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PhD. Thesis

Approximation properties of solutions to fractional equations, Riemann-Liouville fractional Sobolev spaces and a variational problem in Carnot Groups.

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"Karma police, arrest this man. He talks in maths, he buzzes like a fridge, he's like a detuned radio"

Karma Police, Radiohead.

"Non l'ho mai fatto, ma l'ho sempre sognato."

Paolo Villaggio, dal film *"Fantozzi"*.

"La metafora... come dirti... è quando parli di una cosa paragonandola a un'altra... per esempio quando dici "Il cielo piange" che cosa vuol dire?"
'Che... che sta piovendo?'
"Sì, bravo. Questa è una metafora."
'Allora è semplice... ebbè perché ci ha questo nome così complicato?'
"Gli uomini non hanno niente a che vedere con la semplicità o la complessità delle cose."

Philippe Noiret e Massimo Troisi, dal film "*Il Postino*".

Contents

Preface

This doctoral thesis is focused on some problems involving nonlocal operators.

Chapters 1 and 2 are devoted to a general introduction of the topics and to some preliminary notions.

In chapter 3 we give some new results about solutions of fractional equations. Namely we prove that solutions of equations involving a pretty general linear fractional operator, are locally dense among smooth functions. Moreover, we provide some applications of this result. This is a joint work with S. Dipierro and E. Valdinoci [CDV19].

In chapter 4 we analyze some properties of the so called Riemann-Liouville fractional sobolev spaces; in particular, we study what kind of inclusions hold between these spaces and other well known ones such as the Gagliardo-Slobodeckij fractional Sobolev space $W^{s,1}$, and the space BV of functions with bounded variation. This is a joint work with G.E. Comi [CC20].

In chapter 5 we study a minimization problem for nonlocal functionals in Carnot groups. Namely, we prove that, analogously as in the euclidean case, halfspaces are local minimizers for a class of functionals called nonlocal perimeters. Moreover, a partial Γ-convergence result is proved. This is a joint work with S. Don, D. Pallara and A. Pinamonti [CDPP20].

The final appendix A contains some other technical results which are widely used throughout this work; the original results concerning Caputo fractional derivatives have been obtained in a joint paper with S. Dipierro and E. Valdinoci [CDV18].

Chapter 1

Introduction

Nonlocal operators of fractional type present a variety of challenging problems in pure mathematics, also in connections with long-range phase transitions and nonlocal minimal surfaces, and are nowadays commonly exploited in a large number of models describing complex phenomena related to anomalous diffusion and boundary reactions in physics, biology and material sciences (see e.g. [BV16, dlLV11] for several examples, for instance in atom dislocations in crystals, water waves models and quasi-geostrophic equations). Furthermore, anomalous diffusion in the space variables can be seen as the natural counterpart of discontinuous Markov processes (see e.g. [Val09] for a simple explanation on how nonlocal operators and discontinuous Markov processes are related), thus providing important connections with problems in probability and statistics, and several applications to economy and finance (see e.g. [MVN68, Man12] for pioneer works relating anomalous diffusion and financial models).

On the other hand, the development of time-fractional derivatives began at the end of the seventeenth century, also in view of contributions by mathematicians such as Leibniz, Euler, Laplace, Liouville, Abel, Heaviside, and many others, see e.g. [Ros74, Ros77, KR85, Ros92, Fer18] and the references therein for several interesting scientific and historical discussions. From the point of view of the applications, time-fractional derivatives naturally provide a model to comprise memory effects in the description of the phenomena under consideration. The definition of fractional derivative (at least the most exploited) needs the definition of fractional integral; Riemann-Liouville fractional integral is the most celebrated in literature and most used in the applications and the use of Riemann-Liouville fractional integral allow to define Riemann-Liouville fractional Sobolev space $W^{s,p}_{RL,a+}(I)$, for $p \geq 1$, $s \in (0,1)$ and I a open bounded interval. This space is given by functions $u \in L^p(I)$ such that its left Riemann-Liouville $(1-s)$-fractional integral

$$\frac{1}{\Gamma(1-s)} \int_a^x \frac{u(t)}{(x-t)^s} dt,$$

belongs to $W^{1,p}(I)$. We notice that the Riemann-Liouville is a particular case of Volterra operator, with a singular kernel having an L^1-type singularity, and this makes this space intrinsecally different with respect to the Gagliardo-Slobodeckij fractional Sobolev space whose functions have a finite integral seminorm defined through a singular kernel with a non L^1-singularity. Among the others, the book by [SKM93] offers many highlights and applications involving fractional derivatives also of different type beyond the Riemann-Liouville one such as the Caputo and the Marchaud fractional derivative that will be used throughout this work, and other ones such as the Grunwald-Letnikov, the Hadamard and the Weyl fractional derivative. We notice that in the last years many types of fractional derivatives have been introduced, but some of them can be reduced to a derivative of integer order via some computations. We refer to [Tar13, Tar16], where the author points out that the failure of the usual Leibniz rule and of the chain rule are necessary conditions to ensure that we are actually dealing with a derivative of fractional order.

In the variational framework nonlocal functionals arise for example in peridynamics, image processing, shape optimization and nonlocal minimal surfaces [BMC14, BN18, BRS16, CRS10]. A pretty general nonlocal functional has the following expression

$$\mathcal{G}(u,\Omega,\Omega') := \int_{\Omega}\int_{\Omega'} H(u(x)-u(y))K(x-y)dxdy, \tag{1.1}$$

for some $\Omega, \Omega' \subseteq \mathbb{R}^n$, open sets $H : \mathbb{R} \to [0,+\infty)$ convex, and some positive kernel K.
In particular, if we choose in (1.1) $H(z) := |z|$, a functional of the type

$$\mathcal{F}(u,\Omega) := \frac{1}{2}\mathcal{G}(u,\Omega,\Omega) + \mathcal{G}(u,\Omega,\Omega^c) \tag{1.2}$$

is called nonlocal perimeter if the kernel K satisfies additional assumptions such as weighted local integrability, integrability at infinity and radial symmetry; one typical example is given by the fractional kernel $K(z) := |z|^{-n-s}$ for some $s \in (0,1)$.
If we choose $H(z) := z^2$ in (1.1) the functional

$$\mathcal{F}(u,\Omega) := \frac{1}{4}\mathcal{G}(u,\Omega,\Omega) + \int_{\Omega} W(u(x))dx, \tag{1.3}$$

where W is a double-well potential[1] and K is an anisotropic kernel, has been studied in [AB98] in the framework of phase transition problems; in particular, having in mind a two-phase fluid model, in that paper the authors prove that the interface between the two admissible phases tends to zero in a suitable way.
We mention also the paper by Savin and Valdinoci [SV12] in which the authors prove a Γ-convergence result for the rescaled limit of the functional given by (1.2) plus the same potential energy as in (1.3) and with the fractional kernel $K(x) = |x|^{-n-s}$; in particular the novelty is the fact that the Γ-limit is a nonlocal functional when $s \in (0, \frac{1}{2})$, but quite surprisingly is the classical perimeter when $s \in [\frac{1}{2}, 1)$; for this reason the authors refer to this behaviour as "strongly nonlocal regime" in the range $(0, \frac{1}{2})$, and "weakly nonlocal regime" in the range $s \in [\frac{1}{2}, 1)$.
The functional in (1.1) is strictly related even with the Theory of nonlocal Dirichlet forms; in fact if we choose $H(z) := |z|^2$, and $K(z) := |z|^{-n-2s}$ for some $s \in (0,1)$ the free critical points of the energy functional given by the nonlocal quadratic form

$$\mathcal{A}[u] := \frac{c_{n,s}}{4}\mathcal{G}(u, \mathbb{R}^n \setminus \Omega^c, \mathbb{R}^n \setminus \Omega^c) - \int_{\Omega} fudx \qquad f \in L^2(\Omega),$$

are weak solutions of the equation

$$\begin{cases} (-\Delta)^s u = f & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega. \end{cases}$$

[1] A double-well potential with wells in a and b is any nonnegative function $W \in C^2(\mathbb{R})$ such that $W(a) = W(b) = 0$, $W > 0$ in $\mathbb{R} \setminus \{a, b\}$, $W'(a) = W'(b) = 0$, $\min\{W''(a), W''(b)\} > 0$

In this work, we are interested on functionals as in (1.2). Exploiting suitable calibration methods in [Cab19, Pag19] the local minimality of halfspaces for euclidean nonlocal perimeters is proved. Moreover, in [AB98, BP19] the authors prove that the Γ-limit of a suitable rescaled sequence is the classical perimeter, up to a multiplicative constant; these proofs provide the use of density estimates originally introduced in [FM93] for the $\Gamma - \liminf$ inequality, while the $\Gamma - \limsup$ inequality needs some intrinsically euclidean techniques such as polyhedral approximation of the finite perimeter set, or approximation results for the total variation of the gradient of a BV function by means of weighted integrals of the difference quotient.

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Thanks for all the moments shared and for those to be shared.

Chapter 2

Preliminaries

We start with some tools that will be strongly used throughout this work; we notice that Caputo fractional derivative will be indicated with two different notations: here and in Chapter 4 we will use the notation ${}^C D^s_{a+}[u]$ in order to avoid disambiguities with the Riemann-Liouville fractional derivative, while in Chapter 3 we will refer to it with the notation $D^s_{t,a}u$; the two pedices refer respectively to the variable on which the derivative acts and to the initial point (since we will work only with left derivatives we omit the symbol $+$).

Definition 2.1 (Euler's Gamma function). *Let $z \in \mathbb{C}$ with $\Re(z) > 0$. The Euler's Gamma function is given by*

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} dt. \tag{2.1}$$

moreover, since the following identity holds true for any $k \in \mathbb{N}$

$$(z+k-1)(z+k-2)\ldots(z+1)z\Gamma(z) = \Gamma(z+k),$$

definition (2.1) extends to any $z \in \mathbb{C} \setminus \mathbb{Z}_-$.

Definition 2.2 (Beta function). *Let $z, w \in \mathbb{C}$, $\Re(z), \Re(w) > 0$. The Beta function is given by*

$$\beta(z,w) := \int_0^1 t^{z-1}(1-t)^{w-1} dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^{+\infty} \frac{t^{z-1}}{(1+t)^{z+w}} dt$$

Remark 2.3. *For $x, y \in \mathbb{R}$, $y < x$, using the change of variable $s := \frac{t-y}{x-y}$, the following identity immediately follows*

$$\int_y^x (x-t)^{\alpha-1}(t-y)^{\sigma-1} dt = (x-y)^{\alpha+\sigma-1} \int_0^1 s^{\alpha-1}(1-s)^{\sigma-1} ds = \frac{\Gamma(\alpha)\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}(x-y)^{\alpha+\sigma-1}.$$

The results developed in Chapter 4 are all given in dimension one, and here we shall work on bounded open intervals $I = (a,b)$ for some $a, b \in \mathbb{R}$, $a < b$. As it is customary, we denote by $\mathcal{M}(I)$ and $\mathcal{M}(\overline{I})$ the spaces of finite Radon measures over I and $\overline{I} = [a,b]$, respectively. We shall say that $\rho \in C_c^\infty((-1,1))$ is a standard mollifier if $\rho \geq 0$, $\rho(x) = \rho(-x)$ and $\displaystyle\int_{-1}^1 \rho \, dx = 1$. In addition, for any $\varepsilon > 0$, we set $\rho_\varepsilon(x) := \dfrac{1}{\varepsilon}\rho\left(\dfrac{x}{\varepsilon}\right)$.

For the convenience of the reader we recall here definition and some properties of some well known functional spaces.

Definition 2.4. *Let* $1 \leq p < \infty$. *We say that a measurable function* u *belongs to the Marcinkiewicz space* $L^{p,\infty}(I)$ *if*

$$\sup_{t>0} tU(t)^{1/p} < \infty,$$

where for any $t > 0$, $U(t)$ *denotes the indicator function of* u *defined as*

$$U(t) := \mathcal{L}^1(\{x \in I \mid |u(x)| > t\}).$$

2.1 Some facts about BV functions on the real line

Definition 2.5. *We say that* $u \in BV(I)$ *if* $u \in L^1(I)$ *and its weak derivative* Du *is a finite Radon measure; that is, if there exists a finite Radon measure* μ *such that*

$$\int_a^b u(x)\phi'(x)dx = -\int_a^b \phi(x)d\mu(x),$$

for any $\phi \in C_c^1(I)$, *in which case we have* $\mu = Du$ *in* $\mathcal{M}(I)$.

We recall that the space $BV(I)$ is a Banach space when equipped with the norm $\|u\|_{BV(I)} := \|u\|_{L^1(I)} + |Du|(I)$.

For the ease of the reader, we recall here a well-known result on the boundedness of BV functions on segments of the real line.

Lemma 2.6. *We have* $BV(I) \subset L^\infty(I)$ *with a continuous immersion. In particular,*

$$\|u\|_{L^\infty(I)} \leq \max\left\{1, \frac{1}{b-a}\right\} \|u\|_{BV(I)}, \tag{2.2}$$

for any $u \in BV(I)$.

Proof. Thanks to [EG15, Claim 3, Proof of Lemma 5.21], we know that, for any $u \in BV(I)$ and $\mathcal{L}^1$-a.e. $z \in I$,

$$|u(z)| \leq \frac{1}{b-a} \int_a^b |u(x)|\, dx + |Du|(I).$$

Hence, (2.2) follows immediately. □

As a consequence, it is not difficult to show that, if $u \in BV(I)$ and we set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in I, \\ 0 & \text{if } x \in \mathbb{R} \setminus I, \end{cases}$$

then $\tilde{u} \in BV(\mathbb{R})$. In addiction, we may prove that, if $u \in BV(I)$, the approximate limits of u in a from the right, $u(a+)$, and in b from the left, $u(b-)$, exist and they coincide with the precise representative of $\tilde{u}\chi_I$ on those points. In other words, we have

$$u(a+) := \lim_{r\to 0} \frac{1}{r} \int_a^{a+r} u(x)\, dx \quad \text{and} \quad u(b-) := \lim_{r\to 0} \frac{1}{r} \int_{b-r}^{b} u(x)\, dx.$$

In addition, thanks to [AFP00, Corollary 3.80] it is possible to see that, for any standard mollifier ρ, we have

$$(\rho_\varepsilon * u)(a) \to u(a+) \text{ and } (\rho_\varepsilon * u)(b) \to u(b-). \tag{2.3}$$

Finally, it is easy to notice that, consistently with [BLNT17, Remark 4.1],

$$D\tilde{u} = Du \llcorner I + u(a+)\delta_a - u(b-)\delta_b, \tag{2.4}$$

where δ is the Dirac delta measure; while clearly $D\tilde{u} = 0$ in $\mathbb{R} \setminus \overline{I}$.

Now, we recall some known facts in Measure Theory. If $\mu \in \mathcal{M}(I)$, then, according to the Radon-Nikodym Theorem, we can split it into an absolutely continuous part (with respect to the Lebsegue measure) μ_{ac}, and a singular part μ_s, such that $\mu = \mu_{ac} + \mu_s$. Moreover, following [AFP00, Corollary 3.33], we can decompose the singular part μ_s into an atomic measure μ_j and a diffuse measure μ_c; in this way, we have that

$$\mu = \mu_{ac} + \mu_s = \mu_{ac} + \mu_j + \mu_c.$$

In particular, this decomposition induces an analogous decomposition on BV functions on the real line, which does not have a counterpart in the high dimensions. Namely, following [AFP00, Corollary 3.33], for any $u \in BV(I)$ one has that

$$u = u_{ac} + u_j + u_c,$$

where $u_{ac} \in W^{1,1}(I)$, u_j is a jump function and u_c is a Cantor function; that is, they satisfy

$$(Du)_{ac} = u'_{ac}\mathcal{L}^1, \quad (Du)_j = Du_j \text{ and } (Du)_c = Du_c.$$

In particular, the functions $u \in BV(I)$ such that $Du_c = 0$ in I form a special vector subspace of $BV(I)$ known as $SBV(I)$.

2.2 Some known results in fractional Sobolev spaces

In this section we recall the definition of Gagliardo-Slobodeckij fractional Sobolev spaces $W^{s,p}(\Omega)$; in particular, we mention here two results that we will need in the sequel, namely: a fractional Hardy inequality introduced in [Dyd04] and plainly proved for any $1 \leq p < \infty$ in [Lom18], and the density of the set of smooth compactly supported functions $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$.

Definition 2.7. *Let $s \in (0,1)$, $p \in [1,+\infty)$, and Ω be an open set in $\mathbb{R}^n$ for some $n \geq 1$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ as*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega); \quad \frac{u(x) - u(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}.$$

The quantity

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{sp+n}} dxdy \right)^{1/p},$$

is usually called Gagliardo-Slobodeckij seminorm *of u; the space $W^{s,p}(\Omega)$, endowed with the norm*

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)} \right)^{1/p},$$

is a Banach space, which is Hilbert when $p = 2$. See e.g. [DNPV12].

Remark 2.8. *We notice that if $s > 0$, $s = m + \sigma$ for some $m \in \mathbb{N}_0$ and some $\sigma \in (0,1)$, we say that u belongs to the fractional Sobolev space $W^{s,p}(\Omega)$ if $u \in W^{m,p}(\Omega)$ and $D^\alpha u \in W^{\sigma,p}(\Omega)$ for any multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m$.*

Lemma 2.9 ([Lom18], Theorem D.1.4.)**.** *Let $n \geq 1$, $s \in (0,1)$, $p \in [1,+\infty)$ such that $sp < 1$ and let $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary. Then, there exists $c = c(n,s,p,\Omega) \geq 1$ such that*

$$\int_\Omega \frac{|u(x)|^p}{|\delta_\Omega(x)|^{sp}} dx \leq c \, \|u\|^p_{W^{s,p}(\Omega)},$$

for any $u \in W^{s,p}(\Omega)$. The quantity $|\delta_\Omega(x)| := dist(x, \partial\Omega)$, denotes the signed distance from the boundary of Ω. In particular, when Ω is a bounded open interval $I := (a,b) \subseteq \mathbb{R}$, $|\delta_I(x)| = \min\{x-a, b-x\}$.

Theorem 2.10 ([Lom18], Theorem D.2.1.)**.** *Let $n \geq 1$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $1 \leq p < \infty$ such that $sp < 1$.*

Then, we have that $W^{s,p}_0(\Omega) := \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}} = W^{s,p}(\Omega)$, i.e. $C^\infty_c(\Omega)$ is dense in $W^{s,p}(\Omega)$.

Remark 2.11. *As a byproduct of Theorem 2.10, we have that also $C^1_c(\Omega)$ is dense in $W^{s,1}(\Omega)$.*

2.3 Riemann-Liouville fractional operators

2.3.1 Fractional Integrals

Definition 2.12. *Let $u \in L^1(I)$ and $s \in (0,1)$. We define the left and the right Riemann-Liouville s-fractional integral as*

$$I^s_{a+}[u](x) := \frac{1}{\Gamma(s)} \int_a^x \frac{u(t)}{(x-t)^{1-s}} dt, \tag{2.5}$$

and

$$I^s_{b-}[u](x) := \frac{1}{\Gamma(s)} \int_x^b \frac{u(t)}{(t-x)^{1-s}} dt, \tag{2.6}$$

where Γ denotes the Euler's Gamma function (2.1).

Remark 2.13. *Notice that a simple change of variable relates the operators I^s_{a+} and I^s_{b-} through the following formula*

$$I^s_{a+}[u](Q(x)) = I^s_{b-}[u_Q](x),$$

where $Q(x) := b + a - x$ and $u_Q(\cdot) := u(Q(\cdot))$.

Remark 2.14. *It is not difficult to check that definitions (2.5) and (2.6) are well posed for any $u \in L^1(I)$ and $s \in (0,1)$. Indeed, we have*

$$\|I^s_{a+}[u]\|_{L^1(I)} \leq \frac{1}{\Gamma(s)} \int_a^b \int_a^x \frac{|u(t)|}{(x-t)^{1-s}} \, dt \, dx = \frac{1}{\Gamma(s)} \int_a^b \int_t^b \frac{|u(t)|}{(x-t)^{1-s}} \, dx \, dt$$

$$= \frac{1}{s\Gamma(s)} \int_a^b |u(t)|(b-t)^s \, dt \le \frac{(b-a)^s}{\Gamma(s+1)} \|u\|_{L^1(I)},$$

so that $I^s_{a+}[u] \in L^1(I)$ and, in particular, $I^s_{a+}[u](x)$ is well defined for $\mathcal{L}^1$-a.e. $x \in I$. A similar argument shows that also $I^s_{b-}[u] \in L^1(I)$, with

$$\|I^s_{b-}[u]\|_{L^1(I)} \le \frac{(b-a)^s}{\Gamma(s+1)} \|u\|_{L^1(I)},$$

so that $I^s_{b-}[u]$ is well defined almost everywhere in I.

One of the most useful property of the fractional integral, is the following

Lemma 2.15 (Semigroup law)**.** *Let $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \le 1$ and $u \in L^1(I)$. Then, we have*

$$I^{\alpha}_{a+}[I^{\beta}_{a+}[u]] = I^{\alpha+\beta}_{a+}[u],$$

where $I^1_{a+}[u](x) := \displaystyle\int_a^x u(t)dt$.

Proof. It is an easy task to check that

$$\begin{aligned}
I^{\alpha}_{a+}[I^{\beta}_{a+}[u]](x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{dt}{(x-t)^{1-\alpha}} \int_a^t \frac{u(s)}{(t-s)^{1-\beta}} ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x u(s)ds \int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1} dt \qquad (2.7)\\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x \frac{u(s)}{(x-s)^{1-\alpha-\beta}} ds = I^{\alpha+\beta}_{a+}[u](x),
\end{aligned}$$

where the second equality follows by Fubini Theorem, while the third exploits Remark 2.3. □

We establish now a simple duality relation between I^s_{a+} and I^s_{b-} which will be useful in the following.

Lemma 2.16. *Let $u \in L^p(I)$, $v \in L^q(I)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $s \in (0,1)$. Then we have*

$$\int_a^b I^s_{a+}[u](x)\, v(x)\, dx = \int_a^b u(x)\, I^s_{b-}[v](x)\, dx. \qquad (2.8)$$

Proof. By Fubini's Theorem, we have

$$\begin{aligned}
\int_a^b I^s_{a+}[u](x)\, v(x)\, dx &= \frac{1}{\Gamma(s)} \int_a^b \int_a^x \frac{u(t)}{(x-t)^{1-s}}\, v(x)\, dt\, dx \\
&= \frac{1}{\Gamma(s)} \int_a^b \int_t^b \frac{v(x)}{(x-t)^{1-s}}\, u(t)\, dx\, dt \\
&= \int_a^b u(t)\, I^s_{b-}[v](t)\, dt.
\end{aligned}$$

□

We conclude this section by recalling a well known result on the convergence of I^s_{a+} to the identity operator as $s \to 0^+$.

Lemma 2.17. *For any $u \in L^1(I)$ we have $\|I^s_{a+}[u] - u\|_{L^1(I)} \to 0$ as $s \to 0^+$. In particular, if $u \in C^1(\overline{I})$, then $I^s_{a+}[u](x) \to u(x)$ for any $x \in I$ and it holds that*

$$I^s_{a+}[u](x) = \frac{u(a)}{\Gamma(s+1)}(x-a)^s + \frac{1}{\Gamma(s+1)} \int_a^x u'(t)(x-t)^s dt. \tag{2.9}$$

Analogous statements hold for I^s_{b-}.

Proof. We start by assuming that $u \in C^1(\overline{I})$, then, with a simple integration by parts, equality (2.9) immediately follows. Thus, letting $s \to 0^+$ we immediately obtain pointwise convergence, and by Lebesgue dominated convergence Theorem we have convergence in $L^1(I)$. Otherwise, if $u \in L^1(I)$, fixed $\epsilon > 0$ there exists $v \in C^1(\overline{I})$ such that $\|v-u\|_{L^1(I)} \leq \varepsilon$; then

$$\begin{aligned}
\|I^s_{a+}[u] - u\|_{L^1(I)} &\leq \|I^s_{a+}[u-v]\|_{L^1(I)} + \|I^s_{a+}[v] - v\|_{L^1(I)} + \|v-u\|_{L^1(I)} \\
&\leq \max\left\{1, \frac{(b-a)^s}{\Gamma(s+1)}\right\} \|v-u\|_{L^1(I)} + \|I^s_{a+}[v] - v\|_{L^1(I)} \\
&\leq \max\left\{1, \frac{(b-a)^s}{\Gamma(s+1)}\right\} \varepsilon + \|I^s_{a+}[v] - v\|_{L^1(I)}.
\end{aligned}$$

Eventually, for the arbitrariness of ε, sending $s \to 0^+$ the claim is completely proved. □

2.3.2 Continuity of the fractional integral in L^p and Hölder spaces

For the ease of the reader, we summarize in the following Propositions 2.18 and 2.21 some results contained in [SKM93, Chapter 1, Section 3]. From now on, unless otherwise stated, with the notation $X_0(I)$, we will refer to functions $f \in X(I)$ that vanish in the endpoint a, where X denotes some subspace of a Hölder or a Sobolev space.

Proposition 2.18 (Continuity properties of the fractional integral in L^p spaces)**.** *For any $s \in (0,1)$, the fractional integral I^s_{a+} is a continuous operator from*

1. *$L^p(I)$ into $L^p(I)$, for any $p \geq 1$,*
2. *$L^1(I)$ into $L^{\frac{1}{1-s},\infty}(I)$, and so in $L^r(I)$ for any $r \in [1, \frac{1}{1-s})$*
3. *$L^p(I)$ into $L^r(I)$, for any $p \in (1, 1/s)$ and $r \in [1, \frac{p}{1-sp}]$,*
4. *$L^p(I)$ into $C^{0,s-\frac{1}{p}}(I)$ for every $p > 1/s$,*
5. *$L^{1/s}(I)$ into $L^r(I)$ with $r \in [1, +\infty)$,*
6. *$L^\infty(I)$ into $C^{0,s}(I)$.*

where $L^{\frac{1}{1-s},\infty}(I)$ denotes the Marcinkiewicz space defined in 2.4.

Remark 2.19. *We notice that point (i) of Proposition 2.18 is a consequence of a generalized Minkowski inequality as observed in the proof of* [SKM93, Theorem 2.6.]*. In particular the constant of continuity does not depend on p and it is given by $\frac{(b-a)^s}{\Gamma(s+1)}$.*

Corollary 2.20. *For any $s \in (0,1)$, the fractional integral I^s_{a+} is a continuous operator from $BV(I)$ into $C^{0,s}(I)$.*

Proof. Combining Lemma 2.2 and the last point of Proposition 2.18, the claim is completely proved. $\square$

Proposition 2.21 (Continuity properties of the fractional integral in Hölder spaces)**.** *Let $s \in (0,1)$ and $\alpha \in (0,1]$. The fractional integral I^s_{a+} is a continuous operator from*

1. *$C^{0,\alpha}_0(\overline{I})$ onto $C^{0,\alpha+s}_0(\overline{I})$ if $\alpha + s < 1$,*
2. *$C^{0,\alpha}_0(\overline{I})$ onto $H^{1,1}_0(\overline{I})$ if $\alpha + s = 1$,*
3. *$C^{0,\alpha}_0(\overline{I})$ onto $C^{1,\alpha+s-1}_0(\overline{I})$ if $\alpha + s > 1$,*

where the space $H^{1,1}(\overline{I})$ is given by the functions that admit $\omega(h) = |h||\log|h||$ as a local modulus of continuity; namely, for which there exists $C > 0$ such that

$$|f(x+h) - f(x)| \leq C|h||\log|h||, \quad \forall \quad 0 < |h| < 1/2.$$

2.3.3 Fractional Derivatives

Definition 2.22. *Let $s \in (0,1)$. For any $u : I \to \mathbb{R}$ sufficiently smooth, so that $I^{1-s}_{a+}[u]$ is differentiable, we define the left and right Riemann-Liouville fractional derivatives of u as*

$$D^s_{a+}\left[u\right](x) := \frac{d}{dx} I^{1-s}_{a+}\left[u\right](x)\,, \tag{2.10}$$

and

$$D^s_{b-}\left[u\right](x) := -\frac{d}{dx} I^{1-s}_{b-}\left[u\right](x)\,. \tag{2.11}$$

Remark 2.23. *As a consequence of Proposition 2.21, we have that for $0 < s < \alpha < 1$ and $u \in C^{0,\alpha}_0(\overline{I})$, $I^{1-s}_{a+}[u] \in C^{1,\alpha-s}_0(\overline{I})$. Therefore, α-Hölder continuity with $\alpha > s$ is a sufficient condition to ensure the existence of (2.10) and (2.11).*

If one applies Riemann Liouville fractional integrals to the first derivative u', whenever this operation makes sense, one has the following alternative definitions of left and right fractional derivatives originally given by Michele Caputo in [Cap08]

Definition 2.24. *Let $s \in (0,1)$. For any $u \in C^1(\overline{I})$ we define the left and right Caputo fractional derivatives of u as*

$${}^C D^s_{a+}[u](x) := I^{1-s}_{a+}[u'](x) = \frac{1}{\Gamma(1-s)} \int_a^x \frac{u'(t)}{(x-t)^s} dt. \tag{2.12}$$

$${}^C D^s_{b-}[u](x) := -I^{1-s}_{b-}[u'](x) = -\frac{1}{\Gamma(1-s)} \int_x^b \frac{u'(t)}{(t-x)^s} dt. \tag{2.13}$$

The minimal functional spaces in which formulas (2.12) (2.13) are well posed are the space $C^{1,s}_{a+}$ and $C^{1,s}_{b-}$. See Appendix A.1.

For $u \in AC(\overline{I})$, a simple computation relates the Riemann-Liouville and the Caputo fractional derivatives. Indeed, integrating by parts, we have that

$$\int_a^x \frac{u(t)}{(x-t)^s}\,dt = \frac{1}{1-s}\int_a^x u'(t)(x-t)^{1-s}\,dt + u(a)\frac{(x-a)^{1-s}}{1-s}, \tag{2.14}$$

hence, differentiating in x on both sides of (2.14) and dividing by $\Gamma(1-s)$, we obtain the following formula

$$D^s_{a+}[u](x) = {}^C D^s_{a+}u(x) + \frac{u(a)}{\Gamma(1-s)}(x-a)^{-s}. \tag{2.15}$$

Analogously, for right derivatives we have that

$$D^s_{b-}[u](x) = {}^C D^s_{b-}u(x) + \frac{u(b)}{\Gamma(1-s)}(b-x)^{-s}. \tag{2.16}$$

Therefore, the Riemann-Liouville and the Caputo fractional derivative coincide for any $u \in AC(\overline{I})$ that vanishes in the initial point a for left derivatives, or in the final point b for right derivatives.

We also notice that if $u \in AC(\overline{I})$, we can exploit formula (2.15) to obtain another representation of the left Riemann-Liouville fractional derivative

$$\begin{aligned}
D^s_{a+}[u](x) &= \frac{u(a)}{\Gamma(1-s)(x-a)^s} + \frac{1}{\Gamma(1-s)}\int_a^x \frac{u'(t)}{(x-t)^s}dt \\
&= \frac{u(a)}{\Gamma(1-s)(x-a)^s} + \frac{1}{\Gamma(1-s)}\int_a^x u'(t)\left(s\int_{x-t}^{x-a}\xi^{-s-1}d\xi + \frac{1}{(x-a)^s}\right)dt \\
&= \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)}\int_0^{x-a}\frac{d\xi}{\xi^{s+1}}\int_{x-\xi}^x u'(t)\,dt \\
&= \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)}\int_0^{x-a}\frac{u(x)-u(x-\xi)}{\xi^{s+1}}\,d\xi \\
&= \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)}\int_a^x \frac{u(x)-u(t)}{(x-t)^{s+1}}dt.
\end{aligned} \tag{2.17}$$

This different representation formula of the Riemann-Liouville fractional derivative

$${}^M D^s_{a+}[u](x) := \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)}\int_a^x \frac{u(x)-u(t)}{(x-t)^{s+1}}dt,$$

is known as the *Marchaud fractional derivative*; for a precise treatment of this fractional differential operator we refer to [Fer18] and [SKM93].

Now, we recall the notion of L^p-representability. From Proposition 2.18, we have that for any $1 \le p \le \infty$ and any $s \in (0,1)$, $I^s_{a+}(L^p(I)) \subset L^p(I)$, where the inclusion is strict, as it is shown by the following example.

