ON C^2 UMBILICAL HYPERSURFACES

CARLO MANTEGAZZA

ABSTRACT. We show by an elementary argument that the second fundamental form of a connected, totally umbilical hypersurface of class C^2 is a constant multiple of the metric tensor. It follows that the hypersurface is smooth and it is either a piece of a hyperplane or of a sphere.

1. INTRODUCTION

We deal with *totally umbilical*, connected, n–dimensional hypersurfaces S of $\mathbb{R}^{n+1}$, that is, the second fundamental form B satisfies

$$\mathrm{B}_p = \lambda(p) g_p \tag{1.1}$$

at every $p \in S$. Chosen a unit normal vector field ν, the symmetric bilinear form B is defined as $\mathrm{B}(X,Y) = \left\langle \nabla_X^{\mathbb{R}^{n+1}} Y, \nu \right\rangle$, for every pair of tangent vector fields to the hypersurface, g is the induced metric on S and $\lambda : S \to \mathbb{R}$.

If S is of class C^3 at least, hence $\nabla\mathrm{B}$ is well defined, the Codazzi–Mainardi equations must be satisfied (see [9], for instance). In local coordinates,

$$\nabla_i \mathrm{B}_{jk} = \nabla_j \mathrm{B}_{ik}\,.$$

which implies, taking traces, $\mathrm{div}\mathrm{B} = \nabla\mathrm{H}$, that is $\nabla_i\mathrm{H} = g^{jk}\nabla_j\mathrm{B}_{ik}$.
Then, taking the divergence of both sides of equation (1.1), we get

$$\nabla\mathrm{H} = \mathrm{div}\mathrm{B} = \nabla\lambda$$

and taking instead first the trace, then the gradient, we have

$$\nabla\mathrm{H} = n\nabla\lambda\,.$$

Hence, for $n \geq 2$ we conclude that $\lambda = \mathrm{H}/n$ is constant. It follows easily that S is either a piece of a hyperplane ($\lambda = 0$) or of a sphere ($\lambda \neq 0$) in $\mathbb{R}^{n+1}$, see the final part of the proof of the theorem below (or [3, Chapter 7, Theorem 5.1], for instance).

We want to discuss here the case of S only C^2, so Codazzi–Mainardi equations are not (immediately) available. Actually, the C^3–regularity of S can be obtained (as said in [2]), by means of the regularity theory for solutions of elliptic equations, then the Codazzi–Mainardi equations can be used as above. We are going to show the same result by means of an elementary geometric argument.

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Theorem 1.1. *The second fundamental form of a connected, totally umbilical hypersurface of class C^2 is a constant multiple of the metric tensor and S is either a piece of a hyperplane or of a sphere.*

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Note. During the redaction of this document, we were informed by Mario Santilli, that the same result was first proved by P. Hartman [1] in 1947, then the proof was simplified by R. Souam and E. Toubiana [7, (Portuguese)] in 2006 (see also [8]) and, independently, by A. Pauly [4, (german)] in 2008. These proofs deal with surfaces in $\mathbb{R}^3$ but the arguments can be easily extended to any dimension. The same Mario Santilli showed that the conclusion also holds in the $C^{1,1}$ case in [6] (see also [5, Section 3]).

2. Proof of Theorem 1.1

Proof. Referring to equation (1.1), if $\lambda \equiv 0$ we are done, then we are going to show that λ is locally constant on the set where it is nonzero, this implies, as S is connected, that λ is constant.

So, we suppose that for $p \in S$, we have $\lambda(p) \neq 0$ and possibly changing the sign of ν, we can assume it is positive. Then, by choosing suitable coordinates of $\mathbb{R}^{n+1}$, we can assume that locally around p, the hypersurface is given by the graph of a C^2 function $f : \Omega \subseteq T_pS \approx \mathbb{R}^n \to \mathbb{R}$, that is, by the map

$$\Omega \ni (x^1, x^2, \dots, x^n) \mapsto \varphi(x^1, x^2, \dots, x^n) = \big(x^1, x^2, \dots, x^n, f(x^1, x^2, \dots, x^n)\big) \in \mathbb{R}^n \times \mathbb{R}\,,$$

where Ω is a convex open neighborhood of $0 \in \mathbb{R}^n$ and p is the point $0 \in \mathbb{R}^{n+1}$, that is, $f(0,0,\dots,0) = 0$, $\nabla f(0,0,\dots,0) = (0,0,\dots,0)$. The unit normal vector field ν (of class C^1) to S is given by

$$\nu = \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}\,,$$

hence,

$$\frac{\operatorname{Hess} f_{ij}}{\sqrt{1 + |\nabla f|^2}} = \mathrm{B}_{ij} = \lambda g_{ij}\,,$$

which implies that f is strictly convex, as B is positive definite, thus $f \geq 0$ and $f(x) = 0$ if and only if $x = 0$.

