Potential estimates and quasilinear parabolic equations with measure data

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Contents

Abstract

In this memoir, we study the existence and regularity of the quasilinear parabolic equations:

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = B(u,\nabla u) + \mu,$$

in either $\mathbb{R}^{N+1}$ or $\mathbb{R}^N \times (0,\infty)$ or on a bounded domain $\Omega \times (0,T) \subset \mathbb{R}^{N+1}$ where $N \geq 2$. In this paper, we shall assume that the nonlinearity A fulfills standard growth conditions, the function B is a continuous and μ is a radon measure. Our first task is to establish the existence results with $B(u,\nabla u) = \pm|u|^{q-1}u$, for $q > 1$. We next obtain global weighted-Lorentz, Lorentz-Morrey and Capacitary estimates on gradient of solutions with $B \equiv 0$, under minimal conditions on the boundary of domain and on nonlinearity A. Finally, due to these estimates, we solve the existence problems with $B(u,\nabla u) = |\nabla u|^q$ for $q > 1$.

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CHAPTER 1

Introduction and main results

1.1. Introduction

In this memoir, we study a class of quasilinear parabolic equations:

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = B(x,t,u,\nabla u) + \mu \tag{1.1}$$

in $\mathbb{R}^{N+1}$ or $\mathbb{R}^N \times (0,\infty)$ or on a bounded domain $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{N+1}$ where $N \geq 2$. In this paper, μ is a Radon measure, $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies

$$|A(x,t,\zeta)| \leq \Lambda_1 |\zeta| \text{ and} \tag{1.2}$$

$$\langle A(x,t,\zeta) - A(x,t,\lambda), \zeta - \lambda \rangle \geq \Lambda_2 |\zeta - \lambda|^2, \tag{1.3}$$

for every $(\lambda,\zeta) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, for some $\Lambda_1, \Lambda_2 > 0$; and $B : \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is also a Carathéodory function.

The regularity and singularity theory for the parabolic quasilinear equation (1.1) were studied and developed intensely over the past 50 years in [**65, 51, 36, 55, 56, 31, 57, 67, 95, 87, 85**]. Moreover, we also refer to [**23**]-[**26**] for L^p-gradient estimates theory in non-smooth domains and [**71**] for Wiener criteria to existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

First, we are specially interested in the existence of solutions to quasilinear parabolic equations with absorption, source terms and measure data:

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu, \quad \text{(absorption term)} \tag{1.4}$$

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu, \quad \text{(source term)} \tag{1.5}$$

in $\mathbb{R}^{N+1}$ and

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu, \quad u(0) = \sigma, \quad \text{(absorption term)} \tag{1.6}$$

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu, \quad u(0) = \sigma, \quad \text{(source term)} \tag{1.7}$$

in $\mathbb{R}^N \times (0,\infty)$ or a bounded domain $\Omega_T \subset \mathbb{R}^{N+1}$, where $q > 1$ and μ, σ are Radon measures.

The linear case $A(x,t,\nabla u) = \nabla u$ was studied in detail by Fujita, Brezis and Friedman, Baras and Pierre.

For the absorption case, in [**22**], they showed that if $\mu = 0$ and σ is a Dirac mass in Ω, the problem (1.6) in Ω_T (with Dirichlet boundary condition) admits a (unique) solution if and only if $q < (N+2)/N$. Then, optimal results had been obtained in [**6**]. They proved that for any $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$ there exists a unique solution of (1.6) in Ω_T if and only if $\mu \ll \mathrm{Cap}_{2,1,q'}$ and $\sigma \ll \mathrm{Cap}_{\mathbf{G}_{2/q},q'}$ i.e μ, σ are absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,q'}, \mathrm{Cap}_{\mathbf{G}_{2/q},q'}$ (in

$\Omega_T, \Omega)$ respectively. Here q' is the conjugate exponent of q and these two capacities will be defined in section 2.

For the source case, in [**7**], they showed that for any $\mu \in \mathfrak{M}_b^+(\Omega_T)$ and $\sigma \in \mathfrak{M}_b^+(\Omega)$, the problem (1.7) in bounded domain Ω_T has a nonnegative solution if

$$\mu(E) \le C\text{Cap}_{2,1,q'}(E) \quad \text{and} \quad \sigma(O) \le C\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. Here $C = C(N, \text{diam}(\Omega), T)$ is small enough. Conversely, the existence holds then for compact subset $K \subset\subset \Omega$, one find $C_K > 0$ such that

$$\mu(E \cap (K \times [0,T])) \le C_K\text{Cap}_{2,1,q'}(E) \quad \text{and} \quad \sigma(O \cap K) \le C_K\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O)$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. In unbounded domain $\mathbb{R}^N \times (0,\infty)$, in [**36**] they asserted that an inequality

$$u_t - \Delta u \ge u^q, u \ge 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.8}$$

- **i.:** if $q < (N+2)/N$ then the only nonnegative global (in time) solution of above inequality is $u \equiv 0$,
- **ii.:** if $q > (N+2)/N$ then there exists global positive solution of above inequality.

More generally, in [**7**], for $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)$, the equation (1.7) has a nonnegative solution in $\mathbb{R}^N \times (0,\infty)$ (with $A(x,t,\nabla u) = \nabla u$) if and only if

$$\mu(E) \le C\text{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{and} \quad \sigma(O) \le C\text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O) \tag{1.9}$$

hold for every compact set $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$. Here $C = C(N,q)$ is small enough, two capacities $\text{Cap}_{\mathcal{H}_2,q'}, \text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$ will be defined in section 1.2. Note that a necessary and sufficient condition for (1.9) to hold with $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))\backslash\{0\}$ or $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)\backslash\{0\}$ is $q \ge (N+2)/N$. In particular, (1.8) has a (global) positive solution if and only if $q \ge (N+2)/N$. It is known that conditions for data μ, σ in problems with absorption are softer than those in problems with source. Recently, the exponential case, i.e $|u|^{q-1}u$ is replaced by $P(u) \sim \exp(a|u|^q)$, for $a > 0$ and $q \ge 1$ was considered in [**68**].

We consider (1.6) and (1.7) in Ω_T with Dirichlet boundary conditions when $\text{div}(A(x,t,\nabla u))$ is replaced by $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ for $p \in (2 - 1/N, N)$. In [**77**], they showed that for any $q > p - 1$, (1.6) admits a (unique renormalized) solution provided $\sigma \in L^1(\Omega)$ and $\mu \in \mathfrak{M}_b(\Omega_T)$ is diffuse measure i.e absolutely continuous with respect to C_p-parabolic capacity in Ω_T defined on a compact set $K \subset \Omega_T$:

$$C_p(K, \Omega_T) = \inf\left\{||\varphi||_X : \varphi \ge \chi_K, \varphi \in C_c^\infty(\Omega_T)\right\},$$

where $X = \left\{\varphi : \varphi \in L^p(0,T; W_0^{1,p}(\Omega)), \varphi_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))\right\}$ endowed with norm $||\varphi||_X = ||\varphi||_{L^p(0,T;W_0^{1,p}(\Omega))} + ||\varphi_t||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$ and χ_K is the characteristic function of K. An improving result was presented in [**15**] for measures that have good behavior in time, it is based on results of [**17**] relative to the elliptic case. That is, (1.6) has a (renormalized) solution for $q > p-1$ if $\sigma \in L^1(\Omega)$ and $|\mu| \le f + \omega \otimes F$. Here $f \in L^1_+(\Omega_T), F \in L^1_+((0,T))$ and $\omega \in \mathfrak{M}_b^+(\Omega)$ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$ in Ω. Also, (1.7) has a (renormalized) nonnegative solution if $\sigma \in L^\infty_+(\Omega)$, $0 \le \mu \le \omega \otimes \chi_{(0,T)}$ with $\omega \in \mathfrak{M}_b^+(\Omega)$ and

$$\omega(E) \le C_1 \mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) \quad \forall \text{ compact } E \subset \mathbb{R}^N, \quad ||\sigma||_{L^\infty(\Omega)} \le C_2$$

for some C_1, C_2 small enough. Another improving results are also stated in [**16**], especially if $q > p-1$, $p > 2$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$ are absolutely continuous with respect to $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω then (1.6) has a distributional solution.

In [**16**], we also obtain the existence of solutions for the porous medium equation with absorption and data measure. More precisely, for $q > m > \frac{N-2}{N}$, a sufficient condition to have existence of solution to the problem

$$u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \text{ in } \Omega_T,$$
$$u = 0 \text{on } \partial\Omega \times (0,T) \quad \text{ and } u(0) = \sigma \text{ in } \Omega,$$

is $\mu \ll \mathrm{Cap}_{2,1,q'}$, $\sigma \ll \mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ if $m \ge 1$ and $\mu \ll \mathrm{Cap}_{\mathcal{G}_2,\frac{2q}{2(q-1)+N(1-m)}}$, $\sigma \ll \mathrm{Cap}_{\mathbf{G}_{\frac{2-N(1-m)}{q}},\frac{2q}{2(q-1)+N(1-m)}}$ if $\frac{N-2}{N} < m \le 1$. A necessary condition is $\mu \ll \mathrm{Cap}_{2,1,\frac{q}{q-\max\{m,1\}}}$ and $\sigma \ll \mathrm{Cap}_{\mathbf{G}_{\frac{2\max\{m,1\}}{q}},\frac{q}{q-\max\{m,1\}}}$. Moreover, if $\mu = \mu_1 \otimes \chi_{[0,T]}$ with $\mu_1 \in \mathfrak{M}_b(\Omega)$ and $\sigma \equiv 0$ then a condition $\mu_1 \ll \mathrm{Cap}_{\mathbf{G}_2,\frac{q}{q-m}}$ is not only sufficient but also also necessary for existence of solutions to the above problem.

We would like to make a brief survey on quasilinear elliptic equations with absorption, source terms and data measure. Namely, we shall study the following equations:

$$-\Delta_p u + |u|^{q-1}u = \omega, \tag{1.10}$$

$$-\Delta_p u = |u|^{q-1}u + \omega, u \ge 0, \tag{1.11}$$

in Ω with Dirichlet boundary conditions where $1 < p < N$, $q > p-1$. In [**17**], we proved that the existence solution of equation (1.10) holds if $\omega \in \mathfrak{M}_b(\Omega)$ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}$. Moreover, a necessary condition for existence was also showed in [**11, 12**]. For the problem with source term, it was solved in [**4, 7**] for $p = 2$ (also see [**18**]) and [**79**] for any $1 < p \le N$ (also see [**80**]). More precisely, if $\omega \in \mathfrak{M}_b^+(\Omega)$ has compact support in Ω, then a sufficient and necessary condition for the existence of solutions to the problem (1.11) is

$$\omega(E) \le C \mathrm{Cap}_{\mathbf{G}_p,\frac{q}{q-p+1}}(E) \qquad \text{for all compact sets } E \subset \Omega,$$

where C is a constant depending only on N, p, q and $d(\mathrm{supp}(\omega), \partial\Omega)$. Their construction is based upon sharp estimates on the solutions of the problem

$$-\Delta_p u = \omega \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

for nonnegative Radon measures ω in Ω together with a deep analysis of the Wolff potential.
Corresponding results in the case where u^q term is changed by $P(u) \approx \exp(au^\lambda)$ for $a > 0, \lambda > 0$, were given in [**17, 69**].

There are many works for the Riccati equation

$$-\Delta_p u = |\nabla u|^q + \omega \tag{1.12}$$

in Ω with Dirichlet boundary conditions where $1 < p \leq N$ and $q > p-1$. This problem was firstly studied in [**43**] in the case $p = 2$. They proved that the problem has a solution if and only if

$$\omega(E) \leq C\text{Cap}_{\mathbf{G}_1,\frac{q}{q-p+1}}(E) \qquad \text{for all compact sets } E,$$

for some $C > 0$. Then, in [**82, 84, 83**] they extended this result to $2-\frac{1}{N} < p \leq N$, see also [**20**]. Recently, in [**72, 73, 74, 75**], the authors considered this problem in the singular case $1 < p \leq 2 - \frac{1}{N}$.
In [**33**], Duzaar and Mingione gave a local pointwise estimate from above of the solutions to the equation

$$u_t - \text{div}(A(x,t,\nabla u)) = \mu, \tag{1.13}$$

in Ω_T involving the Wolff parabolic potential $\mathbb{I}_2[|\mu|]$ defined by

$$\mathbb{I}_2[|\mu|](x,t) = \int_0^\infty \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} \qquad \text{for all } (x,t) \in \mathbb{R}^{N+1},$$

here $\tilde{Q}_\rho(x,t) := B_\rho(x) \times (t-\rho^2/2, t+\rho^2/2)$. Specifically if $u \in L^2(0,T;H^1(\Omega)) \cap C(\Omega_T)$ is a weak solution to the above equation with data $\mu \in L^2(\Omega_T)$, then

$$|u(x,t)| \leq C \fint_{\tilde{Q}_R(x,t)} |u| + C\int_0^{2R} \frac{|\mu|(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho}, \tag{1.14}$$

for any $Q_{2R}(x,t) := B_{2R}(x) \times (t-(2R)^2, t) \subset \Omega_T$, where a constant C only depends on N and the structure of operator A. In this paper we show that if $u \geq 0, \mu \geq 0$ we also have local pointwise estimate from below:

$$u(y,s) \gtrsim \sum_{k=0}^\infty \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128}r_k^2))}{r_k^N}, \tag{1.15}$$

for any $Q_r(y,s) \subset \Omega_T$, where $r_k = 4^{-k}r$ (see section 2.2).

From the preceding two inequalities, we obtain global pointwise estimates of solution to (1.13). For example, if $\mu \in \mathfrak{M}(\mathbb{R}^{N+1})$ with $\mathbb{I}_2[|\mu|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then there exists a distributional solution to (1.13) in $\mathbb{R}^{N+1}$ such that

$$-K\mathbb{I}_2[\mu^-](x,t) \leq u(x,t) \leq K\mathbb{I}_2[\mu^+](x,t) \quad \text{for a.e } (x,t) \in \mathbb{R}^{N+1}, \tag{1.16}$$

and we emphasize that if $u \geq 0, \mu \geq 0$ then

$$u(x,t) \geq K^{-1}\sum_{k=-\infty}^\infty \frac{\mu(Q_{2^{-2k-3}}(x, t - 35\times 2^{-4k-7}))}{2^{-2Nk}} \quad \text{for a.e } (x,t) \in \mathbb{R}^{N+1},$$

and for $q > 1$,

$$||u||_{L^q(\mathbb{R}^{N+1})} \sim ||\mathbb{I}_2[\mu]||_{L^q(\mathbb{R}^{N+1})}.$$

Where the constant K depends only on N and on the structure of the operator A.

Our first aim is to verify that,

i: problems (1.4) and (1.6) have solutions if μ, σ are absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$, $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ respectively,

ii: problems (1.5) in $\mathbb{R}^{N+1}$ and (1.7) in $\mathbb{R}^N \times (0,\infty)$ with signed measure data μ, σ admit a solution if

$$|\mu|(E) \le C\mathrm{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{and} \quad |\sigma|(O) \le C\mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O) \tag{1.17}$$

holds for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$ for some $C > 0$. Also, the equation (1.7) in a bounded domain Ω_T has a solution if (1.17) holds where capacities $\mathrm{Cap}_{2,1,q'}$, $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ are exploited instead of $\mathrm{Cap}_{\mathcal{H}_2,q'}$, $\mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$.

It is worth mentioning that the solutions of (1.5) in $\mathbb{R}^{N+1}$ and (1.7) in $\mathbb{R}^N \times (0,\infty)$ that we have obtained obey

$$\int_E |u|^q dxdt \lesssim \mathrm{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{for all compact } E \subset \mathbb{R}^{N+1},$$

and we also have an analogous estimate for a solution of (1.7) in Ω_T;

$$\int_E |u|^q dxdt \lesssim \mathrm{Cap}_{2,1,q'}(E) \quad \text{for all compact } E \subset \mathbb{R}^{N+1}.$$

In the case $\mu \equiv 0$, solutions of (1.7) in $\mathbb{R}^N \times (0,\infty)$ and Ω_T verify the decay estimate

$$-Ct^{-\frac{1}{q-1}} \le \inf_x u(x,t) \le \sup_x u(x,t) \le Ct^{-\frac{1}{q-1}} \text{ for any } t > 0.$$

The strategy used to prove the above results is mainly based on some techniques from the two articles [**17, 79**] the global pointwise estimate (1.16) and some delicate estimates on Wolff parabolic potential and the stability theorem see [**14**] (and also Proposition 1.48 of this paper). They will be proved in section 6.

Then, we shall study the global regularity of solutions to quasilinear parabolic equations of the type:

$$\begin{aligned} &u_t - \mathrm{div}\,(A(x,t,\nabla u)) = \mu \text{ in } \Omega_T, \\ &u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad u(0) = \sigma \quad \text{in} \quad \Omega, \end{aligned} \tag{1.18}$$

where the domain Ω_T and the nonlinearity A are as mentioned at the beginning.

Our aim is to find the minimal conditions on the boundary of Ω and on the nonlinearity A so that the following statement holds

$$|||\nabla u|||_{\mathcal{K}} \lesssim ||\mathbb{M}_1[\omega]||_{\mathcal{K}}.$$

Here $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$ and $\mathbb{M}_1$ is the first order fractional Maximal parabolic potential defined by

$$\mathbb{M}_1[\omega](x,t) = \sup_{\rho>0} \frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^{N+1}} \forall (x,t) \in \mathbb{R}^{N+1},$$

the constant C does not depend on u and $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$ and $\mathcal{K}$ is a function space. The same type of question was answered in the elliptic framework (see N. C. Phuc [**82, 83, 84**] for more details).

First, we take $\mathcal{K} = L^{p,s}(\Omega_T)$ for $1 \le p \le \theta$ and $0 < s \le \infty$ under a capacity density condition on the domain Ω where $L^{p,s}(\Omega_T)$ is the Lorentz space. The constant $\theta > 2$ depends on the structure of this condition and on the nonlinearity A. It follows the recent result in [**8**], see remark 1.18. The capacity density condition is that, the complement of Ω satisfies the *uniformly* $2-$*thick* condition, see section 2. We remark that under this condition, the Sobolev embedding $H_0^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$

for $N > 2$ is valid and it is fulfilled by any domain with Lipschitz boundary, or even of corkscrew type. This condition was used in the two papers [**82, 84**]. Also, it is essentially sharp for higher integrability results, presented in [**48**, Remark 3.3]. Furthermore, we also claim that if $\frac{\gamma}{\gamma-1} < p < \theta$, $2 \le \gamma < N+2$, $0 < s \le \infty$ and $\sigma \equiv 0$ then

$$|||\nabla u|||_{L_*^{p,s;(\gamma-1)p}(\Omega_T)} \lesssim ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)},$$

where $L_*^{p,s;(\gamma-1)p}$, $L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}$ are the Lorentz-Morrey spaces involving "calorie" introduced in section 1.2. We would like to refer to [**62**] as the first paper where Lorentz-Morrey estimates for solutions of quasilinear elliptic equations via fractional operators have been obtained.

Next, in order to obtain shaper results, we take $\mathcal{K} = L^{q,s}(\Omega_T, dw)$, the weighted Lorentz spaces with weight in the Muckenhoupht class A_∞ for $q \ge 1$, $0 < s \le \infty$, we require some stricter conditions on the domain Ω and nonlinearity A. The condition on Ω is flat enough in the sense of Reifenberg. It essentially says that at the boundary point and at every scale, the boundary of the domain is between two hyperplanes at both sides (inside and outside) of the domain by a distance which depends on the scale. Conditions on A are that BMO type of A with respect to the $x-$variable is small enough and the derivative of $A(x,t,\zeta)$ with respect to ζ is uniformly bounded. By choosing an appropriate weight we can establish the following important estimates:

a. The Lorentz-Morrey estimates involving "calorie" for $0 < \kappa \le N+2$ is obtained

$$|||\nabla u|||_{L_*^{q,s;\kappa}(\Omega_T)} \lesssim ||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\Omega_T)}.$$

b. Another Lorentz-Morrey estimates is also obtained for $0 < \vartheta \le N$

$$||\mathbb{M}(|\nabla u|)||_{L_{**}^{q,s;\vartheta}(\Omega_T)} \lesssim ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\Omega_T)},$$

where $L_{**}^{q,s;\vartheta}(\Omega_T)$ is introduced in section 2. This estimate implies global Hölder-estimate in space variable and L^q-estimate in time, that is for all ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |\text{osc}_{B_\rho \cap \overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \lesssim \rho^{1-\frac{\vartheta}{q}} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\Omega_T)}$$

provided $0 < \vartheta < \min\{q, N\}$. In particular, there holds

$$\left(\int_0^T |\text{osc}_{B_\rho \cap \overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \lesssim \rho^{1-\frac{\vartheta}{q}} ||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\Omega)} + \rho^{1-\frac{\vartheta}{q}} ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\Omega, L^{q_1}((0,T)))}$$

provided

$$1 < q_1 \le q < 2,$$
$$\max\left\{\frac{2-q}{q-1}, \frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)\right\} < \vartheta \le N.$$

Where $L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\Omega)$ is the standard Morrey space and

$$||\mu||_{L^{q_2;\vartheta}(\Omega,L^{q_1}((0,T)))} = \sup_{\rho>0,x\in\Omega} \rho^{\frac{\vartheta-N}{q_2}} \left(\int_{B_\rho(x)\cap\Omega} \left(\int_0^T |\mu(y,t)|^{q_1} dt \right)^{\frac{q_2}{q_1}} dy \right)^{\frac{1}{q_2}},$$

with $q_2 = \frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q}$. Besides, we also find

$$\left(\int_0^T |\text{osc}_{B_\rho\cap\overline{\Omega}} u(t)|^q dt \right)^{\frac{1}{q}} \lesssim \rho^{1-\frac{\vartheta}{q}} ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\Omega,L^{q_1}((0,T)))}$$

provided

$$\sigma \equiv 0, q \geq 2, 1 < q_1 \leq q,$$
$$\frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right) < \vartheta \leq N.$$

c. A global capacitary estimate is also given

$$\sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1} \\ \text{Cap}_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{\int_K |\nabla u|^q dxdt}{\text{Cap}_{\mathcal{G}_1,q'}(K)} \right) \lesssim \sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1} \\ \text{Cap}_{\mathcal{G}_1,q'}(K)>0}} \left(\frac{|\omega|(K)}{\text{Cap}_{\mathcal{G}_1,q'}(K)} \right)^q.$$

To obtain these estimates we use deep techniques from nonlinear potential theory, see section 2.1 and Theorem 3.18.

We use some ideas (in the quasilinear elliptic framework) in articles of N.C. Phuc [**82, 84, 83**] to establish above estimates. Recently, in [**70, 27**] the author extended these results for (1.18) with distributional data.

We would like to emphasize that above estimates are also true for solutions to equation (1.18) in $\mathbb{R}^{N+1}$ with data μ (of course it is still true for (1.18) in $\mathbb{R}^N \times (0,\infty)$) with data μ, provided $\mathbb{I}_2[|\mu|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$, see Theorem 1.23 and 1.25. Moreover, a global pointwise estimate for the gradient of solutions is obtained when A is independent of the space variable x, that is

$$|\nabla u(x,t)| \lesssim \mathbb{I}_1[|\mu|](x,t) \text{ a.e } (x,t) \in \mathbb{R}^{N+1},$$

see Theorem 1.5.

Our final aim is to obtain existence results for the quasilinear Riccati type parabolic problems (1.1) where $B(x,t,u,\nabla u) = |\nabla u|^q$ for $q > 1$. The strategy we use in order to prove these existence results is a combination of the Schauder Fixed Point Theorem and all above estimates and the stability Theorem see [**14**], Proposition 1.48 in section 3. They will be carried out in section 4.2. The method used in this paper allows to treat more general equations (1.1), namely

$$|B(x,t,u,\nabla u)| \lesssim |u|^{q_1} + |\nabla u|^{q_2}, q_1, q_2 > 1.$$

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1.2. Main Results

Throughout the memoir, we assume that Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$. Besides, we always denote $\Omega_T = \Omega \times (0,T)$, $T_0 = \text{diam}(\Omega) + T^{1/2}$ and $Q_\rho(x,t) = B_\rho(x) \times (t-\rho^2, t)$ $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t - \rho^2/2, t + \rho^2/2)$ for $(x,t) \in \mathbb{R}^{N+1}$ and $\rho > 0$. We always assume that $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory vector valued function, i.e. A is measurable in (x,t) and continuous with respect to ∇u for each fixed (x,t) and satisfies (1.2) and (1.3). This article is divided into three parts. First part, we study the existence problems for the quasilinear parabolic equations with absorption and source terms

$$\begin{cases} u_t - \text{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu & \text{in} \quad \Omega_T, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in} \quad \Omega, \end{cases} \tag{1.19}$$

and

$$\begin{cases} u_t - \text{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu & \text{in} \quad \Omega_T, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in} \quad \Omega, \end{cases} \tag{1.20}$$

where $q > 1$, and μ, σ are Radon measures.

In order to state our results, let us introduce some definitions and notations. If D is either a bounded domain or whole $\mathbb{R}^l$ for $l \in \mathbb{N}$, we denote by $\mathfrak{M}(D)$ (resp. $\mathfrak{M}_b(D)$) the set of Radon measure (resp. bounded Radon measures) in D. Their positive cones are $\mathfrak{M}^+(D)$ and $\mathfrak{M}_b^+(D)$ respectively. For $R \in (0,\infty]$, we define the $R-$truncated Riesz parabolic potential $\mathbb{I}_\alpha$ and fractional maximal parabolic potential $\mathbb{M}_\alpha$, $\alpha \in (0, N+2)$, on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathbb{I}_\alpha^R[\mu](x,t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}} \frac{d\rho}{\rho}, \qquad \mathbb{M}_\alpha^R[\mu](x,t) = \sup_{0<\rho<R} \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}}, \tag{1.21}$$

for all (x,t) in $\mathbb{R}^{N+1}$. If $R = \infty$, we drop it in expressions of (1.21).
We denote by $\mathcal{H}_\alpha$ the Heat kernel of order $\alpha \in (0, N+2)$:

$$\mathcal{H}_\alpha(x,t) = C_\alpha \frac{\chi_{(0,\infty)}(t)}{t^{(N+2-\alpha)/2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for} \quad (x,t) \in \mathbb{R}^{N+1},$$

and $\mathcal{G}_\alpha$ the parabolic Bessel kernel of order $\alpha > 0$:

$$\mathcal{G}_\alpha(x,t) = C_\alpha \frac{\chi_{(0,\infty)}(t)}{t^{(N+2-\alpha)/2}} \exp\left(-t - \frac{|x|^2}{4t}\right) \quad \text{for} \quad (x,t) \in \mathbb{R}^{N+1},$$

see [**5**], where $C_\alpha = \left((4\pi)^{N/2}\Gamma(\alpha/2)\right)^{-1}$. It is known that

$$\mathcal{F}(\mathcal{H}_\alpha)(x,t) = (|x|^2 + it)^{-\alpha/2}, \qquad \mathcal{F}(\mathcal{G}_\alpha)(x,t) = (1 + |x|^2 + it)^{-\alpha/2}.$$

We define the parabolic Riesz potential $\mathcal{H}_\alpha$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathcal{H}_\alpha[\mu](x,t) = (\mathcal{H}_\alpha * \mu)(x,t) \quad \text{for any} \quad (x,t) \in \mathbb{R}^{N+1},$$

the parabolic Bessel potential $\mathcal{G}_\alpha$ of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathcal{G}_\alpha[\mu](x,t) = (\mathcal{G}_\alpha * \mu)(x,t) \quad \text{for any} \quad (x,t) \in \mathbb{R}^{N+1}.$$

We also define $\mathbf{I}_\alpha, \mathbf{G}_\alpha, 0 < \alpha < N$ the Riesz, Bessel potential of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ by

$$\mathbf{I}_\alpha[\mu](x) = \int_0^\infty \frac{\mu(B_\rho(x))}{\rho^{N-\alpha}}\frac{d\rho}{\rho}, \quad \mathbf{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} \mathbf{G}_\alpha(x-y)d\mu(y) \quad \text{for any} \quad x \in \mathbb{R}^N,$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α, see [**2**].
Several different capacities will be used throughout the paper. For $1 < p < \infty$, the $(\mathcal{H}_\alpha, p)$-capacity, $(\mathcal{G}_\alpha, p)$-capacity of a Borel set $E \subset \mathbb{R}^{N+1}$ are defined by

$$\text{Cap}_{\mathcal{H}_\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), \mathcal{H}_\alpha * f \geq \chi_E\right\},$$
$$\text{Cap}_{\mathcal{G}_\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^{N+1}} |f|^p dxdt : f \in L^p_+(\mathbb{R}^{N+1}), \mathcal{G}_\alpha * f \geq \chi_E\right\}.$$

The $W^{2,1}_p-$capacity of compact set $E \subset \mathbb{R}^{N+1}$ is defined by

$$\text{Cap}_{2,1,p}(E) = \inf\left\{||\varphi||^p_{W^{2,1}_p(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of} E\right\},$$

where

$$||\varphi||_{W^{2,1}_p(\mathbb{R}^{N+1})} = ||(\varphi, \frac{\partial\varphi}{\partial t}, \nabla\varphi)||_{L^p(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\ldots,N} ||\frac{\partial^2\varphi}{\partial x_i \partial x_j}||_{L^p(\mathbb{R}^{N+1})}.$$

We remark that thanks to Richard J. Bagby's result (see [**5**]) we obtain the equivalent of capacities $\text{Cap}_{2,1,p}$ and $\text{Cap}_{\mathcal{G}_2,p}$, i.e, for any compact set $K \subset \mathbb{R}^{N+1}$ there holds

$$\text{Cap}_{\mathcal{G}_2,p}(K) \sim \text{Cap}_{2,1,p}(K),$$

see Corollary (2.18) in section 2.1.
The $(\mathbf{I}_\alpha, p)$-capacity, $(\mathbf{G}_\alpha, p)$-capacity of a Borel set $O \subset \mathbb{R}^N$ are defined by

$$\text{Cap}_{\mathbf{I}_\alpha,p}(O) = \inf\left\{\int_{\mathbb{R}^N} |g|^p dx : g \in L^p_+(\mathbb{R}^N), \mathbf{I}_\alpha * g \geq \chi_O\right\},$$
$$\text{Cap}_{\mathbf{G}_\alpha,p}(O) = \inf\left\{\int_{\mathbb{R}^N} |g|^p dx : g \in L^p_+(\mathbb{R}^N), \mathbf{G}_\alpha * g \geq \chi_O\right\}.$$

In this paper, we often use the expression $A \lesssim_{\alpha,\beta} B$ to mean that there exists a universal constant C depending on α, β and on fixed quantities $(N, \Lambda_1, \Lambda_2)$ such that $a \leq Cb$. The same convention is adopted for $\gtrsim_{\alpha,\beta}$ and $\sim_{\alpha,\beta}$.

In our first three Theorems, we present global pointwise potential estimates for solutions to quasilinear parabolic problems

$$\begin{cases} u_t - \text{div}\,(A(x,t,\nabla u)) = \mu \quad \text{in} \quad \Omega_T, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in} \quad \Omega, \end{cases} \tag{1.22}$$

$$\begin{cases} u_t - \text{div}\,(A(x,t,\nabla u)) = \mu \quad \text{in} \quad \mathbb{R}^{N+1}_+, \\ u(0) = \sigma \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.23}$$

and

$$u_t - \text{div}\,(A(x,t,\nabla u)) = \mu \quad \text{in} \quad \mathbb{R}^{N+1}. \tag{1.24}$$

THEOREM 1.1. *There exists a constant K depending on N, Λ_1, Λ_2 such that for any $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$ there is a distributional solution u of* (1.22) *which satisfies*

$$-K\mathbb{I}_2^{2T_0}[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \le u \le K\mathbb{I}_2^{2T_0}[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \quad in \quad \Omega_T. \tag{1.25}$$

REMARK 1.2. Since $\sup_{x\in\mathbb{R}^N} \mathbb{I}_\alpha[\sigma^\pm \otimes \delta_{\{t=0\}}](x,t) \le \frac{\sigma^\pm(\Omega)}{(N+2-\alpha)(2|t|)^{\frac{N+2-\alpha}{2}}}$ for any $t \neq 0$ with $0 < \alpha < N+2$. Thus, if $\mu \equiv 0$, then we obtain the decay estimate:

$$-\frac{K\sigma^-(\Omega)}{N(2t)^{\frac{N}{2}}} \le \inf_{x\in\Omega} u(x,t) \le \sup_{x\in\Omega} u(x,t) \le \frac{K\sigma^+(\Omega)}{N(2t)^{\frac{N}{2}}} \text{ for any } 0 < t < T.$$

THEOREM 1.3. *For any $\mu \in \mathfrak{M}_b^+(\Omega_T), \sigma \in \mathfrak{M}_b^+(\Omega)$, there is a distributional solution u of* (1.22) *satisfying for a.e $(y,s) \in \Omega_T$ and $B_r(y) \subset \Omega$*

$$u(y,s) \gtrsim \sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y, s - \frac{35}{128}r_k^2))}{r_k^N} + \sum_{k=0}^{\infty} \frac{(\sigma \otimes \delta_{\{t=0\}})(Q_{r_k/8}(y, s-\frac{35}{128}r_k^2))}{r_k^N}, \tag{1.26}$$

where $r_k = 4^{-k}r$.

REMARK 1.4. Theorem 1.3 is also true when we replace the assumption (1.3) by the following weaker one

$$\langle A(x,t,\zeta), \zeta\rangle \ge \Lambda_2|\zeta|^2, \langle A(x,t,\zeta) - A(x,t,\lambda), \zeta - \lambda\rangle > 0,$$

for every $(\lambda, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N$, $\lambda \neq \zeta$ and a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

THEOREM 1.5. *Let K be the constant in Theorem 1.1. Let $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ be such that $I_2[|\omega|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$. Then, there is a distributional solution u to* (1.24) *with data $\mu = \omega$ satisfying*

$$-K\mathbb{I}_2[\omega^-] \le u \le K\mathbb{I}_2[\omega^+] \text{ in } \mathbb{R}^{N+1} \tag{1.27}$$

such that the following statements hold.

a: *If $\omega \ge 0$, there holds for a.e $(x,t) \in \mathbb{R}^{N+1}$*

$$u(x,t) \gtrsim \sum_{k=-\infty}^{\infty} \frac{\omega(Q_{2^{-2k-3}}(x, t - 35 \times 2^{-4k-7}))}{2^{-2Nk}}. \tag{1.28}$$

In particular, for any $q > \frac{N+2}{N}$

$$||u||_{L^q(\mathbb{R}^{N+1})} \sim ||\mathcal{H}_2[\omega]||_{L^q(\mathbb{R}^{N+1})}. \tag{1.29}$$

b: *If A is independent of space variable x and satisfies* (1.45)*, then we have*

$$|\nabla u| \lesssim \mathbb{I}_1[|\omega|] \text{ in } \mathbb{R}^{N+1}. \tag{1.30}$$

c: *If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$ and $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distributional solution to* (1.23).

REMARK 1.6. For $q > \frac{N+2}{N}$, we always have the following claim:

$$||\mathcal{H}_2[\mu + \omega \otimes \delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \sim ||\mathcal{H}_2[\mu]||_{L^q(\mathbb{R}^{N+1})} + ||\mathbf{I}_{2/q}[\sigma]||_{L^q(\mathbb{R}^{N+1})},$$

for every $\mu \in \mathfrak{M}^+(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}^+(\mathbb{R}^N)$.

REMARK 1.7. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $0 < \alpha < N+2$ if $\mathbb{I}_\alpha[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then for any $0 < \beta \leq \alpha$, $\mathbb{I}_\beta[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$. However, for $0 < \beta < \alpha < N+2$, one can find $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $\mathbb{I}_\alpha[\omega] \equiv \infty$ and $\mathbb{I}_\beta[\omega] < \infty$ in $\mathbb{R}^{N+1}$, see Appendix.

The next four theorems provide the existence of solutions to quasilinear parabolic equations with absorption and source terms. For convenience, we always denote by q' the conjugate exponent of $q \in (1,\infty)$ i.e $q' = \frac{q}{q-1}$.

THEOREM 1.8. *Let $q > 1$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. Suppose that μ, σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$, $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω_T, Ω respectively. Then there exists a distributional solution u of* (1.19) *satisfying*

$$-K\mathbb{I}_2[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \leq u \leq K\mathbb{I}_2[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \text{ in } \Omega_T.$$

Here the constant K is in Theorem 1.1.

THEOREM 1.9. *Let K be the constant in Theorem 1.1. Let $q > 1$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. There exists a constant $\varepsilon_0 = \varepsilon_0(N, q, \Lambda_1, \Lambda_2, diam(\Omega), T)$ such that if*

$$|\mu|(E) \leq \varepsilon_0 \, Cap_{2,1,q'}(E) \quad \text{and} \quad |\sigma|(O) \leq \varepsilon_0 \, Cap_{\mathbf{G}_{\frac{2}{q}},q'}(O). \tag{1.31}$$

hold for every compact sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, then the problem (1.20) *has a distributional solution u satisfying*

$$-\frac{Kq}{q-1}\mathbb{I}_2[\mu^- + \sigma^- \otimes \delta_{\{t=0\}}] \leq u \leq \frac{Kq}{q-1}\mathbb{I}_2[\mu^+ + \sigma^+ \otimes \delta_{\{t=0\}}] \text{ in } \Omega_T. \tag{1.32}$$

Besides, for every compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$\int_E |u|^q dxdt \lesssim_{T_0,q} Cap_{2,1,q'}(E). \tag{1.33}$$

REMARK 1.10. From (1.33) we get if $q > \frac{N+2}{N}$,

$$\int_{\tilde{Q}_\rho(y,s)} |u|^q dxdt \lesssim_{T_0,q} \rho^{N+2-2q'} \quad \text{for any} \quad \tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1},$$

if $q = \frac{N+2}{N}$,

$$\int_{\tilde{Q}_\rho(y,s)} |u|^q dxdt \lesssim_{T_0,q} (\log(1/\rho))^{-\frac{1}{q-1}} \quad \text{for any} \quad \tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}, 0 < \rho < 1/2,$$

see Remark 2.14.

REMARK 1.11. In the sub-critical case $1 < q < \frac{N+2}{N}$, since the capacity $\text{Cap}_{2,1,q'}, \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ of a single point are positive thus the conditions (1.31) hold for some constant $\varepsilon_0 > 0$ provided $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$. Moreover, in the super-critical case $q > \frac{N+2}{N}$, we have

$$\text{Cap}_{2,1,q'}(E) \gtrsim_q |E|^{1-\frac{2q'}{N+2}} \quad \text{and} \quad \text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}(O) \gtrsim_q |O|^{1-\frac{2}{(q-1)N}},$$

for every Borel sets $E \subset \mathbb{R}^{N+1}$, $O \subset \mathbb{R}^N$, thus if $\mu \in L^{\frac{N+2}{2q'},\infty}(\Omega_T)$ and $\sigma \in L^{\frac{(q-1)N}{2},\infty}(\Omega)$ then (1.31) holds for some constant $\varepsilon_0 > 0$. In addition, if $\mu \equiv 0$, then (1.32) implies for any $0 < t < T$,

$$-c(T_0,q)t^{-\frac{1}{q-1}} \le \inf_{x\in\Omega} u(x,t) \le \sup_{x\in\Omega} u(x,t) \le c(T_0,q)t^{-\frac{1}{q-1}},$$

since $|\sigma|(B_\rho(x)) \lesssim_{T_0,q} \rho^{N-\frac{2}{q-1}}$ for all $x \in \mathbb{R}^N$, $0<\rho<2T_0$.

THEOREM 1.12. *Let K be the constant in Theorem 1.1 and $q > 1$. If $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ is absolutely continuous with respect to the capacity $Cap_{2,1,q'}$ in $\mathbb{R}^{N+1}$, then there exists a distributional solution $u \in L^\gamma_{loc}(\mathbb{R}; W^{1,\gamma}_{loc}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$ to the problem*

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \omega \quad in \quad \mathbb{R}^{N+1}, \tag{1.34}$$

which satisfies

$$-K\mathbb{I}_2[\omega^-] \le u \le K\mathbb{I}_2[\omega^+] \ in\ \mathbb{R}^{N+1}. \tag{1.35}$$

Furthermore, when $\omega = \mu + \sigma\otimes\delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N\times(0,\infty))$, $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N\times(-\infty,0)$ and $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distributional solution to the problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) + |u|^{q-1}u = \mu \quad in \quad \mathbb{R}^N\times(0,\infty), \\ u(0) = \sigma \quad in \quad \mathbb{R}^N. \end{cases} \tag{1.36}$$

REMARK 1.13. The measure $\omega = \mu + \sigma\otimes\delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\mathbb{R}^{N+1}$ if and only if μ,σ are absolutely continuous with respect to the capacities $\text{Cap}_{2,1,q'}$, $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in $\mathbb{R}^{N+1},\mathbb{R}^N$ respectively.

Existence result of the problem (1.20) on $\mathbb{R}^{N+1}$ or on $\mathbb{R}^N\times(0,\infty)$ is similar to Theorem 1.9 presented in the following Theorem, where the capacities $\text{Cap}_{\mathcal{H}_2,q'}$, $\text{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}$ are used in place of respectively $\text{Cap}_{2,1,q'}$, $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$.

THEOREM 1.14. *Let K be the constant in Theorem 1.1 and $q > \frac{N+2}{N}$, $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. There exists a constant $\varepsilon_0 = \varepsilon_0(N,q,\Lambda_1,\Lambda_2)$ such that if*

$$|\omega|(E) \le \varepsilon_0\, Cap_{\mathcal{H}_2,q'}(E), \tag{1.37}$$

for every compact set $E \subset \mathbb{R}^{N+1}$, then the problem

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \omega \quad in \quad \mathbb{R}^{N+1} \tag{1.38}$$

has a distributional solution $u \in L^\gamma_{loc}(\mathbb{R}; W^{1,\gamma}_{loc}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$ satisfying

$$-\frac{Kq}{q-1}\mathbb{I}_2[\omega^-] \le u \le \frac{Kq}{q-1}\mathbb{I}_2[\omega^+] \ in\ \mathbb{R}^{N+1}. \tag{1.39}$$

Moreover, when $\omega = \mu + \sigma\otimes\delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N\times(0,\infty))$, $\sigma \in \mathfrak{M}(\mathbb{R}^N)$ then $u = 0$ in $\mathbb{R}^N\times(-\infty,0)$ and $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distributional solution to the problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = |u|^{q-1}u + \mu \quad in \quad \mathbb{R}^N\times(0,\infty), \\ u(0) = \sigma \quad in \quad \mathbb{R}^N. \end{cases} \tag{1.40}$$

In addition, for any compact set $E \subset \mathbb{R}^{N+1}$ there holds

$$\int_E |u|^q dxdt \lesssim_q Cap_{\mathcal{H}_2,q'}(E). \tag{1.41}$$

REMARK 1.15. The measure $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ satisfies (1.37) if and only if

$$|\mu|(E) \lesssim \mathrm{Cap}_{\mathcal{H}_2,q'}(E) \quad \text{and} \quad |\sigma|(O) \lesssim \mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(O),$$

for every compact sets $E \subset \mathbb{R}^{N+1}$ and $O \subset \mathbb{R}^N$.

REMARK 1.16. If $\omega \in L^{\frac{N+2}{2q'},\infty}(\mathbb{R}^{N+1})$ then (1.37) holds for some constant $\varepsilon_0 > 0$. Moreover, if $\omega = \sigma \otimes \delta_{\{t=0\}}$ with $\sigma \in \mathfrak{M}_b(\mathbb{R}^N)$, then from (1.39) we get the decay estimate:

$$-ct^{-\frac{1}{q-1}} \le \inf_{x\in\mathbb{R}^N} u(x,t) \le \sup_{x\in\mathbb{R}^N} u(x,t) \le ct^{-\frac{1}{q-1}} \text{ for any } t>0,$$

since $|\sigma|(B_\rho(x)) \lesssim_q \rho^{N-\frac{2}{q-1}}$ for any $B_\rho(x) \subset \mathbb{R}^N$.

Second part, we establish global regularity in weighted-Lorentz space and Lorentz-Morrey space for the gradient of solutions to problem (1.22). For this purpose, we need a capacity density condition imposed on Ω. That is, the complement of Ω satisfies *uniformly p-thick with constants* c_0, r_0, i.e, for all $0 < r \le r_0$ and all $x \in \mathbb{R}^N \backslash \Omega$ there holds

$$\mathrm{Cap}_p(\overline{B_r(x)} \cap (\mathbb{R}^N \backslash \Omega), B_{2r}(x)) \ge c_0 \mathrm{Cap}_p(\overline{B_r(x)}, B_{2r}(x)), \tag{1.42}$$

where the involved capacity of a compact set $K \subset B_{2r}(x)$ is given as follows

$$\mathrm{Cap}_p(K, B_{2r}(x)) = \inf\left\{\int_{B_{2r}(x)} |\nabla\phi|^p dy : \phi \in C_c^\infty(B_{2r}(x)), \phi \ge \chi_K\right\}. \tag{1.43}$$

In order to obtain better regularity we need a stricter condition on Ω which is expressed in the following way. We say that Ω is a $(\delta, R_0)-$Reifenberg flat domain for $\delta \in (0,1)$ and $R_0 > 0$ if for every $x_0 \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{z_1, z_2, ..., z_n\}$, which may depend on r and x_0, so that in this coordinate system $x_0 = 0$ and that

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}. \tag{1.44}$$

We remark that this class of flat domains is rather wide since it includes C^1, Lipschitz domains with sufficiently small Lipschitz constants and fractal domains. Besides, it has many important roles in the theory of minimal surfaces and free boundary problems, this class was first appeared in a work of Reifenberg (see [**86**]) in the context of a Plateau problem. Its properties can be found in [**44, 45, 90**].

On the other hand, it is well-known that in general, conditions (1.2) and (1.3) on the nonlinearity $A(x,t,\zeta)$ are not enough to ensure higher integral of gradient of solutions to problem (1.22), we need to assume that A satisfies

$$\langle A_\zeta(x,t,\zeta)\xi, \xi\rangle \ge \Lambda_2 |\xi|^2, |A_\zeta(x,t,\zeta)| \le \Lambda_1, \tag{1.45}$$

for every $(\xi, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{(0,0)\}$ and a.e $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where Λ_1, Λ_2 are constants in (1.2) and (1.3). We also require that the nonlinearity A satisfies a smallness condition of BMO type in the x-variable. We say that $A(x,t,\zeta)$ satisfies a (δ, R_0)-BMO condition for some $\delta, R_0 > 0$ with exponent $s > 0$ if

$$[A]_s^{R_0} := \sup_{(y,s)\in\mathbb{R}^N\times\mathbb{R}, 0<r\le R_0} \left(\fint_{Q_r(y,s)} (\Theta(A, B_r(y))(x,t))^s \, dxdt\right)^{\frac{1}{s}} \le \delta,$$

where

$$\Theta(A, B_r(y))(x,t) := \sup_{\zeta \in \mathbb{R}^N \backslash \{0\}} \frac{|A(x,t,\zeta) - \overline{A}_{B_r(y)}(t,\zeta)|}{|\zeta|},$$

and $\overline{A}_{B_r(y)}(t,\zeta)$ is denoted the average of $A(t,.,\zeta)$ over the cylinder $B_r(y)$, i.e,

$$\overline{A}_{B_r(y)}(t,\zeta) := \fint_{B_r(y)} A(x,t,\zeta)dx = \frac{1}{|B_r(y)|}\int_{B_r(y)} A(x,t,\zeta)dx.$$

The above condition was observed in [**25**]. It is easy to see that the $(\delta, R_0)-$BMO condition on A is satisfied when A is continuous or has small jump discontinuities with respect to x.

In this paper, $\mathbb{M}$ denotes the Hardy-Littlewood maximal function defined for each locally integrable function f in $\mathbb{R}^{N+1}$ by

$$\mathbb{M}(f)(x,t) = \sup_{\rho>0} \fint_{\tilde{Q}_\rho(x,t)} |f(y,s)|dyds \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

It is by now rather standard to verify that $\mathbb{M}$ is bounded operator from $L^1(\mathbb{R}^{N+1})$ to $L^{1,\infty}(\mathbb{R}^{N+1})$ and $L^s(\mathbb{R}^{N+1})$ $(L^{s,\infty}(\mathbb{R}^{N+1}))$ to itself for $s>1$, see [**88, 89**].

We recall that a positive function $w \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ is called an A_∞ weight if there are two positive constants C and ν such that

$$w(E) \le C\left(\frac{|E|}{|Q|}\right)^\nu w(Q),$$

for all cylinder $Q = \tilde{Q}_\rho(x,t)$ and all measurable subsets E of Q. The pair (C,ν) is called the A_∞ constant of w and is denoted by $[w]_{A_\infty}$.

For a weight function $w \in A_\infty$, the weighted Lorentz spaces $L^{q,s}(D,dw)$ with $0<q<\infty$, $0<s\le\infty$ and a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions g on D such that

$$||g||_{L^{q,s}(D,dw)} := \begin{cases} \left(q\int_0^\infty \left(\rho^q w\left(\{(x,t)\in D : |g(x,t)|>\rho\}\right)\right)^{\frac{s}{q}} \frac{d\rho}{\rho}\right)^{1/s} < \infty \text{ if } s<\infty, \\ \sup_{\rho>0} \rho w\left(\{(x,t)\in D : |g(x,t)|>\rho\}\right)^{1/q} < \infty \text{ if } s=\infty. \end{cases}$$

Here we write $w(E) = \int_E w(x,t)dxdt$ for a measurable set $E \subset \mathbb{R}^{N+1}$. Obviously, $||g||_{L^{q,q}(D,dw)} = ||g||_{L^q(D,dw)}$, thus we have $L^{q,q}(D,dw) = L^q(D,dw)$. As usual, when $w\equiv 1$ we simply write $L^{q,s}(D)$ instead of $L^{q,s}(D,dw)$.

We now state the next results of the paper.

THEOREM 1.17. *Let* $\mu \in \mathfrak{M}_b(\Omega_T)$*,* $\sigma \in \mathfrak{M}_b(\Omega)$*, set* $\omega = |\mu| + |\sigma|\otimes\delta_{\{t=0\}}$*. There exists a distributional solution of* (1.22) *with data* μ *and* σ *such that if* $\mathbb{R}^N\backslash\Omega$ *satisfies uniformly* $2-$*thick with constants* c_0, r_0 *then for any* $1\le p<\theta$ *and* $0<s\le\infty$*,*

$$||\mathbb{M}(|\nabla u|)||_{L^{p,s}(\Omega_T)} \lesssim C||\mathbb{M}_1[\omega]||_{L^{p,s}(Q)}. \tag{1.46}$$

Here $\theta = \theta(N,\Lambda_1,\Lambda_1,c_0) > 2$ *and* $C = C(p,s,c_0,T_0/r_0)$ *and* $Q = B_{diam(\Omega)}(x_0)\times(0,T)$ *which* $\Omega \subset B_{diam(\Omega)}(x_0)$*.*
Especially, when $1<p<2$*, then*

$$||\mathbb{M}(|\nabla u|)||_{L^p(\Omega_T)} \lesssim C\left(||\mathcal{G}_1[|\mu|]||_{L^p(\mathbb{R}^{N+1})} + ||\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]||_{L^p(\mathbb{R}^N)}\right). \tag{1.47}$$

REMARK 1.18. If $\frac{N+2}{N+1} < p < 2$, there hold

$$||\mathcal{G}_1[|\mu|]||_{L^p(\mathbb{R}^{N+1})} \lesssim_p ||\mu||_{L^{\frac{p(N+2)}{N+2+p}}(\Omega_T)} \text{ and } ||\mathbf{G}_{\frac{2}{p}-1}[|\sigma|]||_{L^p(\mathbb{R}^N)} \lesssim_p ||\sigma||_{L^{\frac{pN}{N+2-p}}(\Omega)}.$$

From (1.47) we obtain

$$|||\nabla u|||_{L^p(\Omega_T)} \lesssim_p ||\mu||_{L^{\frac{p(N+2)}{N+2+p}}(\Omega_T)} + ||\sigma||_{L^{\frac{pN}{N+2-p}}(\Omega)} \text{ provided } \frac{N+2}{N+1} < p < 2.$$

We should mention that if $\sigma \equiv 0$, then for any $0 < s \leq \infty, p > \frac{N+2}{N+1}$

$$||\mathbb{M}_1[\mu]||_{L^{p,s}(\mathbb{R}^{N+1})} \lesssim_{p,s} ||\mu||_{L^{\frac{p(N+2)}{N+2+p},s}(\Omega_T)},$$

and we get [**8**, Theorem 1.2] from estimate (1.46).

In order to state the next results, we need to introduce Lorentz-Morrey spaces $L_*^{q,s;\theta}(D)$ involving "calorie" with a Borel set $D \subset \mathbb{R}^{N+1}$, is the set of measurable functions g on D such that

$$||g||_{L_*^{q,s;\kappa}(D)} := \sup_{0<\rho<\operatorname{diam}(D),(x,t)\in D} \rho^{\frac{\kappa-N-2}{q}} ||g||_{L^{q,s}(\tilde{Q}_\rho(x,t)\cap D)} < \infty,$$

where $0 < \kappa \leq N+2$, $0 < q < \infty$, $0 < s \leq \infty$. Clearly, $L_*^{q,s;N+2}(D) = L^{q,s}(D)$. Moreover, when $q = s$ the space $L_*^{q,s;\theta}(D)$ will be denoted by $L_*^{q;\theta}(D)$.
The following theorem provides an estimate on gradient in Lorentz-Morrey spaces.

THEOREM 1.19. *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distributional solution of* (1.22) *with data μ and σ such that if $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0 then for any $1 \leq p < \theta$ and $0 < s \leq \infty$, $2 - \gamma_0 < \gamma < N+2$, $\gamma \leq \frac{N+2}{p} + 1$,*

$$\begin{aligned} &||\mathbb{M}\left(|\nabla u|\right)||_{L_*^{p,s;p(\gamma-1)}(\Omega_T)} \lesssim C_1 ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)} \\ &+ C_2 \sup_{0<R\leq T_0,(y_0,s_0)\in\Omega_T} \left(R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \right). \end{aligned} \tag{1.48}$$

Here θ is in Theorem 1.17, $\gamma_0 = \gamma_0(N,\Lambda_1,\Lambda_1,c_0) \in (0,1/2]$ and $C_1 = C_1(p,s,\gamma,c_0,T_0/r_0)$, $C_2 = C_2(p,s,\gamma,c_0)$. Besides, if $\frac{\gamma}{\gamma-1} < p < \theta$, $2 - \gamma_0 < \gamma < N+2$, $0 < s \leq \infty$ and $\mu \in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$, $\sigma \equiv 0$, then u is a unique renormalized solution satisfying

$$||\mathbb{M}\left(|\nabla u|\right)||_{L_*^{p,s;(\gamma-1)p}(\Omega_T)} \lesssim C_1 ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}. \tag{1.49}$$

The following Theorem is about higher regularity for solutions of (1.22). We need more assumptions on boundary of Ω and on the nonlinearity A.

THEOREM 1.20. *Suppose that A satisfies* (1.45). *Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distributional solution of* (1.22) *with data μ, σ such that the following holds. For any $w \in A_\infty$, $1 \leq q < \infty$, $0 < s \leq \infty$ we find $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty}) \in (0,1)$ and $s_0 = s_0(N,\Lambda_1,\Lambda_2) > 0$ such that if Ω is (δ,R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \leq \delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L^{q,s}(\Omega_T,dw)} \lesssim C ||\mathbb{M}_1[\omega]||_{L^{q,s}(\Omega_T,dw)}. \tag{1.50}$$

Here C depends on $q, s, [w]_{A_\infty}$ and T_0/R_0.

Next results are actually consequences of Theorem 1.20. For our purpose, we introduce another Lorentz-Morrey spaces $L_{**}^{q,s;\theta}(O_1\times O_2)$, is the set of measurable functions g on $O_1\times O_2$ such that

$$||g||_{L_{**}^{q,s;\vartheta}(O_1\times O_2)}:=\sup_{0<\rho<\operatorname{diam}(O_1),x\in O_1}\rho^{\frac{\vartheta-N}{q}}||g||_{L^{q,s}((B_\rho(x)\cap O_1)\times O_2))}<\infty,$$

where O_1,O_2 are Borel sets in $\mathbb{R}^N$ and $\mathbb{R}$ respectively, $0<\vartheta\le N$, $0<q<\infty$, $0<s\le\infty$. Obviously, $L_{**}^{q,s;N}(D)=L^{q,s}(D)$. For simplicity of notation, we write $L_{**}^{q;\vartheta}(D)$ instead of $L_{**}^{q,s;\vartheta}(D)$ when $q=s$. Moreover,

$$||g||_{L_{**}^{q,q;\vartheta}(O_1\times O_2)}=||G||_{L^{q;\vartheta}(O_1)},$$

where $G(x)=||g(x,.)||_{L^q(O_1)}$ and $L^{q;\vartheta}(O_1)$ is the usual Morrey space, i.e the spaces of all measurable functions f on O_1 with

$$||f||_{L^{q;\vartheta}(O_1)}:=\sup_{0<\rho<\operatorname{diam}(O_1),y\in O_1}\rho^{\frac{\vartheta-N}{q}}||f||_{L^q(B_\rho(y)\cap O_1)}<\infty.$$

THEOREM 1.21. *Suppose that A satisfies (1.45). Let $\mu\in\mathfrak{M}_b(\Omega_T)$, $\sigma\in\mathfrak{M}_b(\Omega)$, set $\omega=|\mu|+|\sigma|\otimes\delta_{\{t=0\}}$. Let s_0 be in Theorem 1.20. There exists a distributional solution of (1.22) with data μ,σ such that the following holds.*

a: *For any $1\le q<\infty$, $0<s\le\infty$ and $0<\kappa\le N+2$ we find $\delta=\delta(N,\Lambda_1,\Lambda_2,q,s,\kappa)\in(0,1)$ such that if Ω is (δ,R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0}\le\delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L_{*}^{q,s;\kappa}(\Omega_T)}\lesssim C_1||\mathbb{M}_1[|\omega|]||_{L_{*}^{q,s;\kappa}(\Omega_T)},\tag{1.51}$$

for some $C_1=C_1(q,s,\kappa,T_0/R_0)$.

b: *For any $1\le q<\infty$, $0<s\le\infty$ and $0<\vartheta\le N$ we find $\delta=\delta(N,\Lambda_1,\Lambda_2,q,s,\vartheta)\in(0,1)$ such that if Ω is (δ,R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0}\le\delta$ for some R_0 then*

$$||\mathbb{M}(|\nabla u|)||_{L_{**}^{q,s;\vartheta}(\Omega_T)}\lesssim C_2||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\Omega_T)},\tag{1.52}$$

for some $C_2=C_2(q,s,\vartheta,T_0/R_0)$. Especially, when $q=s$ and $0<\vartheta<\min\{N,q\}$, there holds for any ball $B_\rho\subset\mathbb{R}^N$

$$\left(\int_0^T|osc_{B_\rho\cap\overline{\Omega}}u(t)|^qdt\right)^{\frac{1}{q}}\lesssim C_3\rho^{1-\frac{\vartheta}{q}}||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\Omega_T)},\tag{1.53}$$

for some $C_3=C_3(q,\vartheta,T_0/R_0)$.

