

ON THE p -LAPLACIAN WITH ROBIN BOUNDARY CONDITIONS AND BOUNDARY TRACE THEOREMS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, be a $C^{1,1}$ domain whose boundary $\partial\Omega$ is either compact or behaves suitably at infinity. For $p \in (1, \infty)$ and $\alpha > 0$, define

$$\Lambda(\Omega, p, \alpha) := \inf_{\substack{u \in W^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^p dx - \alpha \int_{\partial\Omega} |u|^p d\sigma}{\int_{\Omega} |u|^p dx},$$

where $d\sigma$ is the surface measure on $\partial\Omega$. We show the asymptotics

$$\Lambda(\Omega, p, \alpha) = -(p-1)\alpha^{\frac{p}{p-1}} - (\nu-1)H_{\max} \alpha + o(\alpha), \quad \alpha \rightarrow +\infty,$$

where H_{\max} is the maximum mean curvature of $\partial\Omega$. The asymptotic behavior of the associated minimizers is discussed as well. The estimate is then applied to the study of the best constant in a boundary trace theorem for expanding domains, to the norm estimate for extension operators and to related isoperimetric inequalities.

1. Introduction and main results

1.1. Problem setting. Let $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, be a domain with a sufficiently regular boundary. For $\alpha > 0$ and $p \in (1, \infty)$, consider the quantity

$$(1) \quad \Lambda(\Omega, p, \alpha) := \inf_{\substack{u \in W^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^p dx - \alpha \int_{\partial\Omega} |u|^p d\sigma}{\int_{\Omega} |u|^p dx},$$

where $d\sigma$ is the surface measure on $\partial\Omega$. Standard variational arguments show that under suitable assumptions, e.g. if Ω is bounded with a Lipschitz boundary, the problem (1) has a minimizer, see e.g. Proposition 6.1 below. The respective Euler-Lagrange equation takes the form of a non-linear eigenvalue problem

$$(2) \quad -\Delta_p u = \Lambda |u|^{p-2} u \text{ in } \Omega, \quad |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \alpha |u|^{p-2} u \quad \text{at } \partial\Omega,$$

where Δ_p is the p -Laplacian, $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$, n is the outer unit normal, and $\Lambda = \Lambda(\Omega, p, \alpha)$. In the present paper we work with $C^{1,1}$ domains, either bounded or with a suitable behavior at infinity (see below), and we study the behavior of (1) as α tends to $+\infty$. While the properties of $\Lambda(\Omega, p, \alpha)$ for $\alpha < 0$ are well understood for any $p > 1$, see e.g. [12] and references therein, the same problem for $\alpha \rightarrow +\infty$ was previously studied for the linear case $p = 2$ only. It was shown in [26] that for bounded C^1 domains there holds $\Lambda(\Omega, 2, \alpha) = -\alpha^2 + o(\alpha^2)$ as $\alpha \rightarrow +\infty$. Under additional smoothness assumptions, a more detailed asymptotic expansion

$$\Lambda(\Omega, 2, \alpha) = -\alpha^2 - (\nu-1)H_{\max}(\Omega) \alpha + o(\alpha)$$

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was obtained first in [14, 28] for $\nu = 2$ and then in [30, 31] for the general case, where H_{\max} is the maximal mean curvature of the boundary. Further terms in the asymptotic expansion can be calculated under suitable geometric hypotheses, see e.g. [9, 22, 23, 31]. Non-smooth domains were studied as well, see e.g. [24] and the recent preprint [5]. The above mentioned papers used a number of techniques which are specific for the linear problems, such as the perturbation theory for self-adjoint operators or a separation of variables, which are not available for the p -Laplacian.

In the present paper we are going to modify the existing approaches, which will allow us to feature the variational nature of the problem and to consider arbitrary values of p in a unified way. Furthermore, we work under weaker smoothness conditions when compared to the preceding works, and only $C^{1,1}$ regularity is assumed.

1.2. Main result. Let us pass to the exact formulation of our main result. In the sequel a domain Ω will be called **admissible** if the following conditions are satisfied:

- (i) the boundary $\partial\Omega$ is $C^{1,1}$, i.e. is locally the graph of a function with a Lipschitz gradient,
- (ii) the principal curvatures of $\partial\Omega$ are essentially bounded,
- (iii) for some $\delta > 0$ the map

$$\partial\Omega \times (0, \delta) \ni (s, t) \mapsto s - tn(s) \in \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$$

is bijective.

The mean curvature H of $\partial\Omega$ is the arithmetic mean of the principal curvatures, and we set

$$(3) \quad H_{\max} \equiv H_{\max}(\Omega) := \text{ess sup } H.$$

We remark that we do not assume that this value is attained.

An account of the differential geometry in the $C^{1,1}$ setting, including the precise definition of the curvatures, can be found e.g. in [19, Section 3]. In particular, the assumptions are satisfied for any domain with a compact $C^{1,1}$ boundary. Another obvious example of an admissible domain is given by any $C^{1,1}$ domain coinciding with a half-space outside a ball. Our main result reads as follows:

Theorem 1.1. *For any admissible domain $\Omega \subset \mathbb{R}^\nu$ and any $p \in (1, \infty)$ there holds*

$$(4) \quad \Lambda(\Omega, p, \alpha) = -(p-1)\alpha^{\frac{p}{p-1}} - (\nu-1)H_{\max}(\Omega)\alpha + o(\alpha) \text{ as } \alpha \rightarrow +\infty.$$

Remark 1.2. Remark that the $C^{1,1}$ assumption is a minimal one to define the curvature of the boundary. The asymptotics can be different for domains with a weaker regularity, see e.g. Proposition A.2 below for Lipschitz domains.

Remark 1.3. A significant feature of our method of proving (4) is that it does not assume existence of minimizers. This is important since there exist admissible domains for which problem (1) does not have a minimizer; for example if $p = 2$ and Ω is an infinite cylinder with sufficiently smooth boundary, then it is easily seen that the infimum in (1) is not attained.

The proof of Theorem 1.1 is presented in Sections 3–5 and is organized as follows. In Section 3 we estimate the eigenvalue $\Lambda(\Omega, p, \alpha)$ using some auxiliary operators in a tubular neighborhood of $\partial\Omega$. In Section 4 we obtain an upper bound by a suitable choice of test functions. The lower bound is obtained in Section 5 using an analysis of an auxiliary one-dimensional operator.

It turns out that Theorem 1.1 has various applications to Sobolev boundary trace theorems, extension operators and isoperimetric inequalities. These are described in Section 2.

Apart from the asymptotic behavior of $\Lambda(\Omega, p, \alpha)$ it is natural to address the question of the behavior of the associated eigenfunction, in other words the minimizer of (1). Indeed, in [9] it was shown, for $p = 2$, that as $\alpha \rightarrow \infty$, the eigenfunctions concentrate at the boundary of Ω . In Section 6 we carry this analysis, for a general $p > 1$, a bit further. In particular, we prove an exponential

localization near the boundary, Theorem 6.3, and a localization near the part of the boundary at which the mean curvature attains its maximum, Theorem 6.5.

Some explicitly solvable cases are discussed in Appendix A. In Appendix B we prove an auxiliary elementary inequality used in the proofs, and in Appendix C we show how the remainder estimate in Theorem 1.1 can be improved under stronger regularity assumptions on $\partial\Omega$.

2. Applications

2.1. Best constants for boundary trace theorems. The asymptotic expansion (4) provides a number of consequences for maps between various Sobolev spaces. Note first that a simple scaling argument gives

$$\Lambda(\Omega, p, \mu^{p-1}\alpha) = \mu^p \Lambda(\mu\Omega, p, \alpha), \quad \mu > 0.$$

In particular, as the half-space $\mathbb{R}^{\nu-1} \times \mathbb{R}_+$ is invariant under dilations, the first term on the right-hand side of (4) can be represented as

$$(5) \quad \Lambda(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, p, \alpha) \equiv -(p-1)\alpha^{\frac{p}{p-1}}, \quad \nu \geq 1,$$

see also Appendix A. Theorem 1.1 thus admits the following version for expanding domains:

Corollary 2.1. *For any admissible domain $\Omega \subset \mathbb{R}^\nu$, any $p \in (1, \infty)$ and $\alpha > 0$ one has, as μ tends to $+\infty$,*

$$\Lambda(\mu\Omega, p, \alpha) = \Lambda(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, p, \alpha) - (\nu-1)H_{\max}(\Omega)\alpha\mu^{-1} + o(\mu^{-1}).$$

Furthermore, one checks easily that the function $\mathbb{R}_+ \ni \alpha \mapsto \Lambda(\Omega, p, \alpha)$ is strictly decreasing and continuous with $\Lambda(\Omega, p, 0) = 0$ and $\lim_{\alpha \rightarrow +\infty} \Lambda(\Omega, p, \alpha) = -\infty$ and, hence, defines a bijection between \mathbb{R}_+ and \mathbb{R}_- . Denote by $S(\Omega, p, q)$ the best constant in the trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, which is defined through

$$\frac{1}{S(\Omega, p, q)} = \sup_{\substack{u \in W^{1,p}(\Omega) \\ u \neq 0}} \frac{\left(\int_{\partial\Omega} |u|^q d\sigma \right)^{\frac{p}{q}}}{\int_{\Omega} (|\nabla u|^p + |u|^p) dx}, \quad 1 < q < p_*,$$

with

$$p_* := \begin{cases} \frac{p\nu - p}{\nu - p}, & \text{if } p \in (1, \nu), \\ \infty, & \text{if } p \in [\nu, \infty). \end{cases}$$

The value of $S(\Omega, p, p)$ is then uniquely determined by the implicit equation

$$(6) \quad \Lambda(\Omega, p, S(\Omega, p, p)) = -1.$$

In particular, in view of (5) it holds

$$S(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, p, p) = (p-1)^{\frac{1-p}{p}}, \quad \nu \geq 1,$$

see also [11, Lemmas 3.1 and 3.3]. Various estimates for $S(\Omega, p, q)$ were extensively studied in the literature, see e.g. the review [32]. In particular, it was shown in [10] that for any $q > 2$ there exists a constant $\gamma = \gamma(q, \nu) > 0$ independent of Ω with

$$(7) \quad S(\mu\Omega, 2, q) = S(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, 2, q) - \gamma H_{\max}(\Omega)\mu^{-1} + o(\mu^{-1}) \text{ for } \mu \rightarrow +\infty.$$

Furthermore, by [15, Theorem 1.3], for each Ω there exist positive constants c_1 and c_2 such that for $\mu \rightarrow +\infty$ there holds

$$c_1\mu^\kappa \leq S(\mu\Omega, p, q) \leq c_2\mu^\kappa, \quad \kappa := \begin{cases} \frac{(q-p)(\nu-1)}{q}, & \text{if } 1 < q < p, \\ 0, & \text{if } p \leq q < p_*. \end{cases}$$

In particular, for any $p \in (1, \infty)$ the constant $S(\mu\Omega, p, p)$ remains uniformly bounded and separated from 0 as $\mu \rightarrow +\infty$. The substitution of Corollary 2.1 into (6) gives the following improvement in the spirit of (7):

Corollary 2.2. *For any admissible domain $\Omega \subset \mathbb{R}^\nu$ and any $p \in (1, \infty)$ there holds*

$$(8) \quad S(\mu\Omega, p, p) = S(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, p, p) - \frac{(p-1)^{\frac{2-p}{p}} (\nu-1)}{p} H_{\max}(\Omega) \mu^{-1} + o(\mu^{-1})$$

as μ tends to $+\infty$.