Example 2.25. *Consider*

$$u(x) := \frac{(x-a)^{s-1}}{\Gamma(s)}$$

for some $s \in (0,1)$. Then we have $u \in L^p(I)$ for all $1 \le p < \frac{1}{1-s}$, and, for all $x \in I$, we see that

$$\begin{aligned} I^{1-s}_{a+}[u](x) &= \frac{1}{\Gamma(1-s)\Gamma(s)} \int_a^x (t-a)^{s-1}(x-t)^{-s}\,dt = \frac{1}{\Gamma(1-s)\Gamma(s)} \int_0^1 \sigma^{s-1}(1-\sigma)^{-s} d\sigma \\ &= \frac{\beta(s,1-s)}{\Gamma(1-s)\Gamma(s)} = 1, \end{aligned} \tag{2.18}$$

by the properties of the Euler's beta function β. Therefore, we conclude that

$$D^s_{a+}[u](x) = 0 \quad \textit{for all} \quad x \in I, \tag{2.19}$$

while the left Caputo s-fractional derivative is not well defined. We prove now that the equation

$$I^s_{a+}[f] = u \tag{2.20}$$

has no solution in $L^p(I)$. In fact, suppose by contradiction that there exists $f \in L^p(I)$ satisfying (2.20). If we apply the $(1-s)$-fractional integral on both sides of (2.20), thanks to Lemma 2.15 and (2.18), we get

$$\int_a^x f(t)\,dt = I^1_{a+}[f](x) = I^{1-s}_{a+}[u](x) = 1,$$

for any $x \in I$. Therefore, differentiating on both sides of the equation, we obtain $f = 0$, which is clearly a contradiction.

The next lemma gives a characterization of L^p-representability. We are going prove it only in the case of left fractional integral, the other case being completely analogous.

Definition 2.26. *Let $1 \le q \le \infty$ and $u \in L^q(I)$; we say that u is L^p-representable if $u \in I^s_{a+}(L^p(I))$ or $u \in I^s_{b-}(L^p(I))$ for some $1 \le p \le q$ and $s \in (0,1)$.*

The next lemma gives a characterization of L^p-representability; we prove it only in the case of left fractional integral, the right case is completely analogous.

Lemma 2.27 (L^p-representability criterion)**.** *Let $u \in L^q(I)$, for some $1 \le q \le \infty$ $s \in (0,1)$ and $1 \le p \le q$. We have that $u \in I^s_{a+}(L^p(I))$ if and only if $I^{1-s}_{a+}[u] \in W^{1,p}(I)$ and $I^{1-s}_{a+}[u](a) = 0$.*

Proof. If $u \in I^s_{a+}(L^p(I))$, then $u = I^s_{a+}[f]$ for some $f \in L^p(I)$; therefore, using Lemma 2.15

$$I^{1-s}_{a+}[u](x) = I^{1-s}_{a+}[I^s_{a+}[f]](x) = I^1_{a+}[f](x) = \int_a^x f(t)dt \in W^{1,p}(I),$$

and $I^{1-s}_{a+}[u](a) = I^1_{a+}[f](a) = 0$. On the other hand, if $I^{1-s}_{a+}[u] \in W^{1,p}(I) \subset AC(\overline{I})$ and $I^{1-s}_{a+}[u](a) = 0$, we have that

$$I^{1-s}_{a+}[u](x) = \int_a^x D^s_{a+}[u](t)dt = I^{1-s}_{a+}[I^s_{a+}[D^s_{a+}[u]]](x),$$

therefore, by applying the operator D^{1-s}_{a+} to both sides of the equation we have

$$u(x) = I^s_{a+}[D^s_{a+}[u]](x),$$

with $D^s_{a+}[u] \in L^p(I)$; therefore $u \in I^s_{a+}(L^p(I))$, and this concludes the proof. $\square$

2.4 Riemann-Liouville fractional Sobolev spaces

Now, we got all the necessary tools to introduce the Riemann-Liouville fractional Sobolev spaces

Definition 2.28 (Riemann-Liouville fractional Sobolev spaces)**.** *Let $1 \leq p \leq \infty$, and $s \in (0,1)$. We define the Riemann-Liouville fractional Sobolev spaces as*

$$W^{s,p}_{RL,a+}(I) := \left\{ u \in L^p(I)\,,\ I^{1-s}_{a+}[u] \in W^{1,p}(I) \right\}, \tag{2.21}$$

and

$$W^{s,p}_{RL,b-}(I) := \left\{ u \in L^p(I)\,,\ I^{1-s}_{b-}[u] \in W^{1,p}(I) \right\}. \tag{2.22}$$

It is not difficult to see that the spaces $W^{s,p}_{RL,a+}(I)$ and $W^{s,p}_{RL,b-}(I)$, endowed with the norms

$$\|u\|_{W^{s,p}_{RL,a+}(I)} := \|u\|_{L^p(I)} + \left\|I^{1-s}_{a+}[u]\right\|_{W^{1,p}(I)}, \tag{2.23}$$

and

$$\|u\|_{W^{s,p}_{RL,b-}(I)} := \|u\|_{L^p(I)} + \left\|I^{1-s}_{b-}[u]\right\|_{W^{1,p}(I)}, \tag{2.24}$$

are Banach spaces.

We notice that, in light of the continuity of the fractional integral in L^p given by Proposition 2.18, the norm in (2.23) (analogously for the one in (2.24)), is equivalent to the one given by

$$\|u\| := \|u\|_{L^p(I)} + \left\|D^s_{a+}[u]\right\|_{L^p(I)}$$

therefore, one could define the space $W^{s,p}_{RL,a+}(I)$ and $W^{s,p}_{RL,b-}(I)$ simply requiring that $u \in L^p(I)$ has fractional derivatives in $L^p(I)$; but this definition does not take into account the differentiability properties of the fractional integral which are necessaries for integration-by-parts formulae presented in this chapter and in Chapter 4.

We point out that there is a duality relation between the left Riemann-Liouville fractional derivative and the Caputo right fractional derivative, as shown in the following lemma

Lemma 2.29. *Let $u \in W^{s,1}_{RL,a+}(I)$, $v \in C^{0,1}_c(I)$ and $s \in (0,1)$. Then we have*

$$\int_a^b D^s_{a+}[u](x)\, v(x)\, dx = \int_a^b u(x)\, {}^C D^s_{b-}[v](x)\, dx. \tag{2.25}$$

Proof. Integrating by parts, and using Fubini's theorem, we have

$$\begin{aligned}
\int_a^b D^s_{a+}[u](x)v(x)dx &= -\int_a^b I^{1-s}_{a+}[u](x)\, v'(x)\, dx \\
&= -\frac{1}{\Gamma(1-s)} \int_a^b \int_a^x \frac{u(t)}{(x-t)^s}\, v'(x)\, dt\, dx \\
&= -\frac{1}{\Gamma(1-s)} \int_a^b \int_t^b \frac{v'(x)}{(x-t)^s}\, u(t)\, dx\, dt \\
&= \int_a^b u(t)\, {}^C D^s_{b-}[v](t)\, dt.
\end{aligned}$$

$\square$

In the light of Definition 2.28, we may rephrase Lemma 2.27 in the following way.

Lemma 2.30. *Let $s \in (0,1)$ and $p \in [1,\infty]$. Then, $u \in I^s_{a+}(L^p(I))$ if and only if $u \in W^{s,p}_{RL,a+}(I)$ and $I^{1-s}_{a+}[u](a) = 0$.*

We consider now a version of the fundamental Theorem of Calculus for left Riemann-Liouville fractional derivatives. A similar result was stated in [BI15, Proposition 5], however we provide here a short proof, for completeness.

Lemma 2.31. *Let $s \in (0,1)$ and $u \in L^1(I)$. Then, for $\mathcal{L}^1$-a.e. $x \in I$, we have*

$$u(x) = D^s_{a+}[I^s_{a+}[u]](x). \tag{2.26}$$

If $u \in W^{s,1}_{RL,a+}(I)$, then, for $\mathcal{L}^1$-a.e. $x \in I$, we also have

$$u(x) = I^s_{a+}[D^s_{a+}[u]](x) + \frac{I^{1-s}_{a+}[u](a)}{\Gamma(s)}(x-a)^{s-1}. \tag{2.27}$$

Finally, if $u \in W^{s,1}_{RL,a+}(I) \cap I^s_{a+}(L^1(I))$, then

$$u(x) = D^s_{a+}[I^s_{a+}[u]](x) = I^s_{a+}[D^s_{a+}[u]](x) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in I. \tag{2.28}$$

Proof. If $u \in L^1(I)$, we have $I^s_{a+}[u] \in L^1(I)$, by Remark 2.14, and, by Lemma 2.15,

$$I^{1-s}_{a+}[I^s_{a+}[u]](x) = I^1_{a+}[u](x) = \int_a^x u(t)dt \in W^{1,1}(I).$$

Therefore, for $\mathcal{L}^1$-a.e. $x \in I$, we get

$$D^s_{a+}[I^s_{a+}[u]](x) = \frac{d}{dx} I^{1-s}_{a+}[I^s_{a+}[u]](x) = \frac{d}{dx}(I^1_{a+}[u](x)) = u(x).$$

In order to prove (2.27), we notice that $I^{1-s}_{a+}[u] \in W^{1,1}(I)$ with weak derivative $D^s_{a+}[u] \in L^1(I)$, so that, for $\mathcal{L}^1$-a.e. $x \in I$,

$$\begin{aligned} I^{1-s}_{a+}[u](x) &= \int_a^x D^s_{a+}[u](t)dt + I^{1-s}_{a+}[u](a) \\ &= I^{1-s}_{a+}[I^s_{a+}[D^s_{a+}[u]]](x) + I^{1-s}_{a+}\left[\frac{I^{1-s}_{a+}[u](a)}{\Gamma(s)}(\cdot - a)^{s-1}\right](x), \end{aligned}$$

by (2.18). We notice that, by Remark 2.14, $I^s_{a+}[D^s_{a+}[u]] \in L^1(I)$, since $D^s_{a+}[u] \in L^1(I)$ by assumption. Therefore, we apply D^{1-s}_{a+} to both sides of the equation and use (2.26) to obtain (2.27). Finally, if $u \in W^{s,1}_{RL,a+}(I) \cap I^s_{a+}(L^1(I))$, then, by Lemma 2.27 with $p = q = 1$, we have that $I^{1-s}_{a+}[u](a) = 0$, and this ends the proof. □

Remark 2.32. *Notice that these equalities are stable when $s \to 1^-$ for $u \in C^1(\overline{I})$. Indeed, we have that*

$$\begin{aligned} u(x) &= \lim_{s \to 1^-} D^s_{a+}[I^s_{a+}[u]](x) = \frac{d}{dx}\left(\int_a^x u(t)dt\right) = \int_a^x u'(t)dt + u(a) \\ &= \lim_{s \to 1^-} I^s_{a+}[D^s_{a+}[u]](x) + \frac{I^{1-s}_{a+}[u](a)}{\Gamma(s)}(x-a)^{s-1}, \end{aligned}$$

where the second equality exploits Lemma 2.17.

Remark 2.33. *We notice that if $u \in W^{s,1}_{RL,a+}(I) \setminus I^s_{b-}(L^1(I))$, thanks to Remark 2.13 we have that $u_Q \in W^{s,1}_{RL,b-}(I)$ but this not necessarily implies that $u \in W^{s,1}_{RL,b-}(I)$. Indeed, consider $u(x) := \dfrac{x^{s-1}}{\Gamma(s)}$; we have that $I^{1-s}_{0+}[u](x) = 1$ for any $x \in [0,1]$, hence $I^{1-s}_{0+}[u] \in W^{1,1}((0,1))$. On the other hand, we have that*

$$I^{1-s}_{1-}[u](x) = \frac{1}{\Gamma(1-s)} \int_x^1 t^{s-1}(t-x)^{-s} dt = \frac{1}{\Gamma(1-s)} \int_1^{1/x} \omega^{s-1}(\omega - 1)^{-s} d\omega,$$

and this function belongs to $L^1((0,1)) \setminus W^{1,1}((0,1))$. The check of L^1-summability is an easy task; on the other side, if we compute the first derivative of $I^{1-s}_{1-}[u](x)$ we have that

$$D^s_{1-}[u](x) = -\frac{d}{dx} I^{1-s}_{1-}[u](x) = \frac{1}{\Gamma(1-s)} \frac{1}{x(1-x)^s} \notin L^1(I).$$

2.5 Carnot groups

A connected and simply connected Lie group $(\mathbb{G}, \cdot)$ is said to be a *Carnot group of step s* if its Lie algebra $\mathfrak{g}$ admits a *step s stratification*, i.e., there exist linear subspaces $\mathfrak{g}_1, ..., \mathfrak{g}_s$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_s, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_s \neq \{0\}, \quad [\mathfrak{g}_s, \mathfrak{g}_1] = \{0\} \tag{2.29}$$

where $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$. In the last few years, Carnot groups have been largely studied in several respects, such as differential geometry [CDPT07], subelliptic differential equations [BLU07, Fol73, Fol75, SC84], complex analysis [SS03].

For a general introduction to Carnot groups from the point of view of this chapter and for further examples, we refer, e.g., to [BLU07, Fol75, LD17, SS03].

Fix a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$ and denote by $| \cdot |$ its induced norm. We recall that a curve $\gamma \colon [a, b] \to \mathbb{G}$ is absolutely continuous if it is absolutely continuous as a curve into $\mathbb{R}^n$ via composition with local charts.

Definition 2.34. *An absolutely continuous curve $\gamma \colon [a, b] \to \mathbb{G}$ is said to be* horizontal *if*

$$\gamma'(t) \in \mathfrak{g}_1,$$

for almost every $t \in [a, b]$. The length *of such a curve is given by*

$$L_{\mathbb{G}}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Chow's theorem [BLU07, Theorem 19.1.3] asserts that any two points in a Carnot group can be connected by a horizontal curve. Hence, the following definition is well-posed.

Definition 2.35. *For every $x, y \in \mathbb{G}$, their* Carnot-Carathéodory (CC) distance *is defined by*

$$d(x, y) = \inf \{ L_{\mathbb{G}}(\gamma) \colon \gamma \text{ is a horizontal curve joining } x \text{ and } y \}.$$

We also use the notation $\|x\| = d(x, 0)$ for $x \in \mathbb{G}$.

We denote by

$$B(x,r) = \{y \in \mathbb{G} : \|y^{-1}x\| < r\}$$

the open ball centered at $x \in \mathbb{G}$ with radius $r > 0$ and by $B(r) = B(0,r)$.
It is well-known (see e.g. [Mit85]) that the Hausdorff dimension of the metric space $(\mathbb{G}, d)$ is given by the so-called *homogeneous dimension* Q of $\mathbb{G}$, which is given by

$$Q := \sum_{i=1}^{s} i \dim(\mathfrak{g}_i).$$

The Hausdorff measure $\mathcal{H}^Q$ and the spherical Hausdorff measure $\mathcal{S}^Q$ are all Haar measure on $\mathbb{G}$. We denote by μ one of them, and, for any $f \in L^1(\Omega;\mu)$, we write for shortness

$$\int_\Omega f(x)\, dx := \int_\Omega f(x)\, d\mu(x),$$

for some measurable set Ω.

For any $\lambda > 0$, we denote by $\delta_\lambda^* : \mathfrak{g} \to \mathfrak{g}$ the unique linear map such that

$$\delta_\lambda^* X = \lambda^i X, \qquad \forall X \in \mathfrak{g}_i.$$

The maps $\delta_\lambda^* : \mathfrak{g} \to \mathfrak{g}$ are Lie algebra automorphisms, i.e., $\delta_\lambda^*([X,Y]) = [\delta_\lambda^* X, \delta_\lambda^* Y]$ for all $X, Y \in \mathfrak{g}$. For every $\lambda > 0$, the map δ_λ^* naturally induces an automorphism on the group $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ by the identity $\delta_\lambda(x) = (\exp \circ \delta_\lambda^* \circ \log)(x)$. It is easy to verify that both the families $(\delta_\lambda^*)_{\lambda>0}$ and $(\delta_\lambda)_{\lambda>0}$ are a one-parameter group of automorphisms (of Lie algebra and of groups, respectively), i.e., $\delta_\lambda^* \circ \delta_\eta^* = \delta_{\lambda\eta}^*$ and $\delta_\lambda \circ \delta_\eta = \delta_{\lambda\eta}$ for all $\lambda, \eta > 0$. The maps $\delta_\lambda^*, \delta_\lambda$ are both called *dilation of factor* λ.

Denoting by $\tau_x : \mathbb{G} \to \mathbb{G}$ the *(left) translation* by the element $x \in \mathbb{G}$ defined as

$$\tau_x z := x \cdot z = xz,$$

we remark that the CC distance is homogeneous with respect to dilations and left invariant. More precisely, for every $\lambda > 0$ and for every $x, y, z \in \mathbb{G}$ one has

$$d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x,y), \qquad d(\tau_x y, \tau_x z) = d(y,z).$$

This immediately implies that $\tau_x(B(y,r)) = B(\tau_x y, r)$ and $\delta_\lambda B(y,r) = B(\delta_\lambda y, \lambda r)$.

2.5.1 Perimeter and rectifiability in Carnot Groups

One of the main problems of sub-Riemannian geometry concerns the regularity of the (reduced) boundary of a set of finite perimeter. The solution of this problem in the Euclidean spaces goes back to De Giorgi [DG55]. He proved that the reduced boundary of a set of finite perimeter is $(n-1)$-recifiable, i.e., it can be covered, up to a set of $\mathcal{H}^{n-1}$-measure zero, by a countable family of C^1-hypersurfaces. The validity of such a result has wide consequences in the development of Geometric Measure Theory and Calculus of Variations (see e.g. the monographs [AFP00, EG15]).

The validity of a rectifiability-type theorem in the context of Carnot groups is still not yet known in full generality. However, there are complete results in all Carnot groups of step 2 (see [FSSC01, FSSC03]) and in the so-called Carnot groups of type $\star$ (see [Mar14]). In these papers the authors show that the reduced boundary of a set of finite perimeter in a Carnot group of the chosen class is rectifiable with respect to the intrinsic structure of the group.

We now introduce the notions of perimeter, reduced boundary and rectifiability.

Definition 2.36. *Let Ω be an open set in $\mathbb{G}$ and let $f \in L^1_{\mathrm{loc}}(\Omega)$. We say that f has locally bounded variation in Ω ($f \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$), if, for every $Y \in \mathfrak{g}_1$ and every open set $A \Subset \Omega$, there exists a Radon measure Yf on Ω such that*

$$\int_A fY\varphi \, d\mu = -\int_A \varphi \, d(Yf),$$

for every $\varphi \in C^1_c(A)$. We say that $f \in L^1(\Omega)$ has bounded variation *in Ω ($f \in BV_{\mathbb{G}}(\Omega)$) if f has locally bounded variation in Ω and, for every basis $(X_1, \dots, X_m)$ of $\mathfrak{g}_1$, the total variation $|Df|(\Omega)$ of the measure $Df := (X_1 f, \dots, X_m f)$ is finite. If E is a measurable set in Ω, we say that E has locally finite (resp. finite) perimeter in Ω if $\chi_E \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ (resp. $\chi_E \in BV_{\mathbb{G}}(\Omega)$). In such a case, the measure $|D\chi_E|$ is called* perimeter of *E and it is denoted by $P_{\mathbb{G}}(E; \cdot)$.*

Definition 2.37. *Let $E \subseteq \mathbb{G}$ be a set with locally finite perimeter. We define the* reduced boundary *$\mathcal{F}E$ of E to be the set of points $p \in \mathbb{G}$ such that $P_{\mathbb{G}}(E; B(p,r)) > 0$ for all $r > 0$ and there exists*

$$\lim_{r \to 0} \frac{D\chi_E(B(p,r))}{P_{\mathbb{G}}(E; B(p,r))} = \lim_{r \to 0} \frac{D\chi_E(B(p,r))}{|D\chi_E|(B(p,r))} =: \nu_E(p) \in \mathbb{R}^m,$$

with $|\nu_E(p)| = 1$.

Definition 2.38. *Let $\Omega \subseteq \mathbb{G}$ be an open set in a Carnot group $\mathbb{G}$. We say that a function $f : \Omega \to \mathbb{R}$ is of class $C^1_{\mathbb{G}}$ if f is continuous and, for every $X \in \mathfrak{g}_1$, the derivative Xf in the sense of distributions is represented by a continuous function. Given a basis $X = (X_1, \dots, X_m)$ of $\mathfrak{g}_1$, we also denote by $\nabla_X f \colon \Omega \to \mathbb{R}^m$ the vector valued function defined by*

$$\nabla_X f := (X_1 f, \dots, X_m f).$$

Definition 2.39. *A set $\Sigma \subseteq \mathbb{G}$ is said to be a hypersurface of class $C^1_{\mathbb{G}}$ if, for every $p \in \Sigma$ there exists a neighborhood U of p, and a function $f \colon U \to \mathbb{R}$ of class $C^1_{\mathbb{G}}$ such that*

$$\Sigma \cap U = \{q \in U : f(q) = 0\},$$

and $\inf_U |\nabla_X f| > 0$, for any basis $X = (X_1, \dots, X_m)$ of $\mathfrak{g}_1$.

Definition 2.40. *Let $E \subseteq \mathbb{G}$ be a measurable set. We say that E is $C^1_{\mathbb{G}}$-*rectifiable *(or simply* rectifiable*), if there exists a family $\{\Gamma_j : j \in \mathbb{N}\}$ of $C^1_{\mathbb{G}}$-hypersurfaces such that*

$$\mathcal{H}^{Q-1}\left(E \setminus \bigcup_{j \in \mathbb{N}} \Gamma_j\right) = 0,$$

where Q is the homogeneous dimension of $\mathbb{G}$ and $\mathcal{H}^{Q-1}$ denotes the $(Q-1)-$dimensional Hausdorff measure defined through the Carnot-Carathéodory distance.

Definition 2.41. *For any $\nu \in \mathfrak{g}_1 \setminus \{0\}$, we define the* vertical halfspace with normal ν *by setting*

$$H_\nu := \{x \in \mathbb{G} : \langle \pi_1 \log x, \nu \rangle \geq 0\},$$

where $\pi_1 \colon \mathfrak{g} \to \mathfrak{g}_1$ is the horizontal projection on the Lie algebra and $\log : \mathbb{G} \to \mathfrak{g}$ *is the inverse of the exponential map. Notice that if $x \in \mathbb{G}$ is such that $\langle \pi_1 \log x, \nu \rangle > 0$, then $x^{-1} \in H_\nu^c$.*

Following the notation of [DV19] we introduce the following:

Definition 2.42. *We say that a Carnot group $\mathbb{G}$ satisfies property $\mathcal{R}$ if every set $E \subseteq \mathbb{G}$ of locally finite perimeter in $\mathbb{G}$ has rectifiable reduced boundary.*

As already mentioned before, property $\mathcal{R}$ is satisfied in Euclidean spaces, in all Carnot groups of step 2 and in the so-called Carnot groups of type $\star$.

Remark 2.43. *If $\mathbb{G}$ is a Carnot group satisfying property $\mathcal{R}$ and $E \subseteq \mathbb{G}$ is a set of finite perimeter in $\mathbb{G}$, then, for $\mathcal{H}^{Q-1}$-almost every $p \in \mathcal{F}E$, the family $\delta_{1/r} p^{-1} E$ converges in L^1_{loc} to the halfspace $H_{\nu_E(p)}$. This comes from the fact that C^1-hypersurfaces have flat blow-up (see e.g.* [DV19, Proposition 2.13]*).*

Whenever property $\mathcal{R}$ is not assumed, only partial result about blow-up of sets of finite perimeter are available in the literature. It is proved in [FSSC03] *that, for any set $E \subseteq \mathbb{G}$ with locally finite perimeter and for $\mathcal{H}^{Q-1}$-almost every $p \in \mathcal{F}E$, the family $\delta_{1/r} p^{-1} E$ converges in $L^1_{\mathrm{loc}}(\mathbb{G})$ to a set of* constant horizontal normal *F, namely a set for which there exists $\nu \in \mathfrak{g}_1$ such that*

$$\nu \chi_F \geq 0 \quad \text{and} \quad X \chi_F = 0 \quad \text{for every } X \in \mathfrak{g}_1 \text{ with } X \perp \nu, \tag{2.30}$$

in the sense of distributions.

If in addition $\mathbb{G}$ has step 2, or it is of type $\star$, then it is proved respectively in [FSSC03] *and* [Mar14] *that, up to a left translation, every set of constant horizontal normal is really a vertical halfspace. On the other hand, still in* [FSSC03, Example 3.2]*, it is proved that for general Carnot groups condition* (2.30) *does not characterize vertical halfspaces. The classification of sets with constant horizontal normal is a challenging problem and, as far as we know, the most general result in this direction is* [AKLD09, Theorem 1.2]*: if $E \subset \mathbb{G}$ has locally finite perimeter, then, for $|D\chi_E|$-a.e. $p \in \mathbb{G}$, there exist an infinitesimal sequence of radii (r_j) and a vertical halfspace H such that $\delta_{1/r_j}(p^{-1}E)$ converges in $L^1_{\mathrm{loc}}(\mathbb{G})$ to H, as $j \to \infty$.*

Chapter 3

Local density of solutions to fractional equations

3.1 Introduction and main results

In this chapter, following the recent monograph [CDV19], we prove the local density of functions which annihilate a linear operator built by classical and fractional derivatives, both in space and time, where time-fractional derivatives will be mostly described in terms of the so-called Caputo fractional derivative (see [Cap08]), which induces a natural "direction" in the time variable, distinguishing between "past" and "future".

In particular, the time direction encoded in this setting allows the analysis of "non anticipative systems", namely phenomena in which the state at a given time depends on past events, but not on future ones. The Caputo derivative is also related to other types of time-fractional derivatives, such as the Marchaud fractional derivative, which has applications in modeling anomalous time diffusion, see e.g. [ACV16, AV19, Fer18]. See also [MR93, SKM93] for more details on fractional operators and several applications.

Here, we will take advantadge of the nonlocal structure of a very general linear operator containing fractional derivatives in some variables (say, either time, or space, or both), in order to approximate, in the smooth sense and with arbitrary precision, any prescribed function. Remarkably, *no structural assumption* needs to be taken on the prescribed function: therefore this approximation property reveals a *truly nonlocal behaviour*, since it is in contrast with the rigidity of the functions that lie in the kernel of classical linear operators (for instance, harmonic functions cannot approximate a function with interior maxima or minima, functions with null first derivatives are necessarily constant, and so on).

The approximation results with solutions of nonlocal operators have been first introduced in [DSV17] for the case of the fractional Laplacian, and since then widely studied under different perspectives, including harmonic analysis, see [RS18, GSU16, Rül17, RS17a, RS17b]. The approximation result for the one-dimensional case of a fractional derivative of Caputo type has been considered in [Buc17, CDV18], and operators involving classical time derivatives and additional classical derivatives in space have been studied in [DSV19a].

The great flexibility of solutions of fractional problems established by this type of approximation results has also consequences that go beyond the purely mathematical curiosity. For example, these results can be applied to study the evolution of biological populations, showing how a nonlocal hunting or dispersive strategy can be more convenient than one based on classical diffusion, in order to avoid waste of resources and optimize the search for food in sparse environment, see [MV17, CDV17]. Interestingly, the theoretical descriptions provided in this setting can be compared with a series of concrete biological data and real world experiments, confirming anomalous diffusion behaviours in many biological species, see [VAB+96]. It is worth noticing that the flexible behaviour exhibited by solutions of fractional linear equations is set against the rigidity of nonlocal minimal graphs; see for instance the recent paper [DSV19b].

Another interesting application of time-fractional derivatives arises in neuroscience, for instance in view of the anomalous diffusion which has been experimentally measured in neurons, see e.g. [SWDSA06] and the references therein. In this case, the anomalous diffusion could be seen as the effect of the highly ramified structure of the biological cells taken into account, see [AB91, DV18].

In many applications, it is also natural to consider the case in which different types of diffusion take place in different variables: for instance, classical diffusion in space variables could be naturally combined to anomalous diffusion with respect to variables which take into account genetical information, see [RVD+13, Sef17].

Now, to state the main original results of this work, we introduce some notation. In what follows, we will denote the "local variables" with the symbol x, the "nonlocal variables" with y, the "time-fractional variables" with t. Namely, we consider the variables

$$\begin{aligned} x &= (x_1, \dots, x_n) \in \mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_n}, \\ y &= (y_1, \dots, y_M) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_M} \\ \text{and} \quad t &= (t_1, \dots, t_l) \in \mathbb{R}^{l}, \end{aligned} \tag{3.1}$$

for some $p_1, \dots, p_n$, M, $m_1, \dots, m_M$, $l \in \mathbb{N}$, and we let

$$(x, y, t) \in \mathbb{R}^N, \qquad \text{where } \; N := p_1 + \dots + p_n + m_1 + \dots + m_M + l.$$

When necessary, we will use the notation B_R^k to denote the k-dimensional ball of radius R, centered at the origin in $\mathbb{R}^k$; otherwise, when there are no ambiguities, we will use the usual notation B_R.

Fixed $r = (r_1, \dots, r_n) \in \mathbb{N}^{p_1} \times \dots \times \mathbb{N}^{p_n}$, with $|r_i| \geq 1$ for each $i \in \{1, \dots, n\}$, and $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n) \in \mathbb{R}^n$, we consider the local operator acting on the variables $x = (x_1, \dots, x_n)$ given by

$$\mathfrak{l} := \sum_{i=1}^{n} \mathfrak{a}_i \partial_{x_i}^{r_i}. \tag{3.2}$$

where the multi-index notation has been used.

Furthermore, given $\mathfrak{b} = (\mathfrak{b}_1, \dots, \mathfrak{b}_M) \in \mathbb{R}^M$ and $s = (s_1, \dots, s_M) \in (0, +\infty)^M$, we consider the operator

$$\mathcal{L} := \sum_{j=1}^{M} \mathfrak{b}_j (-\Delta)_{y_j}^{s_j}, \tag{3.3}$$

where each operator $(-\Delta)_{y_j}^{s_j}$ denotes the fractional Laplacian of order $2s_j$ acting on the set of space variables $y_j \in \mathbb{R}^{m_j}$. More precisely, for any $j \in \{1, \dots, M\}$, given $s_j > 0$ and $h_j \in \mathbb{N}$ with $h_j := \min_{q_j \in \mathbb{N}}$ such that $s_j \in (0, q_j)$, in the spirit of [AJS18a], we consider the operator

$$(-\Delta)_{y_j}^{s_j} u\,(x, y, t) := \int_{\mathbb{R}^{m_j}} \frac{\left(\delta_{h_j} u\right)(x, y, t, Y_j)}{|Y_j|^{m_j + 2s_j}}\, dY_j, \tag{3.4}$$

where

$$\left(\delta_{h_j}u\right)(x,y,t,Y_j) := \sum_{k=-h_j}^{h_j}(-1)^k\binom{2h_j}{h_j-k}u\left(x,y_1,\dots,y_{j-1},y_j+kY_j,y_{j+1},\dots,y_M,t\right). \tag{3.5}$$

In particular, when $h_j := 1$, this setting comprises the case of the fractional Laplacian $(-\Delta)^{s_j}_{y_j}$ of order $2s_j \in (0,2)$, given by

$$\begin{aligned}(-\Delta)^{s_j}_{y_j}u\,(x,y,t) := c_{m_j,s_j}\int_{\mathbb{R}^{m_j}}\Big(&2u(x,y,t)-u(x,y_1,\dots,y_{j-1},y_j+Y_j,y_{j+1},\dots,y_M,t)\\&-u(x,y_1,\dots,y_{j-1},y_j-Y_j,y_{j+1},\dots,y_M,t)\Big)\,\frac{dY_j}{|Y_j|^{m_j+2s_j}},\end{aligned}$$

where $s_j \in (0,1)$ and c_{m_j,s_j} denotes a multiplicative normalizing constant (see e.g. formula (3.1.10) in [BV16]).

It is interesting to recall that if $h_j = 2$ and $s_j = 1$ the setting in (3.4) provides a nonlocal representation for the classical Laplacian, see [AV19].

