The geodesic total curvature of spherical curves

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Abstract. The geodesic total curvature of rectifiable spherical curves is analyzed. We extend to the case of high dimension spheres the explicit formula that holds true for curves supported into the 2-sphere. For this purpose, we take advantage of some new integral-geometric formulas concerning both the Euclidean and geodesic total curvature of spherical curves.

Keywords : Spherical curves; geodesic curvature; integral-geometric formulas; non-smooth curves

MSC : 53A04; 49J45

Introduction

The total curvature of curves in Euclidean spaces is defined by J. W. Milnor [6, 7] as the supremum of the rotation $\mathbf{k}^*(P)$ computed among all the inscribed polygonals P. Following J. M. Sullivan [10], we recall that a curve in $\mathbb{R}^{N+1}$ with compact support and finite total curvature is rectifiable. Therefore, its arc-length parameterization $\mathbf{c}$ is a Lipschitz-continuous function, hence it is differentiable a.e., by Rademacher's theorem. Furthermore, denoting by $\mathbf{t} = \dot{\mathbf{c}}$ the tangent indicatrix (or tantrix) of a rectifiable curve $\mathbf{c}$, it turns out that $\mathbf{c}$ has finite total curvature $\mathrm{TC}(\mathbf{c})$ if and only if its tantrix is a function of bounded variation. In this case, moreover, the explicit formula

$$\mathrm{TC}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) \,, \qquad \mathbf{t} = \dot{\mathbf{c}}$$

holds, where the definition of *essential variation* $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$ of the tantrix involves the geodesic distance $d_{\mathbb{S}^N}$ in the Gauss N-sphere $\mathbb{S}^N$ instead of the Euclidean distance in $\mathbb{R}^{N+1}$, see (1.2).

In this paper, we deal with rectifiable curves $\mathbf{c}$ supported in the unit hyper-sphere $\mathcal{S}^N$ of $\mathbb{R}^{N+1}$,

$$\mathcal{S}^N := \{x \in \mathbb{R}^{N+1} \,:\, \|x\| = 1\} \,, \qquad N \geq 2.$$

Notice that we shall make use of a different notation in order to distinguish between the sphere $\mathcal{S}^N$ where the curve is supported and the Gauss sphere $\mathbb{S}^N$ where the tantrix of the curve takes value.

The *geodesic rotation* $\mathbf{k}_{\mathcal{S}^N}(P)$ of a spherical polygonal P in $\mathcal{S}^N$ is the sum of the turning angles between its consecutive geodesic arcs. However, since $\mathcal{S}^N$ has positive sectional curvature, the expected monotonicity formula for the geodesic rotation of inscribed spherical polygonals fails to hold, see [3] and Example 2.1.

Therefore, the good definition of *geodesic total curvature*, say $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c})$, turns out to be the one introduced by Alexandrov-Reshetnyak [1], see Definition 2.2. It involves the *modulus* of spherical polygonals inscribed in $\mathbf{c}$, compare e.g. [5]. Since for polygonals P in $\mathcal{S}^N$ one has $\mathbf{k}^*(P) = \mathbf{k}_{\mathcal{S}^N}(P) + \mathcal{L}(P)$, for a rectifiable curve $\mathbf{c}$ in $\mathcal{S}^N$ one infers that

$$\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty \iff \mathrm{TC}(\mathbf{c}) < \infty \,.$$

Referring to Secs. 3.1–3.2 of [2] for the notation and properties of one-dimensional BV functions, we remark here that the Cantor component of the distributional derivative of the tantrix is orthogonal to $\mathbf{c}$, see (1.4). Moreover, one gets $\dot{\mathbf{t}}(s) \bullet \mathbf{c}(s) = -1$ for a.e. $s \in I_L := (0, L)$, where $L = \mathcal{L}(\mathbf{c})$, whence the vector

$$\dot{\mathbf{t}}^\top(s) := \dot{\mathbf{t}}(s) + \mathbf{c}(s)$$

is tangential to $\mathcal{S}^N$ at $\mathbf{c}(s)$ for a.e. s, and for smooth curves $\mathbf{c}$ its modulus is equal to the geodesic curvature.

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The above facts lead us to introduce the *geodesic curvature energy* functional

$$\mathcal{F}(\mathbf{c}) := \int_{I_L} |\dot{\mathbf{t}}^\top|\, ds + |D^C \mathbf{t}|(I_L) + \sum_{s\in J_{\mathbf{t}}} d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-))\,, \qquad \mathbf{t} = \dot{\mathbf{c}}$$

of a rectifiable curve $\mathbf{c}$ in $\mathcal{S}^N$ satisfying $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty$. Notice that on account of formula (1.2) one has:

$$\mathcal{F}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) - \mathcal{L}(\mathbf{c})\,.$$

In Theorem 3.1, the main result of this paper, we prove in any dimension $N \geq 2$ and for any rectifiable curve $\mathbf{c}$ in $\mathcal{S}^N$ satisfying $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty$ the following representation formula:

$$\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \mathcal{F}(\mathbf{c})\,.$$

When $N = 2$, the latter formula has been obtained in our paper [8], where we exploited the existence of a weak parallel transport along the curve $\mathbf{c}$ whose angle function Θ has bounded variation and satisfies $|D\Theta|(I_L) = \mathcal{F}(\mathbf{c})$. The main feature, that actually holds true for curves supported into (Riemannian) surfaces, is that the angle function has distributional derivative equal to the "weak" signed geodesic curvature of $\mathbf{c}$, whereas a generalized Gauss-Bonnet theorem holds true in this framework. Therefore, our argument from [8] fails to hold for curves supported in high dimension Riemannian manifolds $\mathcal{M}$.

In the proof of Theorem 3.1, we exploit its validity in the case $N = 2$, by making use of some new *integral-geometric formulas* that we now present.

Denote by $G_{j+1}\mathbb{R}^{N+1}$ the Grassmannian of the unoriented $(j+1)$-planes p in $\mathbb{R}^{N+1}$, by μ_{j+1} the corresponding Haar measure, and by π_p the orthogonal projection of $\mathbb{R}^{N+1}$ onto some p in $G_{j+1}\mathbb{R}^{N+1}$. Also, denote by $\eta_p(x)$ the nearest point to $x \in \mathcal{S}^N$ onto the j-dimensional sphere $\mathcal{S}^j_p := \mathcal{S}^N \cap p$, see (2.4).

