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Author for correspondence:
G. Bevilacqua
e-mail:
giulia.bevilacqua1993@gmail.com

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Variational analysis of inextensible elastic curves

G. Bevilacqua[1], L. Lussardi[2], A. Marzocchi[1]

[1]Dipartimento di Matematica e Fisica "N. Tartaglia", Università Cattolica del Sacro Cuore, via della Garzetta 48, I-25133 Brescia, Italy.
[2] Dipartimento di Scienze Matematiche "G.L. Lagrange", Politecnico di Torino, c.so Duca degli Abruzzi 24, I-10129 Torino, Italy.

We minimize elastic energies on framed curves which penalize both curvature and torsion. We also discuss critical points using the infinite dimensional version of the Lagrange multipliers' method. Finally, some examples arising from the applications are discussed and also numerical experiments are presented.

1. Introduction

The study of elastic curves was initiated in 1691 by Jacob Bernoulli and it was continued by Euler who introduced, in his book of 1744, the methods of Calculus of Variations. In his masterpiece, Euler introduced a complete characterization of *elasticae curves*. Since then, the name *elasticae* refers to curves which are critical points for the energy functional

$$\int_{\boldsymbol{r}} \kappa^2 \, ds$$

where κ is the curvature of the curve $\boldsymbol{r}$. Since the fundamental papers by Langer and Singer [17–19], where the equations of elasticae have been integrated, the study of the Euler functional has been widely developed. We refer here, for instance, to a recent and very interesting research on the elastic networks (see [1,11] and references therein).

Elastic energies play an important role in physical applications: we just mention, for instance, the study of slender biological systems, like DNA, knotted or unknotted proteins [7,9,27,29], or the construction of engineering structures, like cables or pipelines [25]. In the literature, we can find also energy functionals that penalize both curvature and torsion. For instance, in 1930 Sadowsky [22,23] (see [15,16] for an English translation) studied the equilibria of a developable Möbius strip by minimizing the bending energy. He argued that when the Möbius strip shrinks to its centerline, the energy reduces to a functional which depends on the curvature and the torsion of the centerline itself. The original form of the Sadowsky energy functional is given by

$$\int_{\boldsymbol{r}} \frac{(\kappa^2+\tau^2)^2}{\kappa^2}\, ds$$

where τ is the torsion of $\boldsymbol{r}$. Later, also Langer and Singer [20] considered an energy functional which penalizes both the curvature and the torsion of the centerline of an elastic rod. Precisely, they considered the functional

$$\lambda_1 \int_{\boldsymbol{r}} ds + \lambda_2 \int_{\boldsymbol{r}} \tau\, ds + \lambda_3 \int_{\boldsymbol{r}} \frac{\kappa^2}{2}\, ds.$$

In this paper, we want to study a more general energy density function depending both on the curvature κ and the torsion τ of the curve. We consider also an explicit dependence on the curvilinear abscissa (see paragraph (a) for some explanation). In other words, we are dealing with the following type of elastic energy functional:

$$\int_{\boldsymbol{r}} f(s,\kappa,\tau)\, ds. \tag{1.1}$$

We are interested in the existence of minimizers of (1.1) among closed curves with fixed length, and in a characterization of its critical points. The corresponding Euler-Lagrange equations without constraints have already been obtained, up to our knowledge, for curves of class at least C^2: for instance, in [6,26] the authors employ the Serret-Frenet frame to describe the geometry of the curves and to compute the first variation. Differently, using our approach, we obtain a system of first-order differential equations which is not in normal form but which embeds the closure of the curve, a fact generally difficult to implement.

In order to introduce a model for an elastic curve as physical as possible [8], we proceed differently from classical approaches [19], where the independent variable of the energy functional is a parametrized curve. Here, we wish to deal with C^1 curves but not necessarily C^2. By this, we cannot adopt the Serret-Frenet frame description, for which the regularity of the curve has to be at least C^2. We adopt then the approach of the *framed curves* with the constraints of C^1-closedness and a natural condition $\boldsymbol{t}' \cdot \boldsymbol{b} = 0$, where here $\boldsymbol{t}$ and $\boldsymbol{b}$ can be thought as the unit tangent vector and the binormal to the given curve, respectively. Framed curves were introduced by Schuricht et al. [14] to describe the physical behavior of elastic curves under additional topological constraints (see also [24]), while the condition $\boldsymbol{t}' \cdot \boldsymbol{b} = 0$ arises in a natural way from a Gamma-limit procedure exploited by Freddi et al. [10] to investigate the dimensional reduction of an elastic Möbius strip.

The paper is organized as follows. First of all, in Section 2, we introduce the mathematical setting of the framed curves showing how to reconstruct a space curve starting from its (weak) curvature and torsion and we introduce the elastic energy functional. Next, in Section 3, we prove the first main result, i.e. the existence of energy minimizers (Theorem 3.1). Even if Theorem 3.1 is very similar to [14, Thm. 1], we will give a complete proof since our set of constraints is different. Then, in Section 4, we find as a second result, the first-order necessary conditions for minimizers (Theorem 4.2) using essentially the infinite-dimensional version of the Lagrange multipliers' method under suitable regularity and growth assumptions on the integrand function. Up to our knowledge, this approach seems to be new at least in this context. We obtain a system of ordinary differential equations (see (4.3)) which is not in normal form, it contains some of the Lagrange multipliers and it makes sense even at points where $\kappa = 0$. We stress the fact that the complete elimination of the Lagrange multipliers from (4.3) needs to work at points where the curvature is not zero. In Theorem 4.4, we report the equations obtained though this elimination, which have already been obtained by Capovilla et al. in [6] (see their Eq. (77) and (78)). Precisely, different from our approach, they derived such a conditions using Serret-Frenet frame and they did not consider the explicit dependence on s. Finally, in Section 5, we consider some examples arising from biological or engineering applications and we perform some numerical examples to visualize the shape of critical points. In particular, we apply Theorem 4.2 to minimizers of Euler elastic energy. Euler functional had been widely studied (see [17–19]) but assuming regularity a priori and using the Serret-Frenet moving frame. Our approach requires less regularity a priori and in order to eliminate the Lagrange multipliers we have to show that the curvature is not zero at almost any point, see Theorem 5.1.

The variational analysis of functionals of type (1.1) is a fundamental preliminary in view to consider the more complicated physical situation where a soap film spans an elastic inextensible curve. This study will be managed in the spirit of the one carried out for the Kirchhoff-Plateau problem (see [3–5,12,13] and references therein) and it will be the content of a forthcoming paper.

2. Framed curves and elastic energy

We introduce framed curves following, up to some variants, the approach presented in [14]. We denote by $SO(3)$ the set of all 3×3 rotation matrices: in other words, $(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})\in SO(3)$ means that $\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. Fix $L>0$ and $p>1$[1]. On a triple $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})\in W^{1,p}((0,L);SO(3))$ we put the following constraints:

$$\boldsymbol{t}'\cdot\boldsymbol{b}=0,\quad \text{a.e. on } (0,L), \tag{2.1}$$

$$\int_0^L \boldsymbol{t}\,ds=0, \tag{2.2}$$

$$\boldsymbol{t}(L)=\boldsymbol{t}(0), \tag{2.3}$$

where $\boldsymbol{t}(L)$ and $\boldsymbol{t}(0)$ are intended in the sense of traces. We let

$$W=\{(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})\in W^{1,p}((0,L);SO(3)) : (2.1)\text{-}(2.2)\text{-}(2.3)\text{ hold true}\}.$$

On W we put the norm of $W^{1,p}((0,L);\mathbb{R}^{3\times 3})$, namely

$$\|A\|_W=(\|A\|_p+\|A'\|_p)^{1/p}$$

where the p-norm of an $X=(x_{ij})\in\mathbb{R}^{3\times 3}$ is given by

$$\|X\|_p=\left(\sum_{i,j=1}^3 |x_{ij}|^p\right)^{1/p}.$$

Let $f\colon[0,L]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be measurable. We define the energy functional $\mathcal{E}\colon W\to\mathbb{R}\cup\{+\infty\}$ as

$$\mathcal{E}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})=\int_0^L f(s,\boldsymbol{t}'\cdot\boldsymbol{n},\boldsymbol{n}'\cdot\boldsymbol{b})\,ds.$$