In our general framework, we take into account also nonlocal operators of time-fractional type. To this end, for any $\alpha > 0$, letting $k := [\alpha]+1$ and $a \in \mathbb{R}\cup\{-\infty\}$, one can introduce the left[1] Caputo fractional derivative of order α and initial point a, defined, for $t > a$, as

$$D^{\alpha}_{t,a}u(t) := \frac{1}{\Gamma(k-\alpha)}\int_a^t\frac{\partial^k_t u\,(\tau)}{(t-\tau)^{\alpha-k+1}}d\tau. \tag{3.6}$$

[2]

In this framework, fixed $\mathfrak{c} = (\mathfrak{c}_1,\dots,\mathfrak{c}_l) \in \mathbb{R}^l$, $\alpha = (\alpha_1,\dots,\alpha_l) \in (0,+\infty)^l$ and $a = (a_1,\dots,a_l) \in (\mathbb{R}\cup\{-\infty\})^l$, we set

$$\mathcal{D}_a := \sum_{h=1}^{l}\mathfrak{c}_h\,D^{\alpha_h}_{t_h,a_h}\,. \tag{3.7}$$

Then, in the notation introduced in (3.2), (3.3) and (3.7), we consider here the superposition of the local, the space-fractional, and the time-fractional operators, that is, we set

$$\Lambda_a := \mathfrak{l}+\mathcal{L}+\mathcal{D}_a. \tag{3.8}$$

With this, the statement of our main result goes as follows:

---

[1] In the literature, one often finds also the notion of right Caputo fractional derivative, defined for $t < a$ by

$$\frac{(-1)^k}{\Gamma\,(k-\alpha)}\int_t^a\frac{\partial^k_t u(\tau)}{(\tau-t)^{\alpha-k+1}}\,d\tau.$$

Since the right time-fractional derivative boils down to the left one (by replacing t with $2a-t$), in this chapter we focus only on the case of left derivatives.

Also, though there are several time-fractional derivatives that are studied in the literature under different perspectives, we focus here on the Caputo derivative, since it possesses well-posedness properties with respect to classical initial value problems, differently than other time-fractional derivatives, such as the Riemann-Liouville derivative, in which the initial value setting involves data containing derivatives of fractional order.

[2] For notational simplicity, we will often denote $\partial^k_t u = u^{(k)}$.

Theorem 3.1. *Suppose that*

$$\begin{aligned}&\textit{either there exists } i\in\{1,\dots,M\} \textit{ such that } \mathpzc{b}_i\neq 0 \textit{ and } s_i\notin\mathbb{N},\\ &\textit{or there exists } i\in\{1,\dots,l\} \textit{ such that } \mathpzc{c}_i\neq 0 \textit{ and } \alpha_i\notin\mathbb{N}.\end{aligned} \tag{3.9}$$

Let $\ell\in\mathbb{N}$, $f:\mathbb{R}^N\to\mathbb{R}$, *with* $f\in C^\ell\big(\overline{B_1^N}\big)$. *Fixed* $\epsilon>0$, *there exist*

$$\begin{aligned}&u=u_\epsilon\in C^\infty\left(B_1^N\right)\cap C\left(\mathbb{R}^N\right),\\ &a=(a_1,\dots,a_l)=(a_{1,\epsilon},\dots,a_{l,\epsilon})\in(-\infty,0)^l,\\ \textit{and}\quad &R=R_\epsilon>1\end{aligned} \tag{3.10}$$

such that

$$\begin{cases}\Lambda_a u=0 & \textit{in } B_1^N,\\ u=0 & \textit{in } \mathbb{R}^N\setminus B_R^N,\end{cases} \tag{3.11}$$

and

$$\|u-f\|_{C^\ell(B_1^N)}<\epsilon. \tag{3.12}$$

We observe that the initial points of the Caputo type operators in Theorem 3.1 also depend on ϵ, as detailed in (3.10) (but the other parameters, such as the orders of the operators involved, are fixed arbitrarily).

We also stress that condition (3.9) requires that the operator Λ_a contains at least one nonlocal operator among its building blocks in (3.2), (3.3) and (3.7). This condition cannot be avoided, since approximation results in the same spirit of Theorem 3.1 cannot hold for classical differential operators.

Theorem 3.1 comprises, as particular cases, the nonlocal approximation results established in the recent literature of this topic. Indeed, when

$$\begin{aligned}&\mathpzc{a}_1=\dots=\mathpzc{a}_n=\mathpzc{b}_1=\dots=\mathpzc{b}_{M-1}=\mathpzc{c}_1=\dots=\mathpzc{c}_l=0,\\ &\mathpzc{b}_M=1,\\ \text{and}\quad &s\in(0,1)\end{aligned}$$

we see that Theorem 3.1 recovers the main result in [DSV17], giving the local density of s-harmonic functions vanishing outside a compact set.

Similarly, when

$$\begin{aligned}&\mathpzc{a}_1=\dots=\mathpzc{a}_n=\mathpzc{b}_1=\dots=\mathpzc{b}_M=\mathpzc{c}_1=\dots=\mathpzc{c}_{l-1}=0,\\ &\mathpzc{c}_l=1,\\ \text{and}\quad &\mathcal{D}_a=D^\alpha_{t,a},\quad \text{for some } \alpha>0,\ a<0\end{aligned}$$

we have that Theorem 3.1 reduces to the main results in [Buc17] for $\alpha\in(0,1)$ and [CDV18] for $\alpha>1$, in which such approximation result was established for Caputo-stationary functions, i.e, functions that annihilate the Caputo fractional derivative.

Also, when

$$\begin{aligned}&p_1=\dots=p_n=1,\\ &\mathpzc{c}_1=\dots=\mathpzc{c}_l=0,\end{aligned}$$

$$\text{and} \qquad s_j \in (0,1), \quad \text{for every } j \in \{1,\ldots,M\},$$

we have that Theorem 3.1 recovers the cases taken into account in [DSV19a], in which approximation results have been established for the superposition of a local operator with a superposition of fractional Laplacians of order $2s_j < 2$.

In this sense, not only Theorem 3.1 comprises the existing literature, but it goes beyond it, since it combines classical derivatives, fractional Laplacians and Caputo fractional derivatives altogether. In addition, it comprises the cases in which the space-fractional Laplacians taken into account are of order greater than 2.

As a matter of fact, this point is also a novelty introduced by Theorem 3.1 here with respect to the previous literature.

Theorem 3.1 was announced in [CDV18], but we also refer to [Kry18] which also considers the case of different, not necessarily fractional, powers of the Laplacian, using a different and innovative methodology.

The rest of the chapter is organized as follows. Section 3.2 focuses on time-fractional operators. More precisely, in Subsections 3.2.1 and 3.3 we study the boundary behaviour of the eigenfunctions of the Caputo derivative and of functions with vanishing Caputo derivative, respectively, detecting their singular boundary behaviour in terms of explicit representation formulas. These type of results are also interesting in themselves and can find further applications.

Section 3.4 is devoted to some properties of the higher order fractional Laplacian. More precisely, Section 3.5 provides some representation formula of the solution of $(-\Delta)^s u = f$ in a ball, with $u = 0$ outside this ball, for all $s > 0$, and extends the Green formula methods introduced in [DG17] and [AJS18b].

Then, in Section 3.6 we study the boundary behaviour of the first Dirichlet eigenfunction of higher order fractional equations, and in Section 3.7 we give some precise asymptotics at the boundary for the first Dirichlet eigenfunction of $(-\Delta)^s$ for any $s > 0$.

Section 3.8 is devoted to the analysis of the asymptotic behaviour of s-harmonic functions, with a "spherical bump function" as exterior Dirichlet datum.

Section 3.9 is devoted to the proof of our main result. To this end, Section 3.10 contains an auxiliary statement, namely Theorem 3.23, which will imply Theorem 3.1. This is technically convenient, since the operator Λ_a depends in principle on the initial point a: this has the disadvantage that if $\Lambda_a u_a = 0$ and $\Lambda_b u_b = 0$ in some domain, the function $u_a + u_b$ is not in principle a solution of any operator, unless $a = b$. To overcome such a difficulty, in Theorem 3.23 we will reduce to the case in which $a = -\infty$, exploiting a polynomial extension introduced and used in [CDV18], and that will be recalled in the Appendix.

In Section 3.11 we make the main step towards the proof of Theorem 3.23. Here, we prove that functions in the kernel of nonlocal operators such as the one in (3.8) span with their derivatives a maximal Euclidean space. This fact is special for the nonlocal case and its proof is based on the boundary analysis of the fractional operators in both time and space. Due to the general form of the operator in (3.8), we have to distinguish here several cases, taking advantage of either the time-fractional or the space-fractional components of the operators.

Finally, in Section 3.12 we complete the proof of Theorem 3.23, using the previous approximation results and suitable rescaling arguments.

3.2 Boundary behaviour of solutions of time-fractional equations

In this section, we give precise asymptotics for the boundary behaviour of solutions of time-fractional equations. The cases of the eigenfunctions and of the Dirichlet problem with vanishing forcing term will be studied in detail (the latter will be often referred to as the time-fractional harmonic case, borrowing a terminology from elliptic equations, with a slight abuse of notation in our case).

3.2.1 Sharp boundary behaviour for the time-fractional eigenfunctions

In this subsection we show that the eigenfunctions of the Caputo fractional derivative in (3.6) have an explicit representation via the Mittag-Leffler function. For this, fixed α, $\beta \in \mathbb{C}$ with $\Re(\alpha) > 0$, for any z with $\Re(z) > 0$, we recall that the Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) := \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}. \tag{3.13}$$

The Mittag-Leffler function plays an important role in equations driven by the Caputo derivatives, replacing the exponential function for classical differential equations, as given by the following well-established result (see [GKMR14] and the references therein):

Lemma 3.2. *Let $\alpha \in (0,1]$, $\lambda \in \mathbb{R}$, and $a \in \mathbb{R} \cup \{-\infty\}$. Then, the unique solution of the boundary value problem*

$$\begin{cases} D^\alpha_{t,a} u(t) = \lambda\, u(t) & \text{for any } t \in (a,+\infty), \\ u(a) = 1 \end{cases}$$

is given by $E_{\alpha,1}(\lambda (t-a)^\alpha)$.

Lemma 3.2 can be actually generalized[3] to any fractional order of differentiation α:

Lemma 3.3. *Let $\alpha \in (0,+\infty)$, with $\alpha \in (k-1,k]$ and $k \in \mathbb{N}$, $a \in \mathbb{R} \cup \{-\infty\}$, and $\lambda \in \mathbb{R}$. Then, the unique continuous solution of the boundary value problem*

$$\begin{cases} D^\alpha_{t,a} u(t) = \lambda\, u(t) & \text{for any } t \in (a,+\infty), \\ u(a) = 1, \\ \partial^m_t u(a) = 0 & \text{for any } m \in \{1,\dots,k-1\} \end{cases} \tag{3.14}$$

is given by $u(t) = E_{\alpha,1}(\lambda (t-a)^\alpha)$.

Proof. For the sake of simplicity we take $a = 0$. Also, the case in which $\alpha \in \mathbb{N}$ can be checked with a direct computation, so we focus on the case $\alpha \in (k-1,k)$, with $k \in \mathbb{N}$.

We let $u(t) := E_{\alpha,1}(\lambda t^\alpha)$. It is straightforward to see that $u(t) = 1 + \mathcal{O}(t^k)$ and therefore

$$u(0) = 1 \quad \text{and} \quad \partial^m_t u(0) = 0 \text{ for any } m \in \{1,\dots,k-1\}. \tag{3.15}$$

[3] It is easily seen that for $k := 1$ Lemma 3.3 boils down to Lemma 3.2.

We also claim that
$$D^{\alpha}_{t,a}u(t) = \lambda\, u(t) \ \text{ for any } \ t \in (0, +\infty). \tag{3.16}$$
To check this, we recall (3.6) and (3.13) (with $\beta := 1$), and we have that
$$\begin{aligned} &D^{\alpha}_{t,a}u\,(t)\\ =\ & \frac{1}{\Gamma\,(k-\alpha)}\int_0^t \frac{u^{(k)}\,(\tau)}{(t-\tau)^{\alpha-k+1}}\,d\tau\\ =\ & \frac{1}{\Gamma\,(k-\alpha)}\int_0^t \left(\sum_{j=1}^{+\infty}\lambda^j\frac{\alpha j\,(\alpha j-1)\ldots(\alpha j-k+1)}{\Gamma\,(\alpha j+1)}\tau^{\alpha j-k}\right)\frac{d\tau}{(t-\tau)^{\alpha-k+1}}\\ =\ & \sum_{j=1}^{+\infty}\lambda^j\frac{\alpha j\,(\alpha j-1)\ldots(\alpha j-k+1)}{\Gamma\,(k-\alpha)\,\Gamma\,(\alpha j+1)}\int_0^t \tau^{\alpha j-k}\,(t-\tau)^{k-\alpha-1}\,d\tau. \end{aligned}$$
Hence, using the change of variable $\tau = t\sigma$, we obtain that
$$D^{\alpha}_{t,a}u\,(t) = \sum_{j=1}^{+\infty}\lambda^j\frac{\alpha j\,(\alpha j-1)\ldots(\alpha j-k+1)}{\Gamma\,(k-\alpha)\,\Gamma\,(\alpha j+1)}t^{\alpha j-\alpha}\int_0^1 \sigma^{\alpha j-k}\,(1-\sigma)^{k-\alpha-1}\,d\tau. \tag{3.17}$$
On the other hand, from the basic properties of the Beta function, it is known that if $\Re(z)$, $\Re(w) > 0$, then
$$\int_0^1 \sigma^{z-1}\,(1-\sigma)^{w-1}\,dt = \frac{\Gamma\,(z)\,\Gamma\,(w)}{\Gamma\,(z+w)}. \tag{3.18}$$
In particular, taking $z := \alpha j-k+1 \in (\alpha-k+1, +\infty) \subseteq (0, +\infty)$ and $w := k-\alpha \in (0, +\infty)$, and substituting (3.18) into (3.17), we conclude that
$$\begin{aligned} D^{\alpha}_{t,a}u\,(t) &= \sum_{j=1}^{+\infty}\lambda^j\frac{\alpha j\,(\alpha j-1)\ldots(\alpha j-k+1)}{\Gamma\,(k-\alpha)\,\Gamma\,(\alpha j+1)}\frac{\Gamma\,(\alpha j-k+1)\,\Gamma\,(k-\alpha)}{\Gamma\,(\alpha j-\alpha+1)}\,t^{\alpha j-\alpha}\\ &= \sum_{j=1}^{+\infty}\lambda^j\frac{\alpha j\,(\alpha j-1)\ldots(\alpha j-k+1)}{\Gamma\,(\alpha j+1)}\frac{\Gamma\,(\alpha j-k+1)}{\Gamma\,(\alpha j-\alpha+1)}\,t^{\alpha j-\alpha}. \end{aligned} \tag{3.19}$$
Now we use the fact that $z\Gamma\,(z) = \Gamma\,(z+1)$ for any $z \in \mathbb{C}$ with $\Re\,(z) > -1$, so, we have
$$\alpha j\,(\alpha j-1)\ldots(\alpha j-k+1)\,\Gamma\,(\alpha j-k+1) = \Gamma\,(\alpha j+1)\,.$$
Plugging this information into (3.19), we thereby find that
$$D^{\alpha}_{t,a}u\,(t) = \sum_{j=1}^{+\infty}\frac{\lambda^j}{\Gamma\,(\alpha j-\alpha+1)}t^{\alpha j-\alpha} = \sum_{j=0}^{+\infty}\frac{\lambda^{j+1}}{\Gamma\,(\alpha j+1)}t^{\alpha j} = \lambda u(t).$$
This proves (3.16).

Then, in view of (3.15) and (3.16) we obtain that u is a solution of (3.14). Hence, to complete the proof of the desired result, we have to show that such a solution is unique. To this end, supposing that we have two solutions of (3.14), we consider their difference w, and we observe that w is a solution of
$$\begin{cases} D^{\alpha}_{t,0}w(t) = \lambda\, w(t) & \text{for any } t \in (0, +\infty),\\ \quad \partial^m_t w(0) = 0 & \text{for any } m \in \{0, \ldots, k-1\}. \end{cases}$$
By Theorem 4.1 in [SZ16], it follows that w vanishes identically, and this proves the desired uniqueness result. □

The boundary behaviour of the Mittag-Leffler function for different values of the fractional parameter α is depicted in Figure 3.1. In light of (3.13), we notice in particular that, near $z = 0$,

$$E_{\alpha,\beta}\left(z\right) = \frac{1}{\Gamma\left(\beta\right)} + \frac{z}{\Gamma\left(\alpha+\beta\right)} + O(z^2)$$

and therefore, near $t = a$,

$$E_{\alpha,1}\left(\lambda\left(t-a\right)^{\alpha}\right) = 1 + \frac{\lambda\left(t-a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)} + O\big(\lambda^2\left(t-a\right)^{2\alpha}\big).$$

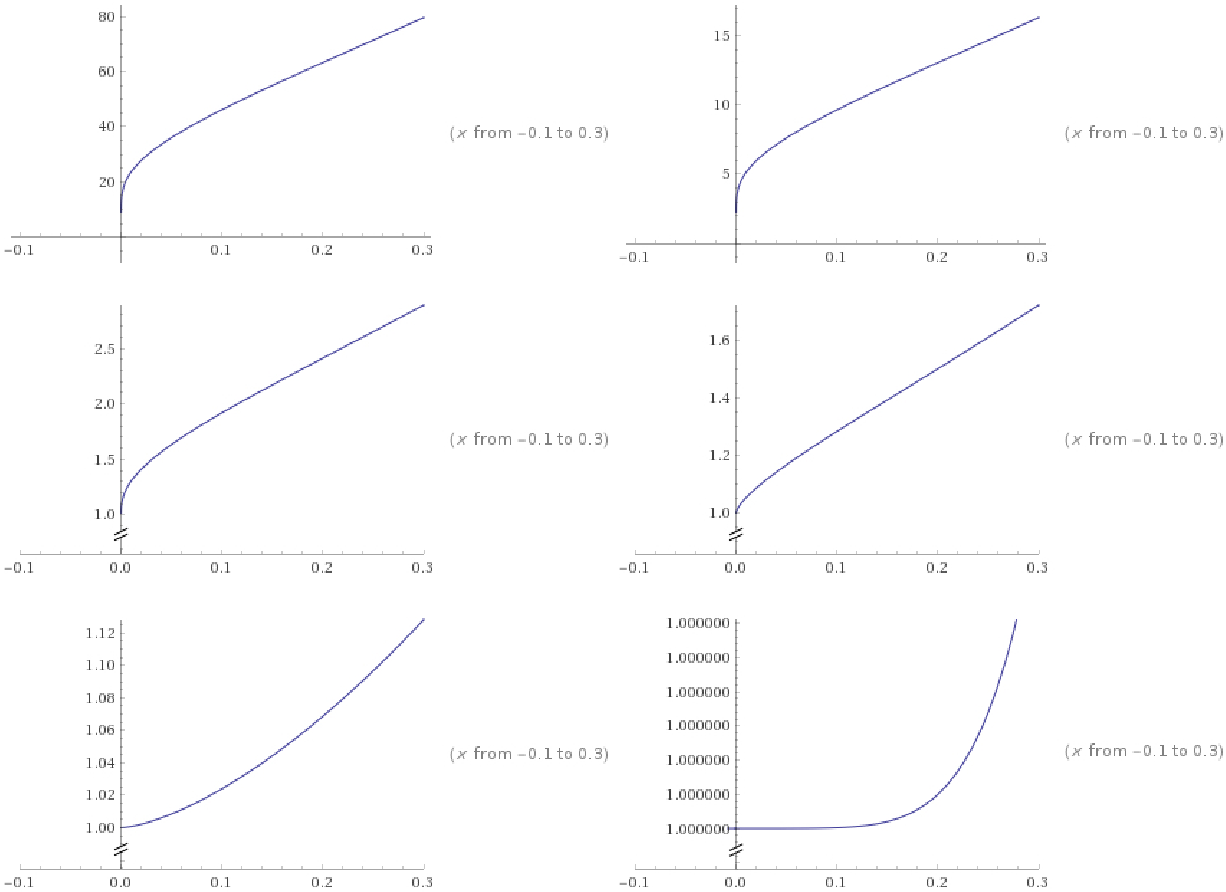

Figure 3.1: *Behaviour of the Mittag-Leffler function* $E_{\alpha,1}\left(t^{\alpha}\right)$ *near the origin for* $\alpha = \frac{1}{100}$, $\alpha = \frac{1}{20}$, $\alpha = \frac{1}{3}$, $\alpha = \frac{2}{3}$ $\alpha = \frac{3}{2}$ *and* $\alpha = \frac{11}{2}$.

3.3 Sharp boundary behaviour for the time-fractional harmonic functions

In this section, we detect the optimal boundary behaviour of time-fractional harmonic functions and of their derivatives. The result that we need for our purposes is the following:

Lemma 3.4. *Let* $\alpha \in (0, +\infty) \setminus \mathbb{N}$. *There exists a function* $\psi : \mathbb{R} \to \mathbb{R}$ *such that* $\psi \in C^{\infty}((1, +\infty))$ *and*

$$D_0^{\alpha}\psi(t) = 0 \qquad \text{for all } t \in (1, +\infty), \tag{3.20}$$

$$\textit{and} \qquad \lim_{\epsilon\searrow 0} \epsilon^{\ell-\alpha}\partial^\ell\psi(1+\epsilon t) = \kappa_{\alpha,\ell}\, t^{\alpha-\ell}, \qquad \textit{for all } \ell\in\mathbb{N}, \tag{3.21}$$

for some $\kappa_{\alpha,\ell}\in\mathbb{R}\setminus\{0\}$*, where* (3.21) *is taken in the sense of distribution for* $t\in(0,+\infty)$*.*

Proof. We use Lemma 2.5 in [CDV18], according to which (see in particular formula (2.16) in [CDV18]) the claim in (3.20) holds true. Furthermore (see formulas (2.19) and (2.20) in [CDV18]), we can write that, for all $t>1$,

$$\psi(t) = -\frac{1}{\Gamma(\alpha)\Gamma([\alpha]+1-\alpha)} \iint_{[1,t]\times[0,3/4]} \partial^{[\alpha]+1}\psi_0(\sigma)\,(\tau-\sigma)^{[\alpha]-\alpha}\,(t-\tau)^{\alpha-1}\,d\tau\,d\sigma, \tag{3.22}$$

for a suitable $\psi_0\in C^{[\alpha]+1}([0,1])$.

In addition, by Lemma 2.6 in [CDV18], we can write that

$$\lim_{\epsilon\searrow 0}\epsilon^{-\alpha}\psi(1+\epsilon) = \kappa, \tag{3.23}$$

for some $\kappa\neq 0$. Now we set

$$(0,+\infty)\ni t\mapsto f_\epsilon(t) := \epsilon^{\ell-\alpha}\partial^\ell\psi(1+\epsilon t).$$

We observe that, for any $\varphi\in C_0^\infty((0,+\infty))$,

$$\begin{aligned}
&\int_0^{+\infty} f_\epsilon(t)\,\varphi(t)\,dt = \epsilon^{\ell-\alpha}\int_0^{+\infty}\partial^\ell\psi(1+\epsilon t)\varphi(t)\,dt\\
&= \epsilon^{-\alpha}\int_0^{+\infty}\frac{d^\ell}{dt^\ell}\big(\psi(1+\epsilon t)\big)\varphi(t)\,dt = (-1)^\ell\,\epsilon^{-\alpha}\int_0^{+\infty}\psi(1+\epsilon t)\,\partial^\ell\varphi(t)\,dt.
\end{aligned} \tag{3.24}$$

Also, in view of (3.22),

$$\begin{aligned}
&\epsilon^{-\alpha}|\psi(1+\epsilon t)|\\
=\;& \left|\frac{\epsilon^{-\alpha}}{\Gamma(\alpha)\Gamma([\alpha]+1-\alpha)}\iint_{[1,1+\epsilon t]\times[0,3/4]}\partial^{[\alpha]+1}\psi_0(\sigma)\,(\tau-\sigma)^{[\alpha]-\alpha}\,(1+\epsilon t-\tau)^{\alpha-1}\,d\tau\,d\sigma\right|\\
\le\;& C\,\epsilon^{-\alpha}\int_{[1,1+\epsilon t]}(1+\epsilon t-\tau)^{\alpha-1}\,d\tau\\
=\;& Ct^\alpha,
\end{aligned}$$

which is locally bounded in t, where $C>0$ here above may vary from line to line.

As a consequence, we can pass to the limit in (3.24) and obtain that

$$\lim_{\epsilon\searrow 0}\int_0^{+\infty} f_\epsilon(t)\,\varphi(t)\,dt = (-1)^\ell\int_0^{+\infty}\lim_{\epsilon\searrow 0}\epsilon^{-\alpha}\psi(1+\epsilon t)\,\partial^\ell\varphi(t)\,dt.$$

This and (3.23) give that

$$\lim_{\epsilon\searrow 0}\int_0^{+\infty} f_\epsilon(t)\,\varphi(t)\,dt = (-1)^\ell\,\kappa\int_0^{+\infty} t^\alpha\,\partial^\ell\varphi(t)\,dt = \kappa\,\alpha\ldots(\alpha-\ell+1)\int_0^{+\infty} t^{\alpha-\ell}\,\varphi(t)\,dt,$$

which establishes (3.21). $\square$

3.4 Boundary behaviour of solutions of space-fractional equations

In this section, we give precise asymptotics for the boundary behaviour of solutions of space-fractional equations. The cases of the eigenfunctions and of the Dirichlet problem with vanishing forcing term will be studied in detail. To this end, we will also exploit useful representation formulas of the solutions in terms of suitable Green functions.

3.5 Green representation formulas and solution of $(-\Delta)^s u = f$ in B_1 with homogeneous Dirichlet datum

Our goal is to provide some representation results on the solution of $(-\Delta)^s u = f$ in a ball, with $u = 0$ outside this ball, for all $s > 0$. Our approach is an extension of the Green formula methods introduced in [DG17] and [AJS18b]: differently from the previous literature, we are not assuming here that f is regular in the whole of the ball, but merely that it is Hölder continuous near the boundary and sufficiently integrable inside. Given the type of singularity of the Green function, these assumptions are sufficient to obtain meaningful representations, which in turn will be useful to deal with the eigenfunction problem in the subsequent section 3.6.

3.5.1 Solving $(-\Delta)^s u = f$ in B_1 for discontinuous f vanishing near ∂B_1

Now, we want to extend the representation results of [DG17] and [AJS18b] to the case in which the right hand side is not Hölder continuous, but merely in a Lebesgue space, but it has the additional property of vanishing near the boundary of the domain. To this end, fixed $s > 0$, we consider the polyharmonic Green function in $B_1 \subset \mathbb{R}^n$, given, for every $x \neq y \in \mathbb{R}^n$, by

$$\begin{aligned}
&\mathcal{G}_s(x,y) := \frac{k(n,s)}{|x-y|^{n-2s}} \int_0^{r_0(x,y)} \frac{\eta^{s-1}}{(\eta+1)^{\frac{n}{2}}}\, d\eta, \\
\text{where}\quad &r_0(x,y) := \frac{\left(1-|x|^2\right)_+ \left(1-|y|^2\right)_+}{|x-y|^2}, \\
\text{with}\quad &k(n,s) := \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\, 4^s \Gamma^2(s)}.
\end{aligned} \tag{3.25}$$

Given $x \in B_1$, we also set

$$d(x) := 1 - |x|. \tag{3.26}$$

In this setting, we have:

Proposition 3.5. *Let $r \in (0,1)$ and $f \in L^2(B_1)$, with $f = 0$ in $\mathbb{R}^n \setminus B_r$. Let*

$$u(x) := \begin{cases} \displaystyle\int_{B_1} \mathcal{G}_s(x,y)\, f(y)\, dy & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases} \tag{3.27}$$

Then:

$$u \in L^1(B_1), \ \text{and} \ \|u\|_{L^1(B_1)} \le C\,\|f\|_{L^1(B_1)}, \tag{3.28}$$

$$\text{for every } R \in (r,1), \quad \sup_{x \in B_1 \setminus B_R} d^{-s}(x)\,|u(x)| \le C_R\,\|f\|_{L^1(B_1)}, \tag{3.29}$$

$$u \text{ satisfies } (-\Delta)^s u = f \text{ in } B_1 \text{ in the sense of distributions,} \tag{3.30}$$

and

$$u \in W^{2s,2}_{loc}(B_1). \tag{3.31}$$

Here above, $C > 0$ is a constant depending on n, s and r, $C_R > 0$ is a constant depending on n, s, r and R and $C_\rho > 0$ is a constant depending on n, s, r and ρ.

When $f \in C^{0,\alpha}(B_1)$ for some $\alpha \in (0,1)$, Proposition 3.5 boils down to the main results of [DG17] and [AJS18b].

Proof of Proposition 3.5. We recall the following useful estimate, see Lemma 3.3 in [AJS18b]: for any $\epsilon \in (0, \min\{n,s\})$, and any $\bar{R}$, $\bar{r} > 0$,

$$\frac{1}{\bar{R}^{n-2s}} \int_0^{\bar{r}/\bar{R}^2} \frac{\eta^{s-1}}{(\eta+1)^{\frac{n}{2}}}\,d\eta \le \frac{2}{s}\,\frac{\bar{r}^{s-(\epsilon/2)}}{\bar{R}^{n-\epsilon}},$$

and so, by (3.25) and (3.26), for every x, $y \in B_1$,

$$\mathcal{G}_s(x,y) \le \frac{C\,d^{s-(\epsilon/2)}(x)\,d^{s-(\epsilon/2)}(y)}{|x-y|^{n-\epsilon}}$$

for some $C > 0$. Hence, recalling (3.27),

$$\begin{aligned}
\int_{B_1} |u(x)|\,dx &\le \int_{B_1} \left(\int_{B_1} \mathcal{G}_s(x,y)\,|f(y)|\,dy \right) dx \\
&\le C \int_{B_1} \left(\int_{B_1} \frac{|f(y)|}{|x-y|^{n-\epsilon}}\,dy \right) dx \\
&= C \int_{B_1} \left(\int_{B_1} \frac{|f(y)|}{|x-y|^{n-\epsilon}}\,dx \right) dy \\
&= C \int_{B_1} |f(y)|\,dy,
\end{aligned}$$

up to renaming $C > 0$ line after line, and this proves (3.28).

Now, if $x \in B_1 \setminus B_R$ and $y \in B_r$, with $0 < r < R < 1$, we have that

$$|x-y| \ge |x| - |y| \ge R - r$$

and accordingly

$$r_0(x,y) \le \frac{2d(x)}{(R-r)^2},$$

which in turn implies that

$$\mathcal{G}_s(x,y) \le \frac{k(n,s)}{|x-y|^{n-2s}} \int_0^{2d(x)/(R-r)^2} \frac{\eta^{s-1}}{(\eta+1)^{\frac{n}{2}}}\,d\eta, \le C\,d^s(x),$$

for some $C > 0$. As a consequence, since f vanishes outside B_r, we see that, for any $x \in B_1 \setminus B_R$,

$$|u(x)| \leq \int_{B_r} \mathcal{G}_s\,(x,y)\;|f(y)|\,dy \leq C\,d^s(x) \int_{B_r} |f(y)|\,dy,$$

which proves (3.29).

Now, we fix $\hat{r} \in (r,1)$ and consider a mollification of f, that we denote by $f_j \in C_0^\infty(B_{\hat{r}})$, with $f_j \to f$ in $L^2(B_1)$ as $j \to +\infty$. We also write $\mathcal{G}_s * f$ as a short notation for the right hand side of (3.27). Then, by [DG17] and [AJS18b], we know that $u_j := \mathcal{G}_s * f_j$ is a (locally smooth, hence distributional) solution of $(-\Delta)^s u_j = f_j$. Furthermore, if we set $\tilde{u}_j := u_j - u$ and $\tilde{f}_j := f_j - f$ we have that

$$\tilde{u}_j = \mathcal{G}_s * (f_j - f) = \mathcal{G}_s * \tilde{f}_j,$$

and therefore, by (3.28),

$$\|\tilde{u}_j\|_{L^1(B_1)} \leq C\,\|\tilde{f}_j\|_{L^1(B_1)},$$

which is infinitesimal as $j \to +\infty$. This says that $u_j \to u$ in $L^1(B_1)$ as $j \to +\infty$, and consequently, for any $\varphi \in C_0^\infty(B_1)$,

$$\begin{aligned}\int_{B_1} u(x)\,(-\Delta)^s\varphi(x)\,dx &= \lim_{j\to+\infty} \int_{B_1} u_j(x)\,(-\Delta)^s\varphi(x)\,dx\\ &= \lim_{j\to+\infty} \int_{B_1} f_j(x)\,\varphi(x)\,dx = \int_{B_1} f(x)\,\varphi(x)\,dx,\end{aligned}$$

thus completing the proof of (3.30).