2.2. Extension operators. Recall that a bounded linear operator E from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^\nu)$ is called an extension operator if Ef coincides with f in Ω for any f . The existence and various estimates for the extension operators in terms of Ω are of interest, see e.g. [7]. We will be concerned with the lower bound for the norms

$$\mathcal{E}(\Omega, p) = \inf \{ \|E\| : E : W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^\nu) \text{ is an extension operator} \}.$$

It is known, in particular, that

$$(9) \quad \mathcal{E}(\Omega, p) \geq \left(1 + \frac{S(\Omega^c, p, p)}{S(\Omega, p, p)} \right)^{\frac{1}{p}}, \quad \Omega^c := \mathbb{R}^\nu \setminus \bar{\Omega}.$$

see [25, Theorem 3.1]. Note that the work [25] deals formally with the case $p = 2$ only, but the proof holds literally for any $p \in (1, \infty)$. Remark also that

$$\mathcal{E}(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, p) = 2^{\frac{1}{p}}.$$

In fact, the lower bound follows from (9), and it is attained by the operator of extension by parity $(Ef)(x_1, \dots, x_{\nu-1}, x_\nu) := f(x_1, \dots, x_{\nu-1}, |x_\nu|)$. We have the following result:

Corollary 2.3. *Assume that both Ω and Ω^c are admissible domains in \mathbb{R}^ν and that $p \in (1, \infty)$, then for $\mu \rightarrow +\infty$ there holds*

$$\mathcal{E}(\mu\Omega, p) \geq \mathcal{E}(\mathbb{R}^{\nu-1} \times \mathbb{R}_+, p) + \frac{(p-1)^{\frac{1}{p}}}{2^{\frac{p-1}{p}} p^2} (\nu-1) (H_{\max}(\Omega) + H_{\min}(\Omega)) \mu^{-1} + o(\mu^{-1}),$$

where $H_{\min}(\Omega) := \text{ess inf } H$.

Proof. We have $H_{\max}(\Omega^c) = -H_{\min}(\Omega)$, and the substitution into (8) and then into (9) gives the result. \square

2.3. Isoperimetric inequalities. Numerous works studied isoperimetric inequalities for the quantities $\Lambda(\Omega, 2, \alpha)$ and $S(\Omega, 2, 2)$. In particular, in [2] it was conjectured that the balls maximize $\Lambda(\Omega, 2, \alpha)$ among all fixed volume domains for any $\alpha > 0$. An analogous question for $S(\Omega, 2, 2)$ was asked e.g. in [33]. The conjecture was supported e.g. by the consideration of the first and second variations of the respective functionals and by showing that the balls are at least local minimizers, see e.g. [16, 33]. It was shown only recently in [18] that the conjecture in the general form is wrong by comparing the eigenvalues of the balls with those for the spherical shells for large α , while it remains true at least in two dimensions for a restricted range of positive α . (It is worth noting that the case $\alpha < 0$ is well understood for any p , see [6, 8].) In fact, the conjecture appears to be closely related to some estimates for the maximum mean curvature H_{\max} as discussed in [30], and the asymptotics (4) and (8) allow us to include into consideration all possible values of p . More precisely, let us recall the following known results:

- (A) The balls *do not* minimize the quantity H_{\max} among the bounded $C^{1,1}$ domains having the same volume. In particular, consider the domains

$$B_\rho := \{x \in \mathbb{R}^\nu : |x| < \rho\}, \quad U_{r,R} := \{x \in \mathbb{R}^\nu : r < |x| < R\}, \\ 0 < r < \rho < R, \quad R^\nu - r^\nu = \rho^\nu,$$

$$\text{then } \text{vol } B_\rho = \text{vol } U_{r,R}, \quad H_{\max}(B_\rho) = \frac{1}{\rho} > \frac{1}{R} = H_{\max}(U_{r,R}).$$

- (B) For $\nu = 2$, the balls are the strict minimizers of H_{\max} among all bounded *simply connected* C^2 domains of a fixed area, see e.g. the discussion in [29].
- (C) For $\nu \geq 3$, the balls are the strict minimizers of H_{\max} among the bounded *star-shaped* C^2 domains of the same volume, see e.g. [30, Theorem 2].
- (D) For $\nu = 3$, the balls *do not* minimize the quantity H_{\max} among the bounded $C^{1,1}$ domains having the same volume *and homeomorphic to a ball*. Moreover, there is no strictly positive lower bound for H_{\max} in terms of the volume. The respective examples were constructed recently in [17].

The combination of (A)–(C) with Theorem 1.1 gives the following observations, with an arbitrary $p \in (1, \infty)$:

- The balls do not maximize $\Lambda(\Omega, p, \alpha)$ among the domains of a fixed volume. In particular, for sufficiently large $\alpha > 0$ there holds $\Lambda(B_\rho, p, \alpha) < \Lambda(U_{r,R}, p, \alpha)$.
- Let $B \subset \mathbb{R}^\nu$ be a ball and $\Omega \subset \mathbb{R}^\nu$ be a simply connected bounded C^2 domain of the same volume, and for $\nu \geq 3$ assume additionally that Ω is star-shaped, then there exists $\alpha_\Omega > 0$ such that $\Lambda(B, p, \alpha) \geq \Lambda(\Omega, p, \alpha)$ for $\alpha > \alpha_\Omega$, with an equality iff Ω is a ball.
- At least for $\nu = 3$, the balls do not maximize $\Lambda(\Omega, p, \alpha)$ among the domains homeomorphic to balls and having a fixed volume.

In a similar way, Corollary 2.2 combined with (A)–(C) gives the following assertions valid for any $p \in (1, \infty)$:

- The balls do not maximize $S(\Omega, p, p)$ among the domains of a fixed volume. In particular, for sufficiently large $\mu > 0$ there holds $S(\mu B_\rho, p, p) < S(\mu U_{r,R}, p, p)$.
- Let $B \subset \mathbb{R}^\nu$ be a ball and $\Omega \subset \mathbb{R}^\nu$ be a simply connected bounded C^2 domain of the same volume, and for $\nu \geq 3$, assume additionally that Ω is star-shaped, then there exists $\mu_\Omega > 0$ such that $S(\mu B, p, p) \geq S(\mu \Omega, p, p)$ for $\mu > \mu_\Omega$, with an equality iff Ω is a ball.
- At least for $\nu = 3$, the balls do not maximize $S(\Omega, p, p)$ among the domains homeomorphic to balls and having a fixed volume.

In view of (7), the same considerations hold for $S(\Omega, 2, q)$ with any $q > 2$.

3. Proof of theorem 1.1: Bracketing and a change of variables

The construction of this section is quite standard and represents a suitable adaptation of [30, Sections 2.2 and 2.3]. For $s \in S := \partial\Omega$, let $n(s)$ be outer unit normal and $L_s := dn(s) : T_s S \rightarrow T_s S$ be the shape operator, which is defined for almost all $s \in S$, see [19, Section 3]. Recall that the eigenvalues $\kappa_1(s) \leq \dots \leq \kappa_{\nu-1}(s)$ of L_s are the so-called principal curvatures at s , and the mean curvature $H(s)$ at s is defined by

$$H(s) = \frac{\kappa_1(s) + \dots + \kappa_{\nu-1}(s)}{\nu - 1} \equiv \frac{1}{\nu - 1} \text{tr } L_s.$$

By assumptions, all κ_j are essentially bounded, and the maximal mean curvature

$$H_{\max} \equiv H_{\max}(\Omega) := \text{ess sup } H$$

is well-defined. In what follows, it will be convenient to use the quantities

$$M(s) := (\nu - 1)H(s), \quad M_{\max} := (\nu - 1)H_{\max}.$$

For $\delta > 0$ denote

$$(10) \quad \Omega_\delta := \{x \in \Omega : \inf_{s \in S} |x - s| < \delta\}, \quad \Theta_\delta := \Omega \setminus \overline{\Omega_\delta},$$

$$\Lambda^N(\Omega, p, \alpha) := \inf_{\substack{u \in W^{1,p}(\Omega_\delta \cup \Theta_\delta) \\ u \neq 0}} \frac{\int_{\Omega_\delta \cup \Theta_\delta} |\nabla u|^p dx - \alpha \int_S |u|^p d\sigma}{\int_{\Omega_\delta \cup \Theta_\delta} |u|^p dx},$$

$$\Lambda^D(\Omega, p, \alpha) := \inf_{\substack{u \in W^{1,p}(\Omega) \\ u=0 \text{ on } \partial\Theta_\delta \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p dx - \alpha \int_S |u|^p d\sigma}{\int_\Omega |u|^p dx},$$

then one clearly has $\Lambda^N(\Omega, p, \alpha) \leq \Lambda(\Omega, p, \alpha) \leq \Lambda^D(\Omega, p, \alpha)$. Furthermore, denote

$$\Lambda^{N,\delta}(\Omega, p, \alpha) := \inf_{\substack{u \in W^{1,p}(\Omega_\delta) \\ u \neq 0}} \frac{\int_{\Omega_\delta} |\nabla u|^p dx - \alpha \int_S |u|^p d\sigma}{\int_{\Omega_\delta} |u|^p dx},$$

$$\widetilde{W}^{1,p}(\Omega_\delta) := \{u \in W^{1,p}(\Omega_\delta) : u = 0 \text{ on } \partial\Omega_\delta \setminus S\},$$

$$\Lambda^{D,\delta}(\Omega, p, \alpha) := \inf_{\substack{u \in \widetilde{W}^{1,p}(\Omega_\delta) \\ u \neq 0}} \frac{\int_{\Omega_\delta} |\nabla u|^p dx - \alpha \int_S |u|^p d\sigma}{\int_{\Omega_\delta} |u|^p dx}.$$

It is easy to check that if $\Lambda^{D,\delta}(\Omega, p, \alpha) \leq 0$, then $\Lambda^N(\Omega, p, \alpha) = \Lambda^{N,\delta}(\Omega, p, \alpha)$ and $\Lambda^D(\Omega, p, \alpha) = \Lambda^{D,\delta}(\Omega, p, \alpha)$, hence,

$$\Lambda^{N,\delta}(\Omega, p, \alpha) \leq \Lambda(\Omega, p, \alpha) \leq \Lambda^{D,\delta}(\Omega, p, \alpha).$$

We will study the quantities $\Lambda^{N,\delta}(\Omega, p, \alpha)$ and $\Lambda^{D,\delta}(\Omega, p, \alpha)$ using a change of variables. By assumption we can choose $\delta > 0$ sufficiently small such that the map

$$(11) \quad \Sigma := S \times (0, \delta) \ni (s, t) \mapsto \Phi(s, t) := s - tn(s) \in \Omega_\delta$$

is bijective and uniformly locally bi-Lipschitz. The metric G on Σ induced by this embedding is

$$(12) \quad G = g \circ (I_s - tL_s)^2 + dt^2,$$

where $I_s : T_s S \rightarrow T_s S$ is the identity map, and g is the metric on S induced by the embedding in \mathbb{R}^ν . The associated volume form $d\Sigma$ on Σ is

$$d\Sigma(s, t) = |\det G(s, t)|^{\frac{1}{2}} ds dt = \varphi(s, t) |\det g(s)|^{\frac{1}{2}} ds dt = \varphi(s, t) d\sigma(s) dt,$$

where $d\sigma(s) = |\det g(s)|^{\frac{1}{2}} ds$ is the induced $(\nu - 1)$ -dimensional volume form on S , and the weight φ is given by

$$(13) \quad \varphi(s, t) := |\det(I_s - tL_s)| = \prod_{j=1}^{\nu-1} (1 - \kappa_j(s)t) = 1 - M(s)t + P(s, t)t^2,$$

where P is a polynomial in t with coefficients which are essentially bounded functions of s , and we assume in addition that $\delta > 0$ is sufficiently small to have $1/2 \leq \varphi \leq 2$ almost everywhere in Σ . In

particular,

$$\Lambda^{N,\delta}(\Omega, p, \alpha) = \frac{\inf_{\substack{u \in W^{1,p}(\Omega_\delta) \\ u \neq 0}} \int_{\Sigma} |(\nabla u) \circ \Phi(s, t)|^p d\Sigma(s, t) - \alpha \int_S |u \circ \Phi(s, 0)|^p d\sigma(s)}{\int_{\Sigma} |u \circ \Phi(s, t)|^p d\Sigma(s, t)},$$