For any integers $2 \leq j \leq N-1$ and any rectifiable curve $\mathbf{c}$ in $\mathcal{S}^N$ with finite total curvature, in Propositions 3.2 and 3.3 we prove the average formulas:

$$\begin{aligned} \mathrm{TC}(\mathbf{c}) &= \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\eta_p(\mathbf{c}))\, d\mu_{j+1}(p) \\ \mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) &= \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}_{\mathcal{S}^j_p}(\eta_p(\mathbf{c}))\, d\mu_{j+1}(p)\,. \end{aligned}$$

We expect that the previous representation formula for the geodesic (or intrinsic) total curvature holds true for curves supported in high dimension Riemannian manifolds $\mathcal{M}$, provided that in the expression of the curvature energy functional $\mathcal{F}(\mathbf{c})$ we take $\dot{\mathbf{t}}^\top(s)$ equal to the projection of the approximate derivative of the tantrix onto the tangent space to $\mathcal{M}$ at $\mathbf{c}(s)$. However, its validity cannot be checked by means of averaging arguments as in Theorem 3.1, when $\mathcal{M}$ fails to be an N-sphere.

We conclude the introduction by describing the content of this paper. In Sec. 1, we recall the main properties concerning the (Euclidean) total curvature, and the related integral-geometric formulas. In Sec. 2, we introduce the notion of geodesic total curvature and collect some relevant average formulas concerning the length of rectifiable curves in $\mathcal{S}^N$ and the geodesic rotation of polygonal curves, Proposition 2.5. In Sec. 3, we finally prove our main results, Theorem 3.1 and Propositions 3.2 and 3.3.

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1 Total curvature

In this section, we recall some properties concerning the (Euclidean) total curvature of curves in $\mathbb{R}^{N+1}$.

TOTAL CURVATURE. The rotation $\mathbf{k}^*(P)$ of a polygonal curve P in $\mathbb{R}^{N+1}$ is the sum of the exterior angles between consecutive segments. If P' is a polygonal inscribed in P, one has $\mathbf{k}^*(P') \leq \mathbf{k}^*(P)$. Therefore, the (Euclidean) *total curvature* $\mathrm{TC}(\mathbf{c})$ of a curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$ is defined, following Milnor [6, 7], as the supremum of the *rotation* $\mathbf{k}^*(P)$ computed among all the polygonals P in $\mathbb{R}^{N+1}$ which are inscribed in $\mathbf{c}$. Then,

$\mathrm{TC}(P) = \mathbf{k}^*(P)$ for each polygonal P. Moreover, if $\mathbf{c}$ has finite total curvature, for each point v in $\mathbf{c}$, small open arcs of $\mathbf{c}$ with an end point equal to v have small total curvature. As a consequence, compare [10], it turns out that the total curvature of $\mathbf{c}$ is equal to the limit of $\mathbf{k}^*(P_h)$ for *any* sequence $\{P_h\}$ of polygonals in $\mathbb{R}^{N+1}$ inscribed in $\mathbf{c}$ and such that $\mathrm{mesh}(P_h) \to 0$.

Integral-geometric formulas. Several properties concerning curves in $\mathbb{R}^{N+1}$ can be obtained once they are checked for e.g. planar curves, by making use of averaging arguments based on the validity of suitable *integral-geometric formulas* that we now recall.

For $0 \le j \le N-1$ integer, denote by $G_{j+1}\mathbb{R}^{N+1}$ the Grassmannian of the unoriented $(j+1)$-planes p in $\mathbb{R}^{N+1}$. It is a compact group, and it can be equipped with a unique rotationally invariant probability measure, that will be denoted by μ_{j+1}. For $p \in G_{j+1}\mathbb{R}^{N+1}$, we denote by π_p the orthogonal projection of $\mathbb{R}^{N+1}$ onto p.

Example 1.1 If $\mathbf{c}$ is a rectifiable curve in $\mathbb{R}^{N+1}$, the following integral-geometric formula for the length $\mathcal{L}(\mathbf{c})$ holds true for any $j = 0, \ldots, N-1$:

$$\mathcal{L}(\mathbf{c}) = \frac{\sigma_j}{\sigma_N} \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}(\pi_p(\mathbf{c}))\, d\mu_{j+1}(p)$$

where σ_j and σ_N are positive constants only depending on j and N, respectively, see e.g. [1, Sec. 4.8].

Also, a classical result that goes back to Fáry [4] shows that the total curvature of a curve (with finite total curvature) is the average of the total curvatures of all its projections onto $(j+1)$-planes:

$$\mathrm{TC}(\mathbf{c}) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\pi_p(\mathbf{c}))\, d\mu_{j+1}(p) \qquad \forall j = 0, \ldots, N-1\,. \tag{1.1}$$

Following [10, Prop. 4.1], it suffices to prove the average formula for an angle, hence for the rotation $\mathbf{k}^*(P)$ of a polygonal P, and then use the monotone convergence theorem.

We e.g. readily check that if a curve $\mathbf{c}$ in $\mathbb{R}^{N+1}$ has compact support and finite total curvature, then $\mathbf{c}$ is a rectifiable curve. In fact, if d is the diameter of $\mathbf{c}$, one has $\mathcal{L}(\pi_p(\mathbf{c})) \le d\,(\mathrm{TC}(\pi_p(\mathbf{c})) + 1)$ for μ_1-a.e. $p \in G_1\mathbb{R}^{N+1}$. Therefore, the previous average formulas (with $j = 0$) yield

$$\mathcal{L}(\mathbf{c}) = \frac{\sigma_0}{\sigma_N} \int_{G_1\mathbb{R}^{N+1}} \mathcal{L}(\pi_p(\mathbf{c}))\, d\mu_1(p) \le \frac{\sigma_0\, d}{\sigma_N} \int_{G_1\mathbb{R}^{N+1}} (\mathrm{TC}(\pi_p(\mathbf{c})) + 1)\, d\mu_1(p) = \frac{\sigma_0\, d}{\sigma_N} (\mathrm{TC}(\mathbf{c}) + 1) < \infty\,.$$