Now, to prove (3.31), we can suppose that $s \in (0,+\infty) \setminus \mathbb{N}$, since the case of integer s is classical, see e.g. [GT01]. First of all, we claim that

$$(3.31) \text{ holds true for every } s \in (0,1). \tag{3.32}$$

For this, we first claim that if $g \in C^\infty(B_1)$ and v is a (locally smooth) solution of $(-\Delta)^s v = g$ in B_1, with $v = 0$ outside B_1, then $v \in W^{2s,2}_{loc}(B_1)$, and, for any $\rho \in (0,1)$,

$$\|v\|_{W^{2s,2}(B_\rho)} \leq C_\rho\,\|g\|_{L^2(B_1)}. \tag{3.33}$$

This claim can be seen as a localization of Lemma 3.1 of [DK12], or a quantification of the last claim in Theorem 1.3 of [BWZ17]. To prove (3.33), we let $R_- < R_+ \in (\rho,1)$, and consider $\eta \in C_0^\infty(B_{R_+})$ with $\eta = 1$ in B_{R_-}. We let $v^* := v\eta$, and we recall formulas (3.2), (3.3) and (A.5) in [BWZ17], according to which

$$\begin{aligned} &(-\Delta)^s v^* - \eta(-\Delta)^s v = g^* \quad \text{in } \mathbb{R}^n,\\ \text{with}\quad &\|g^*\|_{L^2(\mathbb{R}^n)} \leq C\,\|v\|_{W^{s,2}(\mathbb{R}^n)},\end{aligned}$$

for some $C > 0$.

Moreover, using a notation taken from [BWZ17] we denote by $W_0^{s,2}(\overline{B_1})$ the space of functions in $W^{s,2}(\mathbb{R}^n)$ vanishing outside B_1 and we consider the dual space $W_0^{-s,2}(\overline{B_1})$. We remark that if $h \in L^2(B_1)$ we can naturally identify h as an element of $W_0^{-s,2}(\overline{B_1})$ by considering the action of h on any $\varphi \in W_0^{s,2}(\overline{B_1})$ as defined by

$$\int_{B_1} h(x)\,\varphi(x)\,dx.$$

With respect to this, we have that

$$\|h\|_{W_0^{-s,2}(\overline{B_1})} = \sup_{\substack{\varphi\in W_0^{s,2}(\overline{B_1})\\ \|\varphi\|_{W_0^{s,2}(\overline{B_1})}=1}} \int_{B_1} h(x)\,\varphi(x)\,dx \le \|h\|_{L^2(B_1)}. \tag{3.34}$$

We notice also that

$$\|v\|_{W^{s,2}(\mathbb{R}^n)} \le C\,\|g\|_{W^{-s,2}(\overline{B_1})},$$

in light of Proposition 2.1 of [BWZ17]. This and (3.34) give that

$$\|v\|_{W^{s,2}(\mathbb{R}^n)} \le C\,\|g\|_{L^2(B_1)}.$$

Then, by Lemma 3.1 of [DK12] (see in particular formula (3.2) there, applied here with $\lambda := 0$), we obtain that

$$\begin{aligned}\|v^*\|_{W^{2s,2}(\mathbb{R}^n)} &\le C\,\|\eta(-\Delta)^s v + g^*\|_{L^2(\mathbb{R}^n)}\\ &\le C\left(\|(-\Delta)^s v\|_{L^2(B_{R_+})} + \|g^*\|_{L^2(\mathbb{R}^n)}\right)\\ &= C\left(\|g\|_{L^2(B_{R_+})} + \|g^*\|_{L^2(\mathbb{R}^n)}\right)\\ &\le C\left(\|g\|_{L^2(B_1)} + \|v\|_{W^{s,2}(\mathbb{R}^n)}\right)\\ &\le C\,\|g\|_{L^2(B_1)},\end{aligned} \tag{3.35}$$

up to renaming $C > 0$ step by step. On the other hand, since $v^* = v$ in B_ρ,

$$\|v\|_{W^{2s,2}(B_\rho)} = \|v^*\|_{W^{2s,2}(B_\rho)} \le \|v^*\|_{W^{2s,2}(\mathbb{R}^n)}.$$

From this and (3.35), we obtain (3.33), as desired.

Now, we let f_j, $\tilde f_j$, u_j and $\tilde u_j$ as above and make use of (3.33) to write

$$\begin{aligned}&\|u_j\|_{W^{2s,2}(B_\rho)} \le C_\rho\,\|f_j\|_{L^2(B_1)}\\ \text{and}\quad &\|\tilde u_j\|_{W^{2s,2}(B_\rho)} \le C_\rho\,\|\tilde f_j\|_{L^2(B_1)}.\end{aligned} \tag{3.36}$$

As a consequence, taking the limit as $j \to +\infty$ we obtain that

$$\|u\|_{W^{2s,2}(B_\rho)} \le C_\rho\,\|f\|_{L^2(B_1)},$$

that is (3.31) in this case, namely the claim in (3.32).

Now, to prove (3.31), we argue by induction on the integer part of s. When the integer part of s is zero, the basis of the induction is warranted by (3.32). Then, to perform the inductive step, given $s \in (0, +\infty) \setminus \mathbb{N}$, we suppose that (3.31) holds true for $s-1$, namely

$$\mathcal{G}_{s-1} * f \in W^{2s-2,2}_{loc}(B_1). \tag{3.37}$$

Then, following [AJS18b], it is convenient to introduce the notation

$$[x, y] := \sqrt{|x|^2|y|^2 - 2x\cdot y + 1}$$

and consider the auxiliary kernel given, for every $x \ne y \in B_1$, by

$$P_{s-1}(x, y) := \frac{(1-|x|^2)_+^{s-2}(1-|y|^2)_+^{s-1}(1-|x|^2|y|^2)}{[x,y]^n}. \tag{3.38}$$

We point out that if $x \in B_r$ with $r \in (0,1)$, then

$$[x,y]^2 \geq |x|^2|y|^2 - 2|x|\,|y| + 1 = (1-|x|\,|y|)^2 \geq (1-r)^2 > 0. \tag{3.39}$$

Consequently, since f is supported in B_r,

$$P_{s-1} * f \in C^\infty(\mathbb{R}^n). \tag{3.40}$$

Then, we recall that

$$-\Delta_x \mathcal{G}_s(x,y) = \mathcal{G}_{s-1}(x,y) - CP_{s-1}(x,y), \tag{3.41}$$

for some $C \in \mathbb{R}$, see Lemma 3.1 in [AJS18b].

As a consequence, in view of (3.37), (3.40), (3.41), we conclude that

$$-\Delta_x(\mathcal{G}_s * f) = (-\Delta_x \mathcal{G}_s) * f \in W^{2s-2,2}_{loc}(B_1).$$

This and the classical elliptic regularity theory (see e.g. [GT01]) give that $\mathcal{G}_s * f \in W^{2s,2}_{loc}(B_1)$, which completes the inductive proof and establishes (3.31). □

3.5.2 Solving $(-\Delta)^s u = f$ in B_1 for f Hölder continuous near ∂B_1

Our goal is now to extend the representation results of [DG17] and [AJS18b] to the case in which the right hand side is not Hölder continuous in the whole of the ball, but merely in a neighborhood of the boundary. This result is obtained here by superposing the results in [DG17] and [AJS18b] with Proposition 3.5 here, taking advantage of the linear structure of the problem.

Proposition 3.6. *Let $f \in L^2(B_1)$. Let α, $r \in (0,1)$ and assume that*

$$f \in C^{0,\alpha}(B_1 \setminus B_r). \tag{3.42}$$

In the notation of (3.25), let

$$u(x) := \begin{cases} \displaystyle\int_{B_1} \mathcal{G}_s\,(x,y)\; f(y)\,dy & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases} \tag{3.43}$$

Then, in the notation of (3.26), we have that:

$$\text{for every } R \in (r,1), \quad \sup_{x \in B_1 \setminus B_R} d^{-s}(x)\,|u(x)| \leq C_R \left(\|f\|_{L^1(B_1)} + \|f\|_{L^\infty(B_1 \setminus B_r)}\right), \tag{3.44}$$

$$u \text{ satisfies } (-\Delta)^s u = f \text{ in } B_1 \text{ in the sense of distributions,} \tag{3.45}$$

and

$$u \in W^{2s,2}_{loc}(B_1). \tag{3.46}$$

Here above, $C > 0$ is a constant depending on n, s and r, $C_R > 0$ is a constant depending on n, s, r and R and $C_\rho > 0$ is a constant depending on n, s, r and ρ.

Proof. We take $r_1 \in (r,1)$ and $\eta \in C_0^\infty(B_{r_1})$ with $\eta = 1$ in B_r. Let also

$$f_1 := f\eta \qquad \text{and} \qquad f_2 := f - f_1.$$

We observe that $f_1 \in L^2(B_1)$, and that $f_1 = 0$ outside B_{r_1}. Therefore, we are in the position of applying Proposition 3.5 and find a function u_1 (obtained by convolving $\mathcal{G}_s$ against f_1) such that

$$\text{for every } R \in (r_1,1), \quad \sup_{x\in B_1\setminus B_R} d^{-s}(x)\,|u_1(x)| \le C_R\,\|f_1\|_{L^1(B_1)}, \tag{3.47}$$

$$u_1 \text{ satisfies } (-\Delta)^s u_1 = f_1 \text{ in } B_1 \text{ in the sense of distributions}, \tag{3.48}$$

$$\text{and} \qquad u_1 \in W^{2s,2}_{loc}(B_1). \tag{3.49}$$

On the other hand, we have that $f_2 = f(1-\eta)$ vanishes outside $B_1 \setminus B_r$ and it is Hölder continuous. Accordingly, we can apply Theorem 1.1 of [AJS18b] and find a function u_2 (obtained by convolving $\mathcal{G}_s$ against f_2) such that

$$\text{for every } R \in (r_1,1), \quad \sup_{x\in B_1\setminus B_R} d^{-s}(x)\,|u_2(x)| \le C_R\,\|f_2\|_{L^\infty(B_1)}, \tag{3.50}$$

$$u_2 \text{ satisfies } (-\Delta)^s u_2 = f_2 \text{ in } B_1 \text{ in the sense of distributions}, \tag{3.51}$$

$$\text{and} \qquad u_2 \in C^{2s+\alpha}_{loc}(B_1). \tag{3.52}$$

Then, $f = f_1 + f_2$, and thus, in view of (3.43), we have that $u = u_1 + u_2$. Also, u satisfies (3.44), thanks to (3.47) and (3.50), (3.45), thanks to (3.48) and (3.51), and (3.46), thanks to (3.49) and (3.52). □

3.6 Existence and regularity for the first eigenfunction of the higher order fractional Laplacian

The goal of these pages is to study the boundary behaviour of the first Dirichlet eigenfunction of higher order fractional equations.

For this, writing $s = m + \sigma$, with $m \in \mathbb{N}$ and $\sigma \in (0,1)$, we define the energy space

$$H^s_0(B_1) := \{u \in H^s(\mathbb{R}^n);\ u = 0 \text{ in } \mathbb{R}^n \setminus B_1\}, \tag{3.53}$$

endowed with the Hilbert norm

$$\|u\|_{H^s_0(B_1)} := \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|^2_{L^2(B_1)} + \mathcal{E}_s(u,u)\right)^{\frac{1}{2}}, \tag{3.54}$$

where

$$\mathcal{E}_s(u,v) = \int_{\mathbb{R}^n} |\xi|^{2s}\,\mathcal{F}u(\xi)\,\overline{\mathcal{F}v(\xi)}\,d\xi, \tag{3.55}$$

being $\mathcal{F}$ the Fourier transform and using the notation $\overline{z}$ to denote the complex conjugated of a complex number z.

In this setting, we consider $u \in H^s_0(B_1)$ to be such that

$$\begin{cases} (-\Delta)^s u = \lambda_1 u & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{B_1}, \end{cases} \tag{3.56}$$

for every $s > 0$, with λ_1 as small as possible.

The existence of solutions of (3.56) is ensured via variational techniques, as stated in the following result:

Lemma 3.7. *The functional $\mathcal{E}_s(u,u)$ attains its minimum λ_1 on the functions in $H^s_0(B_1)$ with unit norm in $L^2(B_1)$.*

The minimizer satisfies (3.56).

In addition, $\lambda_1 > 0$.

Proof. The proof is based on the direct method in the calculus of variations. We provide some details for completeness. Let $s = m + \sigma$, with $m \in \mathbb{N}$ and $\sigma \in (0,1)$. Let us consider a minimizing sequence $u_j \in H^s_0(B_1) \subseteq H^m(\mathbb{R}^n)$ such that $\|u_j\|_{L^2(B_1)} = 1$ and

$$\lim_{j\to+\infty} \mathcal{E}_s(u_j,u_j) = \inf_{\substack{u\in H^s_0(B_1)\\ \|u\|_{L^2(B_1)}=1}} \mathcal{E}_s(u,u).$$

In particular, we have that u_j is bounded in $H^s_0(B_1)$ uniformly in j, so, up to a subsequence, it converges to some $u_\star$ weakly in $H^s_0(B_1)$ and strongly in $L^2(B_1)$ as $j \to +\infty$.

The weak lower semicontinuity of the seminorm $\mathcal{E}_s(\cdot,\cdot)$ then implies that $u_\star$ is the desired minimizer.

Then, given $\phi \in C^\infty_0(B_1)$, we have that

$$\mathcal{E}_s(u_\star + \epsilon\phi, u_\star + \epsilon\phi) \geq \mathcal{E}_s(u_\star,u_\star),$$

for every $\epsilon \in \mathbb{R}$, and this gives that (3.56) is satisfied in the sense of distributions, and also in the classical sense by the elliptic regularity theory.

Finally, we have that $\mathcal{E}_s(u_\star,u_\star) > 0$, since $u_\star$ (and thus $\mathcal{F}u_\star$) does not vanish identically. Consequently,

$$\lambda_1 = \frac{\mathcal{E}_s(u_\star,u_\star)}{\|u_\star\|^2_{L^2(B_1)}} = \mathcal{E}_s(u_\star,u_\star) > 0,$$

as desired. □

Our goal is now to apply Proposition 3.6 to solutions of (3.56), taking $f := \lambda u$. To this end, we have to check that condition (3.42) is satisfied, namely that solutions of (3.56) are Hölder continuous in $B_1 \setminus B_r$, for any $0 < r < 1$.

To this aim, we prove that polyharmonic operators of any order $s > 0$ always admit a first eigenfunction in the ball which does not change sign and which is radially symmetric. For this, we start discussing the sign property:

Lemma 3.8. *There exists a nontrivial solution of (3.56) that does not change sign.*

Proof. We exploit a method explained in detail in Section 3.1 of [GGS10]. As a matter of fact, when $s \in \mathbb{N}$, the desired result is exactly Theorem 3.7 in [GGS10].

Let u be as in Lemma 3.7. If either $u \geq 0$ or $u \leq 0$, then the desired result is proved. Hence, we argue by contradiction, assuming that u attains strictly positive and strictly negative values. We define

$$\mathcal{K} := \{w : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } \mathcal{E}_s(w,w) < +\infty, \text{ and } w \geq 0 \text{ in } B_1\}.$$

Also, we set

$$\mathcal{K}^\star := \{w \in H^s_0(B_1) \text{ s.t. } \mathcal{E}_s(w,v) \leq 0 \text{ for all } v \in \mathcal{K}\}.$$

We claim that

$$\text{if } w \in \mathcal{K}^\star, \text{ then } w \leq 0. \tag{3.57}$$

To prove this, we recall the notation in (3.25), take $\phi \in C_0^\infty(B_1) \cap \mathcal{K}$, and let

$$v_\phi(x) := \begin{cases} \displaystyle\int_{B_1} \mathcal{G}_s(x,y)\, \phi(y)\, dy & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Then $v_\phi \in \mathcal{K}$ and it satisfies $(-\Delta)^s v_\phi = \phi$ in B_1, thanks to [DG17] or [AJS18b].

Consequently, we can write, for every $x \in B_1$,

$$\phi(x) = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} v_\phi)(x).$$

Hence, for every $w \in \mathcal{K}^\star$,

$$\begin{aligned} 0 &\geq \mathcal{E}_s(w, v_\phi) \\ &= \int_{\mathbb{R}^n} |\xi|^{2s}\, \mathcal{F}v_\phi(\xi)\, \overline{\mathcal{F}w(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}(|\xi|^{2s}\, \mathcal{F}v_\phi)(x)\, w(x)\, dx \\ &= \int_{B_1} \mathcal{F}^{-1}(|\xi|^{2s}\, \mathcal{F}v_\phi)(x)\, w(x)\, dx \\ &= \int_{B_1} \phi(x)\, w(x)\, dx. \end{aligned}$$

Since ϕ is arbitrary and nonnegative, this gives that $w \leq 0$, and this establishes (3.57).

Furthermore, by Theorem 3.4 in [GGS10], we can write

$$u = u_1 + u_2,$$

with $u_1 \in \mathcal{K} \setminus \{0\}$, $u_2 \in \mathcal{K}^\star \setminus \{0\}$, and $\mathcal{E}_s(u_1, u_2) = 0$.

We observe that

$$\mathcal{E}_s(u_1 - u_2, u_1 - u_2) = \mathcal{E}_s(u_1, u_1) + \mathcal{E}_s(u_2, u_2) + 2\mathcal{E}_s(u_1, u_2) = \mathcal{E}_s(u_1, u_1) + \mathcal{E}_s(u_2, u_2).$$

In the same way,

$$\mathcal{E}_s(u, u) = \mathcal{E}_s(u_1 + u_2, u_1 + u_2) = \mathcal{E}_s(u_1, u_1) + \mathcal{E}_s(u_2, u_2),$$

and therefore

$$\mathcal{E}_s(u_1 - u_2, u_1 - u_2) = \mathcal{E}_s(u, u). \tag{3.58}$$

On the other hand,

$$\begin{aligned} \|u_1 - u_2\|^2_{L^2(B_1)} - \|u\|^2_{L^2(B_1)} &= \|u_1 - u_2\|^2_{L^2(B_1)} - \|u_1 + u_2\|^2_{L^2(B_1)} \\ &= -4 \int_{B_1} u_1(x)\, u_2(x)\, dx. \end{aligned}$$

As a consequence, since $u_2 \leq 0$ in view of (3.57), we conclude that

$$\|u_1 - u_2\|^2_{L^2(B_1)} - \|u\|^2_{L^2(B_1)} \geq 0.$$

This and (3.58) say that the function $u_1 - u_2$ is also a minimizer for the variational problem in Lemma 3.7. Since now $u_1 - u_2 \geq 0$, the desired result follows. □

Now, we define the spherical mean of a function v by

$$v_\sharp(x) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} v(\mathcal{R}_\omega\, x)\, d\mathcal{H}^{n-1}(\omega)$$

where $\mathcal{R}_\omega$ is the rotation corresponding to the solid angle $\omega \in \mathbb{S}^{n-1}$, $\mathcal{H}^{n-1}$ is the standard Hausdorff measure, and $|\mathbb{S}^{n-1}| = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. Notice that $v_\sharp(x) = v_\sharp(\mathcal{R}_\varpi x)$ for any $\varpi \in \mathbb{S}^{n-1}$, that is $v_\sharp$ is rotationally invariant.

Then, we have:

Lemma 3.9. *Any positive power of the Laplacian commutes with the spherical mean, that is*

$$\big((-\Delta)^s v\big)_\sharp(x) = (-\Delta)^s v_\sharp(x).$$

Proof. By density, we prove the claim for a function v in the Schwartz space of smooth and rapidly decreasing functions. In this setting, writing $\mathcal{R}_\omega^T$ to denote the transpose of the rotation $\mathcal{R}_\omega$, and changing variable $\eta := \mathcal{R}_\omega^T\, \xi$, we have that

$$\begin{aligned}
(-\Delta)^s v(\mathcal{R}_\omega\, x) &= \int_{\mathbb{R}^n} |\xi|^{2s}\mathcal{F}v(\xi)\, e^{2\pi i \mathcal{R}_\omega\, x\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^{2s}\mathcal{F}v(\xi)\, e^{2\pi i x\cdot \mathcal{R}_\omega^T\, \xi}\, d\xi \qquad (3.59)\\
&= \int_{\mathbb{R}^n} |\eta|^{2s}\mathcal{F}v(\mathcal{R}_\omega\, \eta)\, e^{2\pi i x\cdot\eta}\, d\eta.
\end{aligned}$$

On the other hand, using the substitution $y := \mathcal{R}_\omega^T\, x$,

$$\begin{aligned}
\mathcal{F}v(\mathcal{R}_\omega\,\eta) &= \int_{\mathbb{R}^n} v(x)\, e^{-2\pi i x\cdot\mathcal{R}_\omega\,\eta}\, dx \\
&= \int_{\mathbb{R}^n} v(x)\, e^{-2\pi i \mathcal{R}_\omega^T\, x\cdot\eta}\, dx \\
&= \int_{\mathbb{R}^n} v(\mathcal{R}_\omega\, y)\, e^{-2\pi i y\cdot\eta}\, dy,
\end{aligned}$$

and therefore, recalling (3.59),

$$(-\Delta)^s v(\mathcal{R}_\omega\, x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\eta|^{2s} v(\mathcal{R}_\omega\, y)\, e^{2\pi i(x-y)\cdot\eta}\, dy\, d\eta.$$

As a consequence,

$$\begin{aligned}
\big((-\Delta)^s v\big)_\sharp(x) &= \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} (-\Delta)^s v(\mathcal{R}_\omega\, x)\, d\mathcal{H}^{n-1}(\omega) \\
&= \frac{1}{|\mathbb{S}^{n-1}|}\iiint_{\mathbb{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}^n} |\eta|^{2s} v(\mathcal{R}_\omega\, y)\, e^{2\pi i(x-y)\cdot\eta}\, d\mathcal{H}^{n-1}(\omega)\, dy\, d\eta \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\eta|^{2s} v_\sharp(y)\, e^{2\pi i(x-y)\cdot\eta}\, dy\, d\eta \\
&= \int_{\mathbb{R}^n} |\eta|^{2s}\mathcal{F}(v_\sharp)(\eta)\, e^{2\pi i x\cdot\eta}\, d\eta \\
&= (-\Delta)^s v_\sharp(x),
\end{aligned}$$

as desired. $\square$

It is also useful to observe that the spherical mean is compatible with the energy bounds. In particular we have the following observation:

Lemma 3.10. *We have that*

$$\mathcal{E}_s(v_\sharp, v_\sharp) \le \mathcal{E}_s(v, v). \tag{3.60}$$

Moreover,

$$\textit{if } v \in H^s_0(B_1), \textit{ then so does } v_\sharp. \tag{3.61}$$

Proof. We see that

$$\begin{aligned}
\mathcal{F}(v_\sharp)(\xi) &= \int_{\mathbb{R}^n} v_\sharp(x)\, e^{-2\pi i x\cdot\xi}\, dx \\
&= \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1}\times\mathbb{R}^n} v(\mathcal{R}_\omega\, x)\, e^{-2\pi i x\cdot\xi}\, d\mathcal{H}^{n-1}(\omega)\, dx
\end{aligned}$$

and therefore, taking the complex conjugated,

$$\overline{\mathcal{F}(v_\sharp)(\xi)} = \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1}\times\mathbb{R}^n} v(\mathcal{R}_\omega\, x)\, e^{2\pi i x\cdot\xi}\, d\mathcal{H}^{n-1}(\omega)\, dx.$$

Hence, by (3.55), and exploiting the changes of variables $y := \mathcal{R}_\omega\, x$ and $\tilde y := \mathcal{R}_{\tilde\omega}\, \tilde x$,

$$\begin{aligned}
&\mathcal{E}_s(v_\sharp, v_\sharp) \\
=\;& \int_{\mathbb{R}^n} |\xi|^{2s}\, \mathcal{F}(v_\sharp)(\xi)\, \overline{\mathcal{F}(v_\sharp)(\xi)}\, d\xi \\
=\;& \frac{1}{|\mathbb{S}^{n-1}|^2} \iiiint\!\!\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} |\xi|^{2s} v(\mathcal{R}_\omega\, x)\, v(\mathcal{R}_{\tilde\omega}\, \tilde x)\, e^{2\pi i(\tilde x - x)\cdot\xi}\, d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, dx\, d\tilde x\, d\xi \\
=\;& \frac{1}{|\mathbb{S}^{n-1}|^2} \iiiint\!\!\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} |\xi|^{2s} v(y)\, v(\tilde y)\, e^{2\pi i\tilde y\cdot\mathcal{R}_{\tilde\omega}\,\xi} e^{-2\pi i y\cdot\mathcal{R}_\omega\,\xi}\, d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, dy\, d\tilde y\, d\xi \\
=\;& \frac{1}{|\mathbb{S}^{n-1}|^2} \iiint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n} |\xi|^{2s}\, \mathcal{F}v(\mathcal{R}_\omega\,\xi)\, \overline{\mathcal{F}v(\mathcal{R}_{\tilde\omega}\,\xi)}\, d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, d\xi.
\end{aligned}$$

Consequently, using the Cauchy-Schwarz Inequality, and the substitutions $\eta := \mathcal{R}_\omega\,\xi$ and $\tilde\eta := \mathcal{R}_{\tilde\omega}\,\xi$,

$$\begin{aligned}
\mathcal{E}_s(v_\sharp, v_\sharp) &\le \frac{1}{|\mathbb{S}^{n-1}|^2} \iiint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n} |\xi|^{2s} \left|\mathcal{F}v(\mathcal{R}_\omega\,\xi)\right| \left|\mathcal{F}v(\mathcal{R}_{\tilde\omega}\,\xi)\right|\, d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, d\xi \\
&\le \frac{1}{|\mathbb{S}^{n-1}|^2} \left(\iiint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n} |\xi|^{2s} \left|\mathcal{F}v(\mathcal{R}_\omega\,\xi)\right|^2 d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, d\xi \right)^{\frac12} \\
&\qquad \cdot \left(\iiint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n} |\xi|^{2s} \left|\mathcal{F}v(\mathcal{R}_{\tilde\omega}\,\xi)\right|^2 d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, d\xi \right)^{\frac12} \\
&= \frac{1}{|\mathbb{S}^{n-1}|^2} \left(\iiint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n} |\eta|^{2s} \left|\mathcal{F}v(\eta)\right|^2 d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, d\eta \right)^{\frac12} \\
&\qquad \cdot \left(\iiint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^n} |\tilde\eta|^{2s} \left|\mathcal{F}v(\tilde\eta)\right|^2 d\mathcal{H}^{n-1}(\omega)\, d\mathcal{H}^{n-1}(\tilde\omega)\, d\tilde\eta \right)^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}^n} |\eta|^{2s} \left|\mathcal{F}v(\eta)\right|^2 d\eta\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\tilde\eta|^{2s} \left|\mathcal{F}v(\tilde\eta)\right|^2 d\tilde\eta\right)^{\frac{1}{2}} \\
&= \mathcal{E}_s\,(v,v)\,.
\end{aligned}
$$

This proves (3.60).

Now, we prove (3.61). For this, we observe that

$$
\frac{\partial^\ell v_\sharp}{\partial x_{j_1}\dots\partial x_{j_\ell}}(x) = \frac{1}{|\mathbb{S}^{n-1}|}\sum_{k_1,\dots,k_\ell=1}^{n}\int_{\mathbb{S}^{n-1}}\frac{\partial^\ell v}{\partial x_{k_1}\dots\partial x_{k_\ell}}(\mathcal{R}_\omega\, x)\,\mathcal{R}_\omega^{k_1 j_1}\dots\mathcal{R}_\omega^{k_\ell j_\ell}\,d\mathcal{H}^{n-1}(\omega),
$$

for every $\ell\in\mathbb{N}$ and $j_1,\dots,j_\ell\in\{1,\dots,n\}$, where $\mathcal{R}_\omega^{jk}$ denotes the (j,k) component of the matrix $\mathcal{R}_\omega$. In particular,

$$
\left|\frac{\partial^\ell v_\sharp}{\partial x_{j_1}\dots\partial x_{j_\ell}}(x)\right| \le C\sum_{k_1,\dots,k_\ell=1}^{n}\int_{\mathbb{S}^{n-1}}\left|\frac{\partial^\ell v}{\partial x_{k_1}\dots\partial x_{k_\ell}}(\mathcal{R}_\omega\, x)\right|\,d\mathcal{H}^{n-1}(\omega),
$$

for some $C>0$ only depending on n and ℓ, and hence

$$
\begin{aligned}
\left\|\frac{\partial^\ell v_\sharp}{\partial x_{j_1}\dots\partial x_{j_\ell}}(x)\right\|^2_{L^2(B_1)} &\le C\sum_{k_1,\dots,k_\ell=1}^{n}\iint_{\mathbb{S}^{n-1}\times B_1}\left|\frac{\partial^\ell v}{\partial x_{k_1}\dots\partial x_{k_\ell}}(\mathcal{R}_\omega\, x)\right|^2\,d\mathcal{H}^{n-1}(\omega)\,dx \\
&= C\sum_{k_1,\dots,k_\ell=1}^{n}\iint_{\mathbb{S}^{n-1}\times B_1}\left|\frac{\partial^\ell v}{\partial x_{k_1}\dots\partial x_{k_\ell}}(y)\right|^2\,d\mathcal{H}^{n-1}(\omega)\,dy \\
&= C\sum_{k_1,\dots,k_\ell=1}^{n}\left\|\frac{\partial^\ell v}{\partial x_{k_1}\dots\partial x_{k_\ell}}\right\|^2_{L^2(B_1)},
\end{aligned}
$$

up to renaming C.

This, together with (3.54) and (3.60), gives (3.61), as desired. □

With this preliminary work, we can now find a nontrivial, nonnegative and radial solution of (3.56).

Proposition 3.11. *There exists a solution of* (3.56) *in* $H^s_0(B_1)$ *which is radial, nonnegative and with unit norm in* $L^2(B_1)$.

Proof. Let $u\ge 0$ be a nontrivial solution of (3.56), whose existence is warranted by Lemma 3.8.

Then, we have that $u_\sharp\ge 0$. Moreover,

$$
\begin{aligned}
\int_{B_1} u_\sharp(x)\,dx &= \frac{1}{|\mathbb{S}^{n-1}|}\iint_{\mathbb{S}^{n-1}\times B_1} u(\mathcal{R}_\omega\, x)\,d\mathcal{H}^{n-1}(\omega)\,dx \\
&= \frac{1}{|\mathbb{S}^{n-1}|}\iint_{\mathbb{S}^{n-1}\times B_1} u(y)\,d\mathcal{H}^{n-1}(\omega)\,dy = \int_{B_1} u(y)\,dy>0,
\end{aligned}
$$

and therefore $u_\sharp$ does not vanish identically.

As a consequence, we can define

$$
u_\star := \frac{u_\sharp}{\|u_\sharp\|_{L^2(B_1)}}.
$$

We know that $u_\star \in H^s_0(B_1)$, due to (3.61). Moreover, in view of Lemma 3.9,

$$(-\Delta)^s u_\star = \frac{(-\Delta)^s u_\sharp}{\|u_\sharp\|_{L^2(B_1)}} = \frac{\big((-\Delta)^s u\big)_\sharp}{\|u_\sharp\|_{L^2(B_1)}} = \frac{\lambda_1\, u_\sharp}{\|u_\sharp\|_{L^2(B_1)}} = \lambda_1\, u_\star,$$

which gives the desired result. □

Now, we are in the position of proving the following result.

Lemma 3.12. *Let $s \geq 1$ and $r \in (0,1)$. If $u \in H^s_0(B_1)$ and u is radial, then $u \in C^{0,\alpha}(\mathbb{R}^n \setminus B_r)$ for any $\alpha \in \left[0, \frac{1}{2}\right]$.*

Proof. We write

$$u(x) = u_0(|x|), \qquad \text{for some } \ u_0 : [0, +\infty) \to \mathbb{R} \tag{3.62}$$

and we observe that $u \in H^s_0(B_1) \subset H^1(\mathbb{R}^n)$.

