} k\Big(s, \gamma^{\tau_0+2\tau_1,z_2,v}(s)\Big)\, v(s)\, \mathrm{d}s \right| \le k_{\max}\, \tau_v \le c\, d(z_2, \overline{\mathbb{R}^d \backslash \Omega}),$$

where $\tau_v := \tau^{\tau_0+2\tau_1,z_2,v}$. In the same way, we have

$$|z_2 - x_2| = \left| \int_{\tau_0}^{\tau_0+2\tau_1} k\Big(s, \gamma^{\tau_0,x_2,u^\star}(s)\Big)\, u^\star(s)\, \mathrm{d}s \right| \le 2k_{\max}\, \tau_1 \le c\, |h|.$$

Moreover,

$$|x_0 + \gamma_\tau^{\tau_0+2\tau_1,z_2,v} - 2z_1| \le |x_0 + z_2 - 2z_1| \;+\; |\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - z_2|$$

and

$$|\gamma_\tau^{\tau_0+2\tau_1,z_2,v} - z_2| \le c\, d(z_2, \overline{\mathbb{R}^d \backslash \Omega}).$$

Hence, it remains to prove that

$$d(z_2, \overline{\mathbb{R}^d \backslash \Omega}) \,+\, |x_0 + z_2 - 2z_1| \leq ch^2.$$

Firstly, let us note that

$$d(z_2, \overline{\mathbb{R}^d \backslash \Omega}) \leq |z_2 - 2z_1 + x_0| \,+\, d(2z_1 - x_0, \overline{\mathbb{R}^d \backslash \Omega}).$$

Yet,

$$d(2z_1 - x_0, \overline{\mathbb{R}^d \backslash \Omega}) = d(2z_1 - x_0, \overline{\mathbb{R}^d \backslash \Omega}) \,+\, d(x_0, \overline{\mathbb{R}^d \backslash \Omega}) \,-\, 2d(z_1, \overline{\mathbb{R}^d \backslash \Omega}),$$

as $x_0,\, z_1 \in \partial\Omega$. Hence, by the semi-concavity of the distance function $d(\,.\,, \overline{\mathbb{R}^d \backslash \Omega})$ and, using Proposition 7.9, we get that

$$d(2z_1 - x_0, \overline{\mathbb{R}^d \backslash \Omega}) \leq c|z_1 - x_0|^2 \leq c(|z_1 - x_1| + |x_1 - x_0|)^2 \leq ch^2,$$

since, we have

$$|z_1 - x_1| = \left| \int_{\tau_0}^{\tau_0+\tau_1} k\Big(s, \gamma^{\tau_0, x_1, u}(s)\Big)\, u(s)\, \mathrm{d}s \right| \leq k_{\max}\, \tau_1 \leq c|h|.$$

On the other hand, let $\mathbf{n}$ be the unit outward normal vector at z_1 and let $w := u(\tau_0 + \tau_1) = -\frac{\nabla g(z_1) - \mu \mathbf{n}}{|\nabla g(z_1) - \mu \mathbf{n}|}$ be the unit optimal control vector at z_1, at time $\tau_0 + \tau_1$ (where μ is the unique constant so that $k(\tau_0 + \tau_1, z_1)|\nabla g(z_1) - \mu \mathbf{n}| = 1$; see Lemma 7.14). Then, there is a unit vector e orthogonal to w such that

$$2z_1 - x_0 - z_2 = \alpha \mathbf{n} + \beta e.$$

In fact,

$$|e \cdot \mathbf{n}| = |w \cdot \mathbf{t}| = \left| \frac{\nabla g(z_1) - \mu \mathbf{n}}{|\nabla g(z_1) - \mu \mathbf{n}|} \cdot \mathbf{t} \right| = \left| \frac{\nabla g(z_1)}{|\nabla g(z_1) - \mu \mathbf{n}|} \cdot \mathbf{t} \right| = k(\tau_0 + \tau_1, z_1)|\nabla g(z_1) \cdot \mathbf{t}|,$$

where $\mathbf{t}$ is some unitary tangent vector on the boundary at z_1. And, this implies that

$$|e \cdot \mathbf{n}| \leq \lambda k_{\max} < 1. \tag{7.33}$$

We have

$$(2z_1 - x_0 - z_2) \cdot \mathbf{n} = \alpha + \beta\, e \cdot \mathbf{n}$$

and

$$(2z_1 - x_0 - z_2) \cdot e = \alpha\, e \cdot \mathbf{n} + \beta.$$

Then,

$$\begin{pmatrix} (2z_1 - x_0 - z_2) \cdot \mathbf{n} \\ (2z_1 - x_0 - z_2) \cdot e \end{pmatrix} = \begin{pmatrix} 1 & e \cdot \mathbf{n} \\ e \cdot \mathbf{n} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

or equivalently,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & e \cdot \mathbf{n} \\ e \cdot \mathbf{n} & 1 \end{pmatrix}^{-1} \begin{pmatrix} (2z_1 - x_0 - z_2) \cdot \mathbf{n} \\ (2z_1 - x_0 - z_2) \cdot e \end{pmatrix}.$$

One has

$$|2z_1 - x_0 - z_2| \leq \frac{C}{1 - (e \cdot \mathbf{n})^2} \Big(|(2z_1 - x_0 - z_2) \cdot \mathbf{n}| + |(2z_1 - x_0 - z_2) \cdot e| \Big),$$

where the denominator can be estimated thanks to (7.33). Firstly, we note that

$$z_2 - 2z_1 + x_0$$

$$= x_0 + x_2 - 2x_1 + \int_{\tau_0}^{\tau_0 + 2\tau_1} k(s, \gamma^{\tau_0, x_2, u^\star}(s))\, u^\star(s)\, \mathrm{d}s - 2 \int_{\tau_0}^{\tau_0 + \tau_1} k(s, \gamma^{\tau_0, x_1, u}(s))\, u(s)\, \mathrm{d}s$$

$$= x_0 + x_2 - 2x_1 + \int_{\tau_0}^{\tau_0 + 2\tau_1} k(s, \gamma^{\tau_0, x_2, u^\star}(s))\, u\Big(\frac{s + \tau_0}{2}\Big)\, \mathrm{d}s - 2 \int_{\tau_0}^{\tau_0 + \tau_1} k(s, \gamma^{\tau_0, x_1, u}(s))\, u(s)\, \mathrm{d}s$$

$$= x_0 + x_2 - 2x_1 + 2 \int_{\tau_0}^{\tau_0 + \tau_1} \Big(k(2s - \tau_0, \gamma^{\tau_0, x_2, u^\star}(2s - \tau_0)) - k(s, \gamma^{\tau_0, x_1, u}(s)) \Big)\, u(s)\, \mathrm{d}s.$$

Hence,

$$(z_2 - 2z_1 + x_0) \cdot e$$

$$= (x_0 + x_2 - 2x_1) \cdot e + 2 \int_{\tau_0}^{\tau_0 + \tau_1} \Big(k(2s - \tau_0, \gamma^{\tau_0, x_2, u^\star}(2s - \tau_0)) - k(s, \gamma^{\tau_0, x_1, u}(s)) \Big)\, u(s) \cdot e\, \mathrm{d}s.$$

Yet, from Proposition 7.27, we have

$$|(x_0 + x_2 - 2x_1) \cdot e| \leq |x_0 + x_2 - 2x_1| \leq ch^2.$$

To estimate the second term, let us observe, first, that

$$|u(s) - w| = |u(s) - u(\tau_0 + \tau_1)| = \left| \int_s^{\tau_0+\tau_1} u'(t)\,\mathrm{d}t \right|$$

and then, using (7.25), we get, for all $s \in [\tau_0, \tau_0 + \tau_1]$, that

$$\begin{aligned} |u(s) - w| &= \left| \int_s^{\tau_0+\tau_1} \Big(-\nabla_x k(t, \gamma^{0,x,u}(t)) + (u(t) \cdot \nabla_x k(t, \gamma^{0,x,u}(t)))\, u(t) \Big)\,\mathrm{d}t \right| \\ &\leq c(\tau_0 + \tau_1 - s) \leq c|h|. \end{aligned}$$

This implies that

$$|u(s) \cdot e| = |(u(s) - w) \cdot e| \leq c|h|.$$

Hence,

$$\left| \int_{\tau_0}^{\tau_0+\tau_1} \Big(k(2s - \tau_0, \gamma^{\tau_0,x_2,u^\star}(2s - \tau_0)) - k(s, \gamma^{\tau_0,x_1,u}(s)) \Big)\, u(s) \cdot e\ \mathrm{d}s \right|$$

$$\leq 2\,k_{\max} \int_{\tau_0}^{\tau_0+\tau_1} |u(s) \cdot e|\,\mathrm{d}s \leq c|h|\tau_1 \leq ch^2.$$

Consequently,

$$|(z_2 - 2z_1 + x_0) \cdot e| \leq ch^2.$$

On the other hand, we want to show that

$$|(2z_1 - x_0 - z_2) \cdot \mathbf{n}| \leq ch^2.$$

Let x_0' (resp. z_2') be the projection of x_0 (resp. z_2) into the tangent space on $\partial\Omega$ at z_1.

Then, we have

$$(2z_1 \ - \ x_0 - z_2) \cdot \mathbf{n} = (x_0' - x_0) \cdot \mathbf{n} + (z_2' - z_2) \cdot \mathbf{n}.$$

From (7.10) and the fact that $|z_1 - x_0| \leq c|h|$, it is easy to see that

$$|x_0' - x_0| \leq ch^2.$$

Moreover, as $z_2 \in \Omega$ and $|z_2 - z_1| \leq c|h|$, then

$$(z_2' - z_2) \cdot \mathbf{n} \geq -ch^2.$$

Consequently,

$$|(2z_1 \ - \ x_0 - z_2) \cdot \mathbf{n}| \leq (2z_1 \ - \ x_0 - z_2) \cdot \mathbf{n} + ch^2.$$

Yet,

$$(2z_1 \ - \ x_0 - z_2) \cdot \mathbf{n}$$

$$= -(x_0 + x_2 - 2x_1) \cdot \mathbf{n} - 2\int_{\tau_0}^{\tau_0+\tau_1} \Big(k(2s - \tau_0, \gamma^{\tau_0,x_2,u^\star}(2s - \tau_0)) \ - \ k(s, \gamma^{\tau_0,x_1,u}(s))\Big)\, u(s) \cdot \mathbf{n} \ \mathrm{d}s.$$

From Proposition 7.27, we get again that

$$-(x_0 \ + \ x_2 \ - \ 2x_1) \cdot \mathbf{n} \leq |x_0 \ + \ x_2 \ - \ 2x_1| \leq ch^2.$$

For the second term, we have

$$-\int_{\tau_0}^{\tau_0+\tau_1} \Big(k(2s - \tau_0, \gamma^{\tau_0,x_2,u^\star}(2s - \tau_0)) \ - \ k(s, \gamma^{\tau_0,x_1,u}(s))\Big)\, u(s) \cdot \mathbf{n} \ \mathrm{d}s$$

$$= -\int_{\tau_0}^{\tau_0+\tau_1} \Big(k(2s - \tau_0, \gamma^{\tau_0,x_2,u^\star}(2s - \tau_0)) \ - \ k(s, \gamma^{\tau_0,x_2,u^\star}(2s - \tau_0))\Big)\, u(s) \cdot \mathbf{n} \ \mathrm{d}s$$

$$- \quad \int_{\tau_0}^{\tau_0+\tau_1} \Big(k(s, \gamma^{\tau_0,x_2,u^\star}(2s - \tau_0)) \ - \ k(s, \gamma^{\tau_0,x_1,u}(s))\Big)\, u(s) \cdot \mathbf{n} \ \mathrm{d}s.$$

From (7.3), we have

$$\left| \int_{\tau_0}^{\tau_0+\tau_1} \Big(k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \,-\, k(s,\gamma^{\tau_0,x_1,u}(s)) \Big)\, u(s)\cdot \mathbf{n}\; \mathrm{d}s \right|$$

$$\leq C_1 \int_{\tau_0}^{\tau_0+\tau_1} |\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0) \,-\, \gamma^{\tau_0,x_1,u}(s)|\,\mathrm{d}s.$$

Yet, by Proposition 7.9, one has

$$\begin{aligned}
&\left|\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0) \,-\, \gamma^{\tau_0,x_1,u}(s)\right| \\
&= \left| x_2 \,+\, \int_{\tau_0}^{2s-\tau_0} k(t,\gamma^{\tau_0,x_2,u^\star}(t))\, u^\star(t)\,\mathrm{d}t \,-\, x_1 \,-\, \int_{\tau_0}^{s} k(t,\gamma^{\tau_0,x_1,u}(t))\, u(t)\,\mathrm{d}t \right| \\
&\leq |x_2-x_1| + \int_{\tau_0}^{2s-\tau_0} k(t,\gamma^{\tau_0,x_2,u^\star}(t))\,\mathrm{d}t \,+\, \int_{\tau_0}^{s} k(t,\gamma^{\tau_0,x_1,u}(t))\,\mathrm{d}t \\
&\leq c|h| + 3\,k_{\max}(s-\tau_0).
\end{aligned}$$

Hence, we get

$$\left| \int_{\tau_0}^{\tau_0+\tau_1} \Big(k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \,-\, k(s,\gamma^{\tau_0,x_1,u}(s)) \Big)\, u(s)\cdot \mathbf{n}\; \mathrm{d}s \right|$$

$$\leq \int_{\tau_0}^{\tau_0+\tau_1} (c|h| \,+\, 3\,k_{\max}\,(s-\tau_0))\,\mathrm{d}s \leq ch^2.$$

On the other hand, we have

$$w\cdot\mathbf{n} = -\frac{\nabla g(z_1)-\mu\mathbf{n}}{|\nabla g(z_1)-\mu\mathbf{n}|}\cdot\mathbf{n} = k(\tau_0+\tau_1,z_1)(-\nabla g(z_1)\cdot\mathbf{n}+\mu).$$

Yet,

$$|\nabla g(z_1)-\mu\mathbf{n}|^2 = \mu^2 + |\nabla g(z_1)|^2 - 2\mu\,\nabla g(z_1)\cdot\mathbf{n} = \frac{1}{k(\tau_0+\tau_1,z_1)^2}.$$

Then,

$$2\mu(-\nabla g(z_1)\cdot\mathbf{n}+\mu) = \frac{1}{k(\tau_0+\tau_1,z_1)^2} - |\nabla g(z_1)|^2 + \mu^2 > \frac{1}{{k_{\max}}^2} - \lambda^2 > 0.$$

Finally, using (7.31), one has

$$-\int_{\tau_0}^{\tau_0+\tau_1}\left(k(2s-\tau_0,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\,-\,k(s,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\right)u(s)\cdot\mathbf{n}\;\mathrm{d}s$$

$$=-\int_{\tau_0}^{\tau_0+\tau_1}\left(k(2s-\tau_0,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\,-\,k(s,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\right)w\cdot\mathbf{n}\;\mathrm{d}s$$

$$-\int_{\tau_0}^{\tau_0+\tau_1}\left(k(2s-\tau_0,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\,-\,k(s,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\right)(u(s)-w)\cdot\mathbf{n}\;\mathrm{d}s$$

$$\le\int_{\tau_0}^{\tau_0+\tau_1}\int_s^{2s-\tau_0}-k_t(t,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\,w\cdot\mathbf{n}\,\mathrm{d}t\,\mathrm{d}s\,+\,ch^2\le c(\tau_1^2+h^2)\le ch^2.$$

- $\tau_2 := \tau^{\tau_0,x_2,u^\star} \le 2\tau_1$:

