CONTROL OF HYPERBOLIC AND PARABOLIC EQUATIONS ON NETWORKS AND SINGULAR LIMITS

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Abstract. We study the controllability properties of the transport equation and of parabolic equations with vanishing viscosity posed on a tree. Using a control localized on the exterior nodes, we prove that the hyperbolic and the parabolic systems are null-controllable. The hyperbolic proof relies on the method of characteristics, the parabolic one on duality arguments and Carleman inequalities. We estimate the cost of the null-controllability of transport-diffusion equations with diffusivity $\varepsilon > 0$ and study its asymptotic behavior when $\varepsilon \to 0^+$. We prove that the cost of the controllability decays for a time sufficiently large and explodes for short times. This is done by duality arguments allowing to reduce the problem to obtain observability estimates which depend on the viscosity parameter. These are derived by using Agmon and Carleman inequalities.

1. Introduction

In the past few decades, models based on partial differential equations have been very effective in tackling many problems dealing with flows on networks (e.g. irrigation channels, gas pipelines, blood circulation, vehicular traffic, supply chains, air traffic management – see [11] for a survey of the topic).

In particular, linear advection-diffusion equations on graphs have been employed to describe the flow of a fluid with a dissolved contaminant through a network of one-dimensional cracks (see [66]). Following [31], we represent the network by a finite, directed, and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V} = \{v_1, \dots, v_n\}$ and edges $\mathcal{E} = \{e_1, \dots, e_m\} \subset \mathcal{V} \times \mathcal{V}$. For every edge $e \in \mathcal{E}$, we define an incidence vector $(n^e)_{v\in\mathcal{V}}$ by

$$n^e(v) = -1 \ \text{ if } e = (v, \cdot), \quad n^e(v) = 1 \ \text{ if } e = (\cdot, v), \quad \text{ and } \quad n^e(v) = 0 \ \text{ otherwise.}$$

Here, n^e plays the same role as the outwards normal vector for problems in multi-dimensional domains. For any $v \in \mathcal{V}$, we define the set of incident edges $\mathcal{E}(v) := \{e \in \mathcal{E} : n^e(v) \neq 0\}$; we also distinguish between inner vertices $\mathcal{V}_0 := \{v \in \mathcal{V} : |\mathcal{E}(v)| \geq 2\}$ and boundary or external vertices $\mathcal{V}_\partial := \mathcal{V}\backslash\mathcal{V}_0$. We suppose that for all $v \in \mathcal{V}_0$ there are $e_1, e_2 \in \mathcal{E}(v)$ such that $n^{e_1}(v) = -1$ and $n^{e_2}(v) = 1$. We may model the proposed problem with the system

$$\begin{cases} a^e\partial_t y^e_\varepsilon + b^e\partial_x y^e_\varepsilon - \varepsilon\partial^2_{xx} y^e_\varepsilon = 0, & \text{in } (0,T)\times e,\ e \in \mathcal{E}, \\ y^e_\varepsilon(t,v) = u^v_\varepsilon(t), & \text{on } (0,T)\times\mathcal{V}_\partial, \\ y^{e_1}_\varepsilon(t,v) = y^{e_2}_\varepsilon(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e\in\mathcal{E}(v)}(b^e y^e_\varepsilon - \varepsilon\partial_x y^e_\varepsilon)n^e = 0, & \text{on } (0,T)\times\mathcal{V}_0, \\ y_\varepsilon(0,\cdot) = y_0, & \text{on } \mathcal{E}, \end{cases} \tag{1.1}$$

with $a^e, b^e, \varepsilon > 0$ and for a fixed time-horizon $T > 0$. Each edge is identified with a closed interval $[x_e, \tilde{x}_e]$, and the left end (resp. right end) is identified with the vertex v such that $n^e(v) = -1$ (resp. $n^e(v) = 1$). The boundary condition $(1.1)_3$ is a continuity condition in the internal nodes and $(1.1)_4$ implies that the flux of the mass is null. Here, y^e_ε denotes the concentration of the contaminant, b^e the flow rate in each edge of the graph and u^e_ε is the boundary datum on each boundary vertex of the graph, which we may use as control. We additionally assume

$$\sum_{e\in\mathcal{E}(v)} b^e n^e(v) = 0, \qquad v \in \mathcal{V}_0, \tag{1.2}$$

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which is a balance relation for the flow across a junction. As well as being an assumption used in the parabolic energy estimate to annihilate the terms at junctions, condition (1.2) is motivated by the corresponding hyperbolic problem, for which it ensures that energy does not increase at junctions (see Remark 3.5). Such condition is also pivotal in proving the vanishing viscosity approximation result in [32].

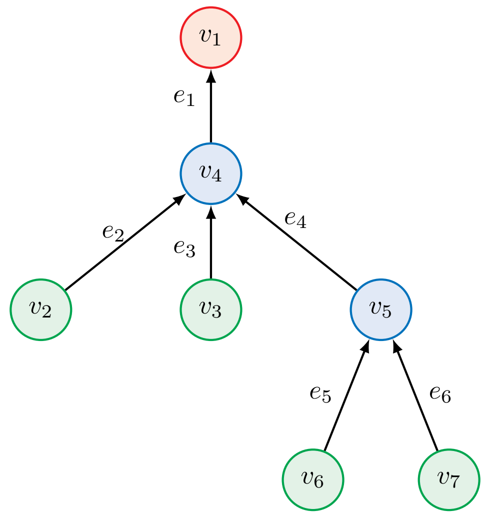

FIGURE 1. An example of tree with edges $e_1 = (v_4, v_1)$, $e_2 = (v_2, v_4)$, $e_3 = (v_3, v_4)$, and $e_4 = (v_5, v_4)$, $e_5 = (v_6, v_5)$, $e_6 = (v_7, v_5)$; inner vertices $\mathcal{V}_0 = \{v_4, v_5\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_3, v_6, v_7\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{in} = \{v_2, v_3, v_6, v_7\}$ (green) and $\mathcal{V}_\partial^{out} = \{v_1\}$ (red), respectively. The set $\mathcal{E}(v_4) = \{e_1, e_2, e_3, e_4\}$ denotes the edges adjacent to the junction v_4. In a similar manner, we can split the set $\mathcal{E}(v_4)$ by $\mathcal{E}^{in}(v_4) = \{e_2, e_3, e_4\}$ and $\mathcal{E}^{out}(v_4) = \{e_1\}$ into edges that go into or out of the vertex v_4. The arrows illustrate the flow direction. The non-zero incidence vectors are $n^{e_1}(v_1) = n^{e_2}(v_4) = n^{e_3}(v_4) = n^{e_4}(v_4) = n^{e_5}(v_5) = n^{e_6}(v_5) = 1$ and $n^{e_1}(v_4) = n^{e_2}(v_2) = n^{e_3}(v_3) = n^{e_4}(v_5) = n^{e_5}(v_6) = n^{e_6}(v_7) = -1$.

In this paper, we study the appropriate boundary data (*boundary control*) in order to lead the solution of the parabolic problem (1.1) to rest, and, in particular, its behavior as the diffusivity parameter ε vanishes. In other words, we are interested in controlling the solution of (1.1) across the network and in studying the cost of controllability as $\varepsilon \to 0^+$, i.e. as the solution of (1.1) approaches the one of the hyperbolic problem

$$\begin{cases} a^e \partial_t y^e + b^e \partial_x y^e = 0, & \text{in } (0,T) \times e,\ e \in \mathcal{E}, \\ y^e(t,v) = u^v(t), & \text{on } (0,T) \times \mathcal{V}_\partial^{in}, \\ y^{e_1}(t,v) = y^{e_2}(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}^{out}(v), \\ \sum_{e \in \mathcal{E}^{in}(v)} b^e y^e = y^{e_1}(t,v) \sum_{e \in \mathcal{E}^{out}(v)} b^e, & \text{on } (0,T) \times \mathcal{V}_0,\ e_1 \in \mathcal{E}^{out}(v), \\ y(0,\cdot) = y_0, & \text{on } \mathcal{E}, \end{cases} \tag{1.3}$$

where the contaminant does not undergo diffusion and is only driven by the (constant) velocity of the liquid flow. Here, $\mathcal{V}_\partial^{in}$ are the vertices from which the flow is coming into the network (i.e. with just outgoing edges), $\mathcal{V}_\partial^{out}$ are the vertices where the flow goes out of the network (i.e. with just incoming edges), $\mathcal{E}^{in}(v)$ are the edges incoming into v, and $\mathcal{E}^{out}(v)$ are the edges outgoing from v (see Figure 1). The junction condition $(1.3)_3$ expresses continuity of the concentrations exiting a junction, and $(1.3)_4$ gives the conservation of the mass. Throughout this paper, we mainly consider tree-shaped networks (that is, networks without loops), an example of which is given in Figure 1.

We remark that, in order to properly approximate the hyperbolic problem, at the parabolic level, we need to choose suitable complementary junction conditions (which amounts to picking a realization of the Laplacian on graphs; see [65]). In particular, for the hyperbolic problem, the number of coupling conditions at each junction $v \in \mathcal{V}_0$ is $|\mathcal{E}^{out}(v)|$, which only suffices to guarantee conservation of mass at the junction and to prescribe the concentrations at the outflow edges. On the other hand, for the parabolic problem, the number of coupling conditions at each junction $v \in \mathcal{V}_0$ is $|\mathcal{E}(v)|$, which allows to guarantee the well-posedness of the system, the continuity of the solution and conservation of mass at the junction.

In order to obtain and understand the limiting behaviour of the controls for (1.1) it is necessary to understand two additional problems that have interest on their own.

- The first problem is the controllability of (1.3) without the assumption (1.2). We show that system (1.3) is controllable for sufficiently large time, and not controllable for small times. In the process, we show that determining the time from which the system is null-controllable is not a simple task (compared to determining it on segments) since it depends on the geometry and metric of the network. Moreover, in networks with loops, applying a null control may not be enough to take the solution to rest. This problem has interest on its own since (1.3) models an inviscid flow on the network.
- The second problem is the study of null-controllability of (1.1). It is known (see [46]) that (1.1) is null controllable by acting on all the external vertices (except at most one), but that it may not be null-controllable if we just act on a smaller subset of the boundary vertices. So what remains open is the study of the behaviour of the cost of the control as $\varepsilon \to 0$.

Our main novel contributions can be then summarized as follows:

- By the method of characteristics, we control the hyperbolic problem (1.3) to zero (for suffiently long times) by acting on the incoming vertices and also discuss the optimal time for which (1.3) is null-controllable, which is not as trivial on graphs as it is on segments.
- We estimate on the cost of controllability for the parabolic problem (1.1), which depends on the time-horizon: for small times, we prove the blow-up of the cost of controllability; for sufficiently long time horizon, we prove its decay.

1.1. **(Linear and nonlinear) flows on networks, control and singular limits.** For scalar conservation laws modeling traffic flow on networks, the convergence of vanishing viscosity approximations was addressed in [22, 3, 20, 21]: under suitable coupling conditions in the junction, the solution of the parabolic approximation exists and, as the viscosity vanishes, it converges to a solution of the original problem (which is entropy admissible – in the sense of [3, 34]). In the context of Hamilton-Jacobi equations, a convergence result for the vanishing viscosity approximation was established in [12] under Kirchhoff-type conditions.

For linear transport equations, further results are available: in [32], suitable coupling condition that guarantee conservation of mass, energy dissipation, and continuity are imposed and a vanishing viscosity convergence result is established; on the other hand, in [40], the equations are coupled by transmission conditions set at the inner node, which do not impose continuity on the unknown and, as the diffusion coefficient vanishes, the family of solutions converges to the unique solution of the first order equations and satisfies suitable transmission conditions at the inner node, which are determined by the parameters appearing in the parabolic transmission conditions. In the present paper, we adopted the coupling conditions of [32].