(a) Geometrical interpretation

Fix $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})\in W$ and $\boldsymbol{x}_0\in\mathbb{R}^3$. We consider the map $\boldsymbol{r}_{\boldsymbol{x}_0}\colon[0,L]\to\mathbb{R}^3$ defined by

$$\boldsymbol{r}_{\boldsymbol{x}_0}(s)=\boldsymbol{x}_0+\int_0^s \boldsymbol{t}\,dr.$$

In other words, $\boldsymbol{r}_{\boldsymbol{x}_0}$ is the curve clamped at the point $\boldsymbol{x}_0$ and *generated* by the orthonormal frame $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$. First of all $\boldsymbol{r}_{\boldsymbol{x}_0}$ is parametrized by the arclength since $|\boldsymbol{r}'_{\boldsymbol{x}_0}|=|\boldsymbol{t}|=1$. Condition (2.2) says that $\boldsymbol{r}_{\boldsymbol{x}_0}$ is a closed curve, that is $\boldsymbol{r}_{\boldsymbol{x}_0}(0)=\boldsymbol{r}_{\boldsymbol{x}_0}(L)$. Moreover, condition (2.3) says that the tangent vector to $\boldsymbol{r}_{\boldsymbol{x}_0}$ is continuous, that is $\boldsymbol{r}'_{\boldsymbol{x}_0}(0)=\boldsymbol{r}'_{\boldsymbol{x}_0}(L)$. We also point out that $\boldsymbol{r}_{\boldsymbol{x}_0}$ belongs to $W^{2,p}((0,L);\mathbb{R}^3)$ but, in general, it does not belong to $W^{3,p}((0,L);\mathbb{R}^3)$ even if

[1]There is no practical reason for interest in $p\neq 2$, but the mathematics is the same, hence we decide to take any $p>1$.

$\boldsymbol{n} \in W^{1,p}((0,L);\mathbb{R}^3)$. Hence, the regularity of admissible curves is in between C^1 and C^2. Finally, we let

$$\kappa = \boldsymbol{t}' \cdot \boldsymbol{n}, \quad \tau = \boldsymbol{n}' \cdot \boldsymbol{b}.$$

Notice that $\kappa, \tau \in L^p(0,L)$. It is easy to see that condition (2.1) implies that the following system holds

$$\begin{cases} \boldsymbol{t}' = \kappa \boldsymbol{n}, \\ \boldsymbol{n}' = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \\ \boldsymbol{b}' = -\tau \boldsymbol{n}, \end{cases} \tag{2.4}$$

which looks like the usual *Serret-Frenet system* of $\boldsymbol{r}_{\boldsymbol{x}_0}$. For that reason, κ and τ can be regarded as the *(signed) weak curvature* of $\boldsymbol{r}_{\boldsymbol{x}_0}$ and the *weak torsion* of $\boldsymbol{r}_{\boldsymbol{x}_0}$ respectively. However, we remark that while in classical Differential Geometry the torsion of a space curve is not defined at a point where the curvature vanishes, for us the quantity $\boldsymbol{n}' \cdot \boldsymbol{b}$ is defined in a weak sense regardless of how big is the set where $\kappa = 0$. We want also to point out that the explicit dependence of f in the variable s is in this case physical because s is the curvilinear abscissa on the solution curve, and also on every other competitor for the minimum, since lengths are conserved. Therefore, a particular feature of f in $s_0 \in [0,L]$ refers to a well-defined material point on the curve. An explicit example of s-dependence could be simply the following one: take an admissible curve $\boldsymbol{r}_0 \in W^{2,p}((0,L);\mathbb{R}^3)$, define κ_0 as its signed curvature and consider the functional

$$\mathcal{E}_0(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) = \int_0^L (\boldsymbol{t}' \cdot \boldsymbol{n} - \kappa_0)^2 \, ds.$$

The curvature κ_0 can be regarded as a sort of *spontaneous curvature*. Of course, $\boldsymbol{r}_0$ is a minimizer for $\mathcal{E}_0$ (in the sense that any moving frame generating $\boldsymbol{r}_0$ is a minimizer). Notice also that in this case we have a minimizer which is C^1 but not necessarily C^2 if $\boldsymbol{r}_0$ is C^1 but not C^2.

3. Minimizers of $\mathcal{E}$

Our first main result is the existence of minimizers for $\mathcal{E}$.

Theorem 3.1. *Assume that:*

$$f(\cdot, a, b) \in L^1(0,L) \text{ for any } a, b \in \mathbb{R}, \tag{3.1}$$

$$f(s, \cdot) \text{ is continuous and convex for any } s \in [0,L], \tag{3.2}$$

$$f(s,a,b) \geq c_1|a|^p + c_2|b|^p + c_3 \text{ for any } a, b \in \mathbb{R}, \tag{3.3}$$

for some $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$. Then $\mathcal{E}$ has a minimizer on W.

Proof. We divide the proof into three steps.

Step 1. We claim that $\inf_W \mathcal{E} < +\infty$. Consider

$$\boldsymbol{t}^*(s) = -\sin\left(\frac{2\pi s}{L}\right) \boldsymbol{e}_1 + \cos\left(\frac{2\pi s}{L}\right) \boldsymbol{e}_2,$$

$$\boldsymbol{n}^*(s) = -\cos\left(\frac{2\pi s}{L}\right) \boldsymbol{e}_1 - \sin\left(\frac{2\pi s}{L}\right) \boldsymbol{e}_2,$$

and

$$\boldsymbol{b}^*(s) = \boldsymbol{e}_3$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. It is easy to see that $(\boldsymbol{t}^*|\boldsymbol{n}^*|\boldsymbol{b}^*) \in W$. Moreover,

$$\kappa = (\boldsymbol{t}^*)' \cdot \boldsymbol{n}^* = \frac{4\pi^2}{L^2}, \quad \tau = (\boldsymbol{n}^*)' \cdot \boldsymbol{b}^* = 0.$$

Hence

$$\mathcal{E}(\boldsymbol{t}^*|\boldsymbol{n}^*|\boldsymbol{b}^*) = \int_0^L f\left(s, \frac{4\pi^2}{L^2}, 0\right)\, ds \overset{(3.1)}{<} +\infty$$

from which the claim follows.

Step 2. We prove that W is sequentially closed with respect to the weak convergence of $W^{1,p}((0,L);\mathbb{R}^{3\times3})$. In order to prove this, let $(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h)$ be a sequence in W that converges weakly in $W^{1,p}((0,L);\mathbb{R}^{3\times3})$ to $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})$. In particular, $\|\boldsymbol{t}_h\|_{1,p}, \|\boldsymbol{n}_h\|_{1,p}, \|\boldsymbol{b}_h\|_{1,p}$ are bounded. Taking into account the Sobolev compact embedding $W^{1,p}((0,L);\mathbb{R}^3) \hookrightarrow C^0([0,L];\mathbb{R}^3)$ we can say that, up to a subsequence not relabeled, $(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h) \to (\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})$ uniformly on $[0,L]$. Now, since for any $s \in [0,L]$ we have $(\boldsymbol{t}_h(s)|\boldsymbol{n}_h(s)|\boldsymbol{b}_h(s)) \in SO(3)$ and $SO(3)$ is closed in $\mathbb{R}^{3\times3}$ we deduce that $(\boldsymbol{t}(s)|\boldsymbol{n}(s)|\boldsymbol{b}(s)) \in SO(3)$ for any $s \in [0,L]$ since the uniform convergence implies the pointwise convergence. For the same reason condition (2.2) is preserved in the limit as well as the constraint (2.3). It remains to prove that (2.1) passes to the limit. Since $\boldsymbol{t}'_h \to \boldsymbol{t}'$ weakly in $L^p(0,L)$ and $\boldsymbol{b}_h \to \boldsymbol{b}$ uniformly on $[0,L]$ we can say that $\boldsymbol{t}'_h \cdot \boldsymbol{b}_h \to \boldsymbol{t}' \cdot \boldsymbol{b}$ weakly in $L^p(0,L)$. Hence

$$\|\boldsymbol{t}' \cdot \boldsymbol{b}\|_p \le \liminf_{h\to+\infty} \|\boldsymbol{t}'_h \cdot \boldsymbol{b}_h\|_p = 0$$

from which we obtain $\boldsymbol{t}' \cdot \boldsymbol{b} = 0$.