Essential variation in the Gauss sphere. On account of the previous remark, since we are interested in curves with support in the unit hyper-sphere of $\mathbb{R}^{N+1}$, we deal with rectifiable curves $\mathbf{c}$ in $\mathbb{R}^{N+1}$. We then tacitly assume that $\mathbf{c}$ is parameterized by arc-length, so that $\mathbf{c} = \mathbf{c}(s)$, with $s \in [0, L] = \overline{I}_L$, where $I_L := (0, L)$ and $L = \mathcal{L}(\mathbf{c})$. Since $\mathbf{c}$ is a Lipschitz function, by Rademacher's theorem (cf. [2, Thm. 2.14]) it is differentiable $\mathcal{L}^1$-a.e. in I_L. Denoting by $\dot f := \frac{d}{ds} f$ the derivative w.r.t. the arc-length parameter s, one has $\mathbf{t}(s) := \dot{\mathbf{c}}(s) \in \mathbb{S}^N$ for a.e. s, where $\mathbb{S}^N$ is the Gauss N-sphere. If $\mathbf{c}$ is smooth and regular, one recovers the formula $\mathrm{TC}(\mathbf{c}) = \int_{I_L} |\mathbf{k}|\, ds$, where $\mathbf{k}(s) := \ddot{\mathbf{c}}(s)$ is the curvature vector. In general, if $\mathrm{TC}(\mathbf{c}) < \infty$ the *tantrix* $\mathbf{t} = \dot{\mathbf{c}} : I_L \to \mathbb{R}^{N+1}$ is a *function of bounded variation*. More precisely, a rectifiable curve $\mathbf{c}$ has finite total curvature if and only its tantrix $\mathbf{t}$ is a function of bounded variation, see [10, Prop. 3.1].

We refer to Secs. 3.1–3.2 of [2] for the notation and properties of one-dimensional BV functions, see also [8]. We only recall that the essential variation $\mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{v})$ of a function $\mathbf{v} \in \mathrm{BV}(I_L, \mathbb{R}^{N+1})$ agrees with the total variation of the distributional derivative $D\mathbf{v}$, namely:

$$\mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{v}) = |D\mathbf{v}|(I_L) = \int_{I_L} |\dot{\mathbf{v}}|\, ds + \sum_{s \in J_{\mathbf{v}}} |\mathbf{v}(s+) - \mathbf{v}(s-)| + |D^C\mathbf{v}|(I_L)$$

where $\dot{\mathbf{v}} \in L^1(I_L, \mathbb{R}^{N+1})$ is the approximate derivative, $\mathbf{v}(s\pm)$ denote the right and left limits of $\mathbf{v}$ at each point $s \in I_L$, the Jump set $J_{\mathbf{v}}$ is the at most countable set of points in I_L where $\mathbf{v}(s-) \ne \mathbf{v}(s+)$, and $D^C\mathbf{v}$ is the Cantor component of the distributional derivative, see Example 1.2.

If $\mathrm{TC}(\mathbf{c}) < \infty$, we thus have $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^N)$. The *essential variation* $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$ *of* $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^N)$ *in* $\mathbb{S}^N$ differs from $\mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{t})$, as its definition involves the *geodesic distance* $d_{\mathbb{S}^N}$ in $\mathbb{S}^N$ instead of the Euclidean distance in $\mathbb{R}^{N+1}$, and one has:

$$\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) = \int_{I_L} |\dot{\mathbf{t}}|\, ds + \sum_{s \in J_{\mathbf{t}}} d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-)) + |D^C \mathbf{t}|(I_L)\,. \tag{1.2}$$

Therefore, in general $\mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{t}) \leq \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$, and equality holds if and only if $\mathbf{t}$ has a continuous representative. Moreover, if a sequence $\{\mathbf{t}_h\} \subset \mathrm{BV}(I_L, \mathbb{S}^N)$ weakly-* converges in BV to $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^N)$, the following lower semicontinuity inequalities hold:

$$\mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{t}) \leq \liminf_{h\to\infty} \mathrm{Var}_{\mathbb{R}^{N+1}}(\mathbf{t}_h)\,, \qquad \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) \leq \liminf_{h\to\infty} \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_h)\,. \tag{1.3}$$

Notice that the Cantor component of the derivative of the tantrix is orthogonal to $\mathbf{c}$. In fact, using that $\mathbf{c}$ is Lipschitz-continuous, and that $\mathbf{t} \bullet \mathbf{c} \equiv 0$, where $\bullet$ is the scalar product in $\mathbb{R}^{N+1}$, we get

$$0 = D^C(\mathbf{t} \bullet \mathbf{c}) = \mathbf{t} \bullet D^C \mathbf{c} + \mathbf{c} \bullet D^C \mathbf{t} = \mathbf{c} \bullet D^C \mathbf{t}\,. \tag{1.4}$$

Example 1.2 If P is a polygonal curve, the tantrix $\mathbf{t}_P$ is a pure Jump function. At each corner point of P one has $d_{\mathbb{S}^N}(\mathbf{t}(s+), \mathbf{t}(s-)) = \theta$, the turning angle, whereas $|\mathbf{t}(s+) - \mathbf{t}(s-)| = 2\sin(\theta/2)$, so that $\mathbf{k}^*(P) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_P)$.

Notice that the Cantor component $D^C \mathbf{t}$ is non-trivial, in general. In fact, let e.g. $\gamma : \overline{I} \to \mathbb{R}^2$, where $I = (0,1)$, denote the Cartesian curve $\gamma(t) := (t, u(t))$ in $\mathbb{R}^2$ given by the graph of the primitive $u(t) := \int_0^t v(\lambda)\, d\lambda$ of the classical Cantor-Vitali function $v : \overline{I} \to \mathbb{R}$ associated to the "middle thirds" Cantor set. It turns out that $\mathbf{t} = (1+v^2)^{-1/2}(1, v)$, whence $\mathbf{t}$ is a Cantor function, i.e., $D^a \mathbf{t} = D^J \mathbf{t} = 0$, and

$$D\mathbf{t}(I) = D^C \mathbf{t}(I) = \int_I \frac{1}{(1+v^2)^{3/2}} (-v, 1)\, dD^C v\,.$$

In particular, one gets $\mathrm{Var}_{\mathbb{R}^2}(\mathbf{t}) = \mathrm{Var}_{\mathbb{S}^1}(\mathbf{t}) = |D^C \mathbf{t}|(I) = \mathrm{TC}(\gamma) = \pi/4$.