REMARK 1.22. Above results also hold when $[A]_s^{R_0}$ is replaced by $\{A\}_s^{R_0}$:

$$\{A\}_s^{R_0}:=\sup_{(y,s)\in\mathbb{R}^N\times\mathbb{R},0<r\le R_0}\left(\fint_{Q_r(y,s)}\left(\Theta(A,Q_r(y,s))(x,t)\right)^s dxdt\right)^{\frac{1}{s}}\le\delta,$$

where

$$\Theta(A,Q_r(y,s))(x,t):=\sup_{\zeta\in\mathbb{R}^N\setminus\{0\}}\frac{|A(x,t,\zeta)-\overline{A}_{Q_r(y,s)}(\zeta)|}{|\zeta|},$$

and $\overline{A}_{Q_r(y,s)}(\zeta)$ is denoted the average of $A(.,.,\zeta)$ over the cylinder $Q_r(y,s)$, i.e,

$$\overline{A}_{Q_r(y,s)}(\zeta):=\fint_{Q_r(y,s)}A(x,t,\zeta)dxdt=\frac{1}{|Q_r(y,s)|}\int_{Q_r(y,s)}A(x,t,\zeta)dxdt.$$

Next results are corresponding estimates of gradient for domain $\mathbb{R}^N \times (0,\infty)$ or whole $\mathbb{R}^{N+1}$.

THEOREM 1.23. *Let $\theta \in (2, N+2)$ be in Theorem 1.17 and $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. There exists a distributional solution u of (1.24) with data $\mu = \omega$ such that the following statements hold*

a: *For any $\frac{N+2}{N+1} < p < \theta$ and $0 < s \le \infty$,*

$$|||\nabla u|||_{L^{p,s}(\mathbb{R}^{N+1})} \lesssim_{p,s} ||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}. \tag{1.54}$$

b: *For any $\frac{N+2}{N+1} < p < \theta$ and $0 < s \le \infty$, $2-\gamma_0 < \gamma < N+2$ and $\gamma \le \frac{N+2}{p} + 1$,*

$$\begin{aligned}|||\nabla u|||_{L_*^{p,s;p(\gamma-1)}(\mathbb{R}^{N+1})} &\lesssim_{p,s,\gamma} ||\mathbb{M}_\gamma[|\omega|]||_{L^\infty(\mathbb{R}^{N+1})}\\ &+ \sup_{R>0,(y_0,s_0)\in\mathbb{R}^{N+1}} \left(R^{\frac{p(\gamma-1)-N-2}{p}} ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}|\omega|]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\right),\end{aligned} \tag{1.55}$$

provided $\mathbb{I}_2[|\omega|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$. Also, if $\omega \in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\mathbb{R}^{N+1})$ with $p > \frac{\gamma}{\gamma-1}$ then

$$|||\nabla u|||_{L_*^{p,s;(\gamma-1)p}(\mathbb{R}^{N+1})} \lesssim_{p,s,\gamma} ||\omega||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\mathbb{R}^{N+1})}, \tag{1.56}$$

for some $\gamma_0 = \gamma_0(N,\Lambda_1,\Lambda_2) \in (0,\frac{1}{2}]$.

c: *The statement **c** in Theorem 1.5 is true.*

REMARK 1.24. Let $s > 1$. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ implies $\mathbb{I}_2[|\omega|] < \infty$ a.e in $\mathbb{R}^{N+1}$ if and only if $s \le N+2$.

THEOREM 1.25. *Suppose that A satisfies (1.45). Let s_0 be in Theorem 1.20. Let $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$ with $\mathbb{I}_2[|\omega|](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$. There exists a distributional solution of (1.24) with data $\mu = \omega$ such that following statements hold.*

a: *For any $w \in A_\infty$, $1 \le q < \infty$, $0 < s \le \infty$ we find $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,[w]_{A_\infty}) \in (0,1)$ such that if $[A]_{s_0}^\infty \le \delta$ then*

$$|||\nabla u|||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \lesssim_{q,s,[w]_{A_\infty}} ||\mathbb{M}_1[|\omega|]||_{L^{q,s}(\mathbb{R}^{N+1},dw)}. \tag{1.57}$$

b: *For any $\frac{N+2}{N+1} < q < \infty$, $0 < s \le \infty$ and $0 < \kappa \le N+2$ we find $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,\kappa) \in (0,1)$ such that if $[A]_{s_0}^\infty \le \delta$ then*

$$|||\nabla u|||_{L_*^{q,s;\kappa}(\mathbb{R}^{N+1})} \lesssim_{q,s,\kappa} ||\mathbb{M}_1[|\omega|]||_{L_*^{q,s;\kappa}(\mathbb{R}^{N+1})}. \tag{1.58}$$

c: *For any $\frac{N+2}{N+1} < q < \infty$, $0 < s \le \infty$ and $0 < \vartheta \le N$ one find $\delta = \delta(N,\Lambda_1,\Lambda_2,q,s,\vartheta) \in (0,1)$ such that if $[A]_{s_0}^\infty \le \delta$ then*

$$|||\nabla u|||_{L_{**}^{q,s;\vartheta}(\mathbb{R}^{N+1})} \lesssim_{q,s,\vartheta} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q,s;\vartheta}(\mathbb{R}^{N+1})}. \tag{1.59}$$

Especially, when $q = s$ and $0 < \vartheta < \min\{N,q\}$, there holds for any ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_{\mathbb{R}} |osc_{B_\rho} u(t)|^q dt\right)^{\frac{1}{q}} \lesssim_{q,s,\vartheta} \rho^{1-\frac{\vartheta}{q}} ||\mathbb{M}_1[|\omega|]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})}. \tag{1.60}$$

d: *The statement **c** in Theorem 1.5 is true.*

We will present some estimates for norms of $\mathbb{M}_1[\omega]$ in $L_*^{q;\kappa}(\mathbb{R}^{N+1})$ and $L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})$ in section 3.2.

Final part, we prove the existence solutions for the quasilinear Riccati type parabolic problems

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = |\nabla u|^q + \mu \text{in } \Omega_T, \\ u = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma \text{in } \Omega, \end{cases} \tag{1.61}$$

$$u_t - \operatorname{div}\left(A(x,t,\nabla u)\right) = |\nabla u|^q + \mu \text{in } \mathbb{R}^{N+1}, \tag{1.62}$$

where $q > 1$.

The following result is considered in subcritical case this means $1 < q < \frac{N+2}{N+1}$, to obtain existence solutions in this case we need data μ, σ to be finite measures and small enough.

THEOREM 1.26. *Let $1 < q < \frac{N+2}{N+1}$ and $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$. There exists $\varepsilon_0 = \varepsilon_0(N,\Lambda_1,\Lambda_2,q) > 0$ such that if*

$$|\Omega_T|^{-1+\frac{q'}{N+2}}\left(|\mu|(\Omega_T) + |\omega|(\Omega)\right) \le \varepsilon_0,$$

the problem (1.61) *has a distribution solution u satisfying*

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \lesssim_q |\mu|(\Omega_T) + |\omega|(\Omega).$$

In the next results are concerned in critical and supercritical case.

THEOREM 1.27. *Suppose that $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let θ be in Theorem 1.17, $q \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$, $\mu \in \mathfrak{M}_b(\Omega_T)$ and $\sigma \in \mathfrak{M}_b(\Omega)$. Assume that $\sigma \equiv 0$ when $q \ge \frac{N+4}{N+2}$. There exists $\varepsilon_0 = \varepsilon_0(N,\Lambda_1,\Lambda_2,q,c_0,T_0/r_0) > 0$ such that if*

$$||\mathbb{I}_1[|\mu|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})} + ||\mathbf{I}_{\frac{2}{(N+2)(q-1)}-1}[|\sigma|]||_{L^{(N+2)(q-1)}(\mathbb{R}^N)} \le \varepsilon_0,$$

then the problem (1.61) *has a distributional solution u satisfying*

$$|||\nabla u|||_{L^{(q-1)(N+2),\infty}(\Omega_T)} \lesssim C, \tag{1.63}$$

for some $C = C(q, c_0, T_0/r_0)$.

We remark that a necessary condition the existence of $\sigma \in \mathfrak{M}_b(\Omega)\backslash\{0\}$ with $\mathbb{M}_1[|\sigma| \otimes \delta_{\{t=0\}}] \in L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})$ is $\frac{N+2}{N+1} \le q < \frac{N+4}{N+2}$.

THEOREM 1.28. *Suppose that A satisfies* (1.45). *Let s_0 be the constant in Theorem 1.20. Let $q \ge \frac{N+2}{N+1}$ and $\mu \in \mathfrak{M}_b(\Omega_T), \sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists $\delta = \delta(N,\Lambda_1,\Lambda_2,q) \in (0,1)$ such that Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 and the following holds. The problem* (1.61) *has a distributional solution u if*

$$\omega(K) \le \varepsilon_0 \, Cap_{\mathcal{G}_1,q'}(K) \quad \forall \text{compact set } K \subset \mathbb{R}^{N+1}, \tag{1.64}$$

with a constant $\varepsilon_0 > 0$ small enough. Moreover, u satisfies

$$\int_K |\nabla u|^q dxdt \lesssim C \, Cap_{\mathcal{G}_1,q'}(K) \quad \forall \text{compact set } K \subset \mathbb{R}^{N+1},$$

where C depends on $q, T_0/R_0, T_0$.

Since $\text{Cap}_{\mathcal{G}_1,s}(B_r(0)\times\{t=0\})=0$ for all $r>0$ and $0<s\leq 2$, see Remark 2.13 thus if there is $\sigma\in\mathfrak{M}_b(\Omega)\backslash\{0\}$ satisfying $(|\sigma|\otimes\delta_{\{t=0\}})(E)\leq \text{Cap}_{\mathcal{G}_1,s}(E)$ for every compact subsets $E\subset\mathbb{R}^{N+1}$ then we must have $s>2$.

We have an **important** Proposition.

PROPOSITION 1.29. All the existence results (in the bounded domain case) that we have obtained in the above Theorems are renormalized solutions if we further assume that $\sigma\in L^1(\Omega)$.

The following theorem is an existence result of solutions to the problem (1.62).

THEOREM 1.30. *Let $\theta\in(2,N+2)$ be in Theorem 1.17, $q\in\left(\frac{N+2}{N+1},\frac{N+2+\theta}{N+2}\right)$ and $\omega\in\mathfrak{M}(\mathbb{R}^{N+1})$. There exists $\varepsilon_0=\varepsilon_0(N,\Lambda_1,\Lambda_2,q)>0$ such that if*

$$||\mathbb{I}_1[|\omega|]||_{L^{(N+2)(q-1),\infty}(\mathbb{R}^{N+1})}\leq\varepsilon_0,$$

then the problem (1.62) has a distributional solution $u\in L^1_{loc}(\mathbb{R};W^{1,1}_{loc}(\mathbb{R}^N))$ such that

$$|||\nabla u|||_{L^{(q-1)(N+2),\infty}(\mathbb{R}^{N+1})}\lesssim_q 1. \tag{1.65}$$

Furthermore, if supp$(\mu)\subset\mathbb{R}^N\times[0,\infty)$ then $u=0$ in $\mathbb{R}^N\times(-\infty,0)$.

THEOREM 1.31. *Suppose that A satisfies (1.45). Let s_0 be the constant in Theorem 1.20. Let $q>\frac{N+2}{N+1}$ and $\mu\in\mathfrak{M}(\mathbb{R}^{N+1})$. There exists $\delta=\delta(N,\Lambda_1,\Lambda_2,q)\in(0,1)$ such that $[A]_{s_0}^{\infty}\leq\delta$ for some R_0 and the following holds. The problem (1.62) has a distributional solution u if*

$$|\mu|(K)\leq\varepsilon_0\,Cap_{\mathcal{H}_1,q'}(K)\quad\forall\text{compact set }K\subset\mathbb{R}^{N+1}, \tag{1.66}$$

with a constant $\varepsilon_0>0$ small enough. Moreover, u satisfies

$$\int_K|\nabla u|^q dxdt\lesssim_q Cap_{\mathcal{H}_1,q'}(K)\quad\forall\text{compact set }K\subset\mathbb{R}^{N+1}.$$

Furthermore, if supp$(\mu)\subset\mathbb{R}^N\times[0,\infty)$ then $u=0$ in $\mathbb{R}^N\times(-\infty,0)$.

1.3. The notion of solutions and some properties

Although the notion of renormalized solutions becomes more and more familiar in the theory of quasilinear parabolic equations with measure data, it is still necessary to present below some main aspects concerning this notion. Let Ω be a bounded domain in $\mathbb{R}^N$, $(a,b)\subset\subset\mathbb{R}$. If $\mu\in\mathfrak{M}_b(\Omega\times(a,b))$, we denote by μ^+ and μ^- respectively its positive and negative part. We denote by $\mathfrak{M}_0(\Omega\times(a,b))$ the space of measures in $\Omega\times(a,b)$ which are absolutely continuous with respect to the C_2-capacity defined on a compact set $K\subset\Omega\times(a,b)$ by

$$C_2(K,\Omega\times(a,b))=\inf\left\{||\varphi||_W:\varphi\geq\chi_K,\varphi\in C_c^\infty(\Omega\times(a,b))\right\}. \tag{1.67}$$

Here $W=\left\{z:z\in L^2(a,b,H^1_0(\Omega)),z_t\in L^2(a,b,H^{-1}(\Omega))\right\}$ is endowed with the norm $||\varphi||_W=||\varphi||_{L^2(a,b,H^1_0(\Omega))}+||\varphi_t||_{L^2(a,b,H^{-1}(\Omega))}$ and χ_K is the characteristic function of K.

We also denote $\mathfrak{M}_s(\Omega\times(a,b))$ the space of measures in $\Omega\times(a,b)$ with support on a set of zero C_2-capacity. Classically, any $\mu\in\mathfrak{M}_b(\Omega\times(a,b))$ can be written in a unique way under the form $\mu=\mu_0+\mu_s$ where $\mu_0\in(\mathfrak{M}_0\cap\mathfrak{M}_b)(\Omega\times(a,b))$ and $\mu_s\in\mathfrak{M}_s(\Omega\times(a,b))$. We recall that any $\mu_0\in(\mathfrak{M}_0\cap\mathfrak{M}_b)(\Omega\times(a,b))$ can be decomposed under the form $\mu_0 = f - \operatorname{div} g + h_t$ where $f \in L^1(\Omega\times(a,b))$, $g \in L^2(\Omega\times(a,b),\mathbb{R}^N)$ and $h \in L^2(a,b,H_0^1(\Omega))$ and (f,g,h) is said to be decomposition of μ_0. Set $\widehat{\mu_0} = \mu_0 - h_t = f - \operatorname{div} g$. In the general case $\widehat{\mu_0} \notin \mathfrak{M}(\Omega\times(a,b))$, but we write, for convenience,

$$\int_{\Omega\times(a,b)} w d\widehat{\mu_0} := \int_{\Omega\times(a,b)} (fw + g.\nabla w)dxdt, \quad \forall w \in L^2(a,b,H_0^1(\Omega))\cap L^\infty(\Omega\times(a,b)).$$

However, for $\sigma \in \mathfrak{M}_b(\Omega)$ and $t_0 \in (a,b)$ then $\sigma \otimes \delta_{\{t=t_0\}} \in \mathfrak{M}_0(\Omega\times(a,b))$ if and only if $\sigma \in L^1(\Omega)$, see [**32**]. We also have that for $\sigma \in \mathfrak{M}_b(\Omega)$, $\sigma \otimes \chi_{[a,b]} \in \mathfrak{M}_0(\Omega\times(a,b))$ if and only if σ is absolutely continuous with respect to the $\mathrm{Cap}_{\mathbf{G}_1,2}$-capacity, see [**32**].

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\},-k\}$. We recall that if u is a measurable function defined and finite a.e. in $\Omega\times(a,b)$, such that $T_k(u) \in L^2(a,b,H_0^1(\Omega))$ for any $k > 0$, there exists a measurable function $v : \Omega\times(a,b) \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\le k} v$ a.e. in $\Omega\times(a,b)$ and for all $k > 0$. We define the gradient of u by $v = \nabla u$.

We recall the definition of a renormalized solution given in [**76**].

Definition 1.32. Suppose that $B \in C(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega\times(a,b))$ and $\sigma \in L^1(\Omega)$. A measurable function u is a **renormalized solution** of

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = B(u,\nabla u) + \mu & \text{in} \quad \Omega\times(a,b),\\ u = 0 \quad \text{on} \quad \partial\Omega\times(a,b), \\ u(a) = \sigma \quad \text{in} \quad \Omega, \end{cases} \tag{1.68}$$

if there exists a decomposition (f,g,h) of μ_0 such that

$$v = u - h \in L^s(a,b,W_0^{1,s}(\Omega)) \cap L^\infty(a,b,L^1(\Omega)) \quad \forall s \in \left[1, \frac{N+2}{N+1}\right),$$
$$T_k(v) \in L^2(a,b,H_0^1(\Omega)) \quad \forall k>0, \quad B(u,\nabla u) \in L^1(\Omega\times(a,b)), \tag{1.69}$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$\begin{aligned} -\int_\Omega S(\sigma)\varphi(a)dx - \int_{\Omega\times(a,b)} \varphi_t S(v)dxdt + \int_{\Omega\times(a,b)} S'(v)A(x,t,\nabla u)\nabla\varphi dxdt \\ + \int_{\Omega\times(a,b)} S''(v)\varphi A(x,t,\nabla u).\nabla v dxdt = \int_{\Omega\times(a,b)} S'(v)\varphi B(u,\nabla u)dxdt \\ + \int_{\Omega\times(a,b)} S'(v)\varphi d\widehat{\mu_0}, \end{aligned} \tag{1.70}$$

for any $\varphi \in L^2(a,b,H_0^1(\Omega)) \cap L^\infty(\Omega\times(a,b))$ such that $\varphi_t \in L^2(a,b,H^{-1}(\Omega)) + L^1(\Omega\times(a,b))$ and $\varphi(.,b) = 0$;

(ii) for any $\phi \in C(\overline{\Omega}\times[a,b])$,

$$\lim_{m\to\infty} \frac{1}{m} \int_{\{m\le v<2m\}} \phi A(x,t,\nabla u)\nabla v dxdt = \int_{\Omega\times(a,b)} \phi d\mu_s^+ \text{ and} \tag{1.71}$$

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{-m\geq v>-2m\}}\phi A(x,t,\nabla u)\nabla v dxdt=\int_{\Omega\times(a,b)}\phi d\mu_s^-. \tag{1.72}$$

REMARK 1.33. If $\mu\in L^1(\Omega\times(a,b))$, then we have the following estimates:

$$||u||_{L^{\frac{N+2}{N},\infty}(\Omega\times(a,b))}\lesssim||\sigma||_{L^1(\Omega)}+|\mu|(\Omega\times(a,b)),$$
$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega\times(a,b))}\lesssim||\sigma||_{L^1(\Omega)}+|\mu|(\Omega\times(a,b)),$$

see [**14**, Remark 4.9]. In particular,

$$||u||_{L^1(\Omega\times(a,b))}\lesssim(\mathrm{diam}(\Omega)+(b-a)^{1/2})^2\left(||\sigma||_{L^1(\Omega)}+|\mu|(\Omega\times(a,b))\right),$$
$$|||\nabla u|||_{L^1(\Omega\times(a,b))}\lesssim(\mathrm{diam}(\Omega)+(b-a)^{1/2})\left(||\sigma||_{L^1(\Omega)}+|\mu|(\Omega\times(a,b))\right).$$

REMARK 1.34. It is easy to see that u is a weak solution to the problem (1.68) in $\Omega\times(a,b)$ with $\mu\in L^2(\Omega\times(a,b))$, $\sigma\in H_0^1(\Omega)$ and $B\equiv 0$ then $U=\chi_{[a,b]}u$ is a unique renormalized solution of

$$\begin{cases} U_t-\mathrm{div}\,(A(x,t,\nabla U))=\chi_{(a,b)}\mu+(\chi_{[a,b)}\sigma)_t & \text{in} \quad \Omega\times(c,b),\\ U=0 \qquad \text{on } \partial\Omega\times(c,b), \\ U(c)=0 \quad \text{ in } \Omega,\end{cases}$$

for any $c<a$.

REMARK 1.35. Let $\Omega'\subset\subset\Omega$ and $a<a'<b'<b$. For a nonnegative function $\eta\in C_c^\infty(\Omega'\times(a',b'))$, from (1.70) we have

$$\begin{aligned}&(\eta S(v))_t-\eta_t S(v)+S'(v)A(x,t,\nabla u)\nabla\eta-\mathrm{div}\,(S'(v)\eta A(x,t,\nabla u))\\&\quad+S''(v)\eta A(x,t,\nabla u)\nabla v=S'(v)\eta f+\nabla\left(S'(v)\eta\right).g-\mathrm{div}\,(S'(v)\eta g)\end{aligned}$$

in $\mathcal{D}'(\Omega'\times(a',b'))$. Thus, $(\eta S(v))_t\in L^2(a',b',H^{-1}(\Omega'))+L^1(D)$ and we have the following estimate:

$$\begin{aligned}&||\left(\eta S(v)\right)_t||_{L^2(a',b',H^{-1}(\Omega'))+L^1(D)}\lesssim||S||_{W^{2,\infty}(\mathbb{R})}\big(||\eta_t v||_{L^1(D)}\\&\quad+|||\nabla u||\nabla\eta|||_{L^1(D)}+||\eta|\nabla u|\chi_{|v|\leq M}||_{L^2(D)}+||\eta|\nabla u||\nabla v|\chi_{|v|\leq M}||_{L^2(D)}\\&\quad+||\eta f||_{L^1(D)}+||\eta|\nabla u|^2\chi_{|v|\leq M|}||_{L^1(D)}+||\eta|g|^2||_{L^1(D)}+||\eta|g|||_{L^2(D)}\big)\end{aligned}\tag{1.73}$$

with $D=\Omega'\times(a',b')$ and $\mathrm{supp}(S')\subset[-M,M]$.

We recall the following important results, see in [**14**].

PROPOSITION 1.36. Let $\{\mu_n\}$ be a bounded in $\mathfrak{M}_b(\Omega\times(a,b))$ and σ_n be a bounded in $L^1(\Omega)$. Let u_n be a renormalized solution of (1.22) with data $\mu_n=\mu_{n,0}+\mu_{n,s}$ relative to a decomposition (f_n,g_n,h_n) of $\mu_{n,0}$ and initial data σ_n. If $\{f_n\}$ is bounded in $L^1(\Omega_T)$, $\{g_n\}$ is bounded in $L^2(\Omega\times(a,b),\mathbb{R}^N)$ and $\{h_n\}$ is convergent in $L^2(a,b,H_0^1(\Omega))$, then, up to a subsequence, $\{u_n\}$ converges to a function u in $L^1(\Omega\times(a,b))$. Moreover, if $\{\mu_n\}$ is a bounded in $L^1(\Omega\times(a,b))$ then $\{u_n\}$ is convergent in $L^s(a,b,W_0^{1,s}(\Omega))$ for any $s\in\left[1,\frac{N+2}{N+1}\right)$.

We say that a sequence of bounded measures $\{\mu_n\}$ in $\Omega\times(a,b)$ converges to a bounded measure μ in $\Omega\times(a,b)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty}\int_{\Omega\times(a,b)}\varphi d\mu_n=\int_{\Omega\times(a,b)}\varphi d\mu \quad \text{for all } \varphi\in C(\Omega\times(a,b))\cap L^\infty(\Omega\times(a,b))).$$

We recall the following fundamental stability result of [**14**].

THEOREM 1.37. *Suppose that* $B \equiv 0$. *Let* $\sigma \in L^1(\Omega)$ *and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathfrak{M}_b(\Omega \times (a,b)),$$

with $f \in L^1(\Omega \times (a,b)), g \in L^2(\Omega \times (a,b), \mathbb{R}^N)$, $h \in L^2(a,b,H_0^1(\Omega))$ *and* $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega \times (a,b))$. *Let* $\sigma_n \in L^1(\Omega)$ *and*

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathfrak{M}_b(\Omega \times (a,b))$$

with $f_n \in L^1(\Omega \times (a,b)), g_n \in L^2(\Omega \times (a,b), \mathbb{R}^N), h_n \in L^2(a,b,H_0^1(\Omega))$, *and* $\rho_n, \eta_n \in \mathfrak{M}_b^+(\Omega \times (a,b))$, *such that*

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(\Omega \times (a,b)), \rho_n^2, \eta_n^2 \in L^2(\Omega \times (a,b), \mathbb{R}^N)$ *and* $\rho_{n,s}, \eta_{n,s} \in \mathfrak{M}_s^+(\Omega \times (a,b))$.
Assume that $\{\mu_n\}$ *is a bounded in* $\mathfrak{M}_b(\Omega \times (a,b))$, $\{\sigma_n\}, \{f_n\}, \{g_n\}, \{h_n\}$ *converge to* σ, f, g, h *in* $L^1(\Omega)$,*weakly in* $L^1(\Omega \times (a,b))$,*in* $L^2(\Omega \times (a,b), \mathbb{R}^N)$,*in* $L^2(a,b,H_0^1(\Omega))$ *respectively and* $\{\rho_n\}, \{\eta_n\}$ *converge to* μ_s^+, μ_s^- *in the narrow topology of measures; and* $\{\rho_n^1\}, \{\eta_n^1\}$ *are bounded in* $L^1(\Omega \times (a,b))$, *and* $\{\rho_n^2\}, \{\eta_n^2\}$ *bounded in* $L^2(\Omega \times (a,b), \mathbb{R}^N)$.

Let $\{u_n\}$ *be a sequence of renormalized solutions of*

$$\left\{ \begin{array}{l} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n \quad \text{in} \quad \Omega \times (a,b), \\ u_n = 0 \quad \text{on } \partial\Omega \times (a,b), \\ u_n(a) = \sigma_n \ \text{in } \Omega, \end{array} \right. \tag{1.74}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ *of* $\mu_{n,0}$. *Let* $v_n = u_n - h_n$. *Then up to a subsequence,* $\{u_n\}$ *converges a.e. in* $\Omega \times (a,b)$ *to a renormalized solution* u *of* (1.68), *and* $\{v_n\}$ *converges a.e. in* $\Omega \times (a,b)$ *to* $v = u - h$. *Moreover,* $\{\nabla u_n\}, \{\nabla v_n\}$ *converge respectively to* $\nabla u, \nabla v$ *a.e in* $\Omega \times (a,b)$, *and* $\{T_k(v_n)\}$ *converges to* $T_k(v)$ *strongly in* $L^2(a,b,H_0^1(\Omega))$ *for any* $k > 0$.

In order to apply above theorem, we need some the following properties concerning approximate measures of $\mu \in \mathfrak{M}_b^+(\Omega \times (a,b))$, see in [**14**].

PROPOSITION 1.38. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b^+(\Omega \times (a,b))$ with $\mu_0 \in \mathfrak{M}_0(\Omega \times (a,b)) \cap \mathfrak{M}_b^+(\Omega \times (a,b))$ and $\mu_s \in \mathfrak{M}_s^+(\Omega \times (a,b))$. Let $\{\varphi_n\}$ be sequence of standard mollifiers in $\mathbb{R}^{N+1}$. Then, there exist a decomposition (f,g,h) of μ_0 and $f_n, g_n, h_n \in C_c^\infty(\Omega \times (a,b))$, $\mu_{n,s} \in (C_c^\infty \cap \mathfrak{M}_b^+)(\Omega \times (a,b))$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega \times (a,b)), L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $L^2(a,b,H_0^1(\Omega))$; $\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s}, \mu_{n,s}$ converge to μ, μ_s in the narrow topology respectively; $0 \leq \mu_n \leq \varphi_n * \mu$ and

$$||f_n||_{L^1(\Omega \times (a,b))} + \|g_n\|_{L^2(\Omega \times (a,b), \mathbb{R}^N)} + ||h_n||_{L^2(a,b,H_0^1(\Omega))} + \mu_{n,s}(\Omega \times (a,b)) \leq 2\mu(\Omega \times (a,b)).$$

PROPOSITION 1.39. Let $\mu = \mu_0 + \mu_s, \mu_n = \mu_{n,0} + \mu_{n,s} \in \mathfrak{M}_b^+(\Omega \times (a,b))$ with $\mu_0, \mu_{n,0} \in (\mathfrak{M}_0 \cap \mathfrak{M}_b^+)(\Omega \times (a,b))$ and $\mu_{n,s}, \mu_s \in \mathfrak{M}_s^+(\Omega \times (a,b))$ such that $\{\mu_n\}$ nondecreasingly converges to μ in $\mathfrak{M}_b(\Omega \times (a,b))$. Then, $\{\mu_{n,s}\}$ is nondecreasing and converging to μ_s in $\mathfrak{M}_b(\Omega \times (a,b))$ and there exist decompositions (f,g,h) of μ_0, (f_n, g_n, h_n) of $\mu_{n,0}$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega \times (a,b)), L^2(\Omega \times (a,b), \mathbb{R}^N)$ and $L^2(a,b,H_0^1(\Omega))$ respectively satisfying

$$||f_n||_{L^1(\Omega \times (a,b))} + \|g_n\|_{L^2(\Omega \times (a,b), \mathbb{R}^N)} + ||h_n||_{L^2(a,b,H_0^1(\Omega))} + \mu_{n,s}(\Omega \times (a,b)) \leq 2\mu(\Omega \times (a,b)).$$

REMARK 1.40. For $0<\rho\leq\frac{1}{3}\min\{\sup_{x\in\Omega}d(x,\partial\Omega),(b-a)^{1/2}\}$, set

$$\Omega_\rho^j=\{x\in\Omega: d(x,\partial\Omega)>j\rho\}\times(a+(j\rho)^2,a+((b-a)^{1/2}-j\rho)^2)\text{ for } j=0,...,k_\rho,$$

where $k_\rho=\left[\frac{\min\{\sup_{x\in\Omega}d(x,\partial\Omega),(b-a)^{1/2}\}}{2\rho}\right]$.
We can choose f_n,g_n,h_n in above two Propositions such that for any $j=1,...,k_\rho$,

$$||f_n||_{L^1(\Omega_\rho^j)}+\|g_n\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)}+|||h_n|+|\nabla h_n|||_{L^2(\Omega_\rho^j)}\leq 2\mu(\Omega_\rho^{j-1})\quad\forall n\in\mathbb{N}\tag{1.75}$$

In fact, set $\mu_j=\chi_{\Omega_\rho^{k_\rho-j}\backslash\Omega_\rho^{k_\rho-j+1}}\mu$ if $j=1,...,k_\rho-1$, $\mu_j=\chi_{\Omega\times(a,b)\backslash\Omega_\rho^1}\mu$ if $j=k_\rho$ and $\mu_j=\chi_{\Omega_\rho^{k_\rho}}\mu$ if $j=0$. From the proof of above two Propositions in [**14**], for any $\varepsilon>0$ we can assume supports of f_n,g_n,h_n containing in $\text{supp}(\mu)+\tilde{Q}_\varepsilon(0,0)$. Thus, for any $\mu=\mu_j$ we have f_n^j,g_n^j,h_n^j correspondingly such that their supports contain in $\Omega_{\rho,T}^{k_\rho-j-1/2}\backslash\Omega_{\rho,T}^{k_\rho-j+3/2}$ if $j=1,...,k_\rho-1$ and $\Omega_T\backslash\Omega_{\rho,T}^{3/2}$ if $j=k_\rho$ and $\Omega_{\rho,T}^{k_\rho-1/2}$ if $j=0$. By $\mu=\sum_{j=0}^{k_\rho}\mu_j$, thus it is allowed to choose $f_n=\sum_{j=0}^{k_\rho}f_n^j$, $f_n=\sum_{j=0}^{k_\rho}g_n^j$ and $h_n=\sum_{j=0}^{k_\rho}h_n^j$ and (1.75) satisfies since

$$\begin{aligned}&||f_n||_{L^1(\Omega_\rho^j)}+\|g_n\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)}+|||h_n|+|\nabla h_n|||_{L^2(\Omega_\rho^j)}\\&\leq\sum_{i=0}^{k_\rho-j+1}\left(||f_n^i||_{L^1(\Omega_\rho^j)}+\left\|g_n^i\right\|_{L^2(\Omega_\rho^j,\mathbb{R}^N)}+|||h_n^i|+|\nabla h_n^i|||_{L^2(\Omega_\rho^j)}\right)\\&\leq\sum_{i=j-1}^{k_\rho-j+1}2\mu_j(\Omega\times(a,b))=2\mu(\Omega_\rho^{j-1}).\end{aligned}$$

DEFINITION 1.41. Let $\mu\in\mathfrak{M}_b(\Omega\times(a,b))$ and $\sigma\in\mathfrak{M}_b(\Omega)$. A measurable function u is a distributional solution to the problem (1.68) if $u\in L^s(a,b,W_0^{1,s}(\Omega))$ for any $s\in\left[1,\frac{N+2}{N+1}\right)$ and $B(u,\nabla u)\in L^1(\Omega\times(a,b))$ such that

$$\begin{aligned}&-\int_{\Omega\times(a,b)}u\varphi_t dxdt+\int_{\Omega\times(a,b)}A(x,t,\nabla u)\nabla\varphi dxdt\\&\qquad=\int_{\Omega\times(a,b)}B(u,\nabla u)\varphi dxdt+\int_{\Omega\times(a,b)}\varphi d\mu+\int_\Omega\varphi(a)d\sigma\end{aligned}$$

for every $\varphi\in C_c^1(\Omega\times[a,b))$.

REMARK 1.42. Let $\sigma'\in\mathfrak{M}_b(\Omega)$ and $a'\in(a,b)$, set $\omega=\mu+\sigma'\otimes\delta_{\{t=a'\}}$. If u is a distributional solution to the problem (1.68) with data ω and $\sigma=0$ such that $\text{supp}(\mu)\subset\overline{\Omega}\times[a',b]$, and $u=0,B(u,\nabla u)=0$ in $\Omega\times(a,a')$, then $\tilde{u}:=u|_{\Omega\times[a',b)}$ is a distributional solution to problem (1.68) in $\Omega\times(a',b)$ with data μ and σ'. Indeed, for any $\varphi\in C_c^1(\Omega\times[a',b))$ we define

$$\tilde{\varphi}(x,t)=\begin{cases}\varphi(x,t)\quad\text{if}\quad(x,t)\in\Omega\times[a',b),\\(1+\varepsilon_0)(t-a')\varphi_t(x,a')+\varphi(x,(1+\varepsilon_0)a'-\varepsilon_0t)\quad\text{if}\quad(x,t)\in\Omega\times[a,a'),\end{cases}$$

where $\varepsilon_0 \in \left(0, \frac{b-a'}{a'-a}\right)$.
Clearly, $\tilde{\varphi} \in C_c^1(\Omega \times [a,b))$, thus we have

$$-\int_{\Omega\times(a,b)} u\tilde{\varphi}_t dxdt + \int_{\Omega\times(a,b)} A(x,t,\nabla u)\nabla\tilde{\varphi} dxdt = \int_{\Omega\times(a,b)} B(u,\nabla u)\tilde{\varphi} dxdt + \int_{\Omega\times(a,b)} \tilde{\varphi} d\omega,$$

which implies

$$-\int_{\Omega\times(a',b)} \tilde{u}\varphi_t dxdt + \int_{\Omega\times(a',b)} A(x,t,\nabla\tilde{u})\nabla\varphi dxdt = \int_{\Omega\times(a',b)} B(\tilde{u},\nabla\tilde{u})\varphi dxdt + \int_{\Omega\times(a',b)} \varphi d\mu + \int_{\Omega} \varphi(a') d\sigma'.$$

Definition 1.43. Let $\mu \in \mathfrak{M}(\mathbb{R}^N \times [a,+\infty))$, for $a \in \mathbb{R}$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. A measurable function u is a distributional solution to problem

$$\left\{\begin{array}{l} u_t - \operatorname{div}(A(x,t,\nabla u)) = B(u,\nabla u) + \mu \quad \text{in} \quad \mathbb{R}^N \times (a,+\infty), \\ u(a) = \sigma \quad in \quad \mathbb{R}^N, \end{array}\right. \tag{1.76}$$

if $u \in L^s_{\mathrm{loc}}(a,\infty,W^{1,s}_{\mathrm{loc}}(\mathbb{R}^N))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and $B(u,\nabla u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times [a,\infty))$ such that

$$-\int_{\mathbb{R}^N\times(a,\infty)} u\varphi_t dxdt + \int_{\mathbb{R}^N\times(a,\infty)} A(x,t,\nabla u)\nabla\varphi dxdt = \int_{\mathbb{R}^N\times(a,\infty)} B(u,\nabla u)\varphi dxdt + \int_{\mathbb{R}^N\times(a,\infty)} \varphi d\mu + \int_{\mathbb{R}^N} \varphi(a) d\sigma$$

for every $\varphi \in C_c^1(\mathbb{R}^N \times [a,\infty))$.

Definition 1.44. Let $\omega \in \mathfrak{M}(\mathbb{R}^{N+1})$. A measurable function u is a distributional solution to problem

$$u_t - \operatorname{div}(A(x,t,\nabla u)) = B(u,\nabla u) + \omega \quad \text{in} \quad \mathbb{R}^{N+1}, \tag{1.77}$$

if $u \in L^s_{\mathrm{loc}}(\mathbb{R}; W^{1,s}_{\mathrm{loc}}(\mathbb{R}^N))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and $B(u,\nabla u) \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$ such that

$$-\int_{\mathbb{R}^{N+1}} u\varphi_t dxdt + \int_{\mathbb{R}^{N+1}} A(x,t,\nabla u)\nabla\varphi dxdt = \int_{\mathbb{R}^{N+1}} B(u,\nabla u)\varphi dxdt + \int_{\mathbb{R}^{N+1}} \varphi d\omega,$$

for every $\varphi \in C_c^1(\mathbb{R}^{N+1})$.

Remark 1.45. Let $\mu \in \mathfrak{M}(\mathbb{R}^N \times [a,+\infty))$, for $a \in \mathbb{R}$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. If u is a distributional solution to problem (1.77) with data $\omega = \mu + \sigma \otimes \delta_{\{t=a\}}$ such that $u = 0, B(u,\nabla u) = 0$ in $\mathbb{R}^N \times (-\infty,a)$, then $\tilde{u} := u|_{\mathbb{R}^N\times[a,\infty)}$ is a distributional solution to problem (1.76) in $\mathbb{R}^N \times (a,\infty)$ with data μ and σ, see Remark 1.42.

To prove the existence distributional solution of problem (1.76) we need the following results. First, we have local estimates of the renormalized solution which get from [**14**, Proposition 2.8].

PROPOSITION 1.46. Let u, v be in Definition 1.32. There holds for $k \geq 1$ and $0 \leq \eta \in C_c^\infty(\Omega \times (a,b))$

$$\int_{|v|\leq k} \eta |\nabla u|^2 dxdt + \int_{|v|\leq k} \eta |\nabla v|^2 dxdt \lesssim kA \tag{1.78}$$

where

$$A = ||v\eta_t||_{L^1(\Omega\times(a,b))} + |||\nabla u||\nabla \eta|||_{L^1(\Omega\times(a,b))} + ||\eta f||_{L^1(\Omega\times(a,b))} + ||\eta |g|^2||_{L^1(\Omega\times(a,b))} + |||\nabla \eta||g|||_{L^1(\Omega\times(a,b))} + ||\eta|\nabla h|^2||_{L^1(\Omega\times(a,b))} + \int_{\Omega\times(a,b)} \eta d|\mu_s|.$$

For our purpose, we recall the Landes-time approximation of functions w belonging to $L^2(a,b,H_0^1(\Omega))$, introduced in [**52**], used in [**29, 21, 9**]. For $\nu > 0$ we define

$$\langle w\rangle_\nu(x,t) = \nu \int_a^{\min\{t,b\}} w(x,s)e^{\nu(s-t)}ds \quad \text{for all } (x,t)\in\Omega\times(a,b).$$

We have that $\langle w\rangle_\nu$ converges to w strongly in $L^2(a,b,H_0^1(\Omega))$ and $||\langle w\rangle_\nu||_{L^q(\Omega\times(a,b))} \leq ||w||_{L^q(\Omega\times(a,b))}$ for every $q\in[1,\infty]$. Moreover,

$$(\langle w\rangle_\nu)_t = \nu\left(w - \langle w\rangle_\nu\right) \quad \text{in the distributional sense.}$$

PROPOSITION 1.47. Let $q_0 > 1$ and $0 < \alpha < 1/2$ such that $q_0 > \alpha + 1$. Let $L : \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing such that $L(0) = 0$. If u is a solution of

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) + L(u) = \mu & \text{in} \quad \Omega\times(a,b), \\ u = 0 \quad \text{on} \quad \partial\Omega\times(a,b), \\ u(a) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.79}$$

with $\mu \in C_c^\infty(\Omega\times(a,b))$ then for $0\leq \eta \in C_c^\infty(D)$

$$\frac{1}{k}\int_D |\nabla T_k(u)|^2\eta + \int_D \frac{|\nabla u|^2\eta}{(|u|+1)^{\alpha+1}} + |||\nabla u||\nabla\eta|||_{L^1(D)} + ||L(u)\eta||_{L^1(D)} \lesssim_{\alpha,q_0} B, \tag{1.80}$$

where $q_1 = \frac{q_0-\alpha-1}{2q_0}$, $D = \Omega'\times(a',b')$, $\Omega'\subset\subset\Omega$ and $a < a' < b' < b$;

$$B = ||\eta_t(|u|+1)||_{L^1(D)} + \int_D (|u|+1)^{q_0}\eta dxdt + \int_D |\nabla\eta^{1/q_1}|^{q_1} dxdt + \int_D \eta d|\mu|.$$

Furthermore, for $T_k(w)\in L^2(a',b',H_0^1(\Omega'))$, the Landes-time approximation $\langle T_k(w)\rangle_\nu$ of the truncated function $T_k(w)$ in D then for any $\varepsilon\in(0,1)$ and $\nu>0$

$$\nu\int_D \eta\left(T_k(w) - \langle T_k(w)\rangle_\nu\right) T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu) dxdt + \int_D \eta A(x,t,\nabla T_k(u))\nabla T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu) dxdt \lesssim \varepsilon(1+k)B. \tag{1.81}$$

PROPOSITION 1.48. Let $q_0 > 1$, $\mu_n = \mu_{n,0} + \mu_{n,s} \in \mathfrak{M}_b(B_n(0)\times(-n^2,n^2))$. Let u_n be a renormalized solution of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n & \text{in } B_n(0)\times(-n^2,n^2), \\ u_n = 0 \quad \text{on} \quad \partial B_n(0)\times(-n^2,n^2), \\ u_n(-n^2) = 0 \text{ in } B_n(0), \end{cases} \tag{1.82}$$

relative to the decomposition (f_n, g_n, h_n) of $\mu_{n,0}$ satisfying (1.81) in Proposition 1.47 with $L \equiv 0$. Assume that for any $m \in \mathbb{N}$ and $\alpha \in (0, 1/2)$, $D_m := B_m(0) \times (-m^2, m^2)$

$$\frac{1}{k}|||\nabla T_k(u)|^2||_{L^1(D_m)} + |||\nabla u|^2(|u|+1)^{-\alpha-1}||_{L^1(D_m)} + |||\nabla u|||_{L^1(D_m)} + |\mu_n|(D_m)$$
$$+ ||f_n||_{L^1(D_m)} + ||g_n||_{L^2(D_m,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(D_m)} + ||u_n||_{L^{q_0}(D_m)} \le C(m, \alpha)$$

for all $n \ge m$ and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. Then, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that u_n converges to u a.e in $\mathbb{R}^{N+1}$ and in $L^s_{\text{loc}}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s \in [1, \frac{N+2}{N+1})$.

Proofs of above two propositions are given in Appendix. The following result is as a consequence of Proposition 1.48.

Corollary 1.49. Let $\mu_n \in L^1(B_n(0) \times (-n^2, n^2))$. Let u_n be a unique renormalized solution to the problem 1.82. Assume that for any $m \in \mathbb{N}$,

$$\sup_{n \ge m} |\mu_n|(B_m(0) \times (-m^2, m^2)) < \infty \quad \text{and} \quad \sup_{n \ge m} \int_{B_m(0)\times(-m^2,m^2)} |u_n|^{q_0} dxdt < \infty.$$

then there exists a subsequence of $\{u_n\}$ converging to u a.e in $\mathbb{R}^{N+1}$ and in $L^s_{\text{loc}}(\mathbb{R}; W^{1,s}_{loc}(\mathbb{R}^N))$ for any $s \in [1, \frac{N+2}{N+1})$.

Finally, we would like to present a technical lemma which will be used several times in the paper, especially in the proof of Theorem 1.17, 1.19 and 1.20. It is a consequence of Vitali Covering Lemma (see [**26, 25, 61**]).

Lemma 1.50. *Let Ω be a (R_0, δ)- Reifenberg flat domain with $\delta < 1/4$ and let w be an A_∞ weight. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^L$ with centers $y_i \in \overline{\Omega}$ and a common radius $r \le R_0/4$ covers Ω. Set $s_i = T - ir^2/2$ for all $i = 0, 1, ..., [\frac{2T}{r^2}]$. Let $E \subset F \subset \Omega_T$ be measurable sets for which there exists $0 < \varepsilon < 1$ such that $w(E) < \varepsilon w(\tilde{Q}_r(y_i, s_j))$ for all $i = 1, ..., L$, $j = 0, 1, ..., [\frac{2T}{r^2}]$; and for all $(x, t) \in \Omega_T$, $\rho \in (0, 2r]$, we have $\tilde{Q}_\rho(x, t) \cap \Omega_T \subset F$ if $w(E \cap \tilde{Q}_\rho(x, t)) \ge \varepsilon w(\tilde{Q}_\rho(x, t))$. Then $w(E) \le C\varepsilon w(F)$ for a constant C depending only on N and $[w]_{A_\infty}$.*

Clearly, the lemma implies the following two consquences that we state as Lemmas. More precisely, we have the following two Lemmas.

Lemma 1.51. *Let $0 < \varepsilon < 1, R > 0$ and consider the cylinder $\tilde{Q}_R := \tilde{Q}_R(x_0, t_0)$ for some $(x_0, t_0) \in \mathbb{R}^{N+1}$ and $w \in A_\infty$. Let $E \subset F \subset \tilde{Q}_R$ be two measurable sets in $\mathbb{R}^{N+1}$ with $w(E) < \varepsilon w(\tilde{Q}_R)$ satisfying the following property: for all $(x, t) \in \tilde{Q}_R$ and $r \in (0, R]$, we have $\tilde{Q}_r(x, t) \cap \tilde{Q}_R \subset F$ provided $w(E \cap \tilde{Q}_r(x, t)) \ge \varepsilon w(\tilde{Q}_r(x, t))$. Then $w(E) \le C\varepsilon w(F)$ for some $C = C(N, [w]_{A_\infty})$.*

Lemma 1.52. *Let $0 < \varepsilon < 1$ and $R > R' > 0$ and let $E \subset F \subset Q = B_R(x_0) \times (a, b)$ be two measurable sets in $\mathbb{R}^{N+1}$ with $|E| < \varepsilon|\tilde{Q}_{R'}|$ which satisfy the following property: for all $(x, t) \in Q$ and $r \in (0, R']$, we have $Q_r(x, t) \cap Q \subset F$ if $|E \cap \tilde{Q}_r(x, t)| \ge \varepsilon|\tilde{Q}_r(x, t)|$. Then $|E| \le C\varepsilon|F|$ for a constant C depending only on N.*

CHAPTER 2

Nonlinear potential theory to parabolic equations

2.1. Estimates on Potential

In this chapter, we will develop the nonlinear potential theory corresponding to quasilinear parabolic equations.

First we introduce the Wolff parabolic potential of $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ by

$$\mathbb{W}_{\alpha,p}^R[\mu](x,t) = \int_0^R \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \quad \forall (x,t) \in \mathbb{R}^{N+1},$$

where $\alpha > 0, 1 < p < \alpha^{-1}(N+2)$ and $0 < R \leq \infty$. For convenience, $\mathbb{W}_{\alpha,p}[\mu] := \mathbb{W}_{\alpha,p}^\infty[\mu]$.

The following result is an extension of [**42**, Theorem 1.1], [**17**, Proposition 2.2] to Parabolic potential.

Theorem 2.1. *Let $\alpha > 0$, $1 < p < \alpha^{-1}(N+2)$ and $w \in A_\infty$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. There exist constants $C > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N, \alpha, p, [w]_{A_\infty}$ such that for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$*

$$w(\{\mathbb{W}_{\alpha,p}^R[\mu] > a\lambda, (\mathbb{M}_{\alpha p}^R[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\}) \leq C \exp(-1/(C\varepsilon)) w(\{\mathbb{W}_{\alpha,p}^R[\mu] > \lambda\}) \tag{2.1}$$

where $a = 2 + 3^{\frac{N+2-\alpha p}{p-1}}$.

Proof of Theorem 2.1. We only consider the case $R < \infty$. Let $\{\tilde{Q}_R(x_j,t_j)\}$ be a cover of $\mathbb{R}^{N+1}$ such that $\sum_j \chi_{\tilde{Q}_R(x_j,t_j)} \leq M$ in $\mathbb{R}^{N+1}$ for some constant $M = M(N) > 0$. It is enough to show that there exist constants $c > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N, \alpha, p, [w]_{A_\infty}$ such that for any $Q \in \{\tilde{Q}_R(x_j,t_j)\}$, $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$

$$w(Q \cap \{\mathbb{W}_{\alpha,p}^R[\mu] > a\lambda, (\mathbb{M}_{\alpha p}^R[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\}) \leq c \exp(-(c\varepsilon)^{-1}) w(Q \cap \{\mathbb{W}_{\alpha,p}^R[\mu] > \lambda\}). \tag{2.2}$$

Fix $\lambda > 0$ and $0 < \varepsilon < 1/10$. We set

$$E = Q \cap \{\mathbb{W}_{\alpha,p}^R[\mu] > a\lambda, (\mathbb{M}_{\alpha p}^R[\mu])^{\frac{1}{p-1}} \leq \varepsilon\lambda\} \text{ and } F = Q \cap \{\mathbb{W}_{\alpha,p}^R[\mu] > \lambda\}.$$

Thanks to Lemma 1.51 we will get (2.2) if we verify the following two claims:

$$w(E) \leq c \exp(-(c\varepsilon)^{-1}) w(Q), \tag{2.3}$$

and for any $(x,t) \in Q$, $0 < r \leq R$,

$$w(E \cap \tilde{Q}_r(x,t)) < c \exp(-(c\varepsilon)^{-1}) w(\tilde{Q}_r(x,t)), \tag{2.4}$$

provided that $\tilde{Q}_r(x,t)\cap Q\cap F^c\neq\emptyset$ and $E\cap\tilde{Q}_r(x,t)\neq\emptyset$, where constants c_3,c_4,c_5 and c_6 depend on N,α,p and $[w]_{A_\infty}$.

Claim (2.3): Set

$$g_k(x,t)=\int_{2^{-k}R}^{2^{-k+1}R}\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}.$$

We have for $m\in\mathbb{N}$ and $(x,t)\in E$

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](x,t)&=\sum_{k=m+1}^{\infty}g_k(x,t)+\int_{2^{-m}R}^{R}\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{d\rho}{\rho}\\&\leq\sum_{k=m+1}^{\infty}g_k(x,t)+m(\mathbb{M}^R_{\alpha p}[\mu](x,t))^{\frac{1}{p-1}}\\&\leq\sum_{k=m+1}^{\infty}g_k(x,t)+m\varepsilon\lambda.\end{aligned}$$

We deduce that for $\beta>0$, $m\in\mathbb{N}$

$$\begin{aligned}|E|&\leq|Q\cap\{\sum_{k=m+1}^{\infty}g_k>(1-m\varepsilon)\lambda\}|\\&=|Q\cap\{\sum_{k=m+1}^{\infty}g_k>\sum_{k=m+1}^{\infty}2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\}|\\&\leq\sum_{k=m+1}^{\infty}|Q\cap\{g_k>2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\}|.\end{aligned}$$

We can assume that $(x_0,t_0)\in Q$, $(\mathbb{M}^R_{\alpha p}[\mu](x_0,t_0))^{\frac{1}{p-1}}\leq\varepsilon\lambda$. Thus, by computing, see [**17**, Proof of Proposition 2.2] we have for any $k\in\mathbb{N}$

$$|Q\cap\{g_k>s\}|\lesssim\frac{1}{s^{p-1}}2^{-k\alpha p}|Q|(\varepsilon\lambda)^{p-1}.$$

Consequently,

$$\begin{aligned}|E|&\lesssim\sum_{k=m+1}^{\infty}\frac{1}{\left(2^{-\beta(k-m-1)}(1-2^{-\beta})(1-m\varepsilon)\lambda\right)^{p-1}}2^{-k\alpha p}|Q|(\varepsilon\lambda)^{p-1}\\&\lesssim 2^{-(m+1)\alpha p}\left(\frac{\varepsilon}{1-m\varepsilon}\right)^{p-1}|Q|\left(1-2^{-\beta}\right)^{-p+1}\sum_{k=m+1}^{\infty}2^{(\beta(p-1)-\alpha p)(k-m-1)}.\end{aligned}$$

If we choose $\varepsilon^{-1}-2<m\leq\varepsilon^{-1}-1$ and $\beta=\beta(\alpha,p)$ so that $\beta(p-1)-\alpha p<0$, we obtain

$$|E|\lesssim\exp(-\alpha p\ln(2)\varepsilon^{-1})|Q|.$$

Thus, we get (2.3).

Claim (2.4). Take $(x,t)\in Q$ and $0<r\leq R$. Now assume that $\tilde{Q}_r(x,t)\cap Q\cap F^c\neq\emptyset$ and $E\cap\tilde{Q}_r(x,t)\neq\emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2)\in\tilde{Q}_r(x,t)\cap Q$ such that $\mathbb{W}^R_{\alpha,p}[\mu](x_1,t_1)\leq\lambda$ and $(\mathbb{M}^R_{\alpha p}[\mu](x_2,t_2))^{\frac{1}{p-1}}\leq\varepsilon\lambda$. We need to prove that

$$w(E\cap\tilde{Q}_r(x,t))<c\exp(-(c\varepsilon)^{-1})w(\tilde{Q}_r(x,t)).$$

To do this, for all $(y,s) \in E \cap \tilde{Q}_r(x,t)$, $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{3\rho}(x_1,t_1)$ if $\rho > r$.
If $r \le R/3$, one has

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](y,s) &= \mathbb{W}^r_{\alpha,p}[\mu](y,s) + \int_r^{R/3} \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\ &\quad + \int_{R/3}^{R} \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\ &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + \int_r^{R/3} \left(\frac{\mu(\tilde{Q}_{3\rho}(x_1,t_1))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} + 2(\mathbb{M}^R_{\alpha p}[\mu](y,s))^{\frac{1}{p-1}} \\ &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + 3^{\frac{N+2-\alpha p}{p-1}}\lambda + 2\varepsilon\lambda.\end{aligned}$$

This gives $\mathbb{W}^r_{\alpha,p}[\mu](y,s) > \lambda$.
If $r \ge R/3$, one has

$$\begin{aligned}\mathbb{W}^R_{\alpha,p}[\mu](y,s) &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + \int_{R/3}^{R} \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\ &\le \mathbb{W}^r_{\alpha,p}[\mu](y,s) + 2\varepsilon\lambda,\end{aligned}$$

This gives $\mathbb{W}^r_{\alpha,p}[\mu](y,s) > \lambda$.
Thus,

$$w(E \cap \tilde{Q}_r(x,t)) \le w(\tilde{Q}_r(x,t) \cap \{\mathbb{W}^r_{\alpha,p}[\mu] > \lambda\}).$$

Since $(x_2,t_2) \in \tilde{Q}_r(x,t)$, $(\mathbb{M}^R_{\alpha p}[\mu](x_2,t_2))^{\frac{1}{p-1}} \le \varepsilon\lambda$, so as above we also obtain

$$w(\tilde{Q}_r(x,t) \cap \{\mathbb{W}^r_{\alpha,p}[\mu] > \lambda\}) \le c\exp(-(c\varepsilon)^{-1}) w(\tilde{Q}_r(x,t)),$$

which implies (2.4). This completes the proof. □

THEOREM 2.2. *Let $\alpha > 0$, $1 < p < \alpha^{-1}(N+2)$, $p-1 < q < \infty$ and $0 < s \le \infty$ and $w \in A_\infty$. There holds*

$$||\mathbb{W}^R_{\alpha,p}[\mu]||_{L^{q,s}(\mathbb{R}^{N+1},dw)} \sim_C ||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||_{L^{q,s}(\mathbb{R}^{N+1},dw)}, \tag{2.5}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ and $R \in (0,\infty]$ where C is a positive constant only depending on α, p, q, s and $[w]_{A_\infty}$.