Step 3. The proof now uses the Direct Method of the Calculus of Variations. Let $(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h)$ be a minimizing sequence for $\mathcal{E}$ on W, that is $(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h) \in W$ for any $h \in \mathbb{N}$ and

$$\lim_{h\to+\infty} \mathcal{E}(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h) = \inf_W \mathcal{E}.$$

Since $\inf_W \mathcal{E} < +\infty$, by (3.3) we can say that $\|(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h)\|_{1,p}$ is bounded. Then, up to a subsequence (not relabeled), we get $(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h) \rightharpoonup (\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \in W$ because of the weak closure of W. As before, notice that $\boldsymbol{t}'_h \cdot \boldsymbol{n}_h \to \boldsymbol{t}' \cdot \boldsymbol{n}$ and $\boldsymbol{n}'_h \cdot \boldsymbol{b}_h \to \boldsymbol{n}' \cdot \boldsymbol{b}$ both weakly in $L^p(0,L)$. Since condition (3.2) guarantees the lower semicontinuity of $\mathcal{E}$ with respect to the weak topology of $L^p(0,L)$ we conclude that

$$\mathcal{E}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \le \liminf_{h\to+\infty} \mathcal{E}(\boldsymbol{t}_h|\boldsymbol{n}_h|\boldsymbol{b}_h) = \inf_W \mathcal{E}$$

which ends the proof. □

4. Critical points of $\mathcal{E}$

In this section we want to find the first-order necessary conditions for minimizers of $\mathcal{E}$. We first recall some notation, see for instance [2, Sec. 1.3]. If X, Y are real Banach spaces we denote by $\mathcal{L}(X,Y)$ the space of all linear and continuous functionals $X \to Y$. On $\mathcal{L}(X,Y)$ we put the operator norm, namely for any $T \in \mathcal{L}(X,Y)$ we let

$$\|T\| = \sup_{\|x\|_X=1} \|T(x)\|_Y.$$

Let $x_0 \in X$. A map $\mathcal{F}\colon X \to Y$ is said to be *Fréchet differentable at* x_0 if there exists $\mathcal{F}'(x_0) \in \mathcal{L}(X,Y)$ such that

$$\lim_{\|v\|\to 0} \frac{\mathcal{F}(x_0+v) - \mathcal{F}(x_0) - \mathcal{F}'(x_0)(v)}{\|v\|} = 0.$$

We also say that $\mathcal{F} \in C^1(X,Y)$ if $x \mapsto \mathcal{F}'(x)$ is continuous as a map $X \to \mathcal{L}(X,Y)$. It turns out that if there exists $L(x_0) \in \mathcal{L}(X,Y)$ such that

$$L(x_0)(v) = \frac{d}{d\sigma}\mathcal{F}(x_0+\sigma v)_{|\sigma=0}$$

and $L\colon X \to \mathcal{L}(X,Y)$ is continuous at x_0 then $\mathcal{F}$ is Fréchet differentable at x_0 and $\mathcal{F}'(x_0) = L(x_0)$. In particular, if L is continuous then $\mathcal{F} \in C^1(X,Y)$.

We now recall the infinite-dimensional version of the Lagrange multipliers' method (see, for instance, [28, Sec. 4.14]).

Theorem 4.1. *Let X, Y be two real Banach spaces, $\mathcal{F} \in C^1(X)$ and $\mathcal{G} \in C^1(X;Y)$. Assume that $\mathcal{G}'(x) \neq 0$ whenever $\mathcal{G}(x) = 0$. Let $x_0 \in X$ be such that*

$$\mathcal{F}(x_0) = \min\{\mathcal{F}(x) : x \in X\} \quad \text{and} \quad \mathcal{G}(x_0) = 0.$$

Then there exists a Lagrange multiplier $\lambda \in \mathcal{L}(Y, \mathbb{R})$ such that

$$\mathcal{F}'(x_0) = \lambda(\mathcal{G}'(x_0)).$$

Applying Theorem 4.1 we are ready to obtain a system of first-order necessary conditions for minimizers under suitable regularity and growth assumptions on f. In the sequel, f_ξ will stand for $\frac{\partial f}{\partial \xi}$ for short.

Theorem 4.2. *Assume that f is of class C^1 and satisfies*

$$f(s,a,b) \le c(1 + |a|^p + |b|^p) \tag{4.1}$$

and

$$|f_a(s,a,b)| \le c(1 + |a|^{p-1} + |b|^{p-1}), \quad |f_b(s,a,b)| \le c(1 + |a|^{p-1} + |b|^{p-1}) \tag{4.2}$$

for all $s \in [0, L]$ and any $a, b \in \mathbb{R}$ and for some $c \ge 0$. Let $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \in W$ be a minimizer of $\mathcal{E}$ and let $\kappa = \boldsymbol{t}' \cdot \boldsymbol{n}, \tau = \boldsymbol{n}' \cdot \boldsymbol{b}$. Then, $f_a, f_b \in W^{1,1}(0,L)$ and there exist $\mu \in L^{p'}(0,L)$ with $\mu' \in L^p(0,L)$ and $\boldsymbol{\lambda} \in \mathbb{R}^3$ such that the following conditions hold a.e. on $(0, L)$:

$$\begin{cases} f_b(s,\kappa,\tau)' = \mu\kappa \\ -f_a(s,\kappa,\tau)' = \mu\tau + \boldsymbol{\lambda}\cdot\boldsymbol{n} \\ \kappa f_b(s,\kappa,\tau) - \tau f_a(s,\kappa,\tau) = -\mu' + \boldsymbol{\lambda}\cdot\boldsymbol{b}. \end{cases} \tag{4.3}$$

Proof. Let $X = W^{1,p}((0,L);\mathbb{R}^{3\times 3})$. The free variable in X will be denoted again by $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})$. We define the functional $\mathcal{F} \colon X \to \mathbb{R}$ by

$$\mathcal{F}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) = \int_0^L f(s, \boldsymbol{t}'\cdot\boldsymbol{n}, \boldsymbol{n}'\cdot\boldsymbol{b})\, ds.$$

Fix $(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3) \in X$. Thanks to (4.2) we can differentiate under the integral sign obtaining

$$\begin{aligned} &\frac{d}{d\sigma}\mathcal{F}(\boldsymbol{t}+\sigma\boldsymbol{\eta}_1|\boldsymbol{n}+\sigma\boldsymbol{\eta}_2|\boldsymbol{b}+\sigma\boldsymbol{\eta}_3)_{|\sigma=0} \\ &= \int_0^L f_a(s,\boldsymbol{t}'\cdot\boldsymbol{n},\boldsymbol{n}'\cdot\boldsymbol{b})(\boldsymbol{n}\cdot\boldsymbol{\eta}_1' + \boldsymbol{t}'\cdot\boldsymbol{\eta}_2)\, ds + \int_0^L f_b(s,\boldsymbol{t}'\cdot\boldsymbol{n},\boldsymbol{n}'\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{\eta}_2' + \boldsymbol{n}'\cdot\boldsymbol{\eta}_3)\, ds \\ &\quad =: L(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\eta}_3), \end{aligned}$$

where L is a linear operator. It is easy to see that

$$|L(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3)| \le m\|(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})\|_X \|(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3)\|_X$$

for a suitable constant $m > 0$ by (4.2), the Hölder inequality and the continuous embeddings

$$W^{1,p}((0,L);\mathbb{R}^3) \hookrightarrow C^0([0,L];\mathbb{R}^3), \quad L^p((0,L);\mathbb{R}^3) \hookrightarrow L^1((0,L);\mathbb{R}^3).$$

