Dirichlet integrals and asymmetry

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1 Introduction and main result

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 1$. Suppose that Ω is Steiner symmetric about some hyperplane H, which, without loss of generality, will be assumed throughout to coincide with the coordinate hyperplane orthogonal to the last coordinate in $\mathbb{R}^n$. Namely, denoted a point in $\mathbb{R}^n$ as $x = (x', y)$, with $x' \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$, we assume that the cross sections

$$\Omega_{x'} = \{y \in \mathbb{R} : (x', y) \in \Omega\}$$

are segments centered at 0. Given a nonnegative measurable function u in Ω, its Steiner symmetral u^s about H is the function from Ω into $[0, +\infty]$ whose one-dimensional restrictions $u^s(x', \cdot)$ to $\Omega_{x'}$ are symmetric about 0, and equidistributed with the restrictions $u(x', \cdot)$. In formulas,

$$u^s(x', y) = \inf\Big\{t > 0 : \mathcal{L}^1(\{u(x', \cdot) > t\}) \leq 2|y|\Big\} \qquad \text{for } (x', y) \in \Omega\,.$$

Hereafter, $\mathcal{L}^k$ denotes the k-dimensional (outer) Lebesgue measure.

A celebrated result, tracing its origins in the work of Pólya and Szegö [PS] and subsequently extended and refined by various authors (including [AFLT, Bae, Br, BZ, Bu, C, CF1, FV, Hi, K1, K2, Ko, S, T, U]), states that if u belongs to the Sobolev space $W_0^{1,p}(\Omega)$ for some $p \geq 1$, then $u^s \in W_0^{1,p}(\Omega)$ as well, and

$$\int_\Omega |\nabla u^s|^p\, dx \leq \int_\Omega |\nabla u|^p\, dx\,. \tag{1.1}$$

Accordingly, we shall call the difference between the right-hand side and the left-hand side of (1.1) the *Pólya–Szegö deficit of* u, and denote it by $D_p(u)$; namely, we set

$$D_p(u) = \int_\Omega |\nabla u|^p\, dx - \int_\Omega |\nabla u^s|^p\, dx\,. \tag{1.2}$$

Obviously, $D_p(u) = 0$ if u is Steiner symmetric about H. Thus, the question can be risen of whether the Pólya–Szegö deficit of an arbitrary function $u \in W_0^{1,p}(\Omega)$ can be used to estimate its asymmetry about H, measured as a distance between u and u^s. Of course, such an estimate should ensure that u is arbitrarily close to u^s, provided that $D_p(u)$ is sufficiently small.

In contrast to the stability of various geometric and functional inequalities known in the literature (see e.g. [Bo, H, HHW, Fu]), the answer to this question is negative. Indeed, it is well known, and easily seen by elementary examples, that functions u, far from being symmetric, exist such that $D_p(u) = 0$. Consider, for instance, the case where $n = 1$. In this case $H = \{0\}$, and Steiner symmetrization agrees with Schwarz spherical symmetrization. Let $\Omega = (-3, 3)$, let v the piecewise affine function displayed in Figure 1 and let $v^\star$ be its Schwarz symmetral.

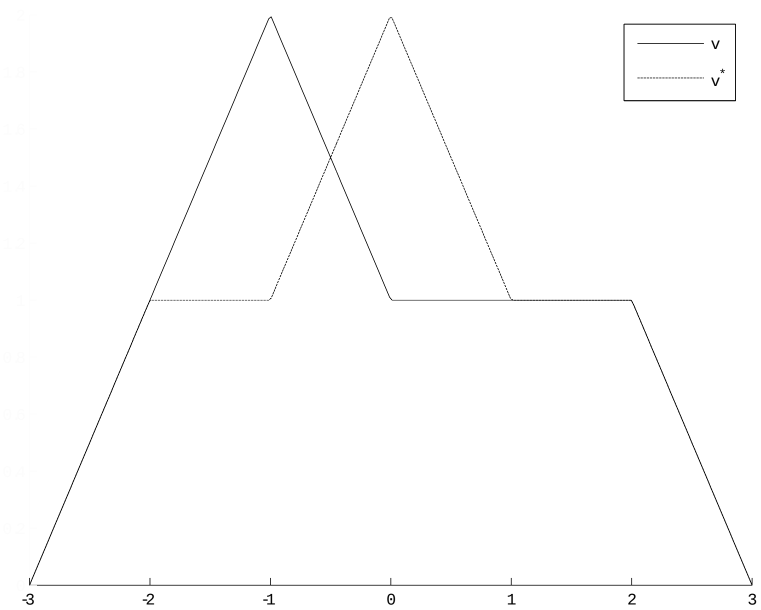

FIG. 1

Since

$$\int_{-3}^{3} |v'(x)|^p \, dx = \int_{-3}^{3} |v^{\star\prime}(x)|^p \, dx \qquad \text{for every } p \geq 1 \,,$$

then $D_p(u) = 0$, but $v \neq v^\star$. It is not difficult to realize that what plays a role here is the plateau (below the top level) in the graph of v. Obviously, this example can be adjusted to produce functions in any dimension n attaining equality in (1.1), which are not Steiner symmetric (see e.g. [CF2, Fig.1]). All these examples, however, must involve functions u whose derivative $\nabla_y u$ vanishes on a set of positive Lebesgue measure. Indeed, in [CF2, Theorem 2.2] we proved that symmetry of extremals in (1.1) can be restored if they are assumed to satisfy

$$\mathcal{L}^n(\{(x', y) \in \Omega : \nabla_y u(x', y) = 0, \, u(x', y) < M(x')\}) = 0 \,.$$

Here,

$$M(x') = \operatorname{esssup}\{u(x', y) : \, y \in \Omega_{x'}\} \qquad \text{for } \, x' \in \pi(\Omega),$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the orthogonal projection onto $\mathbb{R}^{n-1}$. This result can be regarded as a Steiner symmetrization counterpart of Brothers and Ziemer's theorem on the spherical symmetry of minimal rearrangements u, i.e. functions satisfying $\int_{\mathbb{R}^n} |\nabla u|^p \, dx = \int_{\mathbb{R}^n} |\nabla u^\star|^p \, dx$, under the assumption $\mathcal{L}^n(\{\nabla u = 0, \, u < \operatorname{esssup} u\}) = 0$ ([BZ]).

On the other hand, we have recently shown that the result of Brothers and Ziemer is stable under perturbations of such assumption, in the sense that the asymmetry of a minimal rearrangement can be estimated through $\mathcal{L}^n(\{\nabla u = 0, u < \operatorname{esssup} u\})$ ([CF3]).

In view of these facts, it could be conjectured that, although the sole Pólya–Szegö deficit of u is not sufficient to measure the distance of u from u^s, this should be possible if both $D_p(u)$ and $\mathcal{L}^n(\{(x',y) \in \Omega : \nabla_y u(x',y) = 0, u(x',y) < M(x')\})$ are employed. Interestingly enough, not even this is true. Indeed, consider the sequence of functions $\{v_h\}_{h\in\mathbb{N}}$ depicted in Figure 2, and obtained by perturbing the function v in Figure 1.

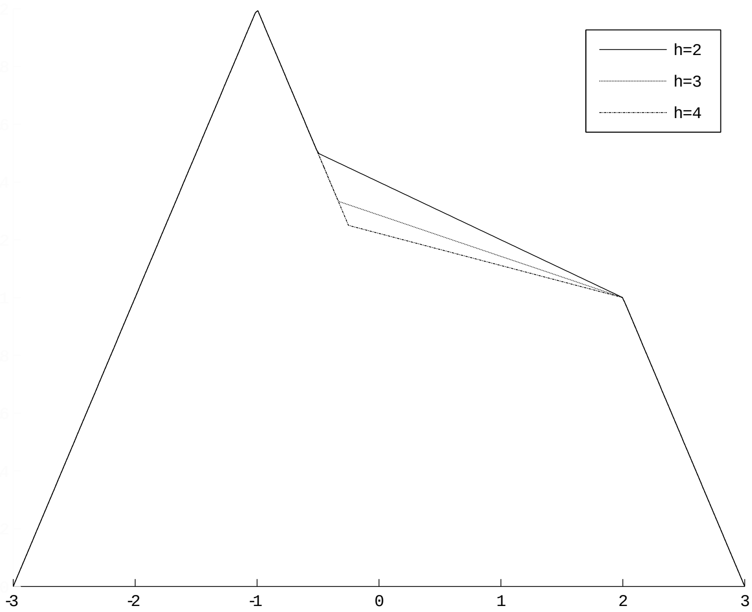

FIG. 2

We have $v_h' \neq 0$ (and $v_h^{\star\prime} \neq 0$) $\mathcal{L}^1$-a.e. for every $h \in \mathbb{N}$, and it is easily verified that

$$\lim_{h\to+\infty} D_p(v_h) = 0 \qquad \text{for every } p \geq 1\,,$$

but $\|v_h - v_h^\star\|_{L^1} \geq const.$ for every $h \in \mathbb{N}$ (*a fortiori*, the same situation occurs if the L^1 norm is replaced by any other rearrangement invariant norm).