Accordingly, for any $0 < r < 1$, we have

$$\infty > \int_{\mathbb{R}^n \setminus B_r} |u(x)|^2\, dx = \int_r^{+\infty} |u_0(\rho)|^2 \rho^{n-1}\, d\rho \geq r^{n-1} \int_r^{+\infty} |u_0(\rho)|^2\, d\rho \tag{3.63}$$

and

$$\infty > \int_{\mathbb{R}^n \setminus B_r} |\nabla u(x)|^2\, dx = \int_r^{+\infty} |u_0'(\rho)|^2 \rho^{n-1}\, d\rho \geq r^{n-1} \int_r^{+\infty} |u_0'(\rho)|^2\, d\rho. \tag{3.64}$$

Thanks to (3.63) and (3.64) we have that $u_0 \in H^1((r, +\infty))$, with $u_0 = 0$ in $[1, +\infty)$.

Then, from the Morrey Embedding Theorem, it follows that $u_0 \in C^{0,\alpha}((r, +\infty))$ for any $\alpha \in \left[0, \frac{1}{2}\right]$, which leads to the desired result. □

Corollary 3.13. *Let $s \in (0, +\infty)$. There exists a radial, nonnegative and nontrivial solution of (3.56) which belongs to $H^s_0(B_1) \cap C^{0,\alpha}(\mathbb{R}^n \setminus B_{1/2})$, for some $\alpha \in (0,1)$.*

Proof. If $s \in (0,1)$, the desired claim follows from Corollary 8 in [DSV19a].

If instead $s \geq 1$, we obtain the desired result as a consequence of Proposition 3.11 and Lemma 3.12. □

3.7 Boundary asymptotics of the first eigenfunctions of $(-\Delta)^s$

In Lemma 4 of [DSV19a], some precise asymptotics at the boundary for the first Dirichlet eigenfunction of $(-\Delta)^s$ have been established in the range $s \in (0,1)$.

Here, we obtain a related expansion in the range $s > 0$ for the eigenfunction provided in Corollary 3.13. The result that we obtain is the following:

Proposition 3.14. *There exists a nontrivial solution ϕ_* of (3.56) which belongs to $H^s_0(B_1) \cap C^{0,\alpha}(\mathbb{R}^n \setminus B_{1/2})$, for some $\alpha \in (0,1)$, and such that, for every $e \in \partial B_1$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$,*

$$\lim_{\epsilon \searrow 0} \epsilon^{|\beta| - s} \partial^\beta \phi_*(e + \epsilon X) = (-1)^{|\beta|} k_*\, s\,(s-1) \dots (s - |\beta| + 1)\, e_1^{\beta_1} \dots e_n^{\beta_n} (-e \cdot X)_+^{s - |\beta|},$$

in the sense of distribution, with $|\beta| := \beta_1 + \dots + \beta_n$ and $k_* > 0$.

The proof of Proposition 3.14 relies on Proposition 3.6 and some auxiliary computations on the Green function in (3.25). We start with the following result:

Lemma 3.15. *Let $0 < r < 1$, $e \in \partial B_1$, $s > 0$, $f \in C^{0,\alpha}(\mathbb{R}^n \setminus B_r) \cap L^2(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, and $f = 0$ outside B_1. Then the integral*

$$\int_{B_1} f(z) \frac{(1-|z|^2)^s}{s|z-e|^n}\, dz \tag{3.65}$$

is finite.

Proof. We denote by I the integral in (3.65). We let

$$I_1 := \int_{B_1 \setminus B_r} f(z) \frac{(1-|z|^2)^s}{s|z-e|^n}\, dz \qquad \text{and} \qquad I_2 := \int_{B_r} f(z) \frac{(1-|z|^2)^s}{s|z-e|^n}\, dz.$$

Then, we have that

$$I = I_1 + I_2. \tag{3.66}$$

Now, if $z \in B_1 \setminus B_r$, we have that

$$f(z) \le |f(z) - f(e)| \le C|z-e|^\alpha,$$

therefore

$$I_1 \le \int_{B_1 \setminus B_r} \frac{(1-|z|^2)^s}{s|z-e|^{n-\alpha}}\, dz < \infty. \tag{3.67}$$

If instead $z \in B_r$,

$$|z-e| \ge 1 - r > 0,$$

and consequently

$$I_2 \le \frac{1}{s\,(1-r)^n} \int_{B_r} f(z)\, dz < \infty. \tag{3.68}$$

The desired result follows from (3.66), (3.67) and (3.68). □

Next result gives a precise boundary behaviour of the Green function for any $s > 0$ (the case in which $s \in (0,1)$ and $f \in C^{0,\alpha}(\mathbb{R}^n)$ was considered in Lemma 6 of [DSV19a], and in fact the proof presented here also simplifies the one in Lemma 6 of [DSV19a] for the setting considered there).

Lemma 3.16. *Let e, $\omega \in \partial B_1$, $\epsilon_0 > 0$ and $r \in (0,1)$. Assume that*

$$e + \epsilon\omega \in B_1, \tag{3.69}$$

for any $\epsilon \in (0, \epsilon_0]$. Let $f \in C^{0,\alpha}(\mathbb{R}^n \setminus B_r) \cap L^2(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, with $f = 0$ outside B_1.
Then

$$\lim_{\epsilon \searrow 0} \epsilon^{-s} \int_{B_1} f(z) \mathcal{G}_s(e + \epsilon\omega, z)\, dz = k(n,s) \int_{B_1} f(z) \frac{(-2e \cdot \omega)^s (1-|z|^2)^s}{s|z-e|^n}\, dz, \tag{3.70}$$

for a suitable normalizing constant $k(n,s) > 0$.

Proof. In light of (3.69), we have that

$$1 > |e + \epsilon\omega|^2 = 1 + \epsilon^2 + 2\epsilon e \cdot \omega,$$

and therefore

$$-e \cdot \omega > \frac{\epsilon}{2} > 0. \tag{3.71}$$

Moreover, if r_0 is as given in (3.25), we have that, for all $z \in B_1$,

$$r_0(e + \epsilon\omega, z) = \frac{\epsilon(-\epsilon - 2e \cdot \omega)(1 - |z|^2)}{|z - e - \epsilon\omega|^2} \leq \frac{3\epsilon}{|z - e - \epsilon\omega|^2}. \tag{3.72}$$

Also, a Taylor series representation allows us to write, for any $t \in (-1, 1)$,

$$\frac{t^{s-1}}{(t+1)^{\frac{n}{2}}} = \sum_{k=0}^{\infty} \binom{-n/2}{k} t^{k+s-1}. \tag{3.73}$$

We also notice that

$$\begin{aligned} \left| \binom{-n/2}{k} \right| &= \left| \frac{-\frac{n}{2}\left(-\frac{n}{2} - 1\right) \ldots \left(-\frac{n}{2} - k + 1\right)}{k!} \right| = \frac{\frac{n}{2}\left(\frac{n}{2} + 1\right) \ldots \left(\frac{n}{2} + (k-1)\right)}{k!} \\ &\leq \frac{n\,(n+1) \ldots (n + (k-1))}{k!} \leq \frac{(n + (k-1))!}{k!} = (k+1) \ldots (n + (k-1)) \\ &\leq (n + k + 1)^{n+1}. \end{aligned} \tag{3.74}$$

This and the Root Test give that the series in (3.73) is uniformly convergent on compact sets in $(-1, 1)$.

As a consequence, if we set

$$r_1(x, z) := \min\left\{\frac{1}{2}, r_0(x, z)\right\}, \tag{3.75}$$

we can switch integration and summation signs and obtain that

$$\int_0^{r_1(x,z)} \frac{t^{s-1}}{(t+1)^{\frac{n}{2}}}\, dt = \sum_{k=0}^{\infty} c_k (r_1(x, z))^{k+s}, \tag{3.76}$$

where

$$c_k := \frac{1}{k+s} \binom{-n/2}{k}.$$

Once again, the bound in (3.74), together with (3.75), give that the series in (3.76) is convergent.

Now, we omit for simplicity the normalizing constant $k(n, s)$ in the definition of the Green function in (3.25), and we define

$$\mathcal{G}(x, z) := |z - x|^{2s-n} \sum_{k=0}^{\infty} c_k (r_1(x, z))^{k+s} \tag{3.77}$$

and

$$g(x, z) := |z - x|^{2s-n} \int_{r_1(x,z)}^{r_0(x,z)} \frac{t^{s-1}}{(t+1)^{\frac{n}{2}}}\, dt.$$

Using (3.25) and (3.76), and dropping dimensional constants for the sake of shortness, we can write

$$\mathcal{G}_s(x,z) = \mathcal{G}(x,z) + g(x,z). \tag{3.78}$$

Now, we show that

$$g(x,z) \le \begin{cases} C\chi(r,z)\,|z-x|^{2s-n} & \text{if} \quad n>2s, \\ C\chi(r,z)\,\log r_0(x,z) & \text{if} \quad n=2s, \\ C\chi(r,z)\,|z-x|^{2s-n}(r_0(x,z))^{s-\frac{n}{2}} & \text{if} \quad n<2s, \end{cases} \tag{3.79}$$

where $\chi(r,z)=1$ if $r_0(x,z)>\frac{1}{2}$ and $\chi(r,z)=0$ if $r_0(x,z)\le\frac{1}{2}$. To check this, we notice that if $r_0(x,z)\le\frac{1}{2}$ we have that $r_1(x,z)=r_0(x,z)$, due to (3.75), and therefore $g(x,z)=0$.

On the other hand, if $r_0(x,z)>\frac{1}{2}$, we deduce from (3.75) that $r_1(x,z)=\frac{1}{2}$, and consequently

$$g(x,z) \le |z-x|^{2s-n}\int_{1/2}^{r_0(x,z)} t^{s-\frac{n}{2}-1}dt \le \begin{cases} C|z-x|^{2s-n} & \text{if} \quad n>2s, \\ C\log r_0(x,z) & \text{if} \quad n=2s, \\ C|z-x|^{2s-n}(r_0(x,z))^{s-\frac{n}{2}} & \text{if} \quad n<2s, \end{cases}$$

for some constant $C>0$. This completes the proof of (3.79).

Now, we exploit the bound in (3.79) when $x=e+\epsilon\omega$. For this, we notice that if $r_0(e+\epsilon\omega,z)>\frac{1}{2}$, recalling (3.72), we find that

$$|z-e-\epsilon\omega|^2 \le 6\epsilon < 9\epsilon, \tag{3.80}$$

and therefore $z\in B_{3\sqrt{\epsilon}}(e+\epsilon\omega)$.

Hence, using (3.79),

$$\begin{aligned} \left|\int_{B_1} f(z)g(e+\epsilon\omega,z)dz\right| &\le \int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |f(z)||g(e+\epsilon\omega,z)|dz \\ &\le \begin{cases} C\displaystyle\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |f(z)||z-e-\epsilon\omega|^{2s-n}dz & \text{if} \quad n>2s, \\ C\displaystyle\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |f(z)|\log r_0(e+\epsilon\omega,z)dz & \text{if} \quad n=2s, \\ C\displaystyle\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |f(z)||z-e-\epsilon\omega|^{2s-n}(r_0(e+\epsilon\omega,z))^{s-\frac{n}{2}}dz & \text{if} \quad n<2s. \end{cases} \end{aligned} \tag{3.81}$$

Now, if $z\in B_{3\sqrt{\epsilon}}(e+\epsilon\omega)$, then

$$|z-e| \le |z-e-\epsilon\omega| + |\epsilon\omega| \le 3\sqrt{\epsilon}+\epsilon < 4\sqrt{\epsilon}. \tag{3.82}$$

Furthermore, for a given $r\in(0,1)$, we have that $B_{3\sqrt{\epsilon}}(e+\epsilon\omega)\subseteq\mathbb{R}^n\setminus B_r$, provided that ϵ is sufficiently small.

Hence, if $z\in B_{3\sqrt{\epsilon}}(e+\epsilon\omega)$, we can exploit the regularity of f and deduce that

$$|f(z)| = |f(z)-f(e)| \le C|z-e|^{\alpha}.$$

This and (3.82) lead to

$$|f(z)| \le C\epsilon^{\frac{\alpha}{2}}, \tag{3.83}$$

for every $z \in B_{3\sqrt{\epsilon}}(e+\epsilon\omega)$.

Thanks to (3.72), (3.81) and (3.83), we have that

$$\begin{aligned}\left|\int_{B_1} f(z)g(e+\epsilon\omega,z)dz\right| &\le \begin{cases} C\epsilon^{\frac{\alpha}{2}}\displaystyle\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |z-e-\epsilon\omega|^{2s-n}dz & \text{if} \quad n>2s,\\ C\epsilon^{\frac{\alpha}{2}}\displaystyle\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} \log\frac{3\epsilon}{|z-e-\epsilon\omega|^2}dz & \text{if} \quad n=2s,\\ C\epsilon^{\frac{\alpha}{2}+s-\frac{n}{2}}\displaystyle\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} dz & \text{if} \quad n<2s \end{cases}\\ &\le C\epsilon^{\frac{\alpha}{2}+s},\end{aligned}$$

up to renaming C.

This and (3.78) give that

$$\int_{B_1} f(z)\mathcal{G}_s(e+\epsilon\omega,z)dz = \int_{B_1} f(z)\mathcal{G}(e+\epsilon\omega,z)dz + o(\epsilon^s). \tag{3.84}$$

Now, we consider the series in (3.77), and we split the contribution coming from the index $k=0$ from the ones coming from the indices $k>0$, namely we write

$$\begin{aligned} &\mathcal{G}(x,z) = \mathcal{G}_0(x,z) + \mathcal{G}_1(x,z),\\ \text{with} \quad &\mathcal{G}_0(x,z) := \frac{|z-x|^{2s-n}}{s}(r_1(x,z))^s\\ \text{and} \quad &\mathcal{G}_1(x,z) := |z-x|^{2s-n}\sum_{k=1}^{+\infty} c_k\,(r_1(x,z))^{k+s}. \end{aligned} \tag{3.85}$$

Firstly, we consider the contribution given by the term $\mathcal{G}_1$. Thanks to (3.75) and (3.83), we have that

$$\begin{aligned} \left|\int_{B_1\cap B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} f(z)\mathcal{G}_1(e+\epsilon\omega,z)dz\right| &\le \int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |f(z)|\mathcal{G}_1(e+\epsilon\omega,z)dz\\ &\le C\epsilon^{\frac{\alpha}{2}}\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |z-e-\epsilon\omega|^{2s-n}\sum_{k=1}^{+\infty}|c_k|\,(r_1(e+\epsilon\omega,z))^{k+s}dz\\ &\le C\epsilon^{\frac{\alpha}{2}}\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |z-e-\epsilon\omega|^{2s-n}\sum_{k=1}^{+\infty}|c_k|\left(\frac{1}{2}\right)^{k+s}dz\\ &\le C\epsilon^{\frac{\alpha}{2}}\int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |z-e-\epsilon\omega|^{2s-n}dz\\ &\le C\epsilon^{\frac{\alpha}{2}+s}, \end{aligned} \tag{3.86}$$

up to renaming the constant C step by step.

On the other hand, for every $z \in \mathbb{R}^n$,

$$|z| = |e+\epsilon\omega+z-e-\epsilon\omega| \ge |e+\epsilon\omega| - |z-e-\epsilon\omega| \ge 1-\epsilon-|z-e-\epsilon\omega|.$$

Therefore, for every $z \in B_1 \setminus \big(B_r \cup B_{3\sqrt{\epsilon}}(e+\epsilon\omega)\big)$, we can take $e_* := \frac{z}{|z|}$ and obtain that

$$\begin{aligned} |f(z)| = |f(z)-f(e_*)| &\le C|z-e_*|^\alpha = C(1-|z|)^\alpha\\ &\le C(\epsilon+|z-e-\epsilon\omega|)^\alpha \le C|z-e-\epsilon\omega|^\alpha, \end{aligned} \tag{3.87}$$

up to renaming $C > 0$.

Also, using (3.72), we see that, for any $k > 0$,

$$(r_0(e+\epsilon\omega, z))^{s+\frac{\alpha}{4}} \left(\frac{1}{2}\right)^{k-\frac{\alpha}{4}} \leq \frac{C\epsilon^{s+\frac{\alpha}{4}}}{2^k|z-e-\epsilon\omega|^{2s+\frac{\alpha}{2}}}. \tag{3.88}$$

This, (3.75) and (3.87) give that if $z \in B_1 \setminus \big(B_r \cup B_{3\sqrt{\epsilon}}(e+\epsilon\omega)\big)$, then

$$\begin{aligned}
|f(z)\mathcal{G}_1(e+\epsilon\omega, z)| &\leq C|z-e-\epsilon\omega|^{\alpha+2s-n} \sum_{k=1}^{+\infty} |c_k| \, (r_1(e+\epsilon\omega, z))^{k+s} \\
&= C|z-e-\epsilon\omega|^{\alpha+2s-n} \sum_{k=1}^{+\infty} |c_k| \, (r_1(e+\epsilon\omega, z))^{s+\frac{\alpha}{4}} (r_1(e+\epsilon\omega, z))^{k-\frac{\alpha}{4}} \\
&\leq C|z-e-\epsilon\omega|^{\alpha+2s-n} \sum_{k=1}^{+\infty} |c_k| \, (r_0(e+\epsilon\omega, z))^{s+\frac{\alpha}{4}} \left(\frac{1}{2}\right)^{k-\frac{\alpha}{4}} \\
&\leq C\epsilon^{s+\frac{\alpha}{4}} |z-e-\epsilon\omega|^{\frac{\alpha}{2}-n} \sum_{k=1}^{+\infty} \frac{|c_k|}{2^k},
\end{aligned}$$

where the latter series is absolutely convergent thanks to (3.74).

This implies that, if we set $E := B_1 \setminus \big(B_r \cup B_{3\sqrt{\epsilon}}(e+\epsilon\omega)\big)$, it holds that

$$\begin{aligned}
\left|\int_E f(z)\mathcal{G}_1(e+\epsilon\omega, z)\,dz\right| &\leq C\epsilon^{s+\frac{\alpha}{4}} \int_E |z-e-\epsilon\omega|^{\frac{\alpha}{2}-n}dz \\
&\leq C\epsilon^{s+\frac{\alpha}{4}} \int_{B_1} |z-e-\epsilon\omega|^{\frac{\alpha}{2}-n}dz \leq C\epsilon^{s+\frac{\alpha}{4}} \int_{B_3} |z|^{\frac{\alpha}{2}-n}dz \leq C\epsilon^{s+\frac{\alpha}{4}}.
\end{aligned} \tag{3.89}$$

Moreover, if $z \in B_r$, we have that

$$|e+\epsilon\omega - z| \geq 1-\epsilon-r,$$

and therefore, recalling (3.88),

$$\begin{aligned}
\sup_{z\in B_r} |\mathcal{G}_1(e+\epsilon\omega, z)| &\leq |z-e-\epsilon\omega|^{2s-n} \sum_{k=1}^{+\infty} |c_k| \, \big(r_1(e+\epsilon\omega, z)\big)^{s+\frac{\alpha}{4}} \big(r_1(e+\epsilon\omega, z)\big)^{k-\frac{\alpha}{4}} \\
&\leq |z-e-\epsilon\omega|^{2s-n} \sum_{k=1}^{+\infty} |c_k| \, \big(r_0(e+\epsilon\omega, z)\big)^{s+\frac{\alpha}{4}} \left(\frac{1}{2}\right)^{k-\frac{\alpha}{4}} \\
&\leq C\,|z-e-\epsilon\omega|^{-n-\frac{\alpha}{2}} \sum_{k=1}^{+\infty} \frac{|c_k|}{2^k} \\
&\leq C(1-\epsilon-r)^{-n-\frac{\alpha}{2}}\,\epsilon^{s+\frac{\alpha}{4}},
\end{aligned}$$

up to renaming C.

As a consequence, we find that

$$\begin{aligned}
\left|\int_{B_r} f(z)\mathcal{G}_1(e+\epsilon\omega, z)\,dz\right| &\leq \sup_{z\in B_r} |\mathcal{G}_1(e+\epsilon\omega, z)| \; \|f\|_{L^1(B_r)} \\
&\leq \|f\|_{L^1(B_r)} \, (1-\epsilon-r)^{-n-\frac{\alpha}{2}}\,\epsilon^{s+\frac{\alpha}{4}} \\
&\leq \|f\|_{L^1(B_r)} \, 2^{n+\frac{\alpha}{2}} (1-r)^{-n-\frac{\alpha}{2}}\,\epsilon^{s+\frac{\alpha}{4}} \\
&= C\epsilon^{s+\frac{\alpha}{4}},
\end{aligned} \tag{3.90}$$

as long as ϵ is suitably small with respect to r, and C is a positive constant which depends on $\|f\|_{L^1(B_r)}$, r, n and α.

Then, by (3.86), (3.89) and (3.90) we conclude that

$$\int_{B_1} f(z)\mathcal{G}_1(e+\epsilon\omega, z)dz = o(\epsilon^s). \tag{3.91}$$

Inserting this information into (3.84), and recalling (3.85), we obtain

$$\int_{B_1} f(z)\mathcal{G}_s(e+\epsilon\omega, z)dz = \int_{B_1} f(z)\mathcal{G}_0(e+\epsilon\omega, z)dz + o(\epsilon^s). \tag{3.92}$$

Now, we define

$$\mathcal{D}_1 := \{z \in B_1 \quad \text{s.t.} \quad r_0(e+\epsilon\omega, z) > 1/2\}$$

and

$$\mathcal{D}_2 := \{z \in B_1 \quad \text{s.t.} \quad r_0(e+\epsilon\omega, z) \le 1/2\}.$$

If $z \in \mathcal{D}_1$, then $z \in B_1 \setminus B_r$, thanks to (3.80), and hence we can use (3.81) and (3.83) and write

$$|f(z)\mathcal{G}_0(e+\epsilon\omega, z)| \le C\epsilon^{\frac{\alpha}{2}}|z-e-\epsilon\omega|^{2s-n}.$$

Then, recalling again (3.81),

$$\left|\int_{\mathcal{D}_1} f(z)\mathcal{G}_1(e+\epsilon\omega, z)dz\right| \le C\epsilon^{\frac{\alpha}{2}} \int_{B_{3\sqrt{\epsilon}}(e+\epsilon\omega)} |z-e-\epsilon\omega|^{2s-n}dz = C\epsilon^{\frac{\alpha}{2}+s}, \tag{3.93}$$

up to renaming the constant $C > 0$. This information and (3.92) give that

$$\int_{B_1} f(z)\mathcal{G}_s(e+\epsilon\omega, z)dz = \int_{\mathcal{D}_2} f(z)\mathcal{G}_0(e+\epsilon\omega, z)dz + o(\epsilon^s).$$

Now, by (3.72) and (3.75), if $z \in \mathcal{D}_2$,

$$\mathcal{G}_0(e+\epsilon\omega, z) = \frac{|z-e-\epsilon\omega|^{2s-n}}{s}(r_0(e+\epsilon\omega))^s = \frac{\epsilon^s(-\epsilon-2e\cdot\omega)^s(1-|z|^2)^s}{s|z-e-\epsilon\omega|^n}.$$

Hence, we have

$$\begin{aligned}
&\lim_{\epsilon\searrow 0} \epsilon^{-s} \int_{B_1} f(z)\mathcal{G}_s(e+\epsilon\omega, z)dz \\
&= \lim_{\epsilon\searrow 0} \epsilon^{-s} \int_{\mathcal{D}_2} f(z)\mathcal{G}_0(e+\epsilon\omega, z)dz \\
&= \lim_{\epsilon\searrow 0} \int_{\{2\epsilon(-\epsilon-2e\cdot\omega)(1-|z|^2)\le|z-e-\epsilon\omega|^2\}} f(z)\frac{(-\epsilon-2e\cdot\omega)^s(1-|z|^2)^s}{s|z-e-\epsilon\omega|^n}dz.
\end{aligned} \tag{3.94}$$

Now we set

$$F_\epsilon(z) := \begin{cases} f(z)\dfrac{(-\epsilon-2e\cdot\omega)^s(1-|z|^2)^s}{s|z-e-\epsilon\omega|^n} & \text{if} \quad 2\epsilon(-\epsilon-2e\cdot\omega)(1-|z|^2) \le |z-e-\epsilon\omega|^2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.95}$$

and we prove that for any $\eta > 0$ there exists $\delta > 0$ independent of ϵ such that, for any $E \subset \mathbb{R}^n$ with $|E| \le \delta$, we have

$$\int_{B_1 \cap E} |F_\epsilon(z)| dz \le \eta. \tag{3.96}$$

To this aim, given η and E as above, we define

$$\rho := \min \left\{ \epsilon(-\epsilon - 2e \cdot \omega),\ \sqrt{2\epsilon(-\epsilon - 2e \cdot \omega)(1-r)},\ \left(\frac{2^{s+\alpha} s^2\, \epsilon^{s+\alpha}\, (-\epsilon - 2e \cdot \omega)^\alpha \eta}{3^{2s}\, \|f\|_{C^{0,\alpha}(B_1 \setminus B_r)}\, |\partial B_1|} \right)^{\frac{1}{2\alpha}} \right\}. \tag{3.97}$$

We stress that the above definition is well-posed, thanks to (3.71). In addition, using the integrability of f, we take $\delta > 0$ such that if $A \subseteq B_1$ and $|A| \le \delta$ then

$$\int_A |f(x)|\, dx \le \frac{s \rho^n \eta}{2 \cdot 3^s}. \tag{3.98}$$

We set

$$E_1 := E \cap B_\rho(e + \epsilon\omega) \qquad \text{and} \qquad E_2 := E \setminus B_\rho(e + \epsilon\omega). \tag{3.99}$$

From (3.95), we see that

$$|F_\epsilon(z)| \le \frac{|f(z)|\, \chi_\star(z)}{2^s s\, \epsilon^s |z - e - \epsilon\omega|^{n-2s}},$$

where

$$\chi_\star(z) := \begin{cases} 1 & \text{if} \quad 2\epsilon(-\epsilon - 2e \cdot \omega)(1 - |z|^2) \le |z - e - \epsilon\omega|^2, \\ 0 & \text{otherwise}, \end{cases}$$

and therefore

$$\int_{B_1 \cap E_1} |F_\epsilon(z)|\, dz \le \int_{B_1 \cap E_1} \frac{|f(z)|\, \chi_\star(z)}{2^s s\, \epsilon^s |z - e - \epsilon\omega|^{n-2s}}\, dz. \tag{3.100}$$

Now, for every $z \in B_1 \cap E_1 \subseteq B_\rho(e + \epsilon\omega)$ for which $\chi_\star(z) \ne 0$, we have that

$$2\epsilon(-\epsilon - 2e \cdot \omega)(1 - |z|^2) \le |z - e - \epsilon\omega|^2 \le \rho^2,$$

and hence

$$|z| \ge \sqrt{1 - \frac{\rho^2}{2\epsilon(-\epsilon - 2e \cdot \omega)}} \ge 1 - \frac{\rho^2}{2\epsilon(-\epsilon - 2e \cdot \omega)},$$

which in turn gives that $|z| \ge r$, recall (3.97).

From this and (3.100) we deduce that

$$\begin{aligned} \int_{B_1 \cap E_1} |F_\epsilon(z)|\, dz &\le \int_{1 - \frac{\rho^2}{2\epsilon(-\epsilon - 2e\cdot\omega)} \le |z| < 1} \frac{\|f\|_{C^{0,\alpha}(B_1 \setminus B_r)}\, (1 - |z|)^\alpha}{2^s s\, \epsilon^s |z - e - \epsilon\omega|^{n-2s}}\, dz \\ &\le \frac{\|f\|_{C^{0,\alpha}(B_1 \setminus B_r)}}{2^s s\, \epsilon^s} \left(\frac{\rho^2}{2\epsilon(-\epsilon - 2e \cdot \omega)} \right)^\alpha \int_{1 - \frac{\rho^2}{2\epsilon(-\epsilon - 2e\cdot\omega)} \le |z| < 1} \frac{dz}{|z - e - \epsilon\omega|^{n-2s}} \\ &\le \frac{\|f\|_{C^{0,\alpha}(B_1 \setminus B_r)}}{2^s s\, \epsilon^s} \left(\frac{\rho^2}{2\epsilon(-\epsilon - 2e \cdot \omega)} \right)^\alpha \int_{B_3} \frac{dx}{|x|^{n-2s}} \\ &= \frac{3^{2s}\, \|f\|_{C^{0,\alpha}(B_1 \setminus B_r)}\, |\partial B_1|}{2^{s+\alpha+1} s^2\, \epsilon^{s+\alpha}\, (-\epsilon - 2e \cdot \omega)^\alpha}\ \rho^{2\alpha} \\ &\le \frac{\eta}{2}, \end{aligned} \tag{3.101}$$

where (3.97) has been exploited in the last inequality.

We also point out that, by (3.95), (3.98) and (3.99),

$$\begin{aligned}\int_{B_1\cap E_2}|F_\epsilon(z)|\,dz &\le \int_{(B_1\setminus B_\rho(e+\epsilon\omega))\cap E}|f(z)|\,\frac{(-\epsilon-2e\cdot\omega)^s(1-|z|^2)^s}{s|z-e-\epsilon\omega|^n}\,dz\\ &\le \frac{3^s}{s\rho^n}\int_{B_1\cap E}|f(z)|\,dz\\ &\le \frac{\eta}{2}.\end{aligned}$$

This, (3.99) and (3.101) give (3.96), as desired.

Notice also that the sequence $F_\epsilon(z)$ converges pointwise to the function

$$F(z) := f(z)\frac{(-2e\cdot\omega)^s(1-|z|^2)^s}{s|z-e|^n}.$$

Hence (3.94), (3.96) and the Vitali Convergence Theorem allow us to conclude that

$$\begin{aligned}\lim_{\epsilon\searrow 0}\int_{B_1} f(z)\mathcal{G}_s(e+\epsilon\omega,z)dz &= \lim_{\epsilon\searrow 0}\int_{B_1} F_\epsilon(z)dz\\ &= \int_{B_1} f(z)\frac{(-2e\cdot\omega)^s(1-|z|^2)^s}{s|z-e|^n}dz,\end{aligned} \tag{3.102}$$

which establishes the claim of Lemma 3.16 (notice that the finiteness of the latter quantity in (3.102) follows from (3.15)). □

With this preliminary work, we can now establish the boundary behaviour of solutions which is needed in our setting. As a matter of fact, from Lemma 3.16 we immediately deduce that:

Corollary 3.17. *Let $e,\omega\in\partial B_1$, $\epsilon_0>0$ and $r\in(0,1)$.*

Assume that $e+\epsilon\omega\in B_1$, for any $\epsilon\in(0,\epsilon_0]$. Let $f\in C^{0,\alpha}(\mathbb{R}^n\setminus B_r)\cap L^2(\mathbb{R}^n)$ for some $\alpha\in(0,1)$, with $f=0$ outside B_1.

Let u be as in (3.43). Then,

$$\lim_{\epsilon\searrow 0}\epsilon^{-s}u(e+\epsilon\omega) = k(n,s)(-2e\cdot\omega)^s\int_{B_1} f(z)\frac{(1-|z|^2)^s}{s|z-e|^n}dz,$$

where $k(n,s)$ denotes a positive normalizing constant.

Now we apply the previous results to detect the boundary growth of a suitable first eigenfunction. For our purposes, the statement that we need is the following:

Corollary 3.18. *There exists a nontrivial solution ϕ_* of (3.56) which belongs to $H^s_0(B_1)\cap C^{0,\alpha}(\mathbb{R}^n\setminus B_{1/2})$, for some $\alpha\in(0,1)$, and such that, for every $e\in\partial B_1$,*

$$\lim_{\epsilon\searrow 0}\epsilon^{-s}\phi_*(e+\epsilon\omega) = k_*\,(-e\cdot\omega)^s_+, \tag{3.103}$$

for a suitable constant $k_*>0$.

Furthermore, for every $R\in(r,1)$, there exists $C_R>0$ such that

$$\sup_{x\in B_1\setminus B_R} d^{-s}(x)\,|\phi_*(x)| \le C_R. \tag{3.104}$$

Proof. Let $\alpha \in (0,1)$ and $\phi \in H^s_0(B_1) \cap C^{0,\alpha}(\mathbb{R}^n \setminus B_{1/2})$ be the nonnegative and nontrivial solution of (3.56), as given in Corollary 3.13.