Therefore, L is also continuous and

$$|L(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3)| \le m\|(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})\|_X$$

whenever $\|(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3)\|_X = 1$. We can then conclude that $\mathcal{F} \in C^1(X)$ and

$$\begin{aligned}&\mathcal{F}'(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3)\\&= \int_0^L f_a(s, \boldsymbol{t}'\cdot\boldsymbol{n}, \boldsymbol{n}'\cdot\boldsymbol{b})(\boldsymbol{n}\cdot\boldsymbol{\eta}_1' + \boldsymbol{t}'\cdot\boldsymbol{\eta}_2)\,ds + \int_0^L f_b(s, \boldsymbol{t}'\cdot\boldsymbol{n}, \boldsymbol{n}'\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{\eta}_2' + \boldsymbol{n}'\cdot\boldsymbol{\eta}_3)\,ds\end{aligned}$$

for any $(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3) \in X$. Next, we consider the constraints. We let

$$Y = L^p(0,L) \times L^p(0,L) \times L^p(0,L) \times L^p((0,L);\mathbb{R}^3) \times L^p(0,L) \times \mathbb{R}^3 \times \mathbb{R}^3$$

equipped with the product topology in order to get a Banach space. We define $\mathcal{G}\colon X \to Y$ as

$$\mathcal{G}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) = \left(\boldsymbol{t}\cdot\boldsymbol{t} - 1, \boldsymbol{n}\cdot\boldsymbol{n} - 1, \boldsymbol{t}\cdot\boldsymbol{n}, \boldsymbol{b} - \boldsymbol{t}\times\boldsymbol{n}, \boldsymbol{t}'\cdot\boldsymbol{b}, \int_0^L \boldsymbol{t}\,ds, \boldsymbol{t}(L) - \boldsymbol{t}(0)\right).$$

Using the same argument as before, we can easily see that $\mathcal{G} \in C^1(X, Y)$ and

$$\begin{aligned}&\mathcal{G}'(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3)\\&= \Big(2\boldsymbol{t}\cdot\boldsymbol{\eta}_1, 2\boldsymbol{b}\cdot\boldsymbol{\eta}_2, \boldsymbol{t}\cdot\boldsymbol{\eta}_2 + \boldsymbol{n}\cdot\boldsymbol{\eta}_1, \boldsymbol{\eta}_3 + \boldsymbol{n}\times\boldsymbol{\eta}_1 - \boldsymbol{t}\times\boldsymbol{\eta}_2, \boldsymbol{b}\cdot\boldsymbol{\eta}_1' + \boldsymbol{t}'\cdot\boldsymbol{\eta}_3,\\&\qquad \int_0^L \boldsymbol{\eta}_1\,ds, \boldsymbol{\eta}_1(L) - \boldsymbol{\eta}_1(0)\Big).\end{aligned}$$

Moreover, $\mathcal{G}'(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \neq 0$ for any $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \in X$ such that $\mathcal{G}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) = 0$. Then, by construction a minimizer of $\mathcal{E}$ is a constrained minimizer of $\mathcal{F}$ on $\{\mathcal{G} = 0\}$. From now on $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})$ will denote such a minimizer and for brevity we also let $f_a = f_a(s, \kappa, \tau)$ and $f_b = f_b(s, \kappa, \tau)$. Applying Theorem 4.1, we can say that there exist Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3 \in L^{p'}(0,L)$, $\boldsymbol{\lambda}_4 \in L^{p'}((0,L);\mathbb{R}^3)$, $\mu \in L^{p'}(0,L)$, $\boldsymbol{\lambda} \in \mathbb{R}^3$ such that

$$\begin{aligned}&\int_0^L f_a\boldsymbol{n}\cdot\boldsymbol{\eta}_1'\,ds + \int_0^L f_a\boldsymbol{t}'\cdot\boldsymbol{\eta}_2\,ds + \int_0^L f_b\boldsymbol{b}\cdot\boldsymbol{\eta}_2'\,ds + \int_0^L f_b\boldsymbol{n}'\cdot\boldsymbol{\eta}_3\,ds\\&= \int_0^L (2\lambda_1\boldsymbol{t} + \lambda_3\boldsymbol{n} + \boldsymbol{\lambda}_4\times\boldsymbol{n} + \boldsymbol{\lambda})\cdot\boldsymbol{\eta}_1\,ds + \int_0^L \mu\boldsymbol{b}\cdot\boldsymbol{\eta}_1'\,ds\\&\quad + \int_0^L (2\lambda_2\boldsymbol{n} + \lambda_3\boldsymbol{t} - \boldsymbol{\lambda}_4\times\boldsymbol{t})\cdot\boldsymbol{\eta}_2\,ds + \int_0^L (\boldsymbol{\lambda}_4 + \mu\boldsymbol{t}')\cdot\boldsymbol{\eta}_3\,ds\end{aligned} \tag{4.4}$$

for any $(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2|\boldsymbol{\eta}_3) \in W_0^{1,p}((0,L);\mathbb{R}^{3\times3})$. Using $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 = 0$ and the arbitrariness of $\boldsymbol{\eta}_3$ we easily obtain

$$\boldsymbol{\lambda}_4 = -\mu\boldsymbol{t}' + f_b\boldsymbol{n}'.$$

Now, using $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_3 = 0$ and $\boldsymbol{\eta}_2 = \varphi\boldsymbol{b}$ with $\varphi \in C_c^1(0,L)$ we deduce that

$$-f_b' = \boldsymbol{\lambda}_4\cdot\boldsymbol{n} = -\mu\,\boldsymbol{t}'\cdot\boldsymbol{n}$$

which is (4.3)$_1$. Next, taking $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_3 = 0$ and $\boldsymbol{\eta}_2 = \varphi\boldsymbol{t}$ we easily get

$$\lambda_3 = 0.$$

Finally, considering $\boldsymbol{\eta}_2 = \boldsymbol{\eta}_3 = 0$ and $\boldsymbol{\eta}_1 = \varphi\boldsymbol{n}$ or $\boldsymbol{\eta}_1 = \varphi\boldsymbol{b}$ we arrive at

$$-f_a' = \boldsymbol{\lambda}\cdot\boldsymbol{n} + \mu\,\boldsymbol{n}'\cdot\boldsymbol{b},$$

which is (4.3)$_2$ and

$$(\boldsymbol{t}'\cdot\boldsymbol{n})f_b - (\boldsymbol{n}'\cdot\boldsymbol{b})f_a = -\mu' + \boldsymbol{\lambda}\cdot\boldsymbol{b}.$$

which is exactly (4.3)$_3$, and the proof is complete. □

Remark 4.3. We point out that from (4.4) we can deduce other conditions that permit us to find λ_1 and λ_2. Actually, we did not consider these relations in the previous proof since they are not necessary to get (4.3).

Assuming a priori regularity we can eliminate the Lagrange multipliers in the system (4.3) obtaining the following result.

Theorem 4.4. *Assume that f is of class C^3 and let $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \in W$ be a solution of the system (4.3) with $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ of class C^4. Then at any point where $\kappa \neq 0$ we have*

$$\begin{cases} 2(\tau f_a)' - \tau' f_a - \left(\dfrac{f_b'}{\kappa}\right)'' + \dfrac{\tau^2}{\kappa} f_b' - (\kappa f_b)' = 0 \\ -\kappa f_a' - \tau f_b' = \left(\dfrac{f_a''}{\kappa} - \dfrac{\tau^2}{\kappa} f_a + \dfrac{2\tau}{\kappa}\left(\dfrac{f_b'}{\kappa}\right)' + \tau' \dfrac{f_b'}{\kappa^2} + \tau f_b\right)' \end{cases} \tag{4.5}$$

where $f_a = f_a(s, \kappa, \tau)$ and $f_b = f_b(s, \kappa, \tau)$.