Though very simple, this counterexample sheds light on this matter, and, loosely speaking, shows that, when $D_p(u) > 0$, not only a large set where $\nabla_y u$ vanishes, but also a large set where $\nabla_y u$ is small, may allow u to be very asymmetric. Consequently, one can hope to control the asymmetry of a function u by means of $D_p(u)$ only in conjunction with a bound on the measure of the sublevel sets of $|\nabla_y u|$. Our main result, contained in Theorem 1.1 below, states that this is actually possible, at least when $p \geq 2$. Indeed, it provides, for every fixed $\sigma > 0$, an estimate for the distance in $L^1(\Omega)$ between u and u^s in terms of $D_p(u)$ and of the quantity $m_u(\sigma)$ defined as

$$m_u(\sigma) = \mathcal{L}^n(\{(x',y) \in \Omega : |\nabla_y u(x',y)| \leq \sigma, u(x',y) < M(x')\}). \tag{1.3}$$

Moreover, such an estimate approaches 0 as $D_p(u)$ and $m_u(\sigma)$ do.
In fact, the conclusion of Theorem 1.1 holds for functions u from a somewhat larger space than $W_0^{1,p}(\Omega)$. Actually, any function from $W^{1,p}(\Omega)$ vanishing, in the appropriate sense, just on $\partial\Omega \cap (\pi(\Omega) \times \mathbb{R})$ is

admissible, since functions from the class

$$W^{1,p}_{0,y}(\Omega) \;=\; \Big\{u:\Omega\to\mathbb{R}:\ \text{for every open set } \omega\subset\subset\pi(\Omega) \text{ the continuation}$$
$$\text{by 0 of } u \text{ outside } \Omega \text{ belongs to } W^{1,p}(\omega\times\mathbb{R})\Big\}$$

are taken into account.

Theorem 1.1 *Let $n\geq 2$, and let Ω be a Steiner symmetric open set in $\mathbb{R}^n$, bounded in the direction y, and such that $\mathcal{L}^{n-1}(\pi(\Omega))<+\infty$. Set*

$$L=\sup\{|y|:(x',y)\in\Omega \text{ for some } x'\in\mathbb{R}^{n-1}\}.$$

Let $p\geq 2$. Then, there exists a constant C depending only on p, $\mathcal{L}^n(\Omega)$ and L such that

$$\begin{aligned}(1.4)\quad \int_\Omega |u(x)-u^s(x)|\,dx \leq C\Big[D_p(u)^{\frac{1}{2p}} \;+\; \sigma\sqrt{m_u(\sigma)}+\|\nabla_y u\|_{L^p(\Omega)}\Big(\frac{D_p(u)^{\frac{1}{2p}}}{\sigma^p}+D_p(u)^{\frac{1}{2p}}+m_u(\sigma)^{\frac{1}{2p'}}\Big)\\ +\|\nabla_y u\|^{3/2}_{L^p(\Omega)}\frac{D_p(u)^{\frac{1}{2p}}}{\sigma^{\frac{p+1}{2}}}\Big]\end{aligned}$$

for every nonnegative function u from $W^{1,p}_{0,y}(\Omega)$ and every $\sigma>0$.

We emphasize that the assumption $p\geq 2$ is indispensable in Theorem 1.1, since no estimate for $\int_\Omega|u-u^s|dx$, approaching 0 when $D_p(u)$ and $m_u(\sigma)$ tend to 0 and $\|\nabla_y u\|_{L^p(\Omega)}$ remains bounded, can hold if $p<2$, as shown by Example 3.6, Section 3. On the other hand, it will be clear from the proof of Theorem 1.1 that suitable alternate choices of the functional in the definition of $D_p(u)$, still with a p-growth, such as $\int_\Omega\sum_{i=1}^n|\nabla_{x_i}u|^p dx$, would lead to an analogous conclusion as in Theorem 1.1 even for $p\in(1,2)$. These facts demonstrate how sensitive the result is to the geometry of the integrand employed in $D_p(u)$.

As a consequence of Theorem 1.1, one can deduce an estimate for $\int_\Omega|u-u^s|dx$ in terms of $m_u(0)$ in the case where equality holds in (1.1). Actually, setting $D_p(u)=0$, letting σ go to 0^+ in (1.4), and making use of the fact that $\lim_{\sigma\to 0^+} m_u(\sigma)=m_u(0)$, yield the following corollary.

Corollary 1.2 *Let Ω, p and u be as in Theorem 1.1. Assume in addition that equality holds in (1.1). Then*

$$\int_\Omega |u(x)-u^s(x)|\,dx\leq C\Big[\mathcal{L}^n(\{(x',y)\in\Omega:\nabla_y u(x',y)=0,\, u(x',y)<M(x')\})\Big]^{\frac{1}{2p'}}\|\nabla_y u\|_{L^p(\Omega)}\,,$$

where C is the same as in Theorem 1.1.

In the special case where $m_u(0)=0$, we recover from Corollary 1.2 the symmetry result from [CF2] to which we alluded above. Notice that, although the result of [CF2] holds for every $p\in(1,+\infty)$, it can be derived from Corollary 1.2 only for $[2,+\infty)$. On the other hand, the present approach has the advantage that the tools exploited are much more elementary.

Corollary 1.3 *Let Ω, p and u be as in Theorem 1.1. Assume in addition that equality holds in (1.1) and that*

$$\mathcal{L}^n(\{(x',y)\in\Omega:\nabla_y u(x',y)=0,\, u(x',y)<M(x')\})=0\,.$$

Then $u=u^s$ a.e. in Ω.

2 The one-dimensional case

This section is devoted to a one-dimensional version of Theorem 1.1. Besides being the first and key step in the proof of Theorem 1.1, the result is of independent interest, since Steiner and Schwarz symmetrization agree in dimension one. Notice that, unlike Theorem 1.1, this result holds for every $p > 1$. In fact, it will be clear from the proof that what prevents Theorem 1.1 from holding also for $p \in (1, 2)$ when $n \geq 2$ is the presence of the transverse derivatives $\nabla_{x'} u$ in the functional $\int_\Omega |\nabla u|^p dx$.
Consistently with (1.2) and (1.3), given $l > 0$, $p > 1$ and $w \in W_0^{1,p}(-l, l)$, throughout this section we set

$$D_p(w) = \int_{-l}^{l} |w'(x)|^p \, dx - \int_{-l}^{l} |w^{\star\prime}(x)|^p \, dx \tag{2.1}$$

and

$$m_w(\sigma) = \mathcal{L}^1 \left(\{x \in (-l, l) : \, |w'(x)| \leq \sigma, w(x) < \operatorname{esssup} w\} \right) \qquad \text{for } \sigma \geq 0\,. \tag{2.2}$$

Theorem 2.1 *Let $l > 0$ and $p > 1$, and let w be a nonnegative function from $W_0^{1,p}(-l, l)$. Then there exists a constant C_1, depending only on p, such that*

$$\begin{aligned}\int_{-l}^{l} |w(x) - w^\star(x)| \, dx \leq\; & C_1 l \Big[l^{\frac{1}{p'}} D_p(w)^{\frac{1}{p}} + \sigma m_w(\sigma) \Big] \\ & + C_1 \operatorname{esssup} w \left[\frac{D_p(w)}{\sigma^p} + m_w(\sigma) + \frac{l^{\frac{1}{p'}} D_p(w)^{\frac{1}{p}}}{\sigma} + \frac{\sqrt{D_p(w)}}{\sigma^{\frac{p+1}{2}}} \sqrt{\operatorname{esssup} w} \right]\end{aligned} \tag{2.3}$$

for every $\sigma > 0$.