Set again

$$z_1 := \gamma_\tau^{\tau_0,x_1,u} \in \partial\Omega.$$

In this case, we have

$$\varphi(0,x-h)\,+\,\varphi(0,x+h)\,-\,2\varphi(0,x)$$

$$\le\varphi(\tau_0,x_2)-2\varphi(\tau_0,x_1)+g(x_0)\le\tau_2+g(\gamma_\tau^{\tau_0,x_2,u^\star})-2\tau_1-2g(z_1)+g(x_0)$$

$$=\tau_2-2\tau_1+2\left(g\left(\frac{x_0+\gamma_\tau^{\tau_0,x_2,u^\star}}{2}\right)-g(z_1)\right)+\quad g(\gamma_\tau^{\tau_0,x_2,u^\star})\,+\,g(x_0)\,-\,2g\left(\frac{x_0+\gamma_\tau^{\tau_0,x_2,u^\star}}{2}\right).$$

From (7.28), we get

$$\varphi(\tau_0,x_2)\,-\,2\varphi(\tau_0,x_1)\,+\,g(x_0)$$

$$\le\tau_2\,-\,2\tau_1\,+\,c\,|\gamma_\tau^{\tau_0,x_2,u^\star}-x_0|^2\,+\,2\left(g\left(\frac{x_0+\gamma_\tau^{\tau_0,x_2,u^\star}}{2}\right)-g(z_1)\right).$$

Using Proposition 7.9, we obtain that

$$|\gamma_\tau^{\tau_0,x_2,u^\star} - x_0| \leq |\gamma_\tau^{\tau_0,x_2,u^\star} - x_2| + |x_2 - x_0| \leq |\gamma_\tau^{\tau_0,x_2,u^\star} - x_2| + c|h|.$$

Yet,

$$|\gamma_\tau^{\tau_0,x_2,u^\star} - x_2| = \left| \int_{\tau_0}^{\tau_0+\tau_2} k(s, \gamma^{\tau_0,x_2,u^\star}(s))\, u^\star(s)\, \mathrm{d}s \right| \leq 2k_{\max}\, \tau_1 \leq c|h|.$$

On the other hand, we have

$$g\left(\frac{x_0 + \gamma_\tau^{\tau_0,x_2,u^\star}}{2}\right) - g(z_1) \leq \frac{1}{2}\, \nabla g(z_1) \cdot (x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1) + O(|x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1|^2).$$

But, it is clear that

$$\begin{aligned} |x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1| \quad &\leq \quad |x_0 + x_2 - 2x_1| + 2\int_{\tau_0}^{\tau_0+\tau_1} |k(s, \gamma^{\tau_0,x_1,u}(s))\, u(s)|\, \mathrm{d}s \\ &+ \quad \int_{\tau_0}^{\tau_0+\tau_2} |k(s, \gamma^{\tau_0,x_2,u^\star}(s))\, u^\star(s)|\, \mathrm{d}s, \end{aligned}$$

which implies, using Proposition 7.27, that

$$|x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1| \leq ch^2 + 4k_{\max}\, \tau_1 \leq c|h|.$$

So, it remains to prove that

$$\tau_2 - 2\tau_1 + \nabla g(z_1) \cdot (x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1) \leq ch^2.$$

Let $\mathbf{n}$ be the unit outward normal vector at z_1. Then, there is a vector e, orthogonal to $\mathbf{n}$, such that

$$\nabla g(z_1) = \alpha \mathbf{n} + \beta e$$

with $\alpha^2 + \beta^2 \leq \lambda^2$. Hence, we get

$$\begin{aligned} \nabla g(z_1) \cdot (x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1) \quad &= \quad (\alpha \mathbf{n} + \beta e) \cdot (x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1) \\ &= \quad \alpha \mathbf{n} \cdot (x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1) + \beta e \cdot (x_0 + \gamma_\tau^{\tau_0,x_2,u^\star} - 2z_1). \end{aligned}$$

Recall that

$$|x_0 - z_1| + |z_1 - \gamma_\tau^{\tau_0,x_2,u^\star}| \leq c|h|.$$

From (7.10), we infer that

$$\alpha\,\mathbf{n}\cdot(x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z_1)\le ch^2.$$

Now, set

$$z:=\gamma^{\tau_0,x_1,u}\Big(\tau_0+\frac{\tau_2}{2}\Big).$$

Then, one has

$$\tau_2\,-\,2\tau_1\,+\,\beta e\cdot(x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z_1)$$
$$=\tau_2\,-\,2\tau_1\,+\,\beta e\cdot(x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z)+\,2\beta e\cdot(z-z_1).$$

Let us observe that

$$|z-z_1|=\Big|\int_{\tau_0+\frac{\tau_2}{2}}^{\tau_0+\tau_1}k(s,\gamma^{\tau_0,x_1,u}(s))\,u(s)\,\mathrm{d}s\Big|\le k_{\max}\Big(\tau_1-\frac{\tau_2}{2}\Big).$$

Using $k_{\max}\,|\beta|\le k_{\max}\,\lambda<1$, we infer that

$$\tau_2\,-\,2\tau_1\,+\,\beta e\,\cdot\,(x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z)\,+\,2\beta e\,\cdot\,(z\,-\,z_1)\,\le\,\beta e\,\cdot\,(x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z).$$

So, the aim, now, is to prove that

$$\beta e\cdot(x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z)\le ch^2.$$

Firstly, let us observe that

$$x_0\,+\,\gamma_\tau^{\tau_0,x_2,u^\star}\,-\,2z$$

$$=x_0\,+\,x_2\,-\,2x_1\,-\,2\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}}k(s,\gamma^{\tau_0,x_1,u}(s))\,u(s)\,\mathrm{d}s+\int_{\tau_0}^{\tau_0+\tau_2}k(s,\gamma^{\tau_0,x_2,u^\star}(s))\,u^\star(s)\,\mathrm{d}s$$

$$=x_0\,+\,x_2\,-\,2x_1\,-\,2\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}}k(s,\gamma^{\tau_0,x_1,u}(s))\,u(s)\,\mathrm{d}s+\int_{\tau_0}^{\tau_0+\tau_2}k(s,\gamma^{\tau_0,x_2,u^\star}(s))\,u\Big(\frac{s+\tau_0}{2}\Big)\,\mathrm{d}s$$

$$=x_0\,+\,x_2\,-\,2x_1\,+\,2\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}}\Big(k(2s-\tau_0,\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))-k(s,\gamma^{\tau_0,x_1,u}(s))\Big)u(s)\,\mathrm{d}s$$

$$= x_0 + x_2 - 2x_1 + 2\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\Big) u(s)\,\mathrm{d}s$$

$$+ 2\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_1,u}(s))\Big) u(s)\,\mathrm{d}s.$$

Recall that

$$|x_2 + x_0 - 2x_1| \leq ch^2.$$

From (7.3), we infer that

$$\left|\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_1,u}(s))\Big) u(s)\,\mathrm{d}s\right|$$

$$\leq L_x \int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \left|\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0) - \gamma^{\tau_0,x_1,u}(s)\right| \mathrm{d}s.$$

Yet, by Proposition 7.9, we have

$$\left|\gamma^{\tau_0,x_2,u^\star}(2s-\tau_0) - \gamma^{\tau_0,x_1,u}(s)\right|$$

$$= \left|x_2 + \int_{\tau_0}^{2s-\tau_0} k(t, \gamma^{\tau_0,x_2,u^\star}(t))\, u^\star(t)\,\mathrm{d}t - x_1 - \int_{\tau_0}^{s} k(t, \gamma^{\tau_0,x_1,u}(t))\, u(t)\,\mathrm{d}t\right|$$

$$\leq |x_2 - x_1| + \left|\int_{\tau_0}^{2s-\tau_0} k(t, \gamma^{\tau_0,x_2,u^\star}(t))\, u^\star(t)\,\mathrm{d}t\right| + \left|\int_{\tau_0}^{s} k(t, \gamma^{\tau_0,x_1,u}(t))\, u(t)\,\mathrm{d}t\right|$$

$$\leq c|h| + 3k_{\max}(s-\tau_0).$$

Consequently, we get

$$\left|\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_1,u}(s))\Big)\, u(s)\,\mathrm{d}s\right| \leq ch^2.$$

On the other hand,

$$\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \Big)\, u(s)\,\mathrm{d}s$$

$$= \int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \Big)\, u(\tau_0+\tau_1)\,\mathrm{d}s$$

$$+ \quad \int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \Big)\, (u(s)-u(\tau_0+\tau_1))\,\mathrm{d}s$$

$$\leq \int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \Big)\, u(\tau_0+\tau_1)\,\mathrm{d}s \,+\, ch^2.$$

We recall that

$$u(\tau_0+\tau_1) = -\frac{\nabla g(z_1) - \mu \mathbf{n}}{|\nabla g(z_1) - \mu \mathbf{n}|},$$

and so, using (7.31), we get

$$\int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \Big)\, u(\tau_0+\tau_1)\cdot \beta e\,\mathrm{d}s$$

$$= -k(\tau_0+\tau_1, z_1)\,\beta^2 \int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \Big(k(2s-\tau_0, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) - k(s, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0)) \Big)\,\mathrm{d}s$$

$$= \; k(\tau_0+\tau_1, z_1)\,\beta^2 \int_{\tau_0}^{\tau_0+\frac{\tau_2}{2}} \int_{s}^{2s-\tau_0} -k_t(t, \gamma^{\tau_0,x_2,u^\star}(2s-\tau_0))\,\mathrm{d}t\,\mathrm{d}s \leq ch^2. \quad \square$$

We finish this section by the following remark

Remark 7.29. *One can give an example showing that a lower bound on the derivative of the dynamic k with respect to t is actually sharp! To see that, let Ω be the unit ball in $\mathbb{R}^d$. Let κ be a differentiable real function with $0 < \kappa_{\min} \leq \kappa \leq \kappa_{\max} < +\infty$. Set $k(t,x) := \kappa(t)$, for every $(t,x) \in \mathbb{R}^+ \times \Omega$. For a given $x \in \Omega$, the optimal trajectory for x, at time 0, will be given by*

$$\gamma'(s) = k(s,\gamma(s))\, e(x) = \kappa(s)\, e(x),$$

where $e(x) := x/|x|$. Let φ be the value function associated with this control problem. Hence, *we observe easily that*

$$\int_0^{\varphi(0,x)} \kappa(s)\,\mathrm{d}s = 1 - |x|.$$

Now, set

$$G(T) := \int_0^T \kappa(s)\,\mathrm{d}s, \quad \textit{for all} \;\; T \geq 0$$

and

$$H := G^{-1}.$$

This yields that

$$\varphi(0,x) = H(1 - |x|).$$

Consequently, we have

$$D^2\varphi(0,x) = H''(1-|x|)\, e(x) \otimes e(x) - \frac{H'(1-|x|)}{|x|}(I - e(x) \otimes e(x)),$$

where

$$H' = \frac{1}{\kappa} \circ H \quad \textit{and} \quad H'' = -\frac{\kappa'}{\kappa^3} \circ H.$$

This shows that $D^2\varphi$ cannot be bounded from above unless κ' is bounded from below.

CHAPTER 8

Minimal time Mean Field Games

Mean field games (MFG) theory has been introduced simultaneously by Lasry and Lions [79, 80, 81] and by Huang, Malhamé and Caines [68, 69] in order to study large population differential games. In this chapter, we are interested in the study of a MFG model motivated by the crowd motion. The understanding of fast exit and evacuation situations in crowd motion research has received a lot of scientific interest in the last decades (see, for instance, [28]). More precisely, we present a MFG model where agents want to leave a given bounded domain in minimal time. Each agent is free to move in any direction, but its maximal speed is assumed to be bounded in terms of the density of agents in order to take into account congestion phenomena. We attack the problem by interpreting equilibria as measures in a space of arcs. In such a relaxed setting, the existence of an equilibrium, in the case of a regular dynamic k, follows by set-valued fixed point arguments (see also [87]). Then, we give the forward-backward system of PDEs which couples a Hamilton-Jacobi equation (for the value function φ of the generic agent) with a continuity equation (for the density ρ of agents). The equilibrium that we will found turns out to be a solution of the following system of PDEs

$$(8.1)\qquad \begin{cases} \partial_t\rho - \nabla\cdot\left(\rho\, k\, \frac{\nabla\varphi}{|\nabla\varphi|}\right) = 0 & \text{in } \mathbb{R}^+\times\Omega, \\ -\partial_t\varphi + k\,|\nabla\varphi| = 1 & \text{in } \mathbb{R}^+\times\Omega, \\ \rho(0)=\rho_0 \ \text{ in } \Omega, \quad \varphi = g & \text{on } \mathbb{R}^+\times\partial\Omega, \end{cases}$$

where $k : \mathcal{P}(\Omega)\times\Omega \mapsto \mathbb{R}^+$ is the dynamic and $g : \partial\Omega \mapsto \mathbb{R}^+$ is a given boundary cost. We note that it is much simpler here to use the Lagrangean approach, instead of using the Schauder fixed point Theorem applied to the Eulerian formulation (i.e. doing a fixed point on the PDE system, as in [38]), to prove existence of a solution for (8.1); this is due to the fact that the velocity $v = -k\frac{\nabla\varphi}{|\nabla\varphi|}$ is a priori non-regular. On the other hand, we are able to give L^p estimates on the density of agents ρ_t, at each time t, as soon as the initial density ρ_0 belongs to L^p. Moreover, thanks to these L^p estimates, we extend the result of existence of an equilibrium to a case where the dynamic k is non-regular.

This chapter will be a part of a joint paper with G. Mazanti, in preparation, [57].

8.1. Existence of equilibria in the regular case

Let Ω be a compact domain in $\mathbb{R}^d$. We denote by $\mathcal{P}(\Omega)$ the family of all Borel probability measures on Ω. Let $k : \mathcal{P}(\Omega)\times\Omega \mapsto \mathbb{R}^+$ be a continuous function. We consider the following optimal-exit problem: agents evolve in Ω, their distribution at time t being given by the probability measure ρ_t. The goal of each agent is to leave the boundary $\partial\Omega$ in minimal time (i.e., paying a minimal cost that is assumed to be given by the time to reach a possible exit-point plus a boundary cost g at this point, where $g : \partial\Omega \mapsto \mathbb{R}^+$ is a given continuous function), but we assume the speed of an agent in a position x at time t to be bounded by $k(\rho_t, x)$. This means that, for a given agent, its trajectory γ depends on the distribution of all agents ρ_t, since the speed of γ should not exceed $k(\rho_t, x)$. On the other hand, the distribution of the agents ρ_t itself depends on how agents choose their trajectories γ. Thus, we are interested here in the equilibrium situations, i.e., in situations where, starting from a time evolution of the density of agents $\rho : \mathbb{R}^+ \to \mathcal{P}(\Omega)$, the trajectories γ chosen by agents induce an evolution of the initial distribution of agents ρ_0 that is precisely given by ρ. For every point $x \in \Omega$, we consider the following problem

$$\inf \left\{ \tau_\gamma + g(\gamma_\tau) \ : \ \gamma \in \Gamma[\rho, x] \right\}, \tag{8.2}$$

where

$$\Gamma[\rho, x] := \left\{ \gamma \in \Gamma,\ \gamma(0) = x,\ |\gamma'(s)| \le k(\rho_s, \gamma(s)) \ \text{ for a.e. } s \in (0, \tau_\gamma) \ \text{ and } \ \gamma'(s) = 0 \ \ \forall \ s > \tau_\gamma \right\},$$

$$\tau_\gamma := \inf\{ s \ge 0 \ : \ \gamma(s) \in \partial\Omega \},$$

$$\gamma_\tau := \gamma(\tau_\gamma) \in \partial\Omega,$$

and Γ is the space of all continuous curves from $\mathbb{R}^+$ to Ω, equipped with the topology of uniform convergence on compact sets, with respect to which Γ is a Polish space (see, for instance, [**17**]).