The study of *uniform controllability problems* for singular perturbations of partial differential equations started with the pioneering works [58, Chapter 3], [55, 56, 63, 62]. In the context of linear advection-diffusion equations in the vanishing viscosity limit, the first result was obtained by Coron and Guerrero in [27], where they made a conjecture on the minimal time needed to achieve uniform controllability. Then, the estimates on this minimal time are improved in [36] with a complex analytic method. The result of [27] was also generalized in several space dimensions and for non-constant transport speed in [41]. For nonlinear transport terms, the only available results have been obtained by Glass and Guerrero (see [37]) for the Burgers equation in the vanishing viscosity limit and later generalized by Léautaud for more general flux functions in [54]. As for other systems, several results have been recently obtained: for the Stokes system (see [7]), for an artificial advection-diffusion problem (see [24, 25]), and for fourth-order parabolic equations (see [15, 64, 47]).

On the other hand, in the network setting, much fewer controllability results are available. The null-controllability of the heat equation on trees with coefficients that depends on the space variable has been established in [4]. In [28], the controllability of several classes of PDEs on networks is considered (wave, Schrödinger, heat, beam, etc.): in particular, in [28, Chapter 8.1] the heat equation with with Kirchhoff-type junction conditions is controlled to zero by the action of a controller under suitable topological assumptions on the graph, which are needed as their proof is based on the transmutation method. For the well-posedness, controllability and stabilization of several hyperbolic problems, we also point to [50, 49, 48, 51] and references therein for models of thermoelastic beams, linked plates, and plate-beam systems). Moreover, we refer to [8, Chapter 1.15], [43, 45, 71, 30, 42, 44] for models arising in water flow, gas transport, etc., and to [29] for the controllability of scalar conservation laws on networks in the context of entropy solutions. Finally, we would like to highlight [5], where the minimal controllability time for the wave equation on metric graphs has been obtained.

Furthermore, results on uniform controllability on networks are, to the best of our knowledge, still open. With the present paper, we aim to address this gap in the literature.

1.2. **Outline of the paper.** The paper is organized as follows.

In Section 2, we introduce the preliminary information on the function spaces used throughout the work and present the known results on the well-posedness of problems (1.3) and (1.1) and on the convergence of (1.1) to (1.3) (which have been obtained in [32]).

In Section 3, we state our main theorems: a controllability result for the hyperbolic problem (1.3) on a tree (for sufficiently long times) and an estimate on the cost of controllability for the parabolic problem (1.1) (which depends on the time-horizon) as the diffusivity parameter vanishes. Moreover, we present some pathological cases to illustrate the sharpness of our results.

In Section 4, we prove the controllability result for the transport equation on a tree by relying on the classical method of characteristics: thanks to the flux conservation condition in (1.3), we are able to argue analogously to the case of a bounded interval, where it suffices to take a null boundary control. We also show that, using non-null boundary controls, we can control the system to zero in a shorter time-span.

Sections 5 and 6 are dedicated to the singular limit problem. In the first one, we prove the blow-up of the cost of controllability for the parabolic problem (1.1) and, in the second one, we prove the decay (for sufficently long time-horizon). Our strategy is based on the H.U.M. and, in particular, on the ideas of [27, 41]. To prove the blow-up, we rely on an Agmon-type inequality. For the proof of the decay, we need to establish a decay property for the L^2-mass of the adjoint system and a Carleman-type inequality.

Section 7 concludes the paper with some open problems for future consideration.

2. Preliminaries

2.1. **Function spaces on a network and parametrization of the edges.** Each edge $e \in \mathcal{E}$ has a positive length l^e and we identify it with the intervals $(0, l^e)$ or, in the case of incoming edges of star graphs, $(-l^e, 0)$.

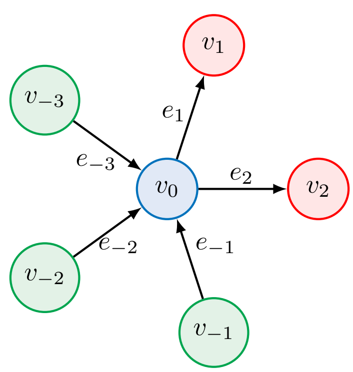

Figure 2. An example of a star-graph (particular case of tree).

Throughout the paper, as in [32], we use the following notation for the space of square-integrable functions:

$$L^2(\mathcal{E}) := L^2(e_1) \times \cdots \times L^2(e_n) = \{w : w^e \in L^2(e),\ e \in \mathcal{E}\},$$

with the norm and scalar product

$$\|w\|^2_{L^2(\mathcal{E})} := \sum_{e\in\mathcal{E}} \|w^e\|^2_{L^2(e)} \quad \text{and} \quad (w_1, w_2)_{L^2(\mathcal{E})} := \sum_{e\in\mathcal{E}} (w_1^e, w_2^e)_{L^2(e)}.$$

Sometimes, we write $\int_{\mathcal{E}} w \,\mathrm{d}x := \sum_{e\in\mathcal{E}} \int_e w^e \,\mathrm{d}x$. We also use the (piecewise) Sobolev space

$$H^s_{pw}(\mathcal{E}) = \{w \in L^2(\mathcal{E}) : w^e \in H^s(e),\ e \in \mathcal{E}\},$$

with

$$\|w\|^2_{H^s_{pw}(\mathcal{E})} := \sum_{e\in\mathcal{E}} \|w^e\|^2_{H^s(e)} \quad \text{and} \quad (w_1, w_2)_{H^s_{pw}(\mathcal{E})} := \sum_{e\in\mathcal{E}} (w_1^e, w_2^e)_{H^s(e)}.$$

Similarly, we define the spaces of (piecewise) k-times differentiable functions $C^k_{pw}(\mathcal{E})$ and the Sobolev spaecs $W^{1,p}_{pw}(\mathcal{E})$. For $s > \frac{1}{2}$, we note that the functions in $H^s_{pw}(\mathcal{E})$ are continuous on $e \in \mathcal{E}$, but may be discontinuous across the junction. For $s > \frac{1}{2}$, we denote by $H^s(\mathcal{E})$, the subspace of functions belonging to $H^s_{pw}(\mathcal{E})$ which are also continuous across the junction. We remark that every $w \in H^1(\mathcal{E})$ has a unique value $w(v)$ at every vertex $v \in \mathcal{V}$ and we use $\ell^2(\mathcal{V})$ to denote the set of possible vertex values.

Also, we define distance between vertices and layers of a tree as follows.

Definition 2.1 (Distance and layers on a graph). *We define the* distance $d(v_1, v_2)$ *between two vertices v_1, v_2 in the graph $\mathcal{G}$ as the minimum number of edges contained in a path joining them (if any, otherwise, $d(v_1, v_2) = \infty$). In addition, for a tree $\mathcal{G}$ with root v, we define the i-th layer of the tree as the vertices at distance i from v.*

Let us now see that given a piecewise continuous function on a tree, we might make it continuous by adding it a constant:

Lemma 2.1 (Continuity of functions in trees). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree and g a piecewise continuous function. Then, for all $e \in \mathcal{E}$ there is $\mathfrak{c}^e \in \mathbb{R}$ such that the function $\tilde{g}$, where $\tilde{g}^e = g^e + \mathfrak{c}^e$ is continuous on $\overline{\mathcal{E}}$.*

Proof. We argue by induction on the number of vertices. The base case, a tree with two vertices (i.e., one edge), is trivial. Let us then assume that the property is true for a tree of N vertices and let us prove that it holds for a tree of $N + 1$ vertices. It is well-known that there is at least one vertex u with degree 1 and with an edge $\tilde{e}$ incident in some vertex $v \in \mathcal{V} \setminus \{u\}$. The graph $(\mathcal{V} \setminus \{u\}, \mathcal{E} \setminus \{\tilde{e}\})$ satisfies the inductive hypothesis, and thus we can define the constants $\mathfrak{c}^e$ for all $e \in \mathcal{E} \setminus \tilde{e}$. Thus, we just have to get a constant in $\tilde{e}$. It suffices to consider:

$$\mathfrak{c}^{\tilde{e}} = -g^e(v) + \tilde{g}_{|(\mathcal{V}\setminus\{u\},\mathcal{E}\setminus\{\tilde{e}\})}(v).$$

□

Throughout the paper, we use c or C to denote positive constants which may change from line to line and that may depend on the network (namely, on parameters of the type a^e, b^e, c^e or l^e).

2.2. **Well-posedness of the parabolic and hyperbolic problems.** We recall a well-posedness result for the parabolic problem (1.1) (see [32, Theorem 3]), which follows from the Lumer-Phillips theorem of semigroup theory (see [67, Chapter 1.4]).

Theorem 2.1 (Well-posedness for the parabolic problem). *For $a^e, b^e > 0$, $y_0 \in H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E})$, and $u_\varepsilon \in C^2([0,T]; \ell_2(\mathcal{V}_\partial))$, the parabolic problem* (1.1) *has a unique classical solution*

$$y_\varepsilon \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E})).$$

Similarly, in [32, Theorem 6], a well-posedness result for the transport problem was obtained.

Theorem 2.2 (Well-posedness for the hyperbolic problem). *For any $y_0 \in H^1_{pw}(\mathcal{E})$ and $u \in C^2([0,T]; \ell_2(\mathcal{V}^{in}_\partial))$, the hyperbolic problem* (1.3) *has a unique classical solution*

$$y \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E})).$$

Finally, the following convergence result holds (see [32, Theorem 10]).

Theorem 2.3 (Error estimate for the vanishing viscosity approximation). *For any $y_0 \in H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E})$ and $u_\varepsilon \in C^2([0,T]; \ell_2(\mathcal{V}_\partial))$. Let*

$$y_\varepsilon \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1(\mathcal{E}) \cap H^2_{pw}(\mathcal{E}))$$

be the solution of problem (1.1) *and*

$$y \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$$

be the solution of the corresponding limit problem (1.3). *Then,*

$$\|y_\varepsilon - y\|_{L^\infty(0,T;L^2(\mathcal{E}))} \le C\sqrt{\varepsilon}, \tag{2.1}$$

with a constant C that depends on the time-horizon T but is independent of the diffusion parameter $0 < \varepsilon \le 1$.

3. Main results

3.1. **Control of the hyperbolic problem.** Our first result concerns the null-controllability of problem (1.3) on a tree. For the sake of simplicity, we consider coefficients that are constant in each edge; however, as explained in Remark 4.2, the results are also valid for strictly positive space-dependent coefficients. In particular, we are interested in controlling the flow across the network by means of controls placed on the inflow vertices $\mathcal{V}_\partial^{in}$.

To this end, we define an upper bound and a lower bound on the time in which the information propagates across the network (i.e., the maximal and minimal travel time of the characteristics across the network).

Definition 3.1 (Propagation time on a network). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree. We define recursively the functions $\hat{T}, \tilde{T} : \mathcal{V} \to \mathbb{R}_+$ as follows:*

$$\begin{aligned}
&\hat{T}(v) = 0, \quad \tilde{T}(v) = 0 && \text{if } v \in \mathcal{V}_\partial^{in}, \\
&\hat{T}(v) = \sup_{e=(v^e,v)\in\mathcal{E}^{in}(v)} \left(\hat{T}(v^e) + \frac{a^e l^e}{b^e}\right) && \text{if } v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{out}, \\
&\tilde{T}(v) = \inf_{e=(v^e,v)\in\mathcal{E}^{in}(v)} \left(\tilde{T}(v^e) + \frac{a^e l^e}{b^e}\right) && \text{if } v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{out}.
\end{aligned}$$

The times $\hat{T}(v)$ and $\tilde{T}(v)$ are respectively the maximal *and* minimal propagation time *required for information to reach $v \in \mathcal{V}$ from a node in $\mathcal{V}_\partial^{in}$.*

Since the network $\mathcal{G}$ has no loops (being a tree), we can prove inductively that $\hat{T}$ and $\tilde{T}$ are well-defined.