A REPRESENTATION FORMULA. Let $\mathbf{c}$ be a rectifiable curve in $\mathbb{R}^{N+1}$ with finite total curvature, and let $\mathbf{t} = \dot{\mathbf{c}}$. For any polygonal P inscribed in $\mathbf{c}$ one has

$$\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_P) \leq \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})\,. \tag{1.5}$$

In fact, assume that P is generated by the consecutive vertexes $\mathbf{c}(s_i)$, where $0 = s_0 < s_1 < \cdots < s_n = L$, and let $\mathbf{v}_i$ be the oriented segment of P from $\mathbf{c}(s_{i-1})$ to $\mathbf{c}(s_i)$. When $\mathbf{c}$ is a *planar curve* (i.e., when $N = 1$), the value of $\mathbf{t}_P \in \mathbb{S}^1$ on the segment $\mathbf{v}_i$ is equal to one of the values of the tantrix $\mathbf{t}$ in the interval $]s_{i-1}, s_i[$, when $\mathbf{t}$ is completed to a continuous curve in $\mathbb{S}^1$ by connecting with geodesic arcs the points $\mathbf{t}(s-)$ and $\mathbf{t}(s+)$ for each $s \in J_{\mathbf{t}}$. Therefore, when $N \geq 2$ the value of the tantrix $\mathbf{t}_P$ in $\mathbf{v}_i$ is an average of the values of the restriction to (s_{i-1}, s_i) of the completed tantrix $\mathbf{t}$, compare [1].

As a consequence of (1.5), one obtains the following representation formula:

$$\mathrm{TC}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})\,, \qquad \mathbf{t} = \dot{\mathbf{c}}\,. \tag{1.6}$$

In fact, if $\{P_h\}$ is an inscribed sequence satisfying $\mathrm{mesh}(P_h) \to 0$, and P_h is parameterized with (piecewise) constant velocity in I_L, then $\{\mathbf{t}_h\}$ converges to $\mathbf{t}$ weakly-* in $\mathrm{BV}(I_L, \mathbb{R}^{N+1})$. The upper bound (1.5) and the lower semicontinuity inequality in (1.3) yield the strict convergence $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_h) \to \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t})$. Using that $\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}_h) = \mathbf{k}^*(P_h) \to \mathrm{TC}(\mathbf{c})$, one concludes with (1.6).

We finally notice that on account of (1.6), one we can re-write the integral-geometric formula (1.1) in terms of the tantrix $\mathbf{t}$, namely:

$$\mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{Var}_{\mathbb{S}^j_p}(\mathbf{t}_{(p)})\, d\mu_{j+1}(p) \qquad \forall j = 0, \ldots, N-1 \tag{1.7}$$

where $\mathbb{S}^j_p := \mathbb{S}^N \cap p$ and $\mathbf{t}_{(p)}$ denotes the tantrix of the projected curve $\pi_p(\mathbf{c})$.

2 Geodesic total curvature of spherical curves

In the sequel we deal with rectifiable curves supported in the unit hyper-sphere $\mathcal{S}^N$ of $\mathbb{R}^{N+1}$,

$$\mathcal{S}^N := \{x \in \mathbb{R}^{N+1} \,:\, \|x\| = 1\}\,, \qquad N \geq 2.$$

Geodesic total curvature. For a general curve $\mathbf{c}$ contained in $\mathcal{S}^N$, we shall denote by $\mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$ the class of polygonals in $\mathcal{S}^N$ which are inscribed in $\mathbf{c}$. The *geodesic rotation* $\mathbf{k}_{\mathcal{S}^N}(P)$ of a polygonal P in $\mathcal{S}^N$ is the sum of the turning angles between the consecutive geodesic arcs of P. One is tempted to define the geodesic total curvature of $\mathbf{c}$ as in the Euclidean case, i.e., by taking the supremum of the geodesic rotation $\mathbf{k}_{\mathcal{S}^N}(P)$ computed among all the polygonals P in $\mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$. However, as observed in [3], the latter definition does not work. In fact, if $P, P' \in \mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$, and P is obtained by adding a vertex in $\mathbf{c}$ to the vertexes of P', then the monotonicity inequality $\mathbf{k}_{\mathcal{S}^N}(P') \leq \mathbf{k}_{\mathcal{S}^N}(P)$ is violated, since $\mathcal{S}^N$ has positive sectional curvature.

Example 2.1 If e.g. $N = 2$ and $\mathbf{c}$ is a parallel which is not a great circle, then the opposite inequality $\mathbf{k}_{\mathcal{S}^2}(P') \geq \mathbf{k}_{\mathcal{S}^2}(P)$ holds, and for any $P \in \mathcal{P}_{\mathcal{S}^2}(\mathbf{c})$ one has $\mathbf{k}_{\mathcal{S}^2}(P) > \int_{\mathbf{c}} |\mathfrak{K}_g|\, ds$, where $\mathfrak{K}_g$ is the geodesic curvature of the parallel $\mathbf{c}$.

Actually, the good definition turns out to be the one introduced by Alexandrov-Reshetnyak [1]. For this purpose, compare e.g. [5], we recall that the *modulus* $\mu_{\mathbf{c}}(P)$ of a polygonal P in $\mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$ is the maximum of the geodesic diameter of the arcs of $\mathbf{c}$ determined by the consecutive vertexes in P. For $\varepsilon > 0$, we also let

$$\Sigma_\varepsilon(\mathbf{c}) := \{P \in \mathcal{P}_{\mathcal{S}^N}(\mathbf{c}) \mid \mu_{\mathbf{c}}(P) < \varepsilon\}\,.$$

Definition 2.2 The *geodesic total curvature* of a curve $\mathbf{c}$ in $\mathcal{S}^N$ is

$$\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) := \lim_{\varepsilon \to 0^+} \sup\{\mathbf{k}_{\mathcal{S}^N}(P) \mid P \in \Sigma_\varepsilon(\mathbf{c})\}\,.$$

Properties. Since for polygonals P in $\mathcal{S}^N$ one has $\mathbf{k}^*(P) = \mathbf{k}_{\mathcal{S}^N}(P) + \mathcal{L}(P)$, for a rectifiable curve $\mathbf{c}$ in $\mathcal{S}^N$ one infers the equivalence:

$$\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty \iff \mathrm{TC}(\mathbf{c}) < \infty\,. \tag{2.1}$$

Remark 2.3 In [1, Thm. 6.3.1] it was erroneously stated that a curve with finite geodesic total curvature has finite Euclidean total curvature, too. This is true if the spherical diameter of the curve is smaller than a dimensional constant c_N. In this case, in fact, for polygonal curves in $\mathcal{S}^N$ one has $\mathbf{k}^*(P) \leq \pi + 2\mathbf{k}_{\mathcal{S}^N}(P)$. Therefore, the previous statement holds true provided that the curve can be divided in a finite number of arcs each one with spherical diameter smaller than c_N. However, the latter property is false, in general, if the curve fails to be rectifiable. If e.g. one takes a curve that winds around an equator of $\mathcal{S}^N$ infinitely many times, its geodesic total curvature is zero but its length and total Euclidean curvature are both infinite.