$$\Lambda^{D,\delta}(\Omega, p, \alpha) = \frac{\inf_{\substack{u \in \widetilde{W}^{1,p}(\Omega_\delta) \\ u \neq 0}} \int_{\Sigma} |(\nabla u) \circ \Phi(s, t)|^p d\Sigma(s, t) - \alpha \int_S |u \circ \Phi(s, 0)|^p d\sigma(s)}{\int_{\Sigma} |u \circ \Phi(s, t)|^p d\Sigma(s, t)},$$

and the map $u \mapsto u \circ \Phi$ defines a bijection between $W^{1,p}(\Omega_\delta)$ and $W^{1,p}(\Sigma)$ as well as between $\widetilde{W}^{1,p}(\Omega_\delta)$ and $\widetilde{W}^{1,p}(\Sigma) := \{v \in W^{1,p}(\Sigma) : v(\cdot, \delta) = 0\}$. Furthermore, for $v = u \circ \Phi$ we have $|(\nabla u) \circ \Phi|^2 = G^{jk} \partial_j v \partial_k v$ with $(G^{jk}) := G^{-1}$, and due to (12) we can estimate, with some $0 < C_- < C_+$,

$$C_- g^{-1} + dt^2 \leq G^{-1} \leq C_+ g^{-1} + dt^2, \quad (g^{\rho\mu}) := g^{-1},$$

which gives $C_- |\nabla_s v|^2 + |\partial_t v|^2 \leq |(\nabla u) \circ \Phi|^2 \leq C_+ |\nabla_s v|^2 + |\partial_t v|^2$ with $|\nabla_s v|^2 := g^{\rho\mu} \partial_\rho v \partial_\mu v$. Hence, with the notation

$$\Lambda^-(p, \alpha) := \frac{\inf_{\substack{u \in W^{1,p}(\Sigma) \\ u \neq 0}} \int_{\Sigma} \left| C_- |\nabla_s u(s, t)|^2 + u_t(s, t)^2 \right|^{\frac{p}{2}} \varphi(s, t) d\sigma(s) dt - \alpha \int_S |u(s, 0)|^p d\sigma(s)}{\int_{\Sigma} |u(s, t)|^p \varphi(s, t) d\sigma(s) dt}$$

and

$$\Lambda^+(p, \alpha) := \frac{\inf_{\substack{u \in \widetilde{W}^{1,p}(\Sigma) \\ u \neq 0}} \int_{\Sigma} \left| C_+ |\nabla_s u(s, t)|^2 + u_t(s, t)^2 \right|^{\frac{p}{2}} \varphi(s, t) d\sigma(s) dt - \alpha \int_S |u(s, 0)|^p d\sigma(s)}{\int_{\Sigma} |u(s, t)|^p \varphi(s, t) d\sigma(s) dt},$$

we conclude that

$$(14) \quad \Lambda^-(p, \alpha) \leq \Lambda(\Omega, p, \alpha) \leq \Lambda^+(p, \alpha)$$

holds true provided $\Lambda^+(p, \alpha) \leq 0$. Now we obtain separately an upper bound for $\Lambda^+(p, \alpha)$, see Section 4, and a lower bound for $\Lambda^-(p, \alpha)$, see Section 5.

4. Proof of theorem 1.1: Upper bound

For an upper bound for $\Lambda^+(p, \alpha)$ we will test on functions of a special form. To have shorter expressions we introduce the parameter

$$(15) \quad \beta := \alpha^{\frac{1}{p-1}}.$$

Pick a C^∞ function $\chi : (0, \delta) \rightarrow [0, 1]$ which equals one in a neighborhood of 0 and zero in a neighborhood of δ , and define $\psi(t) := e^{-\beta t} \chi(t)$. As β tends to $+\infty$,

$$(16) \quad \int_0^\delta \psi(t)^p \varphi(s, t) dt = \frac{1}{p\beta} - \frac{M(s)}{p^2 \beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

$$\int_0^\delta |\psi'(t)|^p \varphi(s, t) dt = \beta^p \left(\frac{1}{p\beta} - \frac{M(s)}{p^2 \beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right),$$

where the remainder estimates are uniform in $s \in S$ due to the essential boundedness of the coefficients of φ . Now we are going to consider two cases separately.

Case I: $p \in (1, 2]$. Using the inequality $(a+b)^q \leq a^q + b^q$ valid for $a, b \geq 0$ and $q \in (0, 1]$ we estimate, with some $C > 0$,

$$\Lambda^+(p, \alpha) \leq \inf_{\substack{u \in \widetilde{W}^{1,p}(\Sigma) \\ u \neq 0}} \left\{ \int_{\Sigma} \left(C |\nabla_s u(s, t)|^p + |u_t(s, t)|^p \right) \varphi(s, t) d\sigma(s) dt - \beta^{p-1} \int_S |u(s, 0)|^p d\sigma(s) \right\} \\ \times \left\{ \int_{\Sigma} |u(s, t)|^p \varphi(s, t) d\sigma(s) dt \right\}^{-1}.$$

Considering the functions u of the form $u(s, t) = v(s)\psi(t)$ with $v \in W^{1,p}(S)$ and using the estimates (16) we arrive, as $\beta \rightarrow +\infty$, at

$$(17) \quad \Lambda^+(p, \alpha) \leq \inf_{\substack{v \in W^{1,p}(S) \\ v \neq 0}} \left\{ C \int_S |\nabla_s v(s)|^p \left[\frac{1}{p\beta} - \frac{M(s)}{p^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right] d\sigma(s) \right. \\ \left. + \beta^p \int_S |v(s)|^p \left[\frac{1}{p\beta} - \frac{M(s)}{p^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right] d\sigma(s) - \beta^{p-1} \int_S |v(s)|^p d\sigma(s) \right\} \\ \times \left\{ \int_S |v(s)|^p \left[\frac{1}{p\beta} - \frac{M(s)}{p^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right] d\sigma(s) \right\}^{-1},$$

where the \mathcal{O} -terms are uniform in $s \in S$ and do not depend on v .

To construct a suitable function v , we adapt the procedure appearing e.g. in [3, Lemma 3.2] for Schrödinger operators with strong potentials. Take an arbitrary $\varepsilon > 0$. By assumption, the set

$$S_\varepsilon := \{s \in S : M_{\max} - \varepsilon \leq M(s) \leq M_{\max}\}$$

has a non-zero measure, and almost any point s of S_ε has density one with respect to the Lebesgue measure, i.e. in our case $\sigma(\mathcal{B}_\rho(s) \cap S_\varepsilon) / \sigma(\mathcal{B}_\rho(s)) \rightarrow 1$ for $\rho \rightarrow 0$, where $\mathcal{B}_\rho(s)$ is the geodesic ball in S centered at s of radius ρ , see e.g. [13, Section 1.7, Corollary 3]. Let us choose any $s \in S_\varepsilon$ with this property. In what follows, we denote by B_r the ball of radius r center at 0 in $\mathbb{R}^{\nu-1}$. Let $y \in B_r$ be the Riemann normal coordinates centered at s , which will be used as local coordinates on S , then for any $v \in W_0^{1,p}(B_r)$, $v \neq 0$, due to (17) one has,

$$(18) \quad \Lambda^+(p, \alpha) \leq \frac{c \int_{B_r} |\nabla v(y)|^p d\sigma(y) + \beta^p \int_{B_r} |v(y)|^p \left[(1-p) - \frac{M(y)}{p\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \right] d\sigma(y)}{\int_{B_r} |v(y)|^p \left[1 - \frac{M(y)}{p\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \right] d\sigma(y)}.$$

Assuming that r is sufficiently small we have $2^{-1}dy \leq d\sigma(y) \leq 2dy$ in B_r . Furthermore, due to the choice of s we have $\text{meas}(B_\rho \cap S_\varepsilon) / \text{meas}(B_\rho) \rightarrow 1$ as $\rho \rightarrow 0$, where meas stands for the Lebesgue measure in $\mathbb{R}^{\nu-1}$. Let $\mu \in (0, 1/2)$, then for sufficiently small $\rho \in (0, r)$ one has $\text{meas}(B_\rho \cap S_\varepsilon) \geq (1 - \mu) \text{meas}(B_\rho)$. Denote

$$\theta \equiv \theta(\varepsilon, \rho) := \text{meas}(B_\rho \setminus S_\varepsilon) / \text{meas}(B_\rho),$$

so that $0 \leq \theta \leq \mu$. For a ball $\omega \subset \mathbb{R}^{\nu-1}$, let Ψ_ω denote a non-negative minimizer of

$$(19) \quad \lambda_\omega := \inf \left\{ \int_\omega |\nabla f|^p dy : f \in W_0^{1,p}(\omega), \int_\omega |f|^p dy = 1 \right\}.$$

By [4], the function Ψ_ω is uniquely defined, and it is radially decreasing. Furthermore, set $\Phi_\omega = c_\omega \Psi_\omega$ with $c_\omega > 0$ chosen such that

$$\int_\omega \Phi_\omega(y)^p d\sigma(y) = 1.$$

Due to the above computations one has $c_\omega \in (2^{-\frac{1}{p}}, 2^{\frac{1}{p}})$. We are going to test in (18) on

$$(20) \quad v = \Phi_{B_\rho} \text{ extended by zero to } B_r.$$

Set $\eta := \theta^{\frac{1}{\nu-1}}$, then $\text{meas}(B_\rho \setminus S_\varepsilon) = \text{meas}(B_{\eta\rho})$. As Φ_{B_ρ} is radially decreasing, we have

$$\int_{B_\rho \setminus S_\varepsilon} \Phi_{B_\rho}^p(y) d\sigma(y) \leq 2 \int_{B_\rho \setminus S_\varepsilon} \Phi_{B_\rho}^p(y) dy \leq 2 \int_{B_{\eta\rho}} \Phi_{B_\rho}^p(y) dy \leq 4 \int_{B_{\eta\rho}} \Psi_{B_\rho}^p(y) dy.$$

Furthermore, using the inequality $\eta \leq \mu^{\frac{1}{\nu-1}}$ and the equality $\Psi_{B_\rho}(y) = \rho^{\frac{1-\nu}{p}} \Psi_{B_1}(y/\rho)$, we have

$$\int_{B_{\eta\rho}} \Psi_{B_\rho}^p(y) dy \leq \int_B \Psi_{B_\rho}^p(y) dy = \rho^{1-\nu} \int_B \Psi_{B_1}^p(y/\rho) dy = \int_B \Psi_{B_1}^p(y) dy =: \gamma(\mu).$$

Putting all together, for an arbitrarily chosen $\mu \in (0, 1/2)$ we can take ρ sufficiently small and make the choice (20), which gives

$$\int_{B_\rho \setminus S_\varepsilon} v(y)^p dy \leq \gamma(\mu) = o(1), \quad \mu \rightarrow 0.$$