Proof. We add the proof for completeness since it is just the computation to eliminate the Lagrange multiplier. Let us take a point where $\kappa \neq 0$. From $(4.3)_1$ we get

$$\mu = \frac{f_b'}{\kappa}.$$

Now, differentiating $(4.3)_3$, using the fact that $\boldsymbol{b}' = -\tau \boldsymbol{n}$ and inserting $(4.3)_2$ we easily get

$$\left(\frac{f_b'}{\kappa}\right)'' = -\kappa' f_b - \kappa f_b' + \tau' f_a + \frac{\tau^2}{\kappa} f_b' + 2\tau f_a'$$

which gives $(4.5)_1$. Next, differentiating $(4.3)_2$, using the fact that $\boldsymbol{n}' = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}$ and inserting $(4.3)_3$ we obtain

$$\lambda \cdot \boldsymbol{t} = \frac{f_a''}{\kappa} - \frac{\tau^2}{\kappa} f_a + \frac{2\tau}{\kappa}\left(\frac{f_b'}{\kappa}\right)' + \tau' \frac{f_b'}{\kappa^2} + \tau f_b$$

and then, since $\boldsymbol{t}' = \kappa \boldsymbol{n}$,

$$-\kappa f_a' - \tau f_b' = \left(\frac{f_a''}{\kappa} - \frac{\tau^2}{\kappa} f_a + \frac{2\tau}{\kappa}\left(\frac{f_b'}{\kappa}\right)' + \tau' \frac{f_b'}{\kappa^2} + \tau f_b\right)'$$

which is $(4.5)_2$. □

5. Some examples

In this section we discuss some explicit examples arising from the applications.

(a) The Euler elastica

As recalled in the introduction, the study of the Euler elastica functional

$$\int_0^L \kappa^2 \, ds$$

has been widely developed. We refer here to the papers by Langer and Singer [17–19]. First of all, they proved that the global (and local) minimizer among all $W^{2,2}$ curves C^1-periodics is essentially unique, and it is represented by any circumference with length L. Concerning critical points, there is essentially another planar closed and C^1-periodic critical point, which is known in literature as *lemniscate* (an eight-figure). Moving to spatial curves, a great variety of space critical points in the same class of admissible curves can be obtained and all of them lie on an embedded torus of revolution. Finally, many other critical points can be found if we do not assume the closedness of the curve (for a complete characterization at least in the plane, we refer to [19]). Langer and Singer in [18], also prove that the general equations of elasticae are given by

$$\begin{cases} 2\kappa'' - 2\kappa\tau^2 + \kappa^3 - c_1\kappa = 0 \\ \kappa^2\tau = c_2 \end{cases} \tag{5.1}$$

where c_1, c_2 are constants (the so-called *free elasticae*, that is critical curves without length constraints, are obtained for $c_1 = 0$). We remark again that curves arising from the integration of (5.1) do not need to be closed curve.

Following our notation, we get the Euler functional choosing $f(s,a,b) = a^2$. In this case condition (3.3) is not satisfied so we are not able to apply Theorem 3.1 in order to get minimizers, at least for space curves. Nevertheless, conditions (4.1) and (4.2) are satisfied, so that, we are able to apply Theorem 4.2, since we know, using Langer and Singer [17–19], that at least a minimizer exists. We point out that Langer and Singer [18] obtained (5.1) in the smooth case using essentially the Serret-Frenet moving frame. Our approach does not require regularity a priori but we have to eliminate the Lagrange multipliers from the system (4.3), and this is not obvious because the elimination of the Lagrange multipliers from (4.3) needs to work at points where $\kappa \neq 0$. For all of these reasons, even if we obtain the same conclusions, we prefer to give the complete proof of the next result.

Theorem 5.1. *Let $\mathcal{E} \colon W \to \mathbb{R} \cup \{+\infty\}$ be given by*

$$\mathcal{E}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) = \int_0^L \kappa^2\, ds.$$

Let $(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b}) \in W$ be a minimizer of $\mathcal{E}$. Let $S = \{s \in [0,L] : \kappa(s) = 0\}$. Then S is a set of zero measure. Moreover, $\kappa \in W^{2,2}(0,L)$, $\tau \in W^{1,2}(0,L)$ and there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{cases} 2\kappa'' - 2\kappa\tau^2 + \kappa^3 - c_1\kappa = 0 \\ \kappa^2\tau = c_2 \end{cases} \tag{5.2}$$

everywhere on $[0,L]$.

Proof. Since $f(s,a,b) = a^2$ we get $f_a = 2a$ and $f_b = 0$. We are in position to apply Theorem 4.2. The system (4.3) reads as

$$\begin{cases} \mu\kappa = 0 \\ \boldsymbol{\lambda}\cdot\boldsymbol{n} = -2\kappa' - \mu\tau \\ \boldsymbol{\lambda}\cdot\boldsymbol{b} = -2\kappa\tau + \mu' \end{cases} \tag{5.3}$$

for some $\mu \in W^{1,2}(0,L)$ and $\boldsymbol{\lambda} \in \mathbb{R}^3$. In particular, $(5.3)_2$ gives $\kappa \in W^{1,2}(0,L)$. Now we divide the proof into some steps.

Step 1. Since $(5.3)_1$ we get $\mu_{|[0,L]\setminus S} = 0$. Hence, from $(5.3)_3$ we obtain $\tau_{|[0,L]\setminus S} \in W^{1,2}([0,L]\setminus S)$.

Step 2. We prove now that $\kappa \in W^{2,2}(0,L)$. Since κ is continuous, S is relatively closed in $[0,L]$. Hence, we can write it as

$$S = \bigcup_{h=0}^{+\infty} S_h$$

where S_h are pairwise disjoint and S_h is either a singleton or a closed interval with non-empty interior. On each $S_h = [a_h, b_h]$ with $a_h < b_h$ we change τ as follows:

$$\bar\tau(s) = \tau(a_h^-) + \frac{\tau(b_h^+) - \tau(a_h^-)}{b_h - a_h}(s - a_h), \quad \forall s \in [a_h, b_h],$$

where $\tau(a_h^-), \tau(b_h^+)$ are the left and right traces of τ respectively at a_h and b_h, with the convention $\tau(0^-) = \tau(L^+) = 0$. By construction, we obtain $\bar\tau \in W^{1,2}(0,L)$ and $\bar\tau = \tau$ on $[0,L]\setminus S$.

Let $(\bar{\boldsymbol{t}}|\bar{\boldsymbol{n}}|\bar{\boldsymbol{b}}) \in W^{1,2}((0,L);\mathbb{R}^3)$ be the unique solution of the Cauchy problem

$$\begin{cases} \bar{\boldsymbol{t}}' = \kappa\bar{\boldsymbol{n}} \\ \bar{\boldsymbol{n}}' = -\kappa\bar{\boldsymbol{t}} + \bar{\tau}\bar{\boldsymbol{b}} \\ \bar{\boldsymbol{b}}' = -\bar{\tau}\bar{\boldsymbol{n}} \\ \bar{\boldsymbol{t}}(0) = \boldsymbol{t}(0) \\ \bar{\boldsymbol{n}}(0) = \boldsymbol{n}(0) \\ \bar{\boldsymbol{b}}(0) = \boldsymbol{b}(0). \end{cases}$$

It is easy to see that $(\bar{\boldsymbol{t}}|\bar{\boldsymbol{n}}|\bar{\boldsymbol{b}}) \in W$ (notice that actually $\bar{\boldsymbol{t}} = \boldsymbol{t}$ everywhere). Moreover, $\mathcal{E}(\bar{\boldsymbol{t}}|\bar{\boldsymbol{n}}|\bar{\boldsymbol{b}}) = \mathcal{E}(\boldsymbol{t}|\boldsymbol{n}|\boldsymbol{b})$, hence $(\bar{\boldsymbol{t}}|\bar{\boldsymbol{n}}|\bar{\boldsymbol{b}})$ is still a minimizer of $\mathcal{E}$. Applying Theorem 4.2 again, we deduce that

$$\begin{cases} \bar{\mu}\kappa = 0 \\ \bar{\boldsymbol{\lambda}} \cdot \bar{\boldsymbol{n}} = -2\kappa' - \bar{\mu}\bar{\tau} \\ \bar{\boldsymbol{\lambda}} \cdot \bar{\boldsymbol{b}} = -2\kappa\bar{\tau} + \bar{\mu}' \end{cases} \tag{5.4}$$

for some $\bar{\mu} \in W^{1,2}(0,L)$ and $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^3$. As a consequence of $(5.4)_2$ we obtain $\kappa \in W^{2,2}(0,L)$.