REMARK 8.1. *If γ belongs to $\Gamma[\rho, x]$, then there is a control $u : \mathbb{R}^+ \mapsto \bar{B}(0,1)$ such that*

$$\begin{cases} \gamma'(t) = k(\rho_t, \gamma(t))\, u(t), & \textit{for a.e. } t, \\ \gamma(0) = x. \end{cases} \tag{8.3}$$

Moreover, (8.3) can be seen as a control system (see Chapter 7) where the dynamic is given by $\tilde{k}(t, x) = k(\rho_t, x)$ for every $(t, x) \in \mathbb{R}^+ \times \Omega$. This point of view allows one to formulate (8.2) as an optimal control problem.

Let us state some assumptions on the data. It is reasonable to suppose that the dynamic k is bounded from above, since if this is not the case, the speed of an agent would be $+\infty$, which is not natural at all. For simplicity, and in order to affirm that there is at least one admissible trajectory γ, starting from a point x, that reaches the boundary in finite time, we want to suppose also that the dynamic k is bounded from below. Thus, we assume (as in Chaptre 7) that

$$0 < k_{\min} := \inf k \le k_{\max} := \sup k < +\infty.$$

Moreover, we suppose, as in the previous chapter, that the cost g is λ-Lipschitz with $\lambda < 1/k_{\max}$. In this way, from Proposition 7.2, we infer that (8.2) reaches a minimum.

Following almost the same ideas proposed in [**11, 34**], we define a "relaxed" notion of MFG equilibria, for which we give existence result. Such a formulation consists of replacing curves of probability measures on Ω with measures on arcs in Ω. For any $t \in \mathbb{R}^+$, we denote by $e_t : \Gamma \mapsto \Omega$ the evaluation map defined by

$$e_t(\gamma) = \gamma(t), \text{ for all } \gamma \in \Gamma.$$

Let $\mathcal{P}(\Gamma)$ be the set of all probability measures on Γ. For any $\eta \in \mathcal{P}(\Gamma)$, we define the curve ρ^η of probability measures on Ω as follows

$$\rho^\eta(t) = (e_t)_\# \eta, \text{ for all } t \in \mathbb{R}^+.$$

Since $e_t : \Gamma \mapsto \Omega$ is continuous, we observe that if $\eta_n, \eta \in \mathcal{P}(\Gamma)$, $n \geq 1$, is such that $\eta_n \rightharpoonup \eta$, then $\rho^{\eta_n}(t) \rightharpoonup \rho^\eta(t)$ for all $t \in \mathbb{R}^+$. For any fixed $\rho_0 \in \mathcal{P}(\Omega)$, we denote by $\mathcal{P}_{\rho_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $(e_0)_\# \eta = \rho_0$.

REMARK 8.2. *We note that $\mathcal{P}_{\rho_0}(\Gamma)$ is nonempty. Indeed, let $j : \Omega \mapsto \Gamma$ be the continuous map defined by*

$$j(x)(t) = x, \quad \textit{for all } t \in \mathbb{R}^+.$$

Then,

$$\eta := j_\# \rho_0 \ \textit{ belongs to } \ \mathcal{P}_{\rho_0}(\Gamma).$$

For all $x \in \Omega$ and $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$, we define

$$\Gamma'[\rho^\eta, x] := \left\{ \gamma \in \Gamma[\rho^\eta, x] \ : \ J(\gamma) = \min_{\Gamma[\rho^\eta, x]} J \right\},$$

where

$$J(\gamma) := \tau_\gamma + g(\gamma_\tau).$$

DEFINITION 8.3. *Let $\rho_0 \in \mathcal{P}(\Omega)$. We say that $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$ is a MFG equilibrium for ρ_0 if*

$$\operatorname{spt}(\eta) \subseteq \bigcup_{x \in \Omega} \Gamma'[\rho^\eta, x].$$

In fact, we are able to prove that $\bigcup_x \Gamma'[\rho^\eta, x]$ is closed and so, $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$ is a MFG equilibrium for ρ_0 if for $\eta-$a.e. $\bar\gamma \in \Gamma$, we have

$$J(\bar\gamma) \leq J(\gamma), \quad \text{for all } \gamma \in \Gamma[\rho^\eta, \bar\gamma(0)].$$

We recall that k is continuous on $\mathcal{P}(\Omega) \times \Omega$. Moreover, we assume that

$$|k(\mu, x_0) - k(\mu, x_1)| \leq C|x_0 - x_1|, \quad \text{for all } x_0, x_1 \in \Omega, \mu \in \mathcal{P}(\Omega).$$

Then, we have the following main result.

THEOREM 8.4. *Under these assumptions, there exists a MFG equilibrium for* ρ_0.

Before proving this theorem, we introduce the following

LEMMA 8.5. *Let* $\eta_n, \eta \in \mathcal{P}_{\rho_0}(\Gamma)$ *be such that* $\eta_n \rightharpoonup \eta$. *Let* $x_n \in \Omega$ *be such that* $x_n \to x$ *and let* $\gamma_n \in \Gamma'[\rho^{\eta_n}, x_n]$ *be such that* $\gamma_n \to \bar{\gamma}$. *Then* $\bar{\gamma} \in \Gamma'[\rho^{\eta}, x]$. *Consequently,* $(\eta, x) \mapsto \Gamma'[\rho^{\eta}, x]$ *has a closed graph.*

PROOF. We set, for simplicity, $\tau_n := \tau_{\gamma_n}$ and $z_n := \gamma_n(\tau_n)$. Using Lemma 7.7 & Proposition 7.8, $(\tau_n)_n$ is bounded and, up to a subsequence, τ_n converges to some $\bar{\tau}$. On the other hand, we see easily that $\bar{\gamma} \in \mathrm{Lip}(\mathbb{R}^+, \Omega)$ with $|\bar{\gamma}'| \leq k_{\max}$. In addition, we have

$$|\gamma_n'(t)| \leq k(\rho^{\eta_n}(t), \gamma_n(t)), \quad \text{for a.e. } t \in (0, \tau_n).$$

Letting $n \to +\infty$, we get that

$$|\bar{\gamma}'(t)| \;\leq\; k(\rho^{\eta}(t), \bar{\gamma}(t)), \quad \text{for a.e. } t \in (0, \bar{\tau}).$$

In the same way, one can prove that $\bar{\gamma}'(t) = 0$ for all $t > \bar{\tau}$. Moreover, we have

$$z_n \to \bar{\gamma}(\bar{\tau}),$$

which implies that $\bar{\gamma}(\bar{\tau}) \in \partial\Omega$ and $\tau := \tau_{\bar{\gamma}} \leq \bar{\tau}$. Define the trajectory $\gamma \in \Gamma[\rho^{\eta}, x]$ as follows

$$\gamma(t) = \begin{cases} \bar{\gamma}(t), & \text{if} \quad t \leq \tau, \\ \bar{\gamma}(\tau), & \text{if} \quad t > \tau. \end{cases}$$

Suppose that $\gamma \notin \Gamma'[\rho^{\eta}, x]$. Then, there is a trajectory $\widehat{\gamma} \in \Gamma'[\rho^{\eta}, x]$ such that $J(\widehat{\gamma}) < J(\gamma)$, which means that we have

$$\tau_{\widehat{\gamma}} + g(\widehat{\gamma}_{\tau}) < \tau + g(\gamma_{\tau}).$$

For each n, let $\tilde{\gamma}_n$ be a geodesic (which is a segment unless $x \in \partial\Omega$) between x_n and x such that $|\tilde{\gamma_n}'| = 1$ and let ϕ_n be a function satisfying

$$\begin{cases} \phi_n'(t) = \frac{k(\rho^{\eta_n}(t), \widehat{\gamma}(\phi_n(t)))}{k(\rho^{\eta}(\phi_n(t)), \widehat{\gamma}(\phi_n(t)))}, \\ \phi_n(k_{\min}^{-1} d(x_n, x)) = 0. \end{cases}$$

Define

$$\widehat{\gamma_n}(t) = \begin{cases} \tilde{\gamma}_n(k_{\min} t) & \text{for all } \; t \in [0, k_{\min}^{-1} d(x_n, x)], \\ \widehat{\gamma}(\phi_n(t)) & \text{else.} \end{cases}$$

It is clear that $\widehat{\gamma_n} \in \Gamma[\rho^{\eta_n}, x_n]$. Moreover, we have

$$J(\widehat{\gamma_n}) = \phi_n^{-1}(\tau_{\widehat{\gamma}}) + g(\widehat{\gamma}_\tau). \tag{8.4}$$

On the other hand, we see that there exists a limit ϕ such that, up to a subsequence, $\phi_n \to \phi$ uniformly on compact sets of $\mathbb{R}^+$. In addition, it is easy to check that for all $t \in \mathbb{R}^+$,

$$\int_0^{\phi_n(t)} k(\rho^\eta(s), \widehat{\gamma}(s))\, \mathrm{d}s = \int_{k_{\min}^{-1} d(x_n, x)}^{t} k(\rho^{\eta_n}(s), \widehat{\gamma}(\phi_n(s)))\, \mathrm{d}s.$$

So, letting $n \to +\infty$, we get

$$\int_0^{\phi(t)} k(\rho^\eta(s), \widehat{\gamma}(s))\, \mathrm{d}s = \int_0^t k(\rho^\eta(s), \widehat{\gamma}(\phi(s)))\, \mathrm{d}s.$$

Set

$$G(\theta) = \int_0^\theta k(\rho^\eta(s), \widehat{\gamma}(s))\, \mathrm{d}s, \ \ \forall\ \theta \in \mathbb{R}^+.$$

One has

$$\begin{aligned} |\phi(t) - t| &= \left| G^{-1}\left(\int_0^t k(\rho^\eta(s), \widehat{\gamma}(\phi(s)))\, \mathrm{d}s \right) - G^{-1}\left(\int_0^t k(\rho^\eta(s), \widehat{\gamma}(s))\, \mathrm{d}s \right) \right| \\ &\leq C \int_0^t |k(\rho^\eta(s), \widehat{\gamma}(\phi(s))) - k(\rho^\eta(s), \widehat{\gamma}(s))|\, \mathrm{d}s \\ &\leq C \int_0^t |\phi(s) - s|\, \mathrm{d}s. \end{aligned}$$

By using Gronwall's Lemma, we get that

$$\phi(t) = t, \ \text{ for all } \ t \in \mathbb{R}^+.$$

Passing to the limit in (8.4), we get

$$\lim_n J(\widehat{\gamma_n}) = \lim_n \phi_n^{-1}(\tau_{\widehat{\gamma}}) + g(\widehat{\gamma}_\tau) = J(\widehat{\gamma}) < J(\gamma). \tag{8.5}$$

As g is $\lambda-$Lip with $\lambda < 1/k_{\max}$, then

$$J(\gamma) \leq \bar{\tau} + g(\bar{\gamma}(\bar{\tau})).$$

Yet,

$$\lim_n J(\gamma_n) = \lim_n \tau_n + g(z_n) = \bar{\tau} + g(\bar{\gamma}(\bar{\tau})).$$

Using (8.5), we infer that, for n large enough,

$$J(\widehat{\gamma_n}) < J(\gamma_n),$$

which is a contradiction, as $\widehat{\gamma_n} \in \Gamma[\rho^{\eta_n}, x_n]$ and $\gamma_n \in \Gamma'[\rho^{\eta_n}, x_n]$. In the same way, we see that $\bar{\tau} = \tau$. Then, $\bar{\gamma} \in \Gamma'[\rho^\eta, x]$. $\square$

Now, we want to prove Theorem 8.4 using a fixed point argument. We introduce the set-valued map $E : \mathcal{P}_{\rho_0}(\Gamma) \rightrightarrows \mathcal{P}_{\rho_0}(\Gamma)$ by defining, for any $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$,

$$E(\eta) = \left\{ \widetilde{\eta} \in \mathcal{P}_{\rho_0}(\Gamma) \ : \ \operatorname{spt}(\widetilde{\eta}) \subseteq \bigcup_{x \in \Omega} \Gamma'[\rho^\eta, x] \right\}.$$

Then, it is immediate to realize that $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$ is a MFG equilibrium for ρ_0 if and only if $\eta \in E(\eta)$. We will therefore show that the set-valued map E has a fixed point. For this purpose, we will apply Kakutani's Theorem [72]. The following lemmas are intended to check that the assumptions of such a theorem are satisfied by E.

LEMMA 8.6. *For any $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$, $E(\eta)$ is a nonempty convex set.*

PROOF. First, we note that $E(\eta)$ is a nonempty set. Indeed, by Lemma 8.5 and [5, 40], we have that $x \mapsto \Gamma'[\rho^\eta, x]$ has a Borel measurable selection $\boldsymbol{\gamma}^\eta : x \mapsto \gamma^\eta_x \in \Gamma'[\rho^\eta, x]$. Thus, the measure $\widehat{\eta}$, defined by $\widehat{\eta} := \boldsymbol{\gamma}^\eta_{\#}\rho_0$ belongs to $E(\eta)$. On the other hand, it is clear that $E(\eta)$ is a convex set. $\square$

LEMMA 8.7. *The multimap $E : \mathcal{P}_{\rho_0}(\Gamma) \rightrightarrows \mathcal{P}_{\rho_0}(\Gamma)$ has a closed graph.*

PROOF. Let $\eta_n, \eta \in \mathcal{P}_{\rho_0}(\Gamma)$ be such that $\eta_n \rightharpoonup \eta$. Let $\widehat{\eta_n} \in E(\eta_n)$ be such that $\widehat{\eta_n} \rightharpoonup \widehat{\eta}$. Since $\widehat{\eta_n} \rightharpoonup \widehat{\eta}$, we have that $\widehat{\eta} \in \mathcal{P}_{\rho_0}(\Gamma)$. For $k \in \mathbb{N}^\star$, let $V_k := \{\gamma \in \Gamma \ : \ d(\gamma, \cup_x \Gamma'[\rho^\eta, x]) \leq \frac{1}{k}\}$. By Lemma 8.5, we see that there exists a neighborhood W of η such that $\bigcup_x \Gamma'[\rho^{\tilde{\eta}}, x] \subset V_k$, for every $\tilde{\eta} \in W$. Then, for n large enough, $\bigcup_x \Gamma'[\rho^{\eta_n}, x] \subset V_k$. Since $\widehat{\eta_n}(\bigcup_x \Gamma'[\rho^{\eta_n}, x]) = 1$, one obtains that $\widehat{\eta_n}(V_k) = 1$, for large n. Yet, $\widehat{\eta_n} \rightharpoonup \widehat{\eta}$ and V_k is closed, it follows that $\widehat{\eta}(V_k) \geq \limsup_n \widehat{\eta_n}(V_k) = 1$ and thus, $\widehat{\eta}(V_k) = 1$. As this holds for every $k \in \mathbb{N}^\star$, then one concludes that $\widehat{\eta}(\bigcup_x \Gamma'[\rho^\eta, x]) = 1$. Hence $\widehat{\eta} \in E(\eta)$, which proves that the graph of E is closed. $\square$

Let us denote by $\Gamma_{k_{\max}}$ the set of trajectories $\gamma \in \Gamma$ such that γ is $k_{\max}-$Lipschitz, i.e.,

$$\Gamma_{k_{\max}} = \{\gamma \in \Gamma \ : \ |\gamma'| \leq k_{\max}\}.$$

By the definition of $E(\eta)$, we deduce that

$$E(\eta) \subset \mathcal{P}_{\rho_0}(\Gamma_{k_{\max}}), \quad \text{for all } \eta \in \mathcal{P}_{\rho_0}(\Gamma).$$

REMARK 8.8. *Notice that $\mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$ is a compact convex subset of $\mathcal{P}_{\rho_0}(\Gamma)$. The convexity of $\mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$ follows immediately. As for compactness, let $(\eta_k)_k \subset \mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$. Since $\Gamma_{k_{\max}}$ is a compact set, $(\eta_k)_k$ is tight. So, by Prokhorov's Theorem, one finds a subsequence which converges weakly to some probability measure $\eta \in \mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$.*

So, we will restrict our domain of interest to $\mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$. Hereafter, we denote by E the restriction $E_{|\mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})}$.