Furthermore, using the fact that ρ was chosen small but fixed and that $p \leq 2$, we have, with suitable $C_j > 0$ and β large enough

$$\begin{aligned} A &:= c \int_{B_r} |\nabla v(y)|^p d\sigma(y) + \beta^p \int_{B_r} v(y)^p \left((1-p) - \frac{M(y)}{p\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \right) d\sigma(y) \\ &= c\lambda_{B_\rho} + (1-p)\beta^p - \frac{M_{\max}}{p}\beta^{p-1} + \frac{\beta^{p-1}}{p} \int_{B_r} (M_{\max} - M(y))v(y)^p d\sigma(y) + C_1\beta^{p-2} \\ &= c\lambda_{B_\rho} + (1-p)\beta^p - \frac{M_{\max}}{p}\beta^{p-1} + \frac{\beta^{p-1}}{p} \int_{B_r \cap S_\varepsilon} (M_{\max} - M(y))v(y)^p d\sigma(y) \\ &\quad + \frac{\beta^{p-1}}{p} \int_{B_r \setminus S_\varepsilon} (M_{\max} - M(y))v(y)^p d\sigma(y) + C_1\beta^{p-2} \\ &\leq (1-p)\beta^p - \frac{M_{\max}}{p}\beta^{p-1} + \frac{\varepsilon}{p}\beta^{p-1} \int_{B_r \cap S_\varepsilon} v(y)^p d\sigma(y) \\ &\quad + \frac{\|M_{\max} - M\|_\infty}{p}\beta^{p-1} \int_{B_r \setminus S_\varepsilon} v(y)^p d\sigma(y) + C_2 \\ &\leq (1-p)\beta^p - \frac{M_{\max}}{p}\beta^{p-1} + \frac{\varepsilon}{p}\beta^{p-1} + \frac{\|M_{\max} - M\|_\infty \gamma(\mu)}{p}\beta^{p-1} + C_2 \\ &= (1-p)\beta^p - \frac{1}{p} \left(M_{\max} - \varepsilon - \|M_{\max} - M\|_\infty \gamma(\mu) \right) \beta^{p-1} + C_2. \end{aligned}$$

In the same way we obtain

$$\begin{aligned} B &:= \int_{B_r} |v(y)|^p \left[1 - \frac{M(y)}{p\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \right] d\sigma(y) \\ &= 1 - \frac{M_{\max}}{p}\beta^{-1} + \frac{1}{p}\beta^{-1} \int_{B_r} (M_{\max} - M(y))v(y)^p d\sigma(y) + \mathcal{O}(\beta^{-2}) \\ &\leq 1 - \frac{1}{p} \left(M_{\max} - \varepsilon - \|M_{\max} - M\|_\infty \gamma(\mu) \right) \beta^{-1} + C_2\beta^{-2}. \end{aligned}$$

For large β one has $A < 0$, and by (18)

$$\Lambda^+(p, \alpha) \leq \frac{A}{B} \leq \frac{(1-p)\beta^p - \frac{1}{p} \left(M_{\max} - \varepsilon - \|M_{\max} - M\|_\infty \gamma(\mu) \right) \beta^{p-1} + C_1}{1 - \frac{1}{p} \left(M_{\max} - \varepsilon - \|M_{\max} - M\|_\infty \gamma(\mu) \right) \beta^{-1} + C_2\beta^{-2}}$$

$$\begin{aligned}
&\leq (1-p)\beta^p - \left(M_{\max} - \varepsilon - \|M_{\max} - M\|_{\infty}\gamma(\mu)\right)\beta^{p-1} + C_3 \\
&= (1-p)\beta^p - M_{\max}\beta^{p-1} + \left(\varepsilon + \|M_{\max} - M\|_{\infty}\gamma(\mu)\right)\beta^{p-1} + C_3,
\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{\alpha \rightarrow +\infty} \frac{\Lambda(\Omega, p, \alpha) - (1-p)\alpha^{\frac{p}{p-1}} + M_{\max}\alpha}{\alpha} &\leq \limsup_{\alpha \rightarrow +\infty} \frac{\Lambda^+(p, \alpha) - (1-p)\alpha^{\frac{p}{p-1}} + M_{\max}\alpha}{\alpha} \\
&= \limsup_{\beta \rightarrow +\infty} \frac{\Lambda^+(p, \alpha) - (1-p)\beta^p + M_{\max}\beta^{p-1}}{\beta^{p-1}} \\
&\leq \varepsilon + \|M_{\max} - M\|_{\infty}\gamma(\mu).
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary and $\gamma(\mu)$ can be made arbitrary small by taking μ arbitrary small, we have the sought estimate $\Lambda(\Omega, p, \alpha) \leq (1-p)\alpha^{\frac{p}{p-1}} - M_{\max}\alpha + o(\alpha)$ for large α .

Case II: $p \in (2, \infty)$. Let $q > 1$, then one can find a constant $c > 0$ such that for all $\varepsilon_0 \in (0, 1)$ and all $a, b \geq 0$ there holds

$$(21) \quad (a+b)^q \leq (1+\varepsilon_0)a^q + \frac{c}{\varepsilon_0^{q-1}}b^q,$$

see e.g. Appendix B. Therefore, with a suitable $C > 0$ and any $\varepsilon \in (0, 1)$ one can estimate

$$\begin{aligned}
\Lambda^+(p, \alpha) &\leq \inf_{\substack{u \in \widetilde{W}^{1,p}(\Sigma) \\ u \neq 0}} \left\{ \int_{\Sigma} \left\{ C\varepsilon_0^{\frac{2-p}{2}} |\nabla_s u(s, t)|^p + (1+\varepsilon_0)|u_t(s, t)|^p \right\} \varphi(s, t) d\sigma(s) dt \right. \\
&\quad \left. - \beta^{p-1} \int_S |u(s, 0)|^p d\sigma(s) \right\} \times \left\{ \int_{\Sigma} |u(s, t)|^p \varphi(s, t) d\sigma(s) dt \right\}^{-1}.
\end{aligned}$$

Considering the functions u of the form $u(s, t) = v(s)\phi(t)$ with $v \in W^{1,p}(S)$ and using the estimates (16) we arrive, as $\beta \rightarrow +\infty$, at

$$\begin{aligned}
(22) \quad \Lambda^+(p, \alpha) &\leq \inf_{\substack{v \in W^{1,p}(S) \\ v \neq 0}} \left\{ C\varepsilon_0^{\frac{2-p}{2}} \int_S |\nabla_s v(s)|^p \left[\frac{1}{p\beta} - \frac{M(s)}{p^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right] d\sigma(s) \right. \\
&\quad \left. + (1+\varepsilon_0)\beta^p \int_S |v(s)|^p \left[\frac{1}{p\beta} - \frac{M(s)}{p^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right] d\sigma(s) - \beta^{p-1} \int_S |v(s)|^p d\sigma(s) \right\} \\
&\quad \times \left\{ \int_S |v(s)|^p \left[\frac{1}{p\beta} - \frac{M(s)}{p^2\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) \right] d\sigma(s) \right\}^{-1},
\end{aligned}$$

where the \mathcal{O} -terms are uniform in $s \in S$ and do not depend on v and ε_0 , and by taking $\varepsilon_0 := \beta^{-\frac{3}{2}}$ and choosing suitable $C_j > 0$ we arrive at

$$\Lambda^+(p, \alpha) \leq \frac{C_1\beta^{\frac{3(p-2)}{4}} \int_S |\nabla_s v(s)|^p d\sigma(s) + \beta^p \int_S \left[(1-p) - \frac{M(s)}{p\beta} + \mathcal{O}\left(\frac{1}{\beta^{\frac{3}{2}}}\right) \right] |v(s)|^p d\sigma(s)}{\int_S |v(s)|^p \left[1 - \frac{M(s)}{p\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \right] d\sigma(s)}.$$

Now using the same notation and the same test function as in the case I we arrive at

$$\begin{aligned}
\Lambda^+(p, \alpha) &\leq \frac{(1-p)\beta^p - \frac{1}{p}\left(M_{\max} - \varepsilon - \|M_{\max} - M\|_{\infty}\gamma(\mu)\right)\beta^{p-1} + C_1\beta^{p-\frac{3}{2}}}{1 - \frac{1}{p}\left(M_{\max} - \varepsilon - \|M_{\max} - M\|_{\infty}\gamma(\mu)\right)\beta^{-1} + C_2\beta^{-2}} \\
&\leq (1-p)\beta^p - M_{\max}\beta^{p-1} + \left(\varepsilon + \|M_{\max} - M\|_{\infty}\gamma(\mu)\right)\beta^{p-1} + C_3\beta^{p-\frac{3}{2}},
\end{aligned}$$

where $C_j > 0$ are suitable constants, and

$$\limsup_{\alpha \rightarrow +\infty} \frac{\Lambda(\Omega, p, \alpha) - (1-p)\alpha^{\frac{p}{p-1}} + M_{\max}\alpha}{\alpha} \leq \varepsilon + \|M_{\max} - M\|_{\infty} \gamma(\mu),$$

while ε and $\gamma(\mu)$ can be chosen arbitrarily small, which gives the result.

5. Proof of theorem 1.1: Lower bound

The minoration $C_- |\nabla_s u(s, t)|^2 + u_t(s, t)^2 \geq u_t(s, t)^2$ gives

$$(23) \quad \Lambda^-(p, \alpha) \geq \inf_{\substack{u \in W^{1,p}(\Sigma) \\ u \neq 0}} \frac{\int_S \left(\int_0^\delta |u_t(s, t)|^p \varphi(s, t) dt - \alpha |u(s, 0)|^p \right) d\sigma(s)}{\int_\Sigma |u(s, t)|^p \varphi(s, t) d\sigma(s) dt}.$$

Denote

$$(24) \quad \lambda(\alpha, p, s) := \inf_{u \in W^{1,p}(0, \delta)} \frac{\int_0^\delta |u'(t)|^p \varphi(s, t) dt - \alpha |u(0)|^p}{\int_0^\delta |u(t)|^p \varphi(s, t) dt},$$

then for a.e. $s \in S$ one has

$$\int_0^\delta |u_t(s, t)|^p \varphi(s, t) dt - \alpha |u(s, 0)|^p \geq \lambda(\alpha, p, s) \int_0^\delta |u(s, t)|^p \varphi(s, t) dt,$$

and (23) implies

$$\Lambda^-(p, \alpha) \geq \inf_{\substack{u \in W^{1,p}(\Sigma) \\ u \neq 0}} \frac{\int_\Sigma \lambda(\alpha, p, s) |u(s, t)|^p \varphi(s, t) d\sigma(s) dt}{\int_\Sigma |u(s, t)|^p \varphi(s, t) d\sigma(s) dt} \geq \operatorname{ess\,inf}_{s \in S} \lambda(\alpha, p, s).$$

Hence the result follows from the following lemma:

Lemma 5.1. *There holds*

$$\lambda \equiv \lambda(\alpha, p, s) = (1-p)\alpha^{\frac{p}{p-1}} - M(s)\alpha + \mathcal{O}(\alpha^{\frac{p-2}{p-1}} \log \alpha) \quad \text{as } \alpha \rightarrow +\infty,$$

where the remainder estimate is uniform in s outside a zero-measure set.

