Gradient flows in Wasserstein spaces and applications to crowd movement

Séminaire X-EDP, October 19th, 2010

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November 17, 2010

Abstract

Starting from a motivation in the modeling of crowd movement, the paper presents the topics of gradient flows, first in $\mathbb{R}^n$, then in metric spaces, and finally in the space of probability measures endowed with the Wasserstein distance (induced by the quadratic transport cost). Differently from the usual theory by Jordan-Kinderlehrer-Otto and Ambrosio-Gigli-Savaré, we propose an approach where the optimality conditions for the minimizers of the optimization problems that one solves at every time step are obtained by looking at perturbation of the form $\rho_\varepsilon = (1-\varepsilon)\rho + \varepsilon\tilde{\rho}$ instead of $\rho_\varepsilon = (id + \varepsilon\xi)_\#\rho$. The ideas to make this approach rigorous are presented in the case of a Fokker-Planck equation, possibly with an interaction term, and then the paper is concluded by a section, where this method is applied to the original problem of crowd motion (referring to a recent paper in collaboration with B. Maury and A. Roudneff-Chupin for the details).

1 Introduction

The goal of this paper, as well of the talk I gave at Séminaire X-EDP is to present some ideas from the theory of Gradient Flows in the space of probability measures, motivated by applications in crowd movement and, more generally, in the motion of fluids under a density constraint (imposing that particles cannot be "too dense").

In many vehicular traffic models with congestion a discrete network is considered and the speed of every vehicle is supposed to decrease with the density of other vehicles nearby, possibly tending to 0 when it approaches a threshold density. For instance one can take $v = (1-\rho)u$, where u is the spontaneous velocity in the absence of other vehicles, v is the true velocity which will be actually realized, and $\rho \leq 1$ is the density. Pedestrian motion models (see [4, 6, 7, 9, 10, 11, 20], just to make a short list of some of the recent models) usually replace the network with a 2D framework, but in many cases the dependence of the velocity on the density stays similar. Both the 1D and 2D cases are studied either under a continuous approach (i.e. looking at the evolutions of densities and velocities as functions of

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$(t,x) \in [0,T] \times \Omega$) or under a discrete one, looking separately at every particle and at its interactions with the others.

A more drastic viewpoint is the one presented by Maury and Venel in [16, 17], where the idea is that particles can move as they want as far as they are not too dense, and if a density constraint is saturated, then their velocity field will change from the spontaneous u to another, less concentrating and typically slower, v. Maury and Venel are concerned with the discrete case, so that the density constraint is interpreted as a non-superposition constraint: particles cannot overlap, but as soon as they are not in touch their motion is unconstrained. The problem that arises is a strange ODE which is studied by the authors both from a theoretical and numerical point of view, putting it in the framework of differential inclusions, maximal monotone operator and gradient-flow in Ω^N (N being the number of particles).

This discrete model will be briefly sketched in the next section, together with its continuous counterpart, which is the main motivation of this paper. The reader may refer to the paper [15] in preparation, which will deal with various aspects of the problem, and in particular with the comparison between the microscopic and the macroscopic versions. What arises in the continuous (macroscopic) model is a PDE of the form

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \left(\rho_t v_t\right) = 0$$

where v_t is defined as the projection of a given field u_t on the set of admissible velocities "infinitesimally preserving" the density constraint $\rho \leq 1$. This set depends on ρ_t (more precisely, on the region where $\rho_t = 1$) and, as a consequence, the velocity field v has globally two regularity problems: first, in general, for fixed ρ_t and u_t, the field v_t is not Lipschitz continuous, since it is obtained through an L^2 projection; second, it does not depend in a smooth way on ρ, and is very sensitive to small perturbations of ρ. This implies that it is very difficult to insert this PDE into a wide classical theory, and leads to the need for some other tools.

It happens, as it has been intensively studied in the last years, that the concept of gradient flow in metric spaces, applied to the case of probability measures endowed with the Wasserstein distance (i.e. the distance induced by the quadratic transport cost in Monge-Kantorovitch theory, see [13, 21, 22]), is very useful for applications to certain nonlinear evolution PDEs. In Section 3, we will give the step-by-step variational interpretation of gradient flows, with the way to adapt it to metric spaces (following De Giorgi and Ambrosio, [8, 1]). After that, we will introduce the tools we need in optimal transport, together with the way of getting some PDEs. The first ideas of this theory date back to [12], and the subject has been systematically studied in [2, 3], but here in this paper there is an important different idea that we will see: actually, when computing the optimality conditions for the variational problems that one gets when discretizing in time, we use *vertical perturbations*

$$\rho_\varepsilon = (1-\varepsilon)\rho + \varepsilon\tilde{\rho} \quad \text{instead of} \quad \rho_\varepsilon = (id + \varepsilon\xi)_{\#}\rho,$$

the latter being also called *horizontal perturbations* or *intrinsic*, because they are the most natural when one looks at the Wasserstein space as a sort of Riemannian manyfold.

For some reasons, it happens that these alternative perturbations have not been that used up to now, especially in evolutionary problems (they actually appear in some papers more related to convex analysis and statical optimization, and an example can be seen in [5]). Yet, they allow very often to get some powerful results for variational or gradient flow problems, and it could be the case that some results are easier to obtain in this way rather than through the usual theory of [2], even if I do not claim that this alternative viewpoint necessarily allows for wider or stronger or better results (but I think it deserves being explored). This is why in Section 4 we will see the procedure giving existence of the solutions to some evolutionary PDEs via this ideas. For the sake of clarity, Section 4 will stick to a sloppy presentation, with ideas from the time-discretized problem, followed by a rigorous proof in the easiest cases. These case include linear terms, like inthe Heat or Focker-Planck equations, but also non-local terms, corresponding to interaction energies, provided some compactness properties are satisfied. I'm also currently studying how to extend these results to other nonlinear cases, where compactness is not straightforward (it is the case for the porous media equation), but this requires a more refined analysis, and will hopefully appear in a forthcoming paper. Let me stress the fact that the only goal of this approach is to prove existence results (under no geodesic convexity assumptions, which are crucial in the theory of [2]) and when I say existence I mean "existence of a weak solution of the associated PDE".