PROOF. Thanks to (2.1) in Theorem (2.1), we have for $0 < s < \infty$

$$\begin{aligned}&||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} = a^s q \int_0^\infty \lambda^s w(\{\mathbb{W}^R_{\alpha,p}[\mu] > a\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\ &\le c\exp(-\frac{1}{c\varepsilon}) q \int_0^\infty \lambda^s w(\{\mathbb{W}^R_{\alpha,p}[\mu] > \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} + cs\int_0^\infty \lambda^s w(\{(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}} > \varepsilon\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\ &= c\exp(-\frac{1}{c\varepsilon})||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} + c\varepsilon^{-s}||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)}.\end{aligned}$$

Choose $0 < \varepsilon < \varepsilon_0$ such that $c\exp(-\frac{1}{c\varepsilon}) < 1/2$ we get

$$||\mathbb{W}^R_{\alpha,p}[\mu]||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)} \lesssim ||(\mathbb{M}^R_{\alpha p}[\mu])^{\frac{1}{p-1}}||^s_{L^{q,s}(\mathbb{R}^{N+1},dw)}.$$

Similarly, we also get above inequality in case $s = \infty$. So, we proved the right-hand side inequality of (2.5).

To complete the proof, we prove the left-hand side inequality of (2.5). Since for every $(x,t)\in\mathbb{R}^{N+1}$

$$(\mathbb{M}^R_{\alpha p}[\mu](x,t))^{\frac{1}{p-1}} \lesssim \mathbb{W}^R_{\alpha,p}[\mu](x,t) + \left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}},$$

$$\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \lesssim \mathbb{W}^R_{\alpha,p}[\mu](x,t).$$

Thus it is enough to show that for any $\lambda>0$

(2.6)

$$w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right) \lesssim w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\gtrsim\lambda\right\}\right).$$

Let $\{Q_j\}=\{\tilde{Q}_{R/4}(x_j,t_j)\}$ be a cover of $\mathbb{R}^{N+1}$ such that for any $Q_j\in\{Q_j\}$, there exist $Q_{j,1},...,Q_{j,M_1}\in\{Q_j\}$ with $\sum_j\sum_{k=1}^{M_1}\chi_{Q_{j,k}}\le M_2$ and $Q_j+\tilde{Q}_{2R}(0,0)\subset\bigcup_{k=1}^{M_1}Q_{j,k}$ for some integer constants $M_i=M_i(N), i=1,2$. Then,

$$w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right) \le \sum_j w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\cap Q_j\right)$$
$$\le \sum_j w\left(\left\{(x,t):\sum_{k=1}^{M_1}\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}>\lambda^{p-1}\right\}\cap Q_j\right)$$
$$\le \sum_j\sum_{k=1}^{M_1} w\left(\left\{(x,t):\left(\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}\cap Q_j\right)$$
$$=\sum_j\sum_{k=1}^{M_1}a_{j,k}w(Q_j),$$

where $a_{j,k}=1$ if $\left(\frac{\mu(Q_{j,k})}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda$ and $a_{j,k}=0$ if otherwise.
Using the strong doubling property of w, there is $c_0=c_0(N,[w]_{A_\infty})$ such that $w(Q_j)\le c_0 w(Q_{j,k})$. On the other hand, if $a_{j,k}=1$ then

$$Q_{j,k}\subset\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}.$$

Therefore,

$$w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{2R}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>\lambda\right\}\right)\le\sum_j\sum_{k=1}^{M_1}c_0a_{j,k}w(Q_{j,k})$$
$$\le\sum_j\sum_{k=1}^{M_1}c_0w\left(\left\{(x,t):\left(\frac{\mu(\tilde{Q}_{R/2}(x,t))}{R^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}>M_1^{-1/(p-1)}\lambda\right\}\cap Q_{j,k}\right),$$

which implies (2.6) since $\sum_j\sum_{k=1}^{M_1}\chi_{Q_{j,k}}\le M_2$ in $\mathbb{R}^{N+1}$. □

THEOREM 2.3. *Let $0 < \alpha p < N+2$ and $w \in A_\infty$. For any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, any cylinder $\tilde{Q}_\rho \subset \mathbb{R}^{N+1}$ there holds*

$$\frac{1}{w(\tilde{Q}_{2\rho})}\int_{\tilde{Q}_{2\rho}} \exp\left(C^{-1}\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}](x,t)\right)dw(x,t) \lesssim_{[w]_{A_\infty}} 1 \tag{2.7}$$

provided $||\mathbb{M}^R_{\alpha p}[\mu_{\tilde{Q}_\rho}]||_{L^\infty(\tilde{Q}_\rho)} \leq 1$, where $\mu_{\tilde{Q}_\rho} = \chi_{\tilde{Q}_\rho}\mu$.

PROOF. Assume that $||\mathbb{M}^R_{\alpha p}[\mu_{\tilde{Q}_\rho}]||_{L^\infty(\tilde{Q}_\rho)} \leq 1$. We apply Theorem (2.1) to $\mu_{\tilde{Q}_\rho}$. Then, choose $\varepsilon = \lambda^{-1}$ for all $\lambda \geq \lambda_0 := \max\{\varepsilon_0^{-1}, \frac{N+2-\alpha p}{p-1}\}$, we obtain

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] > a\lambda\} \cap \tilde{Q}_{2\rho}) \leq c\exp(-\lambda/c)w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] > \lambda\}) \quad \forall \lambda \geq \lambda_0.$$

On the other hand, if $\rho > R$, clearly we have $\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] \equiv 0$ in $\mathbb{R}^{N+1}\backslash\tilde{Q}_{2\rho}$, if $\rho \leq R$, for any $(x,t) \in \mathbb{R}^{N+1}\backslash\tilde{Q}_{2\rho}$

$$\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}](x,t) = \int_\rho^R \left(\frac{\mu_{\tilde{Q}_\rho}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dr}{r} \leq \frac{N+2-\alpha p}{p-1}\left(\frac{\mu(\tilde{Q}_\rho)}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \leq \lambda_0.$$

So, we get $\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] > \lambda\} \subset \tilde{Q}_{2\rho}$ for all $\lambda \geq \lambda_0$. This can be written under the form

$$w(\{\mathbb{W}^R_{\alpha,p}[\mu_{\tilde{Q}_\rho}] > a\lambda\} \cap \tilde{Q}_{2\rho}) \leq \left(\chi_{(0,\lambda_0]} + c\exp(-\lambda/c)\right)w(\tilde{Q}_{2\rho}),$$

for all $\lambda > 0$. Therefore, we get (2.7). □

In what follows, we need some estimates on Wolff parabolic potential:

PROPOSITION 2.4. Let $p > 1, 0 < \alpha p < N+2$ and $q > 1, \alpha pq < N+2$. There hold

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{(N+2)(p-1)}{N+2-\alpha p},\infty}(\mathbb{R}^{N+1})} \lesssim (\mu(\mathbb{R}^{N+1}))^{\frac{1}{p-1}} \quad \forall \mu \in \mathfrak{M}^+_b(\mathbb{R}^{N+1}), \tag{2.8}$$

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha pq},\infty}(\mathbb{R}^{N+1})} \lesssim ||\mu||^{\frac{1}{p-1}}_{L^{q,\infty}(\mathbb{R}^{N+1})} \quad \forall \mu \in L^{q,\infty}(\mathbb{R}^{N+1}), \mu \geq 0, \tag{2.9}$$

$$||\mathbb{W}_{\alpha,p}[\mu]||_{L^{\frac{q(N+2)(p-1)}{N+2-\alpha pq}}(\mathbb{R}^{N+1})} \lesssim ||\mu||^{\frac{1}{p-1}}_{L^{q}(\mathbb{R}^{N+1})} \quad \forall \mu \in L^{q}(\mathbb{R}^{N+1}), \mu \geq 0. \tag{2.10}$$

In particular, for $s > \frac{(p-1)(N+2)}{N+2-\alpha p}$, we define $F(\mu) := (\mathbb{W}_{\alpha,p}[\mu])^s$ for all $\mu \in \mathfrak{M}^+_b(\mathbb{R}^{N+1})$. Then,

$$||F(\mu)||_{L^{\frac{(N+2)(s-p+1)}{\alpha sp}}(\mathbb{R}^{N+1})} \lesssim ||\mu||^{\frac{s}{p-1}}_{L^{\frac{(N+2)(s-p+1)}{\alpha sp}}(\mathbb{R}^{N+1})},$$
$$||F(\mu)||_{L^{\frac{(N+2)(s-p+1)}{\alpha sp},\infty}(\mathbb{R}^{N+1})} \lesssim ||\mu||^{\frac{s}{p-1}}_{L^{\frac{(N+2)(s-p+1)}{\alpha sp},\infty}(\mathbb{R}^{N+1})}.$$

PROOF. Let $s \geq 1$ be such that $\alpha sp < N+2$. It is known that if $\mu \in L^{s,\infty}(\mathbb{R}^{N+1})$ then

$$|\mu|(\tilde{Q}_\rho(x,t)) \lesssim ||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})}\rho^{\frac{N+2}{s'}} \quad \forall \rho > 0.$$

Thus for $\delta = ||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})}^{\frac{s}{N+2}} (\mathbb{M}(\mu)(x,t))^{-\frac{s}{N+2}}$, we have

$$\begin{aligned}\mathbb{W}_{\alpha,p}[\mu](x,t) &= \int_0^\delta \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} + \int_\delta^\infty \left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\ &\lesssim (\mathbb{M}(\mu)(x,t))^{\frac{1}{p-1}} \delta^{\frac{\alpha p}{p-1}} + ||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})}^{\frac{1}{p-1}} \delta^{-\frac{N+2-\alpha sp}{s(p-1)}} \\ &= (\mathbb{M}(\mu)(x,t))^{\frac{N+2-\alpha sp}{(p-1)(N+2)}} ||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})}^{\frac{\alpha sp}{(p-1)(N+2)}}.\end{aligned}$$

So, for any $\lambda > 0$

$$|\{\mathbb{W}_{\alpha,p}[\mu] > \lambda\}| \leq |\{\mathbb{M}(\mu) \gtrsim ||\mu||_{L^{s,\infty}(\mathbb{R}^{N+1})}^{-\frac{\alpha sp}{N+2-\alpha sp}} \lambda^{\frac{(p-1)(N+2)}{N+2-\alpha sp}}\}|.$$

Since $\mathbb{M}$ is bounded from $\mathfrak{M}_b^+(\mathbb{R}^{N+1})$ to $L^{1,\infty}(\mathbb{R}^{N+1})$ and $L^q(\mathbb{R}^{N+1})$ ($L^{q,\infty}(\mathbb{R}^{N+1})$ resp.) to itself, we get the result. $\square$

Remark 2.5. Assume that $\alpha p = N+2$ and $R > 0$. As above we also have for any $\varepsilon > 0$

$$\mathbb{W}_{\alpha,p}^R[\mu](x,t) \lesssim_\varepsilon \max\left\{(|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}, \left((\mathbb{M}(\mu)(x,t))^\varepsilon (|\mu|(\mathbb{R}^{N+1}))^{\frac{\alpha p}{p-1}} R^{\varepsilon \alpha p}\right)^{\frac{1}{\alpha p + \varepsilon(p-1)}}\right\}.$$

Therefore, for any $\lambda \gtrsim_\varepsilon (|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}$,

$$|\{\mathbb{W}_{\alpha,p}^R[\mu] > \lambda\}| \lesssim_\varepsilon \left(\frac{(|\mu|(\mathbb{R}^{N+1}))^{\frac{1}{p-1}}}{\lambda}\right)^{\frac{\alpha p + \varepsilon(p-1)}{\varepsilon}} R^{\alpha p}, \tag{2.11}$$

In particular, if $\mu \in \mathfrak{M}_b^+(\mathbb{R}^{N+1})$ then $\mathbb{W}_{\alpha,p}^R[\mu] \in L^s_{\mathrm{loc}}(\mathbb{R}^{N+1})$ for all $s > 0$.

Remark 2.6. Assume that $p, q > 1, 0 < \alpha pq < N+2$. As in [**66**, Theorem 3], it is easy to prove that if $w \in A_{\frac{q(N+2-\alpha)}{N+2-\alpha pq}}$, i.e, $0 < w \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$ and for any $\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}$

$$\sup_{\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}} \left(\left(\fint_{\tilde{Q}_\rho(y,s)} w dx dt\right) \left(\fint_{\tilde{Q}_\rho(y,s)} w^{-\frac{N+2-\alpha pq}{(q-1)(N+2)}} dx dt\right)^{\frac{(q-1)(N+2)}{N+2-\alpha pq}}\right) = C_1 < \infty,$$

then

$$\left(\int_{\mathbb{R}^{N+1}} (\mathbb{M}_{\alpha p}[|f|])^{\frac{(N+2)q}{N+2-\alpha pq}} w dx dt\right)^{\frac{N+2-\alpha pq}{(N+2)q}} \leq C_2 \left(\int_{\mathbb{R}^{N+1}} |f|^q w^{1-\frac{\alpha pq}{N+2}} dx dt\right)^{\frac{1}{q}},$$

for some a constant $C_2 = C_2(N, \alpha p, q, C_1)$.
Therefore, from (2.5) in Theorem 2.2 we get a weighted version of (2.10):

$$\left(\int_{\mathbb{R}^{N+1}} (\mathbb{W}_{\alpha,p}[|f|])^{\frac{(N+2)(p-1)q}{N+2-\alpha pq}} w dx dt\right)^{\frac{N+2-\alpha pq}{(N+2)q}} \leq C_2 \left(\int_{\mathbb{R}^{N+1}} |f|^p w^{1-\frac{\alpha p}{N+2}} dx dt\right)^{\frac{1}{p}}.$$

In the following proposition, we give another version of (2.10) in the Lorentz-Morrey spaces involving calorie.

Proposition 2.7. Let $p, q > 1$, and $0 < \alpha pq < \theta \leq N+2$. There holds

$$||(\mathbb{W}_{\alpha,p}[|\mu|])^{p-1}||_{L^{\frac{\theta q}{\theta - \alpha pq};\theta}(\mathbb{R}^{N+1})} \lesssim ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} \quad \forall \mu \in L^{q;\theta}(\mathbb{R}^{N+1}). \tag{2.12}$$

PROOF. As the proof of Proposition 2.4 we have

$$\mathbb{W}_{\alpha,p}[|\mu|] \lesssim \left(\mathbb{M}_{\theta/q}[|\mu|]\right)^{\frac{\alpha pq}{\theta(p-1)}} \left(\mathbb{M}[|\mu|]\right)^{\frac{\theta-\alpha pq}{\theta(p-1)}}.$$

Since $\mathbb{M}_{\theta/q}[|\mu|] \lesssim (\mathbb{M}_\theta[|\mu|^q])^{1/q}$, above inequality becomes

$$\mathbb{W}_{\alpha,p}[\mu] \lesssim (\mathbb{M}_\theta[|\mu|^q])^{\frac{\alpha p}{\theta(p-1)}} (\mathbb{M}[\mu])^{\frac{\theta-\alpha pq}{\theta(p-1)}}. \tag{2.13}$$

Take $\tilde{Q}_\rho(y,s) \subset \mathbb{R}^{N+1}$, we have

$$\begin{aligned}\int_{\tilde{Q}_\rho(y,s)} (\mathbb{W}_{\alpha,p}[\mu])^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt &\lesssim \int_{\tilde{Q}_\rho(y,s)} \left(\mathbb{W}_{\alpha,p}[\chi_{\tilde{Q}_{2\rho}(y,s)}\mu]\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &+ \int_{\tilde{Q}_\rho(y,s)} \left(\mathbb{W}_{\alpha,p}[\chi_{(\tilde{Q}_{2\rho}(y,s))^c}\mu]\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &= A + B.\end{aligned}$$

Using (2.13) and boundedness of $\mathbb{M}$ from $L^q(\mathbb{R}^{N+1})$ to itself, yield

$$\begin{aligned}A &\lesssim \int_{\mathbb{R}^{N+1}} (\mathbb{M}_\theta[|\mu|^q])^{\frac{\alpha pq}{\theta-\alpha pq}} \left(\mathbb{M}[\chi_{\tilde{Q}_{2\rho}(y,s)}\mu]\right)^q dxdt \\ &\lesssim ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\alpha pq^2}{\theta-\alpha pq}} \int_{\chi_{\tilde{Q}_{2\rho}(y,s)}} |\mu|^q dxdt \\ &\lesssim ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta}.\end{aligned}$$

On the other hand, since $|\mu|(\tilde{Q}_r(x,t)) \lesssim ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} r^{N+2-\frac{\theta}{q}}$ for all $\tilde{Q}_r(x,t) \subset \mathbb{R}^{N+1}$,

$$\begin{aligned}B &\leq \int_{\tilde{Q}_\rho(y,s)} \left(\int_\rho^\infty \left(\frac{|\mu|(\tilde{Q}_r(x,t))}{r^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\lesssim \int_{\tilde{Q}_\rho(y,s)} \left(\int_\rho^\infty \left(||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})} r^{-\frac{\theta}{q}+\alpha}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right)^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \\ &\lesssim ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta}.\end{aligned}$$

Therefore,

$$\int_{\tilde{Q}_\rho(y,s)} (\mathbb{W}_{\alpha,p}[\mu])^{\frac{\theta q(p-1)}{\theta-\alpha pq}} dxdt \lesssim ||\mu||_{L^{q;\theta}(\mathbb{R}^{N+1})}^{\frac{\theta q}{\theta-\alpha pq}} \rho^{N+2-\theta},$$

which follows (2.12). □

In the next result we state a series of equivalent norms concerning potentials $\mathbb{I}_\alpha, \mathbb{I}_\alpha^R, \mathcal{H}_\alpha, \mathcal{G}_\alpha$.

PROPOSITION 2.8. Let $q > 1$, $0 < \alpha < N+2$ and $R > 0$. Then, the following statements hold

a: for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$

$$||\mathcal{H}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \sim_{\alpha,q} ||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})}, \tag{2.14}$$

$$||\overset{\vee}{\mathcal{H}}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \sim_{\alpha,q} ||\mathbb{I}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})}. \tag{2.15}$$

b: for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$

$$||\mathcal{G}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \sim_{\alpha,q,R} ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}, \tag{2.16}$$

$$||\overset{\vee}{\mathcal{G}}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} \sim_{\alpha,q,R} ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}. \tag{2.17}$$

where $\overset{\vee}{\mathcal{H}}_\alpha[\mu]$ is the backward parabolic Riesz potential, defined by

$$\overset{\vee}{\mathcal{H}}_\alpha[\mu](x,t) = (\overset{\vee}{\mathcal{H}}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{H}_\alpha(y-x, s-t) d\mu(y,s),$$

and $\overset{\vee}{\mathcal{G}}_\alpha[\mu]$ is the backward parabolic Bessel potential:

$$\overset{\vee}{\mathcal{G}}_\alpha[\mu](x,t) = (\overset{\vee}{\mathcal{G}}_\alpha * \mu)(x,t) = \int_{\mathbb{R}^{N+1}} \mathcal{G}_\alpha(y-x, s-t) d\mu(y,s).$$

Proof. **a.** We have:

$$\frac{1}{t^{\frac{N+2-\alpha}{2}}} \chi_{t>0}\chi_{|x|\le 2\sqrt{t}} \lesssim \mathcal{H}_\alpha(x,t) \lesssim \frac{1}{\max\{|x|, \sqrt{2|t|}\}^{N+2-\alpha}},$$

which implies

$$\int_0^\infty \frac{\chi_{B_r(0)\times(\frac{r^2}{4},r^2)}(x,t)}{r^{N+2-\alpha}} \frac{dr}{r} \lesssim \mathcal{H}_\alpha(x,t) \lesssim \int_0^\infty \frac{\chi_{\tilde{Q}_r(0,0)}(x,t)}{r^{N+2-\alpha}} \frac{dr}{r}.$$

Thus,

$$\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2, t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}} \frac{dr}{r} \lesssim \mathcal{H}_\alpha[\mu](x,t) \lesssim \mathbb{I}_\alpha[\mu](x,t). \tag{2.18}$$

Thanks to Theorem 2.2 we will finish the proof of (2.14) if we show that

$$\int_{\mathbb{R}} \left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2, t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}} \frac{dr}{r} \right)^q dt \gtrsim \int_{\mathbb{R}} \int_0^{+\infty} \left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \right)^q \frac{dr}{r} dt.$$

Indeed, we have for $r_k = (\frac{2}{\sqrt{3}})^{-k}$,

$$\left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2, t-r^2/4)\right)}{r^{N+2-\alpha}} \frac{dr}{r} \right)^q \gtrsim \sum_{k=-\infty}^{\infty} \left(\frac{\mu\left(B(x,r_k)\times(t-r_k^2, t-\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}} \right)^q.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_0^\infty \frac{\mu\left(B(x,r)\times(t-r^2, t-\frac{1}{4}r^2)\right)}{r^{N+2-\alpha}} \frac{dr}{r} \right)^q dt \\
&\qquad \gtrsim \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \left(\frac{\mu\left(B(x,r_k)\times(t-\frac{1}{3}r_k^2, t+\frac{1}{3}r_k^2)\right)}{r_k^{N+2-\alpha}} \right)^q dt \\
&\qquad \gtrsim \int_{\mathbb{R}} \int_0^{+\infty} \left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \right)^q \frac{dr}{r} dt.
\end{aligned}$$

Similarly, we also can prove (2.15).
b. Obviously

$$\frac{\exp(-4R^2)}{t^{\frac{N+2-\alpha}{2}}}\chi_{0<t<4R^2}\chi_{|x|\le 2\sqrt{t}} \lesssim \mathcal{G}_\alpha(x,t)$$
$$\lesssim \frac{\chi_{\tilde{Q}_{R/2}(0,0)}(x,t)}{\max\{|x|,\sqrt{2|t|}\}^{N+2-\alpha}} + \frac{\exp\left(-\max\{|x|,\sqrt{2|t|}\}\right)}{R^{N+2-\alpha}}.$$

Thus, we can assert that

$$\int_0^{2R}\frac{\chi_{B_r(0)\times(\frac{r^2}{4},r^2)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r} \lesssim_R \mathcal{G}_\alpha(x,t) \lesssim \int_0^R \frac{\chi_{\tilde{Q}_r(0,0)}(x,t)}{r^{N+2-\alpha}}\frac{dr}{r}$$
$$+ c(R)\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|y|,\sqrt{2|s|}\}\right)\chi_{\tilde{Q}_{R/2}(0,0)}(x-y,t-s)dyds.$$

Immediately, we get
(2.19)

$$\int_0^{2R}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r} \lesssim_R \mathcal{G}_\alpha[\mu](x,t) \lesssim \mathbb{I}_\alpha^R[\mu](x,t) + c(R)\mathbf{F}(x,t),$$

where

$$\mathbf{F}(x,t) = \int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|y|,\sqrt{2|s|}\}\right)\mu\left(\tilde{Q}_{R/2}(x-y,t-s)\right)dyds.$$

As above, we can show that

$$\int_0^\infty\left(\int_0^{2R}\frac{\mu\left(B(x,r)\times(t-r^2,t-\frac{r^2}{4})\right)}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \gtrsim \int_0^\infty\int_0^R\left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\right)^q\frac{dr}{r}.$$

Thus, thanks to Theorem 2.2 we get the left-hand side inequality of (2.16).
To obtain the right-hand side of (2.16), we use $\mu\left(\tilde{Q}_{R/2}(x-y,t-s)\right) \lesssim R^{-(N+2-\alpha)}\mathbb{I}_\alpha^R[\mu](x-y,t-s)$ and Young's inequality

$$\begin{aligned}||\mathcal{G}_\alpha[\mu]||_{L^q(\mathbb{R}^{N+1})} &\lesssim_R ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} + ||\mathbf{F}||_{L^q(\mathbb{R}^{N+1})}\\ &\lesssim_R ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})} + ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}\int_{\mathbb{R}^{N+1}}\exp\left(-\max\{|x|,\sqrt{2|t|}\}\right)dxdt\\ &\lesssim_R ||\mathbb{I}_\alpha^R[\mu]||_{L^q(\mathbb{R}^{N+1})}.\end{aligned}$$

Similarly, we also can obtain (2.17). This completes the proof. □

REMARK 2.9. Assume that $0<\alpha<N+2$. From (2.8) in Proposition 2.4 and $||\mathcal{G}_\alpha[\mu]||_{L^1(\mathbb{R}^{N+1})} \lesssim \mu(\mathbb{R}^{N+1})$ we deduce that for $1\le s<\frac{N+2}{N+2-\alpha}$

$$||\mathcal{G}_\alpha[\mu]||_{L^s(\mathbb{R}^{N+1})} \lesssim \mu(\mathbb{R}^{N+1}) \quad \forall \mu\in\mathfrak{M}_b^+(\mathbb{R}^{N+1}).$$

Next, we introduce the following kernel:

$$E_\alpha^R(x,t) = \max\{|x|,\sqrt{2|t|}\}^{-(N+2-\alpha)}\chi_{\tilde{Q}_R(0,0)}(x,t)$$

where $0<\alpha<N+2$ and $0<R\le\infty$. We denote E_α^∞ by E_α. It is easy to see that $E_\alpha*\mu = (N+2-\alpha)\mathbb{I}_\alpha[\mu]$ and $||E_\alpha^R*\mu||_{L^s(\mathbb{R}^{N+1})}$ is equivalent to $||\mathbb{I}_\alpha^R[\mu]||_{L^s(\mathbb{R}^{N+1})}$ for every $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$ where $1\le s<\infty$.
We obtain equivalences of capacities $\text{Cap}_{E_\alpha,p}$, $\text{Cap}_{E_\alpha^R,p}$, $\text{Cap}_{\mathcal{H}_\alpha,p}$ and $\text{Cap}_{\mathcal{G}_\alpha,p}$.

COROLLARY 2.10. Let $p>1$, $1<\alpha<N+2$ and $R>0$. Then, the following statements hold

a: for any compact $E\subset\mathbb{R}^{N+1}$

$$\mathrm{Cap}_{E_\alpha,p}(E)\lesssim \mathrm{Cap}_{\mathcal{H}_\alpha,p}(E), \tag{2.20}$$

b: for any compact $E\subset\mathbb{R}^{N+1}$

$$\mathrm{Cap}_{E^R_\alpha,p}(E)\lesssim_R \mathrm{Cap}_{\mathcal{G}_\alpha,p}(E), \tag{2.21}$$

c: for any compact $E\subset\mathbb{R}^{N+1}$

$$\mathrm{Cap}_{\mathcal{H}_\alpha,p}(E)\le \mathrm{Cap}_{\mathcal{G}_\alpha,p}(E)\lesssim \mathrm{Cap}_{\mathcal{H}_\alpha,p}(E)+\left(\mathrm{Cap}_{\mathcal{H}_\alpha,p}(E)\right)^{\frac{N+2}{N+2-\alpha p}} \tag{2.22}$$

provided $1<\alpha p<N+2$.

PROOF. By [**2**, Chapter 2], we have

$$\begin{aligned}
&\mathrm{Cap}_{E_\alpha,p}(E)^{1/p}=\sup\{\mu(E):\mu\in\mathfrak{M}^+(E),||E_\alpha*\mu||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\},\\
&\mathrm{Cap}_{E^R_\alpha,p}(E)^{1/p}=\sup\{\mu(E):\mu\in\mathfrak{M}^+(E),||E^R_\alpha*\mu||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\},\\
&\mathrm{Cap}_{\mathcal{H}_\alpha,p}(E)^{1/p}=\sup\{\mu(E):\mu\in\mathfrak{M}^+(E),||\overset{\vee}{\mathcal{H}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\},\\
&\mathrm{Cap}_{\mathcal{G}_\alpha,p}(E)^{1/p}=\sup\{\mu(E):\mu\in\mathfrak{M}^+(E),||\overset{\vee}{\mathcal{G}}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\}.
\end{aligned}$$

Thanks to (2.15), (2.17) in Proposition 2.8 and $\mathbb{I}_\alpha[\mu]=E_\alpha*\mu$ and $||E^R_\alpha*\mu||_{L^s(\mathbb{R}^{N+1})}\sim ||\mathbb{I}^R_\alpha[\mu]||_{L^s(\mathbb{R}^{N+1})}$, we get (2.20) and (2.21).
Since $\mathcal{G}_\alpha\le\mathcal{H}_\alpha$, thus $\mathrm{Cap}_{\mathcal{H}_\alpha,p}(E)\le \mathrm{Cap}_{\mathcal{G}_\alpha,p}(E)$ for any compact $E\subset\mathbb{R}^{N+1}$. Set $\mathrm{Cap}_{E_\alpha,p}(E)=a>0$. We need to prove that

$$\mathrm{Cap}_{E^1_\alpha,p}(E)\lesssim a+a^{\frac{N+2}{N+2-\alpha p}}. \tag{2.23}$$

We will follow a proof of Yu.V. Netrusov in [**2**, Chapter 5]. First, we can find $f\in L^p_+(\mathbb{R}^{N+1})$ such that $||f||_{L^p(\mathbb{R}^{N+1})}\le 2a$ and $E_\alpha*f\ge\chi_E$. Set $F_\alpha=E_\alpha-E^1_\alpha$, we have $c_1F_\alpha\le E^1_\alpha*F_\alpha$ for some $c_1>0$. Thus, $E\subset\{E^1_\alpha*f\ge 1/2\}\cup\{E^1_\alpha*(F_\alpha*f)\ge c_1/2\}$.
Since $||E^1_\alpha||_{L^1(\mathbb{R}^{N+1})}<\infty$, for $c_2=c_1(4||E^1_\alpha||_{L^1(\mathbb{R}^{N+1})})^{-1}$

$$E^1_\alpha*(F_\alpha*f)\le c_1/4+E^1_\alpha*g \text{ with } g=\chi_{F_\alpha*f\ge c_2}F_\alpha*f,$$

which follows $E\subset\{E^1_\alpha*f\ge 1/2\}\cup\{E^1_\alpha*g\ge c_1/4\}$.
Using the subadditivity of capacity, we have

$$\begin{aligned}
\mathrm{Cap}_{E^1_\alpha,p}(E)&\le \mathrm{Cap}_{E^1_\alpha,p}(\{E^1_\alpha*f\ge 1/2\})+\mathrm{Cap}_{E^1_\alpha,p}(\{E^1_\alpha*g\ge c_1/4\})\\
&\lesssim ||f||^p_{L^p(\mathbb{R}^{N+1})}+||g||^p_{L^p(\mathbb{R}^{N+1})}\\
&\lesssim ||f||^p_{L^p(\mathbb{R}^{N+1})}+||E_\alpha*f||^{p*}_{L^{p*}(\mathbb{R}^{N+1})},\ \text{with } p*=\frac{(N+2)p}{N+2-\alpha p}.
\end{aligned}$$

On the other hand, from (2.10) in Proposition 2.4 we have

$$||E_\alpha*f||_{L^{p*}(\mathbb{R}^{N+1})}\lesssim ||f||_{L^p(\mathbb{R}^{N+1})}.$$

Hence, we get (2.23). The proof is complete. $\square$

REMARK 2.11. Since $\mathcal{G}_\alpha \in L^1(\mathbb{R}^{N+1})$,

$$\int_{\mathbb{R}^{N+1}} (\mathcal{G}_\alpha * f)^p \, dxdt \leq ||\mathcal{G}_\alpha||^p_{L^1(\mathbb{R}^{N+1})} \int_{\mathbb{R}^{N+1}} f^p dxdt \forall f \in L^p_+(\mathbb{R}^{N+1}).$$

Thus, for any Borel set $E \subset \mathbb{R}^{N+1}$

$$Cap_{\mathcal{G}_\alpha,p}(E) \geq C|E| \text{with } C = ||\mathcal{G}_\alpha||^{-p}_{L^1(\mathbb{R}^{N+1})}. \tag{2.24}$$

REMARK 2.12. It is well-known that $\mathcal{H}_2$ is the fundamental solution of the heat operator $\frac{\partial}{\partial t} - \Delta$. In [**37**], R. Gariepy and W. P. Ziemer introduced the following capacity:

$$\mathrm{C}_{\mathcal{H}_2}(K) = \sup \left\{ \mu(K) : \mu \in \mathfrak{M}^+(K), \mathcal{H}_2[\mu] \leq 1 \right\},$$

whenever $K \subset \mathbb{R}^{N+1}$ is compact. Thanks to [**2**, Theorem 2.5.5], we obtain

$$\mathrm{Cap}_{\mathcal{H}_1,2}(K) = \mathrm{C}_{\mathcal{H}_2}(K).$$

REMARK 2.13. For any Borel set $E \subset \mathbb{R}^N$, then we always have $\mathrm{Cap}_{\mathcal{G}_1,2}(E \times \{t = 0\}) = 0$ In fact, for $B_1 = B_1(0)$

$$\mathrm{Cap}_{E_1^1,2}(B_1 \times \{t = 0\}) = \sup\{\omega(B_1) : \omega \in \mathfrak{M}^+(B_1), ||E_1^1 * (\omega \otimes \delta_0)||_{L^2(\mathbb{R}^{N+1})} \leq 1\}.$$

Since $||E_1^1 * (\omega \otimes \delta_0)||_{L^2(\mathbb{R}^{N+1})} = \infty$ if $\omega \neq 0$, thus $\mathrm{Cap}_{\mathcal{G}_1,2}(B_1 \times \{t = 0\}) = \mathrm{Cap}_{E_1^1,2}(B_1 \times \{t = 0\}) = 0$. In particular, $\mathrm{Cap}_{\mathcal{G}_1,2}$ is not absolutely continuous with respect to capacity $C_{1,2}(., \Omega \times (a, b))$. This capacity will be defined in next section.

REMARK 2.14. Let $p > 1, \alpha > 0$ and $\tilde{Q}_\rho = \tilde{Q}_\rho(0, 0)$ for $\rho > 0$. Case $\alpha p \geq N+1$, we always have $||\mathcal{H}_\alpha[\mu]||_{L^{p'}(\mathbb{R}^N)} = \infty$ for any $\mu \in \mathfrak{M}^+(\mathbb{R}^N)\backslash\{0\}$. This implies $\mathrm{Cap}_{\mathcal{H}_\alpha,p}(\tilde{Q}_1) = 0$. If $0 < \alpha p < N + 2$, $\mathrm{Cap}_{\mathcal{H}_\alpha,p}(\tilde{Q}_\rho) = c\rho^{N+2-\alpha p}$ for some constant $c > 0$. From (2.22) in Corollary 2.10 we get $\mathrm{Cap}_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho) \sim \rho^{N+2-\alpha p}$ for any $0 < \rho < 1$ if $\alpha p < N + 2$. Since $||\mathcal{G}_\alpha[\delta_{(0,0)}]||_{L^{p'}(\mathbb{R}^{N+1})} < \infty$ thus $\mathrm{Cap}_{\mathcal{G}_\alpha,p}((0, 0)) > 0$ if $\alpha p > N + 2$.

If $\alpha p = N + 2$, $\mathrm{Cap}_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho) \sim (\log(1/\rho))^{1-p}$ for any $0 < \rho < 1/2$. In fact, we can prove that $||\mathbb{I}_\alpha^{1/2}[\mu]||_{L^{p'}(\mathbb{R}^N)} \lesssim 1$ for any $d\mu(x, t) = (\log(1/\rho))^{-1/p'} \rho^{-N-2}\chi_{\tilde{Q}_\rho} dxdt$ it follows $\mathrm{Cap}_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho) \gtrsim (\log(1/\rho))^{1-p}$.

Moreover, for $\mu \in \mathfrak{M}^+(\tilde{Q}_\rho)$, if $||\mathbb{I}^3_\alpha[\mu]||^{p'}_{L^{p'}(\mathbb{R}^{N+1})} \leq 1$,

$$\begin{aligned}
1 &\geq \int_{\tilde{Q}_1\backslash\tilde{Q}_\rho} \left(\int_{2\max\{|x|,|2t|^{1/2}\}}^3 \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \frac{dr}{r} \right)^{p'} dxdt \\
&\geq \int_{\tilde{Q}_1\backslash\tilde{Q}_\rho} \left(\int_{2\max\{|x|,|2t|^{1/2}\}}^3 \frac{1}{r^{N+2-\alpha}} \frac{dr}{r} \right)^{p'} dxdt \mu(\tilde{Q}_\rho)^{p'} \\
&\gtrsim \log(1/\rho)\mu(\tilde{Q}_\rho)^{p'}.
\end{aligned}$$

So $\mathrm{Cap}_{\mathcal{G}_\alpha,p}(\tilde{Q}_\rho) \lesssim \mu(\tilde{Q}_\rho)^p \lesssim (\log(1/\rho))^{1-p}$.

DEFINITION 2.15. The parabolic Bessel potential $\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})$, $\alpha > 0$ and $p > 1$ is defined by

$$\mathcal{L}^p_\alpha(\mathbb{R}^{N+1}) = \left\{ f : f = \mathcal{G}_\alpha * g, g \in L^p(\mathbb{R}^{N+1}) \right\}$$

with the norm $||f||_{\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})} := ||g||_{L^p(\mathbb{R}^{N+1})}$. We denote its dual space by $\left(\mathcal{L}^p_\alpha(\mathbb{R}^{N+1})\right)^*$.

DEFINITION 2.16. Let k be a positive integer, the Sobolev space $W_p^{2k,k}(\mathbb{R}^{N+1})$ is defined by

$$W_p^{2k,k}(\mathbb{R}^{N+1}) = \left\{\varphi : \frac{\partial^{i_1+...+i_N+i}\varphi}{\partial x_1^{i_1}...\partial x_N^{i_N}\partial t^i} \in L^p(\mathbb{R}^{N+1})\text{for any} i_1+...+i_N+2i \leq 2k\right\}$$

with the norm

$$||\varphi||_{W_p^{2k,k}(\mathbb{R}^{N+1})} = \sum_{i_1+...+i_N+2i\leq 2k} ||\frac{\partial^{i_1+...+i_N+i}\varphi}{\partial x_1^{i_1}...\partial x_N^{i_N}\partial t^i}||_{L^p(\mathbb{R}^{N+1})}.$$

We denote its dual space by $\left(W_p^{2k,k}(\mathbb{R}^{N+1})\right)^*$. We also define a corresponding capacity on compact set $E \subset \mathbb{R}^{N+1}$,

$$Cap_{2k,k,p}(E) = \inf\left\{||\varphi||^p_{W_p^{2k,k}(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of} E\right\}.$$

Let us recall Richard J. Bagby's result, proved in [**5**].

THEOREM 2.17. *Let $p > 1$ and k be a positive integer. Then, there holds for any $u \in \mathcal{L}^p_{2k}(\mathbb{R}^{N+1})$,*

$$||u||_{\mathcal{L}^p_{2k}(\mathbb{R}^{N+1})} \sim ||u||_{W_p^{2k,k}(\mathbb{R}^{N+1})}.$$

This Theorem gives the assertion of equivalence of capacity $\text{Cap}_{2k,k,p}, \text{Cap}_{\mathcal{G}_{2k},p}$:

COROLLARY 2.18. Let $p > 1$ and k be a positive integer. There exists a constant C depending on N, k, p such that for any compact set $E \subset \mathbb{R}^{N+1}$

$$\text{Cap}_{\mathcal{G}_{2k},p}(E) \sim \text{Cap}_{2k,k,p}(E).$$

Next result provides some relations of Riesz, Bessel parabolic potential and Riesz, Bessel potential.

PROPOSITION 2.19. Let $q > 1$ and $\frac{2}{q'} < \alpha < N + \frac{2}{q'}$. There hold for any $\omega \in \mathfrak{M}^+(\mathbb{R}^N)$

$$||\mathcal{H}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \sim ||\check{\mathcal{H}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \sim ||\mathbf{I}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)}, \tag{2.25}$$

$$||\mathcal{G}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \sim ||\check{\mathcal{G}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^q(\mathbb{R}^{N+1})} \sim ||\mathbf{G}_{\alpha-\frac{2}{q'}}[\omega]||_{L^q(\mathbb{R}^N)} \tag{2.26}$$

where $\delta_{\{t=0\}}$ is the Dirac mass in time at 0.

PROOF. We have

$$\mathbb{I}_\alpha[\omega\otimes\delta_{\{t=0\}}](x,t) = \int_{\sqrt{2|t|}}^\infty \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}, \mathbb{I}^1_\alpha[\omega\otimes\delta_{\{t=0\}}](x,t) = \int_{\min\{1,\sqrt{2|t|}\}}^1 \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}.$$

By [**17**, Theorem 2.3] and Proposition 2.8, thus it is enough to show that

$$\int_{\mathbb{R}}\left(\int_{\sqrt{2|t|}}^\infty \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \sim \int_0^\infty \left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-\frac{2}{q}}}\right)^q \frac{dr}{r}, \tag{2.27}$$

(2.28)

$$\int_0^{1/2} \left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-\frac{2}{q}}}\right)^q \frac{dr}{r} \lesssim \int_{\mathbb{R}}\left(\int_{\min\{1,\sqrt{2|t|}\}}^1 \frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \lesssim \int_0^1 \left(\frac{\omega(B(x,r))}{r^{N+2-\alpha-\frac{2}{q}}}\right)^q \frac{dr}{r}.$$

Indeed, by changing of variables

$$\int_{-\infty}^{\infty}\left(\int_{\sqrt{2|t|}}^{\infty}\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt = 2\int_0^{\infty} t\left(\int_t^{\infty}\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt.$$

Using Hardy's inequality, we have

$$\int_0^{\infty} t\left(\int_t^{\infty}\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \lesssim \int_0^{\infty} r\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\right)^q dr$$

and using the fact that

$$\int_t^{\infty}\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r} \gtrsim \frac{\omega(B(x,t))}{t^{N+2-\alpha}},$$

we get

$$\int_0^{\infty} t\left(\int_t^{\infty}\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\frac{dr}{r}\right)^q dt \gtrsim \int_0^{\infty} r\left(\frac{\omega(B(x,r))}{r^{N+2-\alpha}}\right)^q dr.$$

Thus, we get (2.27). Likewise, we also obtain (2.28). □

We have comparisons of $\mathrm{Cap}_{\mathcal{H}_\alpha,p}, \mathrm{Cap}_{\mathcal{G}_\alpha,p}, \mathrm{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p}, \mathrm{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}$.

Corollary 2.20. Let $p > 1$ and $\frac{2}{p} < \alpha < N + \frac{2}{p}$. There exists a positive constant C depending on N, q, α such that for any compact set $K \subset \mathbb{R}^N$

$$\mathrm{Cap}_{\mathcal{H}_\alpha,p}(K\times\{0\}) \sim \mathrm{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K), \tag{2.29}$$

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K\times\{0\}) \sim \mathrm{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K). \tag{2.30}$$

Proof. By [**2**, Chapter 2], we have

$$\mathrm{Cap}_{\mathcal{H}_\alpha,p}(K\times\{0\})^{1/p} = \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{H}}_\alpha[\omega\otimes\delta_{\{t=0\}}]||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\},$$

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K\times\{0\})^{1/p} = \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{G}}_\alpha[\omega\otimes\delta_0]||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\},$$

$$\mathrm{Cap}_{\mathbf{I}_{\alpha-\frac{2}{p}},p}(K)^{1/p} = \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\mathbf{I}_{\alpha-\frac{2}{p}}[\omega]||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\},$$

$$\mathrm{Cap}_{\mathbf{G}_{\alpha-\frac{2}{p}},p}(K)^{1/p} = \sup\{\omega(K) : \omega\in\mathfrak{M}^+(K), ||\mathbf{G}_{\alpha-\frac{2}{p}}[\omega]||_{L^{p'}(\mathbb{R}^{N+1})}\le 1\}.$$

Therefore, thanks to Proposition (2.19) we get the results. □

Corollary 2.21. Let $p > 1$ and k be a positive integer such that $2k < N+2/p$. There exists a positive constant C depending on N, k, p such that for any compact set $K \subset \mathbb{R}^N$, we have

$$\mathrm{Cap}_{2k,k,p}(K\times\{0\}) \sim \mathrm{Cap}_{\mathbf{G}_{2k-\frac{2}{p}},p}(K).$$

We also have comparisons of $\mathrm{Cap}_{\mathcal{G}_\alpha,p}, \mathrm{Cap}_{\mathbf{G}_\alpha,p}$.

Proposition 2.22. Let $0 < \alpha < N$, $p > 1$. For $a > 0$ there holds for any compact set $K \subset \mathbb{R}^N$,

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K\times[-a,a]) \sim_a \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K).$$

PROOF. By [**2**], we have

$$\mathrm{Cap}_{\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}},p}(K) \lesssim_a \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K).$$

So, we can find $f \in L^p_+(\mathbb{R}^N)$ such that $\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}} * f \geq \chi_K$ and

$$\int_{\mathbb{R}^N} |f|^p dx \lesssim \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K).$$

Note that $(E_\alpha^{\sqrt{a}} * \tilde{f})(x,t) \gtrsim (\mathbf{I}_\alpha^{\frac{\sqrt{a}}{2}} * f)(x,t)$ for all $(x,t) \in \mathbb{R}^N \times [-a,a]$ where $\tilde{f}(x,t) = f(x)\chi_{[-2a,2a]}(t)$. So,

$$\mathrm{Cap}_{E_\alpha^{\sqrt{a}},p}(K \times [-a,a]) \lesssim \int_{\mathbb{R}^{N+1}} |\tilde{f}|^p dxdt = 2a \int_{\mathbb{R}^N} |f|^p dx.$$

By Corollary 2.10, one has

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \lesssim \mathrm{Cap}_{E_\alpha^{\sqrt{a}},p}(K \times [-a,a]).$$

Thus, we get

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \lesssim \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K).$$

Finally, we prove other one. It is easy to see that

$$||\mathbb{I}_\alpha^{\sqrt{\frac{a}{2}}}[\omega \otimes \chi_{[-a,a]}]||_{L^{p'}(\mathbb{R}^{N+1})} \lesssim ||\mathbf{I}_\alpha^{\sqrt{\frac{a}{2}}}[\omega]||_{L^{p'}(\mathbb{R}^N)} \forall \omega \in \mathfrak{M}^+(\mathbb{R}^N),$$

which implies

$$||\mathcal{G}_\alpha[\omega \otimes \chi_{[-a,a]}]||_{L^{p'}(\mathbb{R}^{N+1})} \lesssim ||\mathbf{G}_\alpha[\omega]||_{L^{p'}(\mathbb{R}^N)} \forall \omega \in \mathfrak{M}^+(\mathbb{R}^{N+1}).$$

It follows,

$$\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K \times [-a,a]) \gtrsim \mathrm{Cap}_{\mathbf{G}_\alpha,p}(K).$$

The proof is complete. □

The following proposition is useful for proving that many operators of classical analysis are bounded in the space of functions f such that

$$\int_K |f|^p dxdt \lesssim \mathrm{Cap}(K)$$

for every compact set $K \subset \mathbb{R}^{N+1}$, $(1 < p < \infty)$, if they are bounded in $L^q(\mathbb{R}^{N+1}, dw)$ with $w \in A_\infty$.

PROPOSITION 2.23. Let $0 < R \leq \infty$, $1 < p \leq \alpha^{-1}(N+2)$, $0 < \delta < \alpha$ and $f, g \in L^1_{loc}(\mathbb{R}^{N+1})$. Suppose that

1.: For any compact set $K \subset \mathbb{R}^{N+1}$

$$\int_K |f| dxdt \lesssim \mathrm{Cap}_{E_\alpha^{R,\delta},p}(K).$$

2.: For all weights $w \in A_1$,

$$\int_{\mathbb{R}^{N+1}} |g| w dxdt \lesssim_{[w]_{A_1}} \int_{\mathbb{R}^{N+1}} |f| w dxdt.$$

Then,

$$\int_K |g| dxdt \lesssim_{\alpha,p,\delta} \mathrm{Cap}_{E_\alpha^{R,\delta},p}(K) \text{ for any compact set } K \subset \mathbb{R}^{N+1}.$$

The capacity is mentioned in Proposition (2.23), that is $(E_\alpha^{R,\delta},p)$-capacity defined by

$$\mathrm{Cap}_{E_\alpha^{R,\delta},p}(E)=\inf\left\{\int_{\mathbb{R}^{N+1}}|f|^pdxdt : f\in L^p_+(\mathbb{R}^{N+1}), E_\alpha^{R,\delta}*f\geq\chi_E\right\},$$

for all measurable sets $E\subset\mathbb{R}^{N+1}$, where $0<R\leq\infty$, $0<\delta<\alpha<N+2$,

$$E_\alpha^{R,\delta}(x,t)=\max\left\{|x|,\sqrt{2|t|}\right\}^{-(N+2-\alpha)}\min\left\{1,\left(\frac{\max\{|x|,\sqrt{2|t|}\}}{R}\right)^{-\delta}\right\}.$$

REMARK 2.24. For $0<\alpha q<N+2$, the inequality (2.10) in Proposition 2.4 implies

$$\left(\int_{\mathbb{R}^{N+1}}\left(E_\alpha^{R,\delta}*f\right)^{\frac{q(N+2)}{N+2-\alpha q}}dxdt\right)^{1-\frac{\alpha q}{N+2}}\lesssim\int_{\mathbb{R}^{N+1}}f^qdxdt\quad\forall f\in L^q(\mathbb{R}^{N+1}), f\geq 0. \tag{2.31}$$

Hence, we get the isoperimetric inequality:

$$|E|^{1-\frac{\alpha p}{N+2}}\lesssim \mathrm{Cap}_{E_\alpha^{R,\delta},p}(E), \tag{2.32}$$

for any measurable set $E\subset\mathbb{R}^{N+1}$.

Also, we recall that a positive function $w\in L^1_{loc}(\mathbb{R}^{N+1})$ is called an A_1 weight, if

$$[w]_{A_1}:=\sup\left(\left(\fint_Q wdyds\right)\underset{(x,t)\in Q}{ess\sup}\frac{1}{w(x,t)}\right)<\infty,$$

where the supremum is taken over all cylinder $Q=\tilde{Q}_R(x,t)\subset\mathbb{R}^{N+1}$. The constant $[w]_{A_1}$ is called the A_1 constant of w.

To prove the Proposition (2.23), we need to introduce the $(R,\delta)-$Wolff parabolic potential,

$$\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t)=\int_0^\infty\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\min\left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\}\frac{d\rho}{\rho}\quad\forall(x,t)\in\mathbb{R}^{N+1},$$

where $p>1$, $0<\alpha p\leq N+2$, $0<\delta<\alpha p'$ and $0<R\leq\infty$ and $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$. It is easy to see that

$$\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t)\lesssim_{\alpha,p,\delta}\sup_{(y,s)\in\mathrm{supp}\mu}\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](y,s). \tag{2.33}$$

REMARK 2.25. We easily verify that the Theorem 2.1 also holds for $\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu]$ and $\mathbb{M}_{\alpha p}^{R,\delta,R_1}[\mu]$:

$$\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu](x,t)=\int_0^{R_1}\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\min\left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\}\frac{d\rho}{\rho},$$

$$\mathbb{M}_{\alpha,p}^{R,\delta/(p-1),R_1}[\mu](x,t)=\sup_{0<\rho<R_1}\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha p}}\min\left\{1,\left(\frac{\rho}{R}\right)^{-\delta(p-1)}\right\}\right),$$

for any $(x,t) \in \mathbb{R}^{N+1}$, where $0 < \delta < \alpha p'$, $1 < p < \alpha^{-1}(N+2)$ and $R_1 > R > 0$. This means, for $w \in A_\infty$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, there exist positive constants $C > 0$ and $\varepsilon_0 \in (0,1)$ depending on $N, \alpha, p, \delta, [w]_{A_\infty}$ such that for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$

$$w(\{\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu] > a\lambda, (\mathbb{M}_{\alpha p}^{R,\delta(p-1),R_1}[\mu])^{\frac{1}{p-1}} \le \varepsilon\lambda\}) \\ \le C\exp(-(C\varepsilon)^{-1}) w(\{\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu] > \lambda\}),$$

where $a = 2 + 3^{\frac{N+2-\alpha p+\delta(p-1)}{p-1}}$.
Therefore, for $q > p-1$

$$||\mathbb{W}_{\alpha,p}^{R,\delta,R_1}[\mu]||_{L^q(\mathbb{R}^{N+1},dw)} \lesssim C_1 ||(\mathbb{M}_{\alpha p}^{R,\delta(p-1),R_1}[\mu])^{\frac{1}{p-1}}||_{L^q(\mathbb{R}^{N+1},dw)},$$

where $C_1 = C_1(\alpha, p, \delta, q)$. Letting $R_1 \to \infty$, we get

$$||\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]||_{L^q(\mathbb{R}^{N+1},dw)} \lesssim C_1 ||(\mathbb{M}_{\alpha p}^{R,\delta(p-1)}[\mu])^{\frac{1}{p-1}}||_{L^q(\mathbb{R}^{N+1},dw)}, \tag{2.34}$$

where $\mathbb{M}_{\alpha p}^{R,\delta(p-1)}[\mu] := \mathbb{M}_{\alpha p}^{R,\delta(p-1),\infty}[\mu]$.

We will need the following three Lemmas to prove the Proposition (2.23).

Lemma 2.26. *Let $0 < p \le \alpha^{-1}(N+2)$ and $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$. There holds for each $\tilde{Q}_r = \tilde{Q}_r(x,t)$*

$$\fint_{\tilde{Q}_r} (\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](y,s))^\beta dyds \lesssim_\delta (\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t))^\beta. \tag{2.35}$$

Proof. We set

$$U_{\alpha,p}^r[\mu](y,s) = \int_r^\infty \left(\frac{|\mu|(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho},$$

$$L_{\alpha,p}^r[\mu](y,s) = \int_0^r \left(\frac{\mu(\tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho}.$$

One has,

$$\fint_{\tilde{Q}_r} (\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](y,s))^\delta dyds \lesssim \fint_{\tilde{Q}_r} (U_{\alpha,p}^r[\mu](y,s))^\delta dyds + \fint_{\tilde{Q}_r} (L_{\alpha,p}^r[\mu](y,s))^\delta dyds.$$

Since for each $(y,s) \in \tilde{Q}_r$ and $\rho \ge r$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2\rho}(x,t)$, thus for each $(y,s) \in \tilde{Q}_r$,

$$U_{\alpha,p}^r[\mu](y,s) \le \int_r^\infty \left(\frac{\mu(\tilde{Q}_{2\rho}(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}} \left(\max\{1, \frac{\rho}{R}\}\right)^{-\delta} \frac{d\rho}{\rho} \lesssim \mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t),$$

which implies

$$\fint_{\tilde{Q}_r} (U_{\alpha,p}^r[\mu](y,s))^\delta dyds \lesssim (\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t))^\delta.$$

Since for each $(y,s) \in \tilde{Q}_r$ and $\rho \le r$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2r}(x,t)$ thus, $L_{\alpha,p}^r[\mu] = L_{\alpha,p}^r[\mu\chi_{\tilde{Q}_{2r}(x,t)}] \le \mathbb{W}_{\alpha,p}^{R,\delta}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]$ in $\tilde{Q}_r(x,t)$. We now consider two cases.

Case 1: $r \le R$. We have for $a > 0$,

$$\fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds \le \fint_{\tilde{Q}_r} (\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s))^\beta dyds$$
$$= \frac{1}{|\tilde{Q}_r|}\beta\int_0^\infty \lambda^{\beta-1}|\{\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] > \lambda\}\cap\tilde{Q}_r|d\lambda$$
$$\lesssim \lambda_0^\beta + r^{-N-2}\int_{\lambda_0}^\infty \lambda^{\beta-1}|\{\mathbb{W}^r_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] > \lambda\}|d\lambda.$$

If $\alpha p = N+2$, we use (2.11) in Remark 2.5 with $\varepsilon = \frac{\alpha p}{\beta}$ and take $\lambda_0 = (\mu(\tilde{Q}_{2r}(x,t)))^{\frac{1}{p-1}}$

$$\fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds \lesssim \lambda_0^\beta + r^{-N-2}\int_{\lambda_0}^\infty \lambda^{\beta-1}\left(\frac{(\mu(\tilde{Q}_{2r}(x,t)))^{\frac{1}{p-1}}}{\lambda}\right)^{\frac{\alpha p+\varepsilon(p-1)}{\varepsilon}} r^{\alpha p}d\lambda$$
$$\lesssim (\mu(\tilde{Q}_{2r}(x,t)))^{\frac{\beta}{p-1}} \lesssim (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta.$$

If $\alpha p < N+2$, we use (2.8) in Proposition 2.4 and take $\lambda_0 = \mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p}{p-1}}$, we get

$$\fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds \lesssim \left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p}{p-1}}\right)^\beta \lesssim (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta.$$

Case 2: $r \ge R$. As above case, we have

$$\fint_{\tilde{Q}_r} (\mathbb{W}_{\alpha-\frac{\delta}{p'},p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}](y,s))^\beta dyds \lesssim \left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}}\right)^\beta.$$

Since $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}] \le R^\delta \mathbb{W}_{\alpha-\frac{\delta}{p'},p}[\mu\chi_{\tilde{Q}_{2r}(x,t)}]$, thus

$$\fint_{\tilde{Q}_r} (L^r_{\alpha,p}[\mu](y,s))^\beta dyds \lesssim \left(\mu(\tilde{Q}_{2r}(x,t))^{\frac{1}{p-1}} r^{-\frac{N+2-\alpha p+\delta(p-1)}{p-1}} R^\delta\right)^\beta \lesssim (\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t))^\beta.$$

Therefore, we get (2.35). The proof is complete. □

REMARK 2.27. It is easy to see that the inequality (2.35) does not hold true for $\mathbb{W}^R_{\alpha,p}[\delta_{(0,0)}]$ where $\delta_{(0,0)}$ is the Dirac mass at $(x,t) = (0,0)$.