In the spirit of (3.43), we define

$$\phi_*(x) := \begin{cases} \lambda_1 \displaystyle\int_{B_1} \mathcal{G}_s\,(x,y)\; \phi(y)\, dy & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases}$$

We stress that we can use Proposition 3.6 in this context, with $f := \lambda_1 \phi$, since condition (3.42) is satisfied in this case.

Then, from (3.44) and (3.46), we know that $\phi_* \in H^s_0(B_1)$ and, from (3.45),

$$(-\Delta)^s \phi_* = \lambda_1\, \phi \text{ in } B_1.$$

In particular, we have that $(-\Delta)^s(\phi - \phi_*) = 0$ in B_1, and $\phi - \phi_* \in H^s_0(B_1)$, which give that $\phi - \phi_*$ vanishes identically. Hence, we can write that $\phi = \phi_*$, and thus ϕ_* is a solution of (3.56).

Now, we check (3.103). For this, we distinguish two cases. If $e \cdot \omega \geq 0$, we have that

$$|e + \epsilon\omega|^2 = 1 + 2\epsilon e \cdot \omega + \epsilon^2 > 1,$$

for all $\epsilon > 0$. Then, in this case $e + \epsilon\omega \in \mathbb{R}^n \setminus B_1$, and therefore $\phi_*(e + \epsilon\omega) = 0$. This gives that, in this case,

$$\lim_{\epsilon \searrow 0} \epsilon^{-s} \phi_*(e + \epsilon\omega) = 0. \tag{3.105}$$

If instead $e \cdot \omega < 0$, we see that

$$|e + \epsilon\omega|^2 = 1 + 2\epsilon e \cdot \omega + \epsilon^2 < 1,$$

for all $\epsilon > 0$ sufficiently small. Hence, we can exploit Corollary 3.17 and find that

$$\lim_{\epsilon \searrow 0} \epsilon^{-s} \phi_*(e + \epsilon\omega) = \lambda_1\, k(n,s)(-2e \cdot \omega)^s \int_{B_1} \phi(z) \frac{(1 - |z|^2)^s}{s|z - e|^n}\, dz, \tag{3.106}$$

with $k(n,s) > 0$. Then, we define

$$k_* := 2^s\, k(n,s) \int_{B_1} \phi(z) \frac{(1 - |z|^2)^s}{s|z - e|^n}\, dz.$$

We observe that k_* is positive by construction, with $k(n,s) > 0$. Also, in light of Lemma 3.15, we know that k_* is finite. Hence, from (3.105) and (3.106) we obtain (3.103), as desired.

It only remains to check (3.104). For this, we use (3.45), and we see that, for every $R \in (r,1)$,

$$\sup_{x \in B_1 \setminus B_R} d^{-s}(x)\, |\phi_*(x)| \leq C_R\, \lambda_1 \left(\|\phi\|_{L^1(B_1)} + \|\phi\|_{L^\infty(B_1 \setminus B_r)} \right),$$

and this gives (3.104) up to renaming C_R. $\square$

Now, we can complete the proof of Proposition 3.14, by arguing as follows.

Proof of Proposition 3.14. Let ψ be a test function in $C_0^\infty(\mathbb{R}^n)$. Let also $R := \frac{r+1}{2} \in (r, 1)$ and

$$g_\epsilon(X) := \epsilon^{-s}\phi_*(e+\epsilon X)\partial^\beta\psi(X).$$

We claim that

$$\sup_{X\in\mathbb{R}^n} |g_\epsilon(X)| \le C, \tag{3.107}$$

for some $C > 0$ independent of ϵ. To prove this, we distinguish three cases. If $e+\epsilon X \in \mathbb{R}^n\setminus B_1$, we have that $\phi_*(e+\epsilon X) = 0$ and thus $g_\epsilon(X) = 0$. If instead $e+\epsilon X \in B_R$, we observe that

$$R > |e+\epsilon X| \ge 1 - \epsilon|X|,$$

and therefore $|X| \ge \frac{1-R}{\epsilon}$. In particular, in this case X falls outside the support of ψ, as long as $\epsilon > 0$ is sufficiently small, and consequently $\partial^\beta\psi(X) = 0$ and $g_\epsilon(X) = 0$.

Hence, to complete the proof of (3.107), we are only left with the case in which $e+\epsilon X \in B_1 \setminus B_R$. In this situation, we make use of (3.104) and we find that

$$\begin{aligned}
|\phi_*(e+\epsilon X)| &\le C\, d^s(e+\epsilon X) = C\,(1-|e+\epsilon X|)^s \\
&\le C\,(1-|e+\epsilon X|)^s(1+|e+\epsilon X|)^s = C\,(1-|e+\epsilon X|^2)^s \\
&= C\,\epsilon^s(-2e\cdot X - \epsilon|X|^2)^s \le C\epsilon^s,
\end{aligned}$$

for some $C > 0$ possibly varying from line to line, and this completes the proof of (3.107).

Now, from (3.107) and the Dominated Convergence Theorem, we obtain that

$$\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^n} \epsilon^{-s}\phi_*(e+\epsilon X)\partial^\beta\psi(X)dX = \int_{\mathbb{R}^n}\lim_{\epsilon\searrow 0}\epsilon^{-s}\phi_*(e+\epsilon X)\partial^\beta\psi(X)dX. \tag{3.108}$$

On the other hand, by Corollary 3.18, used here with $\omega := \frac{X}{|X|}$, we know that

$$\begin{aligned}
\lim_{\epsilon\searrow 0}\epsilon^{-s}\phi_*(e+\epsilon X) &= \lim_{\epsilon\searrow 0}\epsilon^{-s}\phi_*(e+\epsilon|X|\omega) = |X|^s\lim_{\epsilon\searrow 0}\epsilon^{-s}\phi_*(e+\epsilon\omega) \\
&= k_*\,|X|^s\,(-e\cdot\omega)_+^s = k_*\,(-e\cdot X)_+^s.
\end{aligned}$$

Substituting this into (3.108), we thus find that

$$\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^n}\epsilon^{-s}\phi_*(e+\epsilon X)\partial^\beta\psi(X)dX = k_*\int_{\mathbb{R}^n}(-e\cdot X)_+^s\partial^\beta\psi(X)dX.$$

As a consequence, integrating by parts twice,

$$\begin{aligned}
\lim_{\epsilon\searrow 0}\epsilon^{|\beta|-s}&\int_{\mathbb{R}^n}\partial^\beta\phi_*(e+\epsilon X)\psi(X)dX = \lim_{\epsilon\searrow 0}\int_{\mathbb{R}^n}\partial^\beta\Big(\epsilon^{-s}\phi_*(e+\epsilon X)\Big)\psi(X)dX \\
&= (-1)^{|\beta|}\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^n}\epsilon^{-s}\phi_*(e+\epsilon X)\partial^\beta\psi(X)dX \\
&= (-1)^{|\beta|}\,k_*\int_{\mathbb{R}^n}(-e\cdot X)_+^s\partial^\beta\psi(X)dX \\
&= k_*\int_{\mathbb{R}^n}\partial^\beta(-e\cdot X)_+^s\psi(X)dX \\
&= (-1)^{|\beta|}\,k_*\,s(s-1)\dots(s-|\beta|+1)e_1^{\beta_1}\dots e_n^{\beta_n}\int_{\mathbb{R}^n}(-e\cdot X)_+^{s-|\beta|}\psi(X)dX.
\end{aligned}$$

Since the test function ψ is arbitrary, the claim in Proposition 3.14 is proved. $\square$

3.8 Boundary behaviour of s-harmonic functions

In this section we analyze the asymptotic behaviour of s-harmonic functions, with a "spherical bump function" as exterior Dirichlet datum.

The result needed for our purpose is the following:

Lemma 3.19. *Let $s > 0$. Let $m \in \mathbb{N}_0$ and $\sigma \in (0,1)$ such that $s = m + \sigma$.*

Then, there exists

$$\psi \in H^s(\mathbb{R}^n) \cap C_0^s(\mathbb{R}^n) \text{ such that } (-\Delta)^s \psi = 0 \text{ in } B_1, \tag{3.109}$$

and, for every $x \in \partial B_{1-\epsilon}$,

$$\psi(x) = k\,\epsilon^s + o(\epsilon^s), \tag{3.110}$$

as $\epsilon \searrow 0$, for some $k > 0$.

Proof. Let $\overline{\psi} \in C^\infty(\mathbb{R}, [0,1])$ such that $\overline{\psi} = 0$ in $\mathbb{R} \setminus (2,3)$ and $\overline{\psi} > 0$ in $(2,3)$. Let $\psi_0(x) := (-1)^m \overline{\psi}(|x|)$. We recall the Poisson kernel

$$\Gamma_s(x,y) := (-1)^m \frac{\gamma_{n,\sigma}}{|x-y|^n} \frac{(1-|x|^2)_+^s}{(|y|^2-1)^s},$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n \setminus \overline{B_1}$, and a suitable normalization constant $\gamma_{n,\sigma} > 0$ (see formulas (1.10) and (1.30) in [AJS18c]). We define

$$\psi(x) := \int_{\mathbb{R}^n \setminus B_1} \Gamma_s(x,y)\,\psi_0(y)\,dy + \psi_0(x).$$

Notice that $\psi_0 = 0$ in $B_{3/2}$ and therefore we can exploit Theorem in [AJS18c] and obtain that (3.109) is satisfied (notice also that $\psi = \psi_0$ outside B_1, hence ψ is compactly supported).

Furthermore, to prove (3.110) we borrow some ideas from Lemma 2.2 in [DSV17] and we see that, for any $x \in \partial B_{1-\epsilon}$,

$$\begin{aligned}
\psi(x) &= c(-1)^m \int_{\mathbb{R}^n \setminus B_1} \frac{\psi_0(y)(1-|x|^2)^s}{(|y|^2-1)^s |x-y|^n} dy + \psi_0(x) \\
&= c(-1)^m \int_{\mathbb{R}^n \setminus B_1} \frac{\psi_0(y)(1-|x|^2)^s}{(|y|^2-1)^s |x-y|^n} dy \\
&= c\,(1-|x|^2)^s \int_2^3 \left[\int_{\mathbb{S}^{n-1}} \frac{\rho^{n-1}\overline{\psi}(\rho)}{(\rho^2-1)^s |x-\rho\omega|^n} d\omega \right] d\rho \\
&= c\,(2\epsilon-\epsilon^2)^s \int_2^3 \left[\int_{\mathbb{S}^{n-1}} \frac{\rho^{n-1}\overline{\psi}(\rho)}{(\rho^2-1)^s |(1-\epsilon)e_1-\rho\omega|^n} d\omega \right] d\rho \\
&= 2^s c\,\epsilon^s \int_2^3 \left[\int_{\mathbb{S}^{n-1}} \frac{\rho^{n-1}\overline{\psi}(\rho)}{(\rho^2-1)^s |e_1-\rho\omega|^n} d\omega \right] d\rho + o(\epsilon^s) \\
&= c\epsilon^s + o(\epsilon^s),
\end{aligned}$$

where $c > 0$ is a constant possibly varying from line to line, and this establishes (3.110). $\square$

Remark 3.20. As in Proposition 3.14, one can extend (3.110) to higher derivatives (in the distributional sense), obtaining, for any $e \in \partial B_1$ and $\beta \in \mathbb{N}^n$

$$\lim_{\epsilon \searrow 0} \epsilon^{|\beta|-s} \partial^\beta \psi(e+\epsilon X) = k_\beta \, e_1^{\beta_1} \dots e_n^{\beta_n} (-e \cdot X)_+^{s-|\beta|},$$

for some $\kappa_\beta \neq 0$.

Using Lemma 3.19, in the spirit of [DSV17], we can construct a sequence of s-harmonic functions approaching $(x\cdot e)_+^s$ for a fixed unit vector e, by using a blow-up argument. Namely, we prove the following:

Corollary 3.21. *Let $e \in \partial B_1$. There exists a sequence $v_{e,j} \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that $(-\Delta)^s v_{e,j} = 0$ in $B_1(e)$, $v_{e,j} = 0$ in $\mathbb{R}^n \setminus B_{4j}(e)$, and*

$$v_{e,j} \to \kappa (x\cdot e)_+^s \quad \text{in} \quad L^1(B_1(e)),$$

as $j \to +\infty$, for some $\kappa > 0$.

Proof. Let ψ be as in Lemma 3.19 and define

$$v_{e,j}(x) := j^s \psi\left(\frac{x}{j} - e\right).$$

The s-harmonicity and the property of being compactly supported follow by the ones of ψ. We now prove the convergence. To this aim, given $x \in B_1(e)$, we write $p_j := \frac{x}{j} - e$ and $\epsilon_j := 1 - |p_j|$. Recall that since $x \in B_1(e)$, then $|x-e|^2 < 1$, which implies that $|x|^2 < 2x\cdot e$ and $x \cdot e > 0$ for any $x \in B_1(e)$.
As a consequence

$$|p_j|^2 = \left|\frac{x}{j} - e\right|^2 = \frac{|x|^2}{j^2} + 1 - 2\frac{x}{j}\cdot e = 1 - \frac{2}{j}(x\cdot e)_+ + o\left(\frac{1}{j}\right)(x\cdot e)_+^2,$$

and so

$$\epsilon_j = \frac{(1+o(1))}{j}(x\cdot e)_+.$$

Therefore, using (3.110),

$$\begin{aligned} v_{e,j}(x) &= j^s \psi(p_j) \\ &= j^s \kappa(\epsilon_j^s + o(\epsilon_j^s)) \\ &= j^s \left(\frac{\kappa}{j^s}(x\cdot e)_+^s + o\left(\frac{1}{j^s}\right)\right) \\ &= \kappa (x\cdot e)_+^s + o(1). \end{aligned}$$

Integrating over $B_1(e)$, we obtain the desired L^1-convergence. □

Now, we show that, as in the case $s \in (0,1)$ proved in Theorem 3.1 of [DSV17], we can find an s-harmonic function with an arbitrarily large number of derivatives prescribed at some point.

Proposition 3.22. *For any $\beta \in \mathbb{N}^n$, there exist $p \in \mathbb{R}^n$, $R > r > 0$, and $v \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } \; B_r(p), \\ v = 0 & \text{in } \; \mathbb{R}^n \setminus B_R(p), \end{cases} \tag{3.111}$$

$$D^\alpha v(p) = 0 \quad \text{for any} \;\; \alpha \in \mathbb{N}^n \quad \text{with} \;\; |\alpha| \le |\beta| - 1,$$

$$D^\alpha v(p) = 0 \quad \text{for any} \;\; \alpha \in \mathbb{N}^n \quad \text{with} \;\; |\alpha| = |\beta| \quad \text{and} \;\; \alpha \neq \beta$$

and

$$D^\beta v(p) = 1.$$

Proof. Let $\mathcal{Z}$ be the set of all pairs $(v, x) \in (H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)) \times B_r(p)$ that satisfy (3.111) for some $R > r > 0$ and $p \in \mathbb{R}^n$.

To each pair $(v, x) \in \mathcal{Z}$ we associate the vector $(D^\alpha v(x))_{|\alpha| \le |\beta|} \in \mathbb{R}^{K'}$, for some $K' = K'_{|\beta|}$ and consider $\mathcal{V}$ to be the vector space spanned by this construction, namely we set

$$\mathcal{V} := \Big\{ (D^\alpha v(x))_{|\alpha| \le |\beta|}, \quad \text{with} \;\; (v, x) \in \mathcal{Z} \Big\}.$$

We claim that

$$\mathcal{V} = \mathbb{R}^{K'}. \tag{3.112}$$

To check this, we suppose by contradiction that $\mathcal{V}$ lies in a proper subspace of $\mathbb{R}^{K'}$. Then, $\mathcal{V}$ must lie in a hyperplane, hence there exists

$$c = (c_\alpha)_{|\alpha| \le |\beta|} \in \mathbb{R}^{K'} \setminus \{0\} \tag{3.113}$$

which is orthogonal to any vector $(D^\alpha v(x))_{|\alpha| \le |\beta|}$ with $(v, x) \in \mathcal{Z}$, that is

$$\sum_{|\alpha| \le |\beta|} c_\alpha D^\alpha v(x) = 0. \tag{3.114}$$

We notice that the pair $(v_{e,j}, x)$, with v_j as in Corollary 3.21, $e \in \partial B_1$ and $x \in B_1(e)$, belongs to $\mathcal{Z}$. Consequently, fixed $\xi \in \mathbb{R}^n \setminus B_{1/2}$ and set $e := \frac{\xi}{|\xi|}$, we have that (3.114) holds true when $v := v_{e,j}$ and $x \in B_1(e)$, namely

$$\sum_{|\alpha| \le |\beta|} c_\alpha D^\alpha v(x) = 0.$$

Let now $\varphi \in C_0^\infty(B_1(e))$. Integrating by parts, by Corollary 3.21 and the Dominated Convergence Theorem, we have that

$$\begin{aligned} 0 &= \lim_{j \to +\infty} \int_{\mathbb{R}^n} \sum_{|\alpha| \le |\beta|} c_\alpha D^\alpha v_{e,j}(x) \varphi(x)\, dx = \lim_{j \to +\infty} \int_{\mathbb{R}^n} \sum_{|\alpha| \le |\beta|} (-1)^{|\alpha|} c_\alpha v_{e,j}(x) D^\alpha \varphi(x)\, dx \\ &= \kappa \int_{\mathbb{R}^n} \sum_{|\alpha| \le |\beta|} (-1)^{|\alpha|} c_\alpha (x \cdot e)_+^s D^\alpha \varphi(x)\, dx = \kappa \int_{\mathbb{R}^n} \sum_{|\alpha| \le |\beta|} c_\alpha D^\alpha (x \cdot e)_+^s \varphi(x)\, dx. \end{aligned}$$

This gives that, for every $x \in B_1(e)$,

$$\sum_{|\alpha| \le |\beta|} c_\alpha D^\alpha (x \cdot e)_+^s = 0.$$

Moreover, for every $x \in B_1(e)$,

$$D^\alpha (x \cdot e)_+^s = s(s-1)\dots(s-|\alpha|+1)(x\cdot e)_+^{s-|\alpha|} e_1^{\alpha_1}\dots e_n^{\alpha_n}.$$

In particular, for $x = \frac{e}{|\xi|} \in B_1(e)$,

$$D^\alpha (x \cdot e)_+^s \big|_{|x=e/|\xi|} = s(s-1)\dots(s-|\alpha|+1)|\xi|^{-s}\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}.$$

And, using the usual multi-index notation, we write

$$\sum_{|\alpha|\le|\beta|} c_\alpha s(s-1)\dots(s-|\alpha|+1)\xi^\alpha = 0, \tag{3.115}$$

for any $\xi \in \mathbb{R}^n \setminus B_{1/2}$. The identity (3.115) describes a polynomial in ξ which vanishes for any ξ in an open subset of $\mathbb{R}^n$. As a result, the Identity Principle for polynomials leads to

$$c_\alpha s(s-1)\dots(s-|\alpha|+1) = 0,$$

for all $|\alpha| \le |\beta|$.

Consequently, since $s \in \mathbb{R}\setminus\mathbb{N}$, the product $s(s-1)\dots(s-|\alpha|+1)$ never vanishes, and so the coefficients c_α are forced to be null for any $|\alpha| \le |\beta|$. This is in contradiction with (3.113), and therefore the proof of (3.112) is complete.

From this, the desired claim in Proposition 3.22 plainly follows. □

3.9 Proof of the main result

This section is devoted to the proof of the main result in Theorem 3.1. This will be accomplished by an auxiliary result of purely nonlocal type which will allow us to prescribe an arbitrarily large number of derivatives at a point for the solution of a fractional equation.

3.10 A result which implies Theorem 3.1

We will use the notation

$$\Lambda_{-\infty} := \Lambda_{(-\infty,\dots,-\infty)}, \tag{3.116}$$

that is we exploit (3.8) with $a_1 := \dots := a_l := -\infty$. This section presents the following statement:

Theorem 3.23. *Suppose that*

$$\text{either there exists } i \in \{1,\dots,M\} \text{ such that } \mathfrak{b}_i \neq 0 \text{ and } s_i \notin \mathbb{N},$$
$$\text{or there exists } i \in \{1,\dots,l\} \text{ such that } \mathfrak{c}_i \neq 0 \text{ and } \alpha_i \notin \mathbb{N}.$$

Let $\ell \in \mathbb{N}$, $f : \mathbb{R}^N \to \mathbb{R}$, with $f \in C^\ell\left(\overline{B_1^N}\right)$. Fixed $\epsilon > 0$, there exist

$$\begin{aligned} & u = u_\epsilon \in C^\infty\left(B_1^N\right) \cap C\left(\mathbb{R}^N\right), \\ & a = (a_1,\dots,a_l) = (a_{1,\epsilon},\dots,a_{l,\epsilon}) \in (-\infty,0)^l, \\ \text{and} \quad & R = R_\epsilon > 1 \end{aligned}$$

such that:

- *for every* $h \in \{1, \dots, l\}$ *and* $(x, y, t_1, \dots, t_{h-1}, t_{h+1}, \dots, t_l)$

$$\text{the map } \mathbb{R} \ni t_h \mapsto u(x, y, t) \text{ belongs to } C^{k_h, \alpha_h}_{-\infty}, \tag{3.117}$$

in the notation of formula (1.4) of [CDV18],

- *it holds that*

$$\begin{cases} \Lambda_{-\infty} u = 0 & \text{in } B_1^{N-l} \times (-1, +\infty)^l, \\ u(x, y, t) = 0 & \text{if } |(x, y)| \geq R, \end{cases} \tag{3.118}$$

$$\partial^{k_h}_{t_h} u(x, y, t) = 0 \qquad \text{if } t_h \in (-\infty, a_h), \qquad \text{for all } h \in \{1, \dots, l\}, \tag{3.119}$$

and

$$\|u - f\|_{C^\ell(B_1^N)} < \epsilon. \tag{3.120}$$

The proof of Theorem 3.23 will basically occupy the rest of this work, and this will lead us to the completion of the proof of Theorem 3.1. Indeed, we have that:

Lemma 3.24. *If the statement of Theorem 3.23 holds true, then the statement in Theorem 3.1 holds true.*

Proof. Assume that the claims in Theorem 3.23 are satisfied. Then, by (3.117) and (3.119), we are in the position of exploting Lemma A.1 in [CDV18] and conclude that, in $B_1^{N-l} \times (-1, +\infty)^l$,

$$D^{\alpha_h}_{t_h, a_h} u = D^{\alpha_h}_{t_h, -\infty} u,$$

for every $h \in \{1, \dots, l\}$. This and (3.118) give that

$$\Lambda_a u = \Lambda_{-\infty} u = 0 \qquad \text{in } B_1^{N-l} \times (-1, +\infty)^l. \tag{3.121}$$

We also define

$$\underline{a} := \min_{h \in \{1, \dots, l\}} a_h$$

and take $\tau \in C_0^\infty([-\underline{a} - 2, 3])$ with $\tau = 1$ in $[-\underline{a} - 1, 1]$. Let

$$U(x, y, t) := u(x, y, t)\, \tau(t_1) \dots \tau(t_l). \tag{3.122}$$

Our goal is to prove that U satisfies the theses of Theorem 3.1. To this end, we observe that $u = U$ in B_1^N, therefore (3.12) for U plainly follows from (3.120).

In addition, from (3.6), we see that $D^{\alpha_h}_{t_h, a_h}$ at a point $t_h \in (-1, 1)$ only depends on the values of the function between a_h and 1. Since the cutoffs in (3.122) do not alter these values, we see that $D^{\alpha_h}_{t_h, a_h} U = D^{\alpha_h}_{t_h, a_h} u$ in B_1^N, and accordingly $\Lambda_a U = \Lambda_a u$ in B_1^N. This and (3.121) say that

$$\Lambda_a U = 0 \qquad \text{in } B_1^N. \tag{3.123}$$

Also, since u in Theorem 3.23 is compactly supported in the variable (x, y), we see from (3.122) that U is compactly supported in the variables (x, y, t). This and (3.123) give that (3.11) is satisfied by U (up to renaming R). ∎

3.11 A pivotal span result towards the proof of Theorem 3.23

In what follows, we let $\Lambda_{-\infty}$ be as in (3.116), we recall the setting in (3.1), and we use the following multi-indices notations:

$$\begin{gathered}\iota=(i,I,\mathfrak{I})=(i_1,\dots,i_n,I_1,\dots,I_M,\mathfrak{I}_1,\dots,\mathfrak{I}_l)\in\mathbb{N}^N\\ \text{and } \partial^\iota w=\partial^{i_1}_{x_1}\dots\partial^{i_n}_{x_n}\partial^{I_1}_{y_1}\dots\partial^{I_M}_{y_M}\partial^{\mathfrak{I}_1}_{t_1}\dots\partial^{\mathfrak{I}_l}_{t_l}w.\end{gathered}\tag{3.124}$$

Inspired by Lemma 5 of [DSV19a], we consider the span of the derivatives of functions in $\ker\Lambda_{-\infty}$, with derivatives up to a fixed order $K\in\mathbb{N}$. We want to prove that the derivatives of such functions span a maximal vectorial space.

For this, we denote by $\partial^K w(0)$ the vector with entries given, in some prescribed order, by $\partial^\iota w(0)$ with $|\iota|\le K$.

We notice that

$$\partial^K w(0)\in\mathbb{R}^{K'} \text{ for some } K'\in\mathbb{N},\tag{3.125}$$

with K' depending on K.

Now, we adopt the notation in formula (1.4) of [CDV18], and we denote by $\mathcal{A}$ the set of all functions $w=w(x,y,t)$ such that for all $h\in\{1,\dots,l\}$ and all $(x,y,t_1,\dots,t_{h-1},t_{h+1},\dots,t_l)\in\mathbb{R}^{N-1}$, the map $\mathbb{R}\ni t_h\mapsto w(x,y,t)$ belongs to $C^\infty((a_h,+\infty))\cap C^{k_h,\alpha_h}_{-\infty}$, and (3.119) holds true for some $a_h\in(-2,0)$.

We also set

$$\mathcal{H}:=\Big\{w\in C(\mathbb{R}^N)\cap C_0(\mathbb{R}^{N-l})\cap C^\infty(\mathcal{N})\cap\mathcal{A}, \text{ for some neighborhood } \mathcal{N} \text{ of the origin,}\\ \text{such that } \Lambda_{-\infty}w=0 \text{ in } \mathcal{N}\Big\}$$

and, for any $w\in\mathcal{H}$, let $\mathcal{V}_K$ be the vector space spanned by the vector $\partial^K w(0)$.

By (3.125), we know that $\mathcal{V}_K\subseteq\mathbb{R}^{K'}$. In fact, we show that equality holds in this inclusion, as stated in the following[4] result:

Lemma 3.25. *It holds that $\mathcal{V}_K=\mathbb{R}^{K'}$.*

The proof of Lemma 3.25 is by contradiction. Namely, if $\mathcal{V}_K$ does not exhaust the whole of $\mathbb{R}^{K'}$ there exists

$$\theta\in\partial B_1^{K'}\tag{3.126}$$

such that

$$\mathcal{V}_K\subseteq\Big\{\zeta\in\mathbb{R}^{K'} \text{ s.t. } \theta\cdot\zeta=0\Big\}.\tag{3.127}$$

In coordinates, recalling (3.124), we write θ as $\theta_\iota=\theta_{i,I,\mathfrak{I}}$, with $i\in\mathbb{N}^{p_1+\dots+p_n}$, $I\in\mathbb{N}^{m_1+\dots+m_M}$ and $\mathfrak{I}\in\mathbb{N}^l$. We consider

$$\begin{gathered}\text{a multi-index } \overline{I}\in\mathbb{N}^{m_1+\dots+m_M} \text{ such that it maximizes } |I|\\ \text{among all the multi-indexes } (i,I,\mathfrak{I}) \text{ for which } |i|+|I|+|\mathfrak{I}|\le K\\ \text{and } \theta_{i,I,\mathfrak{I}}\neq 0 \text{ for some } (i,\mathfrak{I}).\end{gathered}\tag{3.128}$$

[4]Notice that results analogous to Lemma 3.25 cannot hold for solutions of local operators: for instance, pure second derivatives of harmonic functions have to satisfy a linear equation, so they are forced to lie in a proper subspace. In this sense, results such as Lemma 3.25 here reveal a truly nonlocal phenomenon.

Some comments on the setting in (3.128). We stress that, by (3.126), the set $\mathcal{S}$ of indexes I for which there exist indexes $(i, \mathfrak{I})$ such that $|i| + |I| + |\mathfrak{I}| \le K$ and $\theta_{i,I,\mathfrak{I}} \neq 0$ is not empty. Therefore, since $\mathcal{S}$ is a finite set, we can take

$$S := \sup_{I \in \mathcal{S}} |I| = \max_{I \in \mathcal{S}} |I| \in \mathbb{N} \cap [0, K].$$

Hence, we consider a multi-index $\overline{I}$ for which $|\overline{I}| = S$ to obtain the setting in (3.128). By construction, we have that

- $|i| + |\overline{I}| + |\mathfrak{I}| \le K$,
- if $|I| > |\overline{I}|$, then $\theta_{i,I,\mathfrak{I}} = 0$,
- and there exist multi-indexes i and $\mathfrak{I}$ such that $\theta_{i,\overline{I},\mathfrak{I}} \neq 0$.

As a variation of the setting in (3.128), we can also consider

$$\begin{aligned}&\text{a multi-index } \overline{\mathfrak{I}} \in \mathbb{N}^l \text{ such that it maximizes } |\mathfrak{I}| \\ &\text{among all the multi-indexes } (i, I, \mathfrak{I}) \text{ for which } |i| + |I| + |\mathfrak{I}| \le K \\ &\text{and } \theta_{i,I,\mathfrak{I}} \neq 0 \text{ for some } (i, I).\end{aligned} \tag{3.129}$$

In the setting of (3.128) and (3.129), we claim that there exists an open set of $\mathbb{R}^{p_1+\dots+p_n} \times \mathbb{R}^{m_1+\dots+m_M} \times \mathbb{R}^l$ such that for every $(\underline{x}, \underline{y}, \underline{t})$ in such open set we have that

$$\begin{aligned}\text{either} \qquad 0 &= \sum_{\substack{|i|+|I|+|\mathfrak{I}| \le K \\ |I| = |\overline{I}|}} C_{i,I,\mathfrak{I}}\, \theta_{i,I,\mathfrak{I}}\, \underline{x}^i \underline{y}^I \underline{t}^{\mathfrak{I}}, \qquad \text{with} \qquad C_{i,I,\mathfrak{I}} \neq 0, \\ \text{or} \qquad 0 &= \sum_{\substack{|i|+|I|+|\mathfrak{I}| \le K \\ |\mathfrak{I}| = |\overline{\mathfrak{I}}|}} C_{i,I,\mathfrak{I}}\, \theta_{i,I,\mathfrak{I}}\, \underline{x}^i \underline{y}^I \underline{t}^{\mathfrak{I}}, \qquad \text{with} \qquad C_{i,I,\mathfrak{I}} \neq 0.\end{aligned} \tag{3.130}$$

In our framework, the claim in (3.130) will be pivotal towards the completion of the proof of Lemma 3.25. Indeed, let us suppose for the moment that (3.130) is established and let us complete the proof of Lemma 3.25 by arguing as follows.

Formula (3.130) says that $\theta \cdot \partial^K w(0)$ is a polynomial which vanishes for any triple $(\underline{x}, \underline{y}, \underline{t})$ in an open subset of $\mathbb{R}^{p_1+\dots+p_n} \times \mathbb{R}^{m_1+\dots+m_M} \times \mathbb{R}^l$. Hence, using the identity principle of polynomials, we have that each $C_{i,I,\mathfrak{I}}\, \theta_{i,I,\mathfrak{I}}$ is equal to zero whenever $|i| + |I| + |\mathfrak{I}| \le K$ and either $|I| = |\overline{I}|$ (if the first identity in (3.130) holds true) or $|\mathfrak{I}| = |\overline{\mathfrak{I}}|$ (if the second identity in (3.130) holds true). Then, since $C_{i,I,\mathfrak{I}} \neq 0$, we conclude that each $\theta_{i,I,\mathfrak{I}}$ is zero as long as either $|I| = |\overline{I}|$ (in the first case) or $|\mathfrak{I}| = |\overline{\mathfrak{I}}|$ (in the second case), but this contradicts either the definition of $\overline{I}$ in (3.128) (in the first case) or the definition of $\overline{\mathfrak{I}}$ in (3.129) (in the second case). This would therefore complete the proof of Lemma 3.25.