Proof. Part I Here we consider the case where w is a piecewise affine function in $[-l, l]$, vanishing at $-l$ and l. Note that we may assume, without loss of generality, that w is not constant in any subinterval of $(-l, l)$ where $w < \sup w$. Indeed, if this is not the case, w can be approximated by a sequence $\{w_h\}$ of piecewise affine functions enjoying this additional property, in such a way that each quantity involving w in the statement is the limit, as h goes to $+\infty$, of the same quantity evaluated at w_h. The continuity from the right of the function m_w plays a role in this argument.
Let us denote by $0 = t_0 < t_1 < \ldots < t_m = \sup w$ the levels at which the graph of w has at least one corner, and set $d_i = t_{i+1} - t_i$ for $i = 0, \ldots, m-1$. Then, for each i there exists an even number $2k(i)$ of subintervals $S_{i,k}$ of $(-l, l)$, with $k = 1, \ldots, 2k(i)$, where w is affine and takes values strictly between t_i and t_{i+1}. Let us set $\Delta_{i,k} = \mathcal{L}^1(S_{i,k})$ for $i = 0, \ldots, m-1$ and $k = 1, \ldots, 2k(i)$. Inasmuch as

$$|w'(x)| = \frac{d_i}{\Delta_{i,k}} \qquad \text{if } x \in S_{i,k}\,, \tag{2.4}$$

then

$$\int_{-l}^{l} |w'(x)|^p \, dx = \sum_{i=0}^{m-1} \sum_{k=1}^{2k(i)} \left(\frac{d_i}{\Delta_{i,k}} \right)^p \Delta_{i,k}\,. \tag{2.5}$$

On the other hand, on setting

$$\Delta_i = \sum_{k=1}^{2k(i)} \Delta_{i,k}\,,$$

and observing that

$$|w^{\star\prime}(x)| = \frac{2d_i}{\Delta_i}, \qquad\qquad i = 0, \dots, m-1\,,$$

for every $x \in (-l, l)$ such that $w^\star(x) \in (t_i, t_{i+1})$, we have

$$\int_{-l}^{l} |w^{\star\prime}(x)|^p \, dx = \sum_{i=0}^{m-1} \left(\frac{2d_i}{\Delta_i}\right)^p \Delta_i\,. \tag{2.6}$$

Now, we proceed in steps.

STEP 1 Define

$$J = \{i:\, k(i) \geq 2\}\,.$$

Then a constant c_1, depending only on p, exists such that

$$\sum_{i\in J} d_i \leq c_1 l^{\frac{1}{p'}} D_p(w)^{\frac{1}{p}}\,. \tag{2.7}$$

Owing to Hölder's inequality for sums, we get from (2.5)

$$\int_{-l}^{l} |w'(x)|^p \, dx = \sum_{i=0}^{m-1} d_i \sum_{k=1}^{2k(i)} \left(\frac{d_i}{\Delta_{i,k}}\right)^{p-1} \geq \sum_{i=0}^{m-1} d_i \frac{(2k(i))^p}{\left(\displaystyle\sum_{k=1}^{2k(i)} \frac{\Delta_{i,k}}{d_i}\right)^{p-1}} = \sum_{i=0}^{m-1} d_i^p \frac{(2k(i))^p}{\Delta_i^{p-1}}\,. \tag{2.8}$$

Combining (2.8) and (2.6) yields

$$D_p(w) \geq \sum_{i=0}^{m-1} \frac{(2d_i)^p}{\Delta_i^{p-1}} (k(i)^p - 1)\,. \tag{2.9}$$

Given $\beta > 0$, define

$$H = \left\{i:\, \frac{2d_i}{\Delta_i} > \beta\right\}.$$

Hence, by (2.9),

$$D_p(w) \geq (2^p - 1)\, \beta^{p-1} \sum_{i \in J\cap H} d_i\,. \tag{2.10}$$

On the other hand, on writing $d_i = \frac{1}{2}\Delta_i \frac{2d_i}{\Delta_i}$, one immediately verifies that

$$\sum_{i\notin H} d_i \leq \frac{1}{2} \sum_{i\notin H} \Delta_i \beta \leq l\beta\,. \tag{2.11}$$

Inequalities (2.10) and (2.11) entail that

$$\sum_{i\in J} d_i \leq \frac{D_p(w)}{\beta^{p-1}} + l\beta\,. \tag{2.12}$$

Minimizing the right-hand side of (2.12) with respect to β tells us that

$$\sum_{i\in J} d_i \le \left((p-1)^{\frac{1}{p}} + \frac{1}{(p-1)^{\frac{1}{p'}}}\right) l^{\frac{1}{p'}} D_p(w)^{\frac{1}{p}},$$

whence (2.7) follows.

STEP 2 We have

$$\sum_{i\in J} \Delta_i \le 2m_w(\sigma) + \frac{8}{\sigma} c_1 l^{\frac{1}{p'}} D_p(w)^{\frac{1}{p}} + \frac{2^{p+1} D_p(w)}{\sigma^p} \qquad \text{for } \sigma > 0\,, \tag{2.13}$$

where $c_1(p)$ is the constant appearing in (2.7).
Set

$$N_i = \left\{k \in \{1,\dots,2k(i)\} :\ \frac{d_i}{\Delta_{i,k}} \le \sigma\right\}$$

and

$$\alpha_i = \sum_{k\in N_i} \Delta_{i,k}$$

for $i = 0,\dots,m-1$. It is easily verified that

$$d_i \ge \frac{\sigma}{2k(i)}(\Delta_i - \alpha_i) \qquad \text{for } i = 0,\dots,m-1\,. \tag{2.14}$$

Define

$$P = \left\{i :\ \Delta_i \ge \max\left\{2\alpha_i, \frac{8d_i}{\sigma}\right\}\right\}. \tag{2.15}$$

From (2.9) and (2.14) one obtains that

$$D_p(w) \ge \sum_{i\in J\cap P} \frac{(2d_i)^p}{\Delta_i^{p-1}}(k(i)^p - 1) \ge \sum_{i\in J\cap P} 2^p \frac{d_i^p}{\Delta_i^{p-1}}\left[\frac{\sigma^p(\Delta_i-\alpha_i)^p}{2^p d_i^p} - 1\right] = 2^p \sum_{i\in J\cap P} \Delta_i \left[\frac{\sigma^p}{2^p}\left(1 - \frac{\alpha_i}{\Delta_i}\right)^p - \left(\frac{d_i}{\Delta_i}\right)^p\right].$$

Hence, by definition (2.15),

$$\sum_{i\in J\cap P} \Delta_i \le \frac{2^p D_p(w)}{\left(1 - \frac{1}{2^p}\right)\sigma^p}\,. \tag{2.16}$$

Definition (2.15) again ensures that

$$\sum_{i\in J\setminus P} \Delta_i \le 2 \sum_{i\in J\setminus P} \alpha_i + \frac{8}{\sigma} \sum_{i\in J\setminus P} d_i. \tag{2.17}$$

By (2.2) and (2.4),

$$\sum_{i\in J\setminus P} \alpha_i \le \sum_{i\in J} \alpha_i \le m_w(\sigma)\,. \tag{2.18}$$

Moreover, by (2.7),

$$\sum_{i\in J\setminus P} d_i \le \sum_{i\in J} d_i \le c_1 l^{\frac{1}{p'}} D_p(w)^{\frac{1}{p}}\,. \tag{2.19}$$

Combining (2.16)-(2.19) yields (2.13).

STEP 3 Set

$$I = \{i \in \{0, \ldots, m-1\} : i \notin J\}\ ,$$

so that $k(i) = 1$ if $i \in I$, and define

$$Q = \left\{ i \in I : \text{either } \frac{d_i}{\Delta_{i,1}} \le \sigma \text{ or } \frac{d_i}{\Delta_{i,2}} \le \sigma \right\}.$$

Then

(2.20)
$$\sum_{i \in Q} \Delta_i \le 2 m_w(\sigma)$$

and

(2.21)
$$\sum_{i \in I \setminus Q} |\Delta_{i,1} - \Delta_{i,2}| \le \frac{2^{\frac{p+3}{2}}}{\sqrt{p(p-1)}} \sqrt{\sup w} \frac{\sqrt{D_p(w)}}{\sigma^{\frac{p+1}{2}}}\ .$$

Consider (2.20). For each $i \in Q$, set

$$T_i = \left\{ k \in \{1,2\} : \frac{d_i}{\Delta_{i,k}} \le \sigma \right\}.$$

By (2.2) and (2.4),

(2.22)
$$\sum_{i \in Q} \sum_{k \in T_i} \Delta_{i,k} \le m_w(\sigma)\ .$$

Furthermore, for each i and h such that $i \in Q$ and $h \notin T_i$, one has

(2.23)
$$\Delta_{i,h} < \Delta_{i,k}\ ,$$

where either $k = 1$ or $k = 2$ according to whether $h = 2$ or $h = 1$. Inequalities (2.22)-(2.23) entail that

$$\sum_{i \in Q} \Delta_i = \sum_{i \in Q} \sum_{k=1}^{2} \Delta_{i,k} \le 2 m_w(\sigma)\ ,$$

i.e. (2.20). Let us now prove (2.21). Define $\Phi : (0, +\infty) \to (0, +\infty)$ as $\Phi(s) = s^{1-p}$ for $s > 0$. By (2.5) and (2.6),