Conclusion:

By Remark 8.2 and Remark 8.8, $\mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$ is a nonempty compact convex set. Moreover, by Lemma 8.6, $E(\eta)$ is a nonempty convex set for any $\eta \in \mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$ and, by Lemma 8.7, the set-valued map E has a closed graph. Then, the assumptions of Kakutani's Theorem are satisfied and so, there exists $\eta \in \mathcal{P}_{\rho_0}(\Gamma_{k_{\max}})$ such that $\eta \in E(\eta)$.

We finish this section by characterizing the density $\rho_t := \rho^\eta(t) = (e_t)_\#\eta$, for some equilibrium η, as a solution of a continuity equation of the form $\partial_t\rho + \nabla_x \cdot (\rho\, v) = 0$ for a particular velocity field v. Let φ be the value function (see Chapter 7), associated to the control problem with dynamic $\tilde{k}(t,x) = k(\rho_t, x)$, for all $(t,x) \in \mathbb{R}^+ \times \Omega$. Then, under the assumption that $\tilde{k} \in C^{1,1}$, we have the following:

PROPOSITION 8.9. *Let $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$ be an equilibrium for ρ_0. Then $\rho : t \mapsto \rho^\eta(t)$ is a solution of*

$$\partial_t\rho(t,x) - \nabla \cdot \left(\rho(t,x)\, k(\rho_t,x)\, \frac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|}\right) = 0 \quad \textit{in} \quad (0,\infty) \times \mathring{\Omega}. \tag{8.6}$$

PROOF. Let $\phi \in \mathcal{C}_c^\infty((0,\infty) \times \mathring{\Omega})$. Then, recalling Proposition 7.25 & Corollary 7.26, we have

$$-\int_0^{+\infty}\int_\Omega \partial_t\phi(t,x)\, \mathrm{d}\rho_t(x)\, \mathrm{d}t + \int_0^{+\infty}\int_\Omega k(\rho_t,x)\, \nabla_x\phi(t,x) \cdot \frac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|}\, \mathrm{d}\rho_t(x)\, \mathrm{d}t$$

$$= -\int_0^{+\infty}\int_\Gamma \partial_t\phi(t,\gamma(t))\, \mathrm{d}\eta(\gamma)\, \mathrm{d}t + \int_0^{+\infty}\int_\Gamma k(\rho_t,\gamma(t))\, \nabla_x\phi(t,\gamma(t)) \cdot \frac{\nabla\varphi(t,\gamma(t))}{|\nabla\varphi(t,\gamma(t))|}\, \mathrm{d}\eta(\gamma)\, \mathrm{d}t$$

$$= -\int_0^{+\infty}\int_\Gamma \partial_t\phi(t,\gamma(t))\, \mathrm{d}\eta(\gamma)\, \mathrm{d}t - \int_0^{+\infty}\int_\Gamma \nabla_x\phi(t,\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}\eta(\gamma)\, \mathrm{d}t$$

$$= -\int_\Gamma \int_0^{+\infty} \frac{d}{dt}(\phi(t,\gamma(t)))\,\mathrm{d}t\,\mathrm{d}\eta(\gamma) = 0. \quad \square$$

Let $\eta \in \mathcal{P}_{\rho_0}(\Gamma)$ be an equilibrium, $\rho = \rho^\eta$, i.e. $\rho_t = \rho^\eta(t) = (e_t)_\# \eta$ for every $t \in \mathbb{R}^+$, and φ be the associated value function. Then, by Propositions 7.5 & 8.9, (ρ, φ) solves the following system

$$\begin{cases} \partial_t \rho(t,x) - \nabla \cdot \left(\rho(t,x)\, k(\rho_t, x)\, \dfrac{\nabla_x \varphi(t,x)}{|\nabla_x \varphi(t,x)|} \right) = 0, & (t,x) \in (0,\infty) \times \Omega, \\ -\partial_t \varphi(t,x) + k(\rho_t,x)\, |\nabla_x \varphi(t,x)| = 1, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ \varphi(t,x) = g(x), & (t,x) \in \mathbb{R}^+ \times \partial\Omega, \\ \rho(0,x) = \rho_0(x), & x \in \Omega. \end{cases}$$

8.2. L^p estimates

In this section, we will be interested in the case where the dynamic k is given by the following

$$k(\rho, x) = c\Big(\int_\Omega \chi(x-y)\, \psi(y)\, \mathrm{d}\rho(y) \Big), \text{ for all } (\rho, x) \in \mathcal{P}(\Omega) \times \Omega,$$

where χ is a non-negative $C^{1,1}$ function on $\mathbb{R}^d$, ψ is a *cut-off* function on Ω and c is a positive $C^{1,1}$ decreasing function on $\mathbb{R}^+$. The meaning of this dynamic k is that each agent evaluates an average density of agents around him through the integral term, the convolution kernel χ and the *cut-off* function ψ (which allows us to not take into account agents who have already left the domain, and who remain on $\partial\Omega$), and its maximum speed depends on this evaluation of the density across c.

More precisely, we take $\psi(x) := \alpha(d(x, \partial\Omega))$, for every $x \in \Omega$, where α is a non-negative $C^{1,1}$ increasing function such that $\alpha = 0$ on $[0, \varepsilon/2]$ and $\alpha = 1$ on $[\varepsilon, +\infty)$, where $\varepsilon > 0$ is small enough. As we have $\int_\Omega \chi(x-y)\psi(y)\,\mathrm{d}\rho(y) \le M$, for all $(\rho, x) \in \mathcal{P}(\Omega) \times \Omega$ (we note that the constant M is independent of ε), then

$$0 < c_{\min} := \min_{[0,M]} c \le k \le c_{\max} := \max_{[0,M]} c < +\infty. \tag{8.7}$$

In addition, it is clear that k is (uniformly in ε) $C^{1,1}$ with respect to the variable x and is continuous in ρ. From Theorem 8.4, we infer that, for any $\rho_0 \in \mathcal{P}(\Omega)$, there exists an equilibrium η for ρ_0, associated to our MFG model with the dynamic k. On the other hand, we have $k((e_t)_\#\eta, x) = c(\int_\Gamma \chi(x-\gamma(t))\, \psi(\gamma(t))\, \mathrm{d}\eta(\gamma))$ and so, k is Lipschitz (but, not uniformly in ε) with respect to t. Yet, we want to show that $\partial_t k \ge -C$, for some constant C independent of ε. In fact, one has

$$\partial_t k = c'(\cdot)\Big(-\int_\Gamma \nabla\chi(x-\gamma(t)) \cdot \gamma'(t)\, \psi(\gamma(t))\, \mathrm{d}\eta(\gamma) + \int_\Gamma \chi(x-\gamma(t)) \nabla\psi(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}\eta(\gamma) \Big).$$

We recall that η is concentrated on the optimal trajectories for the control problem associated with the dynamic k. Fix such a trajectory γ (we recall from Corollary 7.17 that γ is $C^{1,1}$) and let u be its associated optimal control. We have

$$\nabla\psi(\gamma(t))\cdot u(t) = \alpha'(d(\gamma(t),\partial\Omega))\,\nabla d(\gamma(t),\partial\Omega)\cdot u(t).$$

From Theorem 7.15, we have

$$u(\tau_\gamma) = -\frac{\nabla g(\gamma(\tau_\gamma)) - \mu\,\mathbf{n}}{|\nabla g(\gamma(\tau_\gamma)) - \mu\,\mathbf{n}|}$$

and so,

$$\nabla d(\gamma(\tau_\gamma),\partial\Omega)\cdot u(\tau_\gamma) = \mathbf{n}\cdot\frac{\nabla g(\gamma(\tau_\gamma)) - \mu\,\mathbf{n}}{|\nabla g(\gamma(\tau_\gamma)) - \mu\,\mathbf{n}|} = k(\tau_\gamma,\gamma(\tau_\gamma))[-\mu + \nabla g(\gamma(\tau_\gamma))\cdot\mathbf{n}].$$

Using Lemma 7.14, we infer that

$$\nabla d(\gamma(\tau_\gamma),\partial\Omega)\cdot u(\tau_\gamma) = \frac{k(\tau_\gamma,\gamma(\tau_\gamma))}{2\mu}\Big(-\mu^2 + |\nabla g(\gamma(\tau_\gamma))|^2 - \frac{1}{k(\tau_\gamma,\gamma(\tau_\gamma))^2}\Big).$$

As $|\nabla g| \le \lambda < \frac{1}{c_{\max}}$ and $\mu \ge \frac{1}{k} - |\nabla g| \ge \frac{1}{c_{\max}} - \lambda$, we get

$$\nabla d(\gamma(\tau_\gamma),\partial\Omega)\cdot u(\tau_\gamma) \le -C,$$

for some constant $C := C(\lambda, c_{\min}, c_{\max}) > 0$. On the other hand, we have

$$\frac{d}{dt}[\nabla d(\gamma(t),\partial\Omega)\cdot u(t)] = k((e_t)_\#\eta,\gamma(t))\,D^2 d(\gamma(t),\partial\Omega)\,u(t)\cdot u(t) + \nabla d(\gamma(t),\partial\Omega)\cdot u'(t).$$

Recalling (7.25), we obtain that, for $\gamma(t)$ close to $\partial\Omega$,

$$\frac{d}{dt}[\nabla d(\gamma(t),\partial\Omega)\cdot u(t)] \ge -M,$$

where M is a constant independent of ε. Hence, we get

$$\nabla d(\gamma(\tau_\gamma),\partial\Omega)\cdot u(\tau_\gamma) - \nabla d(\gamma(t),\partial\Omega)\cdot u(t) \ge -M(\tau_\gamma - t)$$

and so, using Lemma 7.7 & Proposition 7.8, we infer that

$$-C + Md(\gamma(t), \partial\Omega) \geq -C + M(\tau_\gamma - t) \geq \nabla d(\gamma(t), \partial\Omega) \cdot u(t).$$

Consequently, there is some finite constant C, independent of ε, such that $\partial_t k \geq -C$. Thanks to Proposition 7.28, the value function φ, associated to the control problem with the dynamic k, is (uniformly in ε) semi-concave with respect to x.

On the other hand, we see easily that k is $C^{1,1}$ in $\mathbb{R}^+ \times \Omega$. Now, if η is a MFG equilibrium for ρ_0 and $\rho_t = 1_{\mathring{\Omega}} \cdot (e_t)_\# \eta$ for all $t \in \mathbb{R}^+$, then, by Corollary 7.26, we get that the pair (ρ_t, φ) solves the following continuity equation

$$\partial_t \rho_t - \nabla \cdot \left(\rho_t \, k(\rho_t, \cdot) \frac{\nabla \varphi}{|\nabla \varphi|} \right) = 0.$$

So, thanks to the semi-concavity with respect to x of the value function φ, we are able to find L^p estimates on the density of agents ρ_t, at time t. More precisely, we have the following

PROPOSITION 8.10. *For every $t \in \mathbb{R}^+$, $\rho_t \in L^p(\Omega)$ as soon as ρ_0 belongs to $L^p(\Omega)$. Moreover, we have the following estimate*

$$||\rho_t||_{L^p(\Omega)} \leq C ||\rho_0||_{L^p(\Omega)},$$

for some constant C depending only on the semi-concavity constant (w.r.t. x) of the value function φ.

PROOF. First of all, let us define the vector field v as follows: for all $t \in \mathbb{R}^+$, we set

$$v_t(x) := \begin{cases} -k(\rho_t, x) \frac{\nabla \varphi(t,x)}{|\nabla \varphi(t,x)|}, & \text{if } x \in \mathring{\Omega}, \\ 0, & \text{else.} \end{cases}$$

Thanks to the fact that the value function φ is semi-concave with respect to x (see Proposition 7.28), $|\nabla \varphi|$ is bounded from below (see Corollary 7.12), and k is Lipschitz with respect to x, we have $\nabla \cdot v_t \geq -C$. This bound implies a lower bound on the Jacobian determinant J of the flux associated to

$$\begin{cases} X'(t,x) = v_t(X(t,x)), \\ X(0,x) = x. \end{cases}$$

Indeed, if $X \in \mathring{\Omega}$, we have

$$\frac{d}{dt}\nabla X = \nabla v_t(X)\,\nabla X.$$

Setting $J_t := \det(\nabla X(t,\cdot))$, one has

$$\begin{aligned} \frac{d}{dt}J_t(x) &= \nabla \cdot v_t(X(t,x))\,J_t(x) \\ &\geq -C\,J_t(x). \end{aligned}$$

Hence,

$$\log(J_t(x)) - \log(J_0(x)) \geq -Ct.$$

Yet, $J_0 = 1$. So, we get

$$J_t \geq e^{-Ct}, \quad \text{for all } \ t \in \mathbb{R}^+.$$

As η is an equilibrium and $\rho_t = 1_{\mathring{\Omega}} \cdot (e_t)_{\#}\eta$, then, by Corollary 7.26 and the definition of the flow X, we infer that

$$\rho_t = 1_{\mathring{\Omega}} \cdot X(t,\cdot)_{\#}\rho_0,$$

or equivalently,

$$\rho_t = 1_{X_t(\Omega)\cap\mathring{\Omega}} \cdot (\rho_0/J_t) \circ (X_t)^{-1}, \quad \text{for all } \ t \in \mathbb{R}^+.$$

Hence,

$$||\rho_t||^p_{L^p(\Omega)} = \int_{X_t(\Omega)\cap\mathring{\Omega}} \frac{\rho_0((X_t)^{-1}(y))^p}{J_t((X_t)^{-1}(y))^p}\,\mathrm{d}y = \int_{\Omega} \frac{\rho_0(x)^p}{J_t(x)^{p-1}}\,1_{X_t^{-1}(\mathring{\Omega})}(x)\,\mathrm{d}x \leq e^{C(p-1)t}\,||\rho_0||^p_{L^p(\Omega)}.$$

Consequently,

$$||\rho_t||_{L^p(\Omega)} \leq e^{C(1-\frac{1}{p})t}\,||\rho_0||_{L^p(\Omega)}.$$

These estimates are proved, first, for smooth velocity field v and then, by approximation for non-smooth v with $[\nabla \cdot v_t]^- \leq C$ (see [2]), using the a.e. uniqueness of the flow of the non-smooth vector fields v, which comes from the uniqueness of the optimal trajectories in optimal control (see Proposition 7.23). □

8.3. Existence of equilibria for less regular model

Thanks to the previous L^p estimates on ρ_t, we are able to prove existence of an equilibrium η for ρ_0 in the case where the dynamic k is given by

$$k(\rho,x) = c\Big(\int_\Omega \chi(x-y)1_{\overset{\circ}{\Omega}}(y)\,\mathrm{d}\rho(y)\Big), \text{ for all } (\rho,x)\in\mathcal{P}(\Omega)\times\Omega.$$

Notice that the lack of continuity of the dynamic k with respect to ρ prevents us from using the result of Section 8.1. So, the idea will be to consider a sequence of *cut-off* functions $(\psi^\varepsilon)_{\varepsilon>0}$ constructed as in Section 8.2 and converging to $1_{\overset{\circ}{\Omega}}$ in L^q, for all $q<+\infty$, when $\varepsilon\to 0$, and to replace the dynamic k with k_ε, where k_ε is defined as follows

$$k_\varepsilon(\rho,x) = c\Big(\int_\Omega \chi(x-y)\,\psi^\varepsilon(y)\,\mathrm{d}\rho(y)\Big), \text{ for all } (\rho,x)\in\mathcal{P}(\Omega)\times\Omega.$$