In view of the discussion above, it remains to prove (3.130). To this end, we distinguish the following four cases:

1. there exist $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, M\}$ such that $\underline{a}_i \neq 0$ and $\underline{b}_j \neq 0$,
2. there exist $i \in \{1, \dots, n\}$ and $h \in \{1, \dots, l\}$ such that $\underline{a}_i \neq 0$ and $\underline{c}_h \neq 0$,
3. we have that $\underline{a}_1 = \dots = \underline{a}_n = 0$, and there exists $j \in \{1, \dots, M\}$ such that $\underline{b}_j \neq 0$,

4. we have that $a_1 = \cdots = a_n = 0$, and there exists $h \in \{1, \dots, l\}$ such that $c_h \neq 0$.

Notice that cases 1 and 3 deal with the case in which space-fractional diffusion is present (and in case 1 one also has classical derivatives, while in case 3 the classical derivatives are absent).

Similarly, cases 2 and 4 deal with the case in which time-fractional diffusion is present (and in case 2 one also has classical derivatives, while in case 4 the classical derivatives are absent).

Of course, the case in which both space- and time-fractional diffusion occur is already comprised by the previous cases (namely, it is comprised in both cases 1 and 2 if classical derivatives are also present, and in both cases 3 and 4 if classical derivatives are absent).

Proof of (3.130), *case 1.* For any $j \in \{1, \dots, M\}$ we denote by $\tilde{\phi}_{\star,j}$ the first eigenfunction for $(-\Delta)^{s_j}_{y_j}$ vanishing outside $B_1^{m_j}$ given in Corollary 3.13. We normalize it such that $\|\tilde{\phi}_{\star,j}\|_{L^2(\mathbb{R}^{m_j})} = 1$, and we write $\lambda_{\star,j} \in (0, +\infty)$ to indicate the corresponding first eigenvalue (which now depends on s_j), namely we write

$$\begin{cases} (-\Delta)^{s_j}_{y_j} \tilde{\phi}_{\star,j} = \lambda_{\star,j} \tilde{\phi}_{\star,j} & \text{in } B_1^{m_j}, \\ \tilde{\phi}_{\star,j} = 0 & \text{in } \mathbb{R}^{m_j} \setminus \overline{B_1^{m_j}}. \end{cases} \tag{3.131}$$

Up to reordering the variables and/or taking the operators to the other side of the equation, given the assumptions of case 1, we can suppose that

$$a_1 \neq 0 \tag{3.132}$$

and

$$b_M > 0. \tag{3.133}$$

In view of (3.132), we can define

$$R := \left(\frac{1}{|a_1|} \left(\sum_{j=1}^{M-1} |b_j| \lambda_{\star,j} + \sum_{h=1}^{l} |c_h| \right) \right)^{1/|r_1|}. \tag{3.134}$$

Now, we fix two sets of free parameters

$$\underline{x}_1 \in (R+1, R+2)^{p_1}, \dots, \underline{x}_n \in (R+1, R+2)^{p_n}, \tag{3.135}$$

and

$$\underline{t}_{\star,1} \in \left(\frac{1}{2}, 1\right), \dots, \underline{t}_{\star,l} \in \left(\frac{1}{2}, 1\right). \tag{3.136}$$

We also set

$$\lambda_j := \lambda_{\star,j} \text{ for } j \in \{1, \dots, M-1\}, \tag{3.137}$$

where $\lambda_{\star,j}$ is defined as in (3.131), and

$$\lambda_M := \frac{1}{b_M} \left(\sum_{j=1}^{n} |a_j| \, \underline{x}_j^{r_j} - \sum_{j=1}^{M-1} b_j \lambda_j - \sum_{h=1}^{l} c_h \underline{t}_{\star,h} \right). \tag{3.138}$$

Notice that this definition is well-posed, thanks to (3.133). In addition, from (3.135), we can write $\underline{x}_j = (\underline{x}_{j1}, \dots, \underline{x}_{jp_j})$, and we know that $\underline{x}_{j\ell} > R+1$ for any $j \in \{1, \dots, n\}$ and any $\ell \in \{1, \dots, p_j\}$. Therefore,

$$\underline{x}_j^{r_j} = \underline{x}_{j1}^{r_{j1}} \dots \underline{x}_{jp_j}^{r_{jp_j}} \geq 0. \tag{3.139}$$

From this, (3.134) and (3.136), we deduce that

$$\begin{aligned}\sum_{j=1}^{n} |\underline{a}_j| \, \underline{x}_j^{r_j} &\geq |\underline{a}_1| \, \underline{x}_1^{r_1} \geq |\underline{a}_1| \, (R+1)^{|r_1|} > |\underline{a}_1| \, R^{|r_1|} \\ &= \sum_{j=1}^{M-1} |\underline{b}_j| \lambda_j + \sum_{h=1}^{l} |\underline{c}_h| \geq \sum_{j=1}^{M-1} \underline{b}_j \lambda_j + \sum_{h=1}^{l} \underline{c}_h \underline{t}_{\star,h},\end{aligned}$$

and consequently, by (3.138),

$$\lambda_M > 0. \tag{3.140}$$

We also set

$$\omega_j := \begin{cases} 1 & \text{if } j = 1, \dots, M-1, \\ \dfrac{\lambda_{\star,M}^{1/2s_M}}{\lambda_M^{1/2s_M}} & \text{if } j = M. \end{cases} \tag{3.141}$$

Notice that this definition is well-posed, thanks to (3.140). In addition, by (3.131), we have that, for any $j \in \{1, \dots, M\}$, the functions

$$\phi_j(y_j) := \tilde{\phi}_{\star,j}\left(\frac{y_j}{\omega_j}\right) \tag{3.142}$$

are eigenfunctions of $(-\Delta)_{y_j}^{s_j}$ in $B_{\omega_j}^{m_j}$ with external homogenous Dirichlet boundary condition, and eigenvalues λ_j: namely, we can rewrite (3.131) as

$$\begin{cases} (-\Delta)_{y_j}^{s_j} \phi_j = \lambda_j \phi_j & \text{in } B_{\omega_j}^{m_j}, \\ \phi_j = 0 & \text{in } \mathbb{R}^{m_j} \setminus \overline{B_{\omega_j}^{m_j}}. \end{cases} \tag{3.143}$$

Now, we define

$$\psi_{\star,h}(t_h) := E_{\alpha_h,1}(t_h^{\alpha_h}), \tag{3.144}$$

where $E_{\alpha_h,1}$ denotes the Mittag-Leffler function with parameters $\alpha := \alpha_h$ and $\beta := 1$ as defined in (3.13).

Moreover, we consider $a_h \in (-2, 0)$, for every $h = 1, \dots, l$, to be chosen appropriately in what follows (the precise choice will be performed in (3.163)), and, recalling (3.136), we let

$$\underline{t}_h := \underline{t}_{\star,h}^{1/\alpha_h}, \tag{3.145}$$

and we define

$$\psi_h(t_h) := \psi_{\star,h}\big(\underline{t}_h(t_h - a_h)\big) = E_{\alpha_h,1}\big(\underline{t}_{\star,h}(t_h - a_h)^{\alpha_h}\big). \tag{3.146}$$

We point out that, thanks to Lemma 3.3, the function in (3.146), solves

$$\begin{cases} D_{t_h,a_h}^{\alpha_h} \psi_h(t_h) = \underline{t}_{\star,h} \psi_h(t_h) & \text{in } (a_h, +\infty), \\ \psi_h(a_h) = 1, \\ \partial_{t_h}^m \psi_h(a_h) = 0 & \text{for every } m \in \{1, \dots, [\alpha_h]\}. \end{cases} \tag{3.147}$$

Moreover, for any $h \in \{1, \dots, l\}$, we define

$$\psi_h^\star(t_h) := \begin{cases} \psi_h(t_h) & \text{if } t_h \in [a_h, +\infty) \\ 1 & \text{if } t_h \in (-\infty, a_h). \end{cases} \tag{3.148}$$

Thanks to (3.147) and Lemma A.3 in [CDV18] applied here with $b := a_h$, $a := -\infty$, $u := \psi_h$, $u_\star := \psi_h^\star$, we have that $\psi_h^\star \in C_{-\infty}^{k_h,\alpha_h}$, and

$$D_{t_h,-\infty}^{\alpha_h}\psi_h^\star(t_h) = D_{t_h,a_h}^{\alpha_h}\psi_h(t_h) = \lambda_{\star,h}\psi_h(t_h) = \lambda_{\star,h}\psi_h^\star(t_h) \text{ in every interval } I \Subset (a_h, +\infty). \tag{3.149}$$

We observe that the setting in (3.148) is compatible with the ones in (3.117) and (3.119) .

From (3.13) and (3.146), we see that

$$\psi_h(t_h) = \sum_{j=0}^{+\infty} \frac{\lambda_{\star,h}^j (t_h - a_h)^{\alpha_h j}}{\Gamma(\alpha_h j + 1)}.$$

Consequently, for every $\mathfrak{I}_h \in \mathbb{N}$, we have that

$$\partial_{t_h}^{\mathfrak{I}_h}\psi_h(t_h) = \sum_{j=0}^{+\infty} \frac{\lambda_{\star,h}^j \alpha_h j(\alpha_h j - 1)\dots(\alpha_h j - \mathfrak{I}_h + 1)(t_h - a_h)^{\alpha_h j - \mathfrak{I}_h}}{\Gamma(\alpha_h j + 1)}. \tag{3.150}$$

Now, we define, for any $i \in \{1, \dots, n\}$,

$$\overline{a}_i := \begin{cases} \dfrac{a_i}{|a_i|} & \text{if } a_i \neq 0, \\ 1 & \text{if } a_i = 0. \end{cases}$$

We notice that

$$\overline{a}_i \neq 0 \text{ for all } i \in \{1, \dots, n\}, \tag{3.151}$$

and

$$a_i \overline{a}_i = |a_i|. \tag{3.152}$$

Now, for each $i \in \{1, \dots, n\}$, we consider the multi-index $r_i = (r_{i1}, \dots, r_{ip_i}) \in \mathbb{N}^{p_i}$. This multi-index acts on $\mathbb{R}^{p_i}$, whose variables are denoted by $x_i = (x_{i1}, \dots, x_{ip_i}) \in \mathbb{R}^{p_i}$. We let $\overline{v}_{i1}$ be the solution of the Cauchy problem

$$\begin{cases} \partial_{x_{i1}}^{r_{i1}}\overline{v}_{i1} = -\overline{a}_i\overline{v}_{i1} \\ \partial_{x_{i1}}^{\beta_1}\overline{v}_{i1}(0) = 1 & \text{for every } \beta_1 \leq r_{i1} - 1. \end{cases} \tag{3.153}$$

We notice that the solution of the Cauchy problem in (3.153) exists at least in a neighborhood of the origin of the form $[-\rho_{i1}, \rho_{i1}]$ for a suitable $\rho_{i1} > 0$.

Moreover, if $p_i \geq 2$, for any $\ell \in \{2, \dots, p_i\}$, we consider the solution of the following Cauchy problem:

$$\begin{cases} \partial_{x_{i\ell}}^{r_{i\ell}}\overline{v}_{i\ell} = \overline{v}_{i\ell} \\ \partial_{x_{i\ell}}^{\beta_\ell}\overline{v}_{i\ell}(0) = 1 & \text{for every } \beta_\ell \leq r_{i\ell} - 1. \end{cases} \tag{3.154}$$

As above, these solutions are well-defined at least in a neighborhood of the origin of the form $[-\rho_{i\ell}, \rho_{i\ell}]$, for a suitable $\rho_{i\ell} > 0$.

Then, we define

$$\overline{\rho}_i := \min\{\rho_{i1}, \dots, \rho_{ip_i}\} = \min_{\ell \in \{1,\dots,p_i\}} \rho_{i\ell}.$$

In this way, for every $x_i = (x_{i1}, \dots, x_{ip_i}) \in B^{p_i}_{\overline{\rho}_i}$, we set

$$\overline{v}_i(x_i) := \overline{v}_{i1}(x_{i1}) \dots \overline{v}_{ip_i}(x_{ip_i}). \tag{3.155}$$

By (3.153) and (3.154), we have that

$$\begin{cases} \partial^{r_i}_{x_i} \overline{v}_i = -\overline{a}_i \overline{v}_i \\ \\ \partial^{\beta}_{x_i} \overline{v}_i\,(0) = 1 & \begin{array}{c}\text{for every } \beta = (\beta_1, \dots \beta_{p_i}) \in \mathbb{N}^{p_i} \\ \text{such that } \beta_\ell \le r_{i\ell} - 1 \text{ for each } \ell \in \{1, \dots, p_i\}.\end{array} \end{cases} \tag{3.156}$$

Now, we define

$$\rho := \min\{\overline{\rho}_1, \dots \overline{\rho}_n\} = \min_{i \in \{1,\dots,n\}} \overline{\rho}_i.$$

We take

$$\overline{\tau} \in C^\infty_0\left(B^{p_1+\dots+p_n}_{\rho/(R+2)}\right),$$

with $\overline{\tau} = 1$ in $B^{p_1+\dots+p_n}_{\rho/(2(R+2))}$, and, for every $x = (x_1, \dots, x_n) \in \mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_n}$, we set

$$\tau_1\,(x_1, \dots, x_n) := \overline{\tau}\,(\underline{x}_1 \otimes x_1, \dots, \underline{x}_n \otimes x_n)\,. \tag{3.157}$$

We recall that the free parameters $\underline{x}_1, \dots, \underline{x}_n$ have been introduced in (3.135), and we have used here the notation

$$\underline{x}_i \otimes x_i = (\underline{x}_{i1}, \dots, \underline{x}_{ip_i}) \otimes (x_{i1}, \dots, x_{ip_i}) := (\underline{x}_{i1} x_{i1}, \dots, \underline{x}_{ip_i} x_{ip_i}) \in \mathbb{R}^{p_i},$$

for every $i \in \{1, \dots, n\}$.

We also set, for any $i \in \{1, \dots, n\}$,

$$v_i\,(x_i) := \overline{v}_i\,(\underline{x}_i \otimes x_i)\,. \tag{3.158}$$

We point out that if $x_i \in B^{p_i}_{\overline{\rho}_i/(R+2)}$ we have that

$$|\underline{x}_i \otimes x_i|^2 = \sum_{\ell=1}^{p_i} (\underline{x}_{i\ell} x_{i\ell})^2 \le (R+2)^2 \sum_{\ell=1}^{p_i} x_{i\ell}^2 < \overline{\rho}_i^2,$$

thanks to (3.135), and therefore the setting in (3.158) is well-defined for every $x_i \in B^{p_i}_{\overline{\rho}_i/(R+2)}$.

Recalling (3.156) and (3.158), we see that, for any $i \in \{1, \dots, n\}$,

$$\partial^{r_i}_{x_i} v_i(x_i) = \underline{x}_i^{r_i} \partial^{r_i}_{x_i} \overline{v}_i\,(\underline{x}_i \otimes x_i) = -\overline{a}_i \underline{x}_i^{r_i} \overline{v}_i\,(\underline{x}_i \otimes x_i) = -\overline{a}_i \underline{x}_i^{r_i} v_i(x_i). \tag{3.159}$$

We take $e_1, \dots, e_M$, with

$$e_j \in \partial B^{m_j}_{\omega_j}, \tag{3.160}$$

and we introduce an additional set of free parameters $Y_1, \dots, Y_M$ with

$$Y_j \in \mathbb{R}^{m_j} \qquad \text{and} \qquad e_j \cdot Y_j < 0. \tag{3.161}$$

We let $\epsilon > 0$, to be taken small possibly depending on the free parameters e_j, Y_j and $\underline{t}_h$, and we define

$$
\begin{aligned}
w\left(x,y,t\right) :=& \tau_1\left(x\right) v_1\left(x_1\right)\cdot\ldots\cdot v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\cdot\ldots\cdot\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\times\psi_1^\star(t_1)\cdot\ldots\cdot\psi_l^\star(t_l),
\end{aligned}
\tag{3.162}
$$

where the setting in (3.142), (3.148), (3.157) and (3.158) has been exploited.

We also notice that $w\in C\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^{N-l}\right)\cap\mathcal{A}$. Moreover, if

$$
a=(a_1,\ldots,a_l):=\left(-\frac{\epsilon}{\underline{t}_1},\ldots,-\frac{\epsilon}{\underline{t}_l}\right)\in(-\infty,0)^l \tag{3.163}
$$

and (x,y) is sufficiently close to the origin and $t\in(a_1,+\infty)\times\cdots\times(a_l,+\infty)$, we have that

$$
\begin{aligned}
&\Lambda_{-\infty}w\left(x,y,t\right)\\
=&\left(\sum_{i=1}^n\mathfrak{a}_i\partial_{x_i}^{r_i}+\sum_{j=1}^M\mathfrak{b}_j(-\Delta)_{y_j}^{s_j}+\sum_{h=1}^l\mathfrak{c}_hD_{t_h,-\infty}^{\alpha_h}\right)w\left(x,y,t\right)\\
=&\sum_{i=1}^n\mathfrak{a}_iv_1\left(x_1\right)\ldots v_{i-1}\left(x_{i-1}\right)\partial_{x_i}^{r_i}v_i\left(x_i\right)v_{i+1}\left(x_{i+1}\right)\ldots v_n\left(x_n\right)\\
&\qquad\times\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star\left(t_1\right)\ldots\psi_l^\star\left(t_l\right)\\
&+\sum_{j=1}^M\mathfrak{b}_jv_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_{j-1}\left(y_{j-1}+e_{j-1}+\epsilon Y_{j-1}\right)\\
&\qquad\times(-\Delta)_{y_j}^{s_j}\phi_j\left(y_j+e_j+\epsilon Y_j\right)\phi_{j+1}\left(y_{j+1}+e_{j+1}+\epsilon Y_{j+1}\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\qquad\times\psi_1^\star\left(t_1\right)\ldots\psi_l^\star\left(t_l\right)\\
&+\sum_{h=1}^l\mathfrak{c}_hv_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star\left(t_1\right)\ldots\psi_{h-1}^\star\left(t_{h-1}\right)\\
&\qquad\times D_{t_h,-\infty}^{\alpha_h}\psi_h^\star\left(t_h\right)\psi_{h+1}^\star(t_{h+1})\ldots\psi_l^\star\left(t_l\right)\\
=&-\sum_{i=1}^n\mathfrak{a}_i\overline{\mathfrak{a}}_i\underline{x}_i^{r_i}v_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star(t_1)\ldots\psi_l^\star(t_l)\\
&+\sum_{j=1}^M\mathfrak{b}_j\lambda_jv_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star(t_1)\ldots\psi_l^\star(t_l)\\
&+\sum_{h=1}^l\mathfrak{c}_h\underline{t}_{\star,h}v_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star(t_1)\ldots\psi_l^\star(t_l)\\
=&\left(-\sum_{i=1}^n\mathfrak{a}_i\overline{\mathfrak{a}}_i\underline{x}_i^{r_i}+\sum_{j=1}^M\mathfrak{b}_j\lambda_j+\sum_{h=1}^l\mathfrak{c}_h\underline{t}_{\star,h}\right)w(x,y,t),
\end{aligned}
$$

thanks to (3.143), (3.149) and (3.159) .

Consequently, making use of (3.137), (3.138) and (3.152), if (x,y) lies near the origin and $t\in(a_1,+\infty)\times\cdots\times(a_l,+\infty)$, we have that

$$
\Lambda_{-\infty}w\left(x,y,t\right)=\left(-\sum_{i=1}^n|\mathfrak{a}_i|\underline{x}_i^{r_i}+\sum_{j=1}^{M-1}\mathfrak{b}_j\lambda_j+\mathfrak{b}_M\lambda_M+\sum_{h=1}^l\mathfrak{c}_h\underline{t}_{\star,h}\right)w(x,y,t)
$$

$$= \left(-\sum_{i=1}^{n} |a_i| \underline{x}_i^{r_i} + \sum_{j=1}^{M-1} b_j \lambda_{\star,j} + b_M \lambda_M + \sum_{h=1}^{l} c_h \underline{t}_{\star,h} \right) w(x,y,t) = 0.$$

This says that $w \in \mathcal{H}$. Thus, in light of (3.127) we have that

$$0 = \theta \cdot \partial^K w(0) = \sum_{|\iota| \le K} \theta_\iota \partial^\iota w(0) = \sum_{|i|+|I|+|\mathfrak{I}| \le K} \theta_{i,I,\mathfrak{I}}\, \partial_x^i \partial_y^I \partial_t^{\mathfrak{I}} w(0). \tag{3.164}$$

Now, we recall (3.155) and we claim that, for any $j \in \{1, \dots, n\}$, any $\ell \in \{1, \dots, p_j\}$ and any $i_{j\ell} \in \mathbb{N}$, we have that

$$\partial_{x_{j\ell}}^{i_{j\ell}} \overline{v}_{j\ell}(0) \neq 0. \tag{3.165}$$

We prove it by induction over $i_{j\ell}$. Indeed, if $i_{j\ell} \in \{0, \dots, r_{j\ell} - 1\}$, then the initial condition in (3.153) (if $\ell = 1$) or (3.154) (if $\ell \ge 2$) gives that $\partial_{x_{i\ell}}^{i_{j\ell}} \overline{v}_{i\ell}(0) = 1$, and so (3.165) is true in this case.

To perform the inductive step, let us now suppose that the claim in (3.165) still holds for all $i_{j\ell} \in \{0, \dots, i_0\}$ for some i_0 such that $i_0 \ge r_{j\ell} - 1$. Then, using the equation in (3.153) (if $\ell = 1$) or in (3.154) (if $\ell \ge 2$), we have that

$$\partial_{x_{j\ell}}^{i_0+1} \overline{v}_j = \partial_{x_{j\ell}}^{i_0+1-r_{j\ell}} \partial_{x_{j\ell}}^{r_{j\ell}} \overline{v}_j = -\tilde{a}_j \partial_{x_{j\ell}}^{i_0+1-r_{j\ell}} \overline{v}_j, \tag{3.166}$$

with

$$\tilde{a}_j := \begin{cases} \overline{a}_j & \text{if } \ell = 1, \\ -1 & \text{if } \ell \ge 2. \end{cases}$$

Notice that $\tilde{a}_j \neq 0$, in view of (3.151), and $\partial_{x_{j\ell}}^{i_0+1-r_{j\ell}} \overline{v}_j(0) \neq 0$, by the inductive assumption. These considerations and (3.166) give that $\partial_{x_{j\ell}}^{i_0+1} \overline{v}_j(0) \neq 0$, and this proves (3.165).

Now, using (3.155) and (3.165) we have that, for any $j \in \{1, \dots, n\}$ and any $i_j \in \mathbb{N}^{p_j}$,

$$\partial_{x_j}^{i_j} \overline{v}_j(0) \neq 0.$$

This, (3.135) and the computation in (3.159) give that, for any $j \in \{1, \dots, n\}$ and any $i_j \in \mathbb{N}^{p_j}$,

$$\partial_{x_j}^{i_j} v_j(0) = \underline{x}_j^{i_j} \partial_{x_j}^{i_j} \overline{v}_j(0) \neq 0. \tag{3.167}$$

We also notice that, in light of (3.148), (3.162) and (3.164),

$$\begin{aligned} 0 = \sum_{|i|+|I|+|\mathfrak{I}| \le K} & \theta_{i,I,\mathfrak{I}}\, \partial_{x_1}^{i_1} v_1(0) \dots \partial_{x_n}^{i_n} v_n(0)\, \partial_{y_1}^{I_1} \phi_1(e_1 + \epsilon Y_1) \dots \partial_{y_M}^{I_M} \phi_M(e_M + \epsilon Y_M) \\ & \times \partial_{t_1}^{\mathfrak{I}_1} \psi_1(0) \dots \partial^{\mathfrak{I}_l} \psi_l(0). \end{aligned} \tag{3.168}$$

Now, by (3.142) and Proposition 3.14 (applied to $s := s_j$, $\beta := I_j$, $e := \frac{e_j}{\omega_j} \in \partial B_1^{m_j}$, due to (3.160), and $X := \frac{Y_j}{\omega_j}$), we see that, for any $j = 1, \dots, M$,

$$\begin{aligned} \omega_j^{|I_j|} \lim_{\epsilon \searrow 0} \epsilon^{|I_j| - s_j} \partial_{y_j}^{I_j} \phi_j(e_j + \epsilon Y_j) &= \lim_{\epsilon \searrow 0} \epsilon^{|I_j| - s_j} \partial_{y_j}^{I_j} \tilde{\phi}_{\star,j} \left(\frac{e_j + \epsilon Y_j}{\omega_j} \right) \\ &= \kappa_j \frac{e_j^{I_j}}{\omega_j^{|I_j|}} \left(-\frac{e_j}{\omega_j} \cdot \frac{Y_j}{\omega_j} \right)_+^{s_j - |I_j|}, \end{aligned} \tag{3.169}$$

with $\kappa_j \neq 0$, in the sense of distributions (in the coordinates Y_j).

Moreover, using (3.150) and (3.163), it follows that

$$\begin{aligned}
\partial_{t_h}^{\mathfrak{I}_h}\psi_h(0) &= \sum_{j=0}^{+\infty} \frac{\underline{t}_{\star,h}^{j}\,\alpha_h j(\alpha_h j-1)\dots(\alpha_h j-\mathfrak{I}_h+1)(0-a_h)^{\alpha_h j-\mathfrak{I}_h}}{\Gamma(\alpha_h j+1)} \\
&= \sum_{j=0}^{+\infty} \frac{\underline{t}_{\star,h}^{j}\,\alpha_h j(\alpha_h j-1)\dots(\alpha_h j-\mathfrak{I}_h+1)\,\epsilon^{\alpha_h j-\mathfrak{I}_h}}{\Gamma(\alpha_h j+1)\ \underline{t}_h^{\,\alpha_h j-\mathfrak{I}_h}} \\
&= \sum_{j=1}^{+\infty} \frac{\underline{t}_{\star,h}^{j}\,\alpha_h j(\alpha_h j-1)\dots(\alpha_h j-\mathfrak{I}_h+1)\,\epsilon^{\alpha_h j-\mathfrak{I}_h}}{\Gamma(\alpha_h j+1)\ \underline{t}_h^{\,\alpha_h j-\mathfrak{I}_h}}.
\end{aligned}$$

Accordingly, recalling (3.145), we find that

$$\begin{aligned}
&\lim_{\epsilon\searrow 0}\epsilon^{\mathfrak{I}_h-\alpha_h}\partial_{t_h}^{\mathfrak{I}_h}\psi_h(0) = \lim_{\epsilon\searrow 0}\sum_{j=1}^{+\infty} \frac{\underline{t}_{\star,h}^{j}\,\alpha_h j(\alpha_h j-1)\dots(\alpha_h j-\mathfrak{I}_h+1)\,\epsilon^{\alpha_h(j-1)}}{\Gamma(\alpha_h j+1)\ \underline{t}_h^{\,\alpha_h j-\mathfrak{I}_h}} \\
&\quad= \frac{\underline{t}_{\star,h}\,\alpha_h(\alpha_h-1)\dots(\alpha_h-\mathfrak{I}_h+1)}{\Gamma(\alpha_h+1)\ \underline{t}_h^{\,\alpha_h-\mathfrak{I}_h}} = \frac{\underline{t}_h^{\,\mathfrak{I}_h}\,\alpha_h(\alpha_h-1)\dots(\alpha_h-\mathfrak{I}_h+1)}{\Gamma(\alpha_h+1)}.
\end{aligned}\tag{3.170}$$

Also, recalling (3.128), we can write (3.168) as

$$\begin{aligned}
0 = &\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|\le|\overline{I}|}} \theta_{i,I,\mathfrak{I}}\,\partial_{x_1}^{i_1}v_1(0)\dots\partial_{x_n}^{i_n}v_n(0)\,\partial_{y_1}^{I_1}\phi_1(e_1+\epsilon Y_1)\dots\partial_{y_M}^{I_M}\phi_M(e_M+\epsilon Y_M) \\
&\times\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\dots\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0).
\end{aligned}\tag{3.171}$$

Moreover, we define

$$\Xi := |\overline{I}| - \sum_{j=1}^{M} s_j + |\mathfrak{I}| - \sum_{h=1}^{l}\alpha_h.$$

Then, we multiply (3.171) by $\epsilon^{\Xi}\in(0,+\infty)$, and we send ϵ to zero. In this way, we obtain from (3.169), (3.170) and (3.171) that

$$\begin{aligned}
0 &= \lim_{\epsilon\searrow 0}\epsilon^{\Xi}\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|\le|\overline{I}|}} \theta_{i,I,\mathfrak{I}}\,\partial_{x_1}^{i_1}v_1(0)\dots\partial_{x_n}^{i_n}v_n(0)\,\partial_{y_1}^{I_1}\phi_1(e_1+\epsilon Y_1)\dots\partial_{y_M}^{I_M}\phi_M(e_M+\epsilon Y_M) \\
&\qquad\times\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\dots\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0) \\
&= \lim_{\epsilon\searrow 0}\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|\le|\overline{I}|}} \epsilon^{|\overline{I}|-|I|}\theta_{i,I,\mathfrak{I}}\,\partial_{x_1}^{i_1}v_1(0)\dots\partial_{x_n}^{i_n}v_n(0) \\
&\qquad\times\epsilon^{|I_1|-s_1}\partial_{y_1}^{I_1}\phi_1(e_1+\epsilon Y_1)\dots\epsilon^{|I_M|-s_M}\partial_{y_M}^{I_M}\phi_M(e_M+\epsilon Y_M) \\
&\qquad\times\epsilon^{\mathfrak{I}_1-\alpha_1}\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\dots\epsilon^{\mathfrak{I}_l-\alpha_l}\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0) \\
&= \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|=|\overline{I}|}} \tilde{C}_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\partial_{x_1}^{i_1}v_1(0)\dots\partial_{x_n}^{i_n}v_n(0) \\
&\qquad\times e_1^{I_1}\dots e_M^{I_M}\,(-e_1\cdot Y_1)_+^{s_1-|I_1|}\dots(-e_M\cdot Y_M)_+^{s_M-|I_M|}\,\underline{t}_1^{\,\mathfrak{I}_1}\dots\underline{t}_l^{\,\mathfrak{I}_l},
\end{aligned}$$

for a suitable $\tilde{C}_{i,I,\mathfrak{I}} \neq 0$ (strictly speaking, the above identity holds in the sense of distribution with respect to the coordinates Y and $\underline{t}$, but since the left hand side vanishes, we can consider it also a pointwise identity).

Hence, recalling (3.167),

$$\begin{aligned} 0 &= \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|=|\bar{I}|}} C_{i,I,\mathfrak{I}}\, \theta_{i_1,\dots,i_n,I_1,\dots,I_M,\mathfrak{I}_1,\dots,\mathfrak{I}_l}\, \underline{x}_1^{i_1}\dots\underline{x}_n^{i_n} \\ &\qquad\times e_1^{I_1}\dots e_M^{I_M}\,(-e_1\cdot Y_1)_+^{s_1-|I_1|}\dots(-e_M\cdot Y_M)_+^{s_M-|I_M|}\,\underline{t}_1^{\mathfrak{I}_1}\dots\underline{t}_l^{\mathfrak{I}_l} \\ &= (-e_1\cdot Y_1)_+^{s_1}\dots(-e_M\cdot Y_M)_+^{s_M} \\ &\qquad\times \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|=|\bar{I}|}} C_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\underline{x}^i\, e^I\,(-e_1\cdot Y_1)_+^{-|I_1|}\dots(-e_M\cdot Y_M)_+^{-|I_M|}\,\underline{t}^{\mathfrak{I}}, \end{aligned} \tag{3.172}$$

for a suitable $C_{i,I,\mathfrak{I}} \neq 0$.

We observe that the equality in (3.172) is valid for any choice of the free parameters $(\underline{x}, Y, \underline{t})$ in an open subset of $\mathbb{R}^{p_1+\dots+p_n}\times\mathbb{R}^{m_1+\dots+m_M}\times\mathbb{R}^l$, as prescribed in (3.135), (3.136) and (3.161).