Anyway, to come back to the original motivation, in Section 5 we will see a functional whose gradient flow is supposed to give our crowd motion PDE, and then, via vertical perturbations, we show that this is exactly the case. The reader may refer to [14] for all the estimates, which are non-trivial, and we only give the main ideas. The choices of the structure of the functional will be discussed in relation with the modelization goals of this study.

2 Microscopic and Macroscopic models for crowd motion

This section is devoted to the modelization of a density-constrained motion of a particle population. Let us suppose that each particle, if alone, would follow its own velocity u (which could a priori depend on time, position, on the particle itself...). Yet, these particles are modeled by rigid disks that cannot overlap, hence, it is not clear whether the actual velocity can be u, in particular if u tends to concentrate the masses. Hence, we will call v the actual velocity that each particle will have, and the main assumption of the model is that $v = P_{adm(q)}(u)$, where q is the particle configuration, $adm(q)$ is the set of velocities that do not induce overlapping starting from the configuration q, and $P_{adm(q)}$ the projection on this set.

The simplest example is the one where every particle is a disk with the same radius R and center located at q_i. In this case we define the admissible set of configurations K through

$$K := \{q = (q_i)_i \in \Omega^N \ : \ |q_i - q_j| \geq 2R \text{ for all } i \neq j\}.$$

In this way the set of admissible velocities is easily seen to be

$$adm(q) = \{v = (v_i)_i \ : \ (v_i - v_j) \cdot (q_i - q_j) \geq 0 \ \forall (i,j) \ : \ |q_i - q_j| = 2R\}.$$

The evolution equation which has to be solved for following the motion of q is then

$$q'(t) = P_{adm(q(t))} u(t) \tag{1}$$

(with $q(0)$ given).

The typical case is the one where the spontaneous velocity of the $i-$th particle only depends on its position, and in particular we are interested in a gradient structure: $u_i(q) = -\nabla D(q_i)$. In the modelization, the function D is often supposed to be a distance function to a target set, like $D(x) = d(x, \Gamma)$ and Γ is a subset of $\partial\Omega$ representing the exit door that the particles located in the domain Ω aim to reach (and if Ω is not convex it is better to consider as d the geodesic distance).

The equation (1) itself is not easy from a mathematical point of view (proving existence of a solution, uniqueness, finding an algorithm to approximate it, possibly by time discretization...). The main problem is the fact that $q \mapsto P_{adm(q)}u$ is not regular (Lipschitz). On the other hand, in this specific case where $u = -\nabla D$, it may be written in the following way

$$-q'(t) \in \partial(F + I_K)(q(t)),$$

where ∂ denotes the Frechet subgradient and $F(q) := \sum_i D(q_i)$. Notice that, independently of the possible convexity of F, the sum $F + I_K$ (I_K being the indicator function of the set K, equal to 0 on K and to $+\infty$ on its complement) is not convex since the set K itself is not convex. Yet, the set K is not far from being convex, since it is prox-regular (i.e. it admits a neighborhood where the projection on K is well-defined and Lipschitz), and hence it allows for some proofs similar to the convex case. This is the point which has been exploited by Maury and Venel ([16, 17]) to make a complete theory of this equation.

We are now interested in the simplest continuous counterpart of this microscopic model. In this framework, which does not pretend to be any kind of homogenized limit of the discrete particle approaches, but only an easy re-formulation in a density setting, the main ingredients are the following.

- The particles population will be described by a probability density $\rho \in \mathcal{P}(\Omega)$;
- the constraint of non-overlapping will be replaced by a density constraint using the set $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$ (where ρ denotes, by abuse of notation at the same time the probability and its density, since anyway our set K imposes $\rho << \mathcal{L}^d$ and that the density is bounded a.e. by 1);
- for every time t, we consider $u_t : \Omega \to \mathbb{R}^d$ a vector field, possibly depending on time or on ρ;
- the set of admissible velocities will be described by the sign of the divergence on the saturated region $\{\rho = 1\}$: $adm(\rho) = \big\{v : \Omega \to \mathbb{R}^d \,:\, \nabla \cdot v \geq 0 \text{ on } \{\rho = 1\}\big\}$;
- we will consider a projection P, which will be either the projection in $L^2(\mathcal{L}^d)$ or in $L^2(\rho)$ (this will turn out to be the same, since the only relevant zone is $\{\rho = 1\}$);
- we will solve the equation

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \big(\rho_t \left(P_{adm(\rho_t)}u_t\right)\big) = 0. \tag{2}$$

Formula (2) is motivated by the fact that the equation satisfied by the evolution of a density ρ when each particle follows the velocity field v is exactly the *continuity equation* $\frac{\partial}{\partial t}\rho_t + \nabla \cdot \big(\rho_t v_t\big) = 0$ (with $v \cdot n = 0$ on $\partial\Omega$, so that the density does not exit Ω). Here we only insert the fact that v is the projection of u.