REMARK 2.28. From Lemma (2.26), we have, if there exists $(x_0,t_0) \in \mathbb{R}^{N+1}$ such that $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x_0,t_0) < \infty$ then $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \in L^\beta_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta(p-1)}$.

LEMMA 2.29. *Let $R \in (0,\infty]$, $1 < p \le \alpha^{-1}(N+2)$ and $0 < \delta < \alpha p'$. Assume that $\alpha p < N+2$ if $R = \infty$. Then, for any compact set $K \subset \mathbb{R}^{N+1}$ there exists a $\mu \in \mathfrak{M}^+(K)$, called a capacitary measure of K such that*

$$\mu(K) \sim_{\alpha,p,\delta} Cap_{E^{R,\delta/p'}_\alpha,p}(K)$$

and $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu](x,t) \gtrsim_{\alpha,p,\delta} 1$ a.e in K and $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \lesssim_{\alpha,p,\delta} 1$ a.e in $\mathbb{R}^{N+1}$.

PROOF. We consider a measure ν on $M = \mathbb{R}^{N+1}\times\mathbb{Z}$ as follows

$$\nu = m \otimes \sum_{n=-\infty}^{\infty} \delta_n,$$

where m is Lebesgue measure, and δ_n denotes unit mass at n. Thus, $f \in L^p(M, d\nu)$, means $f = \{f_n\}_{-\infty}^{\infty}$, with

$$||f||^p_{L^p(M,d\nu)} = \sum_{n=-\infty}^{\infty} ||f_n||^p_{L^p(\mathbb{R}^{N+1})}.$$

Let $n_R \in \mathbb{Z} \cup \{+\infty\}$ such that $2^{-n_R} \leq R < 2^{-n_R+1}$ if $R < +\infty$ and $n_R = -\infty$ if $R = +\infty$. We define a kernel $\mathbb{P}_\alpha$ in $\mathbb{R}^{N+1} \times M = \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{Z}$ by

$$\mathbb{P}_\alpha(x, t, x', t', n) = \min\{1, 2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}\chi_{\tilde{Q}_{2^{-n}}}(x - x', t - t').$$

If f is $\nu-$measurable and nonnegative and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, the corresponding potentials $\mathcal{P}_\alpha f$, $\overset{\vee}{\mathcal{P}}_\alpha\mu$ and $V^\mu_{\mathbb{P}_\alpha,p}$ are everywhere well defined and given by

$$\begin{aligned}
&(\mathcal{P}_\alpha f)(x,t) = \int_M \mathbb{P}_\alpha(x,t,x',t',n) f(x',t',n) d\nu(x',t',n) \\
&= \sum_{n=-\infty}^{\infty} \min\{1, 2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}(\chi_{\tilde{Q}_{2^{-n}}} * f_n)(x,t), \\
&(\overset{\vee}{\mathcal{P}}_\alpha\mu)(x',t',n) = \int_{\mathbb{R}^{N+1}} \mathbb{P}_\alpha(x,t,x',t',n) d\mu(x,t) \\
&= \min\{1, 2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}(\chi_{\tilde{Q}_{2^{-n}}} * \mu)(x',t'), \\
&V^\mu_{P_\alpha,p}(x,t) = (\mathcal{P}_\alpha(\overset{\vee}{\mathcal{P}}_\alpha\mu)^{p'-1})(x,t) \\
&= \sum_{n=-\infty}^{\infty} \min\{1, 2^{(n-n_R)\delta}\}2^{np'(N+2-\alpha)} \left(\chi_{\tilde{Q}_{2^{-n}}} * \left(\chi_{\tilde{Q}_{2^{-n}}} * \mu\right)^{p'-1}\right)(x,t).
\end{aligned}$$

for any $(x, t, x', t', n) \in \mathbb{R}^{N+1} \times M$.
Since for all $(x, t) \in \mathbb{R}^{N+1}$,

$$\begin{aligned}
|\tilde{Q}_1|2^{-(n+1)(N+2)}(\mu(\tilde{Q}_{2^{-n-1}}(x,t)))^{p'-1} &\leq \left(\chi_{\tilde{Q}_{2^{-n}}} * \left(\chi_{\tilde{Q}_{2^{-n}}} * \mu\right)^{p'-1}\right)(x,t) \\
&\leq |\tilde{Q}_1|2^{-n(N+2)}(\mu(\tilde{Q}_{2^{-n+1}}(x,t)))^{p'-1},
\end{aligned}$$

thus,

$$\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \sim V^\mu_{\mathbb{P}_\alpha,p}. \tag{2.36}$$

We now define the L^p-capacity with $1 < p < \infty$

$$\text{Cap}_{\mathbb{P}_\alpha,p}(E) = \inf\{||f||^p_{L^p(M,d\nu)} : f \in L^p_+(M, d\nu), \mathcal{P}_\alpha f \geq \chi_E\}.$$

for any Borel set $E \subset \mathbb{R}^{N+1}$. By [**2**, Theorem 2.5.1], for any compact set $K \subset \mathbb{R}^{N+1}$

$$\text{Cap}_{\mathbb{P}_\alpha,p}(K)^{1/p} = \sup\{\mu(K) : \mu \in \mathfrak{M}^+(K), ||\overset{\vee}{\mathcal{P}}_\alpha\mu||_{L^{p'}(M,d\nu)} \leq 1\}.$$

By [**2**, Theorem 2.5.6], for any compact set K in $\mathbb{R}^{N+1}$, there exists $\mu \in \mathfrak{M}^+(K)$, called a capacitary measure for K, such that $V^\mu_{\mathbb{P}_\alpha,p} \geq 1$ $\text{Cap}_{\mathbb{P}_\alpha,p}-$q.e. in K, $V^\mu_{P_\alpha,p} \leq 1$ a.e in $\text{supp}(\mu)$ and $\mu(K) = \text{Cap}_{\mathbb{P}_\alpha,p}(K)$. Thanks to (2.36) and (2.33), we have $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \gtrsim 1$ $\text{Cap}_{\mathbb{P}_\alpha,p}-$q.e. in K, $\mathbb{W}^{R,\delta}_{\alpha,p}[\mu] \lesssim 1$ a.e in $\mathbb{R}^{N+1}$ and $\mu(K) = \text{Cap}_{\mathbb{P}_\alpha,p}(K)$.

On the other hand,

$$||\overset{\vee}{\mathcal{P}}_\alpha\mu||^{p'}_{L^{p'}(M,d\nu)} = \sum_{n=-\infty}^{\infty} ||\min\{1,2^{(n-n_R)\delta/p'}\}2^{n(N+2-\alpha)}\chi_{\tilde{Q}_{2^{-n}}}*\mu||^{p'}_{L^{p'}(\mathbb{R}^{N+1})}$$
$$= \sum_{n=-\infty}^{\infty} \min\{1,2^{(n-n_R)\delta}\}2^{np'(N+2-\alpha)}\int_{\mathbb{R}^{N+1}}(\chi_{\tilde{Q}_{2^{-n}}}*\mu)^{p'}dxdt,$$

This quantity is equivalent to

$$\int_{\mathbb{R}^{N+1}}\int_0^\infty\left(\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}}\right)^{p'}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}dxdt.$$

So, thanks to (2.34) in Remark 2.25, we obtain

$$||\overset{\vee}{\mathcal{P}}_\alpha\mu||_{L^{p'}(M,d\nu)} \sim ||E_\alpha^{R,\delta/p'}*\mu||_{L^{p'}(\mathbb{R}^{N+1})}.$$

It follows that the two capacities $\mathrm{Cap}_{\mathbb{P}_\alpha,p}$ and $\mathrm{Cap}_{E_\alpha^{R,\delta/p'},p}$ are equivalent. Therefore, we obtain the desired results. □

Lemma 2.30. *Let $R\in(0,\infty]$, $1<p\le\alpha^{-1}(N+2)$ and $0<\delta<\alpha p'$. Assume that $\alpha p<N+2$ if $R=\infty$. There holds for any $\mu\in\mathfrak{M}_b^+(\mathbb{R}^{N+1})$*

$$(2.37)\qquad Cap_{E_\alpha^{R,\delta/p'},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]>\lambda\}) \lesssim_{\alpha,p,\delta} \lambda^{-p+1}\mu(\mathbb{R}^{N+1})\quad\forall\lambda>0.$$

In particular, $\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]<\infty$ $Cap_{E_\alpha^{R,\delta/p'},p}$−q.e. in $\mathbb{R}^{N+1}$.

Proof. By Lemma 2.29, there is a capacitary measure σ for a compact subset K of $\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]>\lambda\}$ such that $\mathbb{W}_{\alpha,p}^{R,\delta}[\sigma](x,t)\gtrsim_{\alpha,p,\delta}1$ on suppσ and $\mathrm{Cap}_{E_\alpha^{R,\delta/p'},p}(K)\sim\sigma(K)$. Set $\mathbb{M}[\mu,\sigma](x,t)=\sup_{\rho>0}\frac{\mu(\tilde{Q}_\rho(x,t))}{\sigma(\tilde{Q}_{3\rho}(x,t))}$ for any $(x,t)\in$ suppσ. Then, for any $(x,t)\in$ suppσ

$$\lambda<\mathbb{W}_{\alpha,p}^{R,\delta}[\mu](x,t)\le(\mathbb{M}[\mu,\sigma](x,t))^{\frac{1}{p-1}}\int_0^\infty\left(\frac{\sigma(\tilde{Q}_{3\rho}(x,t))}{\rho^{N+2-\alpha p}}\right)^{\frac{1}{p-1}}\min\{1,\left(\frac{\rho}{R}\right)^{-\delta}\}\frac{d\rho}{\rho}$$
$$\lesssim_{\alpha,p,\delta}(\mathbb{M}[\mu,\sigma](x,t))^{\frac{1}{p-1}}.$$

Thus, for any $\lambda>0$, supp$\sigma\subset\{(\mathbb{M}[\mu,\sigma])^{\frac{1}{p-1}}\gtrsim_{\alpha,p,\delta}\lambda\}$. By Vitali Covering Lemma one can cover suppσ with a union of $\tilde{Q}_{3\rho_i}(x_i,t_i)$ for $i=1,...,m(K)$ so that $\tilde{Q}_{\rho_i}(x_i,t_i)$ are disjoint and $\sigma(\tilde{Q}_{3\rho_i}(x_i,t_i))\lesssim\lambda^{-p+1}\mu(\tilde{Q}_{\rho_i}(x_i,t_i))$. It follows that

$$\mathrm{Cap}_{E_\alpha^R,p}(K)\lesssim\sum_{i=1}^{m(K)}\sigma(\tilde{Q}_{3\rho_i}(x_i,t_i))\lesssim\lambda^{-p+1}\sum_{i=1}^{m(K)}\mu(\tilde{Q}_{\rho_i}(x_i,t_i))\lesssim\lambda^{-p+1}\mu(\mathbb{R}^{N+1}).$$

So, for all compact subset K of $\{\mathbb{W}_{\alpha,p}^{R,\delta}[\mu]>\lambda\}$,

$$\mathrm{Cap}_{E_\alpha^{R,\delta/p'},p}(K)\lesssim\lambda^{-p+1}\mu(\mathbb{R}^{N+1}).$$

Therefore, we obtain (2.37). □

Now we are ready to prove Proposition 2.23.

Proof of Proposition 2.23. By Lemma 2.26, 2.29 and 2.30, there exists a capacitary measure μ of a compact subset $K \subset \mathbb{R}^{N+1}$ such that $\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] \gtrsim 1$ a.e in K, $\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] \lesssim 1$ a.e in $\mathbb{R}^{N+1}$ and $\text{Cap}_{E_\alpha^{R,\delta},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] > \lambda\}) \lesssim \lambda^{-p+1} Cap_{E_\alpha^{R,\delta},p}(K)$ for all $\lambda > 0$, $(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta \in A_1$ for any $0 < \beta < \frac{(N+2)(p-1)}{N+2-\alpha p+\delta p}$. From the second assumption we have

$$\int_{\mathbb{R}^{N+1}} |g|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt \lesssim \int_{\mathbb{R}^{N+1}} |f|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt.$$

Thus

$$\begin{aligned}\int_K |g|dxdt &\lesssim \int_{\mathbb{R}^{N+1}} |g|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt \lesssim \int_{\mathbb{R}^{N+1}} |f|(\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu])^\beta dxdt \\ &= \beta \int_0^c \int_{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda} |f|dxdt\lambda^{\beta-1}d\lambda.\end{aligned}$$

By the first assumption we get

$$\int_{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu]>\lambda} |f|dxdt \lesssim \text{Cap}_{E_\alpha^{R,\delta},p}(\{\mathbb{W}_{\alpha,p}^{R,\delta p'}[\mu] > \lambda\}) \lesssim \lambda^{-p+1}\text{Cap}_{E_\alpha^{R,\delta},p}(K).$$

Therefore,

$$\int_K |g|dxdt \lesssim \int_0^c \lambda^{-p+1}\text{Cap}_{E_\alpha^{R,\delta},p}(K)\lambda^{\delta-1}d\lambda \lesssim \text{Cap}_{E_\alpha^{R,\delta},p}(K),$$

since one can choose $\delta > p-1$. This completes the proof. □

Corollary 2.31. Let $f, g \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ be such that

$$\int_{\mathbb{R}^{N+1}} |g|wdxdt \lesssim_{[w]_{A_1}} \int_{\mathbb{R}^{N+1}} |f|wdxdt$$

for any weight $w \in A_1$. Then,

$$\mathbb{I}_\beta[|g|] \lesssim_\beta \mathbb{I}_\beta[|f|] \tag{2.38}$$

for any $\beta \in (0, N+2)$.

The inequality (2.38) for elliptic version was proved in [**81, 83, 72**].

Proof. Let φ_n be the standard mollifiers in $\mathbb{R}^{N+1}$. Thanks to Lemma 2.26, one gets $\mathbb{I}_\beta[\varphi_n] \in A_1$ with $\sup_n [\mathbb{I}_\beta[\varphi_n]]_{A_1} \lesssim_\beta 1$. So, for any $(x_0, t_0) \in \mathbb{R}^{N+1}$,

$$\int_{\mathbb{R}^{N+1}} |g|\mathbb{I}_\beta[\varphi_n((x_0,t_0)+.)] \lesssim_\beta \int_{\mathbb{R}^{N+1}} |f|\mathbb{I}_\beta[\varphi_n((x_0,t_0)+.)].$$

Thus,

$$\int_{\mathbb{R}^{N+1}} \varphi_n((x_0,t_0)+.)\mathbb{I}_\beta[|g|] \lesssim_\beta \int_{\mathbb{R}^{N+1}} \varphi_n((x_0,t_0)+.)\mathbb{I}_\beta[|f|].$$

Letting $n \to \infty$, one has

$$\mathbb{I}_\beta[|g|](x_0,t_0) \lesssim_\beta \mathbb{I}_\beta[|f|](x_0,t_0).$$

This implies (2.38). The proof is complete. □

DEFINITION 2.32. Let $s>1$, $\alpha>0$. We define the space $\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1})$ ($\mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})$ resp.) to be the set of all measure $\mu\in\mathfrak{M}(\mathbb{R}^{N+1})$ such that

$$[\mu]_{\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1})}:=\sup\left\{|\mu|(K)/\mathrm{Cap}_{\mathcal{H}_\alpha,s}(K):\mathrm{Cap}_{\mathcal{H}_\alpha,s}(K)>0\right\}<\infty,$$
$$\left([\mu]_{\mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})}:=\sup\left\{|\mu|(K)/\mathrm{Cap}_{\mathcal{G}_\alpha,s}(K):\mathrm{Cap}_{\mathcal{G}_\alpha,s}(K)>0\right\}<\infty\quad\text{resp.}\right)$$

where the supremum is taken all compact sets $K\subset\mathbb{R}^{N+1}$. For simplicity, we will write $\mathfrak{M}^{\mathcal{H}_\alpha,s},\mathfrak{M}^{\mathcal{G}_\alpha,s}$ to denote $\mathfrak{M}^{\mathcal{H}_\alpha,s}(\mathbb{R}^{N+1}),\mathfrak{M}^{\mathcal{G}_\alpha,s}(\mathbb{R}^{N+1})$ resp.

We see that if $\alpha s\geq N+2$, $\mathfrak{M}^{\mathcal{H}_\alpha,s}=\{0\}$, if $\alpha s<N+2$, $\mathfrak{M}^{\mathcal{H}_\alpha,s}\subset\mathfrak{M}^{\mathcal{G}_\alpha,s}$. On the other hand, $\mathfrak{M}^{\mathcal{G}_\alpha,s}\supset\mathfrak{M}_b$ if $\alpha s>N+2$.

We now have the following two remarks:

REMARK 2.33. Let $s\geq 1$. There holds

$$[f]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}}\lesssim_{\alpha,s}[|f|^s]^{1/s}_{\mathfrak{M}^{\mathcal{G}_\alpha,p}}\text{ for any function } f. \tag{2.39}$$

Indeed, set $a=[|f|^s]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}}$, so for any compact set K in $\mathbb{R}^{N+1}$,

$$\int_K|f|^sdxdt\lesssim_{\alpha,s}\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K).$$

This gives $2a\mathrm{Cap}_{\mathcal{G}_\alpha,p}(K)\gtrsim_{\alpha,s}\int_K(|f|^s+a)\gtrsim_{\alpha,s}a^{1-1/s}\int_K|f|$. It follows (2.39).

REMARK 2.34. Assume that $p>1$ and $\frac{2}{p}<\alpha<N+\frac{2}{p}$. Clearly, from Corollary 2.20 we obtain that for $\omega\in\mathfrak{M}^+(\mathbb{R}^N)$

$$\left[\omega\otimes\delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}}\sim_{\alpha,p}[\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}},\quad\left[\omega\otimes\delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}}\sim_{\alpha,p}[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}}.$$

Here $\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}:=\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}(\mathbb{R}^N)$, $\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}:=\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}(\mathbb{R}^N)$ and

$$[\omega]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}(\mathbb{R}^N)}:=\sup\left\{\omega(K)/\mathrm{Cap}_{\mathbf{I}_{\alpha-2/p},p}(K):\mathrm{Cap}_{\mathbf{I}_{\alpha-2/p},p}(K)>0\right\},$$
$$[\omega]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}(\mathbb{R}^N)}:=\sup\left\{\omega(K)/\mathrm{Cap}_{\mathbf{G}_{\alpha-2/p},p}(K):\mathrm{Cap}_{\mathbf{G}_{\alpha-2/p},p}(K)>0\right\},$$

where the supremum is taken all compact sets $K\subset\mathbb{R}^N$.

Clearly, Theorem 2.2 and Proposition 2.23 lead to the following result.

PROPOSITION 2.35. Let $q>p-1$, $s>1$ and $0<\alpha p<N+2$. Then the following quantities are equivalent

$$\left[\left(\mathbb{W}^R_{\alpha,p}[\mu]\right)^q\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,s}},\left[\left(\mathbb{I}^R_{\alpha p}[\mu]\right)^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,s}}\text{ and }\left[\left(\mathbb{M}^R_{\alpha p}[\mu]\right)^{\frac{q}{p-1}}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,s}},$$

for every $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$ and $0<R\leq\infty$.

In the next result, we present a characterization of the following trace inequality:

$$||E^{R,\delta}_\alpha*f||_{L^p(\mathbb{R}^{N+1},d\mu)}\leq C_1||f||_{L^p(\mathbb{R}^{N+1})}\quad\forall f\in L^p(\mathbb{R}^{N+1}). \tag{2.40}$$

THEOREM 2.36. *Let $0<R\leq\infty$,$1<p<\alpha^{-1}(N+2)$, $0<\delta<\alpha$ and $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$. Then the following statements are equivalent.*

1: *The trace inequality* (2.40) *holds.*
2: *There holds*

$$||E^{R,\delta}_\alpha*f||_{L^p(\mathbb{R}^{N+1},d\omega)}\leq C_2||f||_{L^p(\mathbb{R}^{N+1})}\quad\forall f\in L^p(\mathbb{R}^{N+1}), \tag{2.41}$$

where $d\omega=(\mathbb{I}^{R,\delta}_\alpha\mu)^{p'}dxdt$.

3: *There holds*

$$||E_\alpha^{R,\delta} * f||_{L^{p,\infty}(\mathbb{R}^{N+1},d\mu)} \leq C_3||f||_{L^p(\mathbb{R}^{N+1})} \quad \forall f \in L^p(\mathbb{R}^{N+1}). \tag{2.42}$$

4: *For every compact set* $E \subset \mathbb{R}^{N+1}$,

$$\mu(E) \leq C_4 Cap_{E_\alpha^{R,\delta},p}(E). \tag{2.43}$$

5: $\mathbb{I}_\alpha^{R,\delta}[\mu] < \infty$ *a.e and*

$$\mathbb{I}_\alpha^{R,\delta}[(\mathbb{I}_\alpha^{R,\delta}[\mu])^{p'}] \leq C_5 \mathbb{I}_\alpha^{R,\delta}[\mu] \quad a.e. \tag{2.44}$$

6: *For every compact set* $E \subset \mathbb{R}^{N+1}$,

$$\int_E (\mathbb{I}_\alpha^{R,\delta}[\mu])^{p'} dxdt \leq C_6 Cap_{E_\alpha^{R,\delta},p}(E). \tag{2.45}$$

7: *For every compact set* $E \subset \mathbb{R}^{N+1}$,

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha^{R,\delta}[\mu\chi_E])^{p'} dxdt \leq C_7 \mu(E). \tag{2.46}$$

8: *For every compact set* $E \subset \mathbb{R}^{N+1}$,

$$\int_E (\mathbb{I}_\alpha^{R,\delta}[\mu\chi_E])^{p'} dxdt \leq C_8 \mu(E). \tag{2.47}$$

We can find a simple sufficient condition on μ so that trace inequality (2.40) is satisfied from the isoperimetric inequality (2.32).

Proof of Theorem 2.36. As in [**92**] we can show that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 6 \Leftrightarrow 7$ and $7 \Rightarrow 8, 5 \Rightarrow 2$. Thus, it is enough to show that $8. \Rightarrow 5$.
First, we need to show that

$$\left(\int_r^\infty \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{N+2-\alpha}} \min\{1, \left(\frac{\rho}{R}\right)^{-\delta}\} \frac{d\rho}{\rho}\right)^{p'-1} \lesssim r^{-\alpha}\left(\min\{1, \left(\frac{r}{R}\right)^{-\delta}\}\right)^{-1}. \tag{2.48}$$

We have for any $(y,s) \in \tilde{Q}_r(x,t)$

$$\begin{aligned}
\mathbb{I}_\alpha^{R,\delta}[\mu\chi_{\tilde{Q}_r(x,t)}](y,s) &\geq \int_{2r}^{4r} \frac{\mu(\tilde{Q}_r(x,t) \cap \tilde{Q}_\rho(y,s))}{\rho^{N+2-\alpha}} \min\left\{1, \left(\frac{\rho}{R}\right)^{-\delta}\right\} \frac{d\rho}{\rho} \\
&\gtrsim \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \min\left\{1, \left(\frac{r}{R}\right)^{-\delta}\right\}.
\end{aligned}$$

In (2.47), we take $E = \tilde{Q}_r(x,t)$

$$\mu(\tilde{Q}_r(x,t)) \gtrsim \int_{\tilde{Q}_r(x,t)} (\mathbb{I}_\alpha[\mu\chi_{\tilde{Q}_r(x,t)}])^{p'} \gtrsim \left(\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}} \min\left\{1, \left(\frac{r}{R}\right)^{-\delta}\right\}\right)^{p'} |\tilde{Q}_r|.$$

So $\mu(\tilde{Q}_r(x,t)) \lesssim r^{N+2-\alpha p}\left(\min\left\{1, \left(\frac{r}{R}\right)^{-\delta}\right\}\right)^{-p}$ which implies (2.48).
Next we set

$$L_r[\mu](x,t) = \int_r^{+\infty} \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho} \min\{1, \left(\frac{\rho}{R}\right)^{-\delta}\} \frac{d\rho}{\rho},$$

$$U_r[\mu](x,t) = \int_0^r \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho} \min\{1, \left(\frac{\rho}{R}\right)^{-\delta}\} \frac{d\rho}{\rho},$$

and

$$d\omega = (I_\alpha\mu)^{p'}dxdt, d\sigma_{1,r} = (L_r[\mu])^{p'}\,dxdt, d\sigma_{2,r} = (U_r[\mu])^{p'}\,dxdt.$$

We have $d\omega \le 2^{p'-1}\left(d\sigma_{1,r} + d\sigma_{2,r}\right)$. To prove (2.44) we need to show that

$$\int_0^\infty \frac{\sigma_{1,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\frac{dr}{r} \lesssim \mathbb{I}_\alpha^{R,\delta}[\mu](x,t), \tag{2.49}$$

$$\int_0^\infty \frac{\sigma_{2,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\frac{dr}{r} \lesssim \mathbb{I}_\alpha^{R,\delta}[\mu](x,t). \tag{2.50}$$

Since, for all $r > 0$, $0 < \rho < r$ and $(y,s) \in \tilde{Q}_r(x,t)$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2r}(x,t)$. So,

$$\sigma_{2,r}(\tilde{Q}_r(x,t)) = \int_{\tilde{Q}_r(x,t)} (U_r[\mu])^{p'} = \int_{\tilde{Q}_r(x,t)} \left(U_r[\mu\chi_{\tilde{Q}_{2r}(x,t)}]\right)^{p'}.$$

Thus, from (2.47) we get for $\tilde{Q}_{2r} = \tilde{Q}_{2r}(x,t)$

$$\sigma_{2,r}(\tilde{Q}_r(x,t)) \le \int_{\tilde{Q}_{2r}} \left(U_r[\mu\chi_{\tilde{Q}_{2r}}]\right)^{p'} \le \int_{\tilde{Q}_{2r}} \left(\mathbb{I}_\alpha^{R,\delta}[\mu\chi_{\tilde{Q}_{2r}}]\right)^{p'} \lesssim \mu(\tilde{Q}_{2r}).$$

Therefore, (2.50) follows.

Since, for all $r > 0$, $\rho \ge r$ and $(y,s) \in \tilde{Q}_r(x,t)$ we have $\tilde{Q}_\rho(y,s) \subset \tilde{Q}_{2\rho}(x,t)$. So, for all $(y,s) \in \tilde{Q}_r(x,t)$ we have

$$L_r[\mu](y,s) \le \int_r^{+\infty} \frac{\mu(\tilde{Q}_{2\rho}(x,t))}{\rho^{N+2-\alpha}}\min\left\{1,\left(\frac{\rho}{R}\right)^{-\delta}\right\}\frac{d\rho}{\rho} \lesssim L_r[\mu](x,t).$$

Hence,

$$\sigma_{1,r}(\tilde{Q}_r(x,t)) = \int_{\tilde{Q}_r(x,t)} (L_r[\mu](y,s))^{p'}\,dyds \lesssim r^{N+2}\left(L_r[\mu](x,t)\right)^{p'}.$$

Since $r^{\alpha-1}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\} \le \frac{1}{\alpha-\delta}\frac{d}{dr}\left(r^\alpha\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)$, we deduce that

$$\begin{aligned}
&\int_0^\infty \frac{\sigma_{1,r}(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}\frac{dr}{r}\\
&\quad\lesssim \int_0^\infty r^{\alpha-1}\left(L_r[\mu](x,t)\right)^{p'}\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}dr\\
&\quad\lesssim \int_0^\infty \frac{d}{dr}\left(r^\alpha\min\{1,\left(\frac{r}{R}\right)^{-\delta}\}\right)\left(L_r[\mu](x,t)\right)^{p'}dr\\
&\quad\lesssim \int_0^\infty r^\alpha\left(L_r[\mu](x,t)\right)^{p'-1}\frac{\mu(\tilde{Q}_r(x,t))}{r^{N+2-\alpha}}\min\left\{1,\left(\frac{r}{R}\right)^{-\delta}\right\}^2\frac{dr}{r}.
\end{aligned}$$

Combining this with (2.48), one gets (2.49). The proof is complete. □

REMARK 2.37. It is easy to assert that if **8.** holds then for any $0 < \beta < N+2$

$$\mathbb{I}_\beta\left[\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}\right] \lesssim \mathbb{I}_\beta[\mu]. \tag{2.51}$$

COROLLARY 2.38. Let $p > 1, \alpha > 0$ be such that $0 < \alpha p < N+2$. There holds

$$\left[\left(\mathbb{I}_\alpha[\mu]\right)^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \sim [\mu]^{p'}_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \tag{2.52}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. Furthermore,

$$[\varphi_n * \mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \lesssim [\mu]_{\mathfrak{M}^{\mathcal{H}_{\alpha,p}}} \tag{2.53}$$

for $n \in \mathbb{N}$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ where $\{\varphi_n\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$.

Proof. For $R = \infty$ we have $\mathbb{I}_\alpha^{R,\delta}[\mu] = \mathbb{I}_\alpha[\mu]$ and $E_\alpha^{R,\delta} = E_\alpha$. Thus, by (2.20) in Corollary 2.10 and Theorem 2.36 we get for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\mu(E) \le c\text{Cap}_{\mathcal{H}_\alpha,p}(E)$$

if and only if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E (\mathbb{I}_\alpha[\mu])^{p'}\,dxdt \le c'\text{Cap}_{\mathcal{H}_\alpha,p}(E).$$

It follows (2.52).
Since $\mathbb{I}_\alpha[\varphi_n * \mu] = \varphi_n * \mathbb{I}_\alpha[\mu] \le \mathbb{M}\left(\mathbb{I}_\alpha[\mu]\right)$ and $\mathbb{M}$ is bounded in $L^{p'}(\mathbb{R}^{N+1}, dw)$ with $w \in A_{p'}$,one gets

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha[\varphi_n * \mu])^{p'}\,dw \lesssim_{[w]_{A_{p'}}} \int_{\mathbb{R}^{N+1}} (\mathbb{I}_\alpha[\mu])^{p'}\,dw.$$

Thanks to Proposition 2.23 we have

$$\left[(\mathbb{I}_\alpha[\varphi_n * \mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}} \lesssim \left[(\mathbb{I}_\alpha[\mu])^{p'}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}},$$

which implies (2.53). □

Corollary 2.39. Let $p > 1$, $\alpha > 0$ with $0 < \alpha p \le N + 2$, $0 < \delta < \alpha$ and $R, d > 0$. There holds

$$\left[\left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'}\right]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \lesssim_{d,R} [\mu]^{p'}_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \tag{2.54}$$

for all $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with $\text{diam}(\text{supp}(\mu)) \le d$. Furthermore,

$$[\varphi_n * \mu]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \lesssim_d [\mu]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} \tag{2.55}$$

for $n \in \mathbb{N}$, $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with $\text{diam}(\text{supp}(\mu)) \le d$ where $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Proof. It is easy to see that

$$||E_\alpha^{R,\delta} * \mu||_{L^{p'}(\mathbb{R}^{N+1})} \sim_{d/R} ||E_\alpha^R[\mu]||_{L^{p'}(\mathbb{R}^{N+1})}$$

for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with $\text{diam}(\text{supp}(\mu)) \le d$, thus $\text{Cap}_{E_\alpha^{R,\delta},p}(E) \sim_{d/R} \text{Cap}_{E_\alpha^R,p}(E)$ for every compact set $E \subset \mathbb{R}^{N+1}$, $\text{diam}(E) \le d$. Therefore, by Corollary 2.10 we get $\text{Cap}_{E_\alpha^{R,\delta},p}(E) \sim_{d,R} \text{Cap}_{\mathcal{G}_\alpha,p}(E)$ for every compact set $E \subset \mathbb{R}^{N+1}$, $\text{diam}(E) \le d$. Thus, by Theorem 2.36 and $\text{diam}(\text{supp}(\mu)) \le d$ we get that if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\mu(E) \lesssim_{d,R} \text{Cap}_{\mathcal{G}_\alpha,p}(E),$$

then for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E \left(\mathbb{I}_\alpha^{R,\delta}[\mu]\right)^{p'} dxdt \lesssim_{d,R} \text{Cap}_{E_\alpha^{R,\delta},p}(E) \lesssim_{d,R} \text{Cap}_{\mathcal{G}_\alpha,p}(E).$$

It follows (2.54). As in the Proof of Corollary 2.38 we also have for $w \in A_{p'}$

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{1,\delta}[\varphi_n * \mu]\right)^{p'} dw \lesssim_{[w]_{A_{p'}}} \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{1,\delta}[\mu]\right)^{p'} dw.$$

Thanks to Proposition 2.23 and Theorem 2.36 we obtain (2.55). □

REMARK 2.40. Likewise (see [**83**, Lemma 5.7]), we can verify that for $\frac{2}{p} < \alpha < N + \frac{2}{p}$,

$$[\varphi_{1,n} \star \omega_1]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}} \lesssim [\omega_1]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}, \qquad [\varphi_{1,n} \star \omega_2]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}} \lesssim_d [\omega_2]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}},$$

for $n \in \mathbb{N}$ and $\omega_1, \omega_2 \in \mathfrak{M}^+(\mathbb{R}^N)$ with $\operatorname{diam}(\operatorname{supp}(\omega_2)) \le d$ where $\{\varphi_{1,n}\}$ is a sequence of standard mollifiers in $\mathbb{R}^N$ and $[.]_{\mathfrak{M}^{\mathbf{I}_{\alpha-2/p},p}}, [.]_{\mathfrak{M}^{\mathbf{G}_{\alpha-2/p},p}}$ was defined in Remark 2.34. Hence, we obtain for $\frac{2}{p} < \alpha < N + \frac{2}{p}$

$$\begin{aligned} \left[(\varphi_{1,n} * \omega_1) \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}} &\lesssim \left[\omega_1 \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{H}_\alpha,p}}, \\ \left[(\varphi_{1,n} * \omega_2) \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}} &\lesssim_d \left[\omega_2 \otimes \delta_{\{t=0\}}\right]_{\mathfrak{M}^{\mathcal{G}_\alpha,p}}, \end{aligned}$$

for $n \in \mathbb{N}$ and $\omega_1, \omega_2 \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $\operatorname{diam}(\operatorname{supp}(\mu)) \le d$.

PROPOSITION 2.41. Let $q > 1$, $0 < \alpha q < N + 2$, $0 < R \le \infty, 0 < \delta < \alpha$ and $K > 0$. Let $0 \le f \in L^q_{\text{loc}}(\mathbb{R}^{N+1})$. Let C_4, C_5 be constants in inequalities (2.43) and (2.44) in Theorem 2.36 with $p = q'$. Suppose that $\{u_n\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^{N+1}$ satisfying

$$\begin{aligned} &u_{n+1} \le K\mathbb{I}_\alpha^{R,\delta}[u_n^q] + f \quad \forall n \in \mathbb{N}, \\ &u_0 \le f. \end{aligned} \tag{2.56}$$

Then, if for every compact set $E \subset \mathbb{R}^{N+1}$,

$$\int_E f^q dxdt \le C \operatorname{Cap}_{E_\alpha^{R,\delta},q'}(E) \tag{2.57}$$

with

$$C \le C_4 \left(\frac{2^{-q+1}}{C_5(q-1)} \left(\frac{q-1}{qK2^{q-1}} \right)^q \right)^{q-1}, \tag{2.58}$$

we have

$$u_n \le \frac{Kq2^{q-1}}{q-1} \mathbb{I}_\alpha^{R,\delta}[f^q] + f \quad \forall n \in \mathbb{N}. \tag{2.59}$$

PROOF. From (2.43) and (2.44) in Theorem 2.36, we see that (2.57) implies

$$\mathbb{I}_\alpha^{R,\delta}[(I_\alpha^{R,\delta}[f^q])^q] \le \left(\frac{C}{C_4} \right)^{\frac{1}{q-1}} C_5 \mathbb{I}_\alpha^{R,\delta}[f^q]. \tag{2.60}$$

Now we prove (2.59) by induction. Clearly, (2.59) holds with $n = 0$. Next we assume that (2.59) holds with $n = m$. Then, by (2.58), (2.60) and (2.56) we have

$$\begin{aligned} u_{m+1} &\le K\mathbb{I}_\alpha^{R,\delta}[u_n^q] + f \\ &\le K2^{q-1} \left(\frac{Kq2^{q-1}}{q-1} \right)^q \mathbb{I}_\alpha^{R,\delta}[(\mathbb{I}_\alpha^{R,\delta}[f^q])^q] + K2^{q-1}\mathbb{I}_\alpha^{R,\delta}[f^q] + f \\ &\le K2^{q-1} \left(\frac{Kq2^{q-1}}{q-1} \right)^q \left(\frac{C}{C_4} \right)^{\frac{1}{q-1}} C_5 \mathbb{I}_\alpha^{R,\delta}[f^q] + K2^{q-1}\mathbb{I}_\alpha^{R,\delta}[f^q] + f \\ &\le \frac{Kq2^{q-1}}{q-1} \mathbb{I}_\alpha^{R,\delta}[f^q] + f. \end{aligned}$$

Therefore (2.59) also holds true with $n = m + 1$. The proof is complete. □

COROLLARY 2.42. Let $q > \frac{N+2}{N+2-\alpha}$, $\alpha > 0$ and $f \in L^q_+(\mathbb{R}^{N+1})$. There exists a constant $\varepsilon_0 > 0$ depending on N, α, q such that if for every compact set $E \subset \mathbb{R}^{N+1}$, $\int_E f^q dxdt \leq \varepsilon_0 \mathrm{Cap}_{\mathcal{H}_\alpha, q'}(E)$, then $u = \mathcal{H}_\alpha[u^q] + f$ admits a positive solution $u \in L^q_{loc}(\mathbb{R}^{N+1})$.

PROOF. Consider the sequence $\{u_n\}$ of nonnegative functions defined by $u_0 = f$ and $u_{n+1} = \mathcal{H}_\alpha[u_n^q] + f \forall \geq 0$. It is easy to see that $u_{n+1} \leq c_1 \mathbb{I}_2[u_n^q] + f \forall n \geq 0$. By Proposition 2.41 and Corollary 2.38, there exists a constant $c_2 = c_2(N, \alpha, q) > 0$ such that if for every compact set $E \subset \mathbb{R}^{N+1}$, $\int_E f^q dxdt \leq c_2 \mathrm{Cap}_{\mathcal{H}_\alpha, q'}(E)$ then u_n is well defined and

$$u_n \leq \frac{c_1 q 3^{q-1}}{q-1} \mathbb{I}_\alpha[f^q] + f \forall n \geq 0.$$

Since $\{u_n\}$ is nondecreasing, thus thanks to the dominated convergence theorem we obtain $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ is a solution of $u = \mathcal{H}_\alpha[u^q] + f$ which $u \in L^q_{\mathrm{loc}}(\mathbb{R}^{N+1})$. The proof is complete. □

COROLLARY 2.43. Let $q > 1$, $\alpha > 0$, $0 < R \leq \infty, 0 < \delta < \alpha$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. The following two statements are equivalent.

- **a:** for every compact set $E \subset \mathbb{R}^{N+1}$, we have $\int_E f^q dxdt \leq C \mathrm{Cap}_{E_\alpha^{R,\delta}, q'}(E)$ for some $C > 0$,
- **b:** There exists a function $u \in L^q_{loc}(\mathbb{R}^{N+1})$ such that $u = \mathbb{I}_\alpha^{R,\delta}[u^q] + \varepsilon f$ for some $\varepsilon > 0$.

PROOF. We will prove $b. \Rightarrow a.$ Set $d\omega(x,t) = \left(\left(\mathbb{I}_\alpha^{R,\delta}[u^q]\right)^q + \varepsilon^q f^q \right) dxdt$, thus we have $dw(x,t) \geq \left(I_\alpha^{R,\delta}[\omega]\right)^q dxdt$. Let $\mathbb{M}_\omega$ denote the centered Hardy-littlewoood maximal function which is defined for $g \in L^1_{loc}(\mathbb{R}^{N+1}, d\omega)$,

$$\mathbb{M}_\omega g(x,t) = \sup_{\rho > 0} \frac{1}{\omega(\tilde{Q}_\rho(x,t))} \int_{\tilde{Q}_\rho(x,t)} |g| d\omega.$$

We have for compact set $E \subset \mathbb{R}^{N+1}$

$$\int_{\mathbb{R}^{N+1}} (\mathbb{M}_\omega \chi_E)^q \left(\mathbb{I}_\alpha^{R,\delta}[\omega]\right)^q dxdt \leq \int_{\mathbb{R}^{N+1}} (\mathbb{M}_\omega \chi_E)^q d\omega(x,t).$$

Since $\mathbb{M}_\omega$ is bounded on $L^s(\mathbb{R}^{N+1}, d\omega)$ for $s > 1$ and $(\mathbb{M}_\omega \chi_E)^q \left(\mathbb{I}_\alpha^{R,\delta}[\omega]\right)^q \geq \left(\mathbb{I}_\alpha^{R,\delta}[\omega \chi_E]\right)^q$, thus

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_\alpha^{R,\delta}[\omega \chi_E]\right)^q dxdt \lesssim \omega(E).$$

By Theorem 2.36, we get for any compact set $E \subset \mathbb{R}^{N+1}$

$$\omega(E) \lesssim \mathrm{Cap}_{E_\alpha^{R,\delta}, q'}(E).$$

It follows the result. □

REMARK 2.44. In [**30**], we also use Theorem 2.36 to show the existence of mild solutions to the Navier-Stokes Equations

$$\left\{ \begin{array}{l} \partial_t u - \Delta u + \mathbb{P} \operatorname{div}(u \otimes u) = \mathbb{P} F \text{in } \mathbb{R}^N \times (0, \infty), \\ u(0) = u_0 \text{in } \mathbb{R}^N, \end{array} \right. \tag{2.61}$$

where $u, F \in \mathbb{R}^N$, $\mathbb{P} = id - \nabla \Delta^{-1} \nabla.$ is the Helmholtz Leray projection onto the vector fields of zero divergence, i.e, for $f \in \mathbb{R}^N$, $\mathbb{P} f = f - \nabla u$ and $\Delta u = \operatorname{div} f$.

Namely, there exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that if $\operatorname{div}(u_0) = 0$ and

$$\int_K |D(x,t)|^2 dxdt \le \varepsilon_0 \mathrm{Cap}_{\mathcal{H}_1,2}(K), \tag{2.62}$$

for any compact set $K \subset \mathbb{R}^{N+1}$, where if $(x,t) \in \mathbb{R}^N \times [0,+\infty)$,

$$D(x,t) = (e^{t\Delta}u_0)(x) + \int_0^t (e^{(t-s)\Delta}\mathbb{P}F)(x)ds,$$

and $D(x,t) = 0$ otherwise. Then, the equation (2.61) has globally solution u satisfying

$$|u(x,t)| \le |D(x,t)| + c\mathbb{I}_1[|D|^2](x,t), \tag{2.63}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ for some $c = c(N)$.

2.2. Pointwise estimates of solutions to the parabolic equations

First, we recall Duzzar and Mingione's result [**33**], see also [**49, 50**] which involves local pointwise estimates for solutions of equations (1.22).

THEOREM 2.45. *If $u \in L^2(0,T,H^1(\Omega)) \cap C(\Omega_T)$ is a weak solution to (1.22) with $\mu \in L^2(\Omega_T)$ and $u(0) = 0$, we have*

$$|u(x,t)| \lesssim \fint_{\tilde{Q}_R(x,t)} |u| + \mathbb{I}_2^{2R}[|\mu|](x,t) \tag{2.64}$$

for all $Q_{2R}(x,t) \subset \Omega \times (-\infty,T)$.
Furthermore, if A is independent of space variable x, (1.45) is satisfied and $\nabla u \in C(\Omega_T)$ then

$$|\nabla u(x,t)| \lesssim \fint_{\tilde{Q}_R(x,t)} |\nabla u| dyds + \mathbb{I}_1^{2R}[|\mu|](x,t) \tag{2.65}$$

for all $Q_{2R}(x,t) \subset \Omega \times (-\infty,T)$.

PROOF OF THEOREM 1.1. Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega_T)$, with $\mu_0 \in \mathfrak{M}_0(\Omega_T)$, $\mu_s \in \mathfrak{M}_s(\Omega_T)$. By Proposition 1.38, there exist sequences of nonnegative measures $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$ and $\mu_{n,s,i}$ such that

- $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(\Omega_T)$ strongly converge to some f_i, g_i, h_i in $L^1(\Omega_T)$,$L^2(\Omega_T,\mathbb{R}^N)$ and $L^2(0,T,H_0^1(\Omega))$ respectively;
- $\mu_{n,1}, \mu_{n,2}, \mu_{n,s,1}, \mu_{n,s,2} \in C_c^\infty(\Omega_T)$ converge to $\mu^+, \mu^-, \mu_s^+, \mu_s^-$ resp. in the narrow topology with $\mu_{n,i} = \mu_{n,0,i} + \mu_{n,s,i}$, for $i = 1,2$ and satisfying $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$ and $0 \le \mu_{n,1} \le \varphi_n * \mu^+, 0 \le \mu_{n,2} \le \varphi_n * \mu^-$, where $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$.

Let $\sigma_{1,n}, \sigma_{2,n} \in C_c^\infty(\Omega)$ be convergent to σ^+ and σ^- in the narrow topology and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$ resp. such that $0 \le \sigma_{1,n} \le \varphi_{1,n} * \sigma^+, 0 \le \sigma_{2,n} \le \varphi_{1,n} * \sigma^-$ where $\{\varphi_{1,n}\}$ is a sequence of standard mollifiers in $\mathbb{R}^N$. Set $\mu_n = \mu_{n,1} - \mu_{n,2}$ and $\sigma_n = \sigma_{1,n} - \sigma_{2,n}$.
Let $u_n, u_{n,1}, u_{n,2}$ be solutions of equations

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n & \text{in} \quad \Omega_T, \\ u_n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{on} \quad \Omega, \end{cases} \tag{2.66}$$

$$\begin{cases} (u_{n,1})_t - \operatorname{div}(A(x,t,\nabla u_{n,1})) = \chi_{\Omega_T}\mu_{n,1} \quad \text{in} \quad B_{2T_0}(x_0)\times(0,2T_0^2), \\ u_{n,1} = 0 \quad \text{on} \quad \partial B_{2T_0}(x_0)\times(0,2T_0^2), \\ u_{n,1}(0) = \sigma_{1,n} \quad \text{on} \quad B_{2T_0}(x_0), \end{cases} \tag{2.67}$$

$$\begin{cases} (u_{n,2})_t + \operatorname{div}(A(x,t,-\nabla u_{n,2})) = \chi_{\Omega_T}\mu_{n,2} \quad \text{in} \quad B_{2T_0}(x_0)\times(0,2T_0^2), \\ u_{n,2} = 0 \quad \text{on} \quad \partial B_{2T_0}(x_0)\times(0,2T_0^2), \\ u_{n,2}(0) = \sigma_{2,n} \quad \text{on} \quad B_{2T_0}(x_0), \end{cases} \tag{2.68}$$

where $\Omega \subset B_{T_0}(x_0)$ for $x_0 \in \Omega$.
We see that $u_{n,1}, u_{n,2} \geq 0$ in $B_{2T_0}(x_0)\times(0,2T_0^2)$ and $-u_{n,2} \leq u_n \leq u_{n,1}$ in Ω_T.
Now, we estimate $u_{n,1}$. By Remark 1.34 and Theorem 1.37, a sequence $\{u_{n,1,m}\}$ of solutions to equations

$$\begin{cases} (u_{n,1,m})_t - \operatorname{div}(A(x,t,\nabla u_{n,1,m})) = (g_{n,m})_t + \chi_{\Omega_T}\mu_{n,1} \quad \text{in } B_{2T_0}(x_0)\times(-2T_0^2,2T_0^2), \\ u_{n,1,m} = 0 \quad \text{on} \quad \partial B_{2T_0}(x_0)\times(-2T_0^2,2T_0^2), \\ u_{n,1,m}(-2T_0^2) = 0 \quad \text{on} \quad B_{2T_0}(x_0), \end{cases} \tag{2.69}$$

converges to $u_{n,1}$ in $B_{2T_0}(x_0)\times(0,2T_0^2)$, where $g_{n,m}(x,t) = \sigma_{1,n}(x)\int_{-2T_0^2}^t \varphi_{2,m}(s)ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.
By Remark 1.33, we have

$$||u_{n,1,m}||_{L^1(\tilde{Q}_{2T_0}(x_0,0))} \lesssim T_0^2 A_{n,m}, \tag{2.70}$$

where $A_{n,m} = \mu_{n,1}(\Omega_T) + \int_{\tilde{Q}_{2T_0}(x_0,0)} \sigma_{1,n}(x)\varphi_{2,m}(t)dxdt$.
Hence, thanks to Theorem 2.45 we have for $(x,t)\in\Omega_T$

$$\begin{aligned} u_{n,1,m}(x,t) &\lesssim T_0^{-N-2}||u_{n,1,m}||_{L^1(\tilde{Q}_{2T_0}(x_0,0))} + \mathbb{I}_2[\mu_{n,1}](x,t) + c\mathbb{I}_2[\sigma_{1,n}\varphi_m](x,t) \\ &\lesssim \mathbb{I}_2[\mu_{n,1}](x,t) + \mathbb{I}_2[\sigma_{1,n}\varphi_m](x,t). \end{aligned}$$

Since $0 \leq \mu_{n,1} \leq \varphi_n * \mu^+$, $\sigma_{1,n} \leq \varphi_{1,n} * \sigma^+$,

$$u_{n,1,m}(x,t) \leq c\varphi_n * \mathbb{I}_2[\mu^+](x,t) + c(\varphi_{1,n}\varphi_{2,m}) * \mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}](x,t) \quad \forall (x,t)\in\Omega_T.$$

Letting $m\to\infty$, we get

$$u_{n,1}(x,t) \lesssim \varphi_n * \mathbb{I}_2[\mu^+](x,t) + \varphi_{1,n} * \left(\mathbb{I}_2[\sigma^+ \otimes \delta_{\{t=0\}}](.,t)\right)(x) \quad \forall (x,t)\in\Omega_T.$$

Similarly, we also get

$$u_{n,2}(x,t) \lesssim \varphi_n * \mathbb{I}_2[\mu^-](x,t) + \varphi_{1,n} * \left(\mathbb{I}_2[\sigma^- \otimes \delta_{\{t=0\}}](.,t)\right)(x) \quad \forall (x,t)\in\Omega_T.$$

Consequently, by Proposition 1.36 and Theorem 1.37 , up to a subsequence, $\{u_n\}$ converges to a distributional solution (a renormalized solution if $\sigma \in L^1(\Omega)$) u of (1.22) and satisfied (1.25). □

REMARK 2.46. Obviously, if $\sigma \equiv 0$ and $\operatorname{supp}(\mu) \subset \overline{\Omega}\times[a,T]$, $a>0$ then $u=0$ in $\Omega\times(0,a)$.

REMARK 2.47. If A is independent of space variable x, (1.45) is satisfied then

$$|\nabla u(x,t)| \lesssim_{T_0/d} \mathbb{I}_1^{2T_0}[|\mu| + |\sigma|\otimes\delta_{\{t=0\}}](x,t) \tag{2.71}$$

for any $(x,t)\in\Omega^d\times(0,T)$ and $0 < d \leq \frac{1}{2}\min\{\sup_{x\in\Omega} d(x,\partial\Omega), T_0^{1/2}\}$ where $\Omega^d = \{x\in\Omega : d(x,\partial\Omega) > d\}$. Indeed, by Remark 1.34 and Theorem 1.37, a sequence $\{v_{n,m}\}$ of solutions to equations

$$(2.72)\quad \begin{cases} (v_{n,m})_t - \operatorname{div}(A(t,\nabla u_{n,m})) = (g_{n,m})_t + \chi_{\Omega_T}\mu_n & \text{in} \quad \Omega\times(-2T_0^2,T),\\ v_{n,m}=0 \quad \text{on} \quad \partial\Omega\times(-2T_0^2,T),\\ v_{n,m}(-2T_0^2)=0 \quad \text{on} \quad \Omega, \end{cases}$$

converges to u_n in $L^1(0,T,W_0^{1,1}(\Omega))$, where $g_{n,m}(x,t)=\sigma_n(x)\int_{-2T_0^2}^t \varphi_{2,m}(s)ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.
By Theorem 2.45, we have for any $(x,t)\in\Omega^d\times(0,T)$

$$|\nabla v_{n,m}(x,t)| \lesssim \fint_{\tilde{Q}_{d/2}(x,t)} |\nabla v_{n,m}| + \mathbb{I}_1^d[|\mu_n| + |\sigma_n|\otimes\varphi_{2,m}](x,t).$$

On the other hand, by remark 1.33,

$$|||\nabla v_{n,m}|||_{L^1(\Omega\times(-T_0^2,T))} \lesssim T_0(|\mu_n| + |\sigma_n|\otimes\varphi_{2,m})(\Omega\times(-T_0^2,T)).$$

Therefore, for any $(x,t)\in\Omega^d\times(0,T)$

$$|\nabla v_{n,m}(x,t)| \lesssim_{T_0/d} \mathbb{I}_1[|\mu_n| + |\sigma_n|\otimes\varphi_{2,m}](x,t).$$

Finally, letting $m\to\infty$ and $n\to\infty$ we get for any $(x,t)\in\Omega^d\times(0,T)$

$$|\nabla u(x,t)| \lesssim \mathbb{I}_1[|\mu| + |\sigma|\otimes\delta_{\{t=0\}}](x,t).$$

We conclude (2.71) since $\mathbb{I}_1[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}] \lesssim \mathbb{I}_1^{2T_0}[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}]$ in Ω_T.

Next, we will establish pointwise estimates from below for solutions of equations (1.22).

Theorem 2.48. *If $u\in C(Q_r(y,s))\cap L^2(s-r^2,s,H^1(B_r(y)))$ is a nonnegative weak solution of (1.22) with data $\mu\in\mathfrak{M}^+(Q_r(y,s))$ and $u(s-r^2)\geq 0$, then we have*

$$(2.73)\qquad u(y,s) \gtrsim \sum_{k=0}^{\infty} \frac{\mu(Q_{r_k/8}(y,s-\frac{35}{128}r_k^2))}{r_k^N},$$

where $r_k = 4^{-k}r$.

Proof. It is enough to show that for $\rho\in(0,r)$

$$(2.74)\qquad \frac{\mu(Q_{\rho/8}(y,s-\frac{35}{128}\rho^2))}{\rho^N} \lesssim \inf_{Q_{\rho/4}(y,s)} u - \inf_{Q_\rho(y,s)} u.$$

By [**57**, Theorem 6.18, p. 122], we have for any $\theta\in(0,1+2/N)$,

$$(2.75)\qquad \left(\fint_{Q_{\rho/4}(y,s-\rho^2/4)} (u-a)^\theta\right)^{1/\theta} \lesssim b-a,$$

where $b=\inf_{Q_{\rho/4}(y,s)} u$, $a=\inf_{Q_\rho(y,s)} u$.