(2.24)
$$D_p(w) = \sum_{i=0}^{m-1} d_i^p \left[\sum_{k=1}^{2k(i)} \Phi(\Delta_{i,k}) - 2\Phi\left(\frac{1}{2} \sum_{k=1}^{2k(i)} \Delta_{i,k} \right) \right].$$

Note that, for every $i = 0, \ldots, m-1$,

(2.25)
$$\sum_{k=1}^{2k(i)} \Phi(\Delta_{i,k}) - 2\Phi\left(\frac{1}{2} \sum_{k=1}^{2k(i)} \Delta_{i,k} \right) \ge \sum_{k=1}^{2k(i)} \Phi(\Delta_{i,k}) - 2(k(i))^p \Phi\left(\frac{1}{2} \sum_{k=1}^{2k(i)} \Delta_{i,k} \right) \ge 0\ ,$$

where the first inequality is due to the fact that $k(i) \ge 1$, and the last one holds because $\Phi(s)$ equals s^{1-p} and is convex. From (2.24)-(2.25), we deduce that

(2.26)
$$D_p(w) \ge \sum_{i \in I \setminus Q} d_i^p \left[\Phi(\Delta_{i,1}) + \Phi(\Delta_{i,2}) - 2\Phi\left(\frac{\Delta_{i,1} + \Delta_{i,2}}{2} \right) \right].$$

The convexity of Φ entails that

$$\Phi(s_1) + \Phi(s_2) - 2\Phi\Big(\frac{s_1+s_2}{2}\Big) \geq \Big[\min_{s\in[s_1,s_2]} \Phi''(s)\Big]\frac{(s_1-s_2)^2}{4}$$

if $0 < s_1 < s_2$. Thus, inasmuch as $\Phi''(s) = p(p-1)s^{-p-1}$, a decreasing function, then

$$\Phi(s_1) + \Phi(s_2) - 2\Phi\Big(\frac{s_1+s_2}{2}\Big) \geq \frac{p(p-1)}{4}\frac{1}{(s_1+s_2)^{p+1}}(s_1-s_2)^2 \tag{2.27}$$

for every $s_1, s_2 \in (0, +\infty)$. From (2.26) and (2.27) we infer that

$$D_p(w) \geq \frac{p(p-1)}{4}\sum_{i\in I\setminus Q}\frac{d_i^p}{(\Delta_{i,1}+\Delta_{i,2})^{p+1}}(\Delta_{i,1}-\Delta_{i,2})^2 = \frac{p(p-1)}{4}\sum_{i\in I\setminus Q}\frac{1}{\Big(\frac{\Delta_{i,1}}{d_i}+\frac{\Delta_{i,2}}{d_i}\Big)^{p+1}}\frac{(\Delta_{i,1}-\Delta_{i,2})^2}{d_i}. \tag{2.28}$$

Observe that, if $i \in I \setminus Q$, then $\frac{\Delta_{i,1}}{d_i}+\frac{\Delta_{i,2}}{d_i} < \frac{2}{\sigma}$. Thus, the rightmost side of (2.28) is not less than

$$\frac{p(p-1)}{4}\Big(\frac{\sigma}{2}\Big)^{p+1}\sum_{i\in I\setminus Q}\frac{(\Delta_{i,1}-\Delta_{i,2})^2}{d_i}.$$

Schwarz's inequality ensures that

$$\sum_{i\in I\setminus Q}\frac{(\Delta_{i,1}-\Delta_{i,2})^2}{d_i} \geq \frac{\Big(\sum_{i\in I\setminus Q}|\Delta_{i,1}-\Delta_{i,2}|\Big)^2}{\sum_{i\in I\setminus Q} d_i}.$$

Since $\sum_{i\in I\setminus Q} d_i \leq t_m = \sup w$, we conclude that

$$D_p(w) \geq \frac{p(p-1)}{4}\Big(\frac{\sigma}{2}\Big)^{p+1}\frac{1}{\sup w}\Big(\sum_{i\in I\setminus Q}|\Delta_{i,1}-\Delta_{i,2}|\Big)^2,$$

whence (2.21) follows.

Step 4 Inequality (2.3) holds.
We have

$$\begin{aligned}\int_{-l}^{l}|w(x)-w^\star(x)|\,dx &= \int_{-l}^{l}\Big|\int_0^{\sup w}\Big(\chi_{\{w>t\}}(x)-\chi_{\{w^\star>t\}}(x)\Big)\,dt\Big|dx \\ &\leq \int_0^{\sup w}\Big(\int_{-l}^{l}\Big|\chi_{\{w>t\}}(x)-\chi_{\{w^\star>t\}}(x)\Big|\,dx\Big)dt = \int_0^{\sup w}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt\,,\end{aligned} \tag{2.29}$$

where $\triangle$ stands for symmetric difference of sets. Let us split the last integral as

$$\begin{aligned}\int_0^{\sup w}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt &= \sum_{i\in J}\int_{t_i}^{t_{i+1}}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt \\ &+\sum_{i\in Q}\int_{t_i}^{t_{i+1}}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt + \sum_{i\in I\setminus Q}\int_{t_i}^{t_{i+1}}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt\,.\end{aligned} \tag{2.30}$$

First, one has

$$\sum_{i\in J}\int_{t_i}^{t_{i+1}}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt\le 2l\sum_{i\in J}d_i\le 2c_1 l^{1+\frac{1}{p'}}D_p(w)^{\frac{1}{p}}\,, \tag{2.31}$$

where the second inequality is due to (2.7).
Next, for each $i\in Q$, there exists $k_i\in T_i$ (the set defined in STEP 3) such that

$$\frac{d_i}{\Delta_{i,k_i}}\le\sigma\,. \tag{2.32}$$

From (2.32) and (2.20) one gets that

$$\sum_{i\in Q}\int_{t_i}^{t_{i+1}}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt\le 2l\sum_{i\in Q}d_i\le 2l\sigma\sum_{i\in Q}\Delta_{i,k_i}\le 4l\sigma m_w(\sigma)\,. \tag{2.33}$$

Finally, one can easily verify that, if $t\in(t_i,t_{i+1})$ for some $i\in I$, then, denoted by η_i the distance between the centers of the intervals $\{w>t_i\}$ and $\{w^\star>t_i\}$,

$$\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)\le 2\eta_i+|\Delta_{i,1}-\Delta_{i,2}| \tag{2.34}$$

for every $t\in(t_i,t_{i+1})$. The distance η_i can be estimated by the total length of the intervals $S_{i,k}$ with $i\in J\cup Q$, plus the sum of the asymmetries $|\Delta_{j,1}-\Delta_{j,2}|$ corresponding to the indices $j\in I\setminus Q$ which are smaller than i. Thus, by (2.13) and (2.20) one certainly has

$$\eta_i\le 2m_w(\sigma)+\frac{8}{\sigma}c_1 l^{\frac{1}{p'}}D_p(w)^{\frac{1}{p}}+\frac{2^{p+1}D_p(w)}{\sigma^p}+2m_w(\sigma)+\sum_{\{j\in I\setminus Q:j<i\}}|\Delta_{j,1}-\Delta_{j,2}| \tag{2.35}$$

for every $i\in I\setminus Q$. Moreover, by (2.21),

$$\sum_{i\in I\setminus Q}|\Delta_{i,1}-\Delta_{i,2}|\le\frac{2^{\frac{p+3}{2}}}{\sqrt{p(p-1)}}\sqrt{\sup w}\frac{\sqrt{D_p(w)}}{\sigma^{\frac{p+1}{2}}}\,. \tag{2.36}$$

From (2.34)-(2.36) one infers that

$$\begin{aligned}
&\sum_{i\in I\setminus Q}\int_{t_i}^{t_{i+1}}\mathcal{L}^1\left(\{w>t\}\triangle\{w^\star>t\}\right)dt \\
&\le 2\left[4m_w(\sigma)+\frac{8c_1}{\sigma}l^{\frac{1}{p'}}D_p(w)^{\frac{1}{p}}+\frac{2^{p+1}D_p(w)}{\sigma^p}+\frac{2^{\frac{p+3}{2}}}{\sqrt{p(p-1)}}\sqrt{\sup w}\frac{\sqrt{D_p(w)}}{\sigma^{\frac{p+1}{2}}}\right]\sum_{i\in I\setminus Q}d_i \\
&\le 2\sup w\left[4m_w(\sigma)+\frac{8c_1(p)}{\sigma}l^{\frac{1}{p'}}D_p(w)^{\frac{1}{p}}+\frac{2^{p+1}D_p(w)}{\sigma^p}+\frac{2^{\frac{p+3}{2}}}{\sqrt{p(p-1)}}\sqrt{\sup w}\frac{\sqrt{D_p(w)}}{\sigma^{\frac{p+1}{2}}}\right].
\end{aligned} \tag{2.37}$$

Combining (2.29), (2.30), (2.31), (2.33), and (2.37) yields (2.3).