In this way, if η^ε is a MFG equilibrium for ρ_0, associated to the control problem with the dynamic k_ε, then we prove that $\eta^\varepsilon \rightharpoonup \eta$ where η is, in fact, a MFG equilibrium for ρ_0, associated to the control problem with the dynamic k. First of all, set $\rho_t^\varepsilon = 1_{\overset{\circ}{\Omega}}\cdot(e_t)_\#\eta^\varepsilon$, for all $t\in\mathbb{R}^+$, $\varepsilon>0$. From Section 8.2, we recall that k_ε is uniformly bounded in ε. Moreover, k_ε is uniformly $C^{1,1}$ with respect to x and, we have

$$\partial_t k_\varepsilon \geq -C,$$

where the constant C is, in fact, independent of ε. As a consequence of that, the value function φ_ε, associated to the control problem with the dynamic k_ε, will be (uniformly in ε) semi-concave with respect to x (see Proposition 7.28). Then, the proposition 8.10 implies the following uniform estimates

$$||\rho_t^\varepsilon||_{L^p} \leq C||\rho_0||_{L^p}, \quad \text{for all } t\in\mathbb{R}^+,\, \varepsilon>0.$$

As $\eta^\varepsilon\in\mathcal{P}(\Gamma)$ and $\mathrm{spt}(\eta^\varepsilon)\subset\Gamma_{c_{\max}}$ (we recall that $\Gamma_{c_{\max}}$ is the set of all $c_{\max}$-Lip curves in Γ, which is by the way a compact subset of it), then, up to a subsequence, $\eta^\varepsilon\rightharpoonup\eta$. Set $\rho_t := 1_{\overset{\circ}{\Omega}}\cdot(e_t)_\#\eta$, for every $t\in\mathbb{R}^+$. Hence, we have $\rho_t^\varepsilon\rightharpoonup\rho_t$ in L^p. In addition, $\psi^\varepsilon\to 1_{\overset{\circ}{\Omega}}$ in L^q, for all $q<+\infty$. Using these facts, we get, for every $(t,x)\in\mathbb{R}^+\times\Omega$, that

$$k_\varepsilon((e_t)_\#\eta^\varepsilon,x) = c\Big(\int_\Omega \chi(x-y)\psi^\varepsilon(y)\rho_t^\varepsilon(y)\,\mathrm{d}y\Big) \to c\Big(\int_\Omega \chi(x-y)\rho_t(y)\,\mathrm{d}y\Big) = k((e_t)_\#\eta,x).$$

Moreover, for any $x\in\Omega$, the function $t\mapsto k((e_t)_\#\eta,x)$ is continuous on $\mathbb{R}^+$. Indeed, if $t_n\to t$, then $\rho_{t_n} \rightharpoonup \rho_t$ in L^p and so, we have

$$k((e_{t_n})_{\#}\eta, x) \;=\; c\bigg(\int_\Omega \chi(x\,-\,y)\rho_{t_n}(y)\,\mathrm{d}y\bigg) \;\to\; c\bigg(\int_\Omega \chi(x\,-\,y)\rho_t(y)\,\mathrm{d}y\bigg) \;=\; k((e_t)_{\#}\eta, x).$$

Let us denote by φ the value function associated to the control problem with the dynamic k. Then, we prove that $\varphi_\varepsilon \to \varphi$ uniformly, which will be sufficient to infer that η is a MFG equilibrium for ρ_0, associated to the control problem with the dynamic k. In fact, we have the following

PROPOSITION 8.11. *Let k_ε, $k : \mathcal{P}(\Omega)\times\Omega \mapsto \mathbb{R}^+$ be such that, for every $\varepsilon > 0$, k_ε is continuous on $\mathcal{P}(\Omega)\times\Omega$ and is Lipschitz with respect to the second variable. Let η^ε be a MFG equilibrium, associated to the control problem with dynamic k_ε, and let η be the limit of η^ε in $\mathcal{P}(\Gamma)$. In addition, assume the following:*

- *There exist two constants $c_{\min}$ and $c_{\max}$ such that $0 < c_{\min} \le k_\varepsilon \le c_{\max} < +\infty$.*
- *There exists a constant M independent of ε such that $|\nabla_x k_\varepsilon| \le M$.*
- *For a.e. $t \in \mathbb{R}^+$, we have $k_\varepsilon((e_t)_{\#}\eta^\varepsilon, \cdot) \to k((e_t)_{\#}\eta, \cdot)$ when $\varepsilon \to 0$.*
- *$t \mapsto k((e_t)_{\#}\eta, \cdot)$ is continuous on $\mathbb{R}^+$.*

If φ_ε (resp. φ) is the value function associated to the control problem with dynamic k_ε (resp. k), then $\varphi_\varepsilon \to \varphi$ uniformly in $\mathbb{R}^+\times\Omega$.

PROOF. First of all, let us see that φ_ε converges uniformly to some function $\tilde{\varphi}$ on $\mathbb{R}^+\times\Omega$. Indeed, from Lemma 7.7 & Proposition 7.8, the sequence $(\varphi_\varepsilon)_\varepsilon$ is equibounded. Moreover, by Proposition 7.10, the value function φ_ε is Lipschitz in $\mathbb{R}^+\times\Omega$ with a Lipschitz constant depending only on the Lipschitz constant of the dynamic k_ε with respect to x, which is by the way uniform in ε. Fix $(t,x) \in \mathbb{R}^+\times\Omega$. For every $\varepsilon > 0$, let γ_ε be an optimal trajectory for x, at time t, in the control problem with dynamic k_ε. It is easy to observe that $\gamma_\varepsilon \to \gamma$ uniformly, for some $\gamma \in \Gamma_{c_{\max}}$. Yet, this γ is, in fact, an admissible trajectory for x, at time t, in the control problem with dynamic k. Indeed, for a.e. $s \in (t,\infty)$, we have

$$|\gamma_\varepsilon'(s)| \le k_\varepsilon((e_s)_{\#}\eta^\varepsilon, \gamma_\varepsilon(s)).$$

So, letting $\varepsilon \to 0$, we get

$$|\gamma'(s)| \le k((e_s)_{\#}\eta, \gamma(s)), \quad \text{for a.e. } s.$$

Let u_ε be the optimal control associated with γ_ε and, set

$$\tau_\varepsilon := \tau^{t,x,u_\varepsilon} \quad \text{and} \quad z_\varepsilon := \gamma_\varepsilon(t+\tau_\varepsilon) \in \partial\Omega.$$

It is clear that $\tau_\varepsilon \to \bar{\tau}$ and $z_\varepsilon \to z \in \partial\Omega$. In particular, we have $z = \gamma(t+\bar{\tau})$ and $\tau_\gamma \le \bar{\tau}$. Consequently, $\varphi(t,x) \le \bar{\tau} + g(z) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(t,x) = \tilde{\varphi}(t,x)$.

On the other hand, let γ be an optimal trajectory for x, at time t, in the control problem with dynamic k, and u be the associated optimal control with γ. Let ϕ_ε be a solution of the following

$$\begin{cases} \phi_\varepsilon'(s) = \frac{k_\varepsilon((e_s)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(s)))}{k((e_{\phi_\varepsilon(s)})_\# \eta, \gamma(\phi_\varepsilon(s)))}, \\ \phi_\varepsilon(t) = t. \end{cases} \tag{8.8}$$

Set

$$\gamma_\varepsilon(s) = \gamma(\phi_\varepsilon(s)), \ \text{ for all } s \in [t, \infty).$$

It is clear that γ_ε is admissible for x, at time t, in the control problem with dynamic k_ε (let u_ε be the control associated with γ_ε and set $\tau_\varepsilon = \tau^{t,x,u_\varepsilon}$). Hence, we have

$$\varphi_\varepsilon(t,x) \le \tau_\varepsilon + g(\gamma_\varepsilon(t+\tau_\varepsilon)). \tag{8.9}$$

Yet, we observe easily that $\tau_\varepsilon = \phi_\varepsilon^{-1}(t+\tau) - t$, where $\tau := \tau^{t,x,u}$. From (8.8), we have

$$\int_t^{\phi_\varepsilon(s)} k((e_r)_\# \eta, \gamma(r))\,\mathrm{d}r = \int_t^s k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r)))\,\mathrm{d}r.$$

Set

$$\Psi(\theta) = \int_t^\theta k((e_r)_\# \eta, \gamma(r))\,\mathrm{d}r, \ \text{ for all } \theta \in [t, \infty).$$

We have

$$\begin{aligned} &|\phi_\varepsilon(s) - s| \\ &= \left| \Psi^{-1}\left(\int_t^s k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r)))\,\mathrm{d}r \right) - \Psi^{-1}\left(\int_t^s k((e_r)_\# \eta, \gamma(r))\,\mathrm{d}r \right) \right| \\ &\le C \int_t^s |k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r))) - k((e_r)_\# \eta, \gamma(r))|\,\mathrm{d}r \\ &\le C \int_t^s \Big(|k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r))) - k((e_r)_\# \eta, \gamma(\phi_\varepsilon(r)))| + |k((e_r)_\# \eta, \gamma(\phi_\varepsilon(r))) - k((e_r)_\# \eta, \gamma(r))| \Big)\,\mathrm{d}r \\ &\le C \left(\int_t^s |k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r))) - k((e_r)_\# \eta, \gamma(\phi_\varepsilon(r)))|\,\mathrm{d}r + \int_t^s |\phi_\varepsilon(r) - r|\,\mathrm{d}r \right). \end{aligned}$$

Yet,

$$\begin{aligned} &|k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r))) - k((e_r)_\# \eta, \gamma(\phi_\varepsilon(r)))| \\ &\le \left| \Big(k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(\phi_\varepsilon(r))) - k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(r)) \Big) - \Big(k((e_r)_\# \eta, \gamma(\phi_\varepsilon(r))) - k((e_r)_\# \eta, \gamma(r)) \Big) \right| \\ &\quad + \left| k_\varepsilon((e_r)_\# \eta^\varepsilon, \gamma(r)) - k((e_r)_\# \eta, \gamma(r)) \right| \end{aligned}$$

$$\leq C|\phi_\varepsilon(r)-r| + \left|k_\varepsilon((e_r)_\#\eta^\varepsilon,\gamma(r)) - k((e_r)_\#\eta,\gamma(r))\right|.$$

Hence, one has

$$|\phi_\varepsilon(s)-s|$$
$$\leq C\bigg(\int_t^s \left|k_\varepsilon((e_r)_\#\eta^\varepsilon,\gamma(r)) - k((e_r)_\#\eta,\gamma(r))\right| \mathrm{d}r \,+\, \int_t^s |\phi_\varepsilon(r)-r|\,\mathrm{d}r\bigg).$$

Using Gronwall's inequality, we get

$$|\phi_\varepsilon(s)-s| \quad\leq\quad C\int_t^s e^{C(s-r)}\left|k_\varepsilon((e_r)_\#\eta^\varepsilon,\gamma(r)) - k((e_r)_\#\eta,\gamma(r))\right| \mathrm{d}r.$$

Consequently, $\phi_\varepsilon \to id$, when $\varepsilon \to 0$. In particular, we have $\tau_\varepsilon = \phi_\varepsilon^{-1}(t+\tau)-t \to \tau$. So, passing to the limit in (8.9), we get

$$\tilde{\varphi}(t,x) \leq \tau + g(\gamma(t+\tau)) = \varphi(t,x).$$

This proves that $\varphi_\varepsilon \to \varphi$ uniformly in $\mathbb{R}^+ \times \Omega$. $\square$

Under the same hypotheses of Proposition 8.11, we have the following

Proposition 8.12. *η is an equilibrium for ρ_0.*

Proof. Let $\Gamma[\rho^{\eta^\varepsilon},x]$ (resp. $\Gamma[\rho^\eta,x]$) be the set of all admissible trajectories for x, at time 0, in the MFG model associated to the control problem with dynamic k_ε (resp. k), and let $\Gamma'[\rho^{\eta^\varepsilon},x] \subset \Gamma[\rho^{\eta^\varepsilon},x]$ (resp. $\Gamma'[\rho^\eta,x] \subset \Gamma[\rho^\eta,x]$) be the optimal ones. For $k \in \mathbb{N}^\star$, let $V_k := \{\gamma \in \Gamma \,:\, d(\gamma,\cup_x\Gamma'[\rho^\eta,x]) \leq \frac{1}{k}\}$. We claim that there is some $\varepsilon_0 > 0$ such that, for all $0<\varepsilon\leq\varepsilon_0$, we have $\cup_x\Gamma'[\rho^{\eta^\varepsilon},x] \subset V_k$. Indeed, if this is not the case, then, for all $n \in \mathbb{N}^\star$, there is some optimal trajectory $\gamma_{\varepsilon_n} \in \Gamma'[\rho^{\eta^{\varepsilon_n}},x_{\varepsilon_n}]\backslash V_k$, for some $x_{\varepsilon_n} \in \Omega$. We see that $x_{\varepsilon_n} \to x \in \Omega$ and $\gamma_{\varepsilon_n} \to \gamma$ for some $\gamma \in \Gamma_{c_{\max}}$. Set

$$\tau_{\varepsilon_n} := \tau_{\gamma_{\varepsilon_n}}, \text{ for all } n \in \mathbb{N}^\star.$$

Then, using Lemma 7.7 & Proposition 7.8, we infer that $\tau_{\varepsilon_n} \to \bar{\tau}$ and $\gamma_{\varepsilon_n}(\tau_{\varepsilon_n}) \to \gamma(\bar{\tau}) \in \partial\Omega$, which implies that $\tau_\gamma \leq \bar{\tau}$. Moreover, it is easy to check that γ is admissible in the control problem with dynamic k. Yet, we have

$$\varphi_{\varepsilon_n}(0,x_{\varepsilon_n}) = \tau_{\varepsilon_n} + g(\gamma_{\varepsilon_n}(\tau_{\varepsilon_n})).$$

Then, passing to the limit when $\varepsilon_n \to 0$, we obtain, from Proposition 8.11, that

$$\varphi(0,x) = \bar{\tau} + g(\gamma(\bar{\tau})) \geq \tau_\gamma + g(\gamma(\tau_\gamma)).$$

This implies that $\gamma \in \Gamma'[\rho^\eta, x]$, which is a contradiction. Consequently, we have

$$\eta(V_k) \geq \lim_{\varepsilon} \eta^\varepsilon(V_k) \geq \lim_{\varepsilon} \eta^\varepsilon\left(\bigcup_x \Gamma'[\rho^{\eta^\varepsilon}, x]\right) = 1.$$

Hence, $\eta(V_k) = 1$ and, since k is arbitrary, we infer that $\eta\left(\bigcup_x \Gamma'[\rho^\eta, x]\right) = 1$. This concludes the proof that η is an equilibrium. □

Conclusion:

For any $\rho_0 \in L^p$, there exists a MFG equilibrium η, for ρ_0, associated to the control problem with the following dynamic

$$k(\rho, x) = c\left(\int_\Omega \chi(x-y) 1_{\overset{\circ}{\Omega}}(y)\, \mathrm{d}\rho(y)\right), \text{ for all } (\rho, x) \in \mathcal{P}(\Omega) \times \Omega.$$

CHAPTER 9

Stationary case

This chapter is devoted to the study of the stationary Mean Field Games model of the one considered in Chapter 8. In other words, we consider the following

$$\begin{cases} -\nabla\cdot\left(\rho\, k(\rho,\cdot)\,\frac{\nabla\varphi}{|\nabla\varphi|}\right)=f & \text{in } \Omega \\ k(\rho,\cdot)\,|\nabla\varphi|=1 & \text{in } \Omega \\ \varphi=0 & \text{on } \partial\Omega. \end{cases} \tag{9.1}$$

We prove existence of a solution (ρ,φ) to (9.1) by studying existence of an equilibrium η for a stationary MFG model. For a regular dynamic k, the proof is similar to the one, given in Chapter 8, for the non-stationary case. Moreover, we are able to give L^p estimates on the density ρ, which allow us to extend our result of existence of an equilibrium (or equivalently, of a solution for (9.1)) to some less regular model. In fact, we observe that (9.1) is nothing else than the Monge-Kantorovich system for the transport of f onto the boundary, using the non-uniform Riemannian metric $c=k(\rho,\cdot)^{-1}$ as a transport cost. More precisely, the measure ρ will be, up to multiply it by the dynamic k, the transport density σ between f and its projection on the boundary $P_\# f$ (using the weighted distance d_c in the definition of P). Hence, studying the L^p summability of the density ρ is equivalent to studying the L^p summability of the transport density σ between f and $P_\# f$.