If $\mathbf{c}$ is smooth (and parameterized in arc-length), since $|\mathbf{t}| \equiv 1$ the curvature vector $\mathbf{k}(s) := \dot{\mathbf{t}}(s)$ is orthogonal to $\mathbf{t}(s)$ and decomposes as $\mathbf{k}(s) = -\mathbf{c}(s) + \mathfrak{K}_g(s)\mathbf{u}(s)$, where $\mathbf{u}(s)$ is a tangent unit vector to $\mathcal{S}^N$ at $\mathbf{c}(s)$ and $\mathfrak{K}_g(s)$ is the geodesic curvature at $\mathbf{c}(s)$. In fact, one has $\mathbf{k} \bullet \mathbf{c} = \ddot{\mathbf{c}} \bullet \mathbf{c} = -\dot{\mathbf{c}} \bullet \dot{\mathbf{c}} = -|\mathbf{t}|^2 = -1$.

If $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty$, we have seen that $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^N)$, and in a similar way one gets $\dot{\mathbf{t}}(s) \bullet \mathbf{c}(s) = -1$ for a.e. s, whence the vector

$$\dot{\mathbf{t}}^\top(s) := \dot{\mathbf{t}}(s) + \mathbf{c}(s)$$

is tangential to $\mathcal{S}^N$ at $\mathbf{c}(s)$ for a.e. $s \in I_L$, and in the smooth case one has $|\dot{\mathbf{t}}^\top(s)| = \mathfrak{K}_g(s)$ for every s.

More precisely, the polar decomposition $D\mathbf{t} = \mathbf{u}\,|D\mathbf{t}|$, where $\mathbf{u}$ is the Radon-Nikodym derivative of the measure $D\mathbf{t}$ with respect to its total variation, implies that $\mathbf{u} : I_L \to \mathbb{S}^N$ is a Borel function satisfying

$$\mathbf{u}(s) = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|} \quad \text{and} \quad \mathbf{u}(s) = \frac{\mathbf{t}(s+) - \mathbf{t}(s-)}{|\mathbf{t}(s+) - \mathbf{t}(s-)|}$$

at $\mathcal{L}^1$-a.e. point $s \in I_L$ and at any point $s \in J_{\mathbf{t}}$, respectively. Also, due to the orthogonality condition (1.4) it turns out that $\mathbf{u}$ is tangential to $\mathbb{S}^N$ at $\mathbf{c}(s)$ at $|D^C\mathbf{t}|$-a.e. $s \in I_L$ and actually:

$$D^C\mathbf{t} = (\mathbf{u} \bullet D^C\mathbf{t})\,\mathbf{u}\,.$$

The above facts lead us to introduce the *geodesic curvature energy* functional

$$\mathcal{F}(\mathbf{c}) := \int_{I_L} |\dot{\mathbf{t}}^\top|\, ds + |D^C\mathbf{t}|(I_L) + \sum_{s\in J_{\mathbf{t}}} d_{\mathbb{S}^N}(\mathbf{t}(s+),\mathbf{t}(s-))\,, \qquad \mathbf{t} = \dot{\mathbf{c}} \tag{2.2}$$

for rectifiable curves $\mathbf{c}$ in $\mathcal{S}^N$ satisfying $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty$. Notice that since $|\dot{\mathbf{t}}| = |\dot{\mathbf{t}}^\top| + 1$, by (1.2) one has:

$$\mathcal{F}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^N}(\mathbf{t}) - \mathcal{L}(\mathbf{c})\,. \tag{2.3}$$

For piecewise smooth curves, the geodesic curvature energy functional $\mathcal{F}(\mathbf{c})$ agrees with the sum of the integral of the geodesic curvature (computed separately outside the corner points of $\mathbf{c}$) plus the sum of the turning angles of the tangent vector to $\mathbf{c}$ at the corner points $\mathbf{c}(s)$, that correspond to the values of the parameter s in the finite set $J_{\mathbf{t}}$ of the discontinuity points of $\mathbf{t}$. In particular, for polygonal curves P of $\mathcal{S}^N$ one has $\mathcal{F}(P) = \mathbf{k}_{\mathcal{S}^N}(P)$. Therefore, one expects the validity of the formula

$$\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \mathcal{F}(\mathbf{c})\,.$$

CURVES INTO THE 2-SPHERE. When $N = 2$, the previous representation formula has been obtained in our paper [8], where we exploited the existence of a weak parallel transport along the curve $\mathbf{c}$ whose angle function Θ has bounded variation and satisfies $|D\Theta|(I_L) = \mathcal{F}(\mathbf{c})$. The main feature, that actually holds true for curves supported into (Riemannian) surfaces, is that the angle function has weak derivative equal to the "weak" signed geodesic curvature of $\mathbf{c}$. The validity of the Gauss-Bonnet theorem in this framework, in fact, allowed us to prove the strict convergence of the angle function along sequences of inscribed approximating polygonals, yielding to $\mathbf{k}_{\mathcal{S}^2}(P_h) \to \mathcal{F}(\mathbf{c})$. Since we also know that $\mathbf{k}_{\mathcal{S}^2}(P_h) \to \mathrm{TC}_{\mathcal{S}^2}(\mathbf{c})$ provided that $\{P_h\} \subset \mathcal{P}_{\mathcal{S}^2}(\mathbf{c})$ satisfies $\mu_{\mathbf{c}}(P_h) \to 0$, the previous representation formula holds true for any rectifiable curve $\mathbf{c}$ in $\mathcal{S}^2$ satisfying $\mathrm{TC}_{\mathcal{S}^2}(\mathbf{c}) < \infty$. Notice that the above strategy fails to hold for curves supported in high dimension Riemannian manifolds $\mathcal{M}$, since we do not have a unique way to enclose a loop in $\mathcal{M}$ and hence we cannot rely on arguments based on the Gauss-Bonnet theorem.