PART II We establish inequality (2.3) for every $w\in W_0^{1,p}(-l,l)$.
Fix $w\in W_0^{1,p}(-l,l)$ and $\sigma>0$. Without loss of generality, we may assume that w is continuous, and hence that the set $A=\{x\in(-l,l):w(x)<\max w\}$ is open. Thus, there exist a family $\{I_j\}_{j\in N}$, with $N\subset\mathbb{N}$, of maximal open intervals $I_j\subset A$ such that

$$A=\bigcup_{j\in N}I_j\,.$$

Denote by j_1 and j_2 the indices having the property that $-l$ is the left endpoint of I_{j_1} and l is the right endpoint of I_{j_2}.
Fixed any $\varepsilon > 0$, we set $N_\varepsilon = N$ if N is finite; otherwise, we choose N_ε in such a way that $j_1, j_2 \in N_\varepsilon$ and that

$$\sum_{j\in N\setminus N_\varepsilon}\left[\left(\int_{I_j}|w(x)-\max w|^p\,dx\right)^{1/p}+\left(\int_{I_j}|w'(x)|^p\,dx\right)^{1/p}\right]<\varepsilon\,. \tag{2.38}$$

Consider now a sequence of functions $w_h : \bigcup_{j\in N_\varepsilon}\overline{I}_j \to [0,\max w]$, $h\in\mathbb{N}$, such that, for every $j\in N_\varepsilon$, w_h is piecewise affine in I_j, agrees with w at the endpoints of I_j, and satisfies

$$\|w-w_h\|_{W^{1,p}(I_j)}<\frac{1}{h}\,. \tag{2.39}$$

On passing, if necessary, to a subsequence, we may assume that $w_h'(x)$ converges to $w'(x)$ for $\mathcal{L}^1$-a.e. $x\in\bigcup_{j\in N_\varepsilon} I_j$. Hence,

$$\limsup_{h\to+\infty}\chi_{\{|w_h'|\le\sigma\}}(x)\le\chi_{\{|w'|\le\sigma\}}(x)\qquad\text{for }\mathcal{L}^1\text{-a.e. } x\in\bigcup_{j\in N_\varepsilon}I_j\,.$$

On integrating this inequality, we obtain, by Fatou's lemma, that

$$\limsup_{h\to+\infty}\mathcal{L}^1\Big(\{x\in\cup_{j\in N_\varepsilon}I_j : |w_h'(x)|\le\sigma\}\Big)\le\mathcal{L}^1\Big(\{x\in\cup_{j\in N_\varepsilon}I_j : |w'(x)|\le\sigma\}\Big)\le m_w(\sigma)\,. \tag{2.40}$$

Notice that the last inequality holds since $w<\max w$ in A, and thus in $\cup_{j\in N_\varepsilon}I_j$. For every $h\in\mathbb{N}$, define

$$\overline{w}_h(x)=\begin{cases} w_h(x) & \text{if } x\in\bigcup_{j\in N_\varepsilon}I_j\\ \max w & \text{if } x\in(-l,l)\setminus\bigcup_{j\in N_\varepsilon}I_j\,.\end{cases}$$

Notice that each function $\overline{w}_h$ is piecewice affine and vanishes at $-l$ and l. Moreover, owing to (2.38)-(2.40), there exists h_ε such that, on setting $v_\varepsilon=\overline{w}_{h_\varepsilon}$, one has

$$\|w-v_\varepsilon\|_{W^{1,p}(-l,l)}<2\varepsilon \tag{2.41}$$

and

$$m_{v_\varepsilon}(\sigma)<m_w(\sigma)+\varepsilon\,. \tag{2.42}$$

By (2.41),

$$\lim_{\varepsilon\to0}\int_{-l}^{l}|v_\varepsilon'|^p\,dx=\int_{-l}^{l}|w'|^p\,dx\,. \tag{2.43}$$

Moreover, since we also have

$$\|v_\varepsilon^\star-w^\star\|_{L^p(-l,l)}\le\|v_\varepsilon-w\|_{L^p(-l,l)}\qquad\text{if } 1\le p\le\infty \tag{2.44}$$

(see e.g. [Ch]), it is easily verified that $v_\varepsilon^\star \rightharpoonup w^\star$ weakly in $W^{1,p}(-l,l)$, whence

$$\liminf_{\varepsilon\to0}\int_{-l}^{l}|v_\varepsilon^{\star\prime}|^p\,dx\ge\int_{-l}^{l}|w^{\star\prime}|^p\,dx\,. \tag{2.45}$$

From (2.43) and (2.45) we deduce that

$$\limsup_{\varepsilon\to 0} D_p(v_\varepsilon) \le D_p(w)\,. \tag{2.46}$$

By Part I applied to v_ε, we have that

$$\int_{-l}^{l} |v_\varepsilon(x) - v_\varepsilon^\star(x)|\,dx \le C_1 l \Big[l^{\frac{1}{p'}} D_p(v_\varepsilon)^{\frac{1}{p}} + \sigma m_{v_\varepsilon}(\sigma)\Big] \tag{2.47}$$

$$+ C_1 \sup v_\varepsilon \left[\frac{D_p(v_\varepsilon)}{\sigma^p} + m_{v_\varepsilon}(\sigma) + \frac{l^{\frac{1}{p'}} D_p(v_\varepsilon)^{\frac{1}{p}}}{\sigma} + \frac{\sqrt{D_p(v_\varepsilon)}}{\sigma^{\frac{p+1}{2}}}\sqrt{\sup v_\varepsilon}\right]$$

for every $\varepsilon > 0$. Thanks to (2.41) and (2.44), the left-hand side of (2.47) converges to the left-hand side of (2.3), and, by construction, $\sup v_\varepsilon = \sup w$. Hence, passing to the limit as $\varepsilon \to 0$ in (2.47), and making use of (2.42) and (2.46) imply (2.3). □

3 Proof of Theorem 1.1

Our approach to Theorem 1.1 rests upon a slicing technique, which enables us to apply Theorem 2.1 to the one-dimensional restrictions of u along the cross sections $\Omega_{x'}$. The underlying argument, that ultimately relies on Fubini's theorem and on classical results about one-dimensional restrictions of Sobolev functions, is not, however, entirely straightforward. The difficulties which arise are dealt with in separate lemmas.

A first problem to be faced is how to turn the piece of information contained in the Pólya–Szegö deficit $D_p(u)$ into an estimate for the quantity

$$\int_\Omega |\nabla_y u|^p dx - \int_\Omega |\nabla_y u^s|^p dx$$

involving the sole derivative along the y-axis. The idea is to introduce an auxiliary anisotropic functional, where the role of the derivative $\nabla_y u$ is emphasized, for which a generalized version of the Pólya–Szegö inequality is still available. The resulting inequality yields the desired control on $\int_\Omega |\nabla_y u|^p dx - \int_\Omega |\nabla_y u^s|^p dx$ in terms of $D_p(u)$. This is achieved in Lemma 3.3 below. The point is that, for the generalized Pólya–Szegö inequality to hold, the relevant functional has to be convex. It is exactly at this stage that the assumption $p \ge 2$ comes into play, as demonstrated by the following lemma.

Lemma 3.1 *Let $n \ge 2$ and let $p \ge 2$. Then the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$f(\xi) = |\xi|^p - \gamma|\xi_n|^p \qquad \text{for } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \tag{3.1}$$

is convex for every $\gamma \le \dfrac{1}{(p-1)^2}$.