This chapter is not yet part of a submitted paper. It is based on discussions with P. Pansu and F. Santambrogio.

9.1. Optimal transportation onto the boundary with weighted distances

In this section, we study an optimal transport problem between a given non-negative density $f\in L^1(\Omega)$, in the interior of a domain Ω, and the boundary $\partial\Omega$ in the presence of a non-uniform Riemannian metric d_c, where c is a given positive continuous function on Ω. In others words, we consider the following problem

$$\min\left\{\int_\Omega d_c(x,y)\,\mathrm{d}\lambda\;:\;\lambda\in\mathcal{M}^+(\Omega\times\Omega),\,(\Pi_x)_\#\lambda=f,\,(\Pi_y)_\#\lambda\subset\partial\Omega\right\}, \tag{9.2}$$

where

$$d_c(x,y)=\inf\left\{\int_0^1 c(\gamma(t))|\gamma'(t)|\,\mathrm{d}t\;:\;\gamma\in C^1([0,1],\Omega),\,\gamma(0)=x\ \text{ and }\ \gamma(1)=y\right\},\ \forall\ x,\,y\in\Omega.$$

Since the marginal $(\Pi_y)_\# \lambda$ on $\partial\Omega$ is completely arbitrary, then it is clear that the optimal choice is to take it equal to $P_\# f$, where

$$P(x) = \operatorname{argmin}\{d_c(x,y),\ y \in \partial\Omega\} \quad \text{for all } x \in \Omega,$$

which means that $\lambda := (Id, P)_\# f$ is the unique optimal transport plan for (9.2), which is also the same as

$$\min\left\{\int_{\Omega\times\Omega} d_c(x,y)\,\mathrm{d}\lambda \,:\, \lambda \in \Pi(f, P_\# f)\right\}. \tag{9.3}$$

On the other hand, the following problem

$$\max\left\{\int_{\Omega} u\,\mathrm{d}f \,:\, |\nabla u| \le c,\ u = 0 \ \text{ on } \ \partial\Omega\right\} \tag{9.4}$$

is the dual of (9.2) (we note that $|\nabla u| \le c$ is equivalent to say that u is 1-Lipschitz with respect to the distance d_c). In fact, for every admissible λ in (9.2) and every admissible u in (9.4), we have

$$\int_{\Omega\times\Omega} d_c(x,y)\,\mathrm{d}\lambda \ge \int_{\Omega\times\Omega} (u(x) - u(y))\,\mathrm{d}\lambda = \int_{\Omega} u\,\mathrm{d}f$$

and then, $\sup(9.4) \le \min(9.2)$. Now, taking $\varphi(x) := d_c(x, \partial\Omega)$, for all $x \in \Omega$, we infer that the equality $\sup(9.4) = \min(9.2)$ holds and, φ is in fact a Kantorovich potential for (9.4).

In order to introduce the transport density, as it can be understood recalling (2.2), we have first to define $\mathcal{H}^1 \llcorner \gamma_x$ for a geodesic γ_x between a point x and the boundary, which is the 1-dimensional Hausdorff measure on the path γ_x: formally, if $\phi \in C(\Omega)$, it can be computed by

$$< \mathcal{H}^1 \llcorner \gamma_x, \phi > := \int_0^1 \phi(\gamma_x(t))|\gamma_x'(t)|\,\mathrm{d}t.$$

We can now define the transport density associated to the optimal transport plan λ: the direct generalization of (2.2) turns out to be

$$\sigma := \int_{\Omega} \mathcal{H}^1 \llcorner \gamma_x \,\mathrm{d}f(x)$$

or equivalently,

$$(9.5)\qquad <\sigma,\phi> \;=\; \int_\Omega \mathrm{d}f(x)\int_0^1 \phi(\gamma_x(t))|\gamma_x'(t)|\,\mathrm{d}t \quad \text{for all} \;\; \phi \;\in\; C(\Omega).$$

On the other hand, the Beckmann problem (3.1) becomes

$$(9.6)\qquad \min\left\{\int_\Omega c\,\mathrm{d}|w| \;:\; w\in\mathcal{M}^d(\Omega),\ \nabla\cdot w = f \quad \text{in} \;\; \mathring{\Omega}\right\}.$$

In fact, it is easy to see that $\sup(9.4) \leq \min(9.6)$. Indeed, for any function $u \in C_0^1(\Omega)$ with $|\nabla u| \leq c$ and every vectorial measure $w \in \mathcal{M}^d(\Omega)$ such that $\nabla\cdot w = f$ in $\mathring{\Omega}$, we have

$$(9.7)\qquad \int_\Omega u\,\mathrm{d}f = -\int_\Omega \nabla u\cdot\,\mathrm{d}w \leq \int_\Omega c\,\mathrm{d}|w|.$$

Now, set

$$<w,\xi>= \int_\Omega \mathrm{d}f(x)\int_0^1 \xi(\gamma_x(t))\cdot\gamma_x'(t)\,\mathrm{d}t, \quad \text{for all} \;\; \xi \;\in\; C(\Omega,\mathbb{R}^d).$$

As, for a.e. $x\in\Omega$,

$$(9.8)\qquad \gamma_x'(t) = -\,\varphi(x)\,c^{-1}(\gamma_x(t))\,\frac{\nabla\varphi(\gamma_x(t))}{|\nabla\varphi(\gamma_x(t))|}, \quad \text{for all} \;\; t\in[0,1],$$

then

$$w = -\sigma\,\frac{\nabla\varphi}{|\nabla\varphi|}.$$

Yet, we see that $\int_\Omega c\,\mathrm{d}\sigma = \min(9.2)$. Moreover, it is not difficult to check that $\nabla\cdot w = f$ in $\mathring{\Omega}$. Hence, the vector measure w solves (9.6). In addition, the most complicated version of the system (3.2) becomes

$$(9.9)\qquad \begin{cases} -\nabla\cdot\left(\sigma\,\frac{\nabla\varphi}{|\nabla\varphi|}\right) = f & \text{in} \;\; \Omega,\\ \varphi = 0 & \text{on} \;\; \partial\Omega,\\ |\nabla\varphi| \leq c & \text{in} \;\; \Omega,\\ |\nabla\varphi| = c & \sigma-\text{a.e.} \end{cases}$$

The question that we consider now is whether the transport density σ from f to $P_{\#}f$ (or equivalently, the optimal vector field w in (9.6)) is in $L^p(\Omega)$ or not when $f \in L^p(\Omega)$. Via a symmetrization trick, we have already seen that, in the Euclidean case (see Chapter 3), these L^p estimates hold as soon as one has $f \in L^p$ and Ω satisfies a uniform exterior ball condition. However, this method does not work here, since the transport rays are no more segments but actually geodesics. So, we show that the same L^p result will be true as well in the case where the transport cost is given by a Riemannian metric c, via a different technique which will be described in the next sections.

9.2. Summability of the transport density with weighted distances

Let σ be the transport density associated with the transport of f into $P_{\#}f$. By the definition of σ (see (9.5)), we have, for all $\phi \in C(\Omega)$,

$$< \sigma, \phi > \;=\; \int_\Omega \int_0^1 \phi(\gamma_x(t))\, |\gamma_x'(t)|\, f(x)\, \mathrm{d}t\, \mathrm{d}x.$$

As $c \geq c_{\min} > 0$, then one has

$$\sigma \leq C \int_0^1 f_t\, \mathrm{d}t, \tag{9.10}$$

where

$$< f_t, \phi >:= \int_\Omega \phi(P_t(x))\, d_c(x, \partial\Omega)\, f(x)\, \mathrm{d}x, \quad \text{for all } \phi \in C(\Omega),$$

and

$$P_t(x) := \gamma_x(t), \quad \text{for a.e. } x \in \Omega \text{ and for every } t \in [0, 1].$$

Notice that in the definition of f_t, as we did in Chapter 4, we need to keep the factor $d_c(x, \partial\Omega)$, which will be essential in the estimates. Now, we want to give an explicit formula of f_t in terms of f and P. We have, for all $\phi \in C(\Omega)$,

$$\int_\Omega \phi(y)\, \mathrm{d}f_t(y) = \int_\Omega \phi(P_t(x))\, d_c(x, \partial\Omega)\, f(x)\, \mathrm{d}x.$$

Take a change of variable $y := P_t(x)$. As the image of y and x by P is the same, i.e. $P(y) = P(x)$, then we have

$$d_c(x, \partial\Omega) = (1 - t)^{-1}\, d_c(y, \partial\Omega).$$

Consequently, we get

$$\int_\Omega \phi(y)\, \mathrm{d}f_t(y) = \int_{\Omega_t} \phi(y)\, (1 - t)^{-1} d_c(y, \partial\Omega)\, f\left(P_t^{-1}(y)\right) |J_t(y)|\, \mathrm{d}y,$$

where $\Omega_t := P_t(\Omega)$ and $J_t(y) := \det(DP_t(x))^{-1}$ for all $y = P_t(x)$. Finally, we infer that

$$f_t(y) = (1-t)^{-1} d_c(y, \partial\Omega)\, f\left(P_t^{-1}(y)\right) |J_t(y)|\, 1_{\Omega_t}(y) \text{ for a.e. } y.$$

Notice that $y \in \Omega_t$ is equivalent to $d_c(y, \partial\Omega) \le (1-t)l(y)$, where $l(y)$ is the length of the maximal geodesic curve $\gamma : [0,1] \mapsto \Omega$ containing y such that $P(\gamma(t)) = P(y)$, for all $t \in [0,1]$.

Now, we will introduce the following key proposition, whose proof, for simplicity of exposition, is postponed to Section 9.3.

PROPOSITION 9.1. *Suppose that Ω is a smooth domain with all its curvatures bounded from below by a constant κ, and let c be a smooth positive function on Ω. Then, there exists a positive constant C depending only on d, κ, diam(Ω), $c_{\min}$, $c_{\max}$, $||\nabla c||_\infty$ and $||D^2 c||_\infty$ such that, for a.e. $x \in \Omega$, we have the following estimate*

$$|\det(DP_t(x))| \ge C(1-t).$$

We are now ready to prove the L^p summability of the transport density σ. We recall that a domain Ω satisfies a uniform exterior ball of radius $r > 0$ if for all $y \in \partial\Omega$, there exists some $x \in \mathbb{R}^d$ such that $B(x,r) \cap \Omega = \emptyset$ and $|x-y| = r$ (see Definition 3.6). We note that the existence of a uniform exterior ball of radius r, for a domain Ω, implies that all its curvatures are bounded from below by a constant $\kappa \ge -\frac{1}{r}$.

PROPOSITION 9.2. *Suppose that Ω satisfies a uniform exterior ball of radius $r > 0$, and let c be a $C^{1,1}$ positive function on Ω. Then, the transport density σ, between f and $P_\# f$, belongs to $L^p(\Omega)$ provided $f \in L^p(\Omega)$. In addition, we have the following estimate*

$$||\sigma||_{L^p} \le C||f||_{L^p},$$

where $C = C(d, r, diam(\Omega), c_{\min}, c_{\max}, ||\nabla c||_\infty, ||D^2 c||_\infty) < +\infty$.

These estimates will be similar to the ones given in Chapter 4. But, for completeness, we want to give the proof.

PROOF. From (9.10), we have

$$\begin{aligned}
\| \sigma \|^p_{L^p(\Omega)} &\le C^p \int_\Omega \left(\int_0^1 f_t(y)\,\mathrm{d}t \right)^p \mathrm{d}y \\
&= C^p \int_\Omega \left(\int_0^{1-\frac{d_c(y,\partial\Omega)}{l(y)}} (1-t)^{-1} d_c(y,\partial\Omega)\, f(P_t^{-1}(y))\, J_t(y)\,\mathrm{d}t \right)^p \mathrm{d}y.
\end{aligned}$$

Using Hölder's inequality, we get

$$\| \sigma \|^p_{L^p(\Omega)}$$

$$\leq C^p \int_\Omega \left(\int_0^{1-\frac{d_c(y,\partial\Omega)}{l(y)}} (1-t)^{-p'}\, d_c(y,\partial\Omega)^{p'}\, J_t(y)\, \mathrm{d}t \right)^{\frac{p}{p'}} \left(\int_0^{1-\frac{d_c(y,\partial\Omega)}{l(y)}} f(P_t^{-1}(y))^p\, J_t(y)\, \mathrm{d}t \right) \mathrm{d}y,$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Let us suppose, first, that Ω and c are smooth. Hence, we can apply Proposition 9.1, with $\kappa \geq -\frac{1}{r}$, to deduce that

$$\left(\int_0^{1-\frac{d_c(y,\partial\Omega)}{l(y)}} (1-t)^{-p'}\, d_c(y,\partial\Omega)^{p'}\, J_t(y)\, \mathrm{d}t \right)^{\frac{p}{p'}} \leq C^{-\frac{p}{p'}} \frac{\mathrm{diam}(\Omega)^p}{p'^{\frac{p}{p'}}}.$$

Then,

$$\| \sigma \|_{L^p(\Omega)} \leq C^{-1} \;\| f^+ \|_{L^p(\Omega)},$$

where $C = C(d, r, \mathrm{diam}(\Omega), c_{\min}, c_{\max}, ||\nabla c||_\infty, ||D^2 c||_\infty) > 0$ and, this constant can be taken independent of p. In particular, the estimate also holds for $p=\infty$.