Example 3.1 (Propagation times). *Let us consider the graph in Figure 3 with vertices v_1, v_2, v_3, v_4, and edges $e_1 = [v_1, v_3] \simeq [0, 2]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_3, v_4] \simeq [0, 2]$. We consider the equation* (1.3) *with $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = b^{e_2} = 1$ and $b^{e_3} = 2$. We can compute the maximal travel time to reach $v \in \mathcal{V}$ as follows:*

$$\hat{T}(v_1) = \hat{T}(v_2) = 0, \quad \hat{T}(v_3) = \sup_{i=1,2} \left(\hat{T}(v_i) + l^i\right) = \max\{1, 2\} = 2, \quad \hat{T}(v_4) = \hat{T}(v_3) + \tfrac{l^3}{b^3} = 2 + 1 = 3.$$

Moreover, we compute the minimal travel time to reach $v \in \mathcal{V}$ as follows:

$$\tilde{T}(v_1) = \tilde{T}(v_2) = 0, \quad \tilde{T}(v_3) = \inf_{i=1,2} \left(\tilde{T}(v_i) + l^i\right) = \min\{1, 2\} = 1, \quad \tilde{T}(v_4) = \tilde{T}(v_3) + \tfrac{l^3}{b^3} = 1 + 1 = 2.$$

By relying on this notion of propagation time and on the classical method of characteristics, we can prove the following controllability result:

Theorem 3.1 (Null-controllability for the hyperbolic problem). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree and let y be the solution of* (1.3) *for $u = 0$. Then, $y = 0$ for all $T \geq \max_{v\in\mathcal{V}_\partial^{out}} \hat{T}(v)$.*

The proof of Theorem 3.1 is given in Section 4.

Remark 3.1 (Generalization of the hyperbolic result). *We can actually prove a stronger result: $y^e(T, x) = 0$ for all $x \in e = [v_1, v_2] \simeq [0, l^e]$ and $T \geq \hat{T}(v_1) + \frac{a^e x}{b^e}$.*

Moreover, the following result regarding the minimal propagation time holds.

Proposition 3.1 (Minimal time for the null-controllability for the hyperbolic problem). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree. Then, system* (1.3) *is not null-controllable for $T < \max_{v\in\mathcal{V}_\partial^{out}} \tilde{T}(v)$.*

Proposition 3.1 is also proved in Section 4. We also observe (in Remark 4.3 below) that the time given in Theorem 3.1 may not be optimal if we consider non-null controls. In addition, as we see below in Remark 4.3, the optimal time for null-controllability can be almost any in $[\max_{v\in\mathcal{V}_\partial^{out}} \tilde{T}(v), \max_{v\in\mathcal{V}_\partial^{out}} \hat{T}(v)]$ (by possibly using non-zero controls) by considering the right network, so the upper and lower bound are optimal if we do not add any extra hypothesis on the structure of the network.

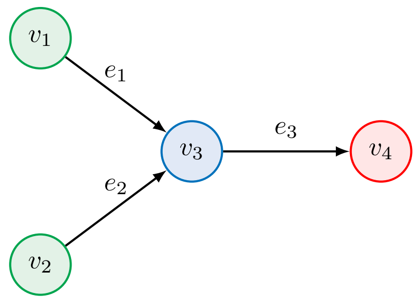

FIGURE 3. Star-shaped graph with edges $e_1 = (v_1, v_3)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$; inner vertices $\mathcal{V}_0 = \{v_3\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_4\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{in} = \{v_1, v_2\}$ (green) and $\mathcal{V}_\partial^{out} = \{v_4\}$ (red), respectively.

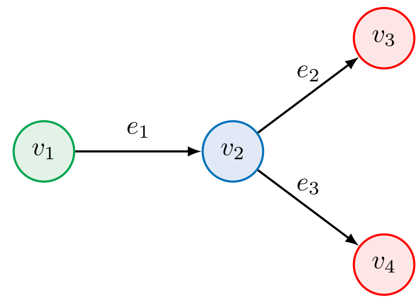

FIGURE 4. Star-shaped graph with edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_4)$; inner vertices $\mathcal{V}_0 = \{v_2\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_3, v_4\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{in} = \{v_1\}$ (green) and $\mathcal{V}_\partial^{out} = \{v_3, v_4\}$ (red), respectively.

3.2. **Control of the parabolic problem.** Our next theorem provides the controllability of the parabolic system (1.1) on tree-shaped networks by acting on the external vertices (except at most one) without any further geometric constraint.

Theorem 3.2 (Controllability of parabolic systems of networks)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree, $a^e, b^e > 0$, and $y_0 \in L^2(\mathcal{E})$. Then, there exists $u \in L^2((0,T) \times \mathcal{V}_\partial)$ such that the solution of* (1.1) *satisfies $y(T, \cdot) = 0$.*

The proof of Theorem 3.2 is essentially given in [46] and is based on the H.U.M. method (see, for example, [57]) – i.e., on the proof of an observability result for the adjoint system

$$\begin{cases} -a^e \partial_t \varphi_\varepsilon^e - b^e \partial_x \varphi_\varepsilon^e - \varepsilon \partial_{xx}^2 \varphi_\varepsilon^e = 0, & \text{in } (0,T) \times e, e \in \mathcal{E}, \\ \varphi_\varepsilon^e(v) = 0, & \text{on } (0,T) \times \mathcal{V}_\partial, \\ \varphi_\varepsilon^{e_1}(t,v) = \varphi_\varepsilon^{e_2}(t,v), & t \in (0,T),\ v \in \mathcal{V}_0,\ \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e} \varphi_\varepsilon^e(t,v) = 0, & \text{on } (0,T) \times \mathcal{V}_0, \\ \varphi_\varepsilon(T, \cdot) = \varphi_T, & \text{on } \mathcal{E}. \end{cases} \tag{3.1}$$

Here, we have used (1.2) to obtain $(3.1)_4$.

Generally, in the parabolic setting, it does not suffice to act on $\mathcal{V}_\partial^{in}$ to drive the system to zero (see Proposition 3.3). However, the estimates in those papers do not take into account how the system behaves when the diffusion decays, as this requires more precise computations.

3.3. **Cost of controllability in the singular limit.** Our final main theorem provides estimates on the cost of controllability as $\varepsilon \to 0^+$ for the parabolic problem (1.1). As in the hyperbolic case, for the sake of simplicity, we consider coefficients that are constant in each edge; but, as explained in Remark 6.4, the results are also valid for strictly positive space-dependent coefficients.

We consider the following quantity to measure the cost of the null-controllability of (1.1):

$$K(\varepsilon, T, a^e, b^e, \mathcal{G}) := \sup_{y_0 \in L^2(\mathcal{E}) \setminus \{0\}} \inf_{u \in \mathcal{U}} \frac{\|u\|_{L^2(0,T)}}{\|y_0\|_{L^2(\mathcal{E})}}, \tag{3.2}$$

where $\mathcal{U}$ denotes the subset of controls in $L^2((0,T) \times \mathcal{V}_\partial)$ such that the solution of (1.1) satisfies $y(T, \cdot) = 0$.

Regarding the cost of controllability, from the hyperbolic result in Theorem 3.1, we expect the following behavior: for small times, $K \to +\infty$ as $\varepsilon \to 0^+$; on the other hand, for T sufficiently large, $K \to 0$ as $\varepsilon \to 0^+$. Our main result concerns a quantitative estimate on these limits.

Theorem 3.3 (Estimates on the cost of controllability)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree and assume that* (1.2) *holds.*

(1) *There exist $\overline{T}, c > 0$ such that, for ε small enough and all $T < \overline{T}$, the following lower bound holds:*

$$K(\varepsilon, T, a^e, b^e, \mathcal{G}) \geq c e^{c/\varepsilon}. \tag{3.3}$$

(2) *There exist $T_0, c, C > 0$ such that, for ε small enough and all $T \geq T_0$, the following upper bound holds:*

$$K(\varepsilon, T, a^e, b^e, \mathcal{G},) \leq C e^{-c/\varepsilon}. \tag{3.4}$$

Remark 3.2 (Acting on all but one boundary vertices in $\mathcal{V}_\partial$)**.** *As in the previous section, we will see that the proof of Item* (2) *is valid if we act in all the boundary vertices except at most one, i.e. if we consider $\mathcal{V}_\partial \setminus \{\tilde{v}\}$ in formula* (3.4)*. As we will establish in Proposition 3.3 below, this result is sharp in some graphs: acting on fewer vertices (even if we act on every vertex in $\mathcal{V}^{in}$) may not suffice.*

The proof of Theorem 3.3 is given in Sections 5 and 6 and is based on the H.U.M. (see, for example, [57]) – that is, on the study of the cost of observation of the adjoint variable, which is given by

$$K(\varepsilon, T, a^e, b^e, \mathcal{G}) = \sup_{\varphi_T \in L^2(\mathcal{E}) \setminus \{0\}} \frac{\|\varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})}}{\left(\iint_{(0,T)\times\mathcal{V}_\partial} |\partial_\nu \varphi_\varepsilon|^2 \,\mathrm{d}t\right)^{1/2}}. \tag{3.5}$$

In order to do some computations, we will often use the following symmetrized system as in [6]. For that, we define the function

$$z_\varepsilon := \varphi_\varepsilon e^{(xb^e + \mathfrak{c}^e)/2\varepsilon}, \tag{3.6}$$

for $\mathfrak{c}^e$ the constants given in Lemma 2.1 that makes the function $xb^e + \mathfrak{c}^e$ continuous on $\overline{\mathcal{E}}$. Note that this argument does not depend of the parametrization of the edges. The function z_ε satisfies:

$$\begin{cases} -a^e \partial_t z_\varepsilon^e - \varepsilon \partial_{xx}^2 z_\varepsilon^e + \frac{|b^e|^2}{4\varepsilon} z_\varepsilon^e = 0, & \text{in } (0,T)\times e, \quad e \in \mathcal{E}, \\ z_\varepsilon^e(v) = 0, & \text{on } (0,T)\times \mathcal{V}_\partial, \\ z_\varepsilon^{e_1}(t,v) = z_\varepsilon^{e_2}(t,v), & t \in (0,T), \ v \in \mathcal{V}_0, \ \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e\in\mathcal{E}(v)} \varepsilon \partial_{n^e} z_\varepsilon^e = 0, & \text{on } (0,T)\times \mathcal{V}_0, \\ z_\varepsilon(T,\cdot) = z_T, & \text{on } \mathcal{E}. \end{cases} \tag{3.7}$$

Here, we relied on (1.2) to obtain $(3.7)_4$.

Roughly speaking, the strategy of our proofs is as follows:

- for Claim (1), we obtain that the cost explodes for small times by considering data for the adjoint problem supported far away from the observation domains and computing its observability cost – this explodes as $\varepsilon \to 0^+$ because for small times the mass remains in the domain, but the mass reaching the observation domain is of order $\exp(C\varepsilon^{-1})$;
- for Claim (2), we show that the cost decays for large times by using first the decay of the free solutions, and then, when the mass of the state is almost null, by exactly observing the mass.

3.4. **Pathological examples and further remarks.**

3.4.1. *Hyperbolic problem.* For particular graphs and choices of the coefficients in (1.1) and (1.3), we can build several pathological examples to illustrate the scope of our controllability results.

Remark 3.3 (Counterexample to exact controllability to any target state $y(T,\cdot) \in C^0_{pw}(\mathcal{E})$)**.** *While we are able to prove null-controllability, and thus controllability to trajectories because of linearity, we may not have exact controllability to any $y \in C^0_{pw}(\mathcal{E})$ – namely, when $|\mathcal{V}_\partial^{out}| > |\mathcal{V}_\partial^{in}|$. In fact, by the method of characteristics, we get an over-determined system on $\mathcal{V}_\partial^{out}$. For example, let us consider the graph in Figure 4, made of the vertices v_1, v_2, v_3, v_4 and the edges $e_1 = [v_1, v_2] \simeq [0,1]$, $e_2 = [v_2, v_3] \simeq [0,1]$, and $e_3 = [v_2, v_4] \simeq [0,1]$. In* (1.3)*, we take $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = 1$, $b^{e_2} = 1/2$, $b^{e_3} = 1/2$. Then, for any $y_0 \in L^2(\mathcal{E})$ and $u \in C^2([0,T]; \ell^2(v_1))$, we have that the solution y of* (1.3) *satisfies the equality $y^{e_2}(t,x) = y^{e_3}(t,x)$ for $t > 1$.*

Next, we illustrate some issues arising from networks with loops.