Proof. Given any $\beta \ge 0$, define $g_\beta : \mathbb{R}^2 \to \mathbb{R}$ as

$$g_\beta(x,y) = \left(\beta^2 + x^2 + y^2\right)^{p/2} - \gamma\left(\beta^2 + y^2\right)^{p/2} \qquad \text{for } (x,y) \in \mathbb{R}^2\,.$$

Assume for a moment that $\beta > 0$. One clearly has that $\dfrac{\partial^2 g_\beta}{\partial x^2}(x,y) > 0$ for every $(x,y) \in \mathbb{R}^2$. Moreover, the Hessian determinant $Hg_\beta(x,y)$ of g_β satisfies

$$
\begin{aligned}
Hg_\beta(x,y) &= p^2\left(\beta^2+x^2+y^2\right)^{p-4}\Big\{\beta^4+(p-1)(x^4+y^4)+2(p-1)x^2y^2+p\beta^2(x^2+y^2) \\
&\qquad -\gamma\frac{(\beta^2+y^2)^{\frac{p-4}{2}}}{(\beta^2+x^2+y^2)^{\frac{p-4}{2}}}\left[\beta^2+(p-1)y^2\right]\left[\beta^2+(p-1)x^2+y^2\right]\Big\} \\
&\geq p^2\left(\beta^2+x^2+y^2\right)^{p-4}\Big\{\beta^4+x^4+y^4+2x^2y^2+2\beta^2x^2+2\beta^2y^2 \\
&\qquad -\gamma(p-1)^2\frac{(\beta^2+y^2)^{\frac{p-4}{2}}}{(\beta^2+x^2+y^2)^{\frac{p-4}{2}}}(\beta^2+y^2)(\beta^2+x^2+y^2)\Big\} \\
&= p^2\left(\beta^2+x^2+y^2\right)^{p-4}\left\{\left(\beta^2+x^2+y^2\right)^2-\gamma(p-1)^2\left(\beta^2+y^2\right)^{\frac{p-2}{2}}\left(\beta^2+x^2+y^2\right)^{\frac{6-p}{2}}\right\} \\
&= p^2\left(\beta^2+x^2+y^2\right)^{\frac{p-2}{2}}\left\{\left(\beta^2+x^2+y^2\right)^{\frac{p-2}{2}}-\gamma(p-1)^2\left(\beta^2+y^2\right)^{\frac{p-2}{2}}\right\}.
\end{aligned}
$$

The last expression in braces is nonnegative for every $(x,y) \in \mathbb{R}^2$ if $\gamma(p-1)^2 \leq 1$. Hence, g_β is convex for every $\gamma \leq \dfrac{1}{(p-1)^2}$ and for every $\beta > 0$. Consequently, g_0 is convex for the same values of γ too. Moreover, for each fixed $y \in \mathbb{R}$, the function $g_0(\cdot, y)$ is strictly increasing in $[0,+\infty)$. Combining these two properties of g_0 entails that the function given by

$$g_0(|\xi'|, \xi_n) \qquad \text{for } \xi \in \mathbb{R}^n\,,$$

where $\xi' = (\xi_1, \ldots, \xi_{n-1})$, is convex. Since $f(\xi) = g_0(|\xi'|, \xi_n)$ for $\xi \in \mathbb{R}^n$, the conclusion follows. $\square$

Remark 3.2 An analogous argument as in the proof of Lemma 3.1 shows that, if $1 < p < 2$, the function f defined by (3.1) is convex only in subsets of $\mathbb{R}^n$ contained in cylinders of the form $\{\xi : |\xi'| \leq R\}$ for some $R > 0$, provided that γ does not exceed a positive number depending on R (and on p). This observation suggests that counterexamples to Theorem 1.1 for $p < 2$ must involve families of functions whose derivatives with respect to the x' variables are not uniformly bounded – see Example 3.6 below.

Lemma 3.3 *Let $p \geq 2$, and let u be any function as in Theorem 1.1. Then*

$$\int_\Omega |\nabla_y u(x)|^p\,dx - \int_\Omega |\nabla_y u^s(x)|^p\,dx \leq D_p(u)(p-1)^2\,. \tag{3.2}$$

Proof. The function f defined as in (3.1), with $\gamma = \dfrac{1}{(p-1)^2}$, is convex by Lemma 3.1, and satisfies $f(0) = 0$ and

$$f(\xi_1, \ldots, \xi_{n-1}, -\xi_n) = f(\xi_1, \ldots, \xi_{n-1}, \xi_n) \qquad \text{for every } \xi \in \mathbb{R}^n\,.$$

Theorem 2.1 of [CF2] then ensures that

$$\int_\Omega f(\nabla u^s)\,dx \leq \int_\Omega f(\nabla u)\,dx\,. \tag{3.3}$$

Inequality (3.2) follows from (3.3) and (1.2). $\square$

The next step consists in deriving from (3.2) an estimate for $\int_{\Omega_{x'}} |\nabla_y u|^p dy - \int_{\Omega_{x'}} |\nabla_y u^s|^p dy$ for every $x' \in \pi(\Omega)$ outside a set on which the integral of u is small. This is the task of the following lemma.

Lemma 3.4 *Let $p \geq 2$, and let u be any function as in Theorem 1.1. Then there exists a measurable set $\Lambda \subset \pi(\Omega)$ such that*

$$\int_{\Omega_{x'}} |\nabla_y u(x',y)|^p \, dy \leq \int_{\Omega_{x'}} |\nabla_y u^s(x',y)|^p \, dy + \sqrt{D_p(u)} \qquad \textit{for every } x' \in \pi(\Omega) \setminus \Lambda \tag{3.4}$$

and

$$\int_{\Lambda \times \mathbb{R}} u(x) \, dx \leq (2L)^{2-\frac{1}{p}} (p-1)^{\frac{2}{p'}} \|\nabla_y u\|_{L^p(\Omega)} D_p(u)^{\frac{1}{2p'}} . \tag{3.5}$$

Proof. A classical result on restrictions of Sobolev functions ensures that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$,

$$u(x', \cdot) \in W_0^{1,p}(\Omega_{x'}) \qquad \text{and} \qquad \nabla_y u(x',y) = \left[\frac{d}{dy} u(x', \cdot)\right](y) \qquad \text{for } \mathcal{L}^1\text{-a.e. } y \in \Omega_{x'} \tag{3.6}$$

(see e.g. [AFP, Theorem 3.107]). Since, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$, $u^s(x',y) = u(x',\cdot)^\star(y)$ for $\mathcal{L}^1$-a.e. $y \in \Omega_{x'}$, then the Pólya–Szegö inequality for Schwarz symmetrization of one-dimensional Sobolev functions tells us that

$$\int_{\Omega_{x'}} |\nabla_y u^s(x',y)|^p \, dy \leq \int_{\Omega_{x'}} |\nabla_y u(x',y)|^p \, dy \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \pi(\Omega) . \tag{3.7}$$

Now, set

$$U = \left\{ x' \in \pi(\Omega) : \int_{\Omega_{x'}} |\nabla_y u(x',y)|^p dy \leq \int_{\Omega_{x'}} |\nabla_y u^s(x',y)|^p dy + \sqrt{D_p(u)} \right\}$$

and choose $\Lambda = \pi(\Omega) \setminus U$. Thus, (3.4) holds by definition.
As for (3.5), we have that

$$\mathcal{L}^{n-1}(\Lambda)\sqrt{D_p(u)} \leq \int_\Lambda dx' \int_{\Omega_{x'}} \Big(|\nabla_y u|^p - |\nabla_y u^s|^p\Big) dy \leq \int_\Omega |\nabla_y u|^p \, dx - \int_\Omega |\nabla_y u^s|^p \, dx \leq D_p(u)(p-1)^2 ,$$

where the second inequality is due to Fubini's theorem and to (3.7), and the third one to Lemma 3.3. Hence, $\mathcal{L}^{n-1}(\Lambda) \leq (p-1)^2\sqrt{D_p(u)}$. Owing to (3.6), we can then easily conclude that

$$\begin{aligned} \int_{\Lambda \times \mathbb{R}} u(x) \, dx &= \int_\Lambda dx' \int_{\Omega_{x'}} u(x',y) \, dy \leq 2L \int_\Lambda dx' \int_{\Omega_{x'}} |\nabla_y u| \, dy \\ &\leq (2L)^{2-\frac{1}{p}} \left(\int_\Omega |\nabla_y u|^p \, dx\right)^{\frac{1}{p}} \left(\mathcal{L}^{n-1}(\Lambda)\right)^{\frac{1}{p'}} , \end{aligned} \tag{3.8}$$

whence (3.5) follows. □

Similarly, Lemma 3.5 below shows how an estimate for $\mathcal{L}^1(\{y \in \Omega_{x'} : |\nabla_y u(x',y)| \leq \sigma, u(x',y) < M(x')\})$ is inherited from (1.3) for every $x' \in \pi(\Omega)$ outside a set where the integral of u is small.