The main difficulty is the fact that the vector field $v = P_{adm(\rho_t)}u_t$ is nor regular (since it is obtained as an L^2 projection, and may only be expected to be L^2 a priori), neither it depends regularly on ρ (it is very sensitive to small changes in the values of ρ: for instance passing from a density 1 to a density $1-\varepsilon$ completely changes the saturated zone, and hence the admissible set of velocities and the projection onto it).

Before entering the ideas which are necessary to overcome these difficulties, we need to make a little bit more precise the definitions above. Actually, instead of considering the divergence of vector fields which are only supposed to be L^2, it is more convenient to give a better description of $adm(\rho)$ by duality :

$$adm(\rho) = \left\{ v \in L^2(\rho) \ : \ \int v \cdot \nabla p \leq 0 \quad \forall p \in H^1(\Omega) \ : \ p \geq 0,\, p(1-\rho) = 0 \right\}.$$

In this way we characterize $v = P_{adm(\rho)}(u)$ through

$$u = v + \nabla p, \quad v \in adm(\rho), \quad \int v \cdot \nabla p = 0,$$
$$p \in press(\rho) := \{p \in H^1(\Omega),\, p \geq 0,\, p(1-\rho) = 0\},$$

where $press(\rho)$ is the space of functions p used as test functions in the dual definition of $adm(\rho)$, which play the role of a pressure affecting the movement. The two cones $\nabla press(\rho)$ and $adm(\rho)$ are orthogonal cones and this allows for an orthogonal decomposition $u_t = v_t + \nabla p_t$. This also gives the alternative expression of Equation (2), i.e.

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \big(\rho_t(u_t - \nabla p_t)\big) = 0. \tag{3}$$

3 Gradient flows in $\mathbb{R}^n$, in metric spaces and in W_2

A gradient-flow in $\mathbb{R}^n$ is nothing but an evolution equation (an ODE) of the form $x'(t) = -\nabla F(x(t))$ (i.e. the trajectories follow the steepest descent lines of a function F). If $F \in C^{1,1}$ this equation falls into the usual theorems for ODE, and in particular Cauchy-Lipschitz theorem. Yet the advantage of those ODE having a gradient structure is the fact that they may be analyzed under much weaker assumptions. For instance, a complete theory of existence and uniqueness results for $\lambda-$convex functions F (i.e. such that $F(x) - \frac{\lambda}{2}|x|^2$ is convex, which means a lower bound on the second derivatives) is available. However, in this paper we are mainly concerned with the existence of a solution (instead of uniqueness or other properties).

To prove such existence, a powerful tool is the time discretization used by Ambrosio and De Giorgi (see [8, 1]), to define the so-called *minimizing movements* in general framework. Actually, if one recursively solves

$$x^\tau_{k+1} \in \operatorname*{argmin}_x F(x) + \frac{1}{2\tau}|x - x^\tau(k)|^2,$$

where $\tau > 0$ is a fixed time step and x_0 is given, he finds a sequence of points $x^\tau(k)$ that may be interpreted as the value of a discrete trajectory at time $k\tau$.

If one looks at the optimality conditions on the optimal point x^τ_{k+1} he finds

$$\frac{x^\tau_{k+1} - x^\tau(k)}{\tau} + \nabla F(x^\tau_{k+1}) = 0,$$

which corresponds to an implicit Euler scheme to solve $x'(t) = -\nabla F(x(t))$. A solution of this ODE will then be found as a limit $\tau \to 0$. It is interesting to notice that, if F is convex but non-smooth, this method provides a solution to $x'(t) \in -\partial F(x(t))$, and is able to provide existence for less regular functions F.

Besides the less-demanding regularity requirements, another advantage of this formulation is the fact that it may easily be adapted to a general metric space. Indeed, one can simply replace $|x - x^\tau(k)|$ with $d(x, x^\tau(k))$ and pass to this more general framework. In particular, it can also be used to study evolution problems for a density ρ when we use the space $\mathcal{P}(\Omega)$ endowed with a suitable distance.

The distance that we consider is the so-called Wasserstein distance, induced by the Monge-Kantorovitch optimal transport problem. If two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ are given on a domain $\Omega \subset \mathbb{R}^d$ (that we take compact for simplicity), such a problem reads

$$\min \left\{ \int_{\Omega\times\Omega} |x-y|^2 \, d\gamma \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

where $\Pi(\mu, \nu)$ is the set of the so-called *transport plans*, i.e.

$$\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}(\Omega \times \Omega) : \, (p^x)_\# \gamma = \mu, \, (p^y)_\# \gamma = \nu, \},$$

p^x and p^y being the two projections of $\Omega \times \Omega$ onto Ω. It is an extension of the Monge problem, which is

$$\inf \left\{ \int |x - T(x)|^2 d\mu \ : \ T : \Omega \to \Omega, \ T_\# \mu = \nu \right\}$$

(in the sense that to any transport map T we can associate a transport plan γ_T by taking $\gamma_T = (id \times T)_\# \mu$, that the cost of T in the Monge problem is the same as that of γ_T in Kantorovitch's one, and that, under some additional assumption on μ, the minimum over the transport plans is realized by a plan of the form γ_T).

However, independently of the fact that the minimum is realized by a transport map or not, we define the distance $W_2(\mu, \nu)$ between two measures μ and ν as the square root of the minimal value. It can be proven that this is a distance on $\mathcal{P}(\Omega)$. The index 2 refers to the quadratic cost, and other distances W_p are possible as well with other exponents $1 \le p \le \infty$.