Remark 3.4 (Networks with loops and controls)**.** *Generalizing Theorem 2.2 to networks with loops is not a straightforward problem, but a challenging open one.*

Let us consider a graph with $\mathcal{V} := \{v_1,\ v_2,\ v_3,\ v_4\}$ *and with edges* $e_1 = [v_1, v_2]$, $e_2 = [v_2, v_3]$, $e_3 = [v_3, v_4]$, $e_4 = [v_4, v_2]$ *(see the left-side picture in Figure 5). In that case, in the free system (i.e., with Dirichlet boundary condition* $u_1 \equiv 0$ *at* v_1*) we can prove that the total mass is constant (which can be done by showing that* $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{E}} u\,\mathrm{d}x = 0$ *with energy methods or by the method of characteristics). And yet, we can use the method of characteristics to prove that the hyperbolic system is null-controllable with a control that is non-null.*

A similar example consists of the same graph with an additional output vertex v_5 *and* $e_5 = [v_3, v_5]$ *(see the right-side picture in Figure 5). In that case, the mass is not conserved, but the null control do not take the system to equilibrium (the mass on the loop decreases exponentially, though). As in the previous example, we can use the method of characteristics to prove that the hyperbolic system is null controllable, with a control that is non-null.*

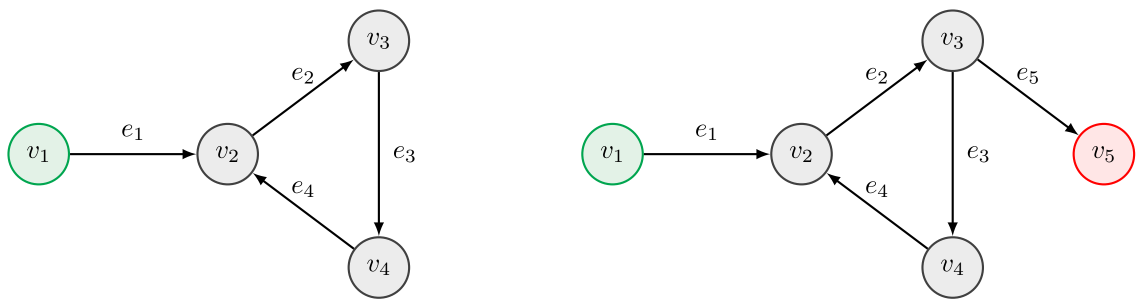

FIGURE 5. Graphs with loops used in Remark 3.4. **Left:** graph with one incoming vertex v_1 (green) and loop made of vertices v_2, v_3, v_4 (gray). **Right:** graph with incoming vertex v_1 (green), outgoing vertex v_5 (red) and loop made of vertices v_2, v_3, v_4 (gray).

3.4.2. *Parabolic problem.* At the parabolic level, we recall the following result that can be found in [46, Remark 3.2].

Proposition 3.2 (Lack of null-controllability for general graphs)**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be the graph in Figure 4, made of the vertices* v_1, v_2, v_3, v_4 *and the edges* $e_1 = [v_1, v_2] \simeq [0,1]$, $e_2 = [v_2, v_3] \simeq [0,1]$, *and* $e_3 = [v_2, v_4] \simeq [0,1]$ *Then, system* (1.1)*, with coefficients* $a = 1$ *and* $b = 0$*, is not approximately controllable by acting only on* v_1 *(i.e., if* $u^{v_3} = u^{v_4} = 0$*).*

Heuristically, the motivation for such result is that, by symmetry, the effect of the control on e_2 and e_3 is identical, so we cannot control both y^{e_2} and y^{e_3} simultaneously (unless some irrationality condition on the length of the edges holds; compare [28, Corollary 8.6]). The proof is made rigorous by a duality argument by considering the eigenfunctions

$$\varphi^{e_1} = 0, \qquad \varphi^{e_2} = e^{-\pi^2 t}\sin(\pi x), \qquad \varphi^{e_3} = -e^{-\pi^2 t}\sin(\pi x).$$

3.4.3. *Singular limit problem.* As in Proposition 3.2, we note that system (1.1) cannot be controlled for any $\varepsilon > 0$ by acting on fewer boundary vertices (compare also [46, Remark 3.2]).

Proposition 3.3 (Lack of null-controllability for general graphs)**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be the graph in Figure 4, made of the vertices* v_1, v_2, v_3, v_4 *and the edges* $e_1 = [v_1, v_2] \simeq [0,1]$, $e_2 = [v_2, v_3] \simeq [0,1]$, *and* $e_3 = [v_2, v_4] \simeq [0,1]$*. Then, system* (1.1) *(with coefficients* $b^{e_1} = a^{e_1} = a^{e_2} = a^{e_3} = 1$ *and* $b^{e_2} = b^{e_3} = \frac{1}{2}$*) is not approximately controllable by acting only on* v_1 *(i.e. if* $u^{v_3} = u^{v_4} = 0$*) for any* $\varepsilon > 0$*.*

Proof. By duality, it suffices to show that there are non-null solutions of (3.1) satisfying $\partial_{n^e}\varphi(\cdot, v_1) = 0$. Considering (3.6), this is equivalent to showing that there are non-null solutions of (3.7) satisfying $\partial_{n^e} z(\cdot, v_1) = 0$. From the spectral decomposition of the Laplacian in the graph we can construct such a solution as follows:

$$z^{e_1} = 0, \qquad z^{e_2} = \exp\left[-\left(\varepsilon\pi^2 + \frac{1}{16\varepsilon}\right)t\right]\sin(\pi x), \qquad z^{e_3} = -\exp\left[-\left(\varepsilon\pi^2 + \frac{1}{16\varepsilon}\right)t\right]\sin(\pi x).$$

□

Finally, we note the relevance of (1.2) for our results.

Remark 3.5 (On the balance relation (1.2))**.** *The balance relation* (1.2) *is needed to ensure that the symmetrized system is dissipative. Indeed, without* (1.2)*, we must replace* $(3.6)_4$ *by*

$$\sum_{e\in\mathcal{E}(v)} \varepsilon\partial_{n^e} z^e_\varepsilon = -\sum_{e\in\mathcal{E}(v)} \frac{n^e b^e}{2}. \tag{3.8}$$

Also, without condition (1.2)*, system* (1.3) *may not be dissipative at junctions. Indeed, from* (1.3)*, we compute*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{E}} a^e (y^e)^2 \,\mathrm{d}x = -\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} n^e b^e (y^e)^2(v), \tag{3.9}$$

which is not non-positive in general. For example, let us consider a simple 1-to-1 junction modeled as follows:

$$\begin{cases} \partial_t y^- + b^-\partial_x y^- = 0, & x\in(-1,0),\\ \partial_t y^+ + b^+\partial_x y^+ = 0, & x\in(0,1),\\ y^-(t,-1)=0, & t>0,\\ b^- y^-(t,0) = b^+ y^+(t,0), & t>0,\\ y^-(0,x) = y_0^-, & x\in(-1,0),\\ y^+(0,x) = y_0^+, & x\in(0,1), \end{cases} \tag{3.10}$$

for $b^-, b^+ > 0$*. Here, the term in* (3.9) *is given by*

$$|y^-(t,0)|^2 (b^-)\left(1 - \frac{b^-}{b^+}\right),$$

so if $b^+ > b^-$ *energy is added on junctions; thus the problem is not dissipative.*

We refer to Section 7 (Item 3) for further comments on this matter.

4. Control of the transport problem

4.1. **Main control result.** We prove Theorem 3.1 by means of the method of characteristics, following [26, Chapter 2.1.2., p. 30] (see Figure 6 for a sample illustration in case of 1-to-1 junctions). More specifically, we use an induction argument to show that, for any tree $\mathcal{G}$, the system (1.3) with $u\equiv 0$ is at rest for any time larger than $\max_{v\in\mathcal{V}} \hat{T}(v)$. For the proof we identify each edge $e=[v_1^e, v_2^e]\simeq[0,l^e]$ as in Remark 3.1.

Proof of Theorem 3.1. The proof is based on an induction of the distance of the vertex v_1^e to the boundary (see Definition 2.1).

The base case is when $v_1^e\in\mathcal{V}_\partial^{in}$. The equality $y(v_1^e)=0$ is satisfied by the boundary condition (recall that $u=0$). Moreover, inside each edge the function y^e behaves like the solution to a classical transport equation. Consequently, as in [26, Chapter 2.1.2., p. 30], $y^e(t,x)=0$ for all $x\in e=[v_1^e,v_2^e]\simeq[0,l^e]$ and $t\geq \frac{a^e x}{b^e}$, and in particular, $y^e(t,v_2^e)=0$ for all $t\geq\hat{T}(v_2^e)$, as $\hat{T}(v_2^e)\geq\frac{a^e l^e}{b^e}$.

Let us now continue with the inductive case; that is, when $v_1^e\in\mathcal{V}_0$. The equality $y^{\tilde{e}}(t,v_1^e)=0$ is satisfied for all $t\geq\hat{T}(v_1^e)$ by the inductive hypothesis. Thus, from $(1.3)_3$ and $(1.3)_4$, we find that $y^e(t,v_1^e)=0$ for all $t\geq\hat{T}(v_1^e)$. Moreover, inside the edge the function y^e behaves like a transport equation in a segment. Consequently, we get that $y^e(T,x)=0$ for all $x\in[0,l^e]$ and $T\geq\hat{T}(v_1^e)+\frac{a^e x}{b^e}$, and in particular, $y^e(T,v_2^e)=0$ for all $T\geq\hat{T}(v_2^e)$, as $\hat{T}(v_2^e)\geq\hat{T}(v_1^e)+\frac{a^e l^e}{b^e}$. $\square$

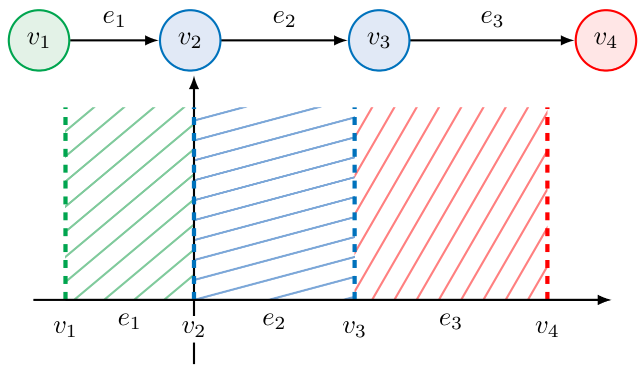

FIGURE 6. For the sake of simplicity, we illustrate the propagation (**bottom**) in a tree made up of three 1-1 junctions (**top**).

Remark 4.1 (Simplified proof for star-graphs). *The same proof as Theorem 3.1 and Proposition 3.1 is valid for star-shaped graphs, with the simplification that instead of an abstract induction argument, a two-step proof suffices.*

Remark 4.2 (Positivity assumption on a and b when they depend on the space variable). *The method of characteristics is valid when a^e and b^e depend on the space variable and $\inf_{\mathcal{E}} a, \inf_{\mathcal{E}} b > 0$, so Theorem 3.1 and Proposition 3.1 remain true under those assumptions. The positivity is needed because if the transport term gets null at some point, the characteristics do not leave the domain and we do not have the null controllability result for the hyperbolic problem. Correspondingly, in that situation the cost of the controllability explodes when $\varepsilon \to 0$.*

4.2. **Optimal control times.** We prove Proposition 3.1 with the method of characteristics.