Then, by the Gauss–Weingarten equations (see [9], for instance)

$$\frac{\partial \nu}{\partial x^i} = -\mathrm{B}_{ij} g^{jk} \frac{\partial \varphi}{\partial x^k} = -\lambda g_{ij} g^{jk} \frac{\partial \varphi}{\partial x^k} = -\lambda \frac{\partial \varphi}{\partial x^i}\,, \tag{2.1}$$

hence, on the $(n+1)$–component, we get

$$\frac{\partial}{\partial x^i} \frac{1}{\sqrt{1 + |\nabla f|^2}} = -\lambda \frac{\partial f}{\partial x^i}\,,$$

for every $i \in \{1, 2, \ldots, n\}$, that is

$$\nabla \frac{1}{\sqrt{1+|\nabla f|^2}} = -\lambda \nabla f \,. \tag{2.2}$$

and

$$\frac{\partial \nu^j}{\partial x^i} = -\lambda \delta_{ij} \tag{2.3}$$

for every $i, j \in \{1, 2, \ldots, n\}$.

Lemma 2.1. *Let $F, G : \Omega \to \mathbb{R}$ be two C^1 functions on an open set $\Omega \subseteq \mathbb{R}^n$ such that $\nabla G = \lambda \nabla F$, with $\nabla F \neq 0$ in Ω, for some continuous function $\lambda : \Omega \to \mathbb{R}$. Then, if the level sets of F are connected, there exists $H : \operatorname{Image} F \to \mathbb{R}$ of class C^1, such that $G = H \circ F$. Moreover, $\lambda(x) = H'(F(x))$, hence λ is constant on the level sets of F.*

Proof. Let us prove that G is constant on every level set of F. Let $x, y \in \Omega$ with $F(x) = F(y) = c$ and $\gamma : [0,1] \to \Omega$ a C^1 curve in the level set $\{F = c\}$ such that $\gamma(0) = x$ and $\gamma(1) = y$ (which always exists by the connectedness of the level sets of F). Then, as $F(\gamma(t)) = c$, we have $\langle \nabla F(\gamma(t)), \dot\gamma(t)\rangle = 0$, for every $t \in [0,1]$ and

$$\begin{aligned} G(y) &= G(\gamma(0)) + \int_0^1 \frac{d}{dt} G(\gamma(t))\, dt = G(x) + \int_0^1 \langle \nabla G(\gamma(t)), \dot\gamma(t)\rangle \, dt \\ &= G(x) + \int_0^1 \lambda(\gamma(t)) \langle \nabla F(\gamma(t)), \dot\gamma(t)\rangle \, dt = G(x)\,. \end{aligned}$$

The existence of the function H is then immediate and the fact that it is C^1 can be shown as follows: let $t_0 \in \mathbb{R}$ in the image of F, that is $F(x_0) = t_0$ and $\sigma : (t_0 - \varepsilon, t_0 + \varepsilon) \to \Omega$ be a C^1 integral curve of the field $\nabla F / |\nabla F|^2$ with $\sigma(t_0) = x_0$, hence $F(\sigma(t_0)) = t_0$, then

$$F(\sigma(t)) = F(\sigma(t_0)) + \int_{t_0}^t \langle \nabla F(\sigma(s)), \dot\sigma(s)\rangle \, dt = t_0 + (t - t_0) = t\,,$$

then, around $t_0 \in \mathbb{R}$, we have $H(t) = H(F(\sigma(t)) = G(\sigma(t))$, which shows that H is of class C^1. The last assertion is trivial. ☐