Proof. Introducing β as in (15) and testing on $u(t) = e^{-\beta t}$ we obtain by a direct computation the upper bound

$$\lambda \leq (1-p)\beta^p - M(s)\beta^{p-1} + \mathcal{O}(\beta^{p-2}) \equiv (1-p)\alpha^{\frac{p}{p-1}} - M(s)\alpha + \mathcal{O}(\alpha^{\frac{p-2}{p-1}}),$$

where the remainder depends on $\|\kappa_j\|_{\infty}$ only, see (13), and, hence, is uniform in s outside a zero-measure set. In particular, for sufficiently large α we have,

$$(25) \quad \lambda \leq \frac{1-p}{2} \beta^p,$$

uniformly in $s \in S$. It follows by standard arguments that problem (24) admits a minimizer v , see e.g. Proposition 6.1 below. Without loss of generality we may assume that

$$(26) \quad v \geq 0, \quad \int_0^\delta v(t)^p \varphi(s, t) dt = \|v\varphi^{\frac{1}{p}}\|_p^p = 1.$$

The Euler-Lagrange equation for v reads

$$(27) \quad (|v'|^{p-2} v' \varphi)' = -\lambda v^{p-1} \varphi,$$

where the prime means the derivative in t , with the boundary conditions

$$(28) \quad |v'(0)|^{p-2}v'(0) = -\alpha v(0)^{p-1}, \quad v'(\delta) = 0.$$

In order to establish suitable decay properties of v in the spirit of Agmon [1], let us take $f \in C^1([0, \delta])$ with $f \geq 0$ and $f(0) = 0$. Multiplying equation (27) by $f^p v$, integrating on $(0, \delta)$ by parts and using the boundary conditions (28) we arrive at

$$(29) \quad \begin{aligned} \lambda \int_0^\delta f(t)^p v(t)^p \varphi(s, t) dt &= - \int_0^\delta (|v'|^{p-2}v'\varphi)'(t) f(t)^p v(t) dt \\ &= \int_0^\delta |v'(t)|^{p-2}v'(t)(f^p v)'(t) \varphi(s, t) dt \\ &= \int_0^\delta |v'(t)|^p f(t)^p \varphi(s, t) dt \\ &\quad + p \int_0^\delta |v'(t)|^{p-2}v'(t) f(t)^{p-1} f'(t) v(t) \varphi(s, t) dt. \end{aligned}$$

An application of the Young inequality

$$(30) \quad |AB| \leq \varepsilon |A|^q + \varepsilon^{-\frac{1}{q-1}} |B|^{\frac{q}{q-1}}, \quad A, B \in \mathbb{R}, \quad \varepsilon > 0, \quad q > 1,$$

with $A = |v'(t)|^{p-2}v'(t)f(t)^{p-1}$, $B = f'(t)v(t)$ and $q = p/(p-1)$ to the second term on the right-hand side of (29) gives

$$-\lambda \int_0^\delta f(t)^p v(t)^p \varphi(s, t) dt \leq (p\varepsilon - 1) \int_0^\delta |v'(t)|^p f(t)^p \varphi(s, t) dt + p\varepsilon^{-(p-1)} \int_0^\delta |f'(t)|^p v(t)^p \varphi(s, t) dt.$$

Taking $\varepsilon = 1/(2p)$ and using (25) we arrive at

$$(31) \quad \int_0^\delta |v'(t)|^p f(t)^p \varphi(s, t) dt + (p-1)\beta^p \int_0^\delta v(t)^p f(t)^p \varphi(s, t) dt \leq (2p)^p \int_0^\delta v(t)^p |f'(t)|^p \varphi(s, t) dt.$$

Choose f in the form $f(t) = \chi(t)e^{\omega t}$, where $\chi \in C^1([0, \delta])$ with

$$\chi(0) = 0, \quad 0 \leq \chi \leq 1, \quad \chi(t) = 1 \text{ for } t \geq \theta, \quad \|\chi'\|_\infty \leq \frac{2}{\theta},$$

and $\omega > 0$ and $\theta \in (0, \delta)$ will be chosen later. Using (21) with $q = p$ and $\varepsilon = 1$ we obtain

$$(32) \quad \begin{aligned} \int_0^\delta v(t)^p |f'(t)|^p \varphi(s, t) dt &= \int_0^\delta v(t)^p e^{p\omega t} |\chi'(t) + \omega\chi(t)|^p \varphi(s, t) dt \\ &\leq c \int_0^\delta v(t)^p |\chi'(t)|^p e^{p\omega t} \varphi(s, t) dt + 2\omega^p \int_0^\delta v(t)^p \chi(t)^p e^{p\omega t} \varphi(s, t) dt \\ &\leq c \int_0^\theta v(t)^p |\chi'(t)|^p e^{p\omega t} \varphi(s, t) dt + 2\omega^p \int_0^\delta v(t)^p \chi(t)^p e^{p\omega t} \varphi(s, t) dt \\ &\leq c_1 \theta^{-p} e^{p\omega\theta} \int_0^\theta v(t)^p \varphi(s, t) dt + 2\omega^p \int_0^\delta v(t)^p \chi(t)^p e^{p\omega t} \varphi(s, t) dt \\ &\leq c_1 \theta^{-p} e^{p\omega\theta} + 2\omega^p \int_0^\delta v(t)^p \chi(t)^p e^{p\omega t} \varphi(s, t) dt, \end{aligned}$$

where $c_1 := 2^p$, and on the last step we used the normalization (26) for v . The substitution into (31) gives

$$\begin{aligned} \int_0^\delta |v'(t)|^p f(t)^p \varphi(s, t) dt + (p-1)\beta^p \int_0^\delta v(t)^p f(t)^p \varphi(s, t) dt &\leq C_1 \theta^{-p} e^{p\omega\theta} \\ &\quad + C_2 \omega^p \int_0^\delta v(t)^p f(t)^p \varphi(s, t) dt \end{aligned}$$

with $C_1 := (2p)^p c_1$ and $C_2 := 2(2p)^p$. Let us set

$$\omega := \kappa\beta \quad \text{with} \quad \kappa := \left(\frac{p-1}{2C_2}\right)^{\frac{1}{p}},$$

so that

$$\int_0^\delta |v'(t)|^p f(t)^p \varphi(s, t) dt + \frac{p-1}{2} \beta^p \int_0^\delta v(t)^p f(t)^p \varphi(s, t) dt \leq C_1 \theta^{-p} e^{p\kappa\beta\theta}.$$

Finally, we put

$$\theta := \frac{1}{\beta}.$$

Then, with a suitable $C_2 > 0$,

$$\int_0^\delta v(t)^p f(t)^p \varphi(s, t) dt \leq C_2, \quad \int_0^\delta |v'(t)|^p f(t)^p \varphi(s, t) dt \leq C_2 \beta^p.$$

Further, as $\varphi_0 := \inf_{(s,t) \in S \times (0,\delta)} \varphi(s, t) > 0$, we have, with $C_3 := C_2/\varphi_0 > 0$,

$$(33) \quad \int_0^\delta v(t)^p f(t)^p dt \leq C_3, \quad \int_0^\delta |v'(t)|^p f(t)^p dt \leq C_3 \beta^p,$$

and (32) gives

$$\int_0^\delta v(t)^p |f'(t)|^p dt \leq C_4 \beta^p$$

for some $C_4 > 0$. Using again (21) with $q = p$ and $\varepsilon = 1$ we conclude with

$$\int_0^\delta |(vf)'(t)|^p dt \leq 2 \int_0^\delta v(t)^p |f'(t)|^p dt + c \int_0^\delta |v'(t)|^p f(t)^p dt \leq C_5 \beta^p.$$

The integral bounds obtained allow us to estimate the values of $v(\delta)$ and $v(0)$ as follows. First,

$$v(\delta)^p f(\delta)^p = p \int_0^\delta (vf)'(t) (vf)^{p-1}(t) dt \leq p \|vf\|_p^{p-1} \|(vf)'\|_p \leq C_6 \beta,$$

implying

$$(34) \quad v(\delta)^p \leq C_6 \beta e^{-p\kappa\delta\beta}.$$

Furthermore,

$$\begin{aligned} v(0)^p &= \varphi(s, 0) v(0)^p \\ &= \varphi(s, \delta) v(\delta)^p - p \int_0^\delta v(t)^{p-1} \varphi(s, t)^{\frac{p-1}{p}} v'(t) \varphi(s, t)^{\frac{1}{p}} dt - \int_0^\delta v(t)^p \partial_t \varphi(s, t) dt \\ &\leq \varphi(s, \delta) v(\delta)^p + p \|v\varphi^{\frac{1}{p}}\|_p^{p-1} \|v'\varphi^{\frac{1}{p}}\|_p - \int_0^\delta v(t)^p \partial_t \varphi(s, t) dt. \end{aligned}$$

Using the normalization of v and the estimate (34) we arrive at

$$(35) \quad v(0)^p \leq p \|v'\varphi^{\frac{1}{p}}\|_p - \int_0^\delta v(t)^p \partial_t \varphi(s, t) dt + \mathcal{O}(\beta^{-2}).$$

In order to estimate the integral on the right-hand side we remark that, for any $b > 0$ and as α is sufficiently large,

$$\begin{aligned} \int_{b\beta^{-1} \log \beta}^\delta v(t)^p \varphi(s, t) dt &\leq f(b\beta^{-1} \log \beta)^{-p} \int_{c\beta^{-1} \log \beta}^\delta v(t)^p f(t)^p \varphi(s, t) dt \\ &\leq \frac{1}{\beta^{p\kappa b}} \int_0^\delta v(t)^p f(t)^p \varphi(s, t) dt \leq \frac{C_3}{\beta^{p\kappa b}}, \end{aligned}$$

where we have used (33). Hence for $b = 2/(p\kappa)$ we obtain

$$\int_{b\beta^{-1} \log \beta}^{\delta} v(t)^p \varphi(s, t) dt = \mathcal{O}(\beta^{-2}).$$

On the other hand the normalization (26) implies

$$\int_0^{b\beta^{-1} \log \beta} v(t)^p \varphi(s, t) dt = 1 - \int_{b\beta^{-1} \log \beta}^{\delta} v(t)^p \varphi(s, t) dt = 1 + \mathcal{O}(\beta^{-2}).$$

Now, as $\partial_t \varphi / \varphi$ and its derivative in t are uniformly bounded in $S \times (0, \delta)$, we have

$$\begin{aligned} \int_0^{\delta} v(t)^p \partial_t \varphi(s, t) dt &= \int_0^{\delta} \frac{\partial_t \varphi(s, t)}{\varphi(s, t)} v(t)^p \varphi(s, t) dt \\ &= \int_0^{b\beta^{-1} \log \beta} \frac{\partial_t \varphi(s, t)}{\varphi(s, t)} v(t)^p \varphi(s, t) dt + \int_{b\beta^{-1} \log \beta}^{\delta} \frac{\partial_t \varphi(s, t)}{\varphi(s, t)} v(t)^p \varphi(s, t) dt \\ &= \int_0^{b\beta^{-1} \log \beta} \left(\frac{\partial_t \varphi(s, 0)}{\varphi(s, 0)} + \mathcal{O}(\beta^{-1} \log \beta) \right) v(t)^p \varphi(s, t) dt + \mathcal{O}(\beta^{-2}) \\ &= \int_0^{b\beta^{-1} \log \beta} \left(-M(s) + \mathcal{O}(\beta^{-1} \log \beta) \right) v(t)^p \varphi(s, t) dt + \mathcal{O}(\beta^{-2}) \\ &= -M(s) + \mathcal{O}(\beta^{-1} \log \beta), \end{aligned}$$

and the substitution into (35) gives $v(0)^p \leq p \|v' \varphi^{\frac{1}{p}}\|_p + M(s) + \mathcal{O}(\beta^{-1} \log \beta)$. Finally, using the definition of λ we infer that

$$\begin{aligned} \lambda &= \|v' \varphi^{\frac{1}{p}}\|_p^p - \alpha v(0)^p \geq \|v' \varphi^{\frac{1}{p}}\|_p^p - p\alpha \|v' \varphi^{\frac{1}{p}}\|_p - \alpha M(s) + \mathcal{O}(\alpha^{\frac{p-2}{p-1}} \log \alpha) \\ &\geq \inf_{x \in \mathbb{R}_+} (x^p - p\alpha x) - \alpha M(s) + \mathcal{O}(\alpha^{\frac{p-2}{p-1}} \log \alpha) = (1-p)\alpha^{\frac{p}{p-1}} - \alpha M(s) + \mathcal{O}(\alpha^{\frac{p-2}{p-1}} \log \alpha), \end{aligned}$$

where the remainder estimate depends again on $\|\kappa_j\|_{\infty}$ only and is uniform for s outside a zero-measure set. \square

6. Behaviour of minimizers: concentration effects

So far we have been dealing only with the asymptotic behavior of the eigenvalue $\Lambda(\Omega, p, \alpha)$. In this section we will discuss some properties of the minimizers, as soon as they exist. In contrast to the most of the paper, for a part of the results we only require that $\partial\Omega$ be Lipschitz. For the sake of completeness, we include the proof of the existence for bounded Lipschitz domains.