By approximation, as the constant C depends on the lower bound of all the curvatures of $\partial\Omega$ and, on $c_{\min}$, $c_{\max}$, $||\nabla c||_\infty$, $||D^2c||_\infty$, we can check again that our result is still true for a domain Ω satisfying a uniform exterior ball condition, with a Riemannian metric $c \in C^{1,1}$. $\square$

9.3. A geometric proof

In this section, we want to prove Proposition 9.1. First of all, let us recall that the distance function $\varphi(x) = d_c(x,\partial\Omega)$ is semi-concave as soon as Ω satisfies a uniform exterior ball condition and c is $C^{1,1}$ (to see that, the reader can refer to Proposition 7.28 in case $\partial\Omega \in C^{1,1}$, or to Theorem 8.2.7 in [**37**]). Let the manifold $\mathbb{R}^d$ be equipped with the conformal metric d_c. Let $[\cdot,\cdot]$ be the Lie bracket and ∇ be the Levi-Civita connection on $(\mathbb{R}^d, d_c)$. Let ν be the unitary inner normal vector on $\partial\Omega$. For every $x \in \Omega$, it is clear that there exist $s \in \partial\Omega$ and $\tau \in [0, l(s)]$, where $l(s)$ is the length of the maximal geodesic γ (with $|\gamma'|=1$) starting from s with $P(\gamma(\tau)) = s$ for all $\tau \in [0, l(s)]$, such that $x = \Psi(s,\tau) := \exp_s \tau\nu(s)$. Moreover, we have that, for every $t \in [0,1]$, $P_t(\Psi(s,\tau)) = \Psi(s,(1-t)\tau)$, for all $s \in \partial\Omega$ and $\tau \in [0, l(s)]$. Then, we get

$$\det(DP_t(\Psi(s,\tau)))\det(D\Psi(s,\tau)) = (1-t)\det(D\Psi(s,(1-t)\tau)). \tag{9.11}$$

Fix $x \in \Omega$ and set $s := P(x)$. Let $(e_1, ..., e_d)$ be an orthonormal basis of $(\mathbb{R}^d, d_c)$ such that $e_1, ..., e_{d-1}$ is an orthonormal basis of the tangent space $T_s\,\partial\Omega$ on $\partial\Omega$ at s and $e_d = \nu(s)$. Consider small variations of s, on $\partial\Omega$, in the directions $e_1, ..., e_{d-1}$, denoted by $s + \delta e_1 + o(\delta), ..., s + \delta e_{d-1} + o(\delta)$. Set

$$J_i(s,\tau) := \frac{d}{d\delta}_{|\delta=0} \Psi(s + \delta e_i + o(\delta), \tau), \text{ for all } i \in \{1, ..., d-1\},$$

and

$$N(s,\tau) := \frac{d}{d\delta}_{|\delta=0} \Psi(s, \tau + \delta).$$

Notice that the vector field J_i, for every $i \in \{1, ..., d-1\}$, have been obtained by differentiating a family of geodesics depending on a parameter (which is, here, δ). Let γ be the maximal geodesic starting from s such that $P(\gamma(\tau)) = s$ for all $\tau \in [0, l(s)]$, and let us parallel-transport along the geodesic γ to define a new family of orthonormal basis $(e_1(\tau), ..., e_d(\tau))$, for all $\tau \in [0, l(s)]$. Set

$$J(s,\tau) := (J_1(s,\tau), ..., J_{d-1}(s,\tau), N(s,\tau)) = D\Psi(s,\tau).$$

The Jacobian of the map Ψ is defined by

$$\mathcal{J}(s,\tau) = \det J(s,\tau).$$

Yet, this Jacobian $\mathcal{J}$ cannot vanish, except possibly at the endpoint of the geodesic γ; this property can be seen as a result of the very special choice of the velocity field ν, which comes from the gradient of a d_c^2−convex function (see, for instance, [112]). So, the formula for the differential, with respect to τ, of the determinant $\mathcal{J}(s,\tau)$ yields

$$\mathcal{J}'(s,\tau) = \operatorname{tr}(J'(s,\tau)\, J(s,\tau)^{-1})\, \mathcal{J}(s,\tau).$$

Set

$$U(\tau) = J'(s,\tau)\, J(s,\tau)^{-1}. \tag{9.12}$$

One has

$$\mathcal{J}'(s,\tau) = \operatorname{tr}(U(\tau))\, \mathcal{J}(s,\tau). \tag{9.13}$$

Let us denote by $d\Psi$ the differential map of Ψ. The fact that $[\partial_{e_i}, \partial_{e_d}] = 0$, for all $i \in \{1, ..., d-1\}$, implies that

$$[J_i, N] = \Big[d\Psi(\partial_{e_i}), d\Psi(\partial_{e_d}) \Big] = d\Psi[\partial_{e_i}, \partial_{e_d}] = 0. \tag{9.14}$$

As

$$J_i(s,\tau) = \sum_{j=1}^{d} J(s,\tau)_{ji}\, e_j(\tau), \text{ for all } i \in \{1, ..., d-1\},$$

then

$$\nabla_N J_i = \nabla_{\gamma'} J_i = \sum_{j=1}^{d} J'(s,\tau)_{ji}\, e_j(\tau) + J(s,\tau)_{ji}\, \nabla_{\gamma'} e_j(\tau) = \sum_{j=1}^{d} J'(s,\tau)_{ji}\, e_j(\tau)$$

since $\nabla_{\gamma'} e_j(\tau) = 0$, for all $j \in \{1, ..., d\}$. On the other hand, from (9.14) and the fact that $[J_i, N] = \nabla_{J_i} N - \nabla_N J_i$, we get that

$$\nabla_N J_i = \nabla_{J_i} N = \sum_{j=1}^{d} J(s,\tau)_{ji} \, \nabla_{e_j(\tau)} N.$$

Now, let V be the matrix, in the basis $(e_j(\tau))_{j=1,\dots,d}$, associated with the endomorphism $X \mapsto \nabla_X N$ (which is by the way the second fundamental form of the submanifold $\{\varphi = \tau\}$ in $(\mathbb{R}^d, d_c)$). Then, one has

$$\sum_{j=1}^{d} J(s,\tau)_{ji} \, \nabla_{e_j(\tau)} N = \sum_{j,k=1}^{d} J(s,\tau)_{ji} \, V_{kj} \, e_k(\tau) = \sum_{k=1}^{d} (VJ)_{ki} \, e_k(\tau).$$

Hence,

$$J' = VJ,$$

which means, recalling (9.12), that $V = U$. Consequently, $U(\tau)$ is the second fundamental form of $\{\varphi = \tau\}$. Yet, from the definitions of Ψ and N, we see easily that $N = \nabla\varphi$. Hence, by the semi-concavity of the distance function φ, we infer that

$$U(\tau) \leq CI$$

for some constant $C := C(d, \kappa, \operatorname{diam}(\Omega), c_{\min}, c_{\max}, ||\nabla c||_\infty, ||D^2 c||_\infty) < +\infty$. As a consequence of this, we obtain

$$\operatorname{tr}(U(\tau)) \leq C,$$

which means, using (9.13), that

$$\frac{\mathcal{J}'(s,\tau)}{\mathcal{J}(s,\tau)} \leq C.$$

Then,

$$\log(\mathcal{J}(s,\tau)) - \log(\mathcal{J}(s,(1-t)\tau)) \leq Ct\tau$$

and

$$\frac{\mathcal{J}(s,(1-t)\tau)}{\mathcal{J}(s,\tau)} \geq e^{-Ct\tau}.$$

Recalling (9.11), we infer that there is a constant $C > 0$ such that

$$\det(DP_t(x)) \geq C(1-t).$$

9.4. Existence of equilibria for stationary MFG

In this section, we want to study the existence of an equilibrium for some regular/non-regular stationary MFG models. We can consider the same MFG, introduced in Chapter 8, adding a density f_t at each time t; this means that we have an initial density ρ_0 of agents evolving in Ω and there is an additional density f_t which is created in Ω at each time t. The goal of each agent is to leave the domain Ω through the boundary $\partial\Omega$ in minimal time under the assumption that the speed of an agent in a position x at time t is bounded by $k(\rho_t, x)$, where ρ_t is the distribution of agents at time t. In this case, the system (8.1) becomes

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\rho\, k \, \frac{\nabla \varphi}{|\nabla \varphi|} \right) = f & \text{in } \ \mathbb{R}^+ \times \Omega, \\ -\partial_t \varphi + k\,|\nabla \varphi| = 1 & \text{in } \ \mathbb{R}^+ \times \Omega, \\ \rho(0) = \rho_0 \ \text{ in } \ \Omega, \ \ \varphi = 0 & \text{on } \ \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{9.15}$$

In order to model (9.15) in the Lagrangian setting, we consider a time dependent measure on curves $\eta : t \mapsto \eta_t \in \mathcal{P}(\Gamma)$ with $(e_0)_\# \eta_0 = \rho_0$, $(e_t)_\# \eta_t = f_t$, for every $t > 0$, where the measure η_t is the distribution of trajectories followed by agents starting at time t. Let us define the curve ρ^η as follows

$$\rho_t^\eta := \int_0^t (e_t)_\# \eta_s \, \mathrm{d}s + (e_t)_\# \eta_0, \ \text{ for all } \ t \in \mathbb{R}^+. \tag{9.16}$$

For such a family η, we consider, for every $(t, x) \in \mathbb{R}^+ \times \Omega$, the following minimal-time exit problem

$$\min \left\{ \tau_\gamma^t \ : \ \gamma \in \Gamma^t[\rho^\eta, x] \right\},$$

where

$$\tau_\gamma^t := \inf\{\tau \geq 0 \ : \ \gamma(t + \tau) \in \partial\Omega\}$$

and

$$\Gamma^t[\rho^\eta, x] := \left\{ \gamma \in \Gamma, \, \gamma(t) = x, \, |\gamma'(s)| \leq k(\rho_s^\eta, \gamma(s)) \ \text{ for a.e. } \ s \in (t, t{+}\tau_\gamma^t), \, \gamma'(s) = 0 \ \ \forall \ \ s > t{+}\tau_\gamma^t \right\}.$$

We may expect that there exists an equilibrium η for the above MFG model with initial density ρ_0 and source f, i.e., there is a curve $\eta : t \mapsto \eta_t \in \mathcal{P}(\Gamma)$ with $(e_0)_\# \eta_0 = \rho_0$, $(e_t)_\# \eta_t = f_t$, for every $t > 0$, such that η_t−a.e. $\gamma \in \Gamma$ is an optimal trajectory for $\gamma(t)$, at time t.

From now on, we assume that f_t is independent of t. We are interested in the study of the stationary MFG of (9.15); this means that we want to find an equilibrium η in such a way that the distribution of agents ρ_t will be constant in t, i.e., $\rho_t = \rho_0$, for all $t \in \mathbb{R}^+$. The stationary version of (9.15) becomes

$$(9.17)\qquad \begin{cases} -\nabla\cdot\left(\rho\, k(\rho,\cdot)\,\frac{\nabla\varphi}{|\nabla\varphi|}\right) = f & \text{in } \Omega,\\ k(\rho,\cdot)\,|\nabla\varphi| = 1 & \text{in } \Omega,\\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us define, first, a stationary MFG equilibrium. Take $\rho \in \mathcal{P}(\Omega)$. So, for every $x \in \Omega$, we consider the following problem

$$(9.18)\qquad \min\left\{\tau_\gamma \,:\, \gamma(0) = x,\, |\gamma'(s)| \le k(\rho,\gamma(s)) \text{ for a.e. } s \in (0,\tau_\gamma) \text{ and } \gamma'(s) = 0 \quad \forall\ s > \tau_\gamma\right\}$$

where

$$\tau_\gamma := \inf\{\tau \ge 0 \,:\, \gamma(\tau) \in \partial\Omega\}.$$

It is clear that if γ is an admissible trajectory in (9.18), then there is a control $u : \mathbb{R}^+ \mapsto \bar{B}(0,1)$ such that

$$(9.19)\qquad \begin{cases} \gamma'(t) = k(\rho,\gamma(t))\,u(t), & \text{for a.e. } t,\\ \gamma(0) = x. \end{cases}$$

In fact, (9.19) can be seen as an autonomous control system (see Chapter 7) where the dynamic is given by $\tilde{k}(x) = k(\rho,x)$, for every $x \in \Omega$, which means that one can formulate (9.18) as an optimal control problem.

We recall, from Lemma 7.7, that $\min\,(9.18) \le k_{\min}^{-1}\operatorname{diam}(\Omega)$. Therefore, after a time $T > k_{\min}^{-1}\operatorname{diam}(\Omega)$, all the agents have already left the domain. For any $\eta \in \mathcal{P}(\Gamma)$, let us define the non-negative measure ρ^η on Ω as follows

$$(9.20)\qquad \rho^\eta = \int_0^T (e_t)_\#\eta \;\mathrm{d}t.$$

Notice that (9.20) can be obtained from (9.16) by taking t large enough. Indeed, if we suppose $\eta_s = \eta$ for all $s > 0$ (since we look for a stationary equilibrium) we have, from (9.16), that

$$\rho_t^\eta = \int_0^t (e_t)_\#\eta_s\,\mathrm{d}s + (e_t)_\#\eta_0 = \int_0^t (e_{t-s})_\#\eta\,\mathrm{d}s + (e_t)_\#\eta_0 = \int_0^t (e_s)_\#\eta\,\mathrm{d}s + (e_t)_\#\eta_0.$$

For t large enough, if we consider just the restriction of the density of agents to the interior of Ω, we get $(e_t)_\#\eta_0 = 0$ and $\int_0^t (e_s)_\#\eta\,\mathrm{d}s = \int_0^T (e_s)_\#\eta\,\mathrm{d}s$, for all $t \ge T$. This yields that, for all $t \ge T$, we have $\rho_t^\eta = \rho_T^\eta = \int_0^T (e_s)_\#\eta\,\mathrm{d}s$.

Let us denote by $\mathcal{P}_f(\Gamma)$ the set of all probability measures on Γ such that $(e_0)_{\#}\eta = f$. We introduce the following

DEFINITION 9.3. *For $f \in \mathcal{P}(\Omega)$, we say that $\eta \in \mathcal{P}_f(\Gamma)$ is a stationary MFG equilibrium with source f if*

$$\operatorname{spt}(\eta) \subseteq \{\gamma \in \Gamma \,:\, \gamma \textit{ is an optimal trajectory for } \gamma(0), \textit{ with } \rho := \rho^\eta\}.$$

9.4.1. Regular case. Suppose that the dynamic k is continuous on $\mathcal{P}(\Omega) \times \Omega$ and, Lipschitz with respect to the second variable x. Then, we have the following

THEOREM 9.4. *There exists a stationary MFG equilibrium with source f.*

PROOF. The strategy of the proof is almost the same as that one given in Chapter 8 (although, it seems to be slightly simpler in this autonomous case). □

Moreover, we can characterize the density ρ^η, whenever η is a stationary MFG equilibrium with source f, using the solution of the equation $-\nabla \cdot (\rho\, v) = f$, for a particular velocity field v. Let φ be the value function (see Chapter 7) associated to the autonomous control problem with dynamic $\tilde{k} = k(\rho^\eta, \cdot)$. Then, under the assumption that $\tilde{k} \in C^{1,1}$, we have the following

PROPOSITION 9.5. *Let $\eta \in \mathcal{P}_f(\Gamma)$ be a stationary MFG equilibrium with source f. Set $\rho := \rho^\eta$ and let φ be the value function with $\tilde{k} = k(\rho, \cdot)$. Then, we have*

$$-\nabla \cdot \left(\rho\; k(\rho, \cdot)\, \frac{\nabla \varphi}{|\nabla \varphi|}\right) = f \quad \textit{in} \quad \mathring{\Omega}.$$

PROOF. Let $\phi \in \mathcal{C}^\infty_c(\mathring{\Omega})$. Then, recalling Proposition 7.25 & Corollary 7.26, we have

$$\int_\Omega k(\rho, x)\, \nabla \phi(x) \cdot \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|}\, \mathrm{d}\rho(x)$$

$$= \int_0^T \int_\Gamma k(\rho, \gamma(t))\, \nabla \phi(\gamma(t)) \cdot \frac{\nabla \varphi(\gamma(t))}{|\nabla \varphi(\gamma(t))|}\, \mathrm{d}\eta(\gamma)\, \mathrm{d}t = -\int_0^T \int_\Gamma \nabla \phi(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}\eta(\gamma)\, \mathrm{d}t$$

$$= -\int_\Gamma \int_0^T \frac{d}{dt}(\phi(\gamma(t)))\, \mathrm{d}t\, \mathrm{d}\eta(\gamma) = \int_\Gamma \phi(\gamma(0))\, \mathrm{d}\eta(\gamma) = \int_\Omega \phi(x)\, \mathrm{d}f(x). \quad \square$$

Consequently, if $\eta \in \mathcal{P}_f(\Gamma)$ is a stationary MFG equilibrium with source f, $\rho = \rho^\eta$ and φ is the associated value function, then, by Propositions 7.5 & 9.5, the pair (ρ, φ) solves the following system

$$\begin{cases} -\nabla \cdot \left(\rho\, k(\rho,\cdot)\, \dfrac{\nabla\varphi}{|\nabla\varphi|}\right) = f & \text{in } \Omega, \\ k(\rho,\cdot)\,|\nabla\varphi| = 1 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

9.4.2. Equilibrium densities are transport densities. Now, we want to explain the relation between the equilibrium density $\rho := \rho^\eta$, for some stationary MFG equilibrium η with source f, and the transport density σ between f and its projection onto the boundary $P_\# f$ using the Riemannian metric $c := k(\rho,\cdot)^{-1}$ as a transport cost. First, let us introduce the following