The main fact that we need to know for the sequel about optimal transport is the following. For any pair (μ, ν) there exists a function $\phi : \Omega \to \mathbb{R}$, called *Kantorovitch potential*, which is Lipschitz continuous (and semi-concave), with the following properties :

- if ϕ is differentiable $\mu-$a.e. (which is the case, for instance, if $\mu << \mathcal{L}^d$), then there is a unique optimal transport plan, which is of the form γ_T, and the optimal map T is given by $T(x) = x - \nabla\phi(x)$;
- the function ϕ also plays the role of the derivative of $\frac{1}{2} W_2^2(\cdot, \nu)$: we have

$$\frac{d}{d\varepsilon} \frac{1}{2} W_2^2(\mu + \varepsilon\chi, \nu)_{|\varepsilon=0} = \int \phi d\chi.$$

Pay attention to the notation for these functional derivatives: given a functional G : $\mathcal{P}(\Omega) \to \mathbb{R}$ we call $\frac{\delta G}{\delta \rho}(\rho)$, if it exists, the only function such that $\frac{d}{d\varepsilon} G(\rho + \varepsilon\chi)_{|\varepsilon=0} = \int \frac{\delta G}{\delta\rho}(\rho) d\chi$ for every perturbation χ such that, at least for $\varepsilon \in [0, \varepsilon_0]$, the measure $\rho + \varepsilon\chi$ belongs to $\mathcal{P}(\Omega)$. Hence, in the case of the Kantorovitch potential, we are saying $\frac{\delta\left(\frac{1}{2}W_2^2(\cdot,\nu)\right)}{\delta\rho} = \phi$

With this distance in mind, we consider now a functional F over $\mathcal{P}(\Omega)$ endowed with the W_2 distance. We consider the time-discretized problem, i.e. we look for

$$\rho^\tau_{k+1} \in \operatorname*{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau(k))}{2\tau}.$$

Which are the discrete optimality conditions? roughly speaking we should have

$$\frac{\delta F}{\delta \rho}(\rho^\tau_{k+1}) + \frac{\phi}{\tau} = const$$

(where the reasons for having a constant instead of 0 is the fact that, in the space of probability measures, only zero-mean densities are considered as admissible perturbations).

More precise statements and proofs of this optimality conditions will be presented in the next section. Here we look at the consequences we can get. Actually, if we combine the fact that the above sum is constant, and that we have $T(x) = x - \nabla\phi(x)$ for the optimal T, we get

$$\frac{T(x) - x}{\tau} = -\frac{\nabla\phi(x)}{\tau} = \nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)(x). \tag{4}$$

We will denote by $-v$ the ratio $\frac{T(x)-x}{\tau}$. Why? because, as a ratio between a displacement and a time step, it has the meaning of a velocity, but since it is the displacement associated to the transport from ρ^τ_{k+1} to ρ^τ_k, it is better to view it rather as a backward velocity (which justifies the minus sign).

Since here we have $v = -\nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)$, this suggests (and we will analyze it in the next section) that at the limit $\tau \to 0$ we will find a solution of

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \big(\rho \nabla\big(\frac{\delta F}{\delta \rho}(\rho)\big)\big) = 0.$$

Before entering some proof details, we want to present some examples of this kind of equations. The three main classes of examples are the functionals considered by McCann in [18], where he proves some convexity properties of such functionals. Consider

$$F(\rho) = \int f(\rho(x))dx, \quad G(\rho) = \int V(x)d\rho, \quad H(\rho) = \int\!\!\int W(x-y)\rho(dx)\rho(dy),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex superlinear function (and the functional F is set to $+\infty$ if ρ is not absolutely continuous w.r.t. the Lebesgue measure) and $V : \Omega \to \mathbb{R}$ and $W : \mathbb{R}^d \to \mathbb{R}$ are regular enough (and W is taken symmetric, i.e. $W(z) = W(-z)$, for simplicity). In this case it is quite easy to realize that we have

$$\frac{\delta F}{\delta \rho}(\rho) = f'(\rho), \quad \frac{\delta G}{\delta \rho}(\rho) = V, \quad \frac{\delta H}{\delta \rho}(\rho) = 2W * \rho.$$

An interesting example is the case $f(t) = t \ln t$. In such a case we have $f'(t) = \ln t + 1$ and $\nabla(f'(\rho)) = \frac{\nabla \rho}{\rho}$: this means that the gradient flow equation associated to the functiona F would be the *Heat Equation* $\frac{\partial \rho}{\partial t} - \Delta\rho = 0$, and that for $F + G$ we would have the *Fokker-Planck Equation* $\frac{\partial \rho}{\partial t} - \Delta\rho - \nabla \cdot (\rho \nabla V) = 0$.

The case of the interaction functional H gives on the contrary a non-local (and nonlinear) equation $\frac{\partial\rho}{\partial t} - \nabla\cdot(\rho(\nabla W * \rho)) = 0$ (where the non-local term may also be combined with the others).

It is a general fact that this formulation/interpretation in terms of gradient flows, through its time-discretization, has several advantages: it allows for easy existence results, and for some uniqueness proofs (more difficult, under some convexity assumptions on the functionals) in a very general framework. Moreover, it also directly provides a time-discretization algorithm. In this short paper we will see how to approach the easiest equations to justify : the Fokker-Planck one (which is linear), with possibly a non-local term (because of compactness properties of the convolution operator). We will only deal with existence issues, independently of convexity assumptions. Wider generalizations to more delicate cases, including possibly the Porous-Media equation (i.e. the case of the functional F with $f(t) = t^m$, see for instance [19]) will be left to a forthcoming study.

4 Getting gradient flows PDEs via vertical perturbations

This section presents some proofs or references to make what presented in Section 3 rigorous.