Proposition 6.1. *If $\Omega \subset \mathbb{R}^{\nu}$ is a bounded Lipschitz domain, then the variational problem (1) has a minimizer for every $\alpha \in \mathbb{R}$.*

Proof. Fix $\alpha \in \mathbb{R}$ and let $\{u_j\}_{j \in \mathbb{N}}$ be a minimizing sequence for $\Lambda(\Omega, p, \alpha)$ normalized to one in $L^p(\Omega)$;

$$(36) \quad \lim_{j \rightarrow \infty} \left(\int_{\Omega} |\nabla u_j|^p dx - \alpha \int_{\partial\Omega} |u_j|^p d\sigma \right) = \Lambda(\Omega, p, \alpha), \quad \|u_j\|_{L^p(\Omega)} = 1 \quad \forall j \in \mathbb{N}.$$

By [21, Thm. 1.5.1.10] for any $\varepsilon \in (0, 1)$ there exists a constant $K_{\varepsilon} > 0$ such that the upper bound

$$(37) \quad \int_{\partial\Omega} |u|^p d\sigma \leq \varepsilon \|\nabla u\|_{L^p(\Omega)}^p + K_{\varepsilon} \|u\|_{L^p(\Omega)}^p$$

holds true for all $u \in W^{1,p}(\Omega)$. Applying this inequality with $u = u_j$ and ε sufficiently small, depending on α , we deduce from (36) that

$$\sup_{j \in \mathbb{N}} \|\nabla u_j\|_{L^p(\Omega)} < \infty.$$

It follows that the sequence $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ and therefore admits a weakly converging subsequence, which we still denote by $\{u_j\}_{j \in \mathbb{N}}$. Let u_α be its weak limit in $W^{1,p}(\Omega)$. The compactness of the embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ implies that there exists another subsequence $\{u_{j_k}\}_{k \in \mathbb{N}}$ of $\{u_j\}_{j \in \mathbb{N}}$ such that

$$\|u_{j_k}\|_{L^p(\Omega)} \rightarrow \|u_\alpha\|_{L^p(\Omega)} \quad \text{and} \quad \|u_{j_k}\|_{L^p(\partial\Omega)} \rightarrow \|u_\alpha\|_{L^p(\partial\Omega)}$$

as $k \rightarrow \infty$. Hence $\|u_\alpha\|_{L^p(\Omega)} = 1$ and using the normalization of u_j and the weak lower semi-continuity of $\int_\Omega |\nabla \cdot|^p$ in (36) we arrive at

$$\Lambda(\Omega, p, \alpha) = \liminf_{k \rightarrow \infty} \left(\int_\Omega |\nabla u_{j_k}|^p dx - \alpha \int_{\partial\Omega} |u_{j_k}|^p d\sigma \right) \geq \int_\Omega |\nabla u_\alpha|^p dx - \alpha \int_{\partial\Omega} |u_\alpha|^p d\sigma.$$

This shows that u_α is a minimizer. \square

We mention the paper [27] discussing further properties of the minimizers such as the uniqueness and the strict positivity. These properties are not used in our estimates below.

The following simple estimate for the eigenvalue is an adaption of a result from [20].

Proposition 6.2. *For any bounded Lipschitz domain Ω one has the inequality*

$$\Lambda(\Omega, p, \alpha) \leq (1-p)\alpha^{\frac{p}{p-1}}$$

for all $\alpha \geq 0$ and $p \in (1, \infty)$.

Proof. Set $\beta := \alpha^{\frac{1}{p-1}}$. Without loss of generality one may assume that Ω is contained in the half-space $x_1 > 0$. Let us test on the function $u(x) = e^{-\beta x_1}$. Consider the vector field $F(x) = (e^{-p\beta x_1}, 0, \dots, 0)$, then the divergence theorem gives

$$\int_{\partial\Omega} u^p dx_1 = \int_{\partial\Omega} e^{-p\beta x_1} d\sigma \geq \int_{\partial\Omega} F \cdot n d\sigma = \int_\Omega \nabla \cdot F dx = p\beta \int_\Omega e^{-p\beta x_1} dx = p\beta \int_\Omega u^p dx,$$

and

$$\begin{aligned} \Lambda(\Omega, p, \alpha) &\leq \frac{\int_\Omega |\nabla u|^p dx - \beta^{p-1} \int_{\partial\Omega} u^p d\sigma}{\int_\Omega u^p dx} \leq \frac{\beta^p \int_\Omega u^p dx - p\beta^p \int_\Omega u^p dx}{\int_\Omega u^p dx} \\ &= (1-p)\beta^p = (1-p)\alpha^{\frac{p}{p-1}}. \end{aligned} \quad \square$$

Since the existence minimizers is not always guaranteed, see Remark 1.3, in the following statements we will include it as an assumption. Similar to the proof of Lemma 5.1, we obtain first an exponential decay with respect to the distance from the boundary using Agmon's approach [1].

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^\nu$ be a Lipschitz domain. Assume that for α large enough the problem (1) admits a minimizer $u \equiv u_\alpha$, which we assume non-negative and normalized by $\|u\|_{L^p(\Omega)} = 1$, and that*

$$\Lambda \equiv \Lambda(\Omega, p, \alpha) \rightarrow -\infty, \quad \alpha \rightarrow +\infty.$$

Then for any $\tau \in (0, 1)$ and any $a > 0$ there holds, with $\gamma := (-\Lambda)^{\frac{1}{p}}$,

$$(38) \quad \int_{\text{dist}(x, \partial\Omega) > \frac{a}{\gamma}} \left(|\nabla u(x)|^p - \Lambda |u(x)|^p \right) \exp(\tau\gamma \text{dist}(x, \partial\Omega)) dx = \mathcal{O}(\gamma^p)$$

as $\alpha \rightarrow +\infty$. Furthermore, if Ω is a bounded Lipschitz domain or an admissible domain, then for any $a > 0$ there holds

$$(39) \quad \int_{\text{dist}(x, \partial\Omega) > a\alpha^{-\frac{1}{p-1}}} \left(|\nabla u(x)|^p + \alpha^{\frac{p}{p-1}} |u(x)|^p \right) \times \exp\left((p-1)^{\frac{1}{p}} \alpha^{\frac{1}{p-1}} \text{dist}(x, \partial\Omega) \right) dx = \mathcal{O}(\alpha^{\frac{2p}{p-1}}),$$

as $\alpha \rightarrow +\infty$.

Proof. For $x \in \Omega$, denote $\rho(x) := \text{dist}(x, \partial\Omega)$, then $|\nabla\rho| \leq 1$. Furthermore, for large $L > 0$ denote $\rho_L(x) := \min\{\rho(x), L\}$, then we have again $|\nabla\rho_L| \leq 1$. The presence of the parameter L is only relevant for unbounded Ω , as for a bounded domain one can take L sufficiently large to have $\rho_L = \rho$.

By standard arguments the minimizer u satisfies (2), which should be understood in the weak sense, i.e.

$$(40) \quad \Lambda(\Omega, p, \alpha) \int_{\Omega} |u|^{p-2} u \phi \, dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx - \alpha \int_{\partial\Omega} |u|^{p-2} u \phi \, d\sigma$$

holds for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\partial\Omega)$. The regularity theory of elliptic equations, see e.g. [34], implies that u is $C^{1,\epsilon}$ inside Ω . Let f be a non-negative bounded uniformly Lipschitz function defined in Ω and vanishing in a neighborhood of $\partial\Omega$, then the equality (40) with $\phi := f^p u$ and an integration by parts give

$$\begin{aligned} \Lambda \int_{\Omega} u^p f^p \, dx &= - \int_{\Omega} \nabla \cdot (|\nabla u|^{p-2} \nabla u) u f^p \, dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (f^p u) \, dx \\ &= \int_{\Omega} |\nabla u|^p f^p \, dx + p \int_{\Omega} |\nabla u|^{p-2} f^{p-1} u \nabla u \cdot \nabla f \, dx. \end{aligned}$$

Applying the Young inequality (30) with $A = \left| |\nabla u|^{p-2} \nabla u f^{p-1} \right|$, $B = |u \nabla f|$ and $q = p/(p-1)$ to the second term on the right-hand side we obtain, for any $\varepsilon \in (0, 1)$,

$$-\Lambda \int_{\Omega} u^p f^p \, dx \leq (p\varepsilon - 1) \int_{\Omega} |\nabla u|^p f^p \, dx + p\varepsilon^{1-p} \int_{\Omega} u^p |\nabla f|^p \, dx.$$

Furthermore, let γ and $\varepsilon_0 \in [0, 1)$ be such that $\gamma \rightarrow +\infty$ for $\alpha \rightarrow +\infty$ and

$$(41) \quad -\Lambda \geq (1 - \varepsilon_0)^p \gamma^p \text{ for large } \alpha.$$

In particular, one can simply take $\gamma := (-\Lambda)^{\frac{1}{p}}$ and $\varepsilon_0 \in (0, 1)$. We thus have

$$(1 - \varepsilon_0)^p \gamma^p \int_{\Omega} u^p f^p \, dx + (1 - p\varepsilon) \int_{\Omega} |\nabla u|^p f^p \, dx \leq p\varepsilon^{1-p} \int_{\Omega} u^p |\nabla f|^p \, dx.$$

Furthermore, we may assume that $\delta_0 := 1 - p\varepsilon > 0$, then

$$(42) \quad (1 - \varepsilon_0)^p \gamma^p \int_{\Omega} u^p f^p \, dx + \delta_0 \int_{\Omega} |\nabla u|^p f^p \, dx \leq \frac{p^p}{(1 - \delta_0)^{p-1}} \int_{\Omega} u^p |\nabla f|^p \, dx.$$

To estimate the term on the right-hand side, let us take a function f of a special form. Namely, we let $a > 0$ and $\chi \in C^\infty(\mathbb{R})$ be such that

$$\chi : \mathbb{R} \rightarrow [0, 1], \quad \chi(t) = 0 \text{ for } t \text{ close to } 0, \quad \chi(t) = 1 \text{ for } t \geq a, \quad c_0 := \|\chi'\|_\infty,$$

and set

$$f(x) := \chi(\gamma\rho(x)) e^{k\gamma\rho_L(x)},$$

where the constant $k > 0$ is to be chosen later. Hence

$$\nabla f(x) = \gamma \left(\chi'(\gamma\rho(x)) e^{k\gamma\rho_L(x)} \nabla\rho(x) + k\chi(\gamma\rho(x)) e^{k\gamma\rho_L(x)} \nabla\rho_L(x) \right),$$

and, in particular, $|\nabla f(x)| \leq \gamma \left(|\chi'(\gamma\rho(x))| e^{k\gamma\rho_L(x)} + k f(x) \right)$. Using Proposition B.1 we have

$$|\nabla f(x)|^p \leq (1 + \varepsilon_1) k^p \gamma^p f(x)^p + \frac{c\gamma^p}{\varepsilon_1^{p-1}} \left| \chi'(\gamma\rho(x)) \right|^p e^{pk\gamma\rho_L(x)}, \quad \varepsilon_1 \in (0, 1),$$

implying

$$\int_{\Omega} u^p |\nabla f|^p \, dx \leq (1 + \varepsilon_1) k^p \gamma^p \int_{\Omega} u^p f^p \, dx + \frac{c\gamma^p}{\varepsilon_1^{p-1}} \int_{\Omega} \left| \chi'(\gamma\rho(x)) \right|^p e^{pk\gamma\rho_L(x)} u(x)^p \, dx$$

$$\begin{aligned} &\leq (1 + \varepsilon_1)k^p\gamma^p \int_{\Omega} u^p f^p dx + \frac{c\gamma^p}{\varepsilon_1^{p-1}} \int_{x \in \Omega; \rho(x) \leq \frac{\alpha}{\gamma}} \left| \chi'(\gamma\rho(x)) \right|^p e^{pk\gamma\rho_L(x)} u(x)^p dx \\ &\leq (1 + \varepsilon_1)k^p\gamma^p \int_{\Omega} u^p f^p dx + \frac{cc_0^p\gamma^p}{\varepsilon_1^{p-1}} e^{pka}, \end{aligned}$$