Proposition 9.6. *There is a unique optimal flow w for (9.6).*

Proof. Similarly to Lemma 2.5, we can prove that for every flow w with $\nabla\cdot w = f^+ - P_\# f^+$, there is an admissible traffic plan Q such that

$$\int_\Omega c\,\mathrm{d}|w - w_Q| + \int_\Omega c\,\mathrm{d}i_Q = \int_\Omega c\,\mathrm{d}|w|.$$

If w is optimal for (9.6), we get, using $|w_Q| \le i_Q$, that $w = w_Q$ and $|w| = i_Q$. Hence,

$$\begin{aligned} \int_\Omega c\,\mathrm{d}|w| &= \int_\Omega c\,\mathrm{d}i_Q = \int_{\mathcal{C}} L_c(\alpha)\,\mathrm{d}Q(\alpha) \ge \int_{\mathcal{C}} d_c(\alpha(0),\alpha(1))\mathrm{d}Q(\alpha) \\ &= \int_{\Omega\times\Omega} d_c(x,y)\mathrm{d}((e_0,e_1)_\# Q)(x,y), \end{aligned}$$

where $L_c(\alpha) := \int_0^1 c(\alpha(t))|\alpha'(t)|\,\mathrm{d}t$, for all $\alpha \in \mathcal{C}$. Yet, min(9.6)= min(9.2) and w is optimal for (9.6). Then, for $Q-$a.e. $\alpha \in \mathcal{C}$, one has $d_c(\alpha(0),\alpha(1)) = L_c(\alpha)$, which means that α is a geodesic. Moreover, $\gamma := (e_0,e_1)_\# Q$ must be an optimal transport plan for (9.2) and so, by the uniqueness of the minimizer for (9.2), we infer that $\gamma = (Id, P)_\# f^+$. Consequently, we get that $w = w_Q = w_\gamma$. □

Proposition 9.7. *Take $f \in \mathcal{P}(\Omega)$ and let η be a stationary MFG equilibrium with source f, and dynamic k. Let $\rho := \rho^\eta$ be an equilibrium density associated with η and σ be the transport density between f and $P_\# f$, where P is the projection map onto the boundary, using the Riemannian metric d_c with $c = k(\rho,\cdot)^{-1}$. Then, $\sigma = k(\rho,\cdot)\,\rho$.*

PROOF. Let φ be the value function, associated to the control problem with dynamic $k(\rho,\cdot)$. Set

$$v := -\rho\, k(\rho,\cdot)\frac{\nabla\varphi}{|\nabla\varphi|}.$$

Thanks to Proposition 9.5, v is admissible in (9.6). Let w be another admissible flow in (9.6). It is clear that we have

$$\frac{|w|}{k(\rho,\cdot)} \geq \frac{|v|}{k(\rho,\cdot)} + \frac{v}{k(\rho,\cdot)|v|}\cdot(w-v) = \frac{|v|}{k(\rho,\cdot)} - \frac{\nabla\varphi}{k(\rho,\cdot)|\nabla\varphi|}\cdot(w-v) = \frac{|v|}{k(\rho,\cdot)} - \nabla\varphi\cdot(w-v).$$

Hence, we get

$$\int_\Omega c|w|\,\mathrm{d}x \geq \int_\Omega c|v|\,\mathrm{d}x - \int_\Omega \nabla\varphi\cdot(w-v)\,\mathrm{d}x = \int_\Omega c|v|\,\mathrm{d}x,$$

where the last equality follows from the fact that $\nabla\cdot v = \nabla\cdot w = f$ in $\mathring{\Omega}$ and $\varphi = 0$ on $\partial\Omega$. Consequently, v is an optimal flow for (9.6). From Proposition 9.6, we infer that $|v|$ is the transport density σ between f and $P_\# f$, i.e. one has $\sigma = k(\rho,\cdot)\,\rho$. $\square$

COROLLARY 9.8. *Suppose that Ω satisfies a uniform exterior ball of radius $r > 0$ and that the dynamic k is $C^{1,1}$ in x with $0 < k_{\min} \leq k \leq k_{\max} < +\infty$. Let η be a stationary MFG equilibrium with source f. Then, the equilibrium density $\rho := \rho^\eta$ belongs to $L^p(\Omega)$ as soon as the source f is in $L^p(\Omega)$. Moreover, we have the following estimate*

$$||\rho||_{L^p} \leq C||f||_{L^p},$$

where the constant C depends only on d, r, diam(Ω), $k_{\min}$, $k_{\max}$, $||\nabla_x k||_\infty$ and $||\nabla_x^2 k||_\infty$.

PROOF. This follows immediately from Propositions 9.2 & 9.7. $\square$

Again, these L^p estimates on ρ allow us to prove existence of a stationary MFG equilibrium in some case where the dynamic k is non-regular.

9.4.3. Non-regular case. Now, we will generalize the result of existence of a stationary MFG equilibrium to the case where the dynamic k is defined as follows

$$k(\rho,x) = h\bigg(\int_\Omega \chi(x-y)1_{\mathring{\Omega}}(y)\,\mathrm{d}\rho(y)\bigg), \text{ for all } (\rho,x)\in\mathcal{P}(\Omega)\times\Omega,$$

where χ is a given non-negative $C^{1,1}$ function, and h is decreasing, positive and $C^{1,1}$. Again as in Chapter 8, one can prove existence of a stationary MFG equilibrium in this non-regular case by using an approximation of the dynamic k with regular ones k_ε $(\varepsilon > 0)$. So, let us define the dynamic k_ε as follows

$$k_\varepsilon(\rho, x) = h\left(\int_\Omega \chi(x-y)\psi_\varepsilon(y)\,\mathrm{d}\rho(y)\right), \text{ for all } (\rho, x) \in \mathcal{P}(\Omega) \times \Omega,$$

where ψ_ε is a cut-off function, which converges (in L^p, for all $p < +\infty$) to $1_{\mathring{\Omega}}$. For every $\varepsilon > 0$, let η^ε be a stationary MFG equilibrium (associated to (9.17) with dynamic k_ε). It is clear that, up to a subsequence, $\eta^\varepsilon \rightharpoonup \eta$, for some $\eta \in \mathcal{P}(\Gamma)$. As a consequence of this, $\rho^\varepsilon := \int_0^T (e_t)_\# \eta^\varepsilon \,\mathrm{d}t \rightharpoonup \rho := \int_0^T (e_t)_\# \eta \,\mathrm{d}t$. Yet, from Corollary 9.8, we have that $(\rho^\varepsilon)_\varepsilon$ is bounded in L^p as soon as $f \in L^p$ and Ω satisfies a uniform exterior ball condition. Hence, $\rho^\varepsilon \rightharpoonup \rho$ in L^p. And, this is sufficient to show that $c_\varepsilon := k_\varepsilon(\rho^\varepsilon, \cdot)^{-1}$ converges uniformly to $c := k(\rho, \cdot)^{-1}$, which also implies that $\varphi_\varepsilon := d_{c_\varepsilon}(\cdot, \partial\Omega)$ converges uniformly to $\varphi := d_c(\cdot, \partial\Omega)$. Recalling Proposition 8.12, we infer that there is a stationary MFG equilibrium η with source f, associated to (9.17) with dynamic k.

9.4.4. Local case. We finish this section by observing that, in this stationary framework, different relations between k and ρ can be considered. Indeed, for the general theory presented in Theorem 9.4 (exactly as for the time-dependent case of Chapter 8), the non-local behavior of k was crucial, so that it was a continuous function of ρ and x. Yet, we can consider for instance the case where the dynamic k is given by $k(\rho, x) = \rho(x)^{-\alpha}$ $(\alpha > 0)$. Of course, this dynamic k is non-regular and we cannot use Theorem 9.4 to infer the existence of a stationary MFG equilibrium in such a case. The system we have to consider is the following:

$$\begin{cases} -\nabla \cdot \left(\rho^{1-\alpha} \frac{\nabla\varphi}{|\nabla\varphi|}\right) = f & \text{in } \Omega, \\ \rho^{-\alpha}\,|\nabla\varphi| = 1 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.21}$$

which becomes

$$\begin{cases} -\nabla \cdot \left(|\nabla\varphi|^{\frac{1}{\alpha}-2}\,\nabla\varphi\right) = f & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation is nothing but the $p-$Laplacian equation with $p = 1/\alpha$ (hence, we need to take $\alpha < 1$), i.e., we have

$$-\Delta_{\frac{1}{\alpha}}\varphi = f.$$

Hence, existence of a stationary MFG equilibrium can be proven just by noting that (9.21) has a solution, by standard PDE arguments. Note the similarity of this stationary congested model with the models related to continuous Wardrop equilibria in [21, 39].

The regularity and the estimates on the solution could of course retrieved from results which are nowadays standard concerning the second order regularity of the solution of the $p-$Laplacian equation, see [**19**, **85**, **111**, **92**, **105**].

A current work with B. Fall consists in a shape optimization problem in order to place an obstacle $\omega \subset \Omega$ and minimize the total evacuation time $\int f\varphi = \int \rho$, but this work is not developped enough to be included in this thesis.

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Titre : Problèmes de transport et de contrôle avec des coûts sur le bord : régularité et sommabilité des densités optimales et d'équilibre

Mots clés : Transport Optimal, Contrôle Optimal, MFG

Résumé : Une première partie de cette thèse est dédiée à l'étude de la régularité de la densité de transport σ dans le problème de Monge entre deux mesures f^+ et f^- sur un domaine Ω. Tout d'abord, on étudie la question de la sommabilité L^p de cette densité de transport entre une mesure f^+ et sa projection sur le bord $(P_{\partial\Omega})_\# f^+$, qui ne découle pas en fait des résultats connus (dus à De Pascale - Evans - Pratelli - Santambrogio) sur la densité de transport entre deux densités L^p, comme dans notre cas la mesure cible est singulière. Par une méthode de symétrisation, dès que Ω est convexe ou satisfait une condition de boule uniforme extérieure, nous prouvons les estimations L^p (si $f^+ \in L^p$, alors $\sigma \in L^p$). En plus, nous analysons le cas où on paye des coûts supplémentaires $g^\pm$ sur le bord, en prouvant que la densité de transport σ est dans L^p dès que $f^\pm \in L^p$, Ω satisfait une condition de boule uniforme extéieure et, $g^\pm$ sont $\lambda^\pm$ −Lipschitiziens avec $\lambda^\pm < 1$ et semi-concaves. Ensuite, on s'attaque à la régularité d'ordre supérieur ($W^{1,p}$, $C^{0,\alpha}$, $BV \cdots$) de la densité de transport σ entre deux densités régulières f^+ et f^-. Plus précisément, nous fournissons une famille de contre-exemples à la régularité supérieure: nous prouvons que la régularité $W^{1,p}$ des mesures source et cible, f^+ et f^-, n'implique pas que la densité de transport est $W^{1,p}$, de même pour la régularité BV, et même $f^\pm \in C^\infty$ n'implique pas que σ est dans $W^{1,p}$, pour p grand. Ensuite, nous étudions la sommabilité L^p de la densité de transport entre deux mesures f^+ et f^- concentrées sur le bord. Plus précisément, nous prouvons que si f^+ et f^- sont dans $L^p(\partial\Omega)$, alors la densité de transport σ entre eux est dans $L^p(\Omega)$ dès que Ω est uniformément convexe et $p \leq 2$; de plus, nous introduisons un contre-exemple montrant que ce résultat n'est plus vrai si $p > 2$. Cela fournit des résultats de régularité $W^{1,p}$ sur la solution u du problème de gradient minimal avec donnée au bord g dans des domaines uniformément convexes (si $g \in W^{1,p}(\partial\Omega) \Rightarrow u \in W^{1,p}(\Omega)$).

Dans une deuxième partie, nous étudions un problème de contrôle optimal motivé par un modèle de jeux à champ moyen. D'abord, nous montrons des résultats de différentiabilité et semi-concavité sur la fonction valeur associée au problème de contrôle (le résultat de semi-concavité est optimal en ce qui concerne les hypothèses sur la régularité en temps). Ensuite, nous démontrons que la densité des agents ρ_t, dans le modèle MFG considéré, est dans L^p dès que la densité initiale $\rho_0 \in L^p$. En plus, nous arrivons à prouver l'existence d'un équilibre pour le problème MFG considéré dans un cas où la dynamique n'est pas régulière.

Dernièrement, nous considérons le problème stationnaire associé au problème MFG. Nous montrons que la densité d'équilibre n'est rien d'autre que la densité de transport entre une densité source f et sa projection sur le bord en utilisant une métrique Riemannienne non-uniforme comme coût de transport. Cela nous permet de démontrer que la densité d'équilibre ρ est dans L^p dès que la densité source $f \in L^p$. Par conséquent, nous arrivons à prouver aussi l'existence d'un équilibre stationnaire dans un cas où la dynamique n'est pas régulière.

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Title: Transport and control problems with boundary costs: regularity and summability of optimal and equilibrium densities

Keywords: Optimal Transport, Optimal Control, MFG

Abstract : A first part of this thesis is dedicated to the study of the regularity of the transport density σ in the Monge problem between two measures f^+ and f^- on a domain Ω. First, we study the question of L^p summability of this transport density between a measure f^+ and its projection on the boundary $(P_{\partial\Omega})_\# f^+$, which does not actually follow from the known results (due to De Pascale, Evans, Pratelli, Santambrogio) on the transport density between two L^p densities, as in our case the target measure is singular. By a symmetrization trick, if Ω is convex or satisfies a uniform exterior ball condition, we prove the L^p estimates (if $f^+ \in L^p$, then $\sigma \in L^p$). In addition, we analyze the case where we pay additional costs $g^\pm$ on the boundary, proving that the transport density σ is in L^p as soon as $f^\pm \in L^p$, Ω satisfies a uniform exterior ball condition and, $g^\pm$ are $\lambda^\pm -$Lip with $\lambda^\pm < 1$ and semi-concave. Then we attack the higher order regularity ($W^{1,p}$, $C^{0,\alpha}$, $BV \cdots$) of the transport density σ between two regular densities f^+ and f^-. More precisely, we provide a family of counter-examples to the higher regularity: we prove that the $W^{1,p}$ regularity of the source and target measures, f^+ and f^- , does not imply that the transport density is in $W^{1,p}$, the same for the BV regularity, and even $f^\pm \in C^\infty$ does not imply that σ is in $W^{1,p}$, for large p. Next, we study the L^p summability of the transport density between two measures, f^+ and f^-, concentrated on the boundary. More precisely, we prove that if f^+ and f^- are in $L^p(\partial\Omega)$, then the transport density σ between them is in $L^p(\Omega)$ as soon as Ω is uniformly convex and $p \leq 2$; moreover, we introduce a counter-example showing that this result is no longer true if $p > 2$. This provides $W^{1,p}$ regularity results on the solution u of the minimal gradient problem with boundary datum g in uniformly convex domains (if $g \in W^{1,p}(\partial\Omega) \Rightarrow u \in W^{1,p}(\Omega)$).

In a second part, we study an optimal control problem motivated by a model of mean field games. First, we show differentiability and semi-concavity results on the value function associated with the control problem (the semi-concavity result will be sharp in regards to the hypotheses on the regularity in time). Then we show that the density of agents ρ_t, in the considered MFG model, is in L^p as soon as the initial density $\rho_0 \in L^p$. In addition, we prove existence of an equilibrium for the considered MFG problem in a case where the dynamic is non-regular.

Lastly, we consider the stationary problem associated with the MFG model. We show that the equilibrium density is nothing but the transport density between a source density f and its projection on the boundary using a non-uniform Riemannian metric as a transport cost. This allows us to show that the equilibrium density ρ is in L^p as soon as the source density $f \in L^p$. Therefore, we also prove existence of a stationary equilibrium in a case where the dynamic is non-regular.