Let $\eta \in C_c^\infty(Q_\rho(y,s))$ such that $0 \le \eta \le 1$, $\mathrm{supp}\eta \subset Q_{\rho/4}(y, s - \frac{1}{4}\rho^2)$, $\eta = 1$ in $Q_{\rho/8}(y, s - \frac{35}{128}\rho^2)$ and $|\nabla \eta| \lesssim 1/\rho$, $|\eta_t| \lesssim 1/\rho^2$. We have

$$\begin{aligned}
\mu(Q_{\rho/8}(y, s - \frac{35}{128}\rho^2)) &\le \int_{Q_\rho(y,s)} \eta^2 d\mu(x,t) \\
&= \int_{Q_\rho(y,s)} u_t \eta^2 dxdt + 2\int_{Q_\rho(y,s)} \eta A(x,t,\nabla u)\nabla\eta dxdt \\
&= -2\int_{Q_\rho(y,s)} (u-a)\eta_t \eta dxdt + 2\int_{Q_\rho(y,s)} \eta A(x,t,\nabla u)\nabla\eta dxdt \\
&\lesssim r^{-2}\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)} (u-a)dxdt + \int_{Q_\rho(y,s)} \eta|\nabla u||\nabla\eta|dxdt \\
&\lesssim r^N(b-a) + \int_{Q_\rho(y,s)} \eta|\nabla u||\nabla\eta|dxdt.
\end{aligned}$$

Here we used (2.75) with $\theta = 1$ in the last inequality. It remains to show that

$$\int_{Q_r(y,s)} \eta|\nabla u||\nabla\eta|dxdt \lesssim r^N(b-a). \tag{2.76}$$

First, we verify that for $\varepsilon \in (0,1)$

$$\int_{Q_\rho(y,s)} |\nabla u|^2 (u-a)^{-\varepsilon-1}\eta^2 dxdt \lesssim \int_{Q_\rho(y,s)} (u-a)^{1-\varepsilon}\left(\eta|\eta_t| + |\nabla\eta|^2\right)dxdt. \tag{2.77}$$

Indeed, for $\delta \in (0,1)$ we choose $\varphi = (u-a+\delta)^{-\varepsilon}\eta^2$ as test function in (1.22),

$$\begin{aligned}
0 &\le \int_{Q_\rho(y,s)} u_t (u-a+\delta)^{-\varepsilon}\eta^2 dxdt + \int_{Q_\rho(y,s)} A(x,t,\nabla u)\nabla\left((u-a+\delta)^{-\varepsilon}\eta^2\right)dxdt \\
&\le 2(1-\varepsilon)\int_{Q_\rho(y,s)} (u-a+\delta)^{1-\varepsilon}|\eta_t|\eta dxdt - \varepsilon\Lambda_2 \int_{Q_\rho(y,s)} |\nabla u|^2 (u-a+\delta)^{-\varepsilon-1}\eta^2 dxdt \\
&+ 2\Lambda_1 \int_{Q_\rho(y,s)} \eta|\nabla u|(u-a+\delta)^{-\varepsilon}|\nabla\eta|dxdt.
\end{aligned}$$

So, we deduce (2.77) from using the Hölder's inequality and letting $\delta \to 0$. Therefore, for $\varepsilon \in (0, 2/N)$ using the Hölder's inequality, we get

$$\begin{aligned}
&\int_{Q_r(y,s)} \eta|\nabla u||\nabla\eta|dxdt \\
&\le \left(\int_{Q_\rho(y,s)} |\nabla u|^2 (u-a)^{-\varepsilon-1}\eta^2 dxdt\right)^{1/2} \left(\int_{Q_\rho(y,s)} (u-a)^{\varepsilon+1}|\nabla\eta|^2 dxdt\right)^{1/2} \\
&\overset{(2.77)}{\lesssim} \left(\int_{Q_\rho(y,s)} (u-a)^{1-\varepsilon}\left(\eta|\eta_t| + |\nabla\eta|^2\right)dxdt\right)^{1/2} \left(\int_{Q_\rho(y,s)} (u-a)^{\varepsilon+1}|\nabla\eta|^2 dxdt\right)^{1/2} \\
&\lesssim \rho^{-2}\left(\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)} (u-a)^{1-\varepsilon}dxdt\right)^{1/2} \left(\int_{Q_{\rho/4}(y,s-\frac{1}{4}\rho^2)} (u-a)^{\varepsilon+1}dxdt\right)^{1/2}.
\end{aligned}$$

Hence, (2.74) follows from this and (2.75). $\square$

Proof of Theorem 1.3. Let $\mu_n \in (C_c^\infty(\Omega_T))^+, \sigma_n \in (C_c^\infty(\Omega))^+$ be in the proof of Theorem 1.1. Let u_n be a weak solution of equation

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n & \text{in} \quad \Omega_T, \\ u_n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{on} \quad \Omega. \end{cases}$$

As the proof of Theorem 1.1, thanks to Theorem 2.48 we get for any $Q_r(y,s) \subset \Omega \times (-\operatorname{diam}(\Omega), T)$ and $r_k = 4^{-k} r$

$$u_n(y,s) \gtrsim \sum_{k=0}^{\infty} \frac{\mu_n(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} + \sum_{k=0}^{\infty} \frac{(\sigma_n \otimes \delta_{\{t=0\}})(Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N}.$$

Finally, by Proposition 1.36 and Theorem 1.37 we get the result. □

Remark 2.49. If $u \in L^q(\Omega_T)$ satisfies (1.26) then $\mathcal{G}_2[\chi_E \mu] \in L^q(\mathbb{R}^{N+1})$ and $\mathbf{G}_{\frac{2}{q}}[\chi_F \sigma] \in L^q(\mathbb{R}^N)$ for every $E \subset\subset \Omega \times [0,T)$ and $F \subset\subset \Omega$. Indeed, for $E \subset\subset \Omega \times [0,T)$, $\varepsilon = \operatorname{dist}(E, (\Omega \times (0,T)) \cup (\Omega \times \{t = T\})) > 0$, we can see that for any $(y,s) \in \Omega_T$, $r_k = 4^{-k}\varepsilon/4$

$$u(y,s) \gtrsim \sum_{k=0}^{\infty} \frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N},$$

where $\tilde{\mu} = \mu + \sigma \otimes \delta_{\{t=0\}}$.
Moreover, for any $(y,s) \notin \Omega_T$

$$\sum_{k=0}^{\infty} \frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} = 0.$$

Thus,

$$\begin{aligned} \infty &> \int_{\mathbb{R}^{N+1}} \sum_{k=0}^{\infty} \left(\frac{\tilde{\mu}(E \cap Q_{r_k/8}(y, s - \frac{35}{128} r_k^2))}{r_k^N} \right)^q dyds \\ &\geq \int_{\mathbb{R}^N} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left(\frac{\tilde{\mu}(E \cap \tilde{Q}_{r_k/8}(y,s))}{r_k^N} \right)^q dsdy \\ &\gtrsim \int_{\mathbb{R}^{N+1}} \int_0^{\varepsilon/64} \left(\frac{\tilde{\mu}(E \cap \tilde{Q}_\rho(y,s))}{\rho^N} \right)^q \frac{d\rho}{\rho} dsdy \gtrsim_\varepsilon \int_{\mathbb{R}^{N+1}} (\mathcal{G}_2[\tilde{\mu}\chi_E])^q \, dsdy. \end{aligned}$$

Thus, from Proposition 2.19, we get the results.

Proof of Theorem 1.5. Set $D_n = B_n(0) \times (-n^2, n^2)$. For $n \geq 4$, by Theorem 1.1, there exists a renormalized solution u_n to problem

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \chi_{D_{n-1}} \omega & \text{in} \quad D_n, \\ u_n = 0 \quad \text{on} \quad \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \quad \text{on} \quad B_n(0). \end{cases}$$

relative to a decomposition (f_n, g_n, h_n) of $\chi_{D_{n-1}} \omega_0$ satisfying

$$-K\mathbb{I}_2[\omega^-](x,t) \leq u_n(x,t) \leq K\mathbb{I}_2[\omega^+](x,t) \quad \forall (x,t) \in D_n. \tag{2.78}$$

From the proof of Theorem 1.1 and Remark 1.40, we can assume that u_n satisfies (1.80) and (1.81) in Proposition 1.47 with $1 < q_0 < \frac{N+2}{N}$, $L \equiv 0$. Moreover, there holds

$$||f_n||_{L^1(D_i)}+||g_n||_{L^2(D_i)}+|||h_n|+|\nabla h_n|||_{L^2(D_i)}\le 2|\omega|(D_{i+1}) \tag{2.79}$$

for any $i=1,...,n-1$ and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$.
On the other hand, by Lemma 2.26 we have for any $s\in(1,\frac{N+2}{N})$
(2.80)

$$\int_{D_m}|u_n|^s dxdt\le K^s\int_{D_m}(I_2[|\omega|])^s dxdt\le K^s\int_{\tilde{Q}_{4m}(x_0,t_0)}(I_2[|\omega|])^s dxdt\lesssim m^{N+2},$$

for $n\ge m\ge |x_0|+|t_0|^{1/2}$. Consequently, we can apply Proposition 1.48 and obtain that u_n converges to some u in $L^1_{loc}(\mathbb{R};W^{1,1}_{loc}(\mathbb{R}^N))$.
Since for any $\alpha\in(0,1/2)$

$$\int_{D_m}\frac{|\nabla u_n|^2}{(|u_n|+1)^{\alpha+1}}dxdt\lesssim_{\alpha,m}1\quad\forall n\ge m,$$

thus using (2.80) and Hölder's inequality, we get for any $1\le s_1<\frac{N+2}{N+1}$

$$\int_{D_m}|\nabla u_n|^{s_1}dxdt\lesssim_{s_1,m}1 \text{ for all } n\ge m\ge|x_0|+|t_0|^{1/2}.$$

This yields $u_n\to u$ in $L^{s_1}_{\text{loc}}(\mathbb{R};W^{1,s_1}_{\text{loc}}(\mathbb{R}^N))$.
Take $\varphi\in C_c^\infty(\mathbb{R}^{N+1})$ and $m_0\in\mathbb{N}$ with $\text{supp}(\varphi)\subset D_{m_0}$, we have for $n\ge m_0+1$

$$-\int_{\mathbb{R}^{N+1}}u_n\varphi_t dxdt+\int_{\mathbb{R}^{N+1}}A(x,t,\nabla u_n)\nabla\varphi dxdt=\int_{\mathbb{R}^{N+1}}\varphi d\omega$$

Letting $n\to\infty$, we conclude that u is a distributional solution to problem (1.24) with data $\mu=\omega$ which satisfies (1.27).
Claim 1. If $\omega\ge 0$. By Theorem 1.3, we have for $n\ge 4^{k_0+1}$, $(y,s)\in B_{4^{k_0}}\times(0,n^2)$

$$u_n(y,s)\gtrsim\sum_{k=0}^{\infty}\frac{\omega(Q_{r_k/8}(y,s-\frac{35}{128}r_k^2)\cap D_{n-1})}{r_k^N},$$

where $r_k=4^{-k+k_0}$. This gives

$$u_n(y,s)\gtrsim\sum_{k=-k_0}^{\infty}\frac{\omega(Q_{2^{-2k-3}}(y,s-35\times 2^{-4k-7})\cap B_{n-1}(0)\times(0,(n-1)^2))}{2^{-2Nk}}.$$

Letting $n\to\infty$ and $k_0\to\infty$ we have (1.28). Finally, thanks to Proposition 2.8 and Theorem 2.2, we will assert (1.29) if we show that for $q>\frac{N+2}{N}$

$$B:=\int_{\mathbb{R}}\left(\sum_{k=-\infty}^{\infty}\frac{\omega(Q_{2^{-2k-3}}(x,t-35\times 2^{-4k-7}))}{2^{-2Nk}}\right)^q dxdt$$
$$\gtrsim\int_{\mathbb{R}}\int_0^{+\infty}\left(\frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^N}\right)^q\frac{d\rho}{\rho}dxdt.$$

Indeed,

$$B\ge\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\left(\frac{\omega(Q_{2^{-2k-3}}(x,t-35\times 2^{-4k-7}))}{2^{-2Nk}}\right)^q dtdx$$
$$=\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\left(\frac{\omega(\tilde{Q}_{2^{-2k-3}}(x,t))}{2^{-2Nk}}\right)^q dt\gtrsim\int_{\mathbb{R}^{N+1}}\int_0^{+\infty}\left(\frac{\omega(\tilde{Q}_\rho(x,t))}{\rho^N}\right)^q\frac{d\rho}{\rho}dxdt.$$

Claim 2. If A is independent of space variable x and (1.45) is satisfied. By Remark 2.47 we get for any $(x,t) \in D_{n/4}$

$$|\nabla u_n(x,t)| \lesssim \mathbb{I}_1[|\omega|](x,t).$$

Letting $n \to \infty$, we get (1.30).

Claim 3. If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by Remark (2.46) we get that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distributional solution to (1.23). The proof is complete. □

Remark 2.50. If $\omega \in \mathfrak{M}_b(\mathbb{R}^{N+1})$ then u satisfies

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\mathbb{R}^{N+1})} \lesssim |\omega|(\mathbb{R}^{N+1}).$$

Moreover, $I_2[|\omega|] \in L^{\frac{N+2}{N},\infty}(\mathbb{R}^{N+1})$ and $I_2[|\omega|] < \infty$ a.e in $\mathbb{R}^{N+1}$.

CHAPTER 3

Global gradient estimates for parabolic equations

3.1. Interior estimates and boundary estimates for parabolic equations

In this section we always assume that $u \in C(-T,T,L^2(\Omega)) \cap L^2(-T,T,H_0^1(\Omega))$ is a solution to equation (1.22) in $\Omega \times (-T,T)$ with $\mu \in L^2(\Omega \times (-T,T))$ and $u(-T) = 0$. We extend u by zero to $\Omega \times (-\infty,-T)$, clearly u is a solution to equation

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = \chi_{(-T,T)}(t)\mu & \text{in} \quad \Omega \times (-\infty,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (-\infty,T). \end{cases} \tag{3.1}$$

3.1.1. Interior Estimates. For each ball $B_{2R} = B_{2R}(x_0) \subset\subset \Omega$ and $t_0 \in (-T,T)$, one considers the unique solution

$$w \in C(t_0 - 4R^2, t_0; L^2(B_{2R})) \cap L^2(t_0 - 4R^2, t_0; H^1(B_{2R})) \tag{3.2}$$

to the following equation

$$\begin{cases} w_t - \operatorname{div}(A(x,t,\nabla w)) = 0 & \text{in} \quad Q_{2R}, \\ w = u & \text{on} \quad \partial_p Q_{2R}, \end{cases} \tag{3.3}$$

where $Q_{2R} = B_{2R} \times (t_0 - 4R^2, t_0)$ and $\partial_p Q_{2R} = \big(\partial B_{2R} \times (t_0 - 4R^2, t_0)\big) \cup \big(B_{2R} \times \{t = t_0 - 4R^2\}\big)$.

Theorem 3.1. *There exist constants $\theta_1 > 2$, $\beta_1 \in (0, \frac{1}{2}]$ such that the following estimates are true*

$$\fint_{Q_{2R}} |\nabla u - \nabla w| dxdt \lesssim \frac{|\mu|(Q_{2R})}{R^{N+1}}, \tag{3.4}$$

$$\left(\fint_{Q_{\rho/2}(y,s)} |\nabla w|^{\theta_1} dxdt \right)^{\frac{1}{\theta_1}} \lesssim \fint_{Q_\rho(y,s)} |\nabla w| dxdt, \tag{3.5}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |w - \overline{w}_{Q_{\rho_1}(y,s)}|^2 dxdt \right)^{1/2} \lesssim \left(\frac{\rho_1}{\rho_2} \right)^{\beta_1} \left(\fint_{Q_{\rho_2}(y,s)} |w - \overline{w}_{Q_{\rho_2}(y,s)}|^2 dxdt \right)^{1/2}, \tag{3.6}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \lesssim \left(\frac{\rho_1}{\rho_2} \right)^{\beta_1 - 1} \left(\fint_{Q_{\rho_2}(y,s)} |\nabla w|^2 dxdt \right)^{1/2} \tag{3.7}$$

for any $Q_\rho(y,s) \subset Q_{2R}$, and $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{2R}$.

PROOF. Inequalities (3.4), (3.5) and (3.6) were proved by Duzaar and Mingione in [**33**]. So, it remains to prove (3.7) in case $\rho_1 \le \rho_2/2$. By the interior Caccioppoli inequality we have

$$\left(\fint_{Q_{\rho_1}(y,s)} |\nabla w|^2 dxdt\right)^{1/2} \lesssim \frac{1}{\rho_1}\left(\fint_{Q_{2\rho_1}(y,s)} |w - \overline{w}_{Q_{2\rho_1}(y,s)}|^2 dxdt\right)^{1/2}.$$

On the other hand, by a Sobolev inequality there holds

$$\left(\fint_{Q_{\rho_2}(y,s)} |w - \overline{w}_{Q_{\rho_2}(y,s)}|^2 dxdt\right)^{1/2} \lesssim \rho_2\left(\fint_{Q_{\rho_2}(y,s)} |\nabla w|^2 dxdt\right)^{1/2}.$$

Therefore, (3.7) follows from (3.6). □

COROLLARY 3.2. Let β_1 be the constant in Theorem 3.1 and $2-\beta_1 < \theta < N+2$. There holds for any $B_\rho(y) \subset B_{\rho_0}(y) \subset\subset \Omega$, $s \in (-T,T)$

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \lesssim_\theta \rho^{N+3-\theta}\left(\left(\frac{T_0}{\rho_0}\right)^{N+3-\theta} + 1\right) ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}. \tag{3.8}$$

PROOF. Take $B_{\rho_2}(y) \subset\subset \Omega$ and $s \in (-T,T)$. Set $Q_\rho := Q_{\rho_1}(y,s)$. For any $Q_{\rho_1} \subset Q_{\rho_2}$ with $\rho_1 \le \rho_2/2$, we take w as in Theorem 3.1 with $Q_{2R} = Q_{\rho_2}(y,s)$. Thus,

$$\int_{Q_{\rho_1}} |\nabla w| dxdt \lesssim \left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2}} |\nabla w| dxdt, \quad \int_{Q_{\rho_2}} |\nabla u - \nabla w| dxdt \lesssim \rho_2 |\mu|(Q_{\rho_2}).$$

It follows that

$$\begin{aligned}\int_{Q_{\rho_1}} |\nabla u| dxdt &\le \int_{Q_{\rho_1}} |\nabla w| dxdt + \int_{Q_{\rho_1}} |\nabla u - \nabla w| dxdt \\ &\lesssim \left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2}} |\nabla w| dxdt + \int_{Q_{\rho_2}} |\nabla u - \nabla w| dxdt \\ &\lesssim \left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2}} |\nabla u| dxdt + \rho_2 |\mu|(Q_{\rho_2}).\end{aligned}$$

This implies

$$\int_{Q_{\rho_1}} |\nabla u| dxdt \lesssim \left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2}} |\nabla u| dxdt + \rho_2^{N+3-\theta} ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}.$$

Since $N+3-\beta < N+\beta_1+1$, applying [**57**, Lemma 4.6, page 54] we obtain

$$\int_{Q_\rho} |\nabla u| dxdt \lesssim \left(\frac{\rho}{\rho_0}\right)^{N+3-\theta} ||\nabla u||_{L^1(\Omega_\times(-T,T))} + \rho^{N+3-\theta} ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))},$$

for any $B_\rho(y) \subset B_{\rho_0}(y) \subset\subset \Omega$, $s \in (-T,T)$. On the other hand, by Remark 1.33

$$||\nabla u||_{L^1(\Omega\times(-T,T))} \lesssim T_0 |\mu|(\Omega\times(-T,T)) \lesssim T_0^{N+3-\theta} ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}.$$

Hence, we get the desired result. □

To continue, we consider the unique solution

$$v \in C(t_0 - R^2, t_0; L^2(B_R)) \cap L^2(t_0 - R^2, t_0; H^1(B_R)) \tag{3.9}$$

to the following equation

$$\begin{cases} v_t - \operatorname{div}\left(\overline{A}_{B_R(x_0)}(t, \nabla v)\right) = 0 & \text{in} \quad Q_R, \\ v = w & \text{on} \quad \partial_p Q_R, \end{cases} \tag{3.10}$$

where $Q_R = B_R(x_0)\times(t_0-R^2, t_0)$ and $\partial_p Q_R = \left(\partial B_R \times (t_0 - R^2, t_0)\right)\cup\left(B_R \times \{t = t_0 - R^2\}\right)$.

LEMMA 3.3. *Let θ_1 be the constant in Theorem 3.1. There holds*

$$\left(\fint_{Q_R} |\nabla w - \nabla v|^2 dxdt\right)^{1/2} \lesssim [A]^R_{s_1} \fint_{Q_{2R}} |\nabla w| dxdt, \tag{3.11}$$

with $s_1 = \frac{2\theta_1}{\theta_1 - 2}$ and

$$\int_{Q_R} |\nabla w|^2 dxdt \sim \int_{Q_R} |\nabla v|^2 dxdt. \tag{3.12}$$

PROOF. We can choose $\varphi = w - v$ as a test function for equations (3.3), (3.10) and since

$$\int_{Q_R} w_t(w-v)dxdt - \int_{Q_R} v_t(w-v)dxdt = \frac{1}{2}\int_{B_R} (w-v)^2(t_0)dx \geq 0,$$

we find

$$-\int_{Q_R} \overline{A}_{B_R(x_0)}(t, \nabla v)\nabla(w-v)dxdt \leq -\int_{Q_R} A(x,t,\nabla w)\nabla(w-v)dxdt.$$

By using inequalities (1.2) and (1.3) together with Hölder's inequality we get

$$\int_{Q_R} |\nabla w|^2 dxdt \sim \int_{Q_R} |\nabla v|^2 dxdt,$$

and we also have

$$\begin{aligned} \Lambda_2 \int_{Q_R} |\nabla w - \nabla v|^2 dxdt &\leq \int_{Q_R} \left(\overline{A}_{B_R(x_0)}(t, \nabla w) - \overline{A}_{B_R(x_0)}(t, \nabla v)\right)(\nabla w - \nabla v)\, dxdt \\ &\leq \int_{Q_R} \left(\overline{A}_{B_R(x_0)}(t, \nabla w) - A(x,t,\nabla w)\right)(\nabla w - \nabla v)\, dxdt \\ &\leq \int_{Q_R} \Theta(A, B_R(x_0))(x,t)|\nabla w||\nabla w - \nabla v| dxdt. \end{aligned}$$

Here we used the definition of $\Theta(A, B_R(x_0))$ in the last inequality. Using Hölder's inequality with exponents $s_1 = \frac{2\theta_1}{\theta_1 - 2}, \theta_1$ and 2 one gets

$$\begin{aligned} \Lambda_2 \fint_{Q_R} |\nabla w - \nabla v|^2 \leq &\left(\fint_{Q_R} \Theta(A, B_R(x_0))(x,t)^{s_1} dxdt\right)^{1/s_1} \left(\fint_{Q_R} |\nabla w|^{\theta_1} dxdt\right)^{1/\theta_1} \\ &\times \left(\fint_{Q_R} |\nabla w - \nabla v|^2 dxdt\right)^{1/2}. \end{aligned}$$

In other words,

$$\left(\fint_{Q_R} |\nabla w - \nabla v|^2 dxdt\right)^{1/2} \lesssim [A]^R_{s_1} \left(\fint_{Q_R} |\nabla w|^{\theta_1} dxdt\right)^{1/\theta_1}.$$

After using the inequality (3.5) in Theorem 3.1 we get (3.11). □

LEMMA 3.4. *Let θ_1 be the constant in Theorem 3.1. There exists a function $v \in C(t_0-R^2, t_0; L^2(B_R)) \cap L^2(t_0-R^2, t_0; H^1(B_R)) \cap L^\infty(t_0-\frac{1}{4}R^2, t_0; W^{1,\infty}(B_{R/2}))$ such that*

$$||\nabla v||_{L^\infty(Q_{R/2})} \lesssim \fint_{Q_{2R}} |\nabla u| dxdt + \frac{|\mu|(Q_{2R})}{R^{N+1}}, \tag{3.13}$$

$$\fint_{Q_R} |\nabla u - \nabla v| dxdt \lesssim \frac{|\mu|(Q_{2R})}{R^{N+1}} + [A]_{s_1}^R \left(\fint_{Q_{2R}} |\nabla u| dxdt + \frac{|\mu|(Q_{2R})}{R^{N+1}} \right), \tag{3.14}$$

where $s_1 = \frac{2\theta_1}{\theta_1 - 2}$.

PROOF. Let w and v be in equations (3.3) and (3.10). By standard interior regularity and inequality (3.5) in Theorem 3.1 and (3.12) in Lemma 3.3 we have

$$||\nabla v||_{L^\infty(Q_{R/2})} \lesssim \left(\fint_{Q_R} |\nabla v|^2 dxdt \right)^{1/2} \lesssim \left(\fint_{Q_R} |\nabla w|^2 dxdt \right)^{1/2} \lesssim \fint_{Q_{2R}} |\nabla w| dxdt.$$

Combining this with (3.4), we get (3.13). On the other hand, (3.11) in Lemma 3.3 and Hölder's inequality yield

$$\fint_{Q_R} |\nabla w - \nabla v| dxdt \lesssim [A]_{s_1}^R \fint_{Q_{2R}} |\nabla w| dxdt.$$

It leads

$$\fint_{Q_R} |\nabla u - \nabla v| dxdt \lesssim \fint_{Q_R} |\nabla u - \nabla w| dxdt + [A]_{s_1}^R \fint_{Q_{2R}} |\nabla w| dxdt.$$

Consequently, we get (3.14) from this and (3.4) in Theorem 3.1. The proof is complete. □

3.1.2. Boundary Estimates. In this subsection, we focus on the corresponding estimates near the boundary.

Let $x_0 \in \partial\Omega$ be a boundary point and for $R > 0$ and $t_0 \in (-T, T)$. We set $\tilde{\Omega}_{6R} = \tilde{\Omega}_{6R}(x_0, t_0) = (\Omega \cap B_{6R}(x_0)) \times (t_0 - (6R)^2, t_0)$ and $Q_{6R} = Q_{6R}(x_0, t_0)$.
We consider the unique solution w to the equation

$$\begin{cases} w_t - \operatorname{div}(A(x, t, \nabla w)) = 0 & \text{in} \quad \tilde{\Omega}_{6R}, \\ w = u & \text{on} \quad \partial_p \tilde{\Omega}_{6R}. \end{cases} \tag{3.15}$$

In what follows we extend μ and u by zero to $(\Omega \times (-\infty, T))^c$ and then extend w by u to $\mathbb{R}^{N+1} \backslash \tilde{\Omega}_{6R}$.

In order to obtain estimates for w as in Theorem 3.1 we require the domain Ω to be satisfied 2−Capacity uniform thickness condition.

3.1.2.1. *2-Capacity uniform thickness domain.* It is well known that if $\mathbb{R}^N \backslash \Omega$ satisfies a uniformly 2−thick condition with constants $c_0, r_0 > 0$, there exist $p_0 \in (\frac{2N}{N+2}, 2)$ and $C = C(N, c_0) > 0$ such that

$$\operatorname{Cap}_{p_0}(\overline{B_r(x)} \cap (\mathbb{R}^N \backslash \Omega), B_{2r}(x)) \geq C r^{N-p_0}, \tag{3.16}$$

for all $0 < r \leq r_0$ and all $x \in \mathbb{R}^N \backslash \Omega$, see [**54, 64**].

THEOREM 3.5. *Suppose that $\mathbb{R}^N\backslash\Omega$ is uniformly $2-$thick with constants c_0, r_0. Let w be in (3.15) with $0 < 6R \le r_0$. There exist constants $\theta_2 > 2$, $\beta_2 \in (0, \frac{1}{2}]$ such that*

$$\fint_{Q_{6R}} |\nabla u - \nabla w| dxdt \lesssim \frac{|\mu|(\tilde{\Omega}_{6R})}{R^{N+1}}, \tag{3.17}$$

$$\left(\fint_{Q_{\rho/2}(z,s)} |\nabla w|^{\theta_2} dxdt\right)^{\frac{1}{\theta_2}} \lesssim \fint_{Q_{3\rho}(z,s)} |\nabla w| dxdt, \tag{3.18}$$

$$\left(\fint_{Q_{\rho_1}(y,s)} |w|^2 dxdt\right)^{1/2} \lesssim \left(\frac{\rho_1}{\rho_2}\right)^{\beta_2} \left(\fint_{Q_{\rho_2}(y,s)} |w|^2 dxdt\right)^{1/2}, \tag{3.19}$$

$$\left(\fint_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt\right)^{1/2} \lesssim \left(\frac{\rho_1}{\rho_2}\right)^{\beta_2 - 1} \left(\fint_{Q_{\rho_2}(z,s)} |\nabla w|^2 dxdt\right)^{1/2}, \tag{3.20}$$

for any $Q_{3\rho}(z,s) \subset Q_{6R}$, $y \in \partial\Omega$, $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s) \subset Q_{6R}$ and $Q_{\rho_1}(z,s) \subset Q_{\rho_2}(z,s) \subset Q_{6R}$.

PROOF. 1. For $\eta \in C_c^\infty([t_0-(6R)^2, t_0))$, $0 \le \eta \le 1$, $\eta_t \le 0$ and $\eta(t_0-(6R)^2) = 1$. Using $\varphi = T_k(u-w)\eta$, for any $k > 0$, as a test function for (3.1) and (3.15), we get

$$\int_{\tilde{\Omega}_{6R}} (u-w)_t T_k(u-w)\eta dxdt$$
$$+ \int_{\tilde{\Omega}_{6R}} \left(A(x,t,\nabla u) - A(x,t,\nabla w)\right) \nabla T_k(u-w)\eta dxdt = \int_{\tilde{\Omega}_{6R}} T_k(u-w)\eta d\mu.$$

Thanks to (1.3), we obtain

$$-\int_{\tilde{\Omega}_{6R}} \overline{T}_k(u-w)\eta_t dxdt + \Lambda_2 \int_{\tilde{\Omega}_{6R}} |\nabla T_k(u-w)|^2 \eta dxdt \le k|\mu|(\tilde{\Omega}_{6R}),$$

where $\overline{T}_k(s) = \int_0^s T_k(\tau)d\tau$. As in [**14**, Proposition 2.8], we also verify that

$$|||\nabla(u-w)|||_{L^{\frac{N+2}{N+1},\infty}(\tilde{\Omega}_{6R})} \lesssim |\mu|(\tilde{\Omega}_{6R}).$$

Hence we get (3.17).
2. We need to prove that

$$\fint_{Q_{r/4}(z,s)} |\nabla w|^2 dxdt \le \frac{1}{2} \fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^2 dxdt + c\left(\fint_{Q_{\frac{26}{10}r}(z,s)} |\nabla w|^{p_0} dxdt\right)^{\frac{2}{p_0}}, \tag{3.21}$$

for all $Q_{\frac{26}{10}r}(z,s) \subset Q_{6R} = Q_{6R}(x_0,t_0)$. Here the constant p_0 is in inequality (3.16). Suppose that $B_r(z) \subset \Omega$. Take $\rho \in (0, r]$. Let $\varphi \in C_c^\infty(B_\rho(z))$, $\eta \in C_c^\infty((s-\rho^2, s])$ be such that $0 \le \varphi, \eta \le 1$, $\varphi = 1$ in $B_{\rho/2}(z)$, $\eta = 1$ in $[s-\rho^2/4, s]$ and $|\nabla\varphi| \le c_1/\rho$, $|\eta_t| \le c_1/\rho^2$. We denote

$$\tilde{w}_{B_\rho(z)}(t) = \left(\int_{B_\rho(z)} \varphi(x)^2 dx\right)^{-1} \int_{B_\rho(z)} w(x,t)\varphi(x)^2 dx.$$

Using $\varphi = (w - \tilde{w}_{B_\rho(z)})\varphi^2\eta^2$ as a test function for the equation (3.15) we have for all $s' \in [s - \rho^2/4, s]$

$$\int_{B_\rho(z)\times(s-\rho^2,s')} (w - \tilde{w}_{B_\rho(z)})_t (w - \tilde{w}_{B_\rho(z)})\varphi^2\eta^2 dxdt$$
$$+ \int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla\left((w - \tilde{w}_{B_\rho(z)})\varphi^2\eta^2\right) dxdt = 0.$$

Here we used the equality $\int_{B_\rho(z)\times(s-\rho^2,s')} \left(\tilde{w}_{B_\rho(z)}\right)_t (w - \tilde{w}_{B_\rho(z)})\varphi^2\eta^2 dxdt = 0$.
Thus, we can write

$$\frac{1}{2}\int_{B_\rho(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + \int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla w\varphi^2\eta^2 dxdt$$
$$= -2\int_{B_\rho(z)\times(s-\rho^2,s')} A(x,t,\nabla w)\nabla\varphi\varphi\eta^2(w - \tilde{w}_{B_\rho(z)})dxdt$$
$$+ \int_{B_\rho(z)\times(s-\rho^2,s')} (w - \tilde{w}_{B_\rho(z)})^2\varphi^2\eta\eta_t dxdt.$$

From conditions (1.2) and (1.3), we get

$$\int_{B_\rho(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + \int_{B_\rho(z)\times(s-\rho^2,s')} |\nabla w|^2\varphi^2\eta^2 dxdt$$
$$\lesssim \int_{B_\rho(z)\times(s-\rho^2,s')} |\nabla w||\nabla\varphi|\varphi\eta^2|w - \tilde{w}_{B_\rho(z)}|dxdt + \frac{1}{\rho^2}\int_{Q_\rho(z,s)} (w - \tilde{w}_{B_\rho(z)})^2 dxdt.$$

Using Hölder's inequality we can verify that

$$\sup_{s'\in[s-\rho^2/4,s]} \int_{B_\rho(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx$$
$$+ \int_{Q_{\rho/2}(z,s)} |\nabla w|^2 dxdt \lesssim \frac{1}{\rho^2}\int_{Q_\rho(z,s)} |w - \tilde{w}_{B_\rho(z)}|^2 dxdt. \tag{3.22}$$

On the other hand, for any $s' \in [s - \rho^2/4, s]$

$$\int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_{\rho/2}(z)}(s'))^2 dx \lesssim \int_{B_\rho(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx, \tag{3.23}$$

where $\varphi_1(x) = \varphi(z + 2(x - z))$ for all $x \in B_{\rho/2}(z)$ and

$$\tilde{w}_{B_{\rho/2}(z)} = \left(\int_{B_{\rho/2}(z)} \varphi_1(x)^2 dx\right)^{-1} \int_{B_{\rho/2}(z)} w(x,t)\varphi_1(x)^2 dx.$$

In fact, since $0 \le \varphi \le 1$ and $\varphi = 1$ in $B_{\rho/2}(z)$, we have

$$\int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_{\rho/2}(z)}(s'))^2 dx$$
$$\lesssim \int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2 dx + (\tilde{w}_{B_{\rho/2}(z)}(s') - \tilde{w}_{B_\rho}(z)(s'))^2|B_{\rho/4}(z)|$$
$$\lesssim \int_{B_\rho(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2\varphi^2 dx + \int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_{\rho/2}(z)}(s'))^2\varphi_1^2 dx$$
$$+ \int_{B_{\rho/2}(z)} (w(s') - \tilde{w}_{B_\rho(z)}(s'))^2\varphi_1^2 dx.$$

which yields (3.23) due to the following inequality

$$\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2\varphi_1^2dx \le \int_{B_{\rho/2}(z)}(w(s')-l)^2\varphi_1^2dx\forall l\in\mathbb{R}.$$

Therefore,

$$\begin{aligned}&\sup_{s'\in[s-\rho^2/4,s]}\int_{B_{\rho/2}(z)}(w(s')-\tilde{w}_{B_{\rho/2}(z)}(s'))^2dx\\ &+\int_{Q_{\rho/2}(z,s)}|\nabla w|^2dxdt \lesssim \frac{1}{\rho^2}\int_{Q_\rho(z,s)}|w-\tilde{w}_{B_\rho(z)}|^2dxdt.\end{aligned}\tag{3.24}$$

Now we use estimate (3.24) for $\rho = r/2$, we have

$$\begin{aligned}\int_{Q_{r/4}(z,s)}|\nabla w|^2dxdt &\lesssim \frac{\,}{r^2}\int_{Q_{r/2}(z,s)}(w-\tilde{w}_{B_{r/2}(z)})^2dxdt\\ &\lesssim \frac{1}{r^2}\left(\sup_{s'\in[s-r^2/4,s]}\int_{B_{r/2}(z)}(w(s')-\tilde{w}_{B_{r/2}(z)}(s'))^2dx\right)^{\frac{2}{N+2}}\\ &\times\int_{s-r^2/4}^{s}\left(\int_{B_{r/2}(z)}(w-\tilde{w}_{B_{r/2}(z)})^2dx\right)^{\frac{N}{N+2}}dt.\end{aligned}$$

After we use estimate (3.24) for $\rho = r$ we get

$$\begin{aligned}\int_{Q_{r/4}(z,s)}|\nabla w|^2dxdt &\lesssim \frac{1}{r^2}\left(\frac{1}{r^2}\int_{Q_r(z,s)}|w-\tilde{w}_{B_r(z)}|^2dxdt\right)^{\frac{2}{N+2}}\\ &\times\int_{s-r^2/4}^{s}\left(\int_{B_{\rho/2}(z)}(w-\tilde{w}_{B_{r/2}(z)})^2dx\right)^{\frac{N}{N+2}}dt.\end{aligned}$$

Thanks to a Sobolev-Poincare inequality, we obtain

$$\int_{Q_{r/4}(z,s)}|\nabla w|^2dxdt \lesssim \frac{1}{r^2}\left(\int_{Q_r(z,s)}|\nabla w|^2dxdt\right)^{\frac{2}{N+2}}\int_{Q_{r/2}(z,s)}|\nabla w|^{\frac{2N}{N+2}}dxdt.$$

Since $p_0\in(\frac{2N}{N+2},2)$, thanks to Hölder's inequality we get (3.21).
Finally, we consider the case $B_r(z)\cap\Omega\neq\emptyset$. In this case we choose $z_0\in\partial\Omega$ such that $|z-z_0| = \text{dist}(z,\partial\Omega)$. Then $|z_0-z|<r$ and thus $\frac{1}{4}r\le\rho_1\le\frac{1}{2}r$,

$$B_{\frac{1}{4}r}(z)\subset B_{\frac{5}{4}r}(z_0)\subset B_{\rho_1+r}(z_0)\subset B_{\rho_1+\frac{11}{10}r}(z_0)\subset B_{\frac{16}{10}r}(z_0)\subset B_{\frac{26}{10}r}(z)\subset B_{6R}(x_0).\tag{3.25}$$

Let $\varphi\in C_c^\infty(B_{\rho_1+\frac{11}{10}r}(z_0))$ be such that $0\le\varphi\le1$, $\varphi=1$ in $B_{\rho_1+r}(z_0)$ and $|\nabla\varphi|\le C/r$. For $\frac{1}{2}r\le\rho_2\le r$, let $\eta\in C_c^\infty((s-\rho_2^2,s])$ be such that $0\le\eta\le1$, $\eta=1$ in $[s-\rho_2^2/4,s]$ and $|\eta_t|\le c/r^2$. Using $\phi = w\varphi^2\eta^2$ as a test function for (3.15) we have for any $s'\in(s-\rho_2^2,s)$

$$\begin{aligned}&\int_{(B_{\rho_1+\frac{11}{10}r}(z_0)\cap\Omega)\times(s-\rho_2^2,s')}w_tw\varphi^2\eta^2dxdt\\ &+\int_{(B_{\rho_1+\frac{11}{10}r}(z_0)\cap\Omega)\times(s-\rho_2^2,s')}A(x,t,\nabla w)\nabla\left(w\varphi^2\eta^2\right)dxdt = 0.\end{aligned}$$

As above we also get

$$\sup_{s'\in[s-\rho_2^2/4,s]}\int_{B_{\rho_1+r}(z_0)} w^2(s')dx + \int_{B_{\rho_1+r}(z_0)\times(s-\rho_2^2/4,s)} |\nabla w|^2 dxdt \lesssim \frac{1}{r^2}\int_{B_{\rho_1+\frac{11}{10}r}(z_0)\times(s-\rho_2^2,s)} w^2 dxdt.$$

In particular, for $\rho_1=\frac{1}{4}r$, $\rho_2=\frac{1}{2}r$ and using (3.25) yields

$$\int_{Q_{\frac{1}{4}r}(z,s)} |\nabla w|^2 dxdt \lesssim \frac{1}{r^2}\int_{B_{\frac{29}{20}r}(z_0)\times(s-r^2/4,s)} w^2 dxdt, \tag{3.26}$$

and $\rho_1=(\frac{1}{4}+\frac{1}{10})r, \rho_2=r$,

$$\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx \lesssim \frac{1}{r^2}\int_{B_{\frac{29}{20}r}(z_0)\times(s-r^2,s)} w^2 dxdt.$$

Set $K_1=\{w=0\}\cap\overline{B}_{\frac{29}{20}r}(z_0)$ and $K_2=\{w=0\}\cap\overline{B}_{\frac{1}{4}r+\frac{11}{10}r}(z_0)$, Since $\mathbb{R}^N\backslash\Omega$ satisfies an uniformly 2−thick, we have the following estimates

$$\mathrm{Cap}_2(K_1,B_{\frac{29}{10}r}(z_0))\gtrsim r^{N-2} \text{ and } \mathrm{Cap}_{p_0}(K_2,B_{\frac{1}{2}r+\frac{11}{5}r}(z_0))\gtrsim r^{N-p_0}.$$

So, by Sobolev-Poincare's inequality we get

$$\fint_{B_{\frac{29}{20}r}(z_0)} w^2 dx \lesssim r^2 \fint_{B_{\frac{5}{2}r}(z)} |\nabla w|^2 dx, \tag{3.27}$$

and

$$\fint_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2 dxdt \lesssim r^2\left(\fint_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} |\nabla w|^{p_0}dx\right)^{\frac{2}{p_0}} \lesssim r^2\left(\fint_{B_{\frac{5}{2}r}(z_0)} |\nabla w|^{p_0}dx\right)^{\frac{2}{p_0}}.$$

Hence,

$$\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx \lesssim \int_{Q_{\frac{5}{2}r}(z,s)} |\nabla w|^2 dxdt, \tag{3.28}$$

and

$$\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(t)dx \lesssim r^{N+2}\left(\fint_{B_{\frac{5}{2}r}(z_0)} |\nabla w|^{p_0}(t)dx\right)^{\frac{2}{p_0}}. \tag{3.29}$$

From (3.26), we have

$$\fint_{Q_{\frac{1}{4}r}(z,s)} |\nabla w|^2 dxdt \lesssim \frac{1}{r^{N+4}}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)\times(s-r^2/4,s)} w^2 dxdt$$

$$\lesssim \frac{1}{r^{N+4}}\left(\sup_{s'\in[s-r^2/4,s]}\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(s')dx\right)^{1-\frac{p_0}{2}}\int_{s-r^2/4}^{s}\left(\int_{B_{\frac{1}{4}r+\frac{11}{10}r}(z_0)} w^2(t)dx\right)^{\frac{p_0}{2}}dt.$$

Using (3.29), (3.28) and Hölder's inequality we get

$$\begin{aligned}\fint_{Q_{\frac{1}{4}r}(z,s)}|\nabla w|^2dxdt &\lesssim \frac{1}{r^{N+4}}\left(\int_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^2dxdt\right)^{1-\frac{p_0}{2}} r^{\frac{N+2}{2}p_0-N}\int_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^{p_0}dxdt\\ &\lesssim \left(\fint_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^2dxdt\right)^{1-\frac{p_0}{2}}\fint_{Q_{\frac{5}{2}r}(z,s)}|\nabla w|^{p_0}dxdt\\ &\le \frac{1}{2}\fint_{Q_{\frac{26}{10}r}(z,s)}|\nabla w|^2dxdt + c\left(\fint_{Q_{\frac{26}{10}r}(z,s)}|\nabla w|^{p_0}dxdt\right)^{\frac{2}{p_0}}.\end{aligned}$$

So we proved (3.21).
Therefore, By Gehring's Lemma (see [**67**]) we get (3.18).
3. Now we prove (3.19). Let $y\in\partial\Omega$, $Q_{\rho_1}(y,s)\subset Q_{\rho_2}(y,s)\subset Q_{6R}$ with $\rho_1\le\rho_2/4$. First, we will show that there exists a constant $\beta_2=\beta_2(N,\Lambda_1,\Lambda_2,c_0)\in(0,1/2]$ such that

$$\operatorname{osc}(w,Q_{\rho_1}(y,s))\lesssim\left(\frac{\rho_1}{\rho_2}\right)^{\beta_2}\operatorname{osc}(w,Q_{\rho_2/2}(y,s)), \tag{3.30}$$

where $\operatorname{osc}(w,O)=\sup_O w-\inf_O w$.
Indeed, since

$$\int_0^1\frac{\mathrm{Cap}_{1,2}(\Omega^c\cap B_r(z),B_{2r}(z))}{r^{N-2}}\frac{dr}{r}=+\infty\forall z\in\partial\Omega.$$

thus by the Wiener criterion (see [**95**]), we have w is continuous up to $\partial_p\tilde{\Omega}_{6R}$. So, we can choose $\varphi=(V-M_{4\rho_1})\,\eta^2\in L^2(-\infty,T;H_0^1(\Omega\cap B_{6R}(x_0)))$ as test function in (3.15), where

- $\eta\in C^\infty(Q_{4\rho_1}(y,s))$, $0\le\eta\le1$ such that $\eta=1$ in $Q_{\rho_1/2}(y,s-\frac{17}{4}\rho_1^2)$, $\operatorname{supp}(\eta)\subset\subset Q_{\rho_1}(y,s-4\rho_1^2)$ and $|\nabla\eta|\le c_{27}/\rho_1$, $|\eta_t|\le c_{28}/\rho_1^2$,
- $M_{4\rho_1}=\sup_{Q_{4\rho_1}(y,s)}w$ and $V=\inf\{M_{4\rho_1}-w,M_{4\rho_1}\}$ in $\tilde{\Omega}_{6R}$, $V=M_{4\rho_1}$ outside $\tilde{\Omega}_{6R}$.

We have

$$\begin{aligned}&\int_{\tilde{\Omega}_{6R}}w_t\,(V-M_{4\rho_1})\,\eta^2dxdt\\ &+\int_{\tilde{\Omega}_{6R}}2\eta A(x,t,\nabla w)\nabla\eta\,(V-M_{4\rho_1})\,dxdt+\int_{\tilde{\Omega}_{6R}}\eta^2A(x,t,\nabla w)\nabla V dxdt=0,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\tilde{\Omega}_{6R}}\eta^2A(x,t,-\nabla V)(-\nabla V)dxdt=\int_{\tilde{\Omega}_{6R}}2\eta A(x,t,-\nabla V)\nabla\eta\,(V-M_{4\rho_1})\,dxdt\\ &-\int_{\tilde{\Omega}_{6R}}(V-M_{4\rho_1})_t\,(V-M_{4\rho_1})\,\eta^2dxdt.\end{aligned}$$

Using (1.2) and (1.3) we get

$$\begin{aligned}
&\Lambda_2 \int_{\tilde{\Omega}_{6R}} \eta^2 |\nabla V|^2 dxdt \\
&\leq 2\Lambda_1 \int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta||V - M_{4\rho_1}| dxdt - 1/2 \int_{\tilde{\Omega}_{6R}} \left((V - M_{4\rho_1})^2 - M_{4\rho_1}^2 \right) (\eta^2)_t dxdt \\
&\leq 2\Lambda_1 M_{4\rho_1} \int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt + 2M_{4\rho_1} \int_{\tilde{\Omega}_{6R}} \eta V |\eta_t| dxdt.
\end{aligned}$$

Since $\mathrm{supp}(|\nabla V|) \cap \mathrm{supp}(\eta) \subset \tilde{\Omega}_{6R}$, thus

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt &\lesssim M_{4\rho_1} \left(\int_{\mathbb{R}^{N+1}} \eta |\nabla V||\nabla \eta| dxdt + \int_{\mathbb{R}^{N+1}} V \left(\eta |\eta_t| + |\nabla \eta|^2 \right) dxdt \right) \\
&\lesssim M_{4\rho_1} \left(\int_{\mathbb{R}^{N+1}} \eta |\nabla V||\nabla \eta| dxdt + \frac{1}{\rho_1^2} \int_{Q_{\rho_1}(y, s-4\rho_1^2)} V dxdt \right). \qquad (3.31)
\end{aligned}$$

By [**57**, Theorem 6.31, p. 132], for any $\sigma \in (0, 1 + 2/N)$ there holds

$$\left(\fint_{Q_{\rho_1}(y, s-4\rho_1^2)} V^\sigma dxdt \right)^{1/\sigma} \lesssim \inf_{Q_{\rho_1}(y,s)} V = M_{4\rho_1} - \sup_{Q_{\rho_1}(y,s)} w = M_{4\rho_1} - M_{\rho_1}. \qquad (3.32)$$

In particular,

$$\frac{1}{\rho_1^2} \int_{Q_{\rho_1}(y, s-4\rho_1^2)} V dxdt \lesssim \rho_1^N (M_{4\rho_1} - M_{\rho_1}). \qquad (3.33)$$

We need to estimate $\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt$. Using Hölder inequality and (3.32), for $\varepsilon \in (0, \min\{2/N, 1\})$ we have

$$\begin{aligned}
\int_{\tilde{\Omega}_{6R}} \eta |\nabla V||\nabla \eta| dxdt &\leq \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2} \left(\int_{\tilde{\Omega}_{6R}} V^{1+\varepsilon} |\nabla \eta|^2 dxdt \right)^{1/2} \\
&\lesssim \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2} \left(\int_{Q_{\rho_1}(y, s-4\rho_1^2)} V^{1+\varepsilon} dxdt \right)^{1/2} \\
&\lesssim \left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2} \rho_1^{\frac{N+2}{2}} (M_{4\rho_1} - M_{\rho_1})^{(1+\varepsilon)/2}.
\end{aligned}$$

To estimate $\left(\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \right)^{1/2}$, we can choose $\varphi = ((V + \delta)^{-\varepsilon} - (M_{4\rho_1} + \delta)^{-\varepsilon})\eta^2$, for $\delta > 0$, as test function in (3.15), we will get

$$\begin{aligned}
&\int_{\tilde{\Omega}_{6R}} \eta^2 (V + \delta)^{-(1+\varepsilon)} |\nabla V|^2 dxdt \\
&\lesssim \int_{\tilde{\Omega}_{6R}} \eta (V + \delta)^{-\varepsilon} |\nabla V||\nabla \eta| dxdt + \int_{\tilde{\Omega}_{6R}} \eta (V + \delta)^{1-\varepsilon} |\eta_t| dxdt.
\end{aligned}$$

Thanks to Hölder's inequality, we obtain

$$\begin{aligned}
\int_{\tilde{\Omega}_{6R}} \eta^2 (V + \delta)^{-(1+\varepsilon)} |\nabla V|^2 dxdt &\lesssim \int_{\tilde{\Omega}_{6R}} (V + \delta)^{1-\varepsilon} \left(\eta |\eta_t| + |\nabla \eta|^2 \right) dxdt \\
&\lesssim \rho_1^2 \int_{Q_{\rho_1}(y, s-4\rho_1^2)} (V + \delta)^{1-\varepsilon} dxdt.
\end{aligned}$$

Letting $\delta \to 0$ and using (3.32), we get

$$\int_{\tilde{\Omega}_{6R}} \eta^2 V^{-(1+\varepsilon)} |\nabla V|^2 dxdt \lesssim \rho_1^{-2} \int_{Q_{\rho_1}(y,s-4\rho_1^2)} V^{1-\varepsilon} dxdt \lesssim \rho_1^N \left(M_{4\rho_1} - M_{\rho_1}\right)^{1-\varepsilon}.$$

Thus,

$$\int_{\tilde{\Omega}_{6R}} \eta |\nabla V| |\nabla \eta| dxdt \lesssim \rho_1^N (M_{4\rho_1} - M_{\rho_1}).$$

Combining this with (3.31) and (3.33),

$$\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt \lesssim \rho_1^N M_{4\rho_1} \left(M_{4\rho_1} - M_{\rho_1}\right).$$

Note that $\eta V = M_{4\rho_1}$ in $\left(\Omega^c \cap B_{\rho_1/2}(y)\right) \times (s - \frac{9}{2}\rho_1^2, s - \frac{17}{4}\rho_1^2)$ thus

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} |\nabla(\eta V)|^2 dxdt &\geq \int_{s-\frac{9}{2}\rho_1^2}^{s-\frac{17}{4}\rho_1^2} \int_{\mathbb{R}^{\mathbb{N}}} |\nabla(\eta V)|^2 dxdt \\ &\geq \int_{s-\frac{9}{2}\rho_1^2}^{s-\frac{17}{4}\rho_1^2} M_{4\rho_1}^2 \mathrm{Cap}_{1,2}(\Omega^c \cap B_{\rho_1/2}(y), B_{\rho_1}(y)) dt \\ &\gtrsim M_{4\rho_1}^2 \rho_1^N.\end{aligned}$$

Here we have used $Cap_{1,2}(\Omega^c \cap B_{\rho_1/2}(y), B_{\rho_1}(y)) \gtrsim \rho_1^{N-2}$ in the last inequality. It follows

$$M_{4\rho_1} \leq c(M_{4\rho_1} - M_{\rho_1}).$$

So

$$\sup_{Q_{\rho_1}(y,s)} w \leq \gamma \sup_{Q_{4\rho_1}(y,s)} w \text{ where } \gamma = \frac{c}{c+1} < 1.$$

Clearly, above estimate is also true when we replace w by $-w$. These give,

$$\mathrm{osc}(w, Q_{\rho_1}(y,s)) \leq \gamma \mathrm{osc}(w, Q_{4\rho_1}(y,s)).$$

It follows (3.30).
We come back the proof of (3.19).
Since $w = 0$ outside Ω_T this leads to

$$\left(\fint_{Q_{\rho_1}(y,s)} |w|^2 dxdt \right)^{1/2} \lesssim \left(\frac{\rho_2}{\rho_1}\right)^{\beta_2} \mathrm{osc}(w, Q_{\rho_2/2}(y,s)).$$

On the other hand, By [**57**, Theorem 6.30, p. 132] we have

$$\sup_{Q_{\rho_2/2}(y,s)} w \lesssim \left(\fint_{Q_{\rho_2}(y,s)} (w^+)^2 \right)^{1/2}, \sup_{Q_{\rho_2/2}(y,s)} (-w) \lesssim \left(\fint_{Q_{\rho_2}(y,s)} (w^-)^2 \right)^{1/2}.$$

Thus, we get (3.19).
Next, we have (3.20) for case $z = y \in \partial\Omega$ since from Caccippoli's inequality,

$$\int_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt \lesssim \frac{1}{\rho_1^2} \int_{Q_{2\rho_1}(z,s)} |w|^2 dxdt,$$

and using Sobolev-Poincare's inequality as in (3.27),

$$\int_{Q_{\rho_2}(z,s)} |w|^2 dxdt \lesssim \rho_2^2 \int_{Q_{\rho_2}(z,s)} |\nabla w|^2 dxdt.$$

We now prove (3.20). Take $Q_{\rho_1}(z,s) \subset Q_{\rho_2}(z,s) \subset Q_{6R}$, it is enough to consider the case $\rho_1 \le \rho_2/20$. Clearly, if $B_{\rho_2/4}(z) \subset \Omega$ then (3.20) follows from (3.7) in Theorem 3.1. We consider $B_{\rho_2/4}(z) \cap \partial\Omega \neq \emptyset$, let $z_0 \in B_{\rho_2/4}(z) \cap \partial\Omega$ such that $|z - z_0| = \text{dist}(z, \partial\Omega) \le \rho_2/4$. Obviously, if $\rho_1 < |z - z_0|/4$ and $z \notin \Omega$, then (3.20) is trivial. If $\rho_1 < |z - z_0|/4$ and $z \in \Omega$, then (3.20) follows from (3.7) in Theorem 3.1.
Now assume $\rho_1 \ge |z - z_0|/4$ then since $Q_{\rho_1}(z,s) \subset Q_{5\rho_1}(z_0,s)$

$$\begin{aligned}\left(\fint_{Q_{\rho_1}(z,s)} |\nabla w|^2 dxdt\right)^{1/2} &\lesssim \left(\fint_{Q_{5\rho_1}(z_0,s)} |\nabla w|^2 dxdt\right)^{1/2} \\ &\lesssim \left(\frac{\rho_1}{\rho_2}\right)^{\beta_2 - 1} \left(\fint_{Q_{\rho_2/4}(z_0,s)} |\nabla w|^2 dxdt\right)^{1/2} \\ &\lesssim \left(\frac{\rho_1}{\rho_2}\right)^{\beta_2 - 1} \left(\fint_{Q_{\rho_2/2}(z,s)} |\nabla w|^2 dxdt\right)^{1/2},\end{aligned}$$

which implies (3.20). $\square$

COROLLARY 3.6. Suppose that $\mathbb{R}^N \backslash \Omega$ satisfies uniformly $2-$thick with constants c_0, r_0. Let β_2 be the constant in Theorem 3.5. For $2 - \beta_2 < \theta < N + 2$, there holds for any $B_\rho(y) \cap \partial\Omega \neq \emptyset$, $s \in (-T,T)$, $0 < \rho \le r_0$

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \lesssim \rho^{N+3-\theta} \left(\left(\frac{T_0}{r_0}\right)^{N+3-\theta} + 1\right) ||\mathbb{M}_\theta[\mu]||_{L^\infty(\Omega\times(-T,T))}, \tag{3.34}$$

where $T_0 = diam(\Omega) + T^{1/2}$.

PROOF. Take $B_{\rho_2/4}(y) \cap \partial\Omega \neq \emptyset$ and $s \in (-T,T)$, $\rho_2 \le 2r_0$. Let $y_0 \in B_{\rho_2/4}(y) \cap \partial\Omega$ be such that $|y - y_0| = dist(y, \partial\Omega) \le \rho_2/4$. Thus $Q_{\rho_2/4}(y,s) \subset Q_{\rho_2/2}(y_0,s)$ For any $Q_{\rho_1}(y,s) \subset Q_{\rho_2}(y,s)$ with $\rho_1 \le \rho_2/4$, we take w as in Theorem 3.5 with $Q_{6R} = Q_{\rho_2/2}(y_0,s)$. Thus,

$$\int_{Q_{\rho_1}(y,s)} |\nabla w| dxdt \lesssim \left(\frac{\rho_1}{\rho_2}\right)^{N+\beta_1+1} \int_{Q_{\rho_2/4}(y,s)} |\nabla w| dxdt,$$
$$\int_{Q_{\rho_2/2}(y_0,s)} |\nabla u - \nabla w| dxdt \lesssim \rho_2 |\mu|(Q_{\rho_2/2}(y_0,s)).$$

As in the proof of Corollary 3.2, we get the result. $\square$

3.1.2.2. *Reifenberg flat domain.* In this subsection, we always assume that A satisfies (1.45). Also, we assume that Ω is a (δ, R_0)- Reifenberg flat domain with $0 < \delta < 1/2$. Fix $x_0 \in \partial\Omega$ and $0 < R < R_0/6$. We have a density estimate

$$|B_t(x) \cap (\mathbb{R}^N \backslash \Omega)| \ge c|B_t(x)| \forall x \in \partial\Omega, 0 < t < R_0, \tag{3.35}$$

with $c = ((1-\delta)/2)^N \ge 4^{-N}$.
In particular, $\mathbb{R}^N \backslash \Omega$ satisfies a uniformly $2-$thick condition with constants $c, r_0 = R_0$.
Next we set $\rho = R(1-\delta)$ so that $0 < \rho/(1-\delta) < R_0/6$. By the definition of

Reifenberg flat domains, there exists a coordinate system $\{y_1, y_2, ..., y_N\}$ with the origin $0 \in \Omega$ such that in this coordinate system $x_0 = (0, ..., 0, -\rho\delta/(1-\delta))$ and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{y = (y_1, y_2, ..., y_N) : y_N > -2\rho\delta/(1-\delta)\}.$$

Since $\delta < 1/2$ we have

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{y = (y_1, y_2, ..., y_N) : y_N > -4\rho\delta\},$$

where $B_\rho^+(0) := B_\rho(0) \cap \{y = (y_1, y_2, ..., y_N) : y_N > 0\}$.
Furthermore we consider the unique solution

$$v \in C(t_0 - \rho^2, t_0; L^2(\Omega \cap B_\rho(0))) \cap L^2(t_0 - \rho^2, t_0; H^1(\Omega \cap B_\rho(0))) \tag{3.36}$$

to the following equation

$$\begin{cases} v_t - \operatorname{div}\left(\overline{A}_{B_\rho(0)}(t, \nabla v)\right) = 0 & \text{in} \quad \tilde{\Omega}_\rho(0), \\ v = w & \text{on} \quad \partial_p \tilde{\Omega}_\rho(0), \end{cases} \tag{3.37}$$

where $\tilde{\Omega}_\rho(0) = (\Omega \cap B_\rho(0)) \times (t_0 - \rho^2, t_0)$ $(-T < t_0 < T)$.
We put $v = w$ outside $\tilde{\Omega}_\rho(0)$. As Lemma 3.3, we have the following Lemma.