There are three main points to analyze: the optimality conditions in the time-discretized problems; the interpolation of the discrete trajectories $\rho^\tau(k)$ and how to get a limit curve ρ that solves the continuity equation together with its velocity v; the fact that v has still the form $-\frac{\delta F}{\delta\rho}(\rho)$, even at the limit $\tau \to 0$.

Our proof will be complete only for the case

$$F(\rho) = \int_\Omega \rho\ln\rho + \int_\Omega V\,d\rho + \int_\Omega\int_\Omega W(x-y)d\rho(x)d\rho(y),$$

where V and W are Lipschitz functions on the compact domain Ω, and a starting measure $\rho_0 \in \mathcal{P}(\Omega)$ such that $F(\rho_0) < +\infty$ is fixed.

Optimality conditions at each time step. We consider the minimization of $F(\rho) + W_2^2(\rho, \rho^\tau(k))/2\tau$ among probability measures $\rho \in \mathcal{P}(\Omega)$. Let us suppose that this minimization problem admits a solution, which is the case for the choice of the functional F above. Actually, the three parts of the functional are weakly-l.s.c., and this is the case for W_2 as well. Hence, Ω being compact, this is sufficient for the existence (notice anyway that more refined considerations could handle the case of unbounded domains as well).

Now, take an optimal measure $\bar\rho$ and compute variations with respect to perturbations of the form $\rho_\varepsilon := (1-\varepsilon)\bar\rho + \varepsilon\tilde\rho$, where $\tilde\rho$ is any other probability measure. This means choosing a perturbation $\chi = \tilde\rho - \bar\rho$, which guarantees that, for $\varepsilon > 0$, the measure ρ_ε is actually a probability over Ω.

We now compute the first variation and, due to optimality, we have

$$0 \le \frac{d}{d\varepsilon}\left(F(\bar\rho + \varepsilon\chi) + \frac{1}{\tau}\frac{W_2^2(\bar\rho + \varepsilon\chi, \rho^\tau(k))}{2}\right)_{|\varepsilon=0} = \int\left(\frac{\delta F}{\delta\rho}(\rho) + \frac{\phi}{\tau}\right)d\chi.$$

If we set for a while $\psi = \frac{\delta F}{\delta\rho}(\rho) + \frac{\phi}{\tau}$ we would have

$$\int\psi\,d\chi \ge 0 \quad \text{i.e.} \int\psi\,d\tilde\rho \ge \int\psi\,d\bar\rho \quad \text{for all } \tilde\rho \in \mathcal{P}(\Omega).$$

This means that $\bar\rho$ also minimizes $\rho \mapsto \int \psi\, d\rho$, and it is clear that the minimizers of such a quantity must be concentrated on $\operatorname{argmin} \psi$. In particular this implies that ψ has a constant value $\rho-$a.e.

For the case we are considering, where the optimal $\bar\rho$ must be absolutely continuous because of the entropy part, one gets, for the density $\bar\rho(x)$

$$\begin{aligned} \ln(\bar\rho(x)) + V(x) + 2(W * \bar\rho)(x) + \frac{\phi(x)}{\tau} &= C \text{ a.e. on } \{\bar\rho > 0\} \\ \ln(\bar\rho(x)) + V(x) + 2(W * \bar\rho)(x) + \frac{\phi(x)}{\tau} &\geq C \text{ everywhere.} \end{aligned}$$

This in particular implies that $\bar\rho$ is actually positive a.e. and that it holds

$$\bar\rho(x) = \exp\left(C - V(x) - 2(W * \bar\rho)(x)) - \frac{\phi(x)}{\tau} \right),$$

which provides Lipschitz regularity for $\bar\rho$. Then, one differentiates and gets the equality (4).

Interpolation between time steps. With this time-discretized method, we have obtained, for each $\tau > 0$, a sequence $(\rho^\tau(k))_k$. We can use it to build at least two interesting curves in the space of measures:

- first we can define some piecewise constant curves, i.e. $\rho^\tau_t := \rho^\tau(k+1)$ for $t \in]k\tau, (k+1)\tau]$; associated to this curve we also define the velocities $v^\tau_t = v^\tau(k+1)$ for $t \in]k\tau, (k+1)\tau]$, where $v^\tau(k)$ is obtained from (4), defining $v^\tau(k) = (id - T)/\tau$, taking as T the optimal transport from ρ^τ_{k+1} to $\rho^\tau(k)$; we also define the momentum variable $E^\tau = \rho^\tau v^\tau$;
- then, we can also consider the densities $\tilde\rho^\tau_t$ that interpolate the discrete values $(\rho^\tau(k))_k$ along geodesics:

$$\tilde\rho^\tau_t = \left(\frac{k\tau - t}{\tau} v^\tau(k) + id \right)_{\#} \rho^\tau(k), \quad \text{for } t \in](k-1)\tau, k\tau[; \tag{5}$$

  the velocities $\tilde v^\tau_t$ are defined so that $(\tilde\rho^\tau, \tilde v^\tau)$ satisfy the continuity equation, taking

$$\tilde v^\tau_t = v^\tau_t \circ \left(\frac{k\tau - t}{\tau} v^\tau(k) + id \right)^{-1};$$

  as before, we define: $\tilde E_\tau = \tilde\rho^\tau \tilde v^\tau$.