Proof of Proposition 3.1. Let us consider the solution of (1.3) with initial value $y_0 = 1$. Then, an inductive argument (as in the proof of Theorem 3.1) yields the following claim: for any control u, any $v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{out}$, and any $e \in \mathcal{E}(v)$, we have $y^e(t, v) > 0$ on $[0, \tilde{T}(v))$. $\square$

Finally, we discuss an improvement on the control time $\max_{v\in\mathcal{V}_\partial^{out}} \hat{T}(v)$ from Theorem 3.1 by means of non-zero boundary controls.

Remark 4.3 (Non zero boundary controls and optimal time). *For all $0 < T_1 \le T_2 < T_3 < T_1 + T_2$ or $T_1 = T_2 = T_3$ we can construct a star-graph such that $\max_{v\in\mathcal{V}_\partial^{out}} \tilde{T} = T_1$, $\min_{v\in\mathcal{V}_\partial^{out}} \tilde{T} = T_3$, and the optimal time in which (1.3) might be driven to 0 is T_2. For the case $T_1 = T_2 = T_3$, it suffices to consider a single edge graph. For the case $0 < T_1 \le T_2 < T_3$, it suffices to consider a star-graph with two incoming edges and an outgoing one (see Figure 3), $a = b = 1$ and $l_{-2} = T_2$, $l_{-1} = T_1 + T_2 - T_3$ and $l_1 = T_1 + T_2 - T_3$. Moreover, the hypothesis $T_3 < T_1 + T_2$ can be removed by considering more complex examples. In particular, if we consider $a = b = 1$, $n = \lfloor \frac{T_3 - T_2}{T_1} \rfloor$ and the graph with vertices $\mathcal{V} = \{v_i\}_{i=-n}^{n+1}$ and $\mathcal{E} = \{e_i\}_{i=1}^{n} \cup \{\tilde{e}_i\}_{i=0}^{n}$, for $e_i = [v_{-i}, v_i]$ and $\tilde{e}_i = [v_i, v_{i+1}]$. It suffices to consider the length $\tilde{l}_0 = T_2$, $l_{-1}, \cdots, l_{n-1} = T_1$, $l_1 = \cdots = l_{-n+1} = T_1$, $l_n = T_3 - T_2 - nT_1$, $l_{-n} = T_1 - l_n$.*

5. Blow-up of the cost of controllability

In this section we prove Claim (1) of Theorem 3.3.

5.1. **Agmon-type inequality.** We start by proving an Agmon-type inequality (see [1, Theorem 5.9]), which gives an exponentially weighted energy estimate.

Lemma 5.1 (Agmon-type inequality). *Let $\zeta \in H^2_{pw}(\mathcal{E}) \cap H^1(\mathcal{E})$ satisfy*

$$a^e \partial_t \zeta^e + b^e \partial_x \zeta^e - |\partial_x \zeta^e|^2 \ge 0. \tag{5.1}$$

Then, any solution φ of the adjoint system (3.1) *satisfies the following Agmon-type inequality:*

$$\begin{aligned}&\sum_{e\in\mathcal{E}}\tfrac{a^e}{2}\int_e |e^{\zeta^e(t,x)/\varepsilon}\varphi^e(t,x)|^2\,\mathrm{d}x+\varepsilon\sum_{e\in\mathcal{E}}\int_t^T\int_e|\partial_x(e^{\zeta^e(s,x)/\varepsilon}\varphi^e(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leq\sum_{e\in\mathcal{E}}\tfrac{a^e}{2}\int_e|e^{\zeta^e(T,x)/\varepsilon}\varphi^e(T,x)|^2\,\mathrm{d}x.\end{aligned}$$

Proof. We compute $\frac{\mathrm{d}}{\mathrm{d}t}\frac{a_e}{2}\int_e|e^{\zeta^e/\varepsilon}\varphi^e(t,x)|^2\,\mathrm{d}x$ as follows:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{a^e}{2}\int_e|e^{\zeta^e/\varepsilon}\varphi^e|^2\,\mathrm{d}x\\ &=a^e\int_e e^{\zeta^e/\varepsilon}\partial_t e^{\zeta^e/\varepsilon}|\varphi^e|^2\,\mathrm{d}x+a^e\int_e e^{2\zeta^e/\varepsilon}\varphi(t,x)\partial_t\varphi^e\,\mathrm{d}x\\ &=a^e\int_e e^{\zeta^e/\varepsilon}\partial_t e^{\zeta^e/\varepsilon}|\varphi^e|^2\,\mathrm{d}x+\int_e e^{2\zeta^e/\varepsilon}\varphi^e\big(-b^e\partial_x\varphi^e-\varepsilon\partial_{xx}^2\varphi^e\big)\,\mathrm{d}x\\ &=a^e\int_e e^{\zeta^e/\varepsilon}\partial_t e^{\zeta^e/\varepsilon}|\varphi^e|^2\,\mathrm{d}x+\frac{b^e}{2}\int_e(\varphi^e)^2\partial_x e^{2\zeta^e/\varepsilon}\,\mathrm{d}x-\frac{b^e}{2}[(\varphi^e)^2e^{2\zeta^e/\varepsilon}]_e\\ &\quad+\varepsilon\int_e(\partial_x\varphi^e)^2e^{2\zeta^e/\varepsilon}\,\mathrm{d}x+\varepsilon\int_e\partial_x\varphi^e\varphi^e\partial_x e^{2\zeta^e/\varepsilon}\,\mathrm{d}x-\varepsilon[\partial_x\varphi^e\varphi^e e^{\zeta^e/\varepsilon}]_e\\ &=\frac{1}{\varepsilon}\int_e e^{2\zeta^e/\varepsilon}(\varphi^e)^2\underbrace{\left(a^e\partial_t\zeta^e+b^e\partial_x\zeta^e-|\partial_x\zeta^e|^2\right)}_{\geq 0}\,\mathrm{d}x+\varepsilon\int_e(\partial_x(\varphi^e e^{\zeta^e/\varepsilon}))^2\,\mathrm{d}x\\ &\quad-\frac{b^e}{2}[(\varphi^e)^2e^{2\zeta^e/\varepsilon}]_e-\varepsilon[\varphi^e\partial_x\varphi^e e^{2\zeta^e/\varepsilon}]_e.\end{aligned}$$

Summing up over $e\in\mathcal{E}$, using Dirichlet boundary conditions, the junction conditions, and the flux balance condition (1.2), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{e\in\mathcal{E}}\frac{a^e}{2}\int_e|e^{\zeta^e/\varepsilon}\varphi^e(t,x)|^2\,\mathrm{d}x&=\frac{1}{\varepsilon}\sum_{e\in\mathcal{E}}\int_e|e^{\zeta^e/\varepsilon}\varphi^e(t,x)|^2\underbrace{\left(a^e\partial_t\zeta^e+b^e\partial_x\zeta^e-|\partial_x\zeta^e|^2\right)}_{\geq 0}\,\mathrm{d}x\\ &\quad+\varepsilon\sum_{e\in\mathcal{E}}\int_e(\partial_x(\varphi^e e^{\zeta^e/\varepsilon}))^2\,\mathrm{d}x.\end{aligned}$$

By integrating this expression in (t,T), we conclude the proof. □

5.2. **Non-degeneracy of the solution.** As a second preliminary tool, we establish that the mass of $\varphi(0,\cdot)$ is bounded away from zero.

Lemma 5.2 (Non-degeneracy of the solution)**.** *Let ω be a subdomain compactly included in an edge $e\in\mathcal{E}$. Then, for all T small enough and a non-null positive functions $\varphi_T\in C^\infty(\Omega)$ such that* $\operatorname{supp}(\varphi_T)\subset\omega$ *there is $c>0$ and such that for all $\varepsilon\in(0,1)$:*

$$\|\varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})}\geq c, \tag{5.2}$$

for φ_ε the solution of (3.1) *with initial value φ_T.*

Proof. We prove (5.2) by contradiction. First of all, we remark that, thanks to the backwards uniqueness of (3.1) (which can be derived from the backwards uniqueness of (3.7)), we obtain that $\|\varphi_\varepsilon(0,\cdot)\|_{L^2(\mathcal{E})}>0$ for all $\varepsilon\in(0,1)$. In addition, since the solution is continuous with respect to ε (which can be proved with the Fourier series solutions of (3.7) as in [7]), the only possibility is that there is $\varepsilon_k\to 0$ such that

$$\|\varphi_{\varepsilon_k}(0,\cdot)\|_{L^2(\mathcal{E})}\to 0.$$

Let us suppose that there exists such a sequence and derive a contradiction. Since φ_{ε_k} is bounded on $L^2((0,T)\times\mathcal{E})$ we may suppose that it converges weakly to some function $\gamma\in L^2((0,T)\times\mathcal{E})$. We can prove by considering the weak solution of parabolic equation that γ is a solution of the adjoint problem of (1.3) with $\varphi(0)=0$ and initial value φ_T, which is impossible by the method of characteristics. □

5.3. **Proof of Claim** (1) **of Theorem 3.3.** Using the tools developed in the previous sections, we complete the proof of Claim (1) of Theorem 3.3.

Proof of Theorem 3.3, Claim (1). Let $e \sim [0, L^e] \in \mathcal{E}$ and let φ^T be a non-null smooth function such that $\operatorname{supp}(\varphi^T) \subset (\frac{L^e}{4}, \frac{3L^e}{4})$ is compactly included in e. Thanks to Lemma 5.2 we know that (5.2) is satisfied uniformly on ε; thus, we have to prove that the observed mass decay exponentially. We now define the auxiliary function ζ by:

$$\zeta^e = b^e \left(x - \frac{L^e}{2}\right)^2 - \frac{(b^e)^2}{a^e}(T-t)\max\{L^e, (L^e)^2\}, \quad \text{in } e$$

and ζ extended by two constants function outside that edge (by $\zeta^e(0)$ on the left and by $\zeta^e(L^e)$ on the right). Using the assumption on $\operatorname{supp} \varphi_T^e$, we deduce:

$$\int_e |e^{\zeta(x,T)/\varepsilon}\varphi_T^e|^2 \,\mathrm{d}x \leq \exp\left(\frac{b^e(L^e)^2}{8\varepsilon}\right)\int_e |\varphi_T^e|^2 \,\mathrm{d}x,$$

$$\exp\left(\frac{b^e(L^e)^2}{2\varepsilon} - \frac{2(b^e)^2 T \max\{L^e, (L^e)^2\}}{a^e\varepsilon}\right)\int_{(0,T)\times\mathcal{V}_\partial^{in}} |\partial_\nu \varphi|^2 \,\mathrm{d}t \leq \int_0^T \int_e |e^{\zeta(t,x)/\varepsilon}\varphi^e|^2 \,\mathrm{d}x \,\mathrm{d}t.$$

Combining these with Lemma 5.1, we deduce:

$$\int_{(0,T)\times\mathcal{V}_\partial^{in}} |\partial_\nu \varphi|^2 \,\mathrm{d}t \leq \exp\left(-\frac{3b^e(L^e)^2}{8\varepsilon} + \frac{2(b^e)^2 T \max\{L^e, (L^e)^2\}}{a^e\varepsilon}\right)\|\varphi_T^e\|^2_{L^2(\mathcal{E})}. \tag{5.3}$$

Plugging (5.2) and (5.3) into (3.5), we deduce that (3.3) holds for T small enough. □

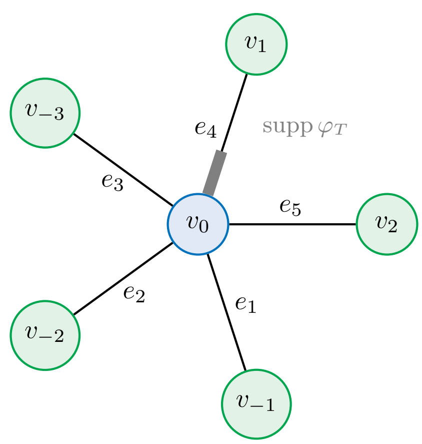

FIGURE 7. Example illustration of the support of φ_T in the case of a star-graph.