LEMMA 3.7. *Let θ_2 be the constant in Theorem 3.5. There holds*

$$\left(\fint_{Q_\rho(0,t_0)} |\nabla w - \nabla v|^2\right)^{1/2} \lesssim [A]_{s_2}^R \fint_{Q_\rho(0,t_0)} |\nabla w| dxdt, \tag{3.38}$$

with $s_2 = \frac{2\theta_2}{\theta_2 - 2}$ *and*

$$\int_{Q_\rho(0,t_0)} |\nabla w|^2 dxdt \sim \int_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt. \tag{3.39}$$

We can see that if the boundary of Ω is bad enough, then the L^∞-norm of ∇v up to $\partial\Omega \cap B_\rho(0) \times (t_0 - \rho^2, t_0)$ could be unbounded. For our purpose, we will consider another equation:

$$\begin{cases} V_t - \operatorname{div}\left(\overline{A}_{B_\rho(0)}(t, \nabla V)\right) = 0 \text{in} Q_\rho^+(0, t_0), \\ V = 0 \qquad \text{on} T_\rho(0, t_0), \end{cases} \tag{3.40}$$

where $Q_\rho^+(0, t_0) = B_\rho^+(0) \times (t_0 - \rho^2, t_0)$ and $T_\rho(0, t_0) = Q_\rho(0, t_0) \cap \{x_N = 0\}$.
A weak solution V of above problem is understood in the following sense: the zero extension of V to $Q_\rho(0, t_0)$ is in $V \in C(t_0 - \rho^2, t_0; L^2(B_\rho(0))) \cap L^2_{\text{loc}}(t_0 - \rho^2, t_0; H^1(B_\rho(0)))$ and for every $\varphi \in C_c^1(Q_\rho^+(0, t_0))$ there holds

$$-\int_{Q_\rho^+(0,t_0)} V\varphi_t dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\varphi dxdt = 0.$$

We have the following gradient L^∞ estimate up to the boundary for V.

LEMMA 3.8 (see [**55, 56**]). *For any weak solution $V \in C(t_0-\rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2_{loc}(t_0 - \rho^2, t_0; H^1(B_\rho^+(0)))$ of (3.40), we have*

$$||\nabla V||_{L^\infty(Q^+_{\rho'/2}(0,t_0))} \lesssim \fint_{Q^+_{\rho'}(0,t_0)} |\nabla V|^2 dxdt \forall 0 < \rho' \leq \rho.$$

Moreover, ∇V is continuous up to $T_\rho(0, t_0)$.

LEMMA 3.9. *If $V \in C(t_0-\rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2(t_0-\rho^2, t_0; H^1(B_\rho^+(0)))$ is a weak solution of (3.40), then its zero extension from $Q_\rho^+(0,t_0)$ to $Q_\rho(0,t_0)$ solves*

$$V_t - \operatorname{div}\left(\overline{A}_{B_\rho(0)}(t, \nabla V)\right) = \frac{\partial F}{\partial x_N}, \tag{3.41}$$

weakly in $Q_\rho(0,t_0)$. Here, $\overline{A}_{B_\rho(0)} = (\overline{A}^1_{B_\rho(0)}, ..., \overline{A}^N_{B_\rho(0)})$ and $F(x,t) = \chi_{x_N<0}\overline{A}^N_{B_\rho(0)}(t, \nabla V(x', 0, t))$ for $(x,t) = (x', x_N, t) \in Q_\rho(0,t_0)$.

PROOF. Let $g \in C^\infty(\mathbb{R})$ with $g = 0$ on $(-\infty, 1/2)$ and $g = 1$ on $(1, \infty)$. Then, for any $\varphi \in C_c^\infty(Q_\rho(0,t_0))$ and $n \in \mathbb{N}$, we have $\varphi_n(x,t) = \varphi_n(x', x_N, t) = g(nx_N)\varphi(x,t) \in C_c^\infty(Q_\rho^+(0,t_0)$. Thus, we get

$$\int_{Q_\rho^+(0,t_0)} V_t \varphi_n dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\left(g(nx_N)\varphi(x,t)\right) dxdt = 0,$$

which implies

$$\int_{Q_\rho^+(0,t_0)} V_t \varphi_n dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\varphi(x,t) g(nx_N) dxdt = -\int_0^\rho G(z) g'(nz) n dz.$$

Here

$$G(z) = \int_{t_0-\rho^2}^{t_0} \int_{|x'|<\sqrt{\rho^2-z^2}} \overline{A}^N_{B_\rho(0)}(t, \nabla V)\varphi(x', z, t) dx'dt \in C([0,\infty)).$$

Letting $n \to \infty$ we get

$$\int_{Q_\rho^+(0,t_0)} V_t \varphi dxdt + \int_{Q_\rho^+(0,t_0)} \overline{A}_{B_\rho(0)}(t, \nabla V)\nabla\varphi(x,t) dxdt = -\int_{Q_\rho(0,t_0)} F \frac{\partial \varphi}{\partial x_N} dxdt.$$

Since $\nabla V = 0, V = 0$ outside Q_ρ^+, therefore we get the result. $\square$

We now consider a scaled version of equation (3.37)

$$\begin{cases} v_t - \operatorname{div}\left(\overline{A}_{B_1(0)}(t, \nabla v)\right) = 0 \text{ in } \tilde{\Omega}_1(0), \\ v = 0 \text{ on } \partial_p \tilde{\Omega}_1(0) \setminus (\Omega \times (-T,T)), \end{cases} \tag{3.42}$$

under assumption

$$B_1^+ \subset \Omega \cap B_1 \subset B_1 \cap \{x_N > -4\delta\} \tag{3.43}$$

with $B_\rho = B_\rho(0)$.

LEMMA 3.10. *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that if $v \in C(t_0-1, t_0; L^2(\Omega \cap B_1)) \cap L^2(t_0-1, t_0; H^1(\Omega \cap B_1))$ is a solution of (3.42) and (3.43) is satisfied and*

$$\fint_{Q_1(0,t_0)} |\nabla v|^2 dxdt \leq 1, \tag{3.44}$$

then there exists a weak solution $V \in C(t_0-1, t_0; L^2(B_1^+)) \cap L^2(t_0-1, t_0; H^1(B_1^+))$ of (3.40) with $\rho = 1$, whose zero extension to $Q_1(0,t_0)$ satisfies

$$\fint_{Q_1(0,t_0)} |v - V|^2 dxdt \leq \varepsilon^2, \tag{3.45}$$

Proof. We argue by contradiction. Suppose that the conclusion is false. Then, there exist a constant $\varepsilon_0 > 0$, $t_0 \in \mathbb{R}$ and a sequence of nonlinearities $\{A_k\}$ satisfying (1.2) and (1.45), a sequence of domains $\{\Omega^k\}$, and a sequence of functions $\{v_k\} \subset C(t_0-1, t_0; L^2(\Omega^k \cap B_1)) \cap L^2(t_0-1, t_0; H^1(\Omega^k \cap B_1))$ such that

$$B_1^+ \subset \Omega^k \cap B_1 \subset B_1 \cap \{x_N > -1/2k\}, \tag{3.46}$$

$$\begin{cases} (v_k)_t - \operatorname{div}\left(\overline{A}_{k,B_1}(t, \nabla v_k)\right) = 0 \text{ in } \tilde{\Omega}_1^k(0), \\ v_k = 0 \text{ on } (\partial_p \tilde{\Omega}_1^k(0)) \backslash (\Omega^k \times (-T,T)), \end{cases} \tag{3.47}$$

and the zero extension of each v_k to $Q_1(0,t_0)$ satisfies

$$\fint_{Q_1(0,t_0)} |\nabla v_k|^2 dxdt \leq 1 \text{ but} \tag{3.48}$$

$$\fint_{Q_1(0,t_0)} |v_k - V_k|^2 dxdt \geq \varepsilon_0^2, \tag{3.49}$$

for any weak solution V_k of

$$\begin{cases} (V_k)_t - \operatorname{div}\left(\overline{A}_{k,B_1}(t, \nabla V_k)\right) = 0, \text{in } Q_1^+(0,t_0), \\ V_k = 0 \text{on} T_1(0,t_0). \end{cases} \tag{3.50}$$

By (3.46) and (3.48) and Poincare's inequality it follows that

$$||v_k||_{L^2(t_0-1,t_0;H^1(B_1))} \lesssim ||\nabla v_k||_{L^2(Q_1(0,t_0))} \lesssim 1,$$
$$||(v_k)_t||_{L^2(t_0-1,t_0;H^{-1}(B_1))} \lesssim \int_{Q_1(0,t_0)} |\nabla v_k|^2 dxdt \lesssim 1.$$

Therefore, using Aubin−Lions Lemma, one can find v_0 and a subsequence of $\{v_k\}$, still denoted by $\{v_k\}$ such that

$$v_k \to v_0 \text{ weakly in } L^2(t_0-1, t_0, H^1(B_1)) \text{ and strongly in } L^2(t_0-1, t_0, L^2(B_1)),$$

and

$$(v_k)_t \to (v_0)_t \text{ weakly in } L^2(t_0-1, t_0, H^{-1}(B_1)).$$

Moreover, $v_0 = 0$ in $Q_1^-(0,t_0) := (B_1 \cap \{x_N < 0\}) \times (1-t_0, 1)$ since $v_k = 0$ on outside $\Omega^k \cap Q_1(0,t_0)$ for all k.
To get a contradiction we take V_k to be the unique solution of $(V_k)_t - \operatorname{div}\left(\overline{A}_{k,B_1}(t, \nabla V_k)\right) = 0$ in $Q_1^+(0,t_0)$ and $V_k - v_0 \in L^2(t_0-1, t_0, H_0^1(B_1^+))$ and $V_k(t_0-1) = v_0(t_0-1)$. As above, one can find V_0 and a subsequence of $\{V_k\}$, still denoted by $\{V_k\}$ such that

$$V_k \to V_0 \text{ weakly in } L^2(t_0-1, t_0, H^1(B_1)) \text{ and strongly in } L^2(t_0-1, t_0, L^2(B_1)),$$

and

$$(V_k)_t \to (V_0)_t \text{ weakly in } L^2(t_0-1, t_0, H^{-1}(B_1)),$$

for some $V_0 \in v_0 + L^2(t_0-1, t_0, H_0^1(B_1^+)$ and $V_0(t_0-1) = v_0(t_0-1)$.
Thanks to (3.49), the proof would be complete if we could show that $v_0 = V_0$. In fact,
Let $\mathcal{J}_k : X \to L^2(Q_1^+(0,t_0), \mathbb{R}^N)$ determine by

$$\mathcal{J}_k(\phi(x,t)) = \overline{A}_{k,B_1}(t, \nabla\phi(x,t)) \text{ for any } \phi \in X,$$

where $X \subset L^2(t_0-1, t_0, H^1(B_1))$ is closures (in the strong topology of $L^2(t_0-1, t_0, H^1(B_1)))$ of convex combinations of $\{v_k\}_{k\geq 1} \cup \{V_k\}_{k\geq 1} \cup \{0\}$.
Since v_k, V_k converge weakly to v_0, V_0 in $L^2(t_0-1, t_0, H^1(B_1))$ resp., thus by Mazur

Theorem, X is compact subset of $L^2(t_0-1,t_0,H^1(B_1))$ and $v_0,V_0\in X$.
Thanks to (1.2) and (1.45), we get $\mathcal{J}_k(0)=0$ and

$$||\mathcal{J}_k(\phi_1)-\mathcal{J}_k(\phi_2)||_{L^2(Q_1^+(0,t_0),\mathbb{R}^N)}\le\Lambda_1||\phi_1-\phi_2||_{L^2(t_0-1,t_0,H^1(B_1))},$$

for every $\phi_1,\phi_2\in X$ and $k\in\mathbb{N}$. Thus, by Ascoli Theorem, there exist $\mathcal{J}\in C(X,L^2(Q_1^+(0,t_0),\mathbb{R}^N))$ and a subsequence of $\{\mathcal{J}_k\}$, still denoted by it, such that

$$\sup_{\phi\in X}||\mathcal{J}_k(\phi)-\mathcal{J}(\phi)||_{L^2(Q_1^+(0,t_0),\mathbb{R}^N)}\to 0\text{ as }k\to\infty,\tag{3.51}$$

and also for any $\phi_1,\phi_2\in X$,

(3.52)
$$\int_{Q_1^+(0,t_0)}(\mathcal{J}(\phi_1)-\mathcal{J}(\phi_2)).(\nabla\phi_1-\nabla\phi_2)\,dxdt\ge\Lambda_2|||\nabla\phi_1-\nabla\phi_2|||_{L^2(Q_1^+(0,t_0))}.$$

From (3.47), we deduce

$$\int_{Q_1^+(0,t_0)}(v_k-V_k)_t(v_0-V_0)dxdt$$
$$+\int_{Q_1^+(0,t_0)}\left(\overline{A}_{k,B_1}(t,\nabla v_k)-\overline{A}_{k,B_1}(t,\nabla V_k)\right).\nabla(v_0-V_0)dxdt=0.$$

We have

$$\int_{Q_1^+(0,t_0)}|\overline{A}_{k,B_1}(\nabla v_k)|^2dxdt\lesssim\int_{Q_1^+(0,t_0)}|\nabla v_k|^2dxdt\lesssim 1,$$
$$\int_{Q_1^+(0,t_0)}|\overline{A}_{k,B_1}(\nabla V_k)|^2dxdt\lesssim\int_{Q_1^+(0,t_0)}|\nabla V_k|^2dxdt\lesssim 1.$$

for every k.
Thus there exist a subsequence, still denoted by $\{\overline{A}_{k,B_1}(t,\nabla v_k),\overline{A}_{k,B_1}(t,\nabla V_k)\}$ and a vector field A_1,A_2 belonging to $L^2(Q_1^+(0,t_0),\mathbb{R}^N)$ such that

$$\overline{A}_{k,B_1}(t,\nabla v_k)\to A_1\text{ and }\overline{A}_{k,B_1}(t,\nabla V_k)\to A_2,$$

weakly in $L^2(Q_1^+(0,t_0),\mathbb{R}^N)$. It follows

$$\int_{Q_1^+(0,t_0)}(v_0-V_0)_t(v_0-V_0)dxdt+\int_{Q_1^+(0,t_0)}(A_1-A_2).\nabla(v_0-V_0)dxdt=0.$$

Since

$$\int_{Q_1^+(0,t_0)}(v_0-V_0)_t(v_0-V_0)dxdt=\int_{B_1^+(0)}(v_0-V_0)^2(t_0)dx\ge 0,$$

we get

$$\int_{Q_1^+(0,t_0)}(A_1-A_2).\nabla(v_0-V_0)dxdt\le 0.\tag{3.53}$$

For our purpose, we need to show that

$$\int_{Q_1^+(0,t_0)}(A_1-\mathcal{J}(v_0)).\nabla(v_0-V_0)dxdt\ge 0,\tag{3.54}$$

$$\int_{Q_1^+(0,t_0)}(A_2-\mathcal{J}(V_0)).\nabla(V_0-v_0)dxdt\ge 0.\tag{3.55}$$

To do this, we fix a function $g \in X$ and any $\varphi \in C_c^1(Q_1^+(0,t_0))$ such that $\varphi \geq 0$. We have

$$\begin{aligned} 0 &\leq \int_{Q_1^+(0,t_0)} \varphi\left(\overline{A}_{k,B_1}(t,\nabla v_k) - \overline{A}_{k,B_1}(t,\nabla g)\right)(\nabla v_k - \nabla g)\,dxdt \\ &= \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1}(t,\nabla v_k)\nabla v_k dxdt - \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1}(t,\nabla v_k)\nabla g dxdt \\ &- \int_{Q_1^+(0,t_0)} \varphi \overline{A}_{k,B_1}(t,\nabla g)(\nabla v_k - \nabla g)\,dxdt \\ &:= L_1 + L_2 + L_3. \end{aligned}$$

It is easy to see that

$$\lim_{k\to\infty} L_2 = -\int_{Q_1^+(0,t_0)} \varphi A_1 \nabla g dxdt \text{ and } \lim_{k\to\infty} L_3 = -\int_{Q_1^+(0,t_0)} \varphi \mathcal{J}(g)(\nabla v_0 - \nabla g)\,dxdt.$$

Moreover, we have

$$L_1 = \frac{1}{2}\int_{Q_1^+(0,t_0)} v_k^2 \varphi_t dxdt - \int_{Q_1^+(0,t_0)} \overline{A}_{k,B_1}(\nabla v_k)\nabla\varphi v_k dxdt.$$

Thus,

$$\begin{aligned} \lim_{k\to\infty} L_1 &= \frac{1}{2}\int_{Q_1^+(0,t_0)} v_0^2 \varphi_t dxdt - \int_{Q_1^+(0,t_0)} A_1 \nabla\varphi v_0 dxdt \\ &= -\int_{Q_1^+(0,t_0)} (v_0)_t \varphi v_0 dxdt - \int_{Q_1^+(0,t_0)} A_1 \nabla(\varphi v_0) dxdt + \int_{Q_1^+(0,t_0)} \varphi A_1 \nabla v_0 dxdt \\ &= \int_{Q_1^+(0,t_0)} \varphi A_1 \nabla v_0 dxdt. \end{aligned}$$

Hence,

$$0 \leq \int_{Q_1^+(0,t_0)} \varphi\left(A_1 - \mathcal{J}(g)\right)(\nabla v_0 - \nabla g)\,dxdt$$

holds for all $\varphi \in C_c^1(Q_1^+(0,t_0))$, $\varphi \geq 0$ and $g \in X$. Now we choose $g = v_0 - \xi(v_0 - V_0) = (1-\xi)v_0 + \xi V_0 \in X$ for $\xi \in (0,1)$, so

$$0 \leq \int_{Q_1^+(0,t_0)} \varphi\left(A - \mathcal{J}(v_0 - \xi(v_0 - V_0))\right)(\nabla v_0 - \nabla V_0)\,dxdt.$$

Letting $\xi \to 0^+$ and $\varphi \to \chi_{Q_1^+(0,t_0)}$, we get (3.54). Similarly, we also obtain (3.55). Thus,

$$\int_{Q_1^+(0,t_0)} (A_1 - A_2)\nabla(v_0 - V_0) dxdt \geq \int_{Q_1^+(0,t_0)} (\mathcal{J}(v_0) - \mathcal{J}(V_0))\nabla(v_0 - V_0) dxdt.$$

Combining this with (3.52), (3.53) and $v_0 - V_0 \in L^2(t_0 - 1, t_0, H_0^1(B_1^+))$ yield $v_0 = V_0$. This completes the proof. □

LEMMA 3.11. *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that if $v \in C(t_0 - 1, t_0; L^2(\Omega \cap B_1)) \cap L^2(t_0 - 1, t_0; H^1(\Omega \cap B_1))$ is a solution of* (3.42) *and* (3.43) *is satisfied and*

$$\fint_{Q_1(0,t_0)} |\nabla v|^2 dxdt \leq 1, \tag{3.56}$$

then there exists a weak solution $V \in C(t_0-1,t_0;L^2(B_1^+))\cap L^2(t_0-1,t_0;H^1(B_1^+))$ *of* (3.40) *with* $\rho=1$*, whose zero extension to* $Q_1(0,t_0)$ *satisfies*

$$||\nabla V||_{L^\infty(Q_{1/4}(0,t_0))} \lesssim 1, \tag{3.57}$$

$$\fint_{Q_{1/8}(0,t_0)} |\nabla v - \nabla V|^2 dxdt \le \varepsilon^2. \tag{3.58}$$

Proof. Given $\varepsilon_1 \in (0,1)$ by applying Lemma 3.10, one finds a small $\delta = \delta(N,\Lambda_1,\Lambda_2,\varepsilon_1)>0$ and a weak solution $V\in C(t_0-1,t_0;L^2(B_1^+(0)))\cap L^2(t_0-1,t_0;H^1(B_1^+(0)))$ of (3.40) with $\rho=1$ such that

$$\fint_{Q_1(0,t_0)} |v-V|^2 dxdt \le \varepsilon_1^2, \tag{3.59}$$

Using $\phi^2 V$ with $\phi \in C_c^\infty(B_1\times(t_0-1,t_0])$, $0\le\phi\le1$ and $\phi=1$ in $Q_{1/2}(0,t_0)$ as test function in (3.40), we obtain

$$\int_{Q_{1/2}(0,t_0)} |\nabla V|^2 dxdt \lesssim \int_{Q_1(0,t_0)} |V|^2 dxdt.$$

This implies

$$\int_{Q_{1/2}(0,t_0)} |\nabla V|^2 dxdt \lesssim \int_{Q_1(0,t_0)} \left(|v-V|^2+|\nabla v|^2\right) dxdt \lesssim 1,$$

since (3.56), (3.59) and Poincaré's inequality. Thus, using Lemma 3.8 we get (3.57). Next, we will prove (3.58). By Lemma 3.9, the zero extension of V to $Q_1(0,t_0)$ satisfies

$$V_t - \operatorname{div}\left(\overline{A}_{B_1}(t,\nabla V)\right) = \frac{\partial F}{\partial x_N} \text{ in weakly } Q_1(0,t_0).$$

where $F(x,t)=\chi_{x_N<0}\overline{A}^N_{B_\rho}(t,\nabla V(x',0,t))$. Thus, we can write

$$\int_{\tilde\Omega_1(0,t_0)} (V-v)_t\varphi dxdt + \int_{\tilde\Omega_1(0,t_0)} \left(\overline{A}_{B_1}(t,\nabla V)-\overline{A}_{B_1}(t,\nabla v)\right)\nabla\varphi dxdt = -\int_{\tilde\Omega_1(0,t_0)} F\frac{\partial\varphi}{\partial x_N}dxdt,$$

for any $\varphi\in L^2(t_0-1,t_0,H_0^1(\Omega\cap B_1))$.
We take $\varphi=\phi^2(V-v)$ where $\phi\in C_c^\infty(B_{1/4}\times(t_0-(1/4)^2,t_0])$, $0\le\phi\le1$ and $\phi=1$ on $\overline{Q}_{1/8}(0,t_0)$, so

$$\begin{aligned}
&\int_{\tilde\Omega_1(0,t_0)} \phi^2\left(\overline{A}_{B_1}(t,\nabla V)-\overline{A}_{B_1}(t,\nabla v)\right)(\nabla V-\nabla v)\,dxdt\\
&= -2\int_{\tilde\Omega_1(0,t_0)} \phi(V-v)\left(\overline{A}_{B_1}(t,\nabla V)-\overline{A}_{B_1}(t,\nabla v)\right)\nabla\phi dxdt\\
&- \int_{\tilde\Omega_1(0,t_0)} \phi^2(V-v)_t(V-v)dxdt\\
&- \int_{\tilde\Omega_1(0,t_0)} \left(\phi^2F\frac{\partial(V-v)}{\partial x_N}+2\phi F(V-v)\frac{\partial\phi}{\partial x_N}\right)dxdt.
\end{aligned}$$

We can rewrite $I_1 = I_2 + I_3 + I_4$.
One has

$$I_1 \gtrsim \int_{\tilde{\Omega}_1(0,t_0)} \phi^2 |\nabla V - \nabla v|^2 dxdt$$

and using Hölder's inequality

$$\begin{aligned} |I_2| &\lesssim \int_{\tilde{\Omega}_1(0,t_0)} \phi |V - v| (|\nabla V| + |\nabla v|) |\nabla \phi| dxdt \\ &\leq \varepsilon_2 \int_{\tilde{\Omega}_1(0,t_0)} \phi^2 (|\nabla V|^2 + |\nabla v|^2) dxdt + c(\varepsilon_2) \int_{\tilde{\Omega}_1(0,t_0)} |V - v|^2 |\nabla \phi|^2 dxdt. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} |I_4| \leq \varepsilon_2 &\int_{\tilde{\Omega}_1(0,t_0)} \phi^2 (|\nabla V|^2 + |\nabla v|^2) dxdt + c(\varepsilon_2) \int_{\tilde{\Omega}_1(0,t_0)} |V - v|^2 |\nabla \phi|^2 dxdt \\ &+ c(\varepsilon_2) \int_{\tilde{\Omega}_1(0,t_0)} |F|^2 \phi^2 dxdt, \end{aligned}$$

and

$$I_3 \leq \int_{\tilde{\Omega}_1(0,t_0)} \phi_t \phi (V - v)^2 dxdt \lesssim \int_{\tilde{\Omega}_{1/4}(0,t_0)} |V - v|^2 dxdt.$$

Hence,

$$\begin{aligned} \int_{\tilde{\Omega}_{1/8}(0,t_0)} |\nabla V - \nabla v|^2 &\lesssim \varepsilon_2 \int_{\tilde{\Omega}_{1/4}(0,t_0)} (|\nabla V|^2 + |\nabla v|^2) + c(\varepsilon_2) \int_{\tilde{\Omega}_{1/4}(0,t_0)} (|V - v|^2 + |F|^2) \\ &\lesssim \varepsilon_2 + c(\varepsilon_2) \left(\varepsilon_1^2 + \int_{\tilde{\Omega}_{1/4}(0,t_0) \cap \{-4\delta < x_N < 0\}} |\nabla V(x', 0, t)|^2 dxdt \right) \\ &\lesssim \varepsilon_2 + c(\varepsilon_2) \left(\varepsilon_1^2 + \delta \right). \end{aligned}$$

Finally, for any $\varepsilon > 0$ by choosing $\varepsilon_2, \varepsilon_1$ and δ appropriately we get (3.58). The proof is complete. ☐

LEMMA 3.12. *For any $\varepsilon > 0$ there exists a small $\delta = \delta(N, \Lambda_1, \Lambda_2, \varepsilon) > 0$ such that if $v \in C(t_0 - \rho^2, t_0; L^2(\Omega \cap B_\rho(0))) \cap L^2(t_0 - \rho^2, t_0; H^1(\Omega \cap B_\rho(0)))$ is a solution of*

$$\left\{ \begin{array}{ll} v_t - \operatorname{div}\left(\overline{A}_{B_\rho(0)}(t, \nabla v) \right) = 0 & in\ \tilde{\Omega}_\rho(0), \\ v = 0 & on \partial_p \tilde{\Omega}_\rho(0) \backslash (\Omega \times (-T, T)), \end{array} \right. \tag{3.60}$$

and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{x_N > -4\rho\delta\}. \tag{3.61}$$

then there exists a weak solution $V \in C(t_0 - \rho^2, t_0; L^2(B_\rho^+(0))) \cap L^2(t_0 - \rho^2, t_0; H^1(B_\rho^+(0)))$ of (3.40), whose zero extension to $Q_1(0, t_0)$ satisfies

$$||\nabla V||^2_{L^\infty(Q_{\rho/4}(0,t_0))} \lesssim \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt \ and \tag{3.62}$$

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V|^2 dxdt \leq \varepsilon^2 \fint_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt. \tag{3.63}$$

PROOF. We set

$$\mathcal{A}(x,t,\xi) = A(\rho x, t_0 + \rho^2(t-t_0), \kappa\xi)/\kappa \text{ and } \tilde{v}(x,t) = v(\rho x, t_0 + \rho^2(t-t_0))/(\rho\kappa)$$

where $\kappa = \left(\frac{1}{|Q_\rho(0,t_0)|}\int_{Q_\rho(0,t_0)} |\nabla v|^2 dxdt\right)^{1/2}$. Then $\mathcal{A}$ satisfies conditions (1.2) and (1.45) with the same constants Λ_1 and Λ_2. Moreover, $\tilde{v}$ is a solution of

(3.64)
$$\begin{cases} \tilde{v}_t - \operatorname{div}\left(\overline{\mathcal{A}}_{B_1(0)}(t,\nabla\tilde{v})\right) = 0 & \text{in} \quad \tilde{\Omega}_1^\rho(0) \\ \tilde{v} = 0 \quad \text{on} \ ((\partial\Omega^\rho \cap B_1(0)) \times (t_0-1,t_0)) \cup ((\Omega^\rho \cap B_1(0)) \times \{t = t_0 - 1\}), \end{cases}$$

where $\Omega^\rho = \{z = x/\rho : x \in \Omega\}$ and satisfies $\fint_{Q_1(0,t_0)} |\nabla\tilde{v}|^2 dxdt = 1$. We also have

$$B_1^+(0) \subset \Omega^\rho \cap B_1(0) \subset B_1(0) \cap \{x_N > -4\delta\}.$$

Therefore, applying Lemma 3.11 for any $\varepsilon > 0$, there exist a constant $\delta = \delta(N,\Lambda_1,\Lambda_2,\varepsilon) > 0$ and $\tilde{V}$ satisfying

$$||\nabla\tilde{V}||_{L^\infty(Q_{1/4}(0,t_0))} \lesssim 1 \quad \text{and} \quad \fint_{Q_{1/8}(0,t_0)} |\nabla\tilde{v} - \nabla\tilde{V}|^2 dxdt \le \varepsilon^2.$$

We complete the proof by choosing $V(x,t) = k\rho\tilde{V}(x/\rho, t_0 + (t-t_0)/\rho^2)$. □

LEMMA 3.13. *Let s_2 be as in Lemma 3.7. For any $\varepsilon \in (0,1)$ there exists a small $\delta = \delta(N,\Lambda_1,\Lambda_2,\varepsilon) > 0$ such that the following holds. If Ω is a (δ,R_0)-Reifenberg flat domain and $u \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is a solution to equation (1.22) with $\mu \in L^2(\Omega \times (-T,T))$ and $u(-T) = 0$, for $x_0 \in \partial\Omega$, $-T < t_0 < T$ and $0 < R < R_0/6$ then there is a function $V \in L^\infty(t_0 - (R/9)^2, t_0; W^{1,\infty}(B_{R/9}(x_0)))$ such that*

$$||\nabla V||_{L^\infty(Q_{R/9}(x_0,t_0))} \lesssim \fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}, \tag{3.65}$$

and

$$\fint_{Q_{R/9}(x_0,t_0)} |\nabla u - \nabla V| dxdt$$
$$\lesssim (\varepsilon + [A]_{s_2}^{R_0}) \fint_{Q_{6R}(x_0,t_0)} |\nabla u| dxdt + (1 + [A]_{s_2}^{R_0}) \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}. \tag{3.66}$$

PROOF. Let $x_0 \in \partial\Omega$, $-T < t_0 < T$ and $\rho = R(1-\delta)$, we may assume that $0 \in \Omega$, $x_0 = (0,...,-\delta\rho/(1-\delta))$ and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{x_N > -4\rho\delta\}. \tag{3.67}$$

We also have

(3.68)
$$Q_{R/9}(x_0,t_0) \subset Q_{\rho/8}(0,t_0) \subset Q_{\rho/4}(0,t_0) \subset Q_\rho(0,t_0) \subset Q_{6\rho}(0,t_0) \subset Q_{6R}(x_0,t_0),$$

provided that $0 < \delta < 1/625$.

Let w and v be in Theorem 3.5 and Lemma 3.7. By Lemma 3.12 for any $\varepsilon > 0$ we can find a small positive $\delta = \delta(N,\alpha,\beta,\varepsilon) < 1/625$ such that there is a function $V \in L^\infty(t_0-\rho^2,t_0;W^{1,\infty}(B_\rho(0)))$ satisfying

$$||\nabla V||^2_{L^\infty(Q_{\rho/4}(0,t_0))} \lesssim \fint_{Q_\rho(0,t_0)} |\nabla v|^2, \quad \fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V|^2 \le \varepsilon^2 \fint_{Q_\rho(0,t_0)} |\nabla v|^2.$$

Then, by (3.39) in Lemma 3.7 and (3.18) in Theorem 3.5 and (3.68) we get

$$||\nabla V||_{L^\infty(Q_{R/9}(x_0,t_0))} \lesssim \left(\fint_{Q_\rho(0,t_0)} |\nabla w|^2\right)^{1/2} \lesssim \fint_{Q_{6R}(x_0,t_0)} |\nabla w|, \tag{3.69}$$

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V| \lesssim \varepsilon \left(\fint_{Q_\rho(0,t_0)} |\nabla w|^2\right)^{1/2} \lesssim \varepsilon \fint_{Q_{6R}(x_0,t_0)} |\nabla w|. \tag{3.70}$$

Therefore, from (3.17) in Theorem 3.5 and (3.69) we get (3.65).
Now we prove (3.66). One has

$$\begin{aligned}\fint_{Q_{R/9}(x_0,t_0)} |\nabla u - \nabla V| &\lesssim \fint_{Q_{\rho/8}(0,t_0)} |\nabla u - \nabla V| \\ &\lesssim \fint_{Q_{\rho/8}(0,t_0)} |\nabla u - \nabla w| + \fint_{Q_{\rho/8}(0,t_0)} |\nabla w - \nabla v| + \fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V|.\end{aligned}$$

From Lemma 3.7 and Theorem 3.5 and (3.70) it follows that

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla u - \nabla w| \lesssim \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}},$$

$$\begin{aligned}\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla w| &\lesssim [A]_{s_2}^{R_0} \fint_{Q_{6\rho}(0,t_0)} |\nabla w| \lesssim [A]_{s_2}^{R_0} \fint_{Q_{6R}(x_0,t_0)} |\nabla w| \\ &\lesssim [A]_{s_2}^{R_0} \left(\fint_{Q_{6R}(x_0,t_0)} |\nabla u| + \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}\right),\end{aligned}$$

$$\fint_{Q_{\rho/8}(0,t_0)} |\nabla v - \nabla V| \lesssim \varepsilon \fint_{Q_{6R}(x_0,t_0)} |\nabla w| \lesssim \varepsilon \left(\fint_{Q_{6R}(x_0,t_0)} |\nabla u| + \frac{|\mu|(Q_{6R}(x_0,t_0))}{R^{N+1}}\right).$$

Hence we get (3.66). The proof is complete.

□

3.2. Global integral gradient bounds for parabolic equations

3.2.1. Global estimates on 2-Capacity uniform thickness domains.

We use the Theorem 3.1 and 3.5 to prove the following theorem.

THEOREM 3.14. *Suppose that $\mathbb{R}^N\backslash\Omega$ satisfies a uniformly 2−thick condition with constants c_0, r_0. Let θ_1, θ_2 be in Theorem 3.1 and 3.5. Set $\theta = \min\{\theta_1,\theta_2\} > 2$ and $T_0 = diam(\Omega) + T^{1/2}$, $Q = B_{diam(\Omega)}(x_0) \times (0,T)$. Let $B_1 = \tilde{Q}_{R_1}(y_0,s_0)$, $B_2 = 4B_1 := \tilde{Q}_{4R_1}(y_0,s_0)$ for $R_1 > 0$. For $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$, there exist a distributional solution u of equation (1.22) with data μ, $u_0 = \sigma$ and constants $\varepsilon_1 = \varepsilon_1(N,\Lambda_1,\Lambda_2,c_0,T_0/r_0), \varepsilon_2 = \varepsilon_2(N,\Lambda_1,\Lambda_2,c_0) > 0$ such that*

$$|\{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q| \lesssim_{c_0,T_0/r_0} \varepsilon|\{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q|, \tag{3.71}$$

for all $\lambda > 0, \varepsilon \in (0,\varepsilon_1)$ and
(3.72)

$$|\{\mathbb{M}(\chi_{B_2}|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\chi_{B_2}\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap B_1| \lesssim_{c_0,T_0/r_0} \varepsilon|\{\mathbb{M}(\chi_{B_2}|\nabla u|) > \lambda\} \cap B_1|,$$

for all $\lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T \cap B_2)} R_2^{-N-2}$, $\varepsilon \in (0,\varepsilon_2)$ with $R_2 = \inf\{r_0, R_1\}/16$. Moreover, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

Proof of Theorem 3.14. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \{\sigma_n\} \subset C_c^\infty(\Omega)$ be as in the proof of Theorem 1.1. We have $|\mu_n| \le \varphi_n * |\mu|$ and $|\sigma_n| \le \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, where $\{\varphi_n\}, \{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$, respectively. Let u_n be solution of equation

$$\left\{\begin{array}{l} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = \mu_n \quad \text{in} \quad \Omega_T, \\ u_n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{in} \quad \Omega. \end{array}\right. \tag{3.73}$$

By Proposition 1.36 and Theorem 1.37, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, converging to a distributional solution u of (1.22) with data $\mu \in \mathfrak{M}_b(\Omega_T)$ and $u_0 = \sigma$ such that $u_n \to u$ in $L^s(0,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$ and if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.
By Remark 1.34 and Theorem 1.37, a sequence $\{u_{n,m}\}_m$ of solutions to equations

$$\left\{\begin{array}{l} (u_{n,m})_t - \operatorname{div}(A(x,t,\nabla u_{n,m})) = \mu_{n,m} \quad \text{in} \quad \Omega \times (-T,T), \\ u_{n,m} = 0 \quad \text{on} \quad \partial\Omega \times (-T,T), \\ u_{n,m}(-T) = 0 \quad \text{on} \quad \Omega, \end{array}\right.$$

converges to $\chi_{\Omega_T} u_n$ in $L^s(-T,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, \frac{N+2}{N+1}\right)$, where $\mu_{n,m} = (g_{n,m})_t + \chi_{\Omega_T}\mu_n$, $g_{n,m}(x,t) = \sigma_n(x)\int_{-T}^t \varphi_{2,m}(s)ds$ and $\{\varphi_{2,m}\}$ is a sequence of mollifiers in $\mathbb{R}$.

Set

$E^1_{\lambda,\varepsilon} = \{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q, F^1_\lambda = \{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q,$

$E^2_{\lambda,\varepsilon} = \{\mathbb{M}(\chi_{B_2}|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\chi_{B_2}\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap B_1, F^2_\lambda = \{\mathbb{M}(\chi_{B_2}|\nabla u|) > \lambda\} \cap B_1,$

for $\varepsilon \in (0,1)$ and $\lambda > 0$.
We verify that

$$|E^1_{\lambda,\varepsilon}| \lesssim_{T_0/r_0} \varepsilon|\tilde{Q}_{R_3}| \quad \forall \lambda > 0, \varepsilon \in (0,1), \tag{3.74}$$

$$|E^2_{\lambda,\varepsilon}| \lesssim \varepsilon|\tilde{Q}_{R_2}| \quad \forall \lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap A)}R_2^{-N-2}, \varepsilon \in (0,1) \tag{3.75}$$

with $R_3 = \inf\{r_0, T_0\}/16$.
In fact, we can assume that $E^1_{\lambda,\varepsilon} \ne \emptyset$. So, $|\mu|(\Omega_T) + |\sigma|(\Omega) \le T_0^{N+1}\varepsilon^{1-\frac{1}{\theta}}\lambda$. We have

$$|E^1_{\lambda,\varepsilon}| \lesssim \frac{1}{\varepsilon^{-1/\theta}\lambda}\int_{\Omega_T}|\nabla u|dxdt.$$

By Remark 1.33, $\int_{\Omega_T}|\nabla u_n|dxdt \lesssim T_0\left(|\mu_n|(\Omega_T) + |\sigma_n|(\Omega)\right)$ for all n. Letting $n \to \infty$ we get $\int_{\Omega_T}|\nabla u|dxdt \lesssim T_0\left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right)$. Thus,

$$|E^1_{\lambda,\varepsilon}| \lesssim \frac{1}{\varepsilon^{-1/\theta}\lambda}T_0\left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right) \lesssim \frac{1}{\varepsilon^{-1/\theta}\lambda}T_0^{N+2}\varepsilon^{1-\frac{1}{\theta}}\lambda = c\varepsilon|\tilde{Q}_{R_3}|.$$

Hence, (3.74) holds.
For any $\lambda > \varepsilon^{-1+\frac{1}{\theta}}||\nabla u||_{L^1(\Omega_T\cap B_2)}R_2^{-N-2}$ we have

$$|E^2_{\lambda,\varepsilon}| \lesssim \frac{1}{\varepsilon^{-1/\theta}\lambda}\int_{\Omega_T}\chi_{B_2}|\nabla u|dxdt \le c\varepsilon|\tilde{Q}_{R_2}|.$$

Hence, (3.75) holds.

Next we verify that for all $(x,t) \in Q$, $r \in (0,R_3]$ and $\lambda > 0, \varepsilon \in (0,1)$, we have $\tilde{Q}_r(x,t)\cap Q \subset F^1_\lambda$ if $|E^1_{\lambda,\varepsilon}\cap \tilde{Q}_r(x,t)| \geq c\varepsilon|\tilde{Q}_r(x,t)|$ where the constant c does not depend on λ and ε. Indeed, take $(x,t)\in Q$ and $0<r\leq R_3$. Now assume that $\tilde{Q}_r(x,t)\cap Q\cap(\mathbb{R}^{N+1}\backslash F^1_\lambda)\neq\emptyset$ and $E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)\neq\emptyset$ i.e, there exist $(x_1,t_1),(x_2,t_2)\in\tilde{Q}_r(x,t)\cap Q$ such that $\mathbb{M}(|\nabla u|)(x_1,t_1)\leq\lambda$ and $\mathbb{M}_1[\omega](x_2,t_2)\leq\varepsilon^{1-\frac{1}{\theta}}\lambda$. We need to prove that

$$|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t))|<c\varepsilon|\tilde{Q}_r(x,t)| \tag{3.76}$$

Obviously, we have for all $(y,s)\in\tilde{Q}_r(x,t)$ there holds

$$\mathbb{M}(|\nabla u|)(y,s)\leq\max\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)(y,s),3^{N+2}\lambda\right\}.$$

So, for all $\lambda>0$ and $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0\leq 3^{-(N+2)\theta}$,

$$E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)=\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)>\varepsilon^{-1/\theta}\lambda,\mathbb{M}_1[\omega]\leq\varepsilon^{1-\frac{1}{\theta}}\lambda\right\}\cap Q\cap\tilde{Q}_r(x,t). \tag{3.77}$$

In particular, $E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)=\emptyset$ if $\overline{B}_{4r}(x)\subset\subset\mathbb{R}^N\backslash\Omega$. Thus, it is enough to consider the case $B_{4r}(x)\subset\subset\Omega$ and $B_{4r}(x)\cap\Omega\neq\emptyset$.
We consider the case $B_{4r}(x)\subset\subset\Omega$. Let $w_{n,m}$ be as in Theorem 3.1 with $Q_{2R}=Q_{4r}(x,t_0)$ and $u=u_{n,m}$ where $t_0=\min\{t+2r^2,T\}$. We have

$$\fint_{Q_{4r}(x,t_0)}|\nabla u_{n,m}-\nabla w_{n,m}|\lesssim\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}}, \tag{3.78}$$

$$\fint_{Q_{2r}(x,t_0)}|\nabla w_{n,m}|^\theta\lesssim\left(\fint_{Q_{4r}(x,t_0)}|\nabla w_{n,m}|\right)^\theta. \tag{3.79}$$

From (3.77), we have

$$\begin{aligned}
&|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|\leq|\{\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla w_{n,m}|\right)>\varepsilon^{-1/\theta}\lambda/4\}\cap\tilde{Q}_r(x,t)\}|\\
&+|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla w_{n,m}|\right)>\varepsilon^{-1/\theta}\lambda/4\}\cap\tilde{Q}_r(x,t)|\\
&+|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla u_n|\right)>\varepsilon^{-1/\theta}\lambda/4\}\cap\tilde{Q}_r(x,t)|\\
&+|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u|\right)>\varepsilon^{-1/\theta}\lambda/4\}\cap\tilde{Q}_r(x,t)|\\
&\lesssim\varepsilon\lambda^{-\theta}\int_{\tilde{Q}_{2r}(x,t)}|\nabla w_{n,m}|^\theta+\varepsilon^{1/\theta}\lambda^{-1}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla w_{n,m}|\\
&+\varepsilon^{1/\theta}\lambda^{-1}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla u_n|+\varepsilon^{1/\theta}\lambda^{-1}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u|.
\end{aligned}$$

Thanks to (3.78) and (3.79) we can continue

$$\begin{aligned}
&|E^1_{\lambda,\varepsilon}\cap\tilde{Q}_r(x,t)|\lesssim\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\fint_{Q_{4r}(x,t_0)}|\nabla u_{n,m}|dxdt\right)^\theta\\
&+\varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)|\left(\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}}\right)^\theta+\varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{|\mu_{n,m}|(Q_{4r}(x,t_0))}{r^{N+1}}\\
&+\varepsilon^{1/\theta}\lambda^{-1}\int_{Q_{2r}(x,t_0)}|\nabla u_{n,m}-\nabla u_n|+\varepsilon^{1/\theta}\lambda^{-1}\int_{Q_{2r}(x,t_0)}|\nabla u_n-\nabla u|.
\end{aligned}$$

Letting $m \to \infty$ and $n \to \infty$, we get

$$|E_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \lesssim \varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)| \left(\fint_{Q_{4r}(x,t_0)} |\nabla u|\right)^{\theta}$$
$$+ \varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)| \left(\frac{\omega(\overline{Q_{4r}(x,t_0)})}{r^{N+1}}\right)^{\theta} + \varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{\omega(\overline{Q_{4r}(x,t_0)})}{r^{N+1}}.$$

Since, $\mathbb{M}(|\nabla u|)(x_1,t_1) \le \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \le \varepsilon^{1-\frac{1}{\theta}}\lambda$ we have

$$\int_{Q_{4r}(x,t_0)} |\nabla u| \le \int_{\tilde{Q}_{9r}(x_1,t_1)} |\nabla u| \le |\tilde{Q}_{9r}(x_1,t_1)|\lambda,$$
$$\omega(\overline{Q_{4r}(x,t_0)}) \le \omega(\tilde{Q}_{8r}(x,t)) \le \omega(\tilde{Q}_{9r}(x_2,t_2)) \le \varepsilon^{1-\frac{1}{\theta}}\lambda(9r)^{N+1}.$$

Thus

$$|E_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \le c\varepsilon|\tilde{Q}_r(x,t)|.$$

Next, we consider the case $B_{4r}(x) \cap \Omega \ne \emptyset$. Let $x_3 \in \partial\Omega$ such that $|x_3 - x| = \text{dist}(x,\partial\Omega)$. Let w_n be as in Theorem 3.5 with $\tilde{\Omega}_{6R} = \tilde{\Omega}_{16r}(x_3,t_0)$ and $u = u_{n,m}$ where $t_0 = \min\{t + 2r^2, T\}$. We have $Q_{12r}(x,t_0) \subset Q_{16r}(x_3,t_0)$,

$$\fint_{Q_{12r}(x,t_0)} |\nabla u_{n,m} - \nabla w_{n,m}| \lesssim \frac{|\mu_{n,m}|(\tilde{\Omega}_{16r}(x_3,t_0))}{r^{N+1}},$$
$$\left(\fint_{Q_{2r}(x,t_0)} |\nabla w_{n,m}|^{\theta}\right)^{\frac{1}{\theta}} \lesssim \fint_{Q_{12r}(x,t_0)} |\nabla w_{n,m}|.$$

As above we also obtain

$$|E^1_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \lesssim \varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)| \left(\fint_{Q_{12r}(x,t_0)} |\nabla u|dxdt\right)^{\theta}$$
$$+ \varepsilon\lambda^{-\theta}|\tilde{Q}_r(x,t)| \left(\frac{\omega(\overline{Q_{16r}(x_3,t_0)})}{r^{N+1}}\right)^{\theta} + \varepsilon^{1/\theta}\lambda^{-1}|\tilde{Q}_r(x,t)|\frac{\omega(\overline{Q_{16r}(x_3,t_0)})}{r^{N+1}}.$$

Since, $\mathbb{M}(|\nabla u|)(x_1,t_1) \le \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \le \varepsilon^{1-\frac{1}{\theta}}\lambda$ we have

$$\int_{Q_{12r}(x,t_0)} |\nabla u|dxdt \le \int_{\tilde{Q}_{24r}(x,t)} |\nabla u|dxdt \le \int_{\tilde{Q}_{25r}(x_1,t_1)} |\nabla u|dxdt \le |\tilde{Q}_{25r}(x_1,t_1)|\lambda,$$
$$\omega(\overline{Q_{16r}(x_3,t_0)}) \le \omega(\tilde{Q}_{32r}(x_3,t)) \le \omega(\tilde{Q}_{36r}(x,t)) \le \omega(\tilde{Q}_{37r}(x_2,t_2)) \le \varepsilon^{1-\frac{1}{\theta}}\lambda(37r)^{N+1}.$$

Thus

$$|E^1_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \le c\varepsilon|\tilde{Q}_r(x,t)|.$$

Hence, (3.76) holds.

Similarly, we also prove that for all $(x,t) \in B_1$ and $r \in (0,R_2]$ and $\lambda > 0, \varepsilon \in (0,1)$ we have $\tilde{Q}_r(x,t) \cap B_1 \subset F^2_\lambda$ if $|E^2_{\lambda,\varepsilon} \cap \tilde{Q}_r(x,t)| \ge c\varepsilon|\tilde{Q}_r(x,t)|$ where the constant c does not depend on λ and ε. We apply Lemma 1.52 for $E = E^1_{\lambda,\varepsilon}, F = F^1_\lambda$ to get (3.71) and for $E = E^2_{\lambda,\varepsilon}, F = F^2_\lambda$ get (3.72). The proof is complete. □

PROOF OF THEOREM 1.17. By theorem 3.14, there exist constants $c > 0$, $0 < \varepsilon_0 < 1$ and a renormalized solution u of equation (1.22) with data μ, $u_0 = \sigma$ such that for any $\varepsilon \in (0,1)$, $\lambda > 0$

$$|\{\mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda, \mathbb{M}_1[\omega] \le \varepsilon^{1-\frac{1}{\theta}}\lambda\} \cap Q| \le c\varepsilon|\{\mathbb{M}(|\nabla u|) > \lambda\} \cap Q|.$$

Therefore, if $0 < s < \infty$

$$\begin{aligned} ||\mathbb{M}(|\nabla u|)||^s_{L^{p,s}(Q)} &= \varepsilon^{-s/\theta} p \int_0^\infty \lambda^s |\{(x,t) \in Q : \mathbb{M}(|\nabla u|) > \varepsilon^{-1/\theta}\lambda\}|^{\frac{s}{p}} \frac{d\lambda}{\lambda} \\ &\le c\varepsilon^{\frac{s(\theta-p)}{\theta p}} p \int_0^\infty \lambda^s |\{(x,t) \in Q : \mathbb{M}(|\nabla u|) > \lambda\}|^{\frac{s}{p}} \frac{d\lambda}{\lambda} \\ &+ \varepsilon^{-s/\theta} p \int_0^\infty \lambda^s |\{(x,t) \in Q : \mathbb{M}_1[\omega] > \varepsilon^{1-\frac{1}{\theta}}\lambda\}|^{\frac{s}{p}} \frac{d\lambda}{\lambda} \\ &= c\varepsilon^{\frac{s(\theta-p)}{\theta p}} ||\mathbb{M}(|\nabla u|)||^s_{L^{p,s}(Q)} + \varepsilon^{-s}||\mathbb{M}_1[\omega]||^s_{L^{p,s}(Q)}. \end{aligned}$$

Since $p < \theta$, we can choose $0 < \varepsilon < \varepsilon_0$ such that $c\varepsilon^{\frac{s(\theta-p)}{\theta p}} \le 1/2$. So, we get the result for case $0 < s < \infty$. Similarly, we also get the result for case $s = \infty$.

Also, we get (1.47) by using (2.16) in Proposition 2.8, (2.26) in Proposition 2.19. The proof is complete. □

REMARK 3.15. Thanks to Proposition 2.4 we have that for any $s \in \left(\frac{N+2}{N+1}, \frac{N+2+\theta}{N+2}\right)$ if $\mu \in L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)$ and $\sigma \equiv 0$ then

$$|||\nabla u|^s||_{L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)} \lesssim_{s,c_0,T_0/r_0} ||\mu||^s_{L^{\frac{(s-1)(N+2)}{s},\infty}(\Omega_T)}.$$

As the proof of Theorem 1.17, we also get

THEOREM 3.16. *Suppose that $\mathbb{R}^N\backslash\Omega$ is uniformly $2-$thick with constants c_0, r_0. Let θ be as in Theorem 3.14. Let $1 \le p < \theta$, $0 < s \le \infty$ and $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distributional solution u of equation* (1.22) *with data μ and $u_0 = \sigma$ such that*

$$\begin{aligned} ||\mathbb{M}(\chi_{\tilde{Q}_{4R}(y_0,s_0)}|\nabla u|)||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} &\lesssim_{p,s,c_0} R^{\frac{N+2}{p}} \inf\{r_0,R\}^{-N-2}||\nabla u||_{L^1(\tilde{Q}_{4R}(y_0,s_0))} \\ &+ ||\mathbb{M}_1[\chi_{\tilde{Q}_{4R}(y_0,s_0)}\omega]||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}, \end{aligned} \tag{3.80}$$

for any $\tilde{Q}_R(y_0,s_0) \subset \mathbb{R}^{N+1}$. Moreover, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

PROOF OF THEOREM 1.19. Let $\{u_{n,m}\}$ and $\mu_{n,m}$ be in the proof of Theorem 3.14. From Corollary 3.2 and 3.6 we assert: for $2 - \inf\{\beta_1, \beta_2\} < \gamma < N + 2$, $0 < \rho \le T_0$ we have

$$\int_{Q_\rho(y,s)} |\nabla u_{n,m}| \lesssim_{c_0,\gamma,T_0/r_0} \rho^{N+3-\gamma}||\mathbb{M}_\gamma[|\mu_{n,m}|]||_{L^\infty(\Omega\times(-T,T))},$$

where β_1, β_2 are constants in Theorem 3.1 and Theorem 3.5. It is easy to see that

$$||\mathbb{M}_\gamma[|\mu_{n,m}|]||_{L^\infty(\Omega\times(-T,T))} \le ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega\times(-T,T))} = ||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)},$$

for any n, m large enough.
Letting $m \to \infty, n \to \infty$, yield

$$\int_{Q_\rho(y,s)} |\nabla u| dxdt \lesssim_{c_0,\gamma,T_0/r_0} \rho^{N+3-\gamma}||\mathbb{M}_\gamma[\omega]||_{L^\infty(\Omega_T)}$$

By Theorem 3.16 we get

$$|||\nabla u|||_{L^{p,s}(\tilde{Q}_R(y_0,s_0)\cap\Omega_T)} \lesssim_{p,s,c_0} c(T_0/r_0)R^{\frac{N+2}{p}+1-\gamma}||\mathbb{M}_\gamma[\omega]|||_{L^\infty(\Omega_T)} + ||\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}\omega]|||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}$$

for any $\tilde{Q}_R(y_0,s_0)\subset\mathbb{R}^{N+1}$ and $0<R\le T_0$. It follows (1.48).

Finally, if $\mu\in L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)$ and $\sigma\equiv 0$, then clearly u is a unique renormalized solution. It suffices to show that

(3.81)
$$||\mathbb{M}_\gamma[|\mu|]|||_{L^\infty(\Omega_T)} \lesssim c_1||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)},$$

(3.82)
$$R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}_1[\chi_{\tilde{Q}_R(y,s_0)}|\mu|]|||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))} \lesssim c_1||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}$$

for any $\tilde{Q}_R(y_0,s_0)\subset\mathbb{R}^{N+1}$ and $0<R\le T_0$, where $c_1=c_1(p,s,\gamma,c_0,T_0/r_0)$.
In fact, for $0<\rho<T_0$ and $(x,t)\in\Omega_T$ we have

$$\begin{aligned}||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)} &\ge ||\mu||_{L_*^{\frac{(\gamma-1)p}{\gamma},\infty;(\gamma-1)p}(\Omega_T)}\\ &\ge \rho^{\frac{(\gamma-1)p-N-2}{\frac{(\gamma-1)p}{\gamma}}}||\mu||_{L^{\frac{(\gamma-1)p}{\gamma},\infty}(\tilde{Q}_\rho(x,t)\cap\Omega_T)}\\ &\gtrsim c_1\rho^{\frac{(\gamma-1)p-N-2}{\frac{(\gamma-1)p}{\gamma}}}|\tilde{Q}_\rho(x,t)|^{-1+\frac{\gamma}{(\gamma-1)p}}|\mu|(\tilde{Q}_\rho(x,t)\cap\Omega_T)\\ &\gtrsim c_1\frac{|\mu|(\tilde{Q}_\rho(x,t)\cap\Omega_T)}{\rho^{N+2-\gamma}},\end{aligned}$$

which obviously implies (3.81).
Next, note that

$$\mathbb{M}_1[\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|](x,t)\lesssim\left(\mathbb{M}\left(\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|\right)(x,t)\right)^{1-\frac{1}{\gamma}}||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}.$$

We derive

$$\begin{aligned}&R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}_1[\chi_{\tilde{Q}_R(y,s_0)}|\mu|]|||_{L^{p,s}(\tilde{Q}_R(y_0,s_0))}\\ &\lesssim R^{\frac{p(\gamma-1)-N-2}{p}}||\mathbb{M}\left(\chi_{\tilde{Q}_R(y_0,s_0)}|\mu|\right)||^{1-\frac{1}{\gamma}}_{L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\tilde{Q}_R(y_0,s_0))}||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}\\ &\lesssim R^{\frac{p(\gamma-1)-N-2}{p}}|||\mu|||^{1-\frac{1}{\gamma}}_{L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\tilde{Q}_R(y_0,s_0))}||\mu||^{\frac{1}{\gamma}}_{L_*^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma};(\gamma-1)p}(\Omega_T)}.\end{aligned}$$

Here we have used the boundedness property of $\mathbb{M}$ in $L^{\frac{(\gamma-1)p}{\gamma},\frac{(\gamma-1)s}{\gamma}}(\mathbb{R}^{N+1})$ for $\frac{(\gamma-1)p}{\gamma}>1$. Therefore, immediately we get (3.82). This completes the proof. □

3.2.2. Global estimates on Reifenberg flat domains. Now we prove results for Reifenberg flat domain. First, we will use Lemma 3.4, 3.13 and Lemma 1.50 to get the following result.