After these definitions we consider some a priori bounds on the curves and the velocities that we defined. We start from some estimates which are standard in the framework of Minimizing Movements. The sequence $(\rho^\tau(k))_k$ satisfies an estimate on its variation which gives a Hölder and H^1 behavior. From the minimality of $\rho^\tau(k)$, compared to $\rho^\tau(k-1)$, one gets

$$W_2^2(\rho^\tau(k), \rho^\tau(k-1)) \leq 2\tau\big(F(\rho^\tau(k)) - F(\rho^\tau(k-1))\big).$$

This also says that $F(\rho^\tau(k))$ is monotone decreasing, which implies a uniform bound on $F(\rho^\tau_t)$ for every t and τ. Morevoer, we also get $W_2^2(\rho^\tau(k), \rho^\tau(k-1)) \leq C\tau$ (discrete Hölder behavior), as well as, if we sum up over k and use $F(\rho_0) < +\infty$,

$$\sum_k \tau \left(\frac{W_2(\rho^\tau(k), \rho^\tau(k-1))}{\tau} \right)^2 \leq C, \tag{6}$$

which is the discrete version of an H^1 estimate. As for $\tilde{\rho}^\tau_t$, it is an absolutely continuous curve in the Wasserstein space and its velocity on the time interval $[(k-1)\tau, k\tau]$ is given by the ratio $W_2(\rho^\tau(k-1), \rho^\tau(k))/\tau$. Hence, the L^2 norm of its velocity on $[0,T]$ is given by

$$\int_0^T |\tilde{\rho}'^\tau|^2_{W_2}(t)dt = \sum_k \frac{W_2^2(\rho^\tau(k), \rho^\tau(k-1))}{\tau}, \tag{7}$$

and, thanks to (6), it admits a uniform bound independent of τ. Here we use the notation $|\sigma'|(t)$ for the metric derivative of a curve σ and $|\sigma'|_{W_2}(t)$ means that this metric derivative is computed according to the distance W_2. In our case, thanks to well-known results on the continuity equation and the Wasserstein metric, this metric derivative is also equal to $||\tilde{v}^\tau_t||_{L^2(\tilde{\rho}^\tau_t)}$. This gives compactness of the curves $\tilde{\rho}^\tau$, as well as an Hölder estimate on their variations (since $H^1 \subset C^{0,1/2}$). The characterization of the velocities v^τ and $\tilde{v}^\tau$ allow to deduce bounds on these vector fields from the bounds on $W_2(\rho^\tau(k-1), \rho^\tau(k))/\tau$.

Considering all these facts, one can collect the following results.

- The norm of v^τ in $L^2((0,T), L^2_{\rho^\tau}(\Omega))$ is τ-uniformly bounded.
- In particular, the bound is valid in L^1 as well, which implies that E^τ is bounded in the space of measures over $[0,T] \times \Omega$.
- The very same estimates are true for $\tilde{v}^\tau$ and $\tilde{E}^\tau$.
- The curves $\tilde{\rho}^\tau$ are bounded in $H^1([0,T], W_2(\Omega))$ and hence compact in $C^0([0,T], W_2(\Omega))$.
- Up to a subsequence, one has $\tilde{\rho}^\tau \to \rho$, as $\tau \to 0$, uniformly according to the W_2 distance.
- From the estimate $W_2(\rho^\tau_t, \tilde{\rho}^\tau_t) \leq C\tau^{1/2}$ one gets that ρ^τ converges to the same limit ρ in the same sense.
- If we denote by E a weak limit of $\tilde{E}^\tau$, since $(\tilde{\rho}^\tau, \tilde{E}^\tau)$ solves the continuity equation, by linearity, passing to the weak limit, also (ρ, E) solves the same equation.
- It is possible to prove (see [14], Section 3.2, Step 1) that the weak limits of $\tilde{E}^\tau$ and E^τ are the same.
- From the bounds in L^2 one gets that also at the limit the measure E is absolutely continuous w.r.t. ρ and has an L^2 density, so that we have for a.e. time t a measure E_t of the form $\rho_t v_t$.
- It is only left to prove that one has $v_t = -\frac{\delta F}{\delta \rho}(\rho_t)$ for a.e. t. This is done (in the following step), by noticing that we have, for every t and τ, the equality $E^\tau_t = -\rho^\tau_t \frac{\delta F}{\delta \rho}(\rho^\tau_t)$, and letting it pass to the limit as $\tau \to 0$. It is crucial in this step to consider the limit of (ρ^τ, E^τ) instead of $(\tilde{\rho}^\tau, \tilde{E}^\tau)$.

Passing to the limit the relation between ρ and v. The fact that (ρ^τ, E^τ) weakly converges to (ρ, E) let easily any linear condition pass to the limit. For non-linear terms, this is more difficult and requires some compactness.

The example that we decided to consider is anyway quite easy, since one can decompose E^τ into two parts: $E^\tau = L^\tau + I^\tau$, where

$$L^\tau := \rho^\tau \nabla \left(\ln \rho^\tau + V\right) = \nabla \rho^\tau + \rho^\tau \nabla V; \quad I^\tau := \rho^\tau \nabla (2W * \rho^\tau) = 2\rho^\tau \left[(\nabla W) * \rho^\tau\right],$$

i.e. we have splitted the momentum into a linear part and an interaction part, which has more compactness properties. Since we had supposed W to be Lipschitz, the function $(\nabla W) * \rho^\tau$ is uniformly bounded, which gives a uniform bound on I^τ in the space of measures. This implies that both parts separately are compact as measures, and hence they converge to L and I, respectively, and $E = L + I$.