6. DECAY OF THE COST OF CONTROLLABILITY

6.1. **The decay property for the parabolic problem.** We start by proving a decay property for the solution of (1.1).

Proposition 6.1 (Decay property). *There are $c, C > 0$ such that the solution of the parabolic problem* (3.1) *satisfies the following decay property for all $\varphi_T \in L^2(\mathcal{E})$ and all $t < T$:*

$$\int_{\mathcal{E}} |\varphi_\varepsilon(t,x)|^2 \,\mathrm{d}x \leq \exp\left(\frac{C - c(T-t)}{\varepsilon}\right)\|\varphi_T\|^2_{L^2(\mathcal{E})}. \tag{6.1}$$

The proof of the decay property is based on classical energy estimates for the symmetrized system (3.7) with Gronwall's inequality.

Proof. In order to prove (6.1), we first obtain a decay property for the symmetrized system (3.7). Multiplying the PDEs in (3.7) by z_ε^e, integrating by parts and summing up all the edges we obtain by using $(3.7)_3$ that:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\int_{\mathcal{E}} a^e|z_\varepsilon|^2\,\mathrm{d}x\right)+\varepsilon\int_{\mathcal{E}}|\partial_x z_\varepsilon|^2\,\mathrm{d}x+\int_{\mathcal{E}}\frac{|b^e|^2}{4\varepsilon}|z_\varepsilon|^2\,\mathrm{d}x=0.$$

Consequently, by using $\inf_{e\in\mathcal{E}} b^e>0$,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{E}} a^e|z_\varepsilon|^2\,\mathrm{d}x\right)\leq-\frac{c}{\varepsilon}\int_{\mathcal{E}}|z_\varepsilon|^2\,\mathrm{d}x.$$

Using backwards Gronwall's inequality in (t,T) we obtain that:

$$\int_{\mathcal{E}}|z_\varepsilon(t,x)|^2\,\mathrm{d}x\leq\exp\left(-\frac{c(T-t)}{\varepsilon}\right)\int_{\mathcal{E}}|z_T|^2\,\mathrm{d}x.$$

Finally, reverting the change of variables (3.6), we obtain the decay property (6.1). $\square$

6.2. **A Carleman inequality.** In order to prove the Carleman estimate, we focus on the viscosity dynamics and do not take into account the information provided by the velocity on the flow of the transport system:

Proposition 6.2 (Carleman inequality)**.** *Let φ_ε be the solution of* (3.1)*. Then there exists a positive constant $C=C(\mathcal{G})$ such that φ_ε satisfies the following inequality:*

$$\begin{aligned}&s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}(\varepsilon^{-2}|\partial_t\varphi_\varepsilon|^2+|\partial^2_{xx}\varphi_\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}t\\&\qquad+s\tau^2\iint_Q e^{-2s\alpha}\xi|\partial_x\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+s^3\tau^4\iint_Q e^{-2s\alpha}\xi^3|\varphi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\&\leq C\sum_{v\in\mathcal{V}_\partial}s\tau\int_0^T e^{-2s\alpha}\xi(t,v)|\partial_{n^e(v)}\varphi^e_\varepsilon(t,v)|^2\,\mathrm{d}t,\end{aligned}\tag{6.2}$$

where $Q:=(0,T)\times\mathcal{E}$, α,ξ are the Fursikov-Imanuvilov weights defined in (6.3)*, $\tau\geq C$ and $s\geq C(T+T^2)\varepsilon^{-1}$.*

The difficulty in the proof arises from the boundary terms at junctions. To suitably deal with them, we define the *Fursikov-Imanuvilov weights* (see [35]) with a piecewise C^2 auxiliary function. Piecewise C^2 weights were first used for proving Carleman inequalities in [9], where the authors dealt with piecewise regular diffusivity; more recently, similar functions have been used to study coupled systems with Kirchhoff-type conditions in [46] and in [10]. For the construction of the auxiliary function, we take the idea from [46], though the construction of the auxiliary function we give is more general as we have sharpened the needed hypothesis as much as possible.

Proof. **Step 0:** *Strategy of the proof and choice of the auxiliary functions.* The main idea is to observe the nodes of the i-th layer with the nodes of the $(i+1)$-th layer (see Definition 2.1). For that, we define an auxiliary function $\eta\in C^2_{pw}\left(\overline{\mathcal{E}}\right)\cap C^0\left(\overline{\mathcal{E}}\right)$ recursively by the edges which joins the i-th and $(i+1)$-th layer of the tree. Let us start with the base case: the edge $e=(v_1,v_2)$, for v_1 the root of the tree. We define η^e as a function satisfying $\partial_x\eta^e(v_1)=1$, $\partial_x\eta^e>0$ in $[v_1,v_2]$ and $\partial_x\eta^e(v_2)=\delta$, for $\delta>0$ a small parameter depending on a to be defined later on. As for the inductive case, if v_1 is on the i-th layer and v_2 on the $(i+1)$-th layer for $i\geq 2$, we define η^e such that $\eta^e(v_1)$ is determined so that $\eta\in C^0(\overline{\mathcal{E}})$, $\partial_x\eta^e(v_1)=1$, $\partial_x\eta^e>0$ in $[v_1,v_2]$ and $\partial_x\eta^e(v_2)=\delta$, for $\delta>0$ a small parameter depending on a, b be fixed later on (see Figure 8 for a sample illustration).

This auxiliary function allows us to define usual Fursikov-Imanuvilov weights:

$$\alpha(t,x):=\frac{e^{8\tau\|\eta\|_\infty}-e^{\tau(6\|\eta\|_\infty+\eta(x))}}{t(T-t)},\quad\xi(t,x):=\frac{e^{\tau(6\|\eta\|_\infty+\eta(x))}}{t(T-t)},\tag{6.3}$$

for $\tau\in\mathbb{R}$ a fixed parameter (and in particular independent of the edge) that will be chosen later. With those weights we consider the change of variable $\psi=e^{-s\alpha}z_\varepsilon$.

From (3.7), we obtain that ψ satisfies the equation

$$L_1\psi+L_2\psi=L_3\psi,\tag{6.4}$$

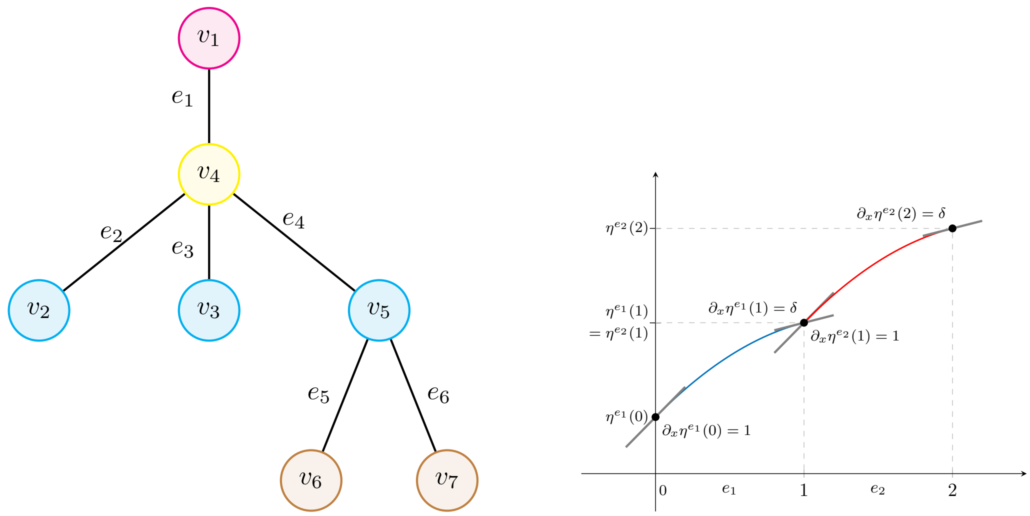

FIGURE 8. Construction of the auxiliary function $\eta \in C^2_{pw}(\bar{\mathcal{E}}) \cap C^0(\bar{\mathcal{E}})$ using the tree from Figure 1. In this tree, we distinguish four layers (**left** picture): the first (magenta) is $\{v_1\}$; the second (yellow) is layer $\{v_4\}$, the third (cyan) is $\{v_2, v_3, v_5\}$; and the fourth (brown) is $\{v_6, v_7\}$. Let us identify $e_1 := [0,1] \simeq [v_1, v_4]$, $e_2 := [1,2] \simeq [v_4, v_2]$, $e_3 := [1,2] \simeq [v_4, v_3]$, $e_4 := [1,2] \simeq [v_4, v_5]$, $e_5 := [2,3] \simeq [v_5, v_6]$, $e_6 := [2, 5/2] \simeq [v_5, v_6]$. The auxiliary function η may be defined as follows: $\eta^{e_1}(x) = x - \frac{1-\delta}{2}x^2 + \frac{1-\delta}{2}$, $\eta^{e_2}(x) = \eta^{e_3}(x) = \eta^{e_4}(x) = (x-1) - \frac{1-\delta}{2}(x-1)^2 + 1$, $\eta^{e_5}(x) = \eta^{e_4}(2) + (x-2) - \frac{1-\delta}{2}(x-2)^2$, $\eta^{e_6}(x) = \eta^{e_4}(2) + (x-2) - (1-\delta)(x-2)^2$. As an example, in the **right** picture, we plot η^{e_1} and η^{e_2} with $\delta = \frac{1}{4}$. In the proof of Proposition 6.2, if $\mathcal{G}$ is the graph illustrated above, we observe the vertex v_1 with v_4; v_4 with v_2, v_3, v_5; and v_5 with v_6, v_7. So, by transitivity, we actually only need to observe using v_2, v_3, v_6, v_7 (i.e. we do not need the vertex in the 1-st layer).

where

$$\begin{cases} L_1\psi := -2\varepsilon s\tau^2|\partial_x\eta|^2\xi\psi - 2\varepsilon s\tau\xi\partial_x\eta\partial_x\psi + a\partial_t\psi, \\ L_2\psi := \varepsilon s^2\tau^2|\partial_x\eta|^2\xi^2\psi + \varepsilon\partial^2_{xx}\psi + sa\partial_t\alpha\psi - \frac{|b|^2}{4\varepsilon}\psi, \\ L_3\psi := \varepsilon s\tau\partial^2_{xx}\eta\xi\psi - \varepsilon s\tau^2|\partial_x\eta|^2\xi\psi. \end{cases} \tag{6.5}$$

Indeed, by using the facts $\partial_x\alpha = -\tau\partial_x\eta\xi$, $\partial^2_{xx}\alpha = -\tau\partial^2_{xx}\eta\xi - (\tau\partial_x\eta)^2\xi$ and (3.7), we obtain

$$\begin{aligned} &-a\partial_t\psi + \varepsilon\partial^2_{xx}\psi - \frac{|b|^2}{4\varepsilon}\psi \\ &\qquad = as\partial_t\alpha\psi + \varepsilon\tau s\partial^2_{xx}\eta\psi + \varepsilon\tau^2 s(\partial_x\eta)^2\xi\psi - \varepsilon s\tau^2(\partial_x\eta)^2\xi\psi + 2\epsilon s\tau e^{-s\alpha}\partial_x z\partial_x\eta. \end{aligned} \tag{6.6}$$

Now, combining (6.6) with the fact that $2\epsilon s\tau e^{-s\alpha}\partial_x z\partial_x\eta\xi = 2\epsilon s\tau\xi\partial_x\eta\partial_x\psi + 2\epsilon s\tau^2\xi^2(\partial_x\eta)^2\psi$, we obtain that ψ satisfies (6.4).