In order to extend the representation formula to the case of high dimension $N \geq 3$, we shall make use of some new integral-geometric formulas relating the geodesic and Euclidean total curvature of a curve $\mathbf{c}$ in $\mathcal{S}^N$ with the geodesic and Euclidean total curvature of its projections onto 2-dimensional spheres, respectively. For this purpose, we collect some more notation, a formula for the length of rectifiable curves, and a formula for the geodesic rotation of polygonal curves.

AVERAGES ON SPHERES. Following [1], for $j = 1,\dots,N$ and $p \in G_{j+1}\mathbb{R}^{N+1}$, we denote by $\eta_p(x)$ the nearest point to $x \in \mathcal{S}^N$ onto the j-dimensional sphere $\mathcal{S}^j_p := \mathcal{S}^N \cap p$. It is well-defined by

$$\eta_p(x) := \frac{\pi_p(x)}{|\pi_p(x)|} \tag{2.4}$$

provided that x is not orthogonal to the $(j+1)$-plane p, i.e., if $x \in \mathcal{S}^N \setminus \mathcal{S}^{j\,\perp}_p$, where $\mathcal{S}^{j\,\perp}_p$ is the $(N-j-1)$-sphere given by the *polar* to $\mathcal{S}^j_p$ in $\mathcal{S}^N$.

The average formula concerning the length of spherical curves was proved in [1, Thm. 4.8.3, p. 108].

Proposition 2.4 *Given a rectifiable curve* $\mathbf{c}$ *in* $\mathcal{S}^N$*, for any integer* $1 \leq j \leq N-1$ *one has*

$$\mathcal{L}(\mathbf{c}) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}(\eta_p(\mathbf{c}))\, d\mu_{j+1}(p)\,.$$

The following integral-geometric formula for the geodesic rotation of spherical polygonal curves was proved in [1, Thm. 6.2.2, p. 190] for $j = 1$. Actually, it holds true for all the ranges of values of j. For the sake of completeness, we report here the proof in the case $j > 1$ taken from [9].

Proposition 2.5 *Given a polygonal curve* γ *in* $\mathcal{S}^N$*, for any integer* $1 \leq j \leq N-1$ *one has*

$$\mathbf{k}_{\mathcal{S}^N}(\gamma) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathbf{k}_{\mathcal{S}^j_p}(\eta_p(\gamma))\, d\mu_{j+1}(p)\,.$$

Proof: Assume $j > 1$. For the sake of simplicity we denote here by $\mathbf{K}_g(\widetilde{\gamma})$ the geodesic rotation of a polygonal $\widetilde{\gamma}$ in a unit sphere of generic dimension. For μ_{j+1}-a.e. $p \in G_{j+1}\mathbb{R}^{N+1}$, the cited integral-geometric formula from [1] implies that the geodesic rotation of the projected curve $\mathbf{K}_g(\eta_p(\gamma))$ is equal to the averaged integral of the geodesic rotation of the projection of the curve $\eta_p(\gamma)$ onto the unit circles corresponding to the 2-planes q of $\mathbb{R}^{N+1}$ that are contained in p, i.e.,

$$\mathbf{K}_g(\eta_p(\gamma)) = \int_{G_2\mathbb{R}_p^{j+1}} \mathbf{K}_g(\eta_q^p(\eta_p(\gamma)))\, d\mu_2^p(q)$$

where μ_2^p is the probability measure corresponding to the Grassmannian $G_2\mathbb{R}_p^{j+1}$, with $\mathbb{R}_p^{j+1} = p$, and η_q^p is the nearest point projection from $\mathcal{S}_p^j$ onto the 1-circle $\mathcal{S}_p^j \cap q$. Therefore, we have:

$$\int_{G_{j+1}\mathbb{R}^{N+1}} \mathbf{K}_g(\eta_p(\gamma))\, d\mu_{j+1}(p) = \int_{G_{j+1}\mathbb{R}^{N+1}} \Big(\int_{G_2\mathbb{R}_p^{j+1}} \mathbf{K}_g(\eta_q^p(\eta_p(\gamma)))\, d\mu_2^p(q) \Big)\, d\mu_{j+1}(p) =: I\,.$$

Moreover, the iterated integral I on the right-hand side is equal to

$$I = \int_{G_2\mathbb{R}^{N+1}} \mathbf{K}_g(\eta_r(\gamma))\, d\mu_2(r)$$

and hence, by applying again the formula from [1], we get $I = \mathbf{K}_g(\gamma)$, as required. □

As a consequence, since $\mathrm{TC}(\eta_p(\gamma)) = \mathcal{L}(\eta_p(\gamma)) + \mathbf{k}_{\mathcal{S}_p^j}(\eta_p(\gamma))$, for a polygonal curve γ in $\mathcal{S}^N$ one also gets:

$$\mathrm{TC}(\gamma) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\eta_p(\gamma))\, d\mu_{j+1}(p)\,, \qquad j = 1, \ldots, N-1\,.$$

3 The explicit formula

In this section, we extend the previous representation formula to the case $N > 2$, by proving the following

Theorem 3.1 *Let $N \geq 3$ and let $\mathbf{c}$ be a rectifiable curve in $\mathcal{S}^N$ with finite geodesic total curvature, $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty$. Then*

$$\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \mathcal{F}(\mathbf{c})$$

where $\mathcal{F}(\mathbf{c})$ is the geodesic curvature energy functional given by (2.2).

The above formula is obtained by exploiting its validity in the case $N = 2$, and by means of the representation result (1.6) for the total Euclidean curvature, through the following new integral-geometric formulas (where we choose $j = 2$), the proof of which is postponed. In both the statements we assume that $\mathbf{c}$ is a rectifiable curve in $\mathcal{S}^N$ and that $2 \leq j \leq N-1$ are integers.