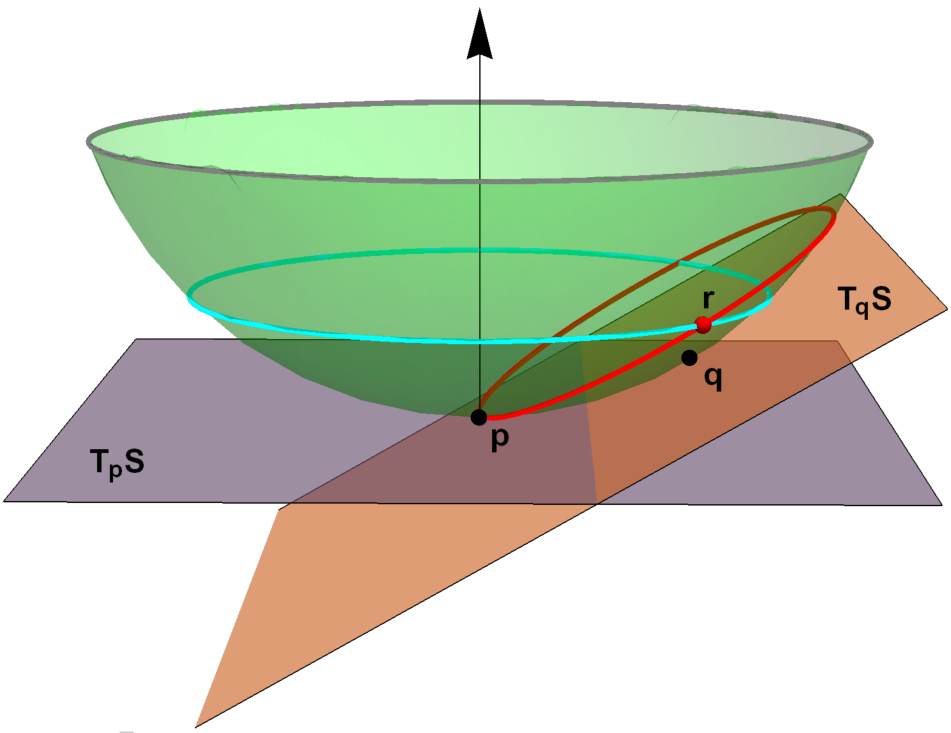
By the strict convexity of f, its only critical point is at $x = 0$, then choosing $F = f$, $G = 1/\sqrt{1+|\nabla f|^2}$ in this lemma and possibly restricting Ω to coincide with the sublevel $\{f < c\} \setminus \{0\}$, for a suitable positive constant c, the level sets of f are connected and we can conclude that λ (in coordinates) is constant on each one of them. That is, $\lambda : S \to \mathbb{R}$ is constant on the $(n-1)$–dimensional submanifolds of $\mathbb{R}^{n+1}$ given by the intersections of the hyperplanes parallel and close enough to T_pM with S.

Choosing a point $q \in S$ close to p and repeating this argument for q, we have a hyperplane Q parallel to T_qS passing by p such that λ is constant, hence equal to $\lambda(p)$, on the intersection $Q \cap S$ (red in the figure below, in the case of surfaces $n = 2$). Since any hyperplane P parallel to T_pS close enough to p intersects $Q \cap S$ at least at some point $r = P \cap Q \cap S$, the function λ is constant and equal to $\lambda(r) = \lambda(p)$ on $P \cap S$ (cyan in the figure). It clearly follows that in a neighborhood of p the function λ is constant.

If λ is constantly zero, by equation (2.1), the unit normal vector field is constant and this easily implies that S is a piece of a hyperplane of $\mathbb{R}^{n+1}$. If $\lambda \neq 0$, up to rescaling, we can assume that $\lambda = 1$ and in the notations above, we consider the function defined by $O(x) = (x, f(x)) + \nu(x)$ for $x \in \Omega$, which represents the point in the "inner" normal direction at distance 1 from the point $(x, f(x))$ of S.
For the $(n+1)$–component of $O(x)$ we have

$$O^{n+1}(x) = f(x) + \frac{1}{\sqrt{1 + |\nabla f(x)|^2}}\,.$$

which is constant equal to 1, by equation (2.2) and $O^{n+1}(0) = 1$.
For the other components $O^j(x) = x^j + \nu^j(x)$, for $j \in \{1, 2, \dots, n\}$, since by equation (2.3) there hold $\nabla O^j(x) = 0$, they are also constant on Ω. We conclude that the map $O : \Omega \to \mathbb{R}^{n+1}$ is constant, which clearly means that S is locally a subset of the sphere of radius one and center $(0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. By the connectedness of S, the last statement of the theorem follows.



$\square$

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Carlo Mantegazza, Dipartimento di Matematica e Applicazioni, Università di Napoli, Via Cintia, Monte S. Angelo 80126 Napoli, Italy

Email address: c.mantegazza@sns.it