Step 3. We claim that for any relatively open interval $I \subseteq ([0,L] \setminus S)$ there exists $c_I \in \mathbb{R}$ such that

$$2\kappa'' - 2\kappa\tau^2 + \kappa^3 - c_I\kappa = 0, \quad \text{on } I. \tag{5.5}$$

First, on I the system (5.3) reduces to

$$\begin{cases} \boldsymbol{\lambda} \cdot \boldsymbol{n} = -2\kappa' \\ \boldsymbol{\lambda} \cdot \boldsymbol{b} = -2\kappa\tau. \end{cases} \tag{5.6}$$

Differentiating $(5.6)_1$ we get $\boldsymbol{\lambda} \cdot \boldsymbol{n}' = -2\kappa''$. Since $\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}$ we obtain

$$2\kappa'' = -\kappa\boldsymbol{\lambda} \cdot \boldsymbol{t} + \tau\boldsymbol{\lambda} \cdot \boldsymbol{b} \in W^{1,2}(I),$$

from which $\kappa \in W^{3,2}(I)$. In particular, $\kappa \in C^2(\bar{I})$. Combining $\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}$ with $(5.6)_2$ we deduce

$$\frac{2\kappa''}{\kappa} - 2\tau^2 = \boldsymbol{\lambda} \cdot \boldsymbol{t}, \quad \text{on } I. \tag{5.7}$$

As a consequence we obtain

$$\left(\frac{2\kappa''}{\kappa} - 2\tau^2 + \kappa^2\right)' = \boldsymbol{\lambda} \cdot \boldsymbol{t}' + 2\kappa\kappa' = \kappa\boldsymbol{\lambda} \cdot \boldsymbol{n} + 2\kappa\kappa' = 0$$

where the last equality follows from $(5.6)_1$. Then, since I is an interval, there exists $c_I \in \mathbb{R}$ such that

$$\frac{2\kappa''}{\kappa} - 2\tau^2 + \kappa^2 = c_I, \quad \text{on } I,$$

which proves the claim.

Step 4. We prove that the measure of S is zero. First of all, it cannot be $\kappa = 0$ everywhere, because of the constraint (2.2). In order to see that the measure of S is zero it is sufficient to show that in the decomposition of $\{S_h\}_{h\in\mathbb{N}}$ there is no S_h with non-empty interior. Assume by contradiction that there exists $S_h = [a_h, b_h]$ with $a_h < b_h$. Then, either $a_h \in (0,L)$ or $b_h \in (0,L)$. Without loss of generality we can assume $b_h \in (0,L)$ (the argument for a_h is the same). Then $\kappa \neq 0$ on $(b_h, b_h + \delta)$

for some $\delta > 0$, so that $\kappa \in C^2([b_h, b_h + \delta])$ and using (5.5) we can say that

$$2\kappa'' - 2\kappa\tau^2 + \kappa^3 - c_h\kappa = 0, \quad \text{on } (b_h, b_h + \delta)$$

for some $c_h \in \mathbb{R}$. As a consequence we deduce that κ is a solution of the Cauchy problem

$$\begin{cases} 2\kappa'' - 2\kappa\tau^2 + \kappa^3 - c_h\kappa = 0, & \text{on } (b_h, b_h + \delta) \\ \kappa(b_h) = 0 \\ \kappa'(b_h) = 0 \end{cases}$$

which means that $\kappa = 0$ on $(b_h, b_h + \delta)$, since the previous boundary values problem has a unique solution, and this is a contradiction.

Step 5. We can now conclude the proof. Since S has zero measure, we immediately deduce that $\tau \in W^{1,2}(0, L)$. It remains to show (5.2). Observe that the function

$$\frac{2\kappa''}{\kappa} - 2\tau^2 + \kappa^2$$

is piecewise constant and it coincides with $\boldsymbol{\lambda} \cdot \boldsymbol{t}$ a.e. $t \in [0, L]$, As a consequence,

$$\frac{2\kappa''}{\kappa} - 2\tau^2 + \kappa^2$$

must be constant, which ends the proof. □

(i) Numerical results

In this paragraph we show some numerical results obtained using the software Mathematica (Wolfram Inc., version 12) concerning solutions of the system (5.2). We do separate analysis for planar curves and for space curves.

Planar curves. In this case $c_2 = 0$. First of all, we pass to the general Cauchy problem for κ, namely

$$\begin{cases} 2\kappa^3\kappa'' + \kappa^6 - c_1\kappa^4 = 0, \\ \kappa(0) = \kappa_0, \\ \kappa'(0) = \kappa_1. \end{cases} \tag{5.8}$$

The idea is to integrate numerically the system (5.8) and then try to reconstruct the shape of the curve. Without loss of generality we can look for the curve $\boldsymbol{r}\colon [0, L] \to \mathbb{R}^2$

$$\boldsymbol{r}(s) = \int_0^s \boldsymbol{t}\, dr$$

where the tangent vector $\boldsymbol{t} = (t^{(1)}, t^{(2)})$ solves the system

$$\begin{cases} (t^{(1)})' = -\kappa t^{(2)} \\ (t^{(2)})' = \kappa t^{(1)} \\ t^{(1)}(0) = 1 \\ t^{(2)}(0) = 0. \end{cases} \tag{5.9}$$

In other words, we require the curve to be clamped at the origin which "starts" with the canonical orthonormal frame. However, we point out that in general it is not necessarily true that $\boldsymbol{r}$ is admissible: for instance we do not have implemented any closedness of $\boldsymbol{r}$. From Differential Geometry, it is known that the necessary and sufficient conditions for a planar curve to be closed

	c_1	κ_0	κ_1	L
Circumference (Fig. 1a)	1.00824	1.01227	0.0003	2π
Lemniscate (Fig. 1b)	0.07911031	0.0442	0.046801	12π

Table 1: Numerical values for the circumference and the lemniscate.

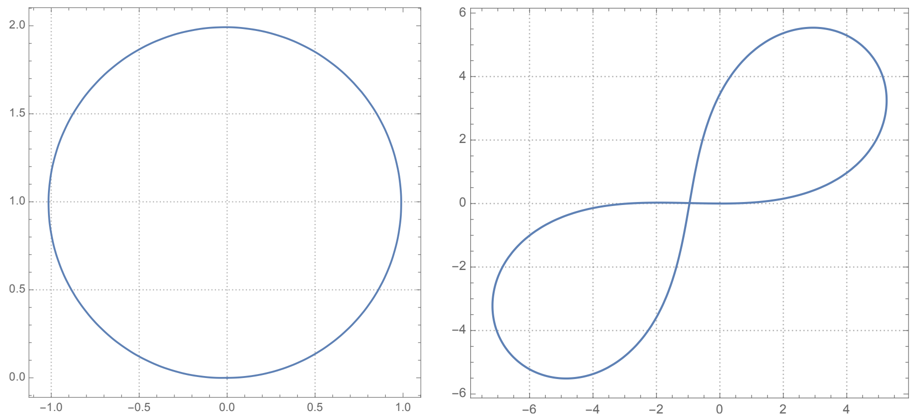
are the following identities

$$\int_0^L \cos\left(\int_0^t \kappa(s)\,ds\right)\,dt = \int_0^L \sin\left(\int_0^t \kappa(s)\,ds\right)\,dt = 0.$$

We also anticipate the fact that for space curves there are no similar conditions on κ, τ in order to guarantee that the curve is closed. For details we refer to [21]. Actually, for numerical reasons we decide to introduce a stop condition in the numeric integration of (5.8)–(5.9): we vary randomly the constants c_1, κ_0, κ_1 until the inequality

$$d = |\boldsymbol{r}(L)| + |\boldsymbol{t}(L) - (1,0)| < 10^{-6} \tag{5.10}$$

is satisfied. Condition (5.10) formally implies that the obtained curve $\boldsymbol{r}$ is *almost closed* as well as its tangent vector.



(a) Circumference (b) Lemniscate

Figure 1: Closed elasticae: the circumference (A) (the only one stable), and the lemniscate (B) (unstable, see [19, Ex. 3]).