We now argue as in [6, 41], but paying extra attention to keep track of the boundary terms at junctions. As usual, we denote $(L_i\psi)_j$ the j-th term in the expression of $L_i\psi$ given above. From (6.4), we have

$$\|L_1\psi + L_2\psi\|^2_{L^2(Q)} = \|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2(L_1\psi, L_2\psi)_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}.$$

In the next two steps, we estimate the product

$$(L_1\psi, L_2\psi)_{L^2(Q)} = \sum_{e\in\mathcal{E}} ((a^e)^{-1/2}(L_1\psi)^e, (a^e)^{-1/2}(L_2\psi)^e)_{L^2((0,T)\times e)}.$$

In particular, we show that the choice of the weights in (6.3) makes it positive up to several terms that can be controlled by the left-hand side for a suitable choice of the parameters τ and s.

Step 1: *Estimates in the interior.* In this step we perform integrations by parts in the spirit of [6, 41], but keeping track of the boundary terms appearing at the vertices.

To begin with,

$$((L_1\psi)_1, (L_2\psi)_1)_{L^2(Q)} = -2\varepsilon^2 s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{6.7}$$

Secondly, we compute

$$\begin{aligned} ((L_1\psi)_2, (L_2\psi)_1)_{L^2(Q)} &= -2\varepsilon^2 s^3\tau^3 \iint_Q a^{-1}|\partial_x\eta|^2\partial_x\eta\xi^3\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= 3\varepsilon^2 s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \underbrace{-\varepsilon^2 s^3\tau^3 \sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \int_0^T (a^e)^{-1}|\partial_x\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t}_{=:J_1} \\ &\quad + o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{6.8}$$

Thirdly, integration by parts (with respect to the time variable) yields, for $\tau \geq C$ and $s \geq C(T+T^2)$,

$$((L_1\psi)_3, (L_2\psi)_1)_{L^2(Q)} = \varepsilon s^2\tau^2 \iint_Q |\partial_x\eta|^2\xi^2\psi\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t = o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \tag{6.9}$$

To continue with,

$$\begin{aligned} ((L_1\psi)_1, (L_2\psi)_2)_{L^2(Q)} &= -2\varepsilon^2 s\tau^2 \iint_Q a^{-1}|\partial_x\eta|^2\xi\psi\partial_{xx}^2\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= 2\varepsilon^2 s\tau^2 \iint_Q a^{-1}|\partial_x\eta|^2\xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \underbrace{-\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} 2\varepsilon^2 s\tau^2 \int_0^T (a^e)^{-1}|\partial_x\eta^e|^2\xi^e\psi^e\partial_{n^e(v)}\psi^e(t,v)\,\mathrm{d}t}_{=:J_2} \\ &\quad + o\left(\varepsilon^2 s^3\tau^4 \iint_Q \xi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right) + o\left(\varepsilon^2 s\tau^2 \iint_Q \xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{6.10}$$

In addition,

$$\begin{aligned} &((L_1\psi)_2, (L_2\psi)_2)_{L^2(Q)} = -2\varepsilon^2 s\tau \iint_Q a^{-1}\partial_x\eta\xi\partial_{xx}^2\psi\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= \varepsilon^2 s\tau^2 \iint_Q a^{-1}|\partial_x\eta|^2\xi|\partial_x\psi|^2 \underbrace{-\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \varepsilon^2 s\tau \int_0^T (a^e)^{-1}\partial_{n^e(v)}\eta^e\xi^e|\partial_x\psi^e|^2(t,v)\,\mathrm{d}t}_{=:J_3} \\ &+ o\left(\varepsilon^2 s\tau^2 \iint_Q \xi|\partial_x\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{6.11}$$

Moreover, we can prove that

$$((L_1\psi)_3, (L_2\psi)_2)_{L^2(Q)} = \varepsilon \iint_Q \partial_t\psi\partial_{xx}^2\psi \,\mathrm{d}x\,\mathrm{d}t = \underbrace{\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} \varepsilon \int_0^T \partial_{n^e(v)}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t}_{=:J_4}. \tag{6.12}$$

Finally,

$$(6.13)\qquad \begin{aligned}(L_1\psi,(L_2\psi)_3+(L_2\psi)_4)_{L^2(Q)} = &\underbrace{\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)} s\tau\int_0^T \xi^e\partial_{n^e(v)}\eta^e\left(s\varepsilon\partial_t\alpha^e+\frac{|b^e|^2}{4}\right)|\psi^e|^2(t,v)\,\mathrm{d}t}_{=:J_5}\\ &+o\left(\varepsilon^2s^3\tau^4\iint_Q\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}$$

Step 2: *Estimation of the boundary terms.* In this part of the proof, we estimate the boundary terms $J_1,\dots,J_5$. In particular, we need to make a distinction between *exterior vertices*, which can be treated as in [6, 41] (since they correspond to the boundary terms appearing in a classical IBVP), and *junctions*, which require new more precise computations. As we are going to see, the terms corresponding to the exterior vertices $v\in\mathcal{V}_\partial$ either vanish (due to the zero Dirichlet boundary condition in (3.7)) or can be moved to the right hand-side of the Carleman estimate (corresponding to the "classical" boundary terms that appear in [41]). The interior junction terms at $v\in\mathcal{V}_0$ are more critical: they are in the left-hand side of the Carleman estimate and we need to show that they are non-negative. To this end, we will rely on the properties of the auxiliary function η and on the Kirchhoff junction condition $(3.7)_4$ (which, in turn, was formulated thanks to (1.2)). At the end of the computations, all these boundary terms at the junction can be absorbed into the expression in the right-hand side of (6.19), which is non-negative.

To begin with, let us deal with the boundary term J_1 in (6.8). If $v\in\mathcal{V}_\partial$, we get that $\psi=0$ from the Dirichlet boundary conditions. Otherwise, for each interior node $v\in\mathcal{V}_0$, we get that

$$(6.14)\qquad -\varepsilon^2s^3\tau^3\int_0^T\xi^3\left(\sum_{e\in\mathcal{E}(v)}(a^e)^{-1}|\partial_x\eta^e|^2\partial_{n^e(v)}\eta^e\right)|\psi^e|^2(t,v)\,\mathrm{d}t\geq c\varepsilon^2s^3\tau^3\int_0^T\xi^3|\psi^e|^2(t,v)\,\mathrm{d}t.$$

Indeed, the function ξ is continuous at junctions, $\partial_{n^{\tilde e}(v)}\eta^{\tilde e}(v)=\delta$ for $\tilde e$ the edge joining the previous layer to v and $\partial_{n^e(v)}\eta^e(v)=-1$ if $e\in\mathcal{E}(v)\setminus\{\tilde e\}$, so we obtain (6.14) by choosing δ small enough. With the right-hand side of (6.14) we can absorb the boundary term J_5 in (6.13) for $s\geq C(T+T^2)\varepsilon^{-1}$ and $\tau\geq C$.

To continue with, let us study the boundary term J_3 given in (6.11) for each $v\in\mathcal{V}$, i.e.

$$-\varepsilon^2s\tau\int_0^T\xi\sum_{e\in\mathcal{E}(v)}(a^e)^{-1}\partial_{n^e(v)}\eta^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.$$

If $v\in\mathcal{V}_\partial$, that term can be moved to the right-hand side of the Carleman estimate. On the other hand, if $v\in\mathcal{V}_0$ there is one edge $\tilde e$ (the edge going joining the previous layer to v) for which $\partial_{n_{\tilde e}}\eta^e=\delta$ and such that $\partial_{n^e}\eta=-1$ for all $e\in\mathcal{E}(v)\setminus\{\tilde e\}$. So, we have to absorb the boundary term of the edge $\tilde e$. For that we use $\psi=ze^{-s\alpha}$ to get:

$$(6.15)\qquad \begin{aligned}&\varepsilon^2s\tau\left(a^{\tilde e}\right)^{-1}\int_0^T\xi^{\tilde e}|\partial_{n^{\tilde e}}\eta^{\tilde e}||\partial_{n^{\tilde e}}\psi^{\tilde e}|^2(t,v)\,\mathrm{d}t\\ &\leq 2\delta\varepsilon^2s\tau\left(a^{\tilde e}\right)^{-1}\int_0^T\xi^{\tilde e}|\partial_{n^{\tilde e}}\alpha^{\tilde e}|^2|\psi|^2(t,v)\,\mathrm{d}t+2\delta s\tau\left(a^{\tilde e}\right)^{-1}\int_0^T\xi^{\tilde e}e^{-2s\alpha^{\tilde e}}|\varepsilon\partial_{n^{\tilde e}}z_\varepsilon|^2(t,v)\,\mathrm{d}t.\end{aligned}$$

We can absorb the first term in the right-hand side of (6.15) by (6.14). As for the second one, it equals

$$(6.16)\qquad \begin{aligned}&\delta\varepsilon^2s\tau(a^{\tilde e})^{-1}\int_0^T\xi^{\tilde e}e^{-2s\alpha^{\tilde e}}\left|\sum_{e\in\mathcal{V}(e)\setminus\tilde e}\partial_{n^e}z^e_\varepsilon\right|^2(t,v)\,\mathrm{d}t\\ &\leq\delta\varepsilon^2s\tau(a^{\tilde e})^{-1}\int_0^T\sum_{e\in\mathcal{V}(e)\setminus\tilde e}\xi^e\left|\partial_{n^e}\psi^e\right|^2(t,v)\,\mathrm{d}t+C\delta\varepsilon^2s^3\tau^3\int_0^T\xi^3|\psi^e|^2(t,v)\,\mathrm{d}t.\end{aligned}$$

Indeed, we use the continuity of functions ξ and α on the junction, the condition on the vertex $(3.7)_4$ and the fact $\partial_x\alpha = -\tau\partial_x\eta\xi$. Taking δ small enough, we can absorb the term (6.16) by (6.14) and

$$\begin{aligned}
&- \varepsilon^2 s\tau \int_0^T \xi \sum_{e\in\mathcal{E}(v)\setminus\tilde{e}} (a^e)^{-1}\partial_{n^e(v)}\eta^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\
&= \varepsilon^2 s\tau \int_0^T \xi \sum_{e\in\mathcal{E}(v)\setminus\tilde{e}} (a^e)^{-1}|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{6.17}$$

With these considerations, we also may absorb J_2 using (6.14) and (6.17) by Cauchy-Schwarz inequality for $s \geq C(T+T^2)\varepsilon^{-1}$ large enough.