Proposition 3.2 *If $\mathrm{TC}(\mathbf{c}) < \infty$, then* $\mathrm{TC}(\mathbf{c}) = \displaystyle\int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}(\eta_p(\mathbf{c}))\, d\mu_{j+1}(p)\,.$

Proposition 3.3 *If $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) < \infty$, then* $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \displaystyle\int_{G_{j+1}\mathbb{R}^{N+1}} \mathrm{TC}_{\mathcal{S}_p^j}(\eta_p(\mathbf{c}))\, d\mu_{j+1}(p)\,.$

Remark 3.4 We recall that in the previous propositions, for rectifiable curves the boundedness of the Euclidean and geodesic total curvature are equivalent properties, see (2.1). Moreover, from the proof of Proposition 3.3, it turns out that if $\mathbf{c}$ is a rectifiable curve in $\mathcal{S}^N$ with finite geodesic total curvature, then $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c})$ is equal to the limit of $\mathbf{k}_{\mathcal{S}^N}(P_h)$ for any sequence $\{P_h\} \subset \mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$ satisfying $\mu_{\mathbf{c}}(P_h) \to 0$.

Proof of Theorem 3.1: We already know that $\mathbf{c}$ has finite Euclidean total curvature $\mathrm{TC}(\mathbf{c})$, see (2.1), whence $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^N)$, where $\mathbf{t} = \dot{\mathbf{c}}$. On account of (2.3) and of the representation formula (1.6) for the Euclidean total curvature, it then suffices to show that

$$\mathcal{L}(\mathbf{c}) + \mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \mathrm{TC}(\mathbf{c})\,. \tag{3.1}$$

By Propositions 2.4 and 3.3, with $j = 2$, we can write

$$\mathcal{L}(\mathbf{c}) + \mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \int_{G_3\mathbb{R}^{N+1}} \big[\mathcal{L}(\eta_p(\mathbf{c})) + \mathrm{TC}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c}))\big]\, d\mu_3(p)$$

where $\mathcal{L}(\eta_p(\mathbf{c})) + \mathrm{TC}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c})) < \infty$ for μ_3-a.e. $p \in G_3\mathbb{R}^{N+1}$. For any such good projection, on account of the result from [8] for curves into the 2-sphere we infer that

$$\mathrm{TC}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c})) = \mathcal{F}(\eta_p(\mathbf{c}))\,, \quad \mathcal{F}(\eta_p(\mathbf{c})) = \mathrm{TC}(\eta_p(\mathbf{c})) - \mathcal{L}(\eta_p(\mathbf{c}))\,.$$

Therefore, we can write

$$\mathcal{L}(\mathbf{c}) + \mathrm{TC}_{\mathcal{S}^N}(\mathbf{c}) = \int_{G_3\mathbb{R}^{N+1}} \mathrm{TC}(\eta_p(\mathbf{c}))\, d\mu_3(p)$$

and hence equality (3.1) follows from Proposition 3.2. □

It remains to prove the integral-geometric formulas stated in Propositions 3.2 and 3.3.

Proof of Proposition 3.2: If P is a polygonal in $\mathbb{R}^{N+1}$ inscribed in $\mathbf{c}$, we know that $\mathbf{k}^*(P) = \mathcal{L}(\mathbf{t}_P)$, where $\mathbf{t}_P$ is the completion of the tantrix of P in $\mathbb{S}^N$. By Proposition 2.4, we thus have:

$$\mathbf{k}^*(P) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathcal{L}(\eta_p(\mathbf{t}_P))\, d\mu_{j+1}(p)\,.$$

We now observe that for μ_{j+1}-a.e. $p \in G_{j+1}\mathbb{R}^{N+1}$, the projection of the completed tantrix $\eta_p(\mathbf{t}_P)$ agrees with the completion of the tantrix of the polygonal $Q = Q(P,p)$ of p inscribed in the spherical curve $\eta_p(\mathbf{c})$, where the ordered vertexes are taken in correspondence to the ordered vertexes of P in $\mathbf{c}$. Therefore, we have $\mathcal{L}(\eta_p(\mathbf{t}_P)) = \mathbf{k}^*(Q(P,q))$ and we can thus write

$$\mathbf{k}^*(P) = \int_{G_{j+1}\mathbb{R}^{N+1}} \mathbf{k}^*(Q(P,p))\, d\mu_{j+1}(p)\,.$$

Now, taking a sequence $\{P_h\}$ with $\mathrm{mesh}(P_h) \to 0$, we have $\mathbf{k}^*(P_h) \nearrow \mathrm{TC}(\mathbf{c})$ and for a.e. p as above we correspondingly have $\mathrm{mesh}(Q(P_h,p)) \to 0$, whence $\mathbf{k}^*(Q(P_h,p)) \nearrow \mathrm{TC}(\eta_p(\mathbf{c}))$, as $h \to \infty$. The assertion then follows from the monotone convergence theorem. □

Proof of Proposition 3.3: We first deal with the case $j = 2$. If $P \in \mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$, by Proposition 2.5 we have

$$\mathbf{k}_{\mathcal{S}^N}(P) = \int_{G_3\mathbb{R}^{N+1}} \mathbf{k}_{\mathcal{S}^2_p}(\eta_p(P))\, d\mu_3(p)\,. \tag{3.2}$$

Moreover, for μ_3-a.e. $p \in G_3\mathbb{R}^{N+1}$ we have $\eta_p(P) \in \mathcal{P}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c}))$.

Denoting again by $Q = Q(P,p)$ the Euclidean polygonal in p inscribed in the spherical curve $\eta_p(\mathbf{c})$, where the ordered vertexes are taken in correspondence to the ordered vertexes of P in $\mathbf{c}$, the geodesic rotation of $\eta_p(P)$ is lower than the rotation of $Q(P,p)$, which is (by definition) lower than the Euclidean total curvature of the spherical curve $\eta_p(\mathbf{c})$.