Lemma 3.5 *Let u be any function as in Theorem 1.1. Then, for every $\sigma > 0$ there exists a measurable set $\Theta_\sigma \subset \pi(\Omega)$ such that*

$$\mathcal{L}^1(\{y \in \Omega_{x'} : |\nabla_y u(x',y)| \leq \sigma, u(x',y) < M(x')\}) \leq \sqrt{m_u(\sigma)} \qquad \textit{for every } x' \in \pi(\Omega) \setminus \Theta_\sigma , \tag{3.9}$$

and

$$\int_{\Theta_\sigma \times \mathbb{R}} u(x) \, dx \leq (2L)^{2-\frac{1}{p}} \|\nabla_y u\|_{L^p(\Omega)} m_u(\sigma)^{\frac{1}{2p'}} . \tag{3.10}$$

Proof. Set $D_\sigma = \{(x', y) \in \Omega : |\nabla_y u(x', y))| \le \sigma, u(x', y) < M(x')\}$, so that $m_u(\sigma) = \mathcal{L}^n(D_\sigma)$, and define

$$V_\sigma = \Big\{x' \in \pi(\Omega) : \mathcal{L}^1((D_\sigma)_{x'}) \le \sqrt{m_u(\sigma)}\Big\}.$$

The choice $\Theta_\sigma = \pi(\Omega) \setminus V_\sigma$ makes (3.9) automatically fulfilled. On the other hand, by Fubini's theorem,

$$\mathcal{L}^{n-1}(\Theta_\sigma)\sqrt{m_u(\sigma)} < \int_{\Theta_\sigma} \mathcal{L}^1((D_\sigma)_{x'})\, dx' \le \int_\Omega \chi_{D_\sigma}(x)\, dx = m_u(\sigma)\,,$$

whence $\mathcal{L}^{n-1}(\Theta_\sigma) \le \sqrt{m_u(\sigma)}$. An analogous chain of inequalities as in (3.8) then yields (3.10). □

We can now accomplish the proof of Theorem 1.1.

Proof of Theorem 1.1. If $D_p(u) \ge 1$, then, by (3.6),

$$\begin{aligned}\int_\Omega |u - u^s| dx \;&\le\; 2\int_\Omega u(x)dx = 2\int_{\pi(\Omega)} dx' \int_{\Omega_{x'}} u(x', y)dy \\ &\le\; 2\int_{\pi(\Omega)} \mathcal{L}^1(\Omega_{x'})dx' \int_{\Omega_{x'}} |\nabla_y u(x', y)|dy \le 4L\int_\Omega |\nabla_y u(x)|dx \le 4L\Big[\mathcal{L}^n(\Omega)\Big]^{\frac{1}{p'}} \|\nabla_y u\|_{L^p(\Omega)},\end{aligned}$$

whence (1.4) follows.
Assume now that $D_p(u) < 1$. From (3.4), (3.9) and the one-dimensional estimate (2.3) applied with $w(\cdot) = u(x', \cdot)$ and $(-l, l) = \Omega_{x'}$, we deduce that there exists a constant c_1, depending only on p, such that

$$\begin{aligned}(3.11)\qquad &\int_{\Omega_{x'}} |u(x', y) - u^s(x', y)|\, dy \le c_1 \mathcal{L}^1(\Omega_{x'}) \Big[\Big(\mathcal{L}^1(\Omega_{x'})\Big)^{\frac{1}{p'}} D_p(u)^{\frac{1}{2p}} + \sigma\sqrt{m_u(\sigma)}\Big] \\ &+ c_1 M(x')\Big(\frac{\sqrt{D_p(u)}}{\sigma^p} + \sqrt{m_u(\sigma)} + \frac{(\mathcal{L}^1(\Omega_{x'}))^{\frac{1}{p'}} D_p(u)^{\frac{1}{2p}}}{\sigma}\Big) + c_1 M(x')^{\frac{3}{2}} \frac{D_p(u)^{\frac{1}{4}}}{\sigma^{\frac{p+1}{2}}}\end{aligned}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega) \setminus (\Lambda \cup \Theta_\sigma)$. Observe that

$$M(x') \le \int_{\Omega_{x'}} |\nabla_y u(x', y)|\, dy \quad \text{and} \quad \mathcal{L}^1(\Omega_{x'}) \le 2L$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$, and

$$\int_{\pi(\Omega)\setminus(\Lambda\cup\Theta_\sigma)} \mathcal{L}^1(\Omega_{x'})\, dx' \le \mathcal{L}^n(\Omega)\,.$$

Thus, on integrating both sides of inequality (3.11) on $\pi(\Omega) \setminus (\Lambda \cup \Theta_\sigma)$ yields

$$\begin{aligned}(3.12)\quad &\int_{[\pi(\Omega)\setminus(\Lambda\cup\Theta_\sigma)]\times\mathbb{R}} |u(x) - u^s(x)|\, dx \le c_2\Big[D_p(u)^{\frac{1}{2p}} + \sigma\sqrt{m_u(\sigma)} \\ &+ \|\nabla_y u\|_{L^p(\Omega)}\Big(\frac{\sqrt{D_p(u)}}{\sigma^p} + \sqrt{m_u(\sigma)} + \frac{D_p(u)^{\frac{1}{2p}}}{\sigma}\Big) + \|\nabla_y u\|^{3/2}_{L^p(\Omega)} \frac{D_p(u)^{\frac{1}{4}}}{\sigma^{\frac{p+1}{2}}}\Big],\end{aligned}$$

for some constant c_2 depending only on p, L and $\mathcal{L}^n(\Omega)$. On the other hand, by (3.5) and (3.10), there exists a constant c_3, depending only on p and L, such that

$$(3.13)\quad \int_{(\Lambda\cup\Theta_\sigma)\times\mathbb{R}} |u(x) - u^s(x)|\, dx \le 2\int_{(\Lambda\cup\Theta_\sigma)\times\mathbb{R}} u(x)\, dx \le c_3 \|\nabla_y u\|_{L^p(\Omega)} \Big(D_p(u)^{\frac{1}{2p'}} + m_u(\sigma)^{\frac{1}{2p'}}\Big)\,.$$

Since $D_p(u) < 1$ and $p \geq 2$, then $D_p(u)^{\frac{1}{2p'}} \leq D_p(u)^{\frac{1}{2p}}$ and $D_p(u)^{\frac{1}{4}} \leq D_p(u)^{\frac{1}{2p}}$. Moreover, $\sqrt{m_u(\sigma)} \leq c_4 m_u(\sigma)^{\frac{1}{2p'}}$, for a suitable constant c_4, depending only on $\mathcal{L}^n(\Omega)$. Therefore, (3.12) and (3.13) entail that

$$\int_\Omega |u(x) - u^s(x)|dx \leq c_5 \Big[D_p(u)^{\frac{1}{2p}} + \sigma\sqrt{m_u(\sigma)}$$
$$+ \|\nabla_y u\|_{L^p(\Omega)} \Big(\frac{\sqrt{D_p(u)}}{\sigma^p} + \frac{D_p(u)^{\frac{1}{2p}}}{\sigma} + D_p(u)^{\frac{1}{2p}} + m_u(\sigma)^{\frac{1}{2p'}} \Big) + \|\nabla_y u\|^{3/2}_{L^p(\Omega)} \frac{D_p(u)^{\frac{1}{2p}}}{\sigma^{\frac{p+1}{2}}} \Big]$$

for some constant c_5 depending only on p, L and $\mathcal{L}^n(\Omega)$. Hence, (1.4) follows, since

$$\frac{\sqrt{D_p(u)}}{\sigma^p} + \frac{D_p(u)^{\frac{1}{2p}}}{\sigma} \leq 2\Big(\frac{D_p(u)^{\frac{1}{2p}}}{\sigma^p} + D_p(u)^{\frac{1}{2p}} \Big),$$

inasmuch as $D_p(u) < 1$. □

We conclude with an example demonstrating the necessity of the condition $p \geq 2$ in Theorem 1.1.