To conclude, let us study the boundary term J_4 in (6.12). If $v\in\mathcal{V}_\partial$, then $\partial_t\psi = 0$ because of the Dirichlet boundary conditions. Otherwise, if $v\in\mathcal{V}_0$, from $(3.7)_4$, we obtain that

$$\begin{aligned}
&\sum_{e\in\mathcal{E}(v)}\int_0^T \varepsilon\partial_{n^e}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t = \sum_{e\in\mathcal{E}(v)}\int_0^T \varepsilon\partial_{n^e}\xi^e e^{-s\alpha^e} z_\varepsilon^e\partial_t(e^{-s\alpha^e}z_\varepsilon^e)(t,v)\,\mathrm{d}t \\
&= s^2\varepsilon \sum_{e\in\mathcal{E}(v)}\int_0^T \partial_{n^e}\xi^e\partial_t\alpha^e e^{-2s\alpha^e}|z_\varepsilon^e|^2(t,v)\,\mathrm{d}t + s\varepsilon\sum_{e\in\mathcal{E}(v)}\int_0^T \partial_{n^e}\xi^e e^{-2s\alpha^e}\frac{\partial_t(|z_\varepsilon^e|^2)}{2}(t,v) \\
&= s^2\varepsilon \sum_{e\in\mathcal{E}(v)}\int_0^T \partial_{n^e}\xi^e\partial_t\alpha^e|\psi^e|^2(t,v)\,\mathrm{d}t - s\varepsilon\sum_{e\in\mathcal{E}(v)}\int_0^T \partial_t(\partial_{n^e}\xi^e)\frac{|\psi^e|^2}{2}(t,v)\,\mathrm{d}t,
\end{aligned} \tag{6.18}$$

which can be absorbed by (6.14). To sum up the result of this step, we have proved that, for $s\geq C(T+T^2)\varepsilon^2$ and $\tau \geq C$, the following estimate holds:

$$\begin{aligned}
\sum_{\ell=1}^5 J_\ell \geq{}& c\sum_{v\in\mathcal{V}_0}\varepsilon^2 s^3\tau^3\int_0^T \xi^3|\psi|^2(t,v)\,\mathrm{d}t + c\sum_{v\in\mathcal{V}_0}\varepsilon^2 s\tau\int_0^T \xi\sum_{e\in\mathcal{E}(v)\setminus\tilde{e}}(a^e)^{-1}|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\
&- C\sum_{v\in\mathcal{V}_\partial}\varepsilon^2 s\tau\int_0^T \xi(a^e)^{-1}\partial_{n^e(v)}\eta^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{6.19}$$

Step 3: *Conclusion of the proof.* Combining (6.7)-(6.19) and the fact that $|\partial_x\eta| > 0$, we obtain the estimate:

$$\begin{aligned}
&\varepsilon^2 s\tau^2\iint_Q \xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3\tau^4\iint_Q e^{-2s\alpha}\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq (L_1\psi, L_2\psi)_{L^2(Q)} + \sum_{v\in\mathcal{V}_\partial}\varepsilon^2 s\tau\int_0^T \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.
\end{aligned} \tag{6.20}$$

From (6.20), it is classical to obtain (6.2) as in [6, 41]. We add $\frac{1}{2}(\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)})$ at both sides of (6.20); we consider that $\|L_1\psi + L_2\psi\|^2_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}$; we absorb $(L_3\psi)_1$ and $(L_3\psi)_2$; and, finally, we estimate the terms on $\partial_t\psi$ and $\partial^2_{xx}\psi$ by considering $(6.5)_1$ and $(6.5)_2$ respectively. □

As an immediate consequence of Proposition 6.2, we deduce the following result.

Corollary 6.1 (Observability inequality in one unit of time). *Let $T>1$. Then, there is $C>0$ independent of T such that for all $\varphi^T\in L^2(\mathcal{E})$ the solution φ_ε of (3.1) satisfies:*

$$\|\varphi_\varepsilon(T-1,\cdot)\|^2_{L^2(\mathcal{E})} \leq e^{C/\varepsilon}\sum_{v\in V_\partial}\int_{T-1}^T |\partial_\nu\varphi_\varepsilon^e(t,v)|^2\,\mathrm{d}t. \tag{6.21}$$

Proof. This can be proved with (6.2), usual bounds on the exponential weights and Theorem 2.1 applied to $\chi(T-t)\varphi_\varepsilon^e$, for χ a cut-off function supported in $(-\infty, 1/2]$. □

6.3. **Proof of Claim** (2) **of Theorem 3.3.** Using the Carleman inequality of the previous section, we now conclude the proof of the main result.

Proof of Theorem 3.3, Claim (2)*.* This result is a direct consequence of Corollary 6.1 and of Proposition 6.1. In fact, combining the decay estimate (6.1) and the observability inequality (6.21), we conclude that

$$\|\varphi_\varepsilon(0,\cdot)\|^2_{L^2(\mathcal{E})} \le e^{(C-c(T-t))/\varepsilon}\|\varphi_\varepsilon(T-1,\cdot)\|^2_{L^2(\mathcal{E})} \le \sum_{v\in V_\partial} e^{(C-c(T-t))/\varepsilon}\int_{T-1}^{T} |\partial_\nu \varphi^e_\varepsilon(t,v)|^2\,\mathrm{d}t.$$

As a consequence, we obtain that (3.4) holds for a sufficiently large time $T>0$. □

Remark 6.1 (Networks with loops)**.** *In networks with loops, the difficulty that must be overcome is to construct an auxiliary function η^e that ensures that J_3 is estimated as in* (6.19)*.*

Remark 6.2 (Reducing the number of vertices in the right-hand side of the Carleman inequality)**.** *The pathological case given in Proposition 3.2 shows that the Carleman inequality may be false if we observe fewer vertices. This shows that the computation* (6.19) *– and, in particular, the estimate of J_3 – is quite sharp and difficult to improve, at least for general trees.*

Remark 6.3 (Special proof for star-shaped graphs)**.** *The Carleman can be simplified when the graph $\mathcal{G}$ is star-shaped. In that situation it suffices to consider an auxiliary function η that increases from the central vertex to the outwards ones. In that case we can replicate the proof in* [27]*. The only difficulty comes from $\iint_Q \partial_t\psi\partial^2_{xx}\psi$; but then the boundary term that appears at the junction can be estimated as in* (6.18)*.*

Remark 6.4 (The case of non-constant coefficients)**.** *The same techniques that we have developed in this work apply to prove analogous results if the coefficients b and a depend on the time and space variables as long as $a \in C^1([0,T]\times\mathcal{E})$ such that $\inf_{\mathcal{E}} a > 0$ and $b \in C^1_{pw}([0,T]\times\mathcal{E})$ satisfying $\sum_{e\in\mathcal{E}(v)} b^e(v) = 0$ in all $v\in\mathcal{V}_0$ and $\inf_{\mathcal{E}} b > 0$ (see Remark 4.2 for the necessity of the positivity of b). In fact, the decay property is proved with the transformation*

$$z^e_\varepsilon = \varphi^e_\varepsilon \exp\left(\frac{\displaystyle\int_0^x b^e(\xi)\,\mathrm{d}\xi + c^e}{2\varepsilon}\right), \tag{6.22}$$

where c^e the right constants given by Lemma 2.1 so that $\int_0^x b(\xi)\,\mathrm{d}\xi + c^e$ is continuous and we used the parametrization of each edge e as a segment. With this change of variables we obtain the system

$$\begin{cases} -a^e\partial_t z^e_\varepsilon + \left(\frac{|b^e|^2}{4\varepsilon} - \frac{\partial_x b^e}{2}\right) z^e_\varepsilon - \varepsilon\partial^2_{xx} z^e_\varepsilon = 0, & \text{in } (0,T)\times e,\ e\in\mathcal{E},\\ z^e_\varepsilon(v) = 0, & \text{on } (0,T)\times\mathcal{V}_\partial,\\ z^{e_1}_\varepsilon(t,v) = z^{e_2}_\varepsilon(t,v), & t\in(0,T),\ v\in\mathcal{V}_0,\ \forall e_1,e_2\in\mathcal{E}^{out}(v),\\ \sum_{e\in\mathcal{E}(v)} \varepsilon\partial_{n^e} z^e_\varepsilon = 0 & \text{on } (0,T)\times\mathcal{V}_0,\\ z_\varepsilon(0,\cdot) = z_0, & \text{on } \mathcal{E}. \end{cases} \tag{6.23}$$

Then, the computations for decay and Carleman estimates are still valid; indeed, we just get some lower order terms that can be easily absorbed by putting them in the source term and taking ε small enough.

7. Conclusions and open problems

In this paper, we have obtained the first results on uniform controllability of linear advection-diffusion equations on a tree (and, in particular, on a star-graph) as the diffusivity parameter vanishes. This extends the classical results contained in [27, 41]. Many related problems remain open. In particular, the following ones are of special relevance.

- **Alternative junction conditions.** In addition to the continuity condition that we have imposed at the junctions, there are alternative transmission conditions that are physically relevant, such as those in [40]. It would be interesting to see if analogous results on the controllability and on the cost could be obtained in the framework of [40]. The main challenge to overcome is that, in our analysis, the continuity condition has been pivotal in estimating the boundary terms in the integrations by parts arising in both the decay property and the Carleman inequality.

- **Balance relation of velocity coefficients in the junction.** The balance relation (1.2), introduced in [32], has been frequently used in many integration by parts throughout the paper and to obtain energy dissipation in the junction (see Remark 3.5). As it concerns a lower order term, (1.2) is not needed for the qualitative controllability properties of the parabolic system (see Theorem 3.2) nor for the well-posedness of either system; it is, however, an interesting question to see if the same convergence results hold if we omit such hypothesis.
- **Networks containing loops.** In our proofs, it is essential to assume that our network contains no loop (i.e. that we are on a tree). Namely, Lemma 2.1 does not hold when there are loops, which would not allow us to do the transformation (3.6). In addition, it can be proved that we do not have the decay property in Proposition 6.1 for cycles, where the mass is preserved, so we cannot expect that the cost of the control decay for general graphs (see Example 3.4). Moreover, in the Carleman estimate, cycles cause problems to define the auxiliary function, as explained in Remark 6.1. It would be interesting to characterize the controllability properties of (1.1) by the topology or metric properties of the network (see, e.g., [28, Chapter 5 & 8]).
- **Control without critical lengths.** An irrationality condition between the lengths of the edges (see [28, Corollary 8.6]) can allow to control a tree-shaped network using a reduced number of controls and may be relevant for the controllability in the presence of loops. Thus, it remains open to prove under which circumstances the cost of the control decays with ε. Actually, this could be studied in parallel with the previous problem.
- **Nonlinear conservation laws.** It would be interesting to extend the analysis of the present paper to nonlinear conservation laws (as in [37, 54]), for which existence results on networks are available in [23, 22, 3, 34, 17].
- **Dispersive effects.** We have considered only a model that includes advection and diffusive phenomena, but the PDEs may also have dispersive effect.In the case of zero dispersion limit, the uniform control properties of the linearized Korteweg-de Vries equation were studied in [38] (on the real line); subsequently, in [39], the authors addressed the case of zero diffusion-dispersion; further recent work on the KdV equation with a vanishing parameter in the diffusive term include [13, 14]. In the setting of a star-graph, well-posedness and controllability results for KdV are also available in the literature (see [16, 19, 18, 2]), but the uniform controllability problem has not been addressed yet., the main challenge being the identification of suitable transmission conditions at the junction to carry out the arguments.
- **Optimal time T for the decay.** Establishing sharp estimates on the time T that separates blow-up and decay is an interesting question, though this problem seems really challenging as it is still open even on networks which consist on a single edge. The first result, in dimension one and for a constant velocity, was obtained in [27]; afterwards, the same problem was studied in [41] in any dimension and with a transport flow belonging to $W^{1,\infty}(\mathbb{R}^+ \times \Omega)$. More recently, better approximations have been given for the optimal time in which the cost of the control decays: the lower bound was improved first in [60], through complex analysis and properties of the entire functions, and in [53] through semi-classical and spectral analysis; and the upper bound was improved in [36, 59] (in the first one through complex analysis and, in the second one, by transforming the original equation into the heat equation without convection terms).
- **Cost of approximate controllability on the heat equation for a sufficiently large time.** For a tree-shaped graph $\mathcal{G}$, the limit system (1.3) is exactly controllable for a sufficiently large time. Consequently, it looks natural that the cost of approximate controllability of (1.1) should decay for a sufficiently large time. We would like to remark that some relevant works for the cost of parabolic equations are [33] (for the heat equation) and [52] (for hypoelliptic equations).
- **Parabolic singular limit of dissipative wave equation.** An interesting extension of this work could be the study of parabolic singular limits – namely, the singular limit of $\varepsilon\partial_{tt}^2 u - \partial_{xx}^2 u + \partial_t u = 0$ as $\varepsilon \to 0^+$ – on networks. Regarding the convergence properties of those systems in domains belonging to $\mathbb{R}^d$, they were first studied in [69], and then in [70]. In addition, the controllability properties of those system, and in particular the asymptotic behaviour, was studied in [63] and [62].
- **Controllability under positivity constraints.** It also remains open to prove that, for positive initial data, we can take the solution to equilibrium without loosing the positivity, as in [61, 68].

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