L and L^τ being linear w.r.t. ρ, it is clear that we must have, as a consequence of the weak convergence of ρ_t^τ to ρ_t, $L_t = \nabla \rho_t + \rho_t \nabla V$. Actually, $\nabla \rho_t^\tau$ converges to $\nabla \rho_t$, at least in the sense of distributions. For the term with ∇V, one can notice that the uniform bound on $F(\rho_t^\tau)$ implies a bound on the entropy $\int \rho_t^\tau \ln \rho_t^\tau$ (since both V and W are supposed bounded from below), and this bound turns the weak convergence $\rho_t^\tau \rightharpoonup \rho_t$ as measures into a weak convergence in L^1 as a consequence of Dunford-Pettis theorem (the densities ρ_t^τ are equi-integrable). Once we have weak convergence in L^1, multiplying times a fixed L^∞ function, i.e. ∇V, preserves the same convergence.

The interaction part I is far from being linear, but the convolution implies more compactness. Actually, ∇W being L^∞, we get a uniform bound in L^∞ on $(\nabla W) * \rho_t^\tau$. Moreover, it is clear that $\rho_t^\tau \rightharpoonup \rho_t$ implies the pointwise convergence of $(\nabla W) * \rho_t^\tau$ to $(\nabla W) * \rho_t$, since

$$\big[(\nabla W) * \rho_t^\tau\big](x) = \int \nabla W(x-y) \rho_t^\tau(y) dy \to \int \nabla W(x-y) \rho_t(y) dy = \big[(\nabla W) * \rho_t\big](x),$$

the convergence of the integrals being justified by the fact that ∇W is L^∞ and $\rho_t^\tau \rightharpoonup \rho_t$ in L^1.

We claim now that the product of an equi-integrable sequence of functions weakly converging in L^1 (here it is $\rho_t^\tau \rightharpoonup \rho_t$) times a sequence uniformly bounded in L^∞ which is pointwisely converging (here it is $(\nabla W) * \rho_t^\tau \to (\nabla W) * \rho_t$) does converge to the product of the limits (in the weak sense of L^1).

This fact is easy to see: take $f_j \to f$ in L^1, $g_j \to g$ pointwisely and suppose that $(f_j)_j$ is equi-integrable and $|g_j|(x) \leq C$; take a test function $\psi \in L^\infty$ and prove $\int f_j g_j \psi \to \int f g \psi$. To do that, use the fact that the equi-integrability of $(f_j)_j$ means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_A |f_j| < \varepsilon$ provided the measure of A is smaller than δ; moreover, every pointwise convergence is uniform, up to a set of arbitrarily small measure. This means that there exists a set A of measure smaller than δ such that $g_j \to g$ uniformly on A. Then one can see that $f_j I_A \rightharpoonup f I_A$, still weakly in L^1, and get

$$\left|\int_\Omega f_j g_j \psi - \int_A f_j g_j \psi\right|, \left|\int_\Omega f g \psi - \int_A f g \psi\right| < ||\psi||_{L^\infty} C \varepsilon, \quad \int_A f_j g_j \psi \to \int_A f g \psi,$$

where the last convergence is due to the weak convergence of $f_j I_A$ and the uniform convergence of $g_j \psi$ to $g \psi$. The number ε being arbitrary, this gives the desired convergence $\int_\Omega f_j g_j \psi \to \int_\Omega f g \psi$.

This allows to let the interaction term pass to the limit in the equation and conclude.

5 Back to crowd motion

In this section we come back to the case we were interested in for crowd motion modelling. Differently from the easier cases of Section 4, here the PDE is neither linear nor based on a convolution (and regularizing) kernel. If the PDE is written in the form given in (3), we have

$$\frac{\partial}{\partial t}\rho_t + \nabla \cdot \big(\rho_t(u_t - \nabla p_t)\big) = 0,$$

where p_t is a pressure, i.e. $p_t \geq 0$ and $p_t(1-\rho_t) = 0$. Thanks to this last relation, the product $\rho_t \nabla p_t$ may also be written simply as ∇p_t, since p_t (and its gradient) vanish where ρ_t is not equal to 1. This reduces the nonlinearity, since there is no more a bilinear term in the continuity equation, but, still, the relation between p and ρ is nonlinear. We will see what is the key point to let it pass to the limit, but first we want to recover the optimality conditions at each step, after properly defining the functional F we will use in the gradient flow procedure.

The functional we choose and some hints on the proof. We consider the case $u = -\nabla D$ and $K = \{\rho \in \mathcal{P}(\Omega) \,:\, \rho \leq 1\}$. We first define the functional F we will consider:

$$F(\rho) = \begin{cases} \int D(x)d\rho & \text{if } \rho \in K, \\ +\infty & \text{if } \rho \notin K. \end{cases}$$

The discrete iterative method is, as usual,

$$\rho^\tau_{k+1} \in \operatorname*{argmin}_\rho F(\rho) + \frac{W_2^2(\rho, \rho^\tau(k))}{2\tau} = \operatorname*{argmin}_{\rho \in K} \int D(x)d\rho + \frac{W_2^2(\rho, \rho^\tau(k))}{2\tau}$$

If we come back to the optimality conditions in terms of $\frac{\delta F}{\delta \rho}$, and we forget for a while the constraint $\rho \in K$, we would get that the optimal $\bar\rho$ is concentrated on argmin $\psi = D + \frac{\phi}{\tau}$. This was a consequence of the fact that $\bar\rho$ also optimized the linear functional $\rho \mapsto \int \psi d\rho$.

Since we need to take into account the constraint $\rho \in K$, and K is a convex set (notice that it is also a *geodesically convex* set, but we do not need this notion here), we can perform variations $\rho_\varepsilon = (1-\varepsilon)\bar\rho + \varepsilon\tilde\rho$, for $\tilde\rho \in K$. In this case we only get $\int \psi d\tilde\rho \geq \int \psi d\rho \quad \forall \tilde\rho \in K$.