THEOREM 3.17. *Suppose that A satisfies* (1.45). *Let s_1, s_2 be in Lemma 3.3 and 3.7, set $s_0 = \max\{s_1, s_2\}$. Let $w \in A_\infty$, $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distributional solution of* (1.22) *with data μ and $u_0 = \sigma$ such that the following holds. For any $\varepsilon > 0, R_0 > 0$ one finds $\delta_1 = \delta_1(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}) \in (0,1)$ and $\delta_2 = \delta_2(N, \Lambda_1, \Lambda_2, \varepsilon, [w]_{A_\infty}, T_0/R_0) \in (0,1)$ and $\Lambda = \Lambda(N, \Lambda_1, \Lambda_2) > 0$ such that if Ω is (δ_1, R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \leq \delta_1$ then*

$$w(\{\mathbb{M}(|\nabla u|) > \Lambda\lambda, \mathbb{M}_1[\omega] \leq \delta_2\lambda\} \cap \Omega_T) \leq C\varepsilon w(\{\mathbb{M}(|\nabla u|) > \lambda\} \cap \Omega_T) \tag{3.83}$$

for all $\lambda > 0$, where the constant C depends only on $N, \Lambda_1, \Lambda_2, T_0/R_0, [w]_{A_\infty}$. Furthermore, if $\sigma \in L^1(\Omega)$ then u is a renormalized solution.

PROOF. Let $\{\mu_n\}, \{\sigma_n\}, \{\mu_{n,m}\}, \{u_n\}, \{u_{n,m}\}, u$ be as in the proof of Theorem 3.14. Let ε be in $(0,1)$. Set $E_{\lambda,\delta_2} = \{\mathbb{M}(|\nabla u|) > \Lambda\lambda, \mathbb{M}_1[\omega] \leq \delta_2\lambda\} \cap \Omega_T$ and $F_\lambda = \{\mathbb{M}(|\nabla u|) > \lambda\} \cap \Omega_T$ for $\varepsilon \in (0,1)$ and $\lambda > 0$. Let $\{y_i\}_{i=1}^L \subset \Omega$ and a ball B_0 with radius $2T_0$ such that

$$\Omega \subset \bigcup_{i=1}^L B_{r_0}(y_i) \subset B_0$$

where $r_0 = \min\{R_0/1080, T_0\}$. Let $s_j = T - jr_0^2/2$ for all $j = 0, 1, ..., [\frac{2T}{r_0^2}]$ and $Q_{2T_0} = B_0 \times (T - 4T_0^2, T)$. So,

$$\Omega_T \subset \bigcup_{i,j} Q_{r_0}(y_i, s_j) \subset Q_{2T_0}.$$

We verify that

$$w(E_{\lambda,\delta_2}) \leq \varepsilon w(\tilde{Q}_{r_0}(y_i, s_j)) \quad \forall \lambda > 0 \tag{3.84}$$

for some δ_2 small enough, depending on $N, p, \alpha, \beta, \epsilon, [w]_{A_\infty}, T_0/R_0$.
In fact, we can assume that $E_{\lambda,\delta_2} \neq \emptyset$, so $|\mu|(\Omega_T) + |\sigma|(\Omega) \leq T_0^{N+1}\delta_2\lambda$. We have

$$|E_{\lambda,\delta_2}| \lesssim \frac{1}{\Lambda\lambda} \int_{\Omega_T} |\nabla u|.$$

We also have

$$\int_{\Omega_T} |\nabla u| \lesssim T_0(|\mu|(\Omega_T) + |\sigma|(\Omega)).$$

Thus,

$$|E_{\lambda,\varepsilon}| \lesssim \frac{1}{\Lambda\lambda} T_0(|\mu|(\Omega_T) + |\sigma|(\Omega)) \lesssim \frac{1}{\Lambda\lambda} T_0^{N+2}\delta_2\lambda = \delta_2|Q_{2T_0}|.$$

which implies

$$w(E_{\lambda,\delta_2}) \lesssim_C \left(\frac{|E_{\lambda,\delta_2}|}{|Q_{2T_0}|}\right)^\nu w(Q_{2T_0}) \leq c\delta_2^\nu w(Q_{2T_0})$$

where (C, ν) is a pair of A_∞ constants of w. It is known that (see, e.g [**39**]) there exist $A_1 = A_1(N, C, \nu)$ and $\nu_1 = \nu_1(N, C, \nu)$ such that

$$\frac{w(\tilde{Q}_{2T_0})}{w(\tilde{Q}_{r_0}(y_i, s_j))} \leq A_1 \left(\frac{|\tilde{Q}_{2T_0}|}{|\tilde{Q}_{r_0}(y_i, s_j)|}\right)^{\nu_1} \forall i, j.$$

So,

$$w(E_{\lambda,\delta_2}) \leq C\,(c\delta_2)^\nu \left(\frac{|\tilde{Q}_{T_0}|}{|\tilde{Q}_{r_0}(y_i, s_j)|}\right)^{\nu_1} w(\tilde{Q}_{r_0}(y_i, s_j)) < \varepsilon w(\tilde{Q}_{r_0}(y_i, s_j)) \quad \forall i, j$$

where $\delta_2 \leq c\left(\frac{\varepsilon}{(T_0 r_0^{-1})^{(N+2)\nu_1}}\right)^{1/\nu}$. It follows (3.84).
Next we verify that for all $(x,t) \in \Omega_T$ and $r \in (0, 2r_0]$ and $\lambda > 0$ we have $\tilde{Q}_r(x,t) \cap \Omega_T \subset F_\lambda$ if $w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) \geq \varepsilon w(Q_r(x,t))$ for some $\delta_2 \leq c\left(\frac{\varepsilon}{(T_0 r_0^{-1})^{(N+2)\nu_1}}\right)^{1/\nu}$.
Indeed, take $(x,t) \in \Omega_T$ and $0 < r \leq 2r_0$. Now assume that $\tilde{Q}_r(x,t) \cap \Omega_T \cap F_\lambda^c \neq \emptyset$ and $E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t) \neq \emptyset$ i.e, there exist $(x_1,t_1), (x_2,t_2) \in \tilde{Q}_r(x,t) \cap \Omega_T$ such that $\mathbb{M}(|\nabla u|)(x_1,t_1) \leq \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \leq \delta_2 \lambda$. We need to prove that

$$w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t))) < \varepsilon w(\tilde{Q}_r(x,t)). \tag{3.85}$$

Clearly,

$$\mathbb{M}(|\nabla u|)(y,s) \leq \max\left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right)(y,s), 3^{N+2}\lambda\right\} \quad \forall (y,s) \in \tilde{Q}_r(x,t).$$

Therefore, for all $\lambda > 0$ and $\Lambda \geq 3^{N+2}$,

$$E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t) = \left\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u|\right) > \Lambda\lambda, \mathbb{M}_1[\omega] \leq \delta_2\lambda\right\} \cap \Omega_T \cap \tilde{Q}_r(x,t). \tag{3.86}$$

In particular, $E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t) = \emptyset$ if $\overline{B}_{8r}(x) \subset\subset \mathbb{R}^N \backslash \Omega$. Thus, it is enough to consider the case $B_{8r}(x) \subset\subset \Omega$ and $B_{8r}(x) \cap \Omega \neq \emptyset$.
We consider the case $B_{8r}(x) \subset\subset \Omega$. Let $v_{n,m}$ be as in Lemma 3.4 with $Q_{2R} = Q_{8r}(x,t_0)$ and $u = u_{n,m}$ where $t_0 = \min\{t + 2r^2, T\}$. We have

$$||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} \lesssim \fint_{Q_{8r}(x,t_0)} |\nabla u_{n,m}| + \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}}, \tag{3.87}$$

$$\begin{aligned}\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m} - \nabla v_{n,m}| \lesssim \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}} + [A]_{s_0}^{R_0}\Bigg(&\fint_{Q_{8r}(x,t_0)} |\nabla u_{n,m}| \\ &+ \frac{|\mu_{n,m}|(Q_{8r}(x,t_0))}{r^{N+1}}\Bigg).\end{aligned}$$

Thanks to $\mathbb{M}(|\nabla u|)(x_1,t_1) \leq \lambda$ and $\mathbb{M}_1[\omega](x_2,t_2) \leq \delta_2\lambda$ with $(x_1,t_1), (x_2,t_2) \in Q_r(x,t)$, we get

$$\begin{aligned}\limsup_{n\to\infty}\limsup_{m\to\infty} ||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t))} &\lesssim \fint_{\tilde{Q}_{17r}(x_1,t_1)} |\nabla u| dxdt + \frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}} \\ &\lesssim \lambda + \delta_2\lambda \lesssim \lambda,\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{m\to\infty} \fint_{Q_{4r}(x,t_0)} |\nabla u_n - \nabla v_n| \\ &\lesssim \frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}} + [A]_{s_0}^{R_0}\left(\fint_{\tilde{Q}_{17r}(x_1,t_1)} |\nabla u| + \frac{\omega(\overline{\tilde{Q}_{17r}(x_2,t_2)})}{r^{N+1}}\right) \\ &\lesssim \delta_2\lambda + [A]_{s_0}^{R_0}(\lambda + \delta_2\lambda) \lesssim (\delta_2 + \delta_1(1+\delta_2))\lambda.\end{aligned}$$

Here we have used $[A]_{s_0}^{R_0} \leq \delta_1$ in the last inequality.
So, we can find n_0 large enough and a sequence $\{k_n\}$ such that

$$||\nabla v_{n,m}||_{L^\infty(\tilde{Q}_{2r}(x,t))} = ||\nabla v_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} \leq c\lambda, \tag{3.88}$$

$$\fint_{Q_{4r}(x,t_0)} |\nabla u_{n,m} - \nabla v_{n,m}| dxdt \le c\,(\delta_2 + \delta_1(1+\delta_2))\,\lambda, \tag{3.89}$$

for all $n \ge n_0$ and $m \ge k_n$.
In view of (3.88) we see that for $\Lambda \ge \max\{3^{N+2}, 8c\}$ and $n \ge n_0$, $m \ge k_n$,

$$|\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla v_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| = 0.$$

Leads to

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \le\, & |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla v_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ & + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n - \nabla u_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ & + |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u - \nabla u_n|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)|.\end{aligned}$$

Therefore, by (3.89) and $\tilde{Q}_{2r}(x,t) \subset Q_{4r}(x,t_0)$ we obtain for any $n \ge n_0$ and $m \ge k_n$

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \le\, & \frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_{n,m} - \nabla v_{n,m}| \\ & + \frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}| + \frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n| \\ & \lesssim (\delta_2 + \delta_1(1+\delta_2))\,|Q_r(x,t)| \\ & + \frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u_n - \nabla u_{n,m}| + \frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)} |\nabla u - \nabla u_n|.\end{aligned}$$

Letting $m \to \infty$ and $n \to \infty$ we get

$$|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \lesssim (\delta_2 + \delta_1(1+\delta_2))\,|\tilde{Q}_r(x,t)|.$$

Thus,

$$\begin{aligned}w(E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)) &\le C\left(\frac{|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)|}{|\tilde{Q}_r(x,t)|}\right)^{\nu} w(\tilde{Q}_r(x,t)) \\ &\lesssim (\delta_2 + \delta_1(1+\delta_2))^{\nu}\, w(\tilde{Q}_r(x,t)) \\ &< \varepsilon w(\tilde{Q}_r(x,t)).\end{aligned}$$

where δ_2, δ_1 are appropriately chosen, (C,ν) is a pair of A_∞ constants of w.
Next we consider the case $B_{8r}(x) \cap \Omega \ne \emptyset$. Let $x_3 \in \partial\Omega$ be such that $|x_3 - x| = \operatorname{dist}(x, \partial\Omega)$. Set $t_0 = \min\{t + 2r^2, T\}$. We have
(3.90)

$$Q_{2r}(x,t_0) \subset Q_{10r}(x_3,t_0) \subset Q_{540r}(x_3,t_0) \subset \tilde{Q}_{1080r}(x_3,t) \subset \tilde{Q}_{1088r}(x,t) \subset \tilde{Q}_{1089r}(x_1,t_1),$$

and

$$Q_{540r}(x_3,t_0) \subset \tilde{Q}_{1080r}(x_3,t) \subset \tilde{Q}_{1088r}(x,t) \subset \tilde{Q}_{1089r}(x_2,t_2). \tag{3.91}$$

Let $V_{n,m}$ be as in Lemma 3.13 with $Q_{6R} = Q_{540r}(x_3,t_0)$, $u = u_{n,m}$ and $\varepsilon = \delta_3 \in (0,1)$. We have

$$||\nabla V_{n,m}||_{L^\infty(Q_{10r}(x_3,t_0))} \lesssim \fint_{Q_{540r}(x_3,t_0)} |\nabla u_{n,m}| + \frac{|\mu_{n,m}|(Q_{540r}(x_3,t_0))}{R^{N+1}}$$

and

$$\fint_{Q_{10r}(x_3,t_0)} |\nabla u_{n,m} - \nabla V_{n,m}|$$
$$\lesssim (\delta_3 + [A]_{s_0}^{R_0}) \fint_{Q_{540r}(x_3,t_0)} |\nabla u_{n,m}| + (1 + [A]_{s_0}^{R_0}) \frac{|\mu_{n,m}|(Q_{540r}(x_3,t_0))}{R^{N+1}}.$$

Since $\mathbb{M}(|\nabla u|)(x_1,t_1) \le \lambda$, $\mathbb{M}_1[\omega](x_2,t_2) \le \delta_2\lambda$ and (3.90), (3.91) we get

$$\begin{aligned}\limsup_{n\to\infty}\limsup_{m\to\infty} ||\nabla V_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} &\lesssim \fint_{Q_{540r}(x_3,t_0)} |\nabla u| + \frac{\omega(\overline{Q_{540r}(x_3,t_0)})}{R^{N+1}} \\ &\lesssim \fint_{\tilde{Q}_{1089r}(x_1,t_1)} |\nabla u| + \frac{\omega(\tilde{Q}_{1089r}(x_2,t_2))}{R^{N+1}} \\ &\lesssim \lambda + \delta_2\lambda \lesssim \lambda\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\limsup_{m\to\infty} \fint_{Q_{2r}(x,t_0)} |\nabla u_{n,m} - \nabla V_{n,m}| dxdt \\ &\lesssim (\delta_3 + [A]_{s_0}^{R_0}) \fint_{Q_{540r}(x_3,t_0)} |\nabla u| dxdt + (1 + [A]_{s_0}^{R_0}) \frac{\omega(\overline{Q_{540r}(x_3,t_0)})}{r^{N+1}} \\ &\lesssim (\delta_3 + [A]_{s_0}^{R_0}) \fint_{\tilde{Q}_{1089r}(x_1,t_1)} |\nabla u| dxdt + (1 + [A]_{s_0}^{R_0}) \frac{\omega(\tilde{Q}_{1089}(x_2,t_2))}{r^{N+1}} \\ &\lesssim (\delta_3 + [A]_{s_0}^{R_0})\lambda + (1 + [A]_{s_0}^{R_0})\delta_2\lambda \\ &\le ((\delta_3 + \delta_1) + (1 + \delta_1)\delta_2)\,\lambda.\end{aligned}$$

Here we used $[A]_s^{R_0} \le \delta_1$ in the last inequality.
So, we can find n_0 large enough and a sequence $\{k_n\}$ such that

$$||\nabla V_{n,m}||_{L^\infty(\tilde{Q}_{2r}(x,t))} = ||\nabla V_{n,m}||_{L^\infty(Q_{2r}(x,t_0))} \le c\lambda, \tag{3.92}$$

$$\fint_{Q_{2r}(x,t_0)} |\nabla u_{n,m} - \nabla V_{n,m}| \le c\,((\delta_3 + \delta_1) + (1 + \delta_1)\delta_2)\,\lambda, \tag{3.93}$$

for all $n \ge n_0$ and $m \ge k_n$.
Now set $\Lambda = \max\{3^{N+2}, 8c\}$. As above we also have for $n \ge n_0$, $m \ge k_n$

$$\begin{aligned}|E_{\lambda,\delta_2} \cap \tilde{Q}_r(x,t)| \le\ & |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m} - \nabla V_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ &+ |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u_n - \nabla u_{n,m}|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)| \\ &+ |\{\mathbb{M}\left(\chi_{\tilde{Q}_{2r}(x,t)}|\nabla u - \nabla u_n|\right) > \Lambda\lambda/4\} \cap \tilde{Q}_r(x,t)|.\end{aligned}$$

Therefore from (3.93) we obtain

$$\begin{aligned}|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)| &\lesssim \frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_{n,m}-\nabla V_{n,m}| \\ &\quad+\frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u_{n,m}|+\frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u-\nabla u_n| \\ &\lesssim ((\delta_3+\delta_1)+(1+\delta_1)\delta_2)\,|\tilde{Q}_r(x,t)| \\ &\quad+\frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u_n-\nabla u_{n,m}|+\frac{1}{\lambda}\int_{\tilde{Q}_{2r}(x,t)}|\nabla u-\nabla u_n|.\end{aligned}$$

Letting $m\to\infty$ and $n\to\infty$ we get

$$|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)|\lesssim((\delta_3+\delta_1)+(1+\delta_1)\delta_2)\,|\tilde{Q}_r(x,t)|.$$

Thus

$$\begin{aligned}w(E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)) &\le C\left(\frac{|E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t)|}{|\tilde{Q}_r(x,t)|}\right)^{\nu}w(\tilde{Q}_r(x,t)) \\ &\le c\,((\delta_3+\delta_1)+(1+\delta_1)\delta_2)^{\nu}\,w(\tilde{Q}_r(x,t)) \\ &< \varepsilon w(\tilde{Q}_r(x,t)),\end{aligned}$$

where $\delta_3,\delta_1,\delta_2$ are appropriately chosen, (C,ν) is a pair of A_∞ constants of w.

Therefore, for all $(x,t)\in\Omega_T$ and $r\in(0,2r_0]$ and $\lambda>0$ if $w(E_{\lambda,\delta_2}\cap\tilde{Q}_r(x,t))\ge\varepsilon w(\tilde{Q}_r(x,t))$ then $\tilde{Q}_r(x,t)\cap\Omega_T\subset F_\lambda$ where $\delta_1=\delta_1(\varepsilon,[w]_{A_\infty})\in(0,1)$ and $\delta_2=\delta_2(\varepsilon,[w]_{A_\infty},T_0/R_0)\in(0,1)$. Thanks to Lemma 1.50 we get the result. □

Proof of Theorem 1.20. As in the proof of Theorem 1.17, we can prove (1.50) by using estimate (3.83) in Theorem 3.17. In particular, thanks to Proposition 2.4 for $q>\frac{N+2}{N+1}$, $\mu\in L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)$ and $\sigma\equiv 0$, one has

$$|||\nabla u|^q||_{L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)}\lesssim_{q,T_0/R_0}||\mu||^q_{L^{\frac{(N+2)(q-1)}{q},\infty}(\Omega_T)}. \tag{3.94}$$

The proof is complete. □

The following corollary is a consequence of Theorem 1.20.

Corollary 3.18. Suppose that A satisfies (1.45). Let $\mu\in\mathfrak{M}_b(\Omega_T)$, $\sigma\in\mathfrak{M}_b(\Omega)$, set $\omega=|\mu|+|\sigma|\otimes\delta_{\{t=0\}}$. Let s_0 be in Theorem 1.20. There exists a distributional solution of (1.22) with data μ,σ such that the following holds. For any $1<q<\infty, 0<\beta<N+2$, we find $\delta=\delta(N,\Lambda_1,\Lambda_2,q)\in(0,1)$ such that if Ω is a (δ,R_0)- Reifenberg flat domain and $[A]_{s_0}^{R_0}\le\delta$ for some R_0 then

$$\mathbb{I}_\beta[|\nabla u|^q\chi_{\Omega_T}]\lesssim C_1\mathbb{I}_\beta[\mathbb{M}_1[\omega]^q\chi_{\Omega_T}], \tag{3.95}$$

(3.96)

$$\sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \mathrm{Cap}_{\mathcal{G}_1,q'}(K)>0}}\left(\frac{\int_{K\cap\Omega_T}|\nabla u|^q dxdt}{\mathrm{Cap}_{\mathcal{G}_1,q'}(K)}\right)\lesssim C_1\sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \mathrm{Cap}_{\mathcal{G}_1,q'}(K)>0}}\left(\frac{\omega(K)}{\mathrm{Cap}_{\mathcal{G}_1,q'}(K)}\right)^q,$$

and if $q > \frac{N+2}{N+1}$,

(3.97)
$$\sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \mathrm{Cap}_{\mathcal{H}_1,q'}(K)>0}} \left(\frac{\int_{K\cap\Omega_T}|\nabla u|^q dxdt}{\mathrm{Cap}_{\mathcal{H}_1,q'}(K)}\right) \lesssim C_2 \sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \mathrm{Cap}_{\mathcal{H}_1,q'}(K)>0}} \left(\frac{\omega(K)}{\mathrm{Cap}_{\mathcal{H}_1,q'}(K)}\right)^q,$$

where $C_1 = C_1(q, R_0, T_0)$ and $C_2 = C_2(q, T_0/R_0)$.

REMARK 3.19. If $1 < q < 2$, we have

$$\sup_{\substack{\text{compact } O\subset\mathbb{R}^{N}\\ \mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}-1},q'}(O)>0}} \frac{|\sigma|(O)}{\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}-1},q'}(O)} \sim_q \sup_{\substack{\text{compact } K\subset\mathbb{R}^{N+1}\\ \mathrm{Cap}_{\mathcal{G}_1,q'}(K)>0}} \frac{(|\sigma|\otimes\delta_{\{t=0\}})(K)}{\mathrm{Cap}_{\mathcal{G}_1,q'}(K)}.$$

If $\frac{N+2}{N+1} < q < 2$, then this estimate is true when two capacities $\mathrm{Cap}_{\mathcal{G}_1,q'}$, ,$\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}-1},q'}$ are replaced by $\mathrm{Cap}_{\mathcal{H}_1,q'}$,$\mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}-1},q'}$ respectively, see Remark 2.34.

PROOF OF COROLLARY 3.18. By Theorem 1.20, there exists a renormalized solution of (1.22) with data μ, $u(0) = \sigma$ satisfying

$$\int_{\Omega_T} |\nabla u|^q dw \lesssim c_1 \int_{\Omega_T} (\mathbb{M}_1[\omega])^q\, dw \tag{3.98}$$

for any $w \in A_\infty$, where $c_1 = c_1(q, T_0/R_0, [w]_{A_\infty})$. Thanks to Corollary 2.31, we obtain (3.95).

We have $\mathbb{M}_1[\omega] \lesssim \mathbb{I}_1^{2T_0,1}[\omega]$ in Ω_T. Thus, (3.98) can be rewritten

$$\int_{\Omega_T} |\nabla u|^q dw \lesssim c_1 \int_{\Omega_T} \left(\mathbb{I}_1^{2T_0,1}[\omega]\right)^q dw.$$

Thanks to Proposition 2.23 and Corollary 2.39 and 2.38 we obtain (3.96) and (3.97). □

As in the proof of Theorem 1.17, we obtain the following corollary by using estimate (3.83) in Theorem 3.17.

COROLLARY 3.20. Suppose that A satisfies (1.45). Let $\mu \in \mathfrak{M}_b(\Omega_T)$, $\sigma \in \mathfrak{M}_b(\Omega)$, set $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$. There exists a distributional solution of (1.22) with data μ, σ such that the following holds. For any $1 \le q < \infty$, we find $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0,1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that if Ω is (δ, R_0)-Reifenberg flat domain Ω and $[A]_{s_0}^{R_0} \le \delta$ for some R_0 then

$$\int_{\lambda_0}^{\infty} \lambda^{q-1} \left|\{\mathbb{M}(|\nabla u|) > \Lambda\lambda\} \cap \Omega_T\right| d\lambda \le C \int_{\lambda_0/C}^{\infty} \lambda^{q-1} \left|\{\mathbb{M}_1[\omega] > \lambda\} \cap \Omega_T\right| d\lambda$$

for any $\lambda_0 \ge 0$. Here C depends on $N, \Lambda_1, \Lambda_2, q$ and T_0/R_0.

In the following, we shall use the classical Minkowski inequality, for convenience we recall it. For any $0 < q_1 \le q_2 < \infty$ there holds

$$\left(\int_X \left(\int_Y |f(x,y)|^{q_1} d\mu_2(y)\right)^{\frac{q_2}{q_1}} d\mu_1(x)\right)^{\frac{1}{q_2}} \le \left(\int_Y \left(\int_X |f(x,y)|^{q_2} d\mu_1(x)\right)^{\frac{q_1}{q_2}} d\mu_2(y)\right)^{\frac{1}{q_1}}$$

for any measure function f in $X \times Y$, where μ_1, μ_2 are nonnegative measures in X and Y respectively.

PROOF OF THEOREM 1.21. We will consider only the case $s \neq \infty$ and leave the case $s = \infty$ to the readers. Take $\kappa_1 \in (0, \kappa)$. It is easy to see that for $(x_0, t_0) \in \Omega_T$ and $0 < \rho < \operatorname{diam}(\Omega) + T^{1/2}$

$$w(x,t) = \min\{\rho^{-N-2+\kappa-\kappa_1}, \max\{|x-x_0|, \sqrt{2|t-t_0|}\}^{-N-2+\kappa-\kappa_1}\} \in A_\infty$$

where $[w]_{A_\infty}$ is independent of (x_0, t_0) and ρ. Thus, from (1.50) in Theorem 1.20 we have

$$\begin{aligned}
||\mathbb{M}(|\nabla u|)||^s_{L^{q,s}(\tilde{Q}_\rho(x_0,t_0)\cap\Omega_T)} &= \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} ||\mathbb{M}(|\nabla u|)||^s_{L^{q,s}(\tilde{Q}_\rho(x_0,t_0)\cap\Omega_T, dw)} \\
&\lesssim \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s}(\Omega_T, dw)} \\
&= q\rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} \int_0^\infty \left(\lambda^q \int_0^\infty |\{\mathbb{M}_1[\omega] > \lambda, w > \tau\} \cap \Omega_T| d\tau\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\
&=: \rho^{\frac{(N+2-\kappa+\kappa_1)s}{q}} A. \qquad (3.99)
\end{aligned}$$

Since $w \leq \rho^{-N-2+\kappa-\kappa_1}$ and $\{\mathbb{M}_1[\omega] > \lambda, w > \tau\} \subset \{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0)$,

$$A \leq q \int_0^\infty \left(\lambda^q \int_0^{\rho^{-N-2+\kappa-\kappa_1}} |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T| d\tau\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}.$$

We divide into two cases.

Case 1: $0 < s \leq q$. We can verify that for any non-increasing function F in $(0, \infty)$ and $0 < a \leq 1$ we have

$$\left(\int_0^\infty F(\tau) d\tau\right)^a \leq 4 \int_0^\infty (\tau F(\tau))^a \frac{d\tau}{\tau}.$$

Hence,

$$\begin{aligned}
A &\leq 4q \int_0^\infty \int_0^{\rho^{-N-2+\kappa-\kappa_1}} \left(\lambda^q \tau |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T|\right)^{\frac{s}{q}} \frac{d\tau}{\tau} \frac{d\lambda}{\lambda} \\
&= 4 \int_0^{\rho^{-N-2+\kappa-\kappa_1}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s}(\tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0,t_0)\cap\Omega_T)} \tau^{\frac{s}{q}} \frac{d\tau}{\tau} \\
&\leq 4 \int_0^{\rho^{-N-2+\kappa-\kappa_1}} ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \tau^{\frac{(N+2-\kappa)s}{(-N-2+\kappa-\kappa_1)q}} \tau^{\frac{s}{q}} \frac{d\tau}{\tau} \\
&\lesssim ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \rho^{-\frac{s\kappa_1}{q}}.
\end{aligned}$$

Case 2: $s > q$. Using the Minkowski inequality, yields

$$\begin{aligned}
A &\lesssim \left(\int_0^{\rho^{-N-2+\kappa-\kappa_1}} \left(\int_0^\infty \left(\lambda^q |\{\mathbb{M}_1[\omega] > \lambda\} \cap \tilde{Q}_{\tau^{\frac{1}{-N-2+\kappa-\kappa_1}}}(x_0, t_0) \cap \Omega_T|\right)^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right)^{\frac{q}{s}} d\tau\right)^{\frac{s}{q}} \\
&\lesssim \left(\int_0^{\rho^{-N-2+\kappa-\kappa_1}} \left(||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \tau^{\frac{(N+2-\kappa)s}{(-N-2+\kappa-\kappa_1)q}}\right)^{\frac{q}{s}} d\tau\right)^{\frac{s}{q}} \\
&= ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)} \rho^{-\frac{s\kappa_1}{q}}.
\end{aligned}$$

Therefore, we always have

$$A \lesssim ||\mathbb{M}_1[\omega]||^s_{L^{q,s;\kappa}(\Omega_T)}\rho^{-\frac{s\kappa_1}{q}}.$$

which implies (1.51) from (3.99).
Similarly, we obtain estimate (3.101) by adapting

$$w(x,t) = \min\{\rho^{-N+\vartheta-\vartheta_1}, |x-x_0|^{-N+\vartheta-\vartheta_1}\} \in A_\infty$$

in above argument, where $0 < \vartheta_1 < \vartheta$, $x_0 \in \Omega$ and $0 < \rho < \operatorname{diam}(\Omega)$ and $[w]_{A_\infty}$ is independent of x_0 and ρ.
Next, to obtain (1.53) we need to show that for any ball $B_\rho \subset \mathbb{R}^N$

$$\left(\int_0^T |\operatorname{osc}_{B_\rho\cap\overline{\Omega}} u(t)|^q dt\right)^{\frac{1}{q}} \lesssim \rho^{1-\frac{\vartheta}{q}} |||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)}. \tag{3.100}$$

Since the extension of u over $(\Omega_T)^c$ is zero and $u \in L^1(0,T,W^{1,1}_0(\Omega))$ thus we have for a.e $t \in (0,T)$, $u(.,t) \in W^{1,1}(\mathbb{R}^N)$. Applying [**38**, Lemma 7.16] to a ball $B_\rho \subset \mathbb{R}^N$, we get for a.e $t \in (0,T)$ and $x \in B_\rho$

$$|u(x,t) - u_{B\rho}(t)| \lesssim \int_{B_\rho} \frac{|\nabla u(y,t)|}{|x-y|^{N-1}} dy \lesssim \int_0^{3\rho} \frac{\int_{B_r(x)} |\nabla u(y,t)| dy}{r^{N-1}} \frac{dr}{r}.$$

Here $u_{B\rho}(t)$ is the average of $u(.,t)$ over B_ρ, i.e $u_{B\rho}(t) = \frac{1}{|B_\rho|}\int_{B_\rho} u(x,t)dx$.
Using the Minkowski and the Hölder inequality, we observe that for a.e $x \in B_\rho$

$$\begin{aligned}\left(\int_0^T |u(x,t) - u_{B\rho}(t)|^q dt\right)^{\frac{1}{q}} &\lesssim \left(\int_0^T \left(\int_0^{3\rho} \frac{\int_{B_r(x)} |\nabla u(y,t)| dy}{r^{N-1}} \frac{dr}{r}\right)^q dt\right)^{\frac{1}{q}} \\ &\lesssim \int_0^{3\rho} \int_{B_r(x)} \left(\int_0^T |\nabla u(y,t)|^q dt\right)^{\frac{1}{q}} dy \frac{dr}{r^N} \\ &\lesssim \int_0^{3\rho} \left(\int_{B_r(x)} \int_0^T |\nabla u(y,t)|^q dt dy\right)^{\frac{1}{q}} |B_r(x)|^{\frac{q-1}{q}} \frac{dr}{r^N} \\ &\lesssim \int_0^{3\rho} r^{\frac{N-\vartheta}{q}} r^{\frac{N(q-1)}{q}} \frac{dr}{r^N} |||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)} \\ &\lesssim \rho^{1-\frac{\vartheta}{q}} |||\nabla u|||_{L^{q;\vartheta}_{**}(\Omega_T)}.\end{aligned}$$

Therefore, we get (3.100). The proof is complete. □

In the following, we give some estimates for the norm of $\mathbb{M}_1[\omega]$ in $L^{q;\kappa}_*(\mathbb{R}^{N+1})$ and in $L^{q;\vartheta}_{**}(\mathbb{R}^{N+1})$.

PROPOSITION 3.21. Let $1 < \kappa \le N+2$, $0 < \vartheta \le N$ and $q, q_1 > 1$. Suppose that $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$. Then $\mathbb{M}_1[\mu] \le 2^{N+2}\mathbb{I}_1[\mu]$ and

a: If $q > \frac{\kappa}{\kappa-1}$ then

$$||\mathbb{I}_1[\mu]||_{L^{q;\kappa}_*(\mathbb{R}^{N+1})} \lesssim_{q,\kappa} ||\mu||_{L_*^{\frac{q\kappa}{q+\kappa};\kappa}(\mathbb{R}^{N+1})}. \tag{3.101}$$

b: If $1 < q < 2$ then

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} \le \mathbf{I}_{\frac{2}{q}-1}[\mu_1](x), \tag{3.102}$$

where μ_1 is a nonnegative radon measure in $\mathbb{R}^N$ defined by $\mu_1(A) = \mu(A \times \mathbb{R})$ for every Borel set $A \subset \mathbb{R}^N$. In particular,

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le ||\mathbf{I}_{\frac{2}{q}-1}[\mu_1]||_{L^{q;\vartheta}(\mathbb{R}^N)}, \tag{3.103}$$

and if $\vartheta > \frac{2-q}{q-1}$ there holds

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \lesssim_{q,\vartheta} ||\mu_1||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)}. \tag{3.104}$$

c: If $\frac{2q}{q+2} < q_1 \le q$ then

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} \le \mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_1}}[\mu_2](x), \tag{3.105}$$

where $d\mu_2(x) = ||\mu(x,.)||_{L^{q_1}(\mathbb{R})}dx$. In particular,

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le ||\mathbf{I}_{\frac{2}{q}+1-\frac{2}{q_1}}[\mu_2]||_{L^{q;\vartheta}(\mathbb{R}^N)}, \tag{3.106}$$

and if $\vartheta > \frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)$ there holds

$$||\mathbb{I}_1[\mu]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \lesssim_{q,\vartheta} ||\mu_2||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N)} = ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))}. \tag{3.107}$$

REMARK 3.22. Let $1 < q < 2$, $0 < \vartheta \le N$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$. From (3.103) and (3.104) in Proposition 3.21 we assert that

$$||\mathbb{I}_1[|\sigma| \otimes \delta_{\{t=0\}}]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \le ||\mathbf{I}_{\frac{2}{q}-1}[|\sigma|]||_{L^{q;\vartheta}(\mathbb{R}^N)},$$

and

$$||\mathbb{I}_1[|\sigma| \otimes \delta_{\{t=0\}}]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \lesssim_{q,\vartheta} ||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} \text{ if } \vartheta > \frac{2-q}{q-1}.$$

Furthermore, from preceding inequality and (3.107) in Proposition 3.21 we can state that

$$||\mathbb{I}_1[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|]||_{L_{**}^{q;\vartheta}(\mathbb{R}^{N+1})} \lesssim_{q,\vartheta} ||\sigma||_{L^{\frac{\vartheta q}{\vartheta+2-q};\vartheta}(\mathbb{R}^N)} + ||\mu||_{L^{\frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q};\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))},$$

provided

$$1 < q_1 \le q < 2,$$
$$\max\left\{\frac{2-q}{q-1}, \frac{1}{q-1}\left(2+q-\frac{2q}{q_1}\right)\right\} < \vartheta \le N.$$

Here

$$||\mu||_{L^{q_2;\vartheta}(\mathbb{R}^N,L^{q_1}(\mathbb{R}))} = \sup_{\rho>0,x\in\mathbb{R}^N} \rho^{\frac{\vartheta-N}{q_2}} \left(\int_{B_\rho(x)} \left(\int_{\mathbb{R}} |\mu(y,t)|^{q_1} dt\right)^{\frac{q_2}{q_1}} dy\right)^{\frac{1}{q_2}},$$

with $q_2 = \frac{\vartheta q q_1}{(\vartheta+2+q)q_1-2q}$.

Proof of Proposition 3.21. Clearly, estimate (3.101) is followed by (2.12) in Proposition 2.7. We want to emphasize that almost every estimates in this proof will be used the Minkowski inequality. For a ball $B_\rho \subset \mathbb{R}^N$, we have for a.e $x \in \mathbb{R}^N$

$$||\mathbb{I}_1[\mu](x,.)||_{L^q(\mathbb{R})} = \left(\int_{-\infty}^{+\infty}\left(\int_0^\infty \frac{\mu(\tilde{Q}_r(x,t))}{r^{N+1}}\frac{dr}{r}\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \int_0^\infty \left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} \frac{dr}{r^{N+2}}. \tag{3.108}$$

Now, we need to estimate $\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}}$.
b. We have

$$\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} = \left(\int_{-\infty}^{+\infty}\left(\int_{\mathbb{R}^{N+1}}\chi_{\tilde{Q}_r(x,t)}(x_1,t_1)d\mu(x_1,t_1)\right)^q dt\right)^{\frac{1}{q}}$$
$$\leq \int_{\mathbb{R}^{N+1}}\left(\int_{-\infty}^{+\infty}\chi_{\tilde{Q}_r(x,t)}(x_1,t_1)dt\right)^{\frac{1}{q}} d\mu(x_1,t_1)$$
$$= r^{\frac{2}{q}}\mu_1(B_r(x)).$$

Combining this with (3.108) we obtain (3.102) and (3.104).
Thus, we also assert (3.104) from [**1**, Theorem 3.1].
c. Set $d\mu_2(x) = ||\mu(x,.)||_{L^{q_1}(\mathbb{R})}dx$. Using Hölder's inequality,

$$\mu(\tilde{Q}_r(x,t)) \leq r^{\frac{2(q_1-1)}{q_1}}\int_{B_r(x)}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(\mu(x_1,t_1))^{q_1}dt_1\right)^{\frac{1}{q_1}} dx_1.$$

Leads to

$$\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} \leq r^{\frac{2(q_1-1)}{q_1}}\int_{B_r(x)}\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(\mu(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}} dt\right)^{\frac{1}{q}} dx_1.$$

Note that

$$\left(\int_{-\infty}^{+\infty}\left(\int_{t-\frac{\rho^2}{2}}^{t+\frac{\rho^2}{2}}(\mu(x_1,t_1))^{q_1}dt_1\right)^{\frac{q}{q_1}} dt\right)^{\frac{q_1}{q}} \leq \rho^{\frac{2q_1}{q}}\int_{-\infty}^{+\infty}(w(x_1,t_1))^{q_1}dt_1.$$

Hence

$$\left(\int_{-\infty}^{+\infty}(\mu(\tilde{Q}_r(x,t)))^q dt\right)^{\frac{1}{q}} \leq r^{\frac{2(q_1-1)}{q_1}+\frac{2}{q}}\int_{B_r(x)}||\mu(x_1,.)||_{L^{q_1}(\mathbb{R})}dx_1 = r^{\frac{2(q_1-1)}{q_1}+\frac{2}{q}}\mu_2(B_r(x)).$$

Consequently, since (3.108) we derive (3.105) and (3.106).
We also obtain (3.107) from [**1**, Theorem 3.1]. □

3.2.3. Global estimates in $\mathbb{R}^N\times(0,\infty)$ and $\mathbb{R}^{N+1}$. Now, we prove Theorem 1.23 and 1.25.

Proof of Theorem 1.23 and Theorem 1.25. For any $n \geq 1$, it is easy to see that

i.: $\mathbb{R}^N \backslash B_n(0)$ satisfies a uniformly 2−thick condition with constants $c_0 = \frac{\text{Cap}_p(B_{1/4}(z_0),B_2(0))}{\text{Cap}_p(B_1(0),B_2(0))}$, $z_0 = (1/2, 0, ..., 0) \in \mathbb{R}^N$ and $r_0 = n$.
ii.: for any $\delta \in (0,1)$, $B_n(0)$ is a $(\delta, 2n\delta)-$ Reifenberg flat domain.
iii.: $[\mathcal{A}]_{s_0}^n \le [\mathcal{A}]_{s_0}^\infty$.

Assume that $||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})} < \infty$. Thus by Remark 1.24 we have

$$\mathbb{I}_2[|\omega|](x,t) < \infty \text{ for a.e } (x,t) \in \mathbb{R}^{N+1}. \tag{3.109}$$

In view of the proof of the Theorem 1.5 and applying Theorem 1.17 to $B_n(0) \times (-n^2, n^2)$ and with data $\chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}\omega$ for any $n \ge 2$, there exists a sequence of renormalized solutions $\{u_n\}$ of

$$\begin{cases} (u_n)_t - \text{div}(A(x,t,\nabla u_n)) = \chi_{B_{n-1}(0)\times(-(n-1)^2,(n-1)^2)}\omega & \text{in} \quad B_n(0) \times (-n^2, n^2), \\ u_n = 0 \quad \text{on} \quad \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \quad \text{in} \quad B_n(0), \end{cases}$$

converging to a distributional solution u in $L^1_{\text{loc}}(\mathbb{R}; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ of 1.24 with data $\mu = \omega$ such that

$$|||\nabla u_n|||_{L^{p,s}(B_n(0)\times(-n^2,n^2))} \lesssim ||\mathbb{M}_1[|\omega|]||_{L^{p,s}(\mathbb{R}^{N+1})}.$$

Here the constant c does not depend on n since $\frac{T_0}{r_0} = \frac{2n+(1+n^2)^{1/2}}{n} \approx 1$.
Using Fatou's Lemma, we get estimate (1.54).
As above, we also obtain (1.55). And similarly, we can prove Theorem 1.25 by this way. The proof is complete. □

Remark 3.23 (sharpness). The inequality (1.57) is optimal in the following sense :

$$|||\nabla \mathcal{H}_2| * |\omega|||_{L^q(\mathbb{R}^N\times(0,\infty))} \sim ||\mathbb{M}_1[|\omega|]||_{L^q(\mathbb{R}^{N+1})}. \tag{3.110}$$

Indeed, we have

$$\nabla \mathcal{H}_2(x,t) = -\frac{C}{2}\frac{\chi_{(0,\infty)}(t)}{t^{(N+1)/2}} \exp(-\frac{|x|^2}{4t})\frac{x}{\sqrt{t}},$$

which leads to

$$\frac{1}{t^{\frac{N+1}{2}}}\chi_{t>0}\chi_{\frac{1}{2}\sqrt{t}\lesssim|x|\le 2\sqrt{t}} \lesssim |\nabla \mathcal{H}_\alpha(x,t)| \lesssim \frac{1}{\max\{|x|, \sqrt{2|t|}\}^{N+1}}.$$

Immediately, we get

$$\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}}\frac{dr}{r} \lesssim |\nabla \mathcal{H}_2| * |\omega|(x,t) \lesssim \mathbb{I}_1[\omega](x,t).$$

Then, Theorem 2.2 gives the right-hand side inequality of (3.110). So, it is enough to show that

$$A := \int_{\mathbb{R}^{N+1}} \left(\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}}\frac{dr}{r}\right)^q dxdt \gtrsim ||\mathbb{M}_1[\omega]||^q_{L^q(\mathbb{R}^{N+1})}. \tag{3.111}$$

To do this, we take $r_k = (3/2)^k$ for $k \in \mathbb{Z}$,

$$\left(\int_0^\infty \frac{\omega\left((B_r(x)\backslash B_{r/2}(x)) \times (t-r^2, t-r^2/4)\right)}{r^{N+1}} \frac{dr}{r}\right)^q$$
$$\gtrsim \sum_{k=-\infty}^{\infty} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}}\right)^q.$$

We deduce that

$$A \gtrsim \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}}\right)^q dxdt.$$

For any k, put $y = x + \frac{7}{8}r_k$ and $s = t - \frac{25}{32}r_k^2$, so $B_{r_k}(x)\backslash B_{3r_k/4}(x) \supset B_{r_k/8}(y)$ and

$$\int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left((B_{r_k}(x)\backslash B_{3r_k/4}(x)) \times (t-r_k^2, t-9r_k^2/16)\right)}{r_k^{N+1}}\right)^q dxdt$$
$$\geq \int_{\mathbb{R}^{N+1}} \left(\frac{\omega\left(B_{r_k/8}(y) \times (s-7r_k^2/32, t+7r_k^2/32)\right)}{r_k^{N+1}}\right)^q dyds.$$

Consequently,

$$A \gtrsim \int_{\mathbb{R}^{N+1}} \sum_{k=-\infty}^{\infty} \left(\frac{\omega\left(B_{r_k/8}(y) \times (s-7r_k^2/32, t+7r_k^2/32)\right)}{r_k^{N+1}}\right)^q dyds.$$

It follows (3.111).

CHAPTER 4

Quasilinear Lane-Emden type and quasilinear riccati type parabolic equations

4.1. Quasilimear Lane-Emden type parabolic equations

4.1.1. Quasilinear Lane-Emden Parabolic Equations in Ω_T. To prove Theorem 1.8 we need the following proposition which was proved in [**7**].

PROPOSITION 4.1. Assume that O is an open subset of $\mathbb{R}^{N+1}$. Let $p>1$ and $\mu \in \mathfrak{M}^+(O)$. If μ is absolutely continuous with respect to $\text{Cap}_{2,1,p}$ in O, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(W_p^{2,1}(\mathbb{R}^{N+1})\right)^*$, with compact support in O which converges to μ weakly in $\mathfrak{M}(O)$. Moreover, if $\mu \in \mathfrak{M}_b^+(O)$ then $||\mu_n - \mu||_{\mathfrak{M}_b(O)} \to 0$ as $n \to \infty$.

REMARK 4.2. By Theorem 2.17, $W_p^{2,1}(\mathbb{R}^{N+1}) = \mathcal{L}_2^p(\mathbb{R}^{N+1})$, it follows $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(\mathcal{L}_2^p(\mathbb{R}^{N+1})\right)^*$. Since $||\mu_n||_{\left(\mathcal{L}_2^p(\mathbb{R}^{N+1})\right)^*} = ||\check{\mathcal{G}}_2[\mu_n]||_{L^{p'}(\mathbb{R}^{N+1})}$, so $\check{\mathcal{G}}_2[\mu_n] \in L^{p'}(\mathbb{R}^{N+1})$.

Consequently, from (2.17) in Proposition 2.8, we obtain $\mathbb{I}_2^R[\mu_n] \in L^{p'}(\mathbb{R}^{N+1})$ for any $n \in \mathbb{N}$ and $R>0$. In particular, $\mathbb{I}_2[\mu_n] \in L^{p'}_{\text{loc}}(\mathbb{R}^{N+1})$ for any $n \in \mathbb{N}$.

REMARK 4.3. As in the proof of Theorem 2.5 in [**17**], we can prove a general version of Proposition 4.1, that is: for $p>1$, if μ is absolutely continuous with respect to $\text{Cap}_{\mathcal{G}_\alpha,p}$ in O, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_b^+(O) \cap \left(\mathcal{L}_\alpha^p(\mathbb{R}^{N+1})\right)^*$, with compact support in O which converges to μ weakly in $\mathfrak{M}(O)$. Furthermore, $\mathbb{I}_\alpha[\mu_n] \in L^{p'}_{\text{loc}}(\mathbb{R}^{N+1})$ for all $n \in \mathbb{N}$. Besides, we also obtain that for $\mu \in \mathfrak{M}_b(O)$ is absolutely continuous with respect to $\text{Cap}_{\mathcal{G}_\alpha,p}$ in O if and only if $\mu = f + \nu$ where $f \in L^1(O)$ and $\nu \in \left(\mathcal{L}_\alpha^p(\mathbb{R}^{N+1})\right)^*$.

PROOF OF THEOREM 1.8. First, assume that $\sigma \in L^1(\Omega)$. Because μ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$, so are μ^+ and μ^-. Applying Proposition 4.1 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω_T which converge to μ^+ and μ^- in $\mathfrak{M}_b(\Omega_T)$ respectively and such that $\mathbb{I}_2[\mu_{1,n}], \mathbb{I}_2[\mu_{2,n}] \in L^q(\Omega_T)$.

For $i=1,2$, set $\tilde{\mu}_{i,1} = \mu_{i,1}$ and $\tilde{\mu}_{i,j} = \mu_{i,j} - \mu_{i,j-1} \geq 0$, so $\mu_{i,n} = \sum_{j=1}^n \tilde{\mu}_{i,j}$. We write $\mu_{i,n} = \mu_{i,n,0} + \mu_{i,n,s}$, $\tilde{\mu}_{i,j} = \tilde{\mu}_{i,j,0} + \tilde{\mu}_{i,j,s}$ with $\mu_{i,n,0}, \tilde{\mu}_{i,n,0} \in \mathfrak{M}_0(\Omega_T)$, $\mu_{i,n,s}, \tilde{\mu}_{i,n,s} \in \mathfrak{M}_s(\Omega_T)$.
As in the proof of Theorem 1.1, for any $j \in \mathbb{N}$ and $i=1,2$, there exist sequences of nonnegative measures $\tilde{\mu}_{m,i,j,0} = (f_{m,i,j}, g_{m,i,j}, h_{m,i,j})$ and $\tilde{\mu}_{m,i,j,s}$ such that

- $f_{m,i,j}, g_{m,i,j}, h_{m,i,j} \in C_c^\infty(\Omega_T)$ strongly converge to some $f_{i,j}, g_{i,j}, h_{i,j}$ in $L^1(\Omega_T), L^2(\Omega_T, \mathbb{R}^N)$ and $L^2(0,T,H_0^1(\Omega))$ respectively;

- $\tilde{\mu}_{m,i,j}, \tilde{\mu}_{m,i,j,s} \in C_c^\infty(\Omega_T)$ converge to $\tilde{\mu}_{i,j}, \tilde{\mu}_{i,j,s}$ resp. in the narrow topology with $\tilde{\mu}_{m,i,j} = \tilde{\mu}_{m,i,j,0} + \tilde{\mu}_{m,i,j,s}$ which satisfy $\tilde{\mu}_{i,j,0} = (f_{i,j}, g_{i,j}, h_{i,j})$ and $0 \le \tilde{\mu}_{m,i,j} \le \varphi_m * \tilde{\mu}_{i,j}$ and

$$||f_{m,i,j}||_{L^1(\Omega_T)} + \|g_{m,i,j}\|_{L^2(\Omega_T,\mathbb{R}^N)} + ||h_{m,i,j}||_{L^2(0,T,H_0^1(\Omega))} + \mu_{m,i,j,s}(\Omega_T) \le 2\tilde{\mu}_{i,j}(\Omega_T). \tag{4.1}$$

Here $\{\varphi_m\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$.

For any $n, k, m \in \mathbb{N}$, let $u_{n,k,m}, u_{1,n,k,m}, u_{2,n,k,m} \in W$ be solutions of problems

$$\begin{cases} (u_{n,k,m})_t - \operatorname{div}(A(x,t,\nabla u_{n,k,m})) + T_k(|u_{n,k,m}|^{q-1}u_{n,k,m}) = \sum_{j=1}^n (\tilde{\mu}_{m,1,j} - \tilde{\mu}_{m,2,j}) \text{ in } \Omega_T, \\ u_{n,k,m} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_{n,k,m}(0) = T_n(\sigma^+) - T_n(\sigma^-) \quad \text{on} \quad \Omega, \end{cases} \tag{4.2}$$

$$\begin{cases} (u_{1,n,k,m})_t - \operatorname{div}(A(x,t,\nabla u_{1,n,k,m})) + T_k(u_{1,n,k,m}^q) = \sum_{j=1}^n \tilde{\mu}_{m,1,j} \text{ in } \Omega_T, \\ u_{1,n,k,m} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_{1,n,k,m}(0) = T_n(\sigma^+) \quad \text{in} \quad \Omega, \end{cases} \tag{4.3}$$

$$\begin{cases} (u_{2,n,k,m})_t - \operatorname{div}(\tilde{A}(x,t,\nabla u_{2,n,k,m})) + T_k(u_{2,n,k,m}^q) = \sum_{j=1}^n \tilde{\mu}_{m,2,j} \text{ in } \Omega_T, \\ u_{2,n,k,m} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_{2,n,k,m}(0) = T_n(\sigma^-) \quad \text{in} \quad \Omega, \end{cases} \tag{4.4}$$

where $\tilde{A}(x,t,\xi) = -A(x,t,-\xi)$ and

$$W = \left\{ z : z \in L^2(0,T,H_0^1(\Omega)), z_t \in L^2(0,T,H^{-1}(\Omega)) \right\}.$$

Thanks to Comparison Principle Theorem and Theorem 1.1, we have that for any m,k the sequences $\{u_{1,n,k,m}\}_n$ and $\{u_{2,n,k,m}\}_n$ are increasing and

$$-K\mathbb{I}_2[T_n(\sigma^-) \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n} * \varphi_m] \le -u_{2,n,k,m} \le u_{n,k,m} \le u_{1,,n,k,m}$$
$$\le K\mathbb{I}_2[\mu_{1,n} * \varphi_m] + K\mathbb{I}_2[T_n(\sigma^+) \otimes \delta_{\{t=0\}}],$$

where a constant K is in Theorem 1.1. Thus,

$$-K\mathbb{I}_2[T_n(\sigma^-) \otimes \delta_{\{t=0\}}] - K\mathbb{I}_2[\mu_{2,n}] * \varphi_m \le -u_{2,n,k,m} \le u_{n,k,m} \le u_{1,,n,k,m}$$
$$\le K\mathbb{I}_2[\mu_{1,n}] * \varphi_m + K\mathbb{I}_2[T_n(\sigma^+) \otimes \delta_{\{t=0\}}].$$

Moreover,

$$\int_{\Omega_T} T_k(u_{i,n,k,m}^q)dxdt \le \int_{\Omega_T} \varphi_m * \mu_{i,n}dxdt + |\sigma|(\Omega) \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

As in [**15**, Proof of Lemma 5.3], thanks to Proposition 1.36 and Theorem 1.37, there exist subsequences of $\{u_{n,k,m}\}_m$ $\{u_{1,n,k,m}\}_m$, $\{u_{2,n,k,m}\}_m$, still denoted them, converging to renormalized solutions $u_{n,k}$ $u_{1,n,k}$, $u_{2,n,k}$ of equations

- (4.2) with data $\mu_{1,n} - \mu_{2,n}$, $u_{n,k}(0) = T_n(\sigma^+) - T_n(\sigma^-)$ and the decomposition $(\sum_{j=1}^n f_{1,j} - \sum_{j=1}^n f_{2,j}, \sum_{j=1}^n g_{1,j} - \sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{1,j} - \sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0} - \mu_{2,n,0}$,
- (4.3) with data $\mu_{1,n}$, $u_{1,n,k}(0) = T_n(\sigma^+)$ and the decomposition $(\sum_{j=1}^n f_{1,j}$, $\sum_{j=1}^n g_{1,j}$, $\sum_{j=1}^n h_{1,j})$ of $\mu_{1,n,0}$,
- (4.4) with data $\mu_{2,n}$, $u_{2,n,k}(0) = T_n(\sigma^-)$ and the decomposition $(\sum_{j=1}^n f_{2,j}$, $\sum_{j=1}^n g_{2,j}, \sum_{j=1}^n h_{2,j})$ of $\mu_{2,n,0}$ respectively,

which satisfy

$$-K\mathbb{I}_2[T_n(\sigma^-)\otimes\delta_{\{t=0\}}]-K\mathbb{I}_2[\mu_{2,n}]\le -u_{2,n,k}\le u_{n,k}\le u_{1,n,k}$$
$$\le K\mathbb{I}_2[\mu_{1,n}]+K\mathbb{I}_2[T_n(\sigma^+)\otimes\delta_{\{t=0\}}].$$

Next, as in [**15**, Proof of Lemma 5.4] since $I_2[\mu_{i,n}]\in L^q(\Omega_T)$ for any n, thanks to Proposition 1.36 and Theorem 1.37, there exist subsequences of $\{u_{n,k}\}_k$ $\{u_{1,n,k}\}_k$, $\{u_{2,n,k}\}_k$, still denoted them, converging to renormalized solutions u_n $u_{1,n}$, $u_{2,n}$ of equations

$$(4.5)\qquad \begin{cases} (u_n)_t-\operatorname{div}(A(x,t,\nabla u_n))+|u_n|^{q-1}u_n=\mu_{1,n}-\mu_{2,n} \quad \text{in} \quad \Omega_T,\\ u_n=0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ u_n(0)=T_n(\sigma^+)-T_n(\sigma^-) \quad \text{in } \Omega,\end{cases}$$

$$(4.6)\qquad \begin{cases} (u_{1,n})_t-\operatorname{div}(A(x,t,\nabla u_{1,n}))+u_{1,n}^q=\mu_{1,n} \quad \text{in} \quad \Omega_T,\\ u_{1,n}=0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ u_{1,n}(0)=T_n(\sigma^+) \quad \text{in} \quad \Omega,\end{cases}$$

$$(4.7)\qquad \begin{cases} (u_{2,n})_t-\operatorname{div}(\tilde{A}(x,t,\nabla u_{2,n}))+u_{2,n}^q=\mu_{2,n} \quad \text{in} \quad \Omega_T,\\ u_{2,n}=0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ u_{2,n}(0)=T_n(\sigma^-) \quad \text{in} \quad \Omega,\end{cases}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j}-\sum_{j=1}^n f_{2,j},\sum_{j=1}^n g_{1,j}-\sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{1,j}-\sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}-\mu_{2,n,0}$, $(\sum_{j=1}^n f_{1,j},\sum_{j=1}^n g_{1,j},\sum_{j=1}^n h_{1,j})$ of $\mu_{1,n,0}$ and $(\sum_{j=1}^n f_{2,j},\sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{2,j})$ of $\mu_{2,n,0}$ respectively, which satisfy

$$-K\mathbb{I}_2[T_n(u_0^-)\otimes\delta_{\{t=0\}}]-K\mathbb{I}_2[\mu_{2,n}]\le -u_{2,n}\le u_n\le u_{1,n}$$
$$\le K\mathbb{I}_2[\mu_{1,n}]+K\mathbb{I}_2[T_n(u_0^+)\otimes\delta_{\{t=0\}}].$$

Moreover, the sequences $\{u_{1,n}\}_n$ and $\{u_{2,n}\}_n$ are increasing and

$$\int_{\Omega_T}u_{i,n}^q dxdt\le|\mu|(\Omega_T)+|\sigma|(\Omega).$$

Note that from (4.1) we have

$$||f_{i,j}||_{L^1(\Omega_T)}+\|g_{i,j}\|_{L^2(\Omega_T,\mathbb{R}^N)}+||h_{i,j}||_{L^2(0,T,H_0^1(\Omega))}\le 2\tilde{\mu}_{i,j}(\Omega_T)$$

which implies

$$\sum_{j=1}^n\Big(||f_{i,j}||_{L^1(\Omega_T)}+||g_{i,j}||_{L^2(\Omega_T,\mathbb{R}^N)}+||h_{i,j}||_{L^2(0,T,H_0^1(\Omega))}\Big)\le 2\mu_{i,n}(\Omega_T)\le 2|\mu|(\Omega_T).$$

Finally, as in [**15**, Proof of Theorem 5.2] thanks to Proposition 1.36, Theorem 1.37 and Monotone Convergence Theorem there exist subsequences of $\{u_n\}_n$, $\{u_{1,n}\}_n$, $\{u_{2,n}\}_n$, still denoted them, converging to renormalized solutions u, u_1, u_2 of equations

- (4.5) with data μ, $u(0)=\sigma$ and the decomposition $(\sum(f_{1,j}-f_{2,j}),\sum(g_{1,j}-g_{2,j}),\sum(h_{1,j}-h_{2,j}))$ of μ_0,
- (4.6) with data μ^+, $u_1(0)=\sigma^+$ and the decomposition $(\sum f_{1,j},\sum g_{1,j},\sum h_{1,j})$ of μ_0^+,
- (4.7) with data μ^-, $u_2(0)=\sigma^-$ and the decomposition $(\sum f_{2,j},\sum g_{2,j},\sum h_{2,j})$ of μ_0^-, respectively

satisfying

$$-K\mathbb{I}_2[\sigma^-\otimes\delta_{\{t=0\}}]-K\mathbb{I}_2[\mu^-]\le -u_2\le u\le u_1\le K\mathbb{I}_2[\mu^+]+K\mathbb{I}_2[\sigma^+\otimes\delta_{\{t=0\}}].$$

Here $\sum:=\sum_{j=1}^\infty$.