where we used the normalization of u on the last step. The substitution into (42) gives

$$(43) \quad \left((1 - \varepsilon_0)^p - \frac{1 + \varepsilon_1}{(1 - \delta_0)^{p-1}} p^p k^p \right) \gamma^p \int_{\Omega} u^p f^p dx + \delta_0 \int_{\Omega} |\nabla u|^p f^p dx \leq \frac{C\gamma^p}{(1 - \delta_0)^{p-1} \varepsilon_1^{p-1}} e^{pka}$$

with $C := cc_0^p p^p$. Note that all the estimates are uniform in the parameter L entering the definition of ρ_L , hence, one can send L to $+\infty$, which means that (43) also holds for

$$f(x) := \chi(\gamma\rho(x)) e^{k\gamma\rho(x)}.$$

Recall that $\varepsilon_0 \in [0, 1)$ must satisfy (41) while $\delta_0 \in (0, 1)$, $\varepsilon_1 \in (0, 1)$ and $k > 0$ are arbitrary. In particular, if $k \in (0, p^{-1})$ is fixed, then one can choose ε_0 , ε_1 and δ_0 positive but sufficiently small to have

$$(1 - \varepsilon_0)^p - \frac{1 + \varepsilon_1}{(1 - \delta_0)^{p-1}} p^p k^p =: b > 0$$

implying

$$b\gamma^p \int_{\Omega} u^p f^p dx + \delta_0 \int_{\Omega} |\nabla u|^p f^p dx \leq C'\gamma^p, \quad C' := \frac{C}{(1 - \delta_0)^{p-1} \varepsilon_1^{p-1}} e^{pka},$$

and (38) follows from

$$\begin{aligned} &\int_{\text{dist}(x, \partial\Omega) > \frac{\alpha}{\gamma}} \left(|\nabla u(x)|^p + \gamma^p |u(x)|^p \right) \exp\left(\tau\gamma \text{dist}(x, \partial\Omega)\right) dx \\ &\leq \int_{\text{dist}(x, \partial\Omega) > \frac{\alpha}{\gamma}} \left(|\nabla u(x)|^p + \gamma^p |u(x)|^p \right) f(x)^p dx, \quad \tau := kp \in (0, 1). \end{aligned}$$

If Ω is a bounded Lipschitz domain, then by Proposition 6.2 the above constructions work with $\gamma = (p-1)^{\frac{1}{p}} \alpha^{\frac{1}{p-1}}$ and $\varepsilon_0 = 0$, then we can set $\delta_0 = \varepsilon_1 = \gamma^{-1}$ and $k = p^{-1} - B\gamma^{-1}$ with $B > 0$ sufficiently large, which gives

$$(44) \quad (1 - \varepsilon_0)^p - \frac{1 + \varepsilon_1}{(1 - \delta_0)^{p-1}} p^p k^p \geq \frac{1}{\gamma}, \quad \alpha \rightarrow +\infty,$$

and one can proceed in the same way to obtain (39).

If Ω is an admissible domain, then by Theorem 1.1 we can take $\gamma = (p-1)^{\frac{1}{p}} \alpha^{\frac{1}{p-1}}$ and $\varepsilon_0 := A\gamma^{-1}$, with a suitable large $A > 0$, then by setting $\delta = \varepsilon_1 = \gamma^{-1}$ and taking $k = p^{-1} - B\gamma^{-1}$ with a suitable large $B > 0$ we obtain the estimate (44) implying (39) again. \square

We mention a simple but important consequence which will be used below. Recall that Ω_δ and Θ_δ are defined in (10).

Corollary 6.4. *Let Ω be an admissible domain such that the problem (1) admits a minimizer u , which we assume to be non-negative and normalized by $\|u\|_{L^p(\Omega)} = 1$, then for any $\delta > 0$ and any $N > 0$ there holds $\|u\|_{W^{1,p}(\Theta_\delta)} = o(\alpha^{-N})$ as $\alpha \rightarrow +\infty$.*

Finally we are in position to prove a weak form of a localization of the minimizer near the set at which the mean curvature of the boundary takes its maximal value.

Theorem 6.5. *Let Ω be an admissible domain such that the problem (1) admits a minimizer $u \equiv u_\alpha$ for large α , which is assumed to be chosen non-negative and normalized by $\|u\|_{L^p(\Omega)} = 1$. Define $\mathcal{H} : \Omega \rightarrow \mathbb{R}$ by $\mathcal{H}(x) = H(s(x))$, where $s(x) \in \partial\Omega$ is given by $\text{dist}(x, \partial\Omega) = |x - s(x)|$, then*

$$(45) \quad \int_{\Omega} (H_{\max} - \mathcal{H}) u^p dx = o(1), \quad \alpha \rightarrow +\infty.$$

Proof. It is well known that $s(x)$ is uniquely defined for almost all $x \in \Omega$. Moreover, in view of Corollary 6.4, it is sufficient to show that

$$(46) \quad \int_{\Omega_\delta} (H_{\max} - \mathcal{H}) u^p dx = o(1), \quad \alpha \rightarrow +\infty,$$

for some $\delta > 0$. We assume that δ is sufficiently small such that the map (11) is bijective, then for $x \in \Omega_\delta$ one has $s = s(x)$ iff $x = \Phi(s, t)$ for some $t \in (0, \delta)$. Furthermore, by the constructions of Section 3 one has

$$(47) \quad \begin{aligned} \Lambda(\Omega, p, \alpha) \|u\|_{L^p(\Omega)}^p &= \int_{\Omega} |\nabla u|^p dx - \alpha \int_{\partial\Omega} u^p d\sigma \geq \int_{\Omega_\delta} |\nabla u|^p dx - \alpha \int_{\partial\Omega} u^p d\sigma \\ &\geq \int_{\Sigma} \left| C_- |\nabla_s v(s, t)|^2 + v_t(s, t)^2 \right|^{\frac{p}{2}} \varphi(s, t) d\sigma(s) dt - \alpha \int_S |v(s, 0)|^p d\sigma(s), \end{aligned}$$

where $v := u \circ \Phi$ and $C_- > 0$. Using Lemma 5.1 we have

$$\begin{aligned} &\int_{\Sigma} \left| C_- |\nabla_s v(s, t)|^2 + v_t(s, t)^2 \right|^{\frac{p}{2}} \varphi(s, t) d\sigma(s) dt - \alpha \int_S |v(s, 0)|^p d\sigma(s) \\ &\geq \int_{\Sigma} |v_t(s, t)|^p \varphi(s, t) d\sigma(s) dt - \alpha \int_S |v(s, 0)|^p d\sigma(s) \\ &\geq \int_{\Sigma} \left[(1-p)\alpha^{\frac{p}{p-1}} - (\nu-1)H(s)\alpha + o(\alpha) \right] v(s, t)^p \varphi(s, t) d\sigma(s) dt \\ &= (1-p)\alpha^{\frac{p}{p-1}} \|u\|_{L^p(\Omega_\delta)}^p - \alpha(\nu-1) \int_{\Omega_\delta} \mathcal{H} u^p dx + o(\alpha) \|u\|_{L^p(\Omega_\delta)}^p. \end{aligned}$$

The substitution into (47) and the asymptotic expansion (4) for $\Lambda(\Omega, p, \alpha)$ give

$$\begin{aligned} (1-p)\alpha^{\frac{p}{p-1}} \|u\|_{L^p(\Omega)}^p - \alpha(\nu-1)H_{\max} \|u\|_{L^p(\Omega)}^p + o(\alpha) \|u\|_{L^p(\Omega)}^p \\ \geq (1-p)\alpha^{\frac{p}{p-1}} \|u\|_{L^p(\Omega_\delta)}^p - \alpha(\nu-1) \int_{\Omega_\delta} \mathcal{H} u^p dx + o(\alpha) \|u\|_{L^p(\Omega_\delta)}^p. \end{aligned}$$

Using $1 = \|u\|_{L^p(\Omega)}^p = \|u\|_{L^p(\Omega_\delta)}^p + \|u\|_{L^p(\Theta_\delta)}^p$ we arrive at

$$\alpha \int_{\Omega_\delta} (H_{\max} - \mathcal{H}) u^p dx \leq o(\alpha) - \frac{p-1}{\nu-1} \alpha^{\frac{1}{p-1}} \|u\|_{L^p(\Theta_\delta)}^p - H_{\max} \|u\|_{L^p(\Theta_\delta)}^p,$$

and the result follows from Corollary 6.4. \square

APPENDIX A. Solvable cases

For the sake of completeness, let us mention some cases in which $\Lambda(\Omega, p, \alpha)$ can be computed explicitly.

Proposition A.1. *For any $p \in (1, \infty)$ and $\alpha > 0$ there holds $\Lambda(\mathbb{R}_+, p, \alpha) = (1-p)\alpha^{\frac{p}{p-1}}$, and the minimizer u_* for Eq. (1) is given by $u_*(t) = \exp(-\alpha^{\frac{1}{p-1}} t)$.*

Proof. By computing the right-hand side of (1) for $u = u_*$ we obtain the inequality $\Lambda(\mathbb{R}_+, p, \alpha) \leq (1-p)\alpha^{\frac{p}{p-1}}$. For the reverse inequality, we remark that $\lim_{x \rightarrow +\infty} u(x) = 0$ for any $u \in W^{1,p}(\mathbb{R}_+)$ and, using the Hölder inequality,

$$|u(0)|^p = -p \int_0^{+\infty} |u|^{p-1} |u'| dt \leq p \int_0^{+\infty} |u|^{p-1} ||u'| dt \leq p \|u\|_p^{p-1} \|u'\|_p \leq p \|u\|_p^{p-1} \|u'\|_p.$$

Therefore,

$$\begin{aligned} \inf_{\substack{u \in W^{1,p}(\mathbb{R}_+) \\ u \neq 0}} \frac{\int_0^{+\infty} |u'(t)|^p dt - \alpha |u(0)|^p}{\int_0^{+\infty} |u(t)|^p dt} &= \inf_{\substack{v \in W^{1,p}(\mathbb{R}_+) \\ \|v\|_p = 1}} \left(\|v'\|_p^p - \alpha |v(0)|^p \right) \\ &\geq \inf_{\substack{v \in W^{1,p}(\mathbb{R}_+) \\ \|v\|_p = 1}} \left(\|v'\|_p^p - \alpha p \|v\|_p \right) \geq \inf_{x \in \mathbb{R}_+} (x^p - p\alpha x) = (1-p)\alpha^{\frac{p}{p-1}}, \end{aligned}$$

which gives the sought result. \square

As observed in [24], the one-dimensional result can be used to study the infinite planar sectors

$$U_\theta := \{(x_1, x_2) \in \mathbb{R}^2 : |\arg(x_1 + ix_2)| < \theta\}, \quad 0 < \theta < \pi.$$

Proceeding literally as in Lemma 2.6 and Lemma 2.8 of [24] one arrives at the following result:

Proposition A.2. *Let $\alpha > 0$ and $p \in (1, +\infty)$, then for $\theta \geq \frac{\pi}{2}$ there holds $\Lambda(U_\theta, p, \alpha) = (1-p)\alpha^{\frac{p}{p-1}}$, while for $\theta < \frac{\pi}{2}$ one has*

$$\Lambda(U_\theta, p, \alpha) = (1-p) \left(\frac{\alpha}{\sin \theta} \right)^{\frac{p}{p-1}} < (1-p)\alpha^{\frac{p}{p-1}},$$

which is attained on

$$u(x_1, x_2) = \exp \left(- \left(\frac{\alpha}{\sin \theta} \right)^{\frac{1}{p-1}} x_1 \right).$$

APPENDIX B. An auxiliary inequality

Proposition B.1. *Let $p > 1$, then for any $\varepsilon \in (0, 1)$ and for all $a, b \geq 0$ there holds*

$$(a+b)^p \leq (1+\varepsilon)a^p + \frac{c}{\varepsilon^{p-1}} b^p, \quad c := \max \left\{ \left(1 - 2^{\frac{1}{1-p}}\right)^{1-p}, 1 \right\}.$$