As expected, in the plane, the only two curves we obtain are the circumference, as the one in Fig. 1a, and the lemniscate, see Fig. 1b, with the values reported in Table 1. Moreover, not imposing Eq. (5.10), we obtain the open planar curves of [19]

Space curves. In this case we do not have $c_2 = 0$, so we have to deal with the complete system (5.2). Again, we can look for the curve $\boldsymbol{r} \colon [0, L] \to \mathbb{R}^3$ given by

$$\boldsymbol{r}(s) = \int_0^s \boldsymbol{t}\,dr$$

	c_1	c_2	κ_0	κ_1	L
Fig. 2a	1.25316	3.92702	1.58313	0.528316	16π
Fig. 2b	0.08	5.06	2.53458	4.04	3π
Fig. 2c	2.06465	4.38778	1.51781	1.47094	16π
Fig. 2d	1.62767	4.08942	2.85503	0.669953	30π

Table 2: Numerical values chosen to plot the elasticae in Fig. 2.

where now the tangent vector $\boldsymbol{t}$ is the solution of

$$\begin{cases} \boldsymbol{t}' = \kappa \boldsymbol{n}, \\ \boldsymbol{n}' = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \\ \boldsymbol{b}' = -\tau \boldsymbol{n}, \\ \boldsymbol{t}(0) = (1,0,0), \\ \boldsymbol{n}(0) = (0,1,0), \\ \boldsymbol{b}(0) = (0,0,1), \end{cases}$$

To find closed spatial curves, we use the stop condition

$$d = |\boldsymbol{r}(L)| + |\boldsymbol{t}(L) - (1,0,0)| < 10^{-6}, \tag{5.11}$$

similar to the one introduced in the plane. The results for closed elasticae are reported in Fig. 2 and in the corresponding Table 2 the values we used to obtain such a figures. Remarkably, eliminating Eq. (5.11), we can even obtain open curves.

(b) The model case in the biological applications

One of the most common energies arising from the applications to Biophyiscs [26] takes the form

$$\int_0^L \frac{\kappa^2 + \tau^2}{2}\, ds. \tag{5.12}$$

Following our notation, let

$$f(s,a,b) = f(a,b) = \frac{a^2+b^2}{2}.$$

It is straightforward to see that f satisfies all the assumptions (3.1)-(3.2)-(3.3)-(4.1)-(4.2) with the choice $p = 2$. As a consequence we have existence of minimizers. Of course, since planar curves have $\tau = 0$ as in the case of the Euler elastica, any circumference of length L is still a minimizer. Concerning critical points, first of all, we notice that f is smooth and we have $f_a = a$ and $f_b = b$. Then the system (4.3) reads as

$$\begin{cases} \tau' = \mu\kappa, \\ \boldsymbol{\lambda}\cdot\boldsymbol{n} = -\kappa' - \mu\tau, \\ \boldsymbol{\lambda}\cdot\boldsymbol{b} = \mu'. \end{cases} \tag{5.13}$$

It is easy to see that the solutions κ, τ of that system are smooth and we can therefore apply Theorem 4.4. Then, on any interval I where $\kappa \neq 0$ we have

$$\begin{cases} \tau\kappa' - \left(\dfrac{\tau'}{\kappa}\right)'' + \dfrac{\tau^2\tau'}{\kappa} = 0 \\ -\kappa\kappa' - \tau\tau' = \left(\dfrac{\kappa''}{\kappa} + \dfrac{2\tau}{\kappa}\left(\dfrac{\tau'}{\kappa}\right)' + \dfrac{(\tau')^2}{\kappa^2}\right)'. \end{cases} \tag{5.14}$$

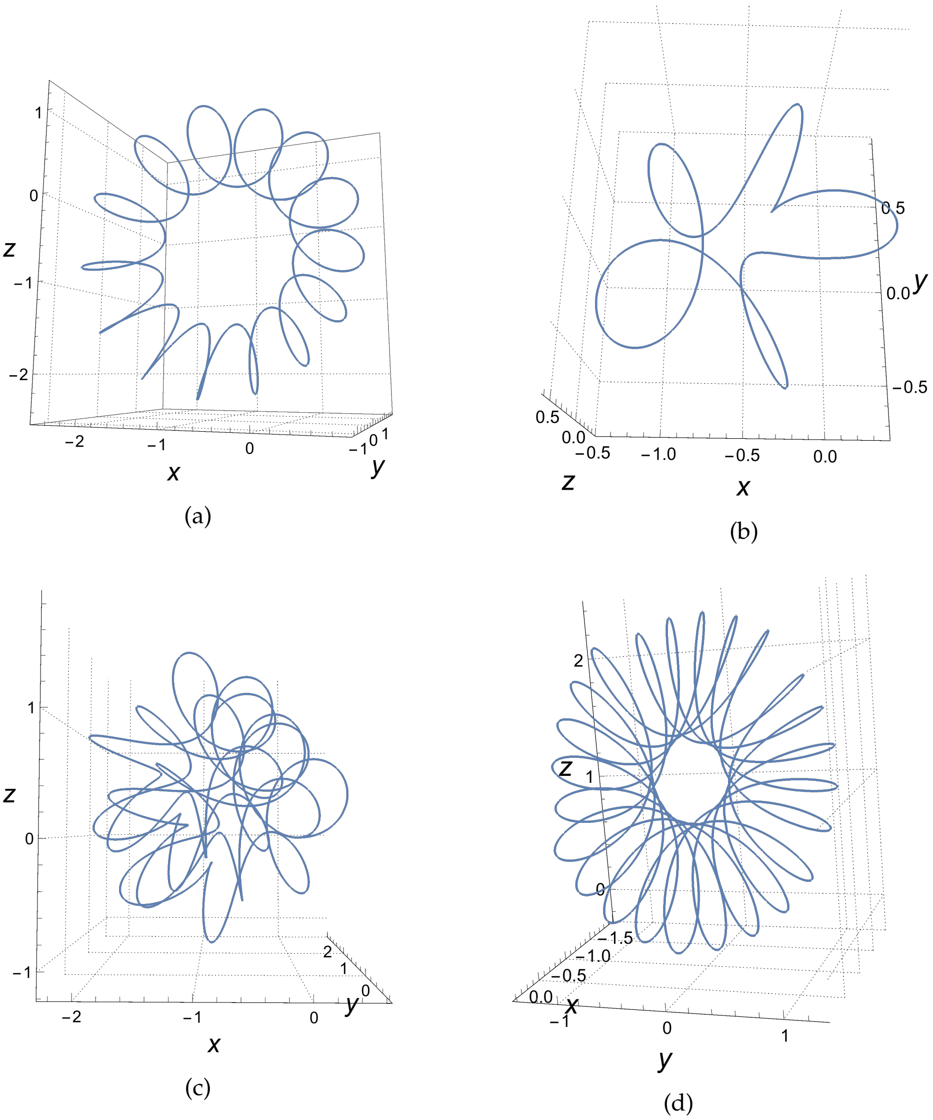


Figure 2: Examples of closed elasticae in the space.

These equations look quite difficult and hold true only when $\kappa \neq 0$. The only fact we put in evidence is that (5.14)$_2$ can be rewritten as

$$-\left(\frac{\kappa^2+\tau^2}{2}\right)' = \left(\frac{\kappa''}{\kappa} + \frac{2\tau}{\kappa}\left(\frac{\tau'}{\kappa}\right)' + \frac{(\tau')^2}{\kappa^2}\right)',$$

that is

$$\frac{\kappa''}{\kappa} + \frac{2\tau}{\kappa}\left(\frac{\tau'}{\kappa}\right)' + \frac{(\tau')^2}{\kappa^2} + \frac{\kappa^2+\tau^2}{2} = c$$

for some constant c. At the end we can reduce the analysis to

$$\begin{cases} \tau\kappa' - \left(\dfrac{\tau'}{\kappa}\right)'' + \dfrac{\tau^2\tau'}{\kappa} = 0, \\[2ex] \dfrac{\kappa''}{\kappa} + \dfrac{2\tau}{\kappa}\left(\dfrac{\tau'}{\kappa}\right)' + \dfrac{(\tau')^2}{\kappa^2} + \dfrac{\kappa^2+\tau^2}{2} = c. \end{cases} \tag{5.15}$$

Concerning the numerical analysis we tried to reproduce the same arguments to solve (5.15). However, we did not succeed in varying randomly the constant c and the initial conditions for κ, τ, since it is very likely that $\kappa = 0$ at some time, thus stopping the numerical procedure.