We now have remark: if $\sigma\equiv 0$ and $\text{supp}(\mu)\subset\overline{\Omega}\times[a,T]$, $a>0$, then $u=u_1=u_2=0$ in $\Omega\times(0,a)$ since $u_{n,k}=u_{1,n,k}=u_{2,n,k}=0$ in $\Omega\times(0,a)$.

Next, we will consider $\sigma\in\mathfrak{M}_b(\Omega)$ such that σ is absolutely continuous with respect to the capacity $\text{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. So, $\chi_{\Omega_T}\mu+\sigma\otimes\delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\Omega\times(-T,T)$. As above, we verify that there exists a renormalized solution u of

(4.8)
$$\begin{cases} u_t-\text{div}(A(x,t,\nabla u))+|u|^{q-1}u=\chi_{\Omega_T}\mu+\sigma\otimes\delta_{\{t=0\}} & \text{in}\quad \Omega\times(-T,T),\\ u=0\quad \text{on}\quad \partial\Omega\times(-T,T),\\ u(-T)=0\quad\text{on}\quad\Omega,\end{cases}$$

satisfying $u=0$ in $\Omega\times(-T,0)$ and

$$-K\mathbb{I}_2[\sigma^-\otimes\delta_{\{t=0\}}]-K\mathbb{I}_2[\mu^-]\le u\le K\mathbb{I}_2[\mu^+]+K\mathbb{I}_2[\sigma^+\otimes\delta_{\{t=0\}}].$$

Finally, from remark 1.42 we get the result. This completes the proof. □

Proof of Theorem 1.9. Let $\{\mu_{n,i}\}\subset C_c^\infty(\Omega_T),\sigma_{i,n}\in C_c^\infty(\Omega)$ for $i=1,2$ be as in the proof of Theorem 1.1. We have $0\le\mu_{n,1}\le\varphi_n*\mu^+,0\le\mu_{n,2}\le\varphi_n*\mu^-,0\le\sigma_{1,n}\le\varphi_{1,n}*\sigma^+,0\le\sigma_{2,n}\le\varphi_{1,n}*\sigma^-$ for any $n\in\mathbb{N}$ where $\{\varphi_n\}$ and $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1},\mathbb{R}^N$ respectively.

We prove that the problem (1.20) has a solution with data $\mu=\mu_{n_0}=\mu_{n_0,1}-\mu_{n_0,2},\sigma=\sigma_{n_0}=\sigma_{1,n_0}-\sigma_{2,n_0}$ for $n_0\in\mathbb{N}$. Put

$$J=\left\{u\in L^q(\Omega_T): u^+\le\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1}+\sigma_{1,n_0}\otimes\delta_{\{t=0\}}]\right.$$
$$\left.\text{and}\quad u^-\le\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2}+\sigma_{2,n_0}\otimes\delta_{\{t=0\}}]\right\},$$

where $\max\{-\frac{N+2}{q'}+2,0\}<\delta<2$.

Clearly, J is closed under the strong topology of $L^q(\Omega_T)$ and convex.

We consider a map $S:J\to J$ defined for each $v\in J$ by $S(v)=u$, where $u\in L^1(\Omega_T)$ is the unique renormalized solution of

$$\begin{cases} u_t-\text{div}(A(x,t,\nabla u))=|v|^{q-1}v+\mu_{n_0,1}-\mu_{n_0,2} & \text{in}\quad\Omega_T,\\ u=0\quad\text{on}\quad\partial\Omega\times(0,T),\\ u(0)=\sigma_{1,n_0}-\sigma_{2,n_0}\quad\text{in}\quad\Omega.\end{cases}\tag{4.9}$$

By Theorem 1.1, we have

$$u^+\le K\mathbb{I}_2^{2T_0}[(v^+)^q]+K\mathbb{I}_2^{2T_0}[\mu_{n_0,1}+\sigma_{1,n_0}\otimes\delta_{\{t=0\}}],$$
$$u^-\le K\mathbb{I}_2^{2T_0}[(v^-)^q]+K\mathbb{I}_2^{2T_0}[\mu_{n_0,2}+\sigma_{2,n_0}\otimes\delta_{\{t=0\}}],$$

where K is the constant in Theorem 1.1. Thus,

$$u^+\le K\left(\frac{qK}{q-1}\right)^q\mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1}+\sigma_{1,n_0}\otimes\delta_{\{t=0\}}]\right)^q\right]+K\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1}+\sigma_{1,n_0}\otimes\delta_{\{t=0\}}],$$
$$u^-\le K\left(\frac{qK}{q-1}\right)^q\mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2}+\sigma_{2,n_0}\otimes\delta_{\{t=0\}}]\right)^q\right]+K\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2}+\sigma_{2,n_0}\otimes\delta_{\{t=0\}}].$$

Thus, thanks to Theorem 2.36 there exists $\varepsilon_0 = \varepsilon_0(N, K, \delta, q)$ such that if for every compact sets $E \subset \mathbb{R}^{N+1}$,

$$|\mu_{n_0,i}|(E) + (|\sigma_{i,n_0}| \otimes \delta_{\{t=0\}})(E) \leq \varepsilon_0 \mathrm{Cap}_{E_2^{2T_0,\delta},q'}(E). \tag{4.10}$$

then $\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \in L^q(\mathbb{R}^{N+1})$ and

$$\mathbb{I}_2^{2T_0,\delta}\left[\left(\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}]\right)^q\right] \leq \frac{(q-1)^{q-1}}{(Kq)^q}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}]$$

for $i = 1, 2$ which implies $u \in L^q(\Omega_T)$ and

$$u^+ \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0}[\mu_{n_0,1} + \sigma_{1,n_n} \otimes \delta_{\{t=0\}}], \quad u^- \leq \frac{qK}{q-1}\mathbb{I}_2^{2T_0}[\mu_{n_0,2} + \sigma_{2,n_0} \otimes \delta_{\{t=0\}}].$$

Now we assume that (4.10) is satisfied, so S is well defined. Therefore, if we can show that the map $S : J \to J$ is continuous and $S(J)$ is pre-compact under the strong topology of $L^q(\Omega_T)$ then by the Schauder Fixed Point Theorem, S has a fixed point on J. Hence the problem (1.20) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$.

Now we show that **S is continuous**. Let $\{v_n\}$ be a sequence in J such that v_n converges strongly in $L^q(\Omega_T)$ to a function $v \in J$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^q(\Omega_T)$.

By Proposition 1.36, there exists a subsequence of $\{u_n\}$, still denoted by it, converging to u a.e in Ω_T. Since

$$|u_n| \leq \sum_{i=1,2}\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,i} + \sigma_{i,n_0} \otimes \delta_{\{t=0\}}] \in L^q(\Omega_T) \quad \forall \in \mathbb{N}$$

Thanks to the Dominated Convergence Theorem, we have $u_n \to u$ in $L^q(\Omega_T)$. Hence, thanks to Theorem 1.37 we get $u = S(v)$.
Next we show that **S is pre-compact**. Indeed if $\{u_n\} = \{S(v_n)\}$ is a sequence in $S(J)$. By Proposition 1.36, there exists a subsequence of $\{u_n\}$, still denoted by it, converging to u a.e in Ω_T. Again, the Dominated Convergence Theorem we get $u_n \to u$ in $L^q(\Omega_T)$. So **S is pre-compact**.

Next, thanks to Corollary 2.39 and Remark 2.40 we have

$$[\mu_{n,i} + \sigma_{i,n} \otimes \delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \lesssim [|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \quad \forall n \in \mathbb{N}, i = 1, 2.$$

In addition, by the proof of Corollary 2.39 we get

$$\mathrm{Cap}_{E_2^{2T_0,\delta},q'}(E) \lesssim_{T_0} \mathrm{Cap}_{\mathcal{G}_2,q'}(E)$$

for every compact set E with $\mathrm{diam}(E) \leq 2T_0$. Thus, there is $\varepsilon_1 = \varepsilon_1(N, K, \delta, q, T_0)$ such that if

$$[|\mu| + |\sigma| \otimes \delta_{\{t=0\}}]_{\mathfrak{M}^{\mathcal{G}_2,q'}} \leq \varepsilon_1, \tag{4.11}$$

then (4.10) holds for any $n_0 \in \mathbb{N}$.

Now we suppose that (4.11) holds, it is equivalent to (1.31), by Remark 2.34. Therefore, for any $n \in \mathbb{N}$ there exists a renormalized solution u_n of

$$\begin{cases} (u_n)_t - \mathrm{div}(A(x, t, \nabla u_n)) = |u_n|^{q-1}u_n + \mu_{n,1} - \mu_{n,2} & \text{in} \quad \Omega_T, \\ u_n = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ u_n(0) = \sigma_{1,n} - \sigma_{2,n} \quad \text{in} \quad \Omega, \end{cases} \tag{4.12}$$

which satisfies

$$-\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n,2}+\sigma_{2,n}\otimes\delta_{\{t=0\}}]\le u_n\le\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n,1}+\sigma_{1,n}\otimes\delta_{\{t=0\}}].$$

Thus, for every $(x,t)\in\Omega_T$,

$$\begin{aligned}-\frac{qK}{q-1}\varphi_n*\mathbb{I}_2^{2T_0,\delta}[\mu^-](x,t)-\frac{qK}{q-1}\varphi_{1,n}*(\mathbb{I}_2^{2T_0,\delta}[\sigma^-\otimes\delta_{\{t=0\}}](.,t))(x)\le u_n(x,t)\\ \le\frac{qK}{q-1}\varphi_n*(\mathbb{I}_2^{2T_0,\delta}[\mu^-])(x,t)+\frac{qK}{q-1}\varphi_{1,n}*(\mathbb{I}_2^{2T_0,\delta}[\sigma^-\otimes\delta_{\{t=0\}}](.,t))(x).\end{aligned}$$

Since $\varphi_n*\mathbb{I}_2^{2T_0,\delta}[\mu^\pm](x,t),\varphi_{1,n}*(\mathbb{I}_2^{2T_0,\delta}[\sigma^\pm\otimes\delta_{\{t=0\}}](.,t))(x)$ converge to $\mathbb{I}_2^{2T_0,\delta}[\mu^\pm](x,t),\mathbb{I}_2^{2T_0,\delta}[\sigma^\pm\otimes\delta_{\{t=0\}}](x,t)$ in $L^q(\mathbb{R}^{N+1})$ as $n\to\infty$, respectively, so $|u_n|^q$ is equi-integrable.
By Proposition 1.36, there exists a subsequence of $\{u_n\}$, still denoted by its, converging to u a.e in Ω_T. It follows $|u_n|^{q-1}u_n\to|u|^{q-1}u$ in $L^1(\Omega_T)$.
Consequently, by Proposition 1.36 and Theorem 1.37, we obtain that u is a distributional solution (a renormalized solution if $\sigma\in L^1(\Omega)$) of (1.20) with data μ, σ, and satisfies (1.32). Furthermore, by Corollary 2.39 we have

$$\left[\left(\mathbb{I}_2^{2T_0,\delta}[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}]\right)^q\right]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\sim_{T_0}\left[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}\right]^q_{\mathfrak{M}^{\mathcal{G}_2,q'}}$$

which implies $[|u|^q]_{\mathfrak{M}^{\mathcal{G}_2,q'}}\lesssim_{T_0}1$ and we get (1.33). The proof is complete. □

Remark 4.4. In view of above proof, we can see that

i: The Theorem 1.9 also holds when we replace assumption (1.31) by

$$|\mu|(E)\le\varepsilon_0\mathrm{Cap}_{\mathcal{H}_2,q'}(E)\quad\text{and}\quad|\sigma|(F)\le\varepsilon_0\mathrm{Cap}_{\mathbf{I}_{\frac{2}{q}},q'}(F),$$

for every compact sets $E\subset\mathbb{R}^{N+1},F\subset\mathbb{R}^N$ and $\varepsilon_0>0$ small enough.

ii: If $\sigma\equiv0$ and $\mathrm{supp}(\mu)\subset\overline{\Omega}\times[a,T]$, $a>0$, then we can show that the solution u in Theorem 1.9 satisfies $u=0$ in $\Omega\times(0,a)$ since we can replace the set E by E':

$$\begin{aligned}E'=\Big\{u\in L^q(\Omega_T):u=0\text{ in }\Omega\times(0,a)\quad\text{and}\quad u^+\le\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,1}+\sigma_{1,n_0}\otimes\delta_{\{t=0\}}],\\ u^-\le\frac{qK}{q-1}\mathbb{I}_2^{2T_0,\delta}[\mu_{n_0,2}+\sigma_{2,n_0}\otimes\delta_{\{t=0\}}]\Big\}.\end{aligned}$$

4.1.2. Quasilinear Lane-Emden Parabolic Equations in $\mathbb{R}^N\times(0,\infty)$ and $\mathbb{R}^{N+1}$. This section is devoted to prove Theorem 1.12 and Theorem 1.14.

Proof of the Theorem 1.12. Since ω is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,q'}$ in $\mathbb{R}^{N+1}$, so does $|\omega|$. Set $D_n=B_n(0)\times(-n^2,n^2)$. From the proof of Theorem 1.8, there exist renormalized solutions u_n,v_n of

$$\begin{cases}(u_n)_t-\mathrm{div}(A(x,t,\nabla u_n))+|u_n|^{q-1}u_n=\chi_{D_n}\omega & \text{in}\quad D_n,\\ u_n=0\quad\text{on}\quad\partial B_n(0)\times(-n^2,n^2),\\ u_n(-n^2)=0\quad\text{in}\quad B_n(0),\end{cases}$$

and

$$\begin{cases}(v_n)_t-\mathrm{div}(A(x,t,\nabla v_n))+v_n^q=\chi_{D_n}|\omega| & \text{in}\quad D_n,\\ v_n=0\quad\text{on}\quad\partial B_n(0)\times(-n^2,n^2),\\ v_n(-n^2)=0\quad\text{in}\quad B_n(0),\end{cases}$$

relative to decompositions (f_n, g_n, h_n) of $\chi_{D_n}\omega_0$ and $(\overline{f}_n, \overline{g}_n, \overline{h}_n)$ of $\chi_{B_n(0)\times(0,n^2)}|\omega_0|$, satisfied (1.80), (1.81) in Proposition 1.47 with $1 < q_0 < q$, $L(u_n) = |u_n|^{q-1}u_n$, $L(v_n) = v_n^q$ and μ is replaced by $\chi_{D_n}\omega$ and $\chi_{D_n}|\omega|$ respectively. Moreover, there hold

$$-KI_2[\omega^-] \le u_n \le KI_2[\omega^+], 0 \le v_n \le KI_2[|\omega|] \quad \text{in} \quad D_n, \tag{4.13}$$

and $v_{n+1} \ge v_n$, $|u_n| \le v_n$ in D_n.
By Remark 1.40, we can assume that

$$||f_n||_{L^1(D_i)} + ||g_n||_{L^2(D_i,\mathbb{R}^N)} + |||h_n| + |\nabla h_n|||_{L^2(D_i)} \le 2|\omega|(D_{i+1}),$$
$$||\overline{f}_n||_{L^1(D_i)} + ||\overline{g}_n||_{L^2(D_i,\mathbb{R}^N)} + |||\overline{h}_n| + |\nabla \overline{h}_n|||_{L^2(D_i)} \le 2|\omega|(D_{i+1}),$$

for any $i = 1, ..., n-1$ and $h_n, \overline{h}_n$ are convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$. On the other hand, since u_n, v_n satisfy (1.80) in Proposition 1.47 with $1 < q_0 < q$, $L(u_n) = |u_n|^{q-1}u_n$, $L(v_n) = v_n^q$ and thanks to Hölder inequality: for any $\varepsilon \in (0, 1)$

$$(|u_n| + 1)^{q_0} \le \varepsilon|u_n|^q + c(\varepsilon), (|v_n| + 1)^{q_0} \le \varepsilon|v_n|^q + c(\varepsilon),$$

we get

$$\int_{D_i} |u_n|^q dxdt + \int_{D_i} |u_n|^{q_0} dxdt + \int_{D_i} v_n^q dxdt + \int_{D_i} v_n^{q_0} dxdt \le C(i) + c|\omega|(D_{i+1}). \tag{4.14}$$

for $i = 1, ..., n-1$, where the constant $C(i)$ depends on $N, \Lambda_1, \Lambda_2, q_0, q$ and i.
Consequently, we can apply Proposition 1.48 with $\mu_n = -|u_n|^{q-1}u_n + \chi_{D_n}\omega, -v_n^q + \chi_{D_n}|\omega|$ and obtain that there are subsequences of u_n, v_n, still denoted by them, converging to some u, v in $L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$ resp. So, $\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}} \in L^1_{loc}(\mathbb{R}^{N+1})$ for all $\alpha > 0$ and $u \in L^q_{loc}(\mathbb{R}^{N+1})$ satisfies (1.35). In addition, using Hölder inequality we get $u \in L^\gamma_{\text{loc}}(\mathbb{R}; W^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$.

Thanks to (4.14) and the Monotone Convergence Theorem we get $v_n \to v$ in $L^q_{loc}(\mathbb{R}^{N+1})$. After, we also have $u_n \to u$ in $L^q_{loc}(\mathbb{R}^{N+1})$ by $|u_n| \le v_n$ and the Dominated Convergence Theorem.
Consequently, u is a distributional solution of problem (1.34) which satisfies (1.35). If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0, \infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by the proof of Theorem 1.8 we obtain that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N\times[0,\infty)}$ is a distributional solution to (1.36). This completes the proof. □

Proof of the Theorem 1.14. By the proof of Theorem 1.9 and Remark 4.4, 2.34, there exists a constant $\varepsilon_0 = \varepsilon_0(N, q, \Lambda_1, \Lambda_2)$ such that if ω satisfies for every compact set $E \subset \mathbb{R}^{N+1}$,

$$|\omega|(E) \le \varepsilon_0 \text{Cap}_{\mathcal{H}_2,q'}(E), \tag{4.15}$$

then there is a renormalized solution u_n of

$$\begin{cases} (u_n)_t - \text{div}(A(x, t, \nabla u_n)) = |u_n|^{q-1}u_n + \chi_{D_n}\omega \quad \text{in} \quad D_n \\ u_n = 0 \quad \text{on} \quad \partial B_n(0) \times (-n^2, n^2), \\ u_n(-n^2) = 0 \quad \text{in} \quad B_n(0), \end{cases}$$

relative to a decomposition (f_n, g_n, h_n) of $\chi_{D_n}\omega_0$, satisfying (1.80), (1.81) in Proposition 1.47 with $q_0 = q$, $L \equiv 0$ and μ is replaced by $|u_n|^{q-1}u_n + \chi_{D_n}\omega$ and

$$-\frac{qK}{q-1}\mathbb{I}_2[\omega^-](x,t) \le u_n \le \frac{qK}{q-1}\mathbb{I}_2[\omega^+](x,t) \tag{4.16}$$

for a.e (x,t) in D_n and $I_2[\omega^\pm] \in L^q_{loc}(\mathbb{R}^{N+1})$.
Besides, thanks to Remark 1.40, we can assume that f_n, g_n, h_n satisfies (2.79) in proof of Theorem (1.5) and h_n is convergent in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$.

Consequently, we can apply Proposition 1.48 and obtain that there exist a subsequence of u_n, still denoted by it, converging to some u a.e in $\mathbb{R}^{N+1}$ and in $L^1_{loc}(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}^N))$. Also, $u_n \to u$ in $L^q_{\text{loc}}(\mathbb{R}^{N+1})$ by Dominated Convergence Theorem, $\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}} \in L^1_{loc}(\mathbb{R}^{N+1})$ for all $\alpha > 0$. Using Hölder inequality we get $u \in L^\gamma_{\text{loc}}(\mathbb{R}; W^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))$ for any $1 \le \gamma < \frac{2q}{q+1}$.

Thus we obtain that u is a distributional solution of (1.38) which satisfies (1.39). Since (4.15) holds, thus by Theorem 2.36 we get

$$[(\mathbb{I}_2[|\omega|])^q]_{\mathfrak{M}^{\mathcal{H}_2,q'}} \sim [|\omega|]^q_{\mathfrak{M}^{\mathcal{H}_2,q'}},$$

so we have $[|u|^q]_{\mathfrak{M}^{\mathcal{H}_2,q'}} \lesssim 1$. It follows (1.41).

If $\omega = \mu + \sigma \otimes \delta_{\{t=0\}}$ with $\mu \in \mathfrak{M}(\mathbb{R}^N \times (0,\infty))$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then by Remark 4.4 we obtain that $u_n = 0$ in $B_n(0) \times (-n^2, 0)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. Therefore, clearly $u|_{\mathbb{R}^N \times [0,\infty)}$ is a distributional solution to (1.40). The proof is complete. □

4.2. Quasilinear riccati type parabolic equations

4.2.1. Quasilinear Riccati Type Parabolic Equation in Ω_T. We provide below only the proof of Theorem 1.26, 1.28. The proof of Theorem 1.27 can be proceeded by a similar argument.

Proof of Theorem 1.26. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T)$ be as in the proof of Theorem 1.1. We have $|\mu_n|(\Omega_T) \le |\mu|(\Omega_T)$ for any $n \in \mathbb{N}$. Let $\sigma_n \in C_c^\infty(\Omega)$ converge to σ in the narrow topology of measures and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$ such that $||\sigma_n||_{L^1(\Omega)} \le |\sigma|(\Omega)$. For $n_0 \in \mathbb{N}$, we prove that the problem (1.61) has a solution with data $\mu = \mu_{n_0}$ and $\sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E}_\Lambda = \left\{ u \in L^1(0,T,W^{1,1}_0(\Omega)) : |||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \le \Lambda \right\},$$

where $\Lambda > 0$, $L^{\frac{N+2}{N+1},\infty}(\Omega_T)$ is the Lorentz space with norm

$$||f||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} := \sup_{0<|D|<\infty} \left(|D|^{-\frac{1}{N+2}} \int_{D\cap\Omega_T} |f| \right).$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W^{1,1}_0(\Omega))$ and convex.
We consider a map $S : \mathbf{E}_\Lambda \to \mathbf{E}_\Lambda$ defined for each $v \in \mathbf{E}_\Lambda$ by $S(v) = u$, where $u \in L^1(0,T,W^{1,1}_0(\Omega))$ is the unique solution of

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = |\nabla v|^q + \mu_{n_0} & \text{in} \quad \Omega_T, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = \sigma_{n_0} \quad \text{in } \Omega. \end{cases} \tag{4.17}$$

By Remark 1.33, we have

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \lesssim |||\nabla v|^q||_{L^1(\Omega_T)} + |\mu_{n_0}|(\Omega_T) + ||\sigma_{n_0}||_{L^1(\Omega)}.$$

It leads to

$$\begin{aligned} |||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} &\lesssim |\Omega_T|^{1-\frac{q(N+1)}{N+2}} |||\nabla v|||^q_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} + |\mu|(\Omega_T) + |\sigma|(\Omega) \\ &\lesssim |\Omega_T|^{1-\frac{q(N+1)}{N+2}} \Lambda^q + |\mu|(\Omega_T) + |\sigma|(\Omega). \end{aligned}$$

Thus, we now suppose that

$$|\Omega_T|^{-1+\frac{q'}{N+2}} \left(|\mu|(\Omega_T) + |\sigma|(\Omega)\right) \leq \varepsilon_0,$$

then

$$|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda := c(|\mu|(\Omega) + |\sigma|(\Omega)),$$

for some $\varepsilon_0 > 0, c > 0$. So, S is well defined.
Now we show that S **is continuous**. Let $\{v_n\}$ be a sequence in $\mathbf{E}_\Lambda$ such that v_n converges strongly in $L^1(0,T,W_0^{1,1}(\Omega))$ to a function $v \in \mathbf{E}_\Lambda$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$. We have

$$\left\{ \begin{array}{l} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = |\nabla v_n|^q + \mu_{n_0} \quad \text{in} \quad \Omega_T, \\ u_n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_n(0) = \sigma_{n_0} \quad \text{in } \Omega, \end{array} \right. \tag{4.18}$$

satisfying

$$|||\nabla u_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda, |||\nabla v_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda.$$

Thus, $|\nabla v_n|^q \to |\nabla v|^q$ in $L^1(\Omega_T)$. Therefore, it is easy to see that we get $u_n \to S(v)$ in $L^1(0,T,W_0^{1,1}(\Omega))$ by Theorem 1.37.
Next we show that S **is pre-compact**. Indeed if $\{u_n\} = \{S(v_n)\}$ is a sequence in $S(\mathbf{E}_\Lambda)$. By Proposition 1.36, there exists a subsequence of $\{u_n\}$ converging to some u in $L^1(0,T,W_0^{1,1}(\Omega))$. Consequently, by the Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}_\Lambda$. So, the problem (1.61) has a solution with data μ_{n_0}, σ_{n_0}.

Therefore, for any $n \in \mathbb{N}$, there exists a renormalized solution u_n of

$$\left\{ \begin{array}{l} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) = |\nabla u_n|^q + \mu_n \quad \text{in} \quad \Omega_T, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{in } \Omega, \end{array} \right. \tag{4.19}$$

which satisfies

$$|||\nabla u_n|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda.$$

Thanks to Proposition 1.36, there exists a subsequence of $\{u_n\}$ converging to u in $L^1(0,T,W_0^{1,1}(\Omega))$. So, $|||\nabla u|||_{L^{\frac{N+2}{N+1},\infty}(\Omega_T)} \leq \Lambda$ and $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$ since $\{|\nabla u_n|^q\}$ is equi-integrable. It follows the result by Proposition 1.36 and Theorem 1.37. ☐

Proof of Theorem 1.28. Let $\{\mu_n\} \subset C_c^\infty(\Omega_T), \sigma_n \in C_c^\infty(\Omega)$ be as in the proof of Theorem 1.1. We have $|\mu_n| \leq \varphi_n * |\mu|, |\sigma_n| \leq \varphi_{1,n} * |\sigma|$ for any $n \in \mathbb{N}$, $\{\varphi_n\}$, $\{\varphi_{1,n}\}$ are sequences of standard mollifiers in $\mathbb{R}^{N+1}, \mathbb{R}^N$ respectively. Set $\omega_n = |\mu_n| + |\sigma_n| \otimes \delta_{\{t=0\}}$ and $\omega = |\mu| + |\sigma| \otimes \delta_{\{t=0\}}$.

For $n_0 \in \mathbb{N}$, $\varepsilon > 0$, we prove that the problem (1.61) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$. Now we put

$$\mathbf{E} = \left\{ u \in L^q(0,T,W_0^{1,q}(\Omega)) : \mathbb{I}_1[|\nabla u|^q \chi_{\Omega_T}] \le \mathbb{I}_1[\omega] \right\}.$$

By Fatou's lemma, $\mathbf{E}_\Lambda$ is closed under the strong topology of $L^1(0,T,W_0^{1,1}(\Omega))$ and convex.

We consider a map $S : \mathbf{E} \to \mathbf{E}$ defined for each $v \in \mathbf{E}$ by $S(v) = u$, where $u \in L^q(0,T,W_0^{1,q}(\Omega))$ is the unique solution of problem (4.17). By Corollary 3.18 and 3.20, there exist $\delta = \delta(N,\Lambda_1,\Lambda_2,q) \in (0,1)$ and $s_0 = s_0(N,\Lambda_1,\Lambda_2) > 0$ such that Ω is (δ,R_0)- Reifenberg flat domain and $[\mathcal{A}]_{s_0}^{R_0} \le \delta$ for some R_0 we have

$$\mathbb{I}_1[|\nabla u|^q \chi_{\Omega_T}] \lesssim \mathbb{I}_1[\mathbb{M}_1[|\nabla v|^q + \omega_{n_0}]^q \chi_{\Omega_T}], \tag{4.20}$$

and

$$\int_{\lambda_0}^{\infty} \lambda^{q-1} \left|\{\mathbb{M}(|\nabla u|) > \Lambda\lambda\} \cap \Omega_T\right| d\lambda \le c \int_{\lambda_0/c}^{\infty} \lambda^{q-1} \left|\{\mathbb{M}_1[|\nabla v|^q + \omega_{n_0}] > \lambda\} \cap \Omega_T\right| d\lambda \tag{4.21}$$

for any $\lambda_0 \ge 0$ where $c = c(N,\Lambda_1,\Lambda_2,q,T_0/R_0,T_0)$.

We will prove that if

$$[\omega]_{\mathfrak{M}^{\mathcal{G}_{1,q'}}} \le \varepsilon_0 \tag{4.22}$$

for some $\varepsilon_0 > 0$ small enough, then S is well defined.

By Corollary 2.39,

$$[\omega_{n_0}]_{\mathfrak{M}^{\mathcal{G}_{1,q'}}} \lesssim [\omega]_{\mathfrak{M}^{\mathcal{G}_{1,q'}}} \lesssim \varepsilon_0. \tag{4.23}$$

Thus, thanks to Theorem 2.36

$$\mathbb{I}_1^{2T_0,1}[\mathbb{I}_1^{2T_0,1}[\omega]^q] \lesssim \varepsilon_0^{q-1} \mathbb{I}_1^{2T_0,1}[\omega], \qquad \mathbb{I}_1^{2T_0,1}[\mathbb{I}_1^{2T_0,1}[\omega_{n_0}]^q] \lesssim \varepsilon_0^{q-1} \mathbb{I}_1^{2T_0,1}[\omega_{n_0}].$$

Note that

$$\mathbb{I}_1^{2T_0,1}[\omega_{n_0}] \le \varphi_{n_0} \star \mathbb{I}_1^{2T_0,1}[\omega] \overset{(2.35)}{\lesssim} \mathbb{I}_1^{2T_0,1}[\omega].$$

Thus, (4.20) and $\mathbb{I}_1[|\nabla v|^q \chi_{\Omega_T}] \le \mathbb{I}_1[\omega]$ imply

$$\begin{aligned} \mathbb{I}_1[|\nabla u|^q \chi_{\Omega_T}] &\lesssim \mathbb{I}_1[\mathbb{I}_1[\omega]^q \chi_{\Omega_T}] + \mathbb{I}_1[\mathbb{M}_1[\omega_{n_0}]^q \chi_{\Omega_T}] \\ &\lesssim \mathbb{I}_1^{2T_0,1}[\mathbb{I}_1^{2T_0,1}[\omega]^q] + \mathbb{I}_1^{2T_0,1}[\mathbb{I}_1^{2T_0,1}[\omega_{n_0}]^q] \\ &\lesssim \varepsilon_0^{q-1} \mathbb{I}_1^{2T_0,1}[\omega] \\ &\le \mathbb{I}_1[\omega] \end{aligned}$$

for $\varepsilon_0 > 0$ small enough. Hence, S is well defined.

Moreover, it follows from (4.21) that

$$\int_{\lambda_0}^{\infty} \lambda^{q-1} \left|\{\mathbb{M}(|\nabla S(v)|) > \lambda\} \cap \Omega_T\right| d\lambda \le c \int_{\lambda_0/c}^{\infty} \lambda^{q-1} \left|\{\mathbb{I}_1[\omega] > \lambda\} \cap \Omega_T\right| d\lambda \tag{4.24}$$

for any $\lambda_0 \ge 0$ and $v \in \mathbf{E}$.

Now we assume (4.22). We show that S **is continuous**. Let $\{v_n\}$ be a sequence in $\mathbf{E}$ such that v_n converges strongly in $L^q(0,T,W_0^{1,q}(\Omega))$ to a function $v \in \mathbf{E}$. Set $u_n = S(v_n)$. We need to show that $u_n \to S(v)$ in $L^q(0,T,W_0^{1,q}(\Omega))$. Thanks to Proposition 1.36, $u_n \to S(v)$ a.e. By (4.24), $\{|\nabla u_n|^q\}$ is equi-integrable. Thus, $u_n \to S(v)$ in $L^q(0,T,W_0^{1,q}(\Omega))$. Similarly, we also obtain that S **is pre-compact**.

Thus, by the Schauder Fixed Point Theorem, S has a fixed point on $\mathbf{E}$. Hence the problem (1.61) has a solution with data $\mu = \mu_{n_0}, \sigma = \sigma_{n_0}$. This means, for any $n \in \mathbb{N}$, there exists a solution $u_n \in \mathbf{E}$ of problem (4.19). Since $u_n \in \mathbf{E}$ satisfies (4.24) with $S(v) = u_n$, so, $\{|\nabla u_n|^q\}$ is equi-integrable. By Proposition 1.36, there exists a subsequence of $\{u_n\}$ converging to some function u in $L^1(0, T, W_0^{1,1}(\Omega))$. Thus, $|\nabla u_n|^q \to |\nabla u|^q$ in $L^1(\Omega)$. The results follow by Proposition 1.36 and Theorem 1.37. The proof is complete. $\square$

4.2.2. Quasilinear Riccati Type Parabolic Equation in $\mathbb{R}^N \times (0, \infty)$ and $\mathbb{R}^{N+1}$. In this section, we provide the proofs of Theorem 1.31. We shall follow the same strategy as the proof of Theorem 1.30.

Proof of Theorem 1.31. Let $D_n = B_n(0) \times (-n^2, n^2)$, $\mu_n = \varphi_n * (\chi_{D_{n-1}} \mu)$ for any $n \geq 2$. Here $\{\varphi_n\}$ is a sequence of standard mollifiers in $\mathbb{R}^{N+1}$. We have $\mu_n \in C_c^\infty(\mathbb{R}^{N+1})$ with $\operatorname{supp}(\mu_n) \subset D_n$ and $\mu_n \to \mu$ weakly in $\mathfrak{M}(\mathbb{R}^{N+1})$.
Assume that

$$[\mu]_{\mathfrak{M}^{\mathcal{H}_1, q'}} \leq \varepsilon_0. \tag{4.25}$$

By Corollary 2.39,

$$[\mu_n]_{\mathfrak{M}^{\mathcal{H}_1, q'}} \lesssim [\mu]_{\mathfrak{M}^{\mathcal{H}_1, q'}} \lesssim \varepsilon_0.$$

Therefore, thanks to Theorem 1.28, there exist $\delta = \delta(N, \Lambda_1, \Lambda_2, q) \in (0, 1)$ and $s_0 = s_0(N, \Lambda_1, \Lambda_2) > 0$ such that if $[\mathcal{A}]_{s_0}^\infty \leq \delta$, then for any n, the problem

$$\begin{cases} u_t - \operatorname{div}(A(t, x, \nabla u)) = |\nabla u|^q + \mu_n & \text{in} \quad D_n, \\ u = 0 \quad \text{on} \quad \partial B_n(0) \times (-n^2, n^2), \\ u(-n^2) = 0 \quad \text{in} \quad B_n(0), \end{cases}$$

has a solution $u_n \in \mathbf{E}_n$ with data $\mu = \mu_n$ provided $\varepsilon_0 > 0$ small enough. Here

$$\mathbf{E}_n = \left\{ v \in L^q(-n^2, n^2, W_0^{1,q}(B_n(0))) : \mathbb{I}_1[|\nabla v|^q \chi_{D_n}] \leq \mathbb{I}_1[\mu] \right\}.$$

Moreover, u_n satisfies

$$\int_{K \cap D_n} |\nabla u_n|^q dxdt \lesssim \operatorname{Cap}_{\mathcal{H}_1, q'}(K) \quad \forall \text{ compact set } K \subset \mathbb{R}^{N+1}.$$

By (4.24), one has

$$\int_{\lambda_0}^\infty \lambda^{q-1} \left| \{\mathbb{M}(|\nabla u_n|) > \lambda\} \cap D_n \right| d\lambda \leq c \int_{\lambda_0/c}^\infty \lambda^{q-1} \left| \{\mathbb{I}_1[\mu] > \lambda\} \right| d\lambda, \tag{4.26}$$

for some $c > 0$. So, $\{|\nabla u_n|^q\}_{n \geq n_0}$ is equi-integrable in D_{n_0} for any $n_0 > 0$.
Hence, by Corollary 1.49, there exists a subsequence of $\{u_n\}$ converging to a distributional solution u of (1.62) satisfying $\mathbb{I}_1[|\nabla u|^q] \leq \mathbb{I}_1[\mu]$ and

$$\int_K |\nabla u_n|^q dxdt \lesssim \operatorname{Cap}_{\mathcal{H}_1, q'}(K) \quad \forall \text{ compact set } K \subset \mathbb{R}^{N+1}.$$

Furthermore, if $\operatorname{supp}(\mu) \subset \mathbb{R}^N \times [0, \infty]$ and $\sigma \in \mathfrak{M}(\mathbb{R}^N)$, then $u_n = 0$ in $B_n(0) \times (-n^2, -\frac{2}{n})$ where $\operatorname{supp}(\omega_n) \subset \mathbb{R}^N \times (-\frac{2}{n}, \infty)$. So, $u = 0$ in $\mathbb{R}^N \times (-\infty, 0)$. The proof is complete. $\square$

CHAPTER 5

Appendix

Proof of the Remark 1.7. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $0 < \alpha < N+2$ if $\mathbb{I}_\alpha[\omega](x_0,t_0) < \infty$ for some $(x_0,t_0) \in \mathbb{R}^{N+1}$ then for any $0 < \beta \leq \alpha$, $\mathbb{I}_\beta[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$. Indeed, by Remark 2.28 we have $\mathbb{I}_\alpha[\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$ for any $0 < s < \frac{N+2}{N+2-\beta}$.
Take $0 < \beta \leq \alpha$ and $0 < s < \frac{N+2}{N+2-\beta}$. For $R > 0$, by Proposition 2.4 we have $\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega] \in L^s_{\text{loc}}(\mathbb{R}^{N+1})$. Thus,

$$\begin{aligned}
&\int_{\tilde{Q}_R(0,0)} (\mathbb{I}_\beta[\omega](x,t))^s \,dxdt \\
&\quad\lesssim \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega](x,t)\right)^s dxdt + \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)^c}\omega](x,t)\right)^s dxdt \\
&\quad\lesssim \int_{\tilde{Q}_R(0,0)} \left(\mathbb{I}_\beta[\chi_{\tilde{Q}_{2R}(0,0)}\omega](x,t)\right)^s dxdt + R^{-s(\alpha-\beta)} \int_{\tilde{Q}_R(0,0)} (\mathbb{I}_\alpha[\omega](x,t))^s \,dxdt \\
&\quad< \infty.
\end{aligned}$$

For $0 < \beta < \alpha < N+2$, we consider

$$\omega(x,t) = \sum_{k=4}^{\infty} \frac{a_k}{|\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)|} \chi_{\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)}(x,t),$$

where $a_k = 2^{n(N+2-\theta)}$ if $k = 2^n$ and $a_k = 0$ otherwise with $\theta \in (\beta,\alpha]$.
It is easy to see that $\mathbb{I}_\alpha[\omega] \equiv \infty$ and $\mathbb{I}_\beta[\omega] < \infty$ in $\mathbb{R}^{N+1}$. □

Proof of the Remark 1.24. For $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, since $\mathbb{I}_2[\omega] \lesssim I_1[I_1[\omega]]$ thus:
If $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ with $1 < s < N+2$, then by Proposition 2.4

$$||\mathbb{I}_2[\omega]||_{L^{\frac{s(N+1)}{N+2-s},\infty}(\mathbb{R}^{N+1})} \lesssim ||\mathbb{I}_1[\omega]||_{L^{s,\infty}(\mathbb{R}^{N+1})} < \infty.$$

If $\mathbb{I}_1[\omega] \in L^{N+2,\infty}(\mathbb{R}^{N+1})$, then by Theorem 2.3, one has $\mathbb{I}_2[\omega] \in L^{s_0}_{\text{loc}}(\mathbb{R}^{N+1}) \forall s_0 > 1$. So, $\mathbb{I}_2[\omega] < \infty$ a.e in $\mathbb{R}^{N+1}$ if $\mathbb{I}_1[\omega] \in L^{s,\infty}(\mathbb{R}^{N+1})$ with $1 < s \leq N+2$.
For $s > N+2$, there exists $\omega \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $\mathbb{I}_2[\omega] \equiv \infty$ in $\mathbb{R}^{N+1}$ and $\mathbb{I}_1[\omega] \in L^s(\mathbb{R}^{N+1})$. Indeed, consider

$$\omega(x,t) = \sum_{k=1}^{\infty} \frac{k^{N-1}}{|\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)|} \chi_{\tilde{Q}_{k+1}(0,0)\backslash\tilde{Q}_k(0,0)}(x,t).$$

We have for $(x,t)\in\mathbb{R}^{N+1}$ and $n_0\in\mathbb{N}$ with $n_0>\log_2(\max\{|x|,\sqrt{2|t|}\})$

$$\begin{aligned}\mathbb{I}_2[\omega](x,t)&\gtrsim\sum_{n_0}^{\infty}\frac{\omega(\tilde{Q}_{2^n}(x,t))}{2^{nN}}\gtrsim\sum_{n_0}^{\infty}\frac{\omega(\tilde{Q}_{2^{n-1}}(0,0))}{2^{nN}}\\&\gtrsim\sum_{n_0}^{\infty}\frac{\sum_{k=1}^{2^{n-1}-1}k^{N-1}}{2^{nN}}\gtrsim\sum_{k=1}^{\infty}\left(\sum_{n_0}^{\infty}\chi_{k\le 2^{n-1}-1}\frac{1}{2^{nN}}\right)k^{N-1}\\&\gtrsim\sum_{k=n_0}^{\infty}k^{-1}=\infty.\end{aligned}$$

On the other hand, for $s_1>\frac{N+2}{2}$

$$\int_{\mathbb{R}^{N+1}}\omega^{s_1}dxdt=c\sum_{k=1}^{\infty}\frac{k^{s(N-1)}}{((k+1)^{N+2}-k^{N+2})^{s_1-1}}\lesssim\sum_{k=1}^{\infty}\frac{k^{s_1(N-1)}}{k^{(s_1-1)(N+1)}}<\infty,$$

since $(s_1-1)(N+1)-s_1(N-1)>1$. Thus,

$$||\mathbb{I}_1[\omega]||_{L^s(\mathbb{R}^{N+1})}\lesssim||\omega||_{L^{\frac{s(N+2)}{N+2+s}}(\mathbb{R}^{N+1})}<\infty.$$

□

Proof of the Proposition 1.47. We will use an idea in [**10, 11**] to prove 1.80. For $S'\in W^{1,\infty}(\mathbb{R})$ with $S(0)=0$, $S''\ge0$, $S'(\tau)\tau\ge0$ for all $\tau\in\mathbb{R}$ and $||S'||_{L^\infty(\mathbb{R})}\le1$ we have

$$\begin{aligned}&-\int_D\eta_tS(u)dxdt+\int_DS'(u)A(x,t,\nabla u)\nabla\eta dxdt\\&\qquad+\int_DS''(u)\eta A(x,t,\nabla u)\nabla udxdt+\int_DS'(u)\eta L(u)dxdt=\int_DS'(u)\eta d\mu.\end{aligned}$$

Thus,

$$\begin{aligned}&\Lambda_2\int_DS''(u)\eta|\nabla u|^2dxdt+\int_DS'(u)\eta L(u)dxdt\\&\qquad\le\Lambda_1\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt.\end{aligned}$$

a. We choose $S'\equiv\varepsilon^{-1}T_\varepsilon$ for $\varepsilon>0$ and let $\varepsilon\to0$ we will obtain

$$\int_D\eta|L(u)|dxdt\le\Lambda_1\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt. \tag{5.1}$$

b. for $S'(u)=(1-(|u|+1)^{-\alpha})\text{sign}(u)$ for $\alpha>0$ then

$$\int_D\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt\lesssim\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt.$$

Using Hölder's inequality, we have

$$\int_D|\nabla u||\nabla\eta|dxdt+\int_D\frac{|\nabla u|^2}{(|u|+1)^{\alpha+1}}\eta dxdt\lesssim B. \tag{5.2}$$

c. for $S'(u)=\frac{-k+\delta+|u|}{2\delta}\text{sign}(u)\chi_{k-\delta<|u|<k+\delta}+\text{sign}(u)\chi_{|u|\ge k+\delta}$, $0<\delta\le k$ then

$$\frac{1}{2\delta}\int_{k-\delta<|u|<k+\delta}|\nabla u|^2\eta dxdt\lesssim\int_D|\nabla u||\nabla\eta|dxdt+\int_D\eta d|\mu|+\int_D|\eta_t||u|dxdt. \tag{5.3}$$

In particular,
(5.4)
$$\frac{1}{k}\int_D |\nabla T_k(u)|^2\eta dxdt \lesssim \int_D |\nabla u||\nabla\eta|dxdt + \int_D \eta d|\mu| + \int_D |\eta_t||u|dxdt \quad \forall k>0.$$

Consequently, we deduce (1.80) from (5.1)-(5.4).
Next, take $\varphi \in C_c^\infty(D)$ and $S'(u) = \chi_{|u|\le k-\delta} + \frac{k+\delta-|u|}{2\delta}\chi_{k-\delta<|u|<k+\delta}$, $S(0)=0$ we have

$$\begin{aligned}
-\int_D \varphi_t\eta S(u)dxdt + \int_D S'(u)\eta A(x,t,\nabla u)\nabla\varphi dxdt + \int_D S'(u)\varphi A(x,t,\nabla u)\nabla\eta dxdt\\
-\frac{1}{2\delta}\int_{k-\delta<|u|<k+\delta} \text{sign}(u)\varphi\eta A(x,t,\nabla u)\nabla u dxdt + \int_D S'(u)\varphi\eta L(u)dxdt\\
= \int_D S'(u)\varphi\eta d\mu + \int_D \varphi\eta_t S(u)dxdt.
\end{aligned}$$

Combining with (5.1), (5.2) and (5.3), we get

$$-\int_D \varphi_t\eta S(u)dxdt + \int_D S'(u)\eta A(x,t,\nabla u)\nabla\varphi dxdt \lesssim ||\varphi||_{L^\infty(D)}B.$$

Letting $\delta\to 0$, we get

$$-\int_D \varphi_t\eta T_k(u)dxdt + \int_D \eta A(x,t,\nabla T_k(u))\nabla\varphi dxdt \lesssim ||\varphi||_{L^\infty(D)}B.$$

We take $\varphi = T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)$,

$$\begin{aligned}
-\int_D \frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt\\
+\int_D \eta A(x,t,\nabla T_k(u))\nabla T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)dxdt \lesssim \varepsilon B.
\end{aligned}$$

Using integration by part, we have

$$\begin{aligned}
-\int_D \frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt = \frac{1}{2}\int_D (T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu))^2\eta_t dxdt\\
+\int_D T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)\langle T_k(w)\rangle_\nu\eta_t dxdt\\
+\nu\int_D \eta(T_k(w) - \langle T_k(w)\rangle_\nu)T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)dxdt.
\end{aligned}$$

Thus,

$$\begin{aligned}
-\int_D \frac{\partial}{\partial t}\left(T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)\right)\eta T_k(u)dxdt\\
\ge -\varepsilon(1+k)||\eta_t||_{L^1(D)} + \nu\int_D \eta\left(T_k(w) - \langle T_k(w)\rangle_\nu\right)T_\varepsilon(T_k(u) - \langle T_k(w)\rangle_\nu)dxdt,
\end{aligned}$$

which follows (1.81). □

Proof of the proposition 1.48. Let $S_k \in W^{2,\infty}(\mathbb{R})$ such that $S_k(z) = z$ if $|z|\le k$ and $S_k(z) = sign(z)2k$ if $|z|>2k$. For $m\in\mathbb{N}$, let η_m be the cut off function on D_m with respect to D_{m+1}. It is easy to see that from the assumption and Remark 1.35, Proposition 1.46 we get $U_{m,n}=\eta_m S_k(v_n)$, $v_n=u_n-h_n$

$$\sup_{n\geq m+1}\Big(||\left(U_{m,n}\right)_t||_{L^2(-m^2,m^2,H^{-1}(B_m(0)))+L^1(D_m)}+||U_{m,n}||_{L^2(-m^2,m^2,H^1_0(B_m(0)))}$$
$$+||u_n||_{L^1(D_m)}+||v_n||_{L^1(D_m)}\Big)\leq M_m<\infty.$$

Thus, $\{U_{m,n}\}_{n\geq m+1}$ is relatively compact in $L^1(D_m)$. On the other hand, for any $n_1,n_2\geq m+1$

$$\begin{aligned}|\{|v_{n_1}-v_{n_2}|>\lambda\}\cap D_m| &= |\{|\eta_m v_{n_1}-\eta_m v_{n_2}|>\lambda\}\cap D_m|\\ &\leq \frac{1}{k}\left(||v_{n_1}||_{L^1(D_m)}+||v_{n_2}||_{L^1(D_m)}\right)+\frac{1}{\lambda}||\eta_m S_k(v_{n_1})-\eta_m S_k(v_{n_2})||_{L^1(D_m)}\\ &\leq \frac{2M_m}{k}+\frac{1}{\lambda}||U_{m,n_1}-U_{m,n_2}||_{L^1(D_m)},\end{aligned}$$

and h_n is convergent in $L^1_{\rm loc}(\mathbb{R}^{N+1})$. So, for any $m\in\mathbb{N}$ there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ is a Cauchy sequence (in measure) in D_m. Therefore, there is a subsequence of $\{u_n\}$ converging to some function u a.e in $\mathbb{R}^{N+1}$. Clearly, $u\in L^1_{\rm loc}(\mathbb{R};W^{1,1}_{loc}(\mathbb{R}^N))$. Now, we prove that $\nabla u_n\to\nabla u$ a.e in $\mathbb{R}^{N+1}$.
From (1.81) with $D=D_{m+2}$, $\eta=\eta_m$ and $T_k(w)=T_k(\eta_{m+1}u)$ we have

$$\begin{aligned}\nu\int_{D_{m+2}}&\eta_m\left(T_k(\eta_{m+1}u)-\langle T_k(\eta_{m+1}u)\rangle_\nu\right)T_\varepsilon(T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu)dxdt\\ &+\int_{D_{m+2}}\eta_m A(x,t,\nabla T_k(u_n))\nabla T_\varepsilon(T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu)dxdt\\ &\lesssim\varepsilon(1+k)C(n,m)\forall n\geq m+2,\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}C(n,m)&=||(\eta_m)_t(|u_n|+1)||_{L^1(D_{m+2})}\\ &+\int_{D_{m+2}}(|u_n|+1)^{q_0}\eta dxdt+\int_{D_{m+2}}|\nabla\eta_m^{1/q_1}|^{q_1}dxdt+\int_{D_{m+2}}\eta_m d|\mu_n|,\end{aligned}$$

with $q_1<\frac{q_0-1}{2q_0}$. By the assumption, we verify that the right hand side of (5.5) is bounded by $c\varepsilon$, where c does not depend on n.
Since $\{\eta_m T_k(u_n)\}_{n\geq m+2}$ is bounded in $L^2(-(m+2)^2,(m+2)^2;H^1_0(B_{m+2}(0)))$, thus there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that

$$\lim_{n\to\infty}\int_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|\leq\varepsilon}\eta_m A(x,t,\nabla T_k(u))\nabla\left(T_k(u_n)-T_k(u)\right)dxdt=0.$$

Therefore, thanks to $u_n\to u$ a.e in D_{m+2} and $\langle T_k(\eta_{m+1}u)\rangle_\nu\to T_k(\eta_{m+1}u)$ in $L^2(-(m+2)^2,(m+2)^2;H^1_0(B_{m+2}(0)))$, we get

$$\limsup_{\nu\to\infty}\limsup_{n\to\infty}\int_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu|\leq\varepsilon}\eta_{1,m}\Phi_{n,k}dxdt\lesssim\varepsilon\forall\varepsilon\in(0,1),$$

where $\Phi_{n,k}=\left(A(x,t,T_k(u_n))-A(x,t,T_k(u))\right)\nabla\left(T_k(u_n)-T_k(u)\right)$.

Using Hölder's inequality, one has

$$\begin{aligned}\int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} dxdt &= \int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} \chi_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu| \le \varepsilon} dxdt \\ &+ \int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} \chi_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu| > \varepsilon} dxdt \\ &\le ||\eta_{1,m}||_{L^1(D_{m+2})}^{1/2} \left(\int\limits_{|T_k(u_n)-\langle T_k(\eta_{m+1}u)\rangle_\nu| \le \varepsilon} \eta_m \Phi_{n,k} dxdt \right)^{1/2} \\ &+ |\{|T_k(u_n) - \langle T_k(\eta_{m+1}u)\rangle_\nu| > \varepsilon\} \cap D_{m+1}|^{1/2} \left(\int_{D_{m+2}} \eta_m^2 \Phi_{k,n} dxdt \right)^{1/2} \\ &= A_{n,\nu,\varepsilon}.\end{aligned}$$

Clearly, $\limsup_{\varepsilon\to 0} \limsup_{\nu\to\infty} \limsup_{n\to\infty} A_{n,\nu,\varepsilon} = 0$. It follows

$$\limsup_{n\to\infty} \int_{D_{m+2}} \eta_m \Phi_{k,n}^{1/2} dxdt = 0.$$

Since $\Phi_{n,k} \ge \Lambda_2 |\nabla T_k(u_n) - \nabla T_k(u)|^2$, thus $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(D_m)$. Note that

$$\begin{aligned}|\{|\nabla u_{n_1} - \nabla u_{n_2}| > \lambda\} \cap D_m| &\le \frac{1}{k} \left(||u_{n_1}||_{L^1(D_m)} + ||u_{n_2}||_{L^1(D_m)} \right) \\ &+ \frac{1}{\lambda} ||\nabla T_k(u_{n_1}) - \nabla T_k(u_{n_2})||_{L^1(D_m)} \\ &\le \frac{2M_m}{k} + \frac{1}{\lambda} |||\nabla T_k(u_{n_1}) - \nabla T_k(u_{n_2})|||_{L^1(D_m)}.\end{aligned}$$

Thus, we can show that there is a subsequence of $\{\nabla u_n\}$ converging ∇u a.e in $\mathbb{R}^{N+1}$. The proof is complete. □

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