Example 3.6 Let $n = 2$ and let $\Omega = (-1, 1) \times (-1, 1)$. Let $a : \mathbb{R} \to [0, 1]$ be the function given by $a(x) = 1 - |x|$ for $|x| \leq 1$, and extended periodically outside $[-1, 1]$. Let $b : [-1, 1] \to \mathbb{R}$ be defined by $b(y) = 1 - |y|$ if $1/2 \leq |y| \leq 1$, $b(y) = -2y + 3/2$ if $1/4 \leq y \leq 1/2$ and $b(y) = 2y/3 + 5/6$ if $-1/2 \leq y \leq 1/4$. Note that $b(y) = b^\star(y)$ if $y \in [-2, -1/2] \cup [1/2, 1]$, and $b(y) \neq b^\star(y)$ if $y \in [-1/2, 1/2]$, but

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} b(y)^p \, dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} b^\star(y)^p \, dy \qquad \text{for every } p \geq 1\,. \tag{3.14}$$

Consider the sequence $\{v_h\}_{h\in\mathbb{N}}$ of functions $v_h : \Omega \to [0, +\infty)$ defined as

$$v_h(x, y) = (2x + 3 + a(hx))b(y) \qquad \text{for } (x, y) \in \Omega\,.$$

Clearly, v_h is nonnegative and belongs to $W^{1,p}_{0,y}(\Omega)$ for every $p \geq 1$ and every $h \in \mathbb{N}$. Moreover,

$$v_h^s(x, y) = (2x + 3 + a(hx))b^\star(y) \qquad \text{for } (x, y) \in \Omega\,.$$

We claim that

$$\lim_{h\to+\infty} D_p(v_h) = 0 \qquad \text{if } p < 2\,. \tag{3.15}$$

Indeed

$$\begin{aligned}
(3.16)\; D_p(v_h) &= \iint_\Omega |\nabla v_h|^p \, dxdy - \iint_\Omega |\nabla v_h^s|^p \, dxdy \\
&= \iint_\Omega \Big[(2 + ha'(hx))^2 b(y)^2 + (2x + 3 + a(hx))^2 b'(y)^2 \Big]^{\frac{p}{2}} dxdy \\
&\quad - \iint_\Omega \Big[(2 + ha'(hx))^2 b^\star(y)^2 + (2x + 3 + a(hx))^2 b^{\star\prime}(y)^2 \Big]^{\frac{p}{2}} dxdy \\
&= \iint_{(-1,1)\times(-\frac{1}{2},\frac{1}{2})} |2 + ha'(hx)|^p b(y)^p \Big[1 + \frac{(2x + 3 + a(hx))^2 b'(y)^2}{(2 + ha'(hx))^2 b(y)^2} \Big]^{\frac{p}{2}} dxdy \\
&\quad - \iint_{(-1,1)\times(-\frac{1}{2},\frac{1}{2})} |2 + ha'(hx)|^p b^\star(y)^p \Big[1 + \frac{(2x + 3 + a(hx))^2 b^{\star\prime}(y)^2}{(2 + ha'(hx))^2 b^\star(y)^2} \Big]^{\frac{p}{2}} dxdy\,.
\end{aligned}$$

Observe that a constant c_1, depending only on b, exists such that for every $h > 2$

$$(3.17)\qquad \frac{(2x+3+a(hx))|b'(y)|}{|2+ha'(hx)|b(y)} \leq \frac{c_1}{h} \qquad \text{and} \qquad \frac{(2x+3+a(hx))|b^{\star\prime}(y)|}{|2+ha'(hx)|b^\star(y)} \leq \frac{c_1}{h} \qquad \text{if } y \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

From (3.16)-(3.17) we deduce that

$$D_p v_h(u) \leq \iint_{(-1,1)\times(-\frac{1}{2},\frac{1}{2})} |2+ha'(hx)|^p [b(y)^p - b^\star(y)^p] dxdy + \frac{c_2}{h^{2-p}}$$

for some positive constant c_2 independent of h. Since, by (3.14),

$$\iint_{(-1,1)\times(-\frac{1}{2},\frac{1}{2})} |2+ha'(hx)|^p [b(y)^p - b^\star(y)^p] dxdy = \int_{-1}^{1} |2+ha'(hx)|^p dx \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} b(y)^p\, dy - \int_{-\frac{1}{2}}^{\frac{1}{2}} b^\star(y)^p\, dy \right) = 0\,,$$

then (3.15) follows. Now, we have

$$\{(x,y) \in \Omega : |\nabla_y v_h(x,y)| \leq \sigma\} \subset (-1,1) \times \{y \in (-1,1) : |b'(y)| \leq \sigma\} \qquad \text{for } \sigma \geq 0\,,$$

whence $m_{v_h}(\sigma) = 0$ if $\sigma < 2/3$. Furthermore,

$$\iint_\Omega |\nabla_y v_h|^p\, dxdy = \iint_\Omega (2x+3+a(hx))^p |b'(y)|^p\, dxdy \leq 2\cdot 6^p \int_{-1}^{1} |b'(y)|^p\, dy$$

for every $h \in \mathbb{N}$, and therefore $\|\nabla_y v_h\|_{L^p(\Omega)}$ is uniformly bounded for $h \in \mathbb{N}$. On the other hand,

$$\iint_\Omega |v_h - v_h^s|\, dxdy = \iint_\Omega (2x+3+a(hx))|b(y) - b^\star(y)|\, dxdy \geq 2\int_{-1}^{1} |b(y) - b^\star(y)|\, dy > 0\,,$$

independently of h. Consequently, if $p < 2$ no estimate for the left-hand side of (1.4), which approaches 0 when $D_p(u)$ and $m_u(\sigma)$ go to 0 and $\|\nabla_y u\|_{L^p(\Omega)}$ remains bounded, can hold.

References

[AFLT] A.Alvino, V.Ferone, P.L.Lions & G.Trombetti, *Convex symmetrization and rearrangements,* Ann. Inst. H. Poincaré, Anal. Non Linéaire **14** (1997), 275–293.

[AFP] L.Ambrosio, N.Fusco & D.Pallara, *Functions of bounded variation and free discontinuity problems,* Oxford University Press, Oxford (2000)

[Bae] A.Baernstein II, *A unified approach to symmetrization,* in Partial differential equations of ellyptic type, A. Alvino, E. Fabes & G. Talenti eds., Symposia Math. 35, Cambridge Univ. Press, (1994)

[Bo] T.Bonnesen, *Über die isoperimetrische Defizit ebener Figuren,* Math. Ann., **91** (1924), 252–268

[Br] F.Brock, *Weighted Dirichlet-type inequalities for Steiner Symmetrization,* Calc. Var., **8** (1999), 15–25.

[BZ] J.Brothers & W.Ziemer, *Minimal rearrangements of Sobolev functions,* J. Reine Angew. Math. **384** (1988), 153–179

[Bu] A.Burchard, *Steiner symmetrization is continuous in $W^{1,p}$,* Geom. Funct. Anal. **7** (1997), 823–860

[Ch] G.Chiti, *Rearrangements of functions and convergence in Orlicz spaces*, Appl. Anal. **9** (1979), 23–27

[C] A.Cianchi, *Second order derivatives and rearrangements*, Duke Math. J. **105** (2000), 355–385

[CF1] A.Cianchi & N.Fusco, *Functions of bounded variations and rearrangements,* Arch. Rat. Mech. Anal. **165** (2002), 1–40

[CF2] A.Cianchi & N.Fusco, *Steiner symmetric extremals in Pólya–Szegö type inequalities,* preprint (2003)

[CF3] A.Cianchi & N.Fusco, *Minimal rearrangements, strict convexity and critical points,* Appl. Anal., to appear.

[FV] A.Ferone & R.Volpicelli, *Convex symmetrization: the equality case in the Pólya–Szegö inequality,* Calc. Var. and PDE's, to appear

[Fu] B.Fuglede, *Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$,* Trans. Amer. Math. Soc. **314** (1989), 619–638

[H] R.R.Hall *A quantitative isoperimetric inequality in n-dimensional space*, J. reine angew. Math. **428** (1992), 161–196

[HHW] R.R.Hall, W.K.Hayman & A.W. Weitsman *On asymmetry and capacity*, J. Anal. Math. **56** (1991), 87–123

[Hi] K. Hilden, *Symmetrization of functions in Sobolev spaces and the isoperimetric inequality*, Manus. Math. **18** (1976), 215–235

[K1] B.Kawohl, *Rearrangements and convexity of level sets in PDE,* Lecture Notes in Math. **1150**, Springer-Verlag, Berlin (1985)

[K2] B.Kawohl, *On the isoperimetric nature of a rearrangement inequality and its consequences for some variational problems* , Arch. Rat. Mech. Anal. **94** (1986), 227–243

[Ko] V.I.Kolyada, *Rearrangements of functions and embedding theorems,* Uspekhi Mat. Nauk **44** (1989), 61–95; english translation in: Russian Math. Surveys **44** (1989), 73–117.

[PS] G.Pòlya & G.Szegö, *Isoperimetric inequalities in Mathematical Physics,* Ann. of Math. Studies, **27**, Princeton University Press, Princeton, (1951)

[S] E. Sperner, *Symmetrisierung für Funktionen mehrerer reeller Variablen,* Manus. Math. **11** (1974), 159-170.

[T] G.Talenti, *Best constant in Sobolev inequality,* Ann. Mat. Pura Appl. **110** (1976), 353–372

[U] A.Uribe, *Minima of the Dirichlet norm and Toeplitz operators,* preprint (1985)