Once a function ψ is given, which are the measures that optimize $\int \psi d\rho$ in K? the answer is easy: it is sufficient to concentrate ρ on a level set of ψ, and to put the maximal possible density (i.e .1) on it. This means that there is a constant t such that

$$\bar\rho = \begin{cases} 1 & \text{on } \psi < t, \\ 0 & \text{on } \psi > t, \\ \in [0,1] & \text{on } \psi = t \end{cases}$$

It is useful to define the function $p := (t-\psi)_+$. This function satisfies $p \geq 0$ and $p(1-\bar\rho) = 0$. This means that it is an admissible pressure! Afterwards, it is easy to pass to the gradients $\bar\rho$−a.e. and get $\nabla p = -\nabla\psi = -\nabla D - \frac{\nabla\phi}{\tau} = u - v$ a.e.. This gives $v = u - \nabla p$, which is the desired velocity field. Hence, this discrete scheme is consistent with the equation that we want to consider, even if the fact that, at the limit, this brings a solution of the PDE is still to be established.

The details are contained in [14]: here we only give an idea of the proof. Most of the ingredients are the same as in the previous section. By using the fact that $\rho^\tau\nabla p^\tau = \nabla p^\tau$ and taking a weak limit p of p^τ it is easy to see that the PDE is satisfied by ρ, p and u at the limit. We are only left to prove that $p_t(1-\rho_t) = 0$ for a.e. (t, x).

This is not trivial since a priori we have a product of two weak convergences. Yet, the key point is the fact that we have an L^2 bound on v^τ (coming from $\int_0^T \int |v^\tau|^2\rho^\tau = \int_0^T \int |\tilde{v}^\tau|^2\tilde{\rho}^\tau = \int_0^T |(\tilde{\rho}^\tau)'|^2_{W_2}dt \le C$) and that $v^\tau = u^\tau - \nabla p^\tau$. Since u^τ is supposed to be bounded (take D Lipschitz continuous), this implies an L^2 bound on ∇p^τ, and gives

$$\int_0^T \int |\nabla p^\tau|^2 = \int |\nabla p^\tau|^2\rho^\tau \le C.$$

Hence, p^τ satisfies an H^1 bound, and its weak convergence turns into a strong L^2 convergence! Moreover, the convergence $\rho^\tau \rightharpoonup \rho$ is weak as measures but also weak-* in L^∞, since they are all bounded by the same constant, and this should allow to conclude the proof.

The only delicate point is the fact that we do not have an H^1 bound in $[0, T] \times \Omega$, but only an H^1 bound in space, integrated in time. This makes the possibility of exploiting this compactness property more tricky, but in the end it works for almost any time t. The details are, again, in [14].

Modify the functional for modelization purposes If we consider the model above as the description of a crowd leaving a panic area, we choose $D(x) = d(x, \Gamma)$, where $\Gamma \subset \partial\Omega$ represents the door.

The density evolves by minimizing this mean distance to the door…but never leaves Ω. For $t \to \infty$ it is likely to fill a neighborhood of the door with density $\rho = 1$, which is the configuration that minimizes F. In particular, if the particles stand for people trying to reach the door so as to escape from a fire, they will all die! this requires a modification of the model, in order to allow the particles to leave Ω.

A first possibility could be to consider a larger domain, such as $\mathbb{R}^d \setminus (\partial\Omega \setminus \Gamma)$, i.e. the whole space without the part of the boundary of Ω which stands for the hard walls (the door Γ being included in the domain). This has some problems, since in particular a domain like that will not be convex, and most of the analysis is easier in convex domains. Some new developments are in progress concerning non-convex domains, but anyway they could not concern this kind of domain whose complement is too "thin" (more precisely, the results of the previous paragraph may be extended to closed non-convex domains, but the closure of the previous domain is the whole $\mathbb{R}^d$, which is not what we want to consider).

Hence, both for mathematical and modeling reasons, we decided to consider a different situation, where we give a new definition of the admissible set K:

$$K := \{\rho \in \mathcal{P}(\Omega) \ : \ \rho = \rho_\Gamma + \rho_\Omega, \ \rho_\Omega \le 1\,, \mathrm{supp}(\rho_\Gamma) \subset \Gamma\}$$

The reason for this choice is the following: we consider that as soon as a particle reaches Γ, it is safe ($D = 0$), and then, instead of following its movement after Γ, we leave it on Γ. This is done for simplicity, but it only means that we are no longer concerned with what happens to the particles that have reached Γ, not that they are really blocked on the door. Obviously, we need to withdraw the density constraint on Γ, so as to let particles stay on it (think that usually $|\Gamma| = 0$, so that the density constraint would prevent ρ to give mass to Γ), and also to represent the fact that Γ stands actually for everything that happens at the door *and beyond.*

The mathematical problem with this modified Γ is much trickier. One of the difficulties, that prevent the usual theory to be applied, is the fact that the set K loses some of the properties that it had previously (in particular it is no more geodesically convex: the geodesic - for the W_2 distance - between two points of K could go out of K).

To appreciate this approach one needs to think that the general theory mainly deals with geodesically convex functionals even if, for existence purposes, something more can be said when the slope of the functional is l.s.c. (see [2] for these concepts). However, this seems difficult to check here since it is not evident what the slope of the constrained functional F is. This is why the vertical perturbation method turns out to be useful in this setting, and it provides a proof for the existence of a solution for Ω convex, with or without the exit door Γ, thanks to the limit $\tau \to 0$.

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