Proof. By homogeneity, it is sufficient to show that $(1+t)^p \leq (1+\varepsilon)t^p + c\varepsilon^{1-p}$ for all $t \geq 0$. Denote $f_\varepsilon(t) := (1+t)^p - (1+\varepsilon)t^p$, then one simply needs an upper estimate for $C(\varepsilon) := \varepsilon^{p-1} \sup_{t \in \mathbb{R}_+} f_\varepsilon(t)$. We have $f_\varepsilon(0) = 1$ and $f_\varepsilon(+\infty) = -\infty$. The equation $f'_\varepsilon(t) = 0$ has a unique solution

$$t = t_\varepsilon = \frac{1}{(1+\varepsilon)^{\frac{1}{p-1}} - 1}, \quad f(t_\varepsilon) = \frac{1+\varepsilon}{\left((1+\varepsilon)^{\frac{1}{p-1}} - 1\right)^{p-1}},$$

implying

$$C(\varepsilon) = \max \left\{ \frac{\varepsilon^{p-1}(1+\varepsilon)}{\left((1+\varepsilon)^{\frac{1}{p-1}} - 1\right)^{p-1}}, 1 \right\}.$$

Denote $\sigma := (1+\varepsilon)^{\frac{1}{p-1}} - 1$, then

$$\sup_{\varepsilon \in (0,1)} \frac{\varepsilon^{p-1}(1+\varepsilon)}{\left((1+\varepsilon)^{\frac{1}{p-1}} - 1\right)^{p-1}} = \sup_{\sigma \in (0, \sigma_1)} \left(\frac{(1+\sigma)^p - (1+\sigma)}{\sigma} \right)^{p-1}, \quad \sigma_1 := 2^{\frac{1}{p-1}} - 1.$$

Using the convexity of $\varphi(\sigma) = (1+\sigma)^p - (1+\sigma)$ we have

$$\varphi(\sigma) \leq \varphi(0) + \frac{\varphi(\sigma_1) - \varphi(0)}{\sigma_1} \sigma \equiv \frac{2^{\frac{1}{p-1}}}{2^{\frac{1}{p-1}} - 1} \sigma, \quad \sigma \in (0, \sigma_1),$$

which gives the sought inequality. \square

APPENDIX C. Remainder estimates for more regular domains

If a stronger regularity of Ω is imposed, the remainder in Theorem 1.1 can be made more explicit.

Proposition C.1. *In Theorem 1.1 assume additionally that the boundary of Ω is C^3 smooth and that the mean curvature attains the maximum value H_{\max} , then the remainder estimate in (4) can be improved to $\mathcal{O}(\alpha^{1-\kappa})$ with*

$$(48) \quad \kappa = \begin{cases} \frac{1}{p+1}, & p \in (1, 2], \\ \frac{1}{3(p-1)}, & p \in (2, \infty). \end{cases}$$

If, in addition, Ω is C^4 smooth, then one can take

$$(49) \quad \kappa = \begin{cases} \frac{2}{p+2}, & p \in (1, 2], \\ \frac{1}{2(p-1)}, & p \in (2, \infty). \end{cases}$$

Proof. Remark first that the result of section 5 imply

$$\Lambda(\Omega, p, \alpha) \geq (1-p)\alpha^{\frac{p}{p-1}} - (\nu-1)H_{\max}\alpha + \mathcal{O}(\alpha^{\frac{p}{p-1}} \log \alpha),$$

and $\alpha^{\frac{p}{p-1}} \log \alpha = o(\alpha^{1-\kappa})$ for κ given by (48) or (49). Therefore, it is sufficient to show the upper bound, which will be done by taking another test function in the computations of Section 4.

Let $s_0 \in S$ be such that $M(s_0) = M_{\max}$. As $\partial\Omega$ is C^3 smooth, then M is C^1 , and for some $m \geq 0$ we have $-M_{\max} \leq -M(s) \leq -M_{\max} + md(s, s_0)$ with $d(\cdot, \cdot)$ standing for the geodesic distance on S and for s sufficiently close to s_0 ,

Let us choose a function $f \in C_c^\infty(\mathbb{R})$ which equals 1 in a neighborhood of the origin and consider the functions $v \in W^{1,p}(S)$ given by

$$(50) \quad v(s) = f\left(\frac{d(s, s_0)}{\mu}\right),$$

where μ is a positive parameter which tends to 0 as $\beta \rightarrow +\infty$ and will be chosen later. One can estimate, for large β ,

$$(51) \quad \begin{aligned} A &:= \int_S |v(s)|^p d\sigma(s) = a\mu^{\nu-1} + \mathcal{O}(\mu^\nu), \quad A^{-1} = \mathcal{O}(\mu^{1-\nu}), \\ \int_S (-M(s))|v(s)|^p d\sigma(s) &= -M_{\max}A + \mathcal{O}(\mu^\nu), \\ \int_S |\nabla v(s)|^p d\sigma(s) &= \mathcal{O}(\mu^{\nu-p-1}). \end{aligned}$$

Consider first the case $p \in (1, 2]$. The substitution into (17) gives

$$\begin{aligned} \Lambda^+(p, \alpha) &\leq \left\{ \mathcal{O}(\mu^{\nu-p-1}\beta^{-1}) + \frac{1}{p}\beta^{p-1}A - \frac{1}{p^2}\beta^{p-2}M_{\max}A + \mathcal{O}(\mu^\nu\beta^{p-2}) + \mathcal{O}(\mu^{\nu-1}\beta^{p-3}) - \beta^{p-1}A \right\} \\ &\quad \times \left\{ \frac{1}{p\beta}A - \frac{M_{\max}}{p^2\beta^2}A + \mathcal{O}(\mu^\nu\beta^{-2}) + \mathcal{O}(\mu^{\nu-1}\beta^{-3}) \right\}^{-1} \\ &= \left\{ \frac{1-p}{p}\beta^{p-1}A - \frac{1}{p^2}M_{\max}\beta^{p-2}A + \mathcal{O}(\mu^{\nu-p-1}\beta^{-1} + \mu^\nu\beta^{p-2} + \mu^{\nu-1}\beta^{p-3}) \right\} \\ &\quad \times \left\{ \frac{A}{p\beta} \left(1 - \frac{M_{\max}}{p\beta} + \mathcal{O}(\mu\beta^{-1} + \beta^{-2}) \right) \right\}^{-1} \\ &= \left((1-p)\beta^p - \frac{1}{p}M_{\max}\beta^{p-1} + \mathcal{O}(\mu^{-p} + \mu\beta^{p-1} + \beta^{p-2}) \right) \left(1 + \frac{M_{\max}}{p\beta} + \mathcal{O}(\mu\beta^{-1} + \beta^{-2}) \right) \end{aligned}$$

$$= (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\mu^{-p} + \mu\beta^{p-1} + \beta^{p-2}).$$

The remainder is optimized by $\mu = \beta^{-\frac{p-1}{p+1}}$, and we arrive

$$\begin{aligned} \Lambda^+(p, \alpha) &\leq (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\beta^{p-1-\frac{p-1}{p+1}}) \\ &= (1-p)\alpha^{\frac{p}{p+1}} - M_{\max}\alpha + \mathcal{O}(\alpha^{1-\frac{1}{p+1}}). \end{aligned}$$

If, in addition, Ω is C^4 smooth, then M is C^2 smooth, and for some $m \geq 0$ we have $-M_{\max} \leq -M(s) \leq -M_{\max} + md(s, s_0)^2$ as s is sufficiently close to s_0 , which allows one to replace the second estimate in (51) by

$$(52) \quad \int_S (-M(s))|v(s)|^p d\sigma(s) = -M_{\max}A + \mathcal{O}(\mu^{\nu+1}),$$

and a similar computation gives

$$\Lambda^+(p, \alpha) \leq (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\mu^{-p} + \mu^2\beta^{p-1} + \beta^{p-2}).$$

Hence, taking $\mu = \beta^{-\frac{p-1}{p+2}}$ we arrive at

$$\begin{aligned} \Lambda^+(p, \alpha) &\leq (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\beta^{p-1-\frac{2(p-1)}{p+2}}) \\ &= (1-p)\alpha^{\frac{p}{p+1}} - M_{\max}\alpha + \mathcal{O}(\alpha^{1-\frac{2}{p+2}}). \end{aligned}$$

For the case $p \in (2, +\infty)$, the substitution of (51) into (22) gives

$$\begin{aligned} \Lambda^+(p, \alpha) &\leq \left\{ \frac{1-p}{p}\beta^{p-1}A - \frac{M_{\max}}{p^2}\beta^{p-2}A + \mathcal{O}(\varepsilon_0^{\frac{2-p}{2}}\mu^{\nu-p-1}\beta^{-1} + \mu^\nu\beta^{p-2} \right. \\ &\quad \left. + \mu^{\nu-1}\beta^{p-3} + \varepsilon_0\mu^{\nu-1}\beta^{p-1}) \right\} \times \left\{ \frac{A}{p\beta} \left(1 - \frac{M_{\max}}{p\beta} + \mathcal{O}(\mu\beta^{-1} + \beta^{-2}) \right) \right\}^{-1} \\ &= \left((1-p)\beta^p - \frac{1}{p}M_{\max}\beta^{p-1} + \mathcal{O}(\varepsilon_0^{\frac{2-p}{2}}\mu^{-p} + \mu\beta^{p-1} + \beta^{p-2} + \varepsilon_0\beta^p) \right) \\ &\quad \times \left(1 + \frac{M_{\max}}{p\beta} + \mathcal{O}(\mu\beta^{-1} + \beta^{-2}) \right) \\ &= (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\varepsilon_0^{\frac{2-p}{2}}\mu^{-p} + \mu\beta^{p-1} + \beta^{p-2} + \varepsilon_0\beta^p). \end{aligned}$$

In order to optimize we solve $\varepsilon_0^{\frac{2-p}{2}}\mu^{-p} = \mu\beta^{p-1} = \varepsilon_0\beta^p$, which gives

$$\mu = \beta^{-1/3}, \quad \varepsilon_0 = \beta^{-4/3}, \quad \mathcal{O}(\varepsilon_0^{\frac{2-p}{2}}\mu^{-p} + \mu\beta^{p-1} + \beta^{p-2} + \varepsilon_0\beta^p) = \mathcal{O}(\beta^{p-\frac{4}{3}}),$$

and, therefore,

$$\Lambda^+(p, \alpha) \leq (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\beta^{p-\frac{4}{3}}) = (1-p)\alpha^{\frac{p}{p+1}} - M_{\max}\alpha + \mathcal{O}(\alpha^{1-\frac{1}{3(p+1)}}).$$

For C^4 domains, a similar computation using (52) gives

$$\Lambda^+(p, \alpha) \leq (1-p)\beta^p - M_{\max}\beta^{p-1} + \mathcal{O}(\varepsilon_0^{\frac{2-p}{2}}\mu^{-p} + \mu^2\beta^{p-1} + \beta^{p-2} + \varepsilon_0\beta^p).$$

The remainder is optimized by

$$\mu = \beta^{-1/4}, \quad \varepsilon_0 = \beta^{-3/2}, \quad \mathcal{O}(\varepsilon_0^{\frac{2-p}{2}}\mu^{-p} + \mu\beta^{p-1} + \beta^{p-2} + \varepsilon_0\beta^p) = \mathcal{O}(\beta^{p-\frac{3}{2}}),$$

which gives the sought result. We remark that for $p = 2$ in (49) we obtain $\kappa = 1/2$, which is optimal, see [22, 31]. \square

Finally, an easy revision of the proof of Theorem 6.5 gives the following result:

Corollary C.2. *Under the assumptions of Proposition C.1, the right-hand-side of (45) can be improved to $\mathcal{O}(\alpha^{-\kappa})$.*

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