(c) The Sadowsky energy functional

Following our notation, the Sadowsky energy functional takes the form

$$\int_0^L \frac{(\kappa^2+\tau^2)^2}{\kappa^2}\, ds.$$

As said in the Introduction, Sadowsky [15,16,22,23] obtained such an energy functional as a limit of an elastic Möbius which reduces to its centerline. Some authors tried to give a rigorous justification of such a limit process. We mention only the paper by Freddi et al. [10] (further references can be found therein), where the authors studied the Γ-limit of the bending energy on the Möbius strip with respect to a topology that ensures the convergence of the minimizers. In this way, they obtain as Γ-limit the functional

$$\int_0^L f(\kappa, \tau)\, ds$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(a,b) = \begin{cases} \dfrac{(a^2+b^2)^2}{a^2} & \text{if } |a| > |b|, \\[2ex] 4b^2 & \text{if } |a| \le |b|. \end{cases} \tag{5.16}$$

Such a functional turns out to be the *corrected version* of the Sadowsky functional. It is easy to see that f is continuous and convex: in order to see the convexity, we notice that that if $f_1 : \mathbb{R}^2 \setminus \{(a,b) \in \mathbb{R}^2 : a > 0\} \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$f_1(a,b) = \frac{(a^2+b^2)^2}{a^2}, \quad f_2(a,b) = 4b^2,$$

then f_1, f_2 are both of class C^1 and convex and $\nabla f_1(a,b) = \nabla f_2(a,b)$ whenever $a = |b| > 0$. Then, (3.2) is satisfied. Moreover,

$$\frac{(a^2+b^2)^2}{a^2} = a^2 + 2b^2 + \frac{b^4}{a^2} \ge a^2 + 2b^2$$

while if $|a| \le |b|$ we get $4b^2 \ge 2b^2 + 2a^2 \ge a^2 + 2b^2$. As a consequence,

$$f(a,b) \ge a^2 + 2b^2$$

which shows (3.3) with $p = 2$. We are therefore in position to apply Theorem 3.1, hence we have existence of minimizers.

Remark 5.2. The system of critical points for the Sadowsky functional takes a very complicated form due to the complexity of the energy density Eq. (5.16), hence we are not going to write explicitly the first-order conditions for minimizers.

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References

1. A. Dall'Acqua, M. Novaga and A. Pluda, *Minimal elastic networks*, Indiana Univ. Math. J. 69 (2020), 1909–1932.
2. M. Badiale and E. Serra, *Semilinear Elliptic Equations for Beginners*, Springer-Verlag London, 2011.
3. G. Bevilacqua, L. Lussardi, A. Marzocchi, *Soap film spanning electrically repulsive elastic protein links*, Proceedings of School & Research Workshop *Mathematical Modeling of Self-Organizations in Medicine, Biology and Ecology: from micro to macro*, Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur. 96 (2018), suppl. 3, A1, 13pp.
4. G. Bevilacqua, L. Lussardi, A. Marzocchi, *Soap film spanning an elastic link*, Quart. Appl. Math. 77 (3) (2019), 507–523.
5. G. Bevilacqua, L. Lussardi, A. Marzocchi, *Dimensional reduction of the Kirchhoff-Plateau problem*, J. Elasticity 140, 135–148 (2020).
6. R. Capovilla, C. Chryssomalakos and J. Guven, *Hamiltonians for curves*, J. Phys. A: Math. Gen. 35 (2002), 6571–6587.
7. B. Coleman, and D. Swigon, *Theory of supercoiled elastic rings with self-contact and its application to DNA plasmids*, Journal of Elasticity and the Physical Science of Solids 60(3), 173 (2000).
8. K. A. Dill, S. B. Ozkan, M. S. Shell, and T. R. Weikl, *The Protein Folding Problem*, Annu. Rev. Biophys., 37 (2008), 289–316.
9. A. Feoli, V. V. Nesterenko, and G. Scarpetta, *Functionals linear in curvature and statistics of helical proteins*, Nuclear Physics B 705(3), (2005), 577–592.
10. L. Freddi, P. Hornung, M.G. Mora and R. Paroni, *A corrected Sadowsky functional for inextensible elastic ribbons*, J. Elasticity 123 (2016), 125–136.
11. H. Garcke, J. Menzel and A. Pluda, *Long time existence of solutions to an elastic flow of networks*, Comm. Partial Differential Equations 45 (2020), 1253–1305.
12. G.G. Giusteri, P. Franceschini and E. Fried, *Instability paths in the Kirchhoff-Plateau problem*, J. Nonlinear Sci. 26 (2016), 1097–1132.
13. G.G. Giusteri, L. Lussardi, E. Fried, *Solution of the Kirchhoff-Plateau problem*, J. Nonlinear Sci. 27 (2017), 1043–1063.
14. O. Gonzalez, J. H. Maddocks, F. Schuricht and H. von der Mosel, *Global curvature and self-contact of nonlinearly elastic curves and rods*, Calc. Var. 14 (2002), 29–68.
15. D.F. Hinz and E. Fried, *Translation of Michael Sadowsky's Paper "An Elementary Proof for the Existence of a Developable Möbius Band and the Attribution of the Geometric Problem to a Variational Problem"*, J. Elasticity 119 (2015), 3–6.
16. D.F. Hinz and E. Fried, *Translation and Interpretation of Michael Sadowsky's Paper "Theory of Elastically Bendable Inextensible Bands with Applications to the Möbius Band"*, J. Elasticity 119 (2015), 7–17.
17. J. Langer and D.A. Singer, *Knotted elastic curves*, J. London. Math. Soc. 30 (1984), 512–520.
18. J. Langer and D.A. Singer, *The total squared curvature of closed curves*, J. Diff. Geom. 20 (1984), 1–22.
19. J. Langer and D.A. Singer, *Curve straightening and a minimax argument for closed elastic curves*, Topology, 24 (1985), 75–88.
20. J. Langer and D.A. Singer, *Lagrangian aspects of the Kirchhoff elastic rod*, SIAM Rev. 38 (1996), 605–618.
21. Y.A. Nikolaevskii, *On Fenchel's problem*, Mathematical Notes 56 (1994), 1158–1164.
22. M. Sadowsky, *Ein elementarer Beweis für die Existenz eines abwickelbaren Möbiuschen Bandes und die Zurückführung des geometrischen Problems auf ein Variationsproblem*, Sitzungsber. Preuss. Akad. Wiss. 1930, Mitteilung vom 26. Juni, 412–415.

23. M. Sadowsky, *Theorie der elastisch biegsamen undehnbaren Bänder mit Anwendungen auf das Möbiussche Band*, Verhandl. des 3. Intern. Kongr. f. Techn. Mechanik 2 (1930), 444–451.
24. F. Schuricht, *Global injectivity and topological constraints for spatial nonlinearly elastic rods*, J. Nonlinear Sci. 12 (2002), 423–444.
25. E.L. Starostin and G.H.M. van der Heijden, *The shape of a Möbius strip*, Nature materials 6(8) (2007), 563–567.
26. E.L. Starostin and G.H.M. van der Heijden, *Force and moment balance equations for geometric variational problems on curves*, Phys. Rev. E 79 (2009), 066602.
27. N. Thamwattana, J. McCoy and J. Hill, *Energy density functions for protein structures*, Q. J. Mech. Appl. Math. 61(3) (2008), 431–451.
28. E. Zeidler, *Applied Functional Analysis, Main Principles and Their Applications*, Springer-Verlag, New York, 1995.
29. S. Zhang, X. Zuo, M. Xia, S. Zhao and E. Zhang, *General equilibrium shape equations of polymer chains*, Phys. Rev. E 70 (2004), 051902.