We now show that for μ_3-a.e. $p \in G_3\mathbb{R}^{N+1}$ the Euclidean total curvature of $\eta_p(\mathbf{c})$ is lower than the Euclidean total curvature of the curve $\pi_p(\mathbf{c})$ where, we recall, π_p is the orthogonal projection onto p. In fact, let $\widetilde{P}$ denote a polygonal in $\mathbb{R}^{N+1}$ inscribed in $\mathbf{c}$ and generated by the consecutive vertexes $\mathbf{c}(s_i)$, where $0 = s_0 < s_1 < \cdots < s_n = L$, and assume that both $\mathbf{c}$ and $\widetilde{P}$ do not intersect the $(N-3)$-sphere ${\mathcal{S}^2_p}^{\perp}$ given by the polar to $\mathcal{S}^2_p$ in $\mathcal{S}^N$. Notice that since $\mathbf{c}$ is a rectifiable curve, for any choice of the vertexes $\mathbf{c}(s_i)$ this is a condition verified for μ_3-a.e. $p \in G_3\mathbb{R}^{N+1}$, compare [1]. Denoting by P_1, P_2 the polygonals in p inscribed in $\eta_p(\mathbf{c})$ and $\pi_p(\mathbf{c})$ and generated by the consecutive vertexes $\eta_p(\mathbf{c}(s_i))$ and $\pi_p(\mathbf{c}(s_i))$, respectively, since $\eta_p(\mathbf{c}(s_i)) = \pi_p(\mathbf{c}(s_i))/|\pi_p(\mathbf{c}(s_i))|$ for each i, see (2.4), it turns out that $\mathbf{k}^*(P_1) \le \mathbf{k}^*(P_2)$.

We have definitely obtained for μ_3-a.e. $p \in G_3\mathbb{R}^{N+1}$:

$$\mathbf{k}_{\mathcal{S}^2_p}(\eta_p(P)) \le \mathbf{k}^*(Q(P,p)) \le \mathrm{TC}(\eta_p(\mathbf{c})) \le \mathrm{TC}(\pi_p(\mathbf{c})) =: g(p) \tag{3.3}$$

whereas by the integral-geometric formula (1.1)

$$\int_{G_3\mathbb{R}^{N+1}} g(p)\, d\mu_3(p) = \int_{G_3\mathbb{R}^{N+1}} \mathrm{TC}(\pi_p(\mathbf{c}))\, d\mu_3(p) = \mathrm{TC}(\mathbf{c}) < \infty\,,$$

whence $g(p)$ is a non-negative summable function in $L^1(G_3\mathbb{R}^{N+1}, \mu_3)$.

Choose now a sequence $\{P_h\} \subset \mathcal{P}_{\mathcal{S}^N}(\mathbf{c})$ with $\mu_{\mathbf{c}}(P_h) \to 0$. For a.e. p as above we have $\{\eta_p(P_h)\} \subset \mathcal{P}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c}))$, with $\mu_{\eta_p(\mathbf{c})}(\eta_p(P_h)) \to 0$. On account of Propositions 2.4 and 3.2, we have $\mathcal{L}(\eta_p(\mathbf{c})) < \infty$ and $\mathrm{TC}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c})) \le \mathrm{TC}(\eta_p(\mathbf{c})) < \infty$. Therefore, by the cited result from [8], we know that $\mathbf{k}_{\mathcal{S}^2_p}(\eta_p(P_h)) \to \mathrm{TC}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c}))$ as $h \to \infty$ for a.e. p, whereas $\mathbf{k}_{\mathcal{S}^2_p}(\eta_p(P_h)) \le g(p)$ for each h and a.e. p, see (3.3). As a consequence, by the dominated convergence theorem we get from (3.2)

$$\lim_{h\to\infty} \mathbf{k}_{\mathcal{S}^N}(P_h) = \int_{G_3\mathbb{R}^{N+1}} \lim_{h\to\infty} \mathbf{k}_{\mathcal{S}^2_p}(\eta_p(P_h))\, d\mu_3(p) = \int_{G_3\mathbb{R}^{N+1}} \mathrm{TC}_{\mathcal{S}^2_p}(\eta_p(\mathbf{c}))\, d\mu_3(p)\,.$$

Since the above limit holds true for any sequence $\{P_h\}$ as above, the assertion in the case $j = 2$ readily follows from the definition of geodesic total curvature $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c})$.

Finally, when $N \ge 4$, the case $j > 2$ of the integral-geometric formula for $\mathrm{TC}_{\mathcal{S}^N}(\mathbf{c})$ can be obtained from the case $j = 2$, by using the same argument that we followed in the proof of Proposition 2.5, this time taking $\gamma = \mathbf{c}$ and $\mathbf{K}_g(\widetilde{\gamma})$ equal to the geodesic total curvature of a curve $\widetilde{\gamma}$ in a unit sphere of generic dimension. We omit any further detail. □

References

[1] A. D. Alexandrov, Yu. G. Reshetnyak: *General theory of irregular curves.* Mathematics and its Applications. Soviet series. Kluwer Academic Publishers, Dordrecht, 1989.

[2] L. Ambrosio, N. Fusco, and D. Pallara: *Functions of bounded variaton and free discontinuity problems.* Oxford University Press, Oxford, 2000.

[3] M. Castrillón Lopez, V. Fernandéz Mateos, and J. Muñoz Masqué: Total curvatures of curves in Riemannian manifolds. *Diff. Geom. Appl.* **28** (2010), 140–147.

[4] I. Fáry: Sur la courbure totale d'une courbe gauche faisant un nœud. *Bull. Soc. Math. France* **77** (1949), 128–138.

[5] C. Maneesawarng, Y. Lenbury: Total curvature and length estimate for curves in CAT(K) spaces. *Diff. Geom. Appl.* **19** (2003), 211–222.

[6] J. W. Milnor: On the total curvature of knots. *Ann. of Math.* **52** (1950), 248–257.

[7] J. W. Milnor: On total curvatures of closed space curves. *Math. Scand.* **1** (1953), 289–296.

[8] D. Mucci and A. Saracco: The total intrinsic curvature of curves in Riemannian surfaces. *Rend. Circ. Mat. Palermo, II. Ser.* **70** (2021), 521–557. Erratum to: *Rend. Circ. Mat. Palermo, II. Ser.* (2021).

[9] D. Mucci and A. Saracco: Weak curvatures of irregular curves in high dimension Euclidean spaces. Preprint: https://arxiv.org/abs/2008.02336 (2020). To appear on *Annals of Global Analysis and Geometry.*

[10] J. M. Sullivan: Curves of finite total curvature. In: *Discrete Differential Geometry* (Bobenko, Schröder, Sullivan, and Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkäuser, 2008.