Now, we take new free parameters, $\underline{y}_1,\dots,\underline{y}_M$ with $\underline{y}_j \in \mathbb{R}^{m_j}\setminus\{0\}$, and we define

$$e_j := \frac{\omega_j \underline{y}_j}{|\underline{y}_j|} \quad \text{and} \quad Y_j := -\frac{\underline{y}_j}{|\underline{y}_j|^2}. \tag{3.173}$$

We stress that the setting in (3.173) is compatible with that in (3.161), since

$$e_j\cdot Y_j = -\frac{\omega_j \underline{y}_j}{|\underline{y}_j|}\cdot\frac{\underline{y}_j}{|\underline{y}_j|^2} = -\frac{\omega_j}{|\underline{y}_j|} < 0,$$

thanks to (3.141). We also notice that, for all $j\in\{1,\dots,M\}$,

$$e_j^{I_j}\,(-e_j\cdot Y_j)_+^{-|I_j|} = \frac{\omega_j^{|I_j|}\underline{y}_j^{I_j}}{|\underline{y}_j|^{|I_j|}}\,\frac{|\underline{y}_j|^{|I_j|}}{\omega_j^{|I_j|}} = \underline{y}_j^{I_j},$$

and hence

$$e^I\,(-e_1\cdot Y_1)_+^{-|I_1|}\dots(-e_M\cdot Y_M)_+^{-|I_M|} = \underline{y}^I.$$

Plugging this into formula (3.172), we obtain the first identity in (3.130), as desired. Hence, the proof of (3.130) in case 1 is complete. □

Proof of (3.130), *case 2.* Thanks to the assumptions given in case 2, we can suppose that formula (3.132) still holds, and also that

$$\underline{c}_l > 0. \tag{3.174}$$

In addition, for any $j\in\{1,\dots,M\}$, we consider λ_j and ϕ_j as in (3.143).

Then, we define

$$R := \left(\frac{1}{|\underline{a}_1|}\left(\sum_{h=1}^{l-1}|\underline{c}_h| + \sum_{j=1}^{M}|\underline{b}_j|\lambda_j\right)\right)^{1/|r_1|}. \tag{3.175}$$

We notice that, in light of (3.132), the setting in (3.175) is well-defined.

Now, we fix two sets of free parameters $\mathscr{x}_1, \dots, \mathscr{x}_n$ as in (3.135) and $\mathscr{t}_{\star,1}, \dots, \mathscr{t}_{\star,l}$ as in (3.136), here taken with R as in (3.175). Moreover, we define

$$\lambda := \frac{1}{\mathscr{c}_l \mathscr{t}_{\star,l}} \left(\sum_{j=1}^{n} |\mathscr{a}_j| \mathscr{x}_j^{r_j} - \sum_{j=1}^{M} \mathscr{b}_j \lambda_j - \sum_{h=1}^{l-1} \mathscr{c}_h \mathscr{t}_{\star,h} \right). \tag{3.176}$$

We notice that (3.176) is well-defined, thanks to (3.136) and (3.174). Furthermore, recalling (3.135), (3.139) and (3.175), we find that

$$\begin{aligned} \sum_{i=1}^{n} |\mathscr{a}_i| \mathscr{x}_i^{r_i} &\geq |\mathscr{a}_1| \mathscr{x}_1^{r_1} > |\mathscr{a}_1| (R+1)^{|r_1|} > |\mathscr{a}_1| R^{|r_1|} \\ &= \sum_{h=1}^{l-1} |\mathscr{c}_h| + \sum_{j=1}^{M} |\mathscr{b}_j| \lambda_j \geq \sum_{h=1}^{l-1} \mathscr{c}_h \mathscr{t}_{\star,h} + \sum_{j=1}^{M} \mathscr{b}_j \lambda_j. \end{aligned}$$

Consequently, by (3.176),

$$\lambda > 0. \tag{3.177}$$

Hence, we can define

$$\overline{\lambda} := \lambda^{1/\alpha_l}. \tag{3.178}$$

Moreover, we consider $a_h \in (-2, 0)$, for every $h \in \{1, \dots, l\}$, to be chosen appropriately in what follows (the exact choice will be performed in (3.185)), and, using the notation in (3.144) and (3.145), we define

$$\psi_h(t_h) := \psi_{\star,h}\big(\mathscr{t}_h(t_h - a_h)\big) = E_{\alpha_h,1}\big(\mathscr{t}_{\star,h}(t_h - a_h)^{\alpha_h}\big) \quad \text{if } h \in \{1, \dots, l-1\} \tag{3.179}$$

and

$$\psi_l(t_l) := \psi_{\star,l}\big(\overline{\lambda}\, \mathscr{t}_l(t_l - a_l)\big) = E_{\alpha_l,1}\big(\lambda \mathscr{t}_{\star,l}(t_l - a_l)^{\alpha_l}\big). \tag{3.180}$$

We recall that, thanks to Lemma 3.3, the function in (3.179) solves (3.147) and satisfies (3.150) for any $h \in \{1, \dots, l-1\}$, while the function in (3.180) solves

$$\begin{cases} D^{\alpha_l}_{t_l,a_l} \psi_l(t_l) = \lambda \mathscr{t}_{\star,l} \psi_l(t_l) & \text{in } (a_l, +\infty), \\ \psi_l(a_l) = 1, & \\ \partial^m_{t_l} \psi_l(a_l) = 0 & \text{for every } m \in \{1, \dots, [\alpha_l]\}. \end{cases} \tag{3.181}$$

As in (3.148), we extend the functions ψ_h constantly in $(-\infty, a_h)$, calling $\psi_h^\star$ this extended function. In this way, Lemma A.3 in [CDV18] translates (3.181) into

$$D^{\alpha_h}_{t_h,-\infty} \psi^\star_h(t_h) = \mathscr{t}_{\star,h} \psi_h(t_h) = \mathscr{t}_{\star,h} \psi^\star_h(t_h) \text{ in every interval } I \Subset (a_h, +\infty). \tag{3.182}$$

Now, we let $\epsilon > 0$, to be taken small possibly depending on the free parameters, and we exploit the functions defined in (3.157) and (3.158), provided that one replaces the positive constant R defined in (3.134) with the one in (3.175), when necessary.

With this idea in mind, for any $j \in \{1, \dots, M\}$, we let[5]

$$e_j \in \partial B_1^{m_j}, \tag{3.183}$$

[5] Comparing (3.183) with (3.160), we observe that (3.160) reduces to (3.183) with the choice $\omega_j := 1$.

and we define

$$
\begin{aligned}
w\left(x,y,t\right) :=& \tau_1\left(x\right) v_1\left(x_1\right)\cdot\ldots\cdot v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\cdot\ldots\cdot\phi_M\left(y_M+e_M+\epsilon Y_M\right) \\
&\times\psi_1^\star(t_1)\cdot\ldots\cdot\psi_l^\star(t_l),
\end{aligned}
\tag{3.184}
$$

where the setting in (3.143), (3.157), (3.158), (3.161), (3.179) and (3.180) has been exploited.

We also notice that $w\in C\left(\mathbb{R}^N\right)\cap C_0(\mathbb{R}^{N-l})\cap\mathcal{A}$. Moreover, if

$$
a=(a_1,\ldots,a_l):=\left(-\frac{\epsilon}{\underline{\mathscr{t}}_1},\ldots,-\frac{\epsilon}{\underline{\mathscr{t}}_l}\right)\in(-\infty,0)^l \tag{3.185}
$$

and (x,y) is sufficiently close to the origin and $t\in(a_1,+\infty)\times\cdots\times(a_l,+\infty)$, we have that

$$
\begin{aligned}
&\Lambda_{-\infty}w\left(x,y,t\right)\\
=\;&\left(\sum_{i=1}^n\mathscr{a}_i\partial_{x_i}^{r_i}+\sum_{j=1}^M\mathscr{b}_j(-\Delta)_{y_j}^{s_j}+\sum_{h=1}^l\mathscr{c}_hD_{t_h,-\infty}^{\alpha_h}\right)w\left(x,y,t\right)\\
=\;&\sum_{i=1}^n\mathscr{a}_iv_1\left(x_1\right)\ldots v_{i-1}\left(x_{i-1}\right)\partial_{x_i}^{r_i}v_i\left(x_i\right)v_{i+1}\left(x_{i+1}\right)\ldots v_n\left(x_n\right)\\
&\qquad\times\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star\left(t_1\right)\ldots\psi_{l-1}^\star\left(t_{l-1}\right)\psi_l^\star\left(t_l\right)\\
&+\sum_{j=1}^M\mathscr{b}_jv_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_{j-1}\left(y_{j-1}+e_{j-1}+\epsilon Y_{j-1}\right)\\
&\qquad\times(-\Delta)_{y_j}^{s_j}\phi_j\left(y_j+e_j+\epsilon Y_j\right)\phi_{j+1}\left(y_{j+1}+e_{j+1}+\epsilon Y_{j+1}\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\qquad\times\psi_1^\star\left(t_1\right)\ldots\psi_{l-1}^\star\left(t_{l-1}\right)\psi_l^\star\left(t_l\right)\\
&+\sum_{h=1}^l\mathscr{c}_hv_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi_1^\star\left(t_1\right)\ldots\psi_{h-1}^\star\left(t_{h-1}\right)\\
&\qquad\times D_{t_h,-\infty}^{\alpha_h}\psi_h^\star\left(t_h\right)\psi_{h+1}^\star(t_{h+1})\ldots\psi_{l-1}^\star\left(t_{l-1}\right)\psi_l^\star\left(t_l\right)\\
=\;&-\sum_{i=1}^n\mathscr{a}_i\bar{\mathscr{a}}_i\underline{\mathscr{x}}_i^{r_i}v_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\qquad\times\psi_1^\star(t_1)\ldots\psi_{l-1}^\star(t_{l-1})\psi_l^\star(t_l)\\
&+\sum_{j=1}^M\mathscr{b}_j\lambda_jv_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\qquad\times\psi_1^\star(t_1)\ldots\psi_{l-1}^\star\left(t_{l-1}\right)\psi_l^\star(t_l)\\
&+\sum_{h=1}^{l-1}\mathscr{c}_h\underline{\mathscr{t}}_{\star,h}v_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\qquad\times\psi_1^\star(t_1)\ldots\psi_{l-1}^\star(t_{l-1})\psi_l^\star(t_l)\\
&+\mathscr{c}_l\lambda\underline{\mathscr{t}}_{\star,l}v_1\left(x_1\right)\ldots v_n\left(x_n\right)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\\
&\qquad\times\psi_1^\star(t_1)\ldots\psi_{l-1}^\star(t_{l-1})\psi_l^\star(t_l)\\
=\;&\left(-\sum_{i=1}^n\mathscr{a}_i\bar{\mathscr{a}}_i\underline{\mathscr{x}}_i^{r_i}+\sum_{j=1}^M\mathscr{b}_j\lambda_j+\sum_{h=1}^{l-1}\mathscr{c}_h\underline{\mathscr{t}}_{\star,h}+\mathscr{c}_l\lambda\underline{\mathscr{t}}_{\star,l}\right)w(x,y,t),
\end{aligned}
$$

thanks to (3.143), (3.147), (3.159) and (3.182).

Consequently, making use of (3.152) and (3.176), when (x, y) is near the origin and $t \in (a_1, +\infty) \times \cdots \times (a_l, +\infty)$, we have that

$$\Lambda_{-\infty} w\,(x, y, t) = \left(-\sum_{i=1}^{n} |\underline{a}_i| \underline{x}_i^{r_i} + \sum_{j=1}^{M} \underline{b}_j \lambda_j + \sum_{h=1}^{l-1} \underline{c}_h \underline{t}_{\star,h} + \lambda \underline{c}_l \underline{t}_{\star,l} \right) w(x, y, t) = 0.$$

This says that $w \in \mathcal{H}$. Thus, in light of (3.127) we have that

$$0 = \theta \cdot \partial^K w\,(0) = \sum_{|\iota| \le K} \theta_\iota \partial^\iota w\,(0) = \sum_{|i|+|I|+|\mathfrak{I}| \le K} \theta_{i,I,\mathfrak{I}}\, \partial_x^i \partial_y^I \partial_t^{\mathfrak{I}} w\,(0)\,.$$

Hence, in view of (3.167) and (3.184),

$$\begin{aligned} 0 &= \sum_{|i|+|I|+|\mathfrak{I}| \le K} \theta_{i,I,\mathfrak{I}}\, \partial_{x_1}^{i_1} v_1(0) \dots \partial_{x_n}^{i_n} v_n(0) \\ &\qquad \times \partial_{y_1}^{I_1} \phi_1(e_1 + \epsilon Y_1) \dots \partial_{y_M}^{I_M} \phi_M(e_M + \epsilon Y_M)\, \partial_{t_1}^{\mathfrak{I}_1} \psi_1(0) \dots \partial_{t_l}^{\mathfrak{I}_l} \psi_l(0) \\ &= \sum_{|i|+|I|+|\mathfrak{I}| \le K} \theta_{i,I,\mathfrak{I}}\, \underline{x}_1^{r_1} \dots \underline{x}_n^{r_n}\, \partial_{x_1}^{i_1} \overline{v}_1(0) \dots \partial_{x_n}^{i_n} \overline{v}_n(0) \\ &\qquad \times \partial_{y_1}^{I_1} \phi_1(e_1 + \epsilon Y_1) \dots \partial_{y_M}^{I_M} \phi_M(e_M + \epsilon Y_M)\, \partial_{t_1}^{\mathfrak{I}_1} \psi_1(0) \dots \partial_{t_l}^{\mathfrak{I}_l} \psi_l(0). \end{aligned} \tag{3.186}$$

Moreover, using (3.13), (3.180) and (3.185), it follows that

$$\begin{aligned} \partial_{t_l}^{\mathfrak{I}_l} \psi_l(0) &= \sum_{j=0}^{+\infty} \frac{\lambda^j\, \underline{t}_{\star,l}^j\, \alpha_l j(\alpha_l j - 1) \dots (\alpha_l j - \mathfrak{I}_l + 1)(0 - a_l)^{\alpha_l j - \mathfrak{I}_l}}{\Gamma\,(\alpha_l j + 1)} \\ &= \sum_{j=0}^{+\infty} \frac{\lambda^j\, \underline{t}_{\star,l}^j\, \alpha_l j(\alpha_l j - 1) \dots (\alpha_l j - \mathfrak{I}_l + 1)\, \epsilon^{\alpha_l j - \mathfrak{I}_l}}{\Gamma\,(\alpha_l j + 1)\ \underline{t}_l^{\,\alpha_l j - \mathfrak{I}_l}} \\ &= \sum_{j=1}^{+\infty} \frac{\lambda^j\, \underline{t}_{\star,l}^j\, \alpha_l j(\alpha_l j - 1) \dots (\alpha_l j - \mathfrak{I}_l + 1)\, \epsilon^{\alpha_l j - \mathfrak{I}_l}}{\Gamma\,(\alpha_l j + 1)\ \underline{t}_l^{\,\alpha_l j - \mathfrak{I}_l}}. \end{aligned}$$

Accordingly, by (3.145), we find that

$$\begin{aligned} \lim_{\epsilon \searrow 0} \epsilon^{\mathfrak{I}_l - \alpha_l} \partial_{t_l}^{\mathfrak{I}_l} \psi_l(0) &= \lim_{\epsilon \searrow 0} \sum_{j=1}^{+\infty} \frac{\lambda^j\, \underline{t}_{\star,l}^j\, \alpha_l j(\alpha_l j - 1) \dots (\alpha_l j - \mathfrak{I}_l + 1)\, \epsilon^{\alpha_l (j-1)}}{\Gamma\,(\alpha_l j + 1)\ \underline{t}_l^{\,\alpha_l j - \mathfrak{I}_l}} \\ &= \frac{\lambda\, \underline{t}_{\star,l}\, \alpha_l(\alpha_l - 1) \dots (\alpha_l - \mathfrak{I}_l + 1)}{\Gamma\,(\alpha_l + 1)\ \underline{t}_l^{\,\alpha_l - \mathfrak{I}_l}} = \frac{\lambda\, \underline{t}_l^{\mathfrak{I}_l}\, \alpha_l(\alpha_l - 1) \dots (\alpha_l - \mathfrak{I}_l + 1)}{\Gamma\,(\alpha_l + 1)}. \end{aligned} \tag{3.187}$$

Hence, recalling (3.129), we can write (3.186) as

$$\begin{aligned} 0 = &\sum_{\substack{|i|+|I|+|\mathfrak{I}| \le K \\ |\mathfrak{I}| \le |\overline{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}}\, \underline{x}_1^{r_1} \dots \underline{x}_n^{r_n}\, \partial_{x_1}^{i_1} \overline{v}_1(0) \dots \partial_{x_n}^{i_n} \overline{v}_n(0) \\ &\qquad \times \partial_{y_1}^{I_1} \phi_1(e_1 + \epsilon Y_1) \dots \partial_{y_M}^{I_M} \phi_M(e_M + \epsilon Y_M) \partial_{t_1}^{\mathfrak{I}_1} \psi_1(0) \dots \partial_{t_l}^{\mathfrak{I}_l} \psi_l(0). \end{aligned} \tag{3.188}$$

Moreover, we define

$$\Xi := |\overline{\mathfrak{I}}| - \sum_{h=1}^{l} \alpha_h + |I| - \sum_{j=1}^{M} s_j.$$

Then, we multiply (3.188) by $\epsilon^{\Xi} \in (0,+\infty)$, and we send ϵ to zero. In this way, we obtain from (3.170), used here for $h \in \{1,\dots,l-1\}$, (3.187) and (3.188) that

$$\begin{aligned}
0 &= \lim_{\epsilon \searrow 0} \epsilon^{\Xi} \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |\mathfrak{I}|\le|\overline{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}}\, \underline{x}_1^{r_1}\dots\underline{x}_n^{r_n}\, \partial_{x_1}^{i_1}\overline{v}_1(0)\dots\partial_{x_n}^{i_n}\overline{v}_n(0)\\
&\qquad \times \partial_{y_1}^{I_1}\phi_1(e_1+\epsilon Y_1)\dots\partial_{y_M}^{I_M}\phi_M(e_M+\epsilon Y_M)\\
&\qquad \times \partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\dots\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0)\\
&= \lim_{\epsilon \searrow 0} \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |\mathfrak{I}|\le|\overline{\mathfrak{I}}|}} \epsilon^{|\overline{\mathfrak{I}}|-|\mathfrak{I}|}\theta_{i,I,\mathfrak{I}}\, \underline{x}_1^{r_1}\dots\underline{x}_n^{r_n}\, \partial_{x_1}^{i_1}\overline{v}_1(0)\dots\partial_{x_n}^{i_n}\overline{v}_n(0)\\
&\qquad \times \epsilon^{|I_1|-s_1}\partial_{y_1}^{I_1}\phi_1(e_1+\epsilon Y_1)\dots\epsilon^{|I_M|-s_M}\partial_{y_M}^{I_M}\phi_M(e_M+\epsilon Y_M)\\
&\qquad \times \epsilon^{\mathfrak{I}_1-\alpha_1}\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\dots\epsilon^{\mathfrak{I}_l-\alpha_l}\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0)\\
&= \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |\mathfrak{I}|=|\overline{\mathfrak{I}}|}} \lambda\,\tilde{C}_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\, \underline{x}_1^{r_1}\dots\underline{x}_n^{r_n}\, \partial_{x_1}^{i_1}\overline{v}_1(0)\dots\partial_{x_n}^{i_n}\overline{v}_n(0)\\
&\qquad \times e_1^{I_1}\dots e_M^{I_M}\,(-e_1\cdot Y_1)_+^{s_1-|I_1|}\dots(-e_M\cdot Y_M)_+^{s_M-|I_M|}\,\underline{t}_1^{\mathfrak{I}_1}\dots\underline{t}_l^{\mathfrak{I}_l},
\end{aligned}$$

for a suitable $\tilde{C}_{i,I,\mathfrak{I}}$. We stress that $\tilde{C}_{i,I,\mathfrak{I}} \neq 0$, thanks also to (3.169), applied here with $\omega_j := 1$, $\tilde{\phi}_{\star,j} := \phi_j$ and e_j as in (3.183) for any $j \in \{1,\dots,M\}$.

Hence, recalling (3.177),

$$\begin{aligned}
0 &= \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |\mathfrak{I}|=|\overline{\mathfrak{I}}|}} C_{i,I,\mathfrak{I}}\,\theta_{i_1,\dots,i_n,I_1,\dots,I_M,\mathfrak{I}_1,\dots,\mathfrak{I}_l}\,\underline{x}_1^{i_1}\dots\underline{x}_n^{i_n}\\
&\qquad \times e_1^{I_1}\dots e_M^{I_M}\,(-e_1\cdot Y_1)_+^{s_1-|I_1|}\dots(-e_M\cdot Y_M)_+^{s_M-|I_M|}\,\underline{t}_1^{\mathfrak{I}_1}\dots\underline{t}_l^{\mathfrak{I}_l}\\
&= (-e_1\cdot Y_1)_+^{s_1}\dots(-e_M\cdot Y_M)_+^{s_M}\\
&\qquad \times \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |\mathfrak{I}|=|\overline{\mathfrak{I}}|}} C_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\underline{x}^{i}\,e^{I}\,(-e_1\cdot Y_1)_+^{-|I_1|}\dots(-e_M\cdot Y_M)_+^{-|I_M|}\,\underline{t}^{\mathfrak{I}},
\end{aligned} \tag{3.189}$$

for a suitable $C_{i,I,\mathfrak{I}} \neq 0$.

We observe that the equality in (3.189) is valid for any choice of the free parameters $(\underline{x}, Y, \underline{t})$ in an open subset of $\mathbb{R}^{p_1+\dots+p_n} \times \mathbb{R}^{m_1+\dots+m_M} \times \mathbb{R}^l$, as prescribed in (3.135), (3.136) and (3.161).

Now, we take new free parameters $\underline{y}_j$ with $\underline{y}_j \in \mathbb{R}^{m_j} \setminus \{0\}$ for any $j = 1,\dots,M$, and perform in (3.189) the same change of variables done in (3.173), obtaining that

$$0 = \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |\mathfrak{I}|=|\overline{\mathfrak{I}}|}} C_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\underline{x}^{i}\,\underline{y}^{I}\,\underline{t}^{\mathfrak{I}},$$

for some $C_{i,I,\mathfrak{I}} \neq 0$.

Hence, the second identity in (3.130) is obtained as desired, and the proof of Lemma 3.25 in case 2 is completed. □

Proof of (3.130), *case 3.* We divide the proof of case 3 into two subcases, namely either

$$\text{there exists } h \in \{1,\dots,l\} \text{ such that } \underline{c}_h \neq 0, \tag{3.190}$$

or

$$c_h = 0 \text{ for every } h \in \{1, \dots, l\}. \tag{3.191}$$

We start by dealing with the case in (3.190). Up to relabeling and reordering the coefficients c_h, we can assume that

$$c_1 \neq 0. \tag{3.192}$$

Also, thanks to the assumptions given in case 3, we can suppose that

$$b_M < 0, \tag{3.193}$$

and, for any $j \in \{1, \dots, M\}$, we consider $\lambda_{\star,j}$ and $\tilde{\phi}_{\star,j}$ as in (3.131). Then, we take $\omega_j := 1$ and ϕ_j as in (3.142), so that (3.143) is satisfied. In particular, here we have that

$$\lambda_j = \lambda_{\star,j} \qquad \text{and} \qquad \phi_j = \tilde{\phi}_{\star,j}. \tag{3.194}$$

We define

$$R := \frac{1}{|c_1|} \sum_{j=1}^{M-1} |b_j| \lambda_{\star,j}. \tag{3.195}$$

We notice that, in light of (3.192), the setting in (3.195) is well-defined.

Now, we fix a set of free parameters

$$t_{\star,1} \in (R+1, R+2), \dots t_{\star,l} \in (R+1, R+2). \tag{3.196}$$

Moreover, we define

$$\lambda_M := \frac{1}{b_M} \left(-\sum_{j=1}^{M-1} b_j \lambda_{\star,j} - \sum_{h=1}^{l} |c_h| t_{\star,h} \right). \tag{3.197}$$

We notice that (3.197) is well-defined thanks to (3.193). From (3.195) we deduce that

$$\begin{aligned} \sum_{h=1}^{l} |c_h| t_{\star,h} + \sum_{j=1}^{M-1} b_j \lambda_{\star,j} &\geq |c_1| t_{\star,1} - \sum_{j=1}^{M-1} |b_j| \lambda_{\star,j} \\ &> |c_1| R - \sum_{j=1}^{M-1} |b_j| \lambda_{\star,j} = 0. \end{aligned}$$

Consequently, by (3.193) and (3.197),

$$\lambda_M > 0. \tag{3.198}$$

Now, we define, for any $h \in \{1, \dots, l\}$,

$$\bar{c}_h := \begin{cases} \dfrac{c_h}{|c_h|} & \text{if } c_h \neq 0, \\ 1 & \text{if } c_h = 0. \end{cases}$$

We notice that

$$\bar{c}_h \neq 0 \text{ for all } h \in \{1, \dots, l\}, \tag{3.199}$$

and

$$c_h \bar{c}_h = |c_h|. \tag{3.200}$$

Moreover, we consider $a_h \in (-2, 0)$, for every $h = 1, \dots, l$, to be chosen appropriately in what follows (see (3.208) for a precise choice).

Now, for every $h \in \{1, \dots, l\}$, we define

$$\psi_h(t_h) := E_{\alpha_h,1}(\overline{c}_h \underline{t}_{\star,h}(t_h - a_h)^{\alpha_h}), \tag{3.201}$$

where $E_{\alpha_h,1}$ denotes the Mittag-Leffler function with parameters $\alpha := \alpha_h$ and $\beta := 1$ as defined in (3.13). By Lemma 3.3, we know that

$$\begin{cases} D^{\alpha_h}_{t_h,a_h}\psi_h(t_h) = \overline{c}_h \underline{t}_{\star,h}\psi_h(t_h) & \text{in } (a_h, +\infty), \\ \psi_h(a_h) = 1, \\ \partial^m_{t_h}\psi_h(a_h) = 0 \quad \text{for any } m = 1, \dots, [\alpha_h], \end{cases} \tag{3.202}$$

and we consider again the extension $\psi^\star_h$ given in (3.148). By Lemma A.3 in [CDV18], we know that (3.202) translates into

$$D^{\alpha_h}_{t_h,-\infty}\psi^\star_h(t_h) = \overline{c}_h \underline{t}_{\star,h}\psi^\star_h(t_h) \text{ in every interval } I \Subset (a_h, +\infty). \tag{3.203}$$

Now, we consider auxiliary parameters $\underline{t}_h$, e_j and Y_j as in (3.145), (3.160) and (3.161). Moreover, we introduce an additional set of free parameters

$$\underline{x} = (\underline{x}_1, \dots, \underline{x}_n) \in \mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_n}. \tag{3.204}$$

We let $\epsilon > 0$, to be taken small possibly depending on the free parameters. We take $\tau \in C^\infty(\mathbb{R}^{p_1+\dots+p_n}, [0 + \infty))$ such that

$$\tau(x) := \begin{cases} \exp\left(\underline{x} \cdot x\right) & \text{if } x \in B_1^{p_1+\dots+p_n}, \\ 0 & \text{if } x \in \mathbb{R}^{p_1+\dots+p_n} \setminus B_2^{p_1+\dots+p_n}, \end{cases} \tag{3.205}$$

where

$$\underline{x} \cdot x := \sum_{j=1}^{n} \underline{x}_i \cdot x_i$$

denotes the standard scalar product.

We notice that, for any $i \in \mathbb{N}^{p_1} \times \dots \times \mathbb{N}^{p_n}$,

$$\partial^i_x \tau(0) = \partial^{i_1}_{x_1} \dots \partial^{i_n}_{x_n} \tau(0) = \underline{x}^{i_{11}}_{11} \dots \underline{x}^{i_{1p_1}}_{1p_1} \dots \underline{x}^{i_{n1}}_{n1} \dots \underline{x}^{i_{np_n}}_{np_n} = \underline{x}^i. \tag{3.206}$$

We define

$$w\,(x, y, t) := \tau(x)\phi_1\,(y_1 + e_1 + \epsilon Y_1) \cdot \ldots \cdot \phi_M\,(y_M + e_M + \epsilon Y_M)\,\psi^\star_1(t_1) \cdot \ldots \cdot \psi^\star_l(t_l), \tag{3.207}$$

where the setting in (3.143) has also been exploited.

We also notice that $w \in C\left(\mathbb{R}^N\right) \cap C_0\left(\mathbb{R}^{N-l}\right) \cap \mathcal{A}$. Moreover, if

$$a = (a_1, \dots, a_l) := \left(-\frac{\epsilon}{\underline{t}_1}, \dots, -\frac{\epsilon}{\underline{t}_l}\right) \in (-\infty, 0)^l \tag{3.208}$$

and (x, y) is sufficiently close to the origin and $t \in (a_1, +\infty) \times \dots \times (a_l, +\infty)$, we have that

$$\Lambda_{-\infty} w\,(x, y, t)$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{M}\mathcal{b}_j(-\Delta)^{s_j}_{y_j}+\sum_{h=1}^{l}\mathcal{c}_h D^{\alpha_h}_{t_h,-\infty}\right) w\left(x,y,t\right)\\
&= \sum_{j=1}^{M}\mathcal{b}_j\tau(x)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_{j-1}\left(y_{j-1}+e_{j-1}+\epsilon Y_{j-1}\right)(-\Delta)^{s_j}_{y_j}\phi_j\left(y_j+e_j+\epsilon Y_j\right)\\
&\qquad\qquad \times\phi_{j+1}\left(y_{j+1}+e_{j+1}+\epsilon Y_{j+1}\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi^\star_1\left(t_1\right)\ldots\psi^\star_l\left(t_l\right)\\
&\quad +\sum_{h=1}^{l}\mathcal{c}_h\tau(x)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi^\star_1\left(t_1\right)\ldots\psi^\star_{h-1}\left(t_{h-1}\right)\\
&\qquad\qquad \times D^{\alpha_h}_{t_h,-\infty}\psi^\star_h\left(t_h\right)\psi^\star_{h+1}(t_{h+1})\ldots\psi^\star_l\left(t_l\right)\\
&= \sum_{j=1}^{M}\mathcal{b}_j\lambda_j\tau(x)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi^\star_1(t_1)\ldots\psi^\star_l(t_l)\\
&\quad +\sum_{h=1}^{l}\mathcal{c}_h\bar{\mathcal{c}}_h\underline{\mathcal{t}}_{\star,h}\tau(x)\phi_1\left(y_1+e_1+\epsilon Y_1\right)\ldots\phi_M\left(y_M+e_M+\epsilon Y_M\right)\psi^\star_1(t_1)\ldots\psi^\star_l(t_l)\\
&= \left(\sum_{j=1}^{M}\mathcal{b}_j\lambda_j+\sum_{h=1}^{l}\mathcal{c}_h\bar{\mathcal{c}}_h\underline{\mathcal{t}}_{\star,h}\right) w(x,y,t),
\end{aligned}
$$

thanks to (3.143) and (3.203).

Consequently, making use of (3.194), (3.197) and (3.200), if (x,y) is near the origin and $t\in(a_1,+\infty)\times\cdots\times(a_l,+\infty)$, we have that

$$
\Lambda_{-\infty}w\left(x,y,t\right)=\left(\sum_{j=1}^{M}\mathcal{b}_j\lambda_{\star,j}+\mathcal{b}_M\lambda_M+\sum_{h=1}^{l}|\mathcal{c}_h|\underline{\mathcal{t}}_{\star,h}\right) w(x,y,t)=0.
$$

This says that $w\in\mathcal{H}$. Thus, in light of (3.127) we have that

$$
0=\theta\cdot\partial^K w\left(0\right)=\sum_{|\iota|\le K}\theta_\iota\partial^\iota w\left(0\right)=\sum_{|i|+|I|+|\mathfrak{I}|\le K}\theta_{i,I,\mathfrak{I}}\,\partial^i_x\partial^I_y\partial^{\mathfrak{I}}_t w\left(0\right).
$$

From this and (3.207), we obtain that

$$
0=\sum_{|i|+|I|+|\mathfrak{I}|\le K}\theta_{i,I,\mathfrak{I}}\,\partial^i_x\tau(0)\partial^{I_1}_{y_1}\phi_1(e_1+\epsilon Y_1)\ldots\partial^{I_M}_{y_M}\phi_M(e_M+\epsilon Y_M)\,\partial^{\mathfrak{I}_1}_{t_1}\psi_1(0)\ldots\partial^{\mathfrak{I}_l}_{t_l}\psi_l(0). \tag{3.209}
$$

Moreover, using (3.201) and (3.208), it follows that, for every $\mathfrak{I}_h\in\mathbb{N}$

$$
\begin{aligned}
\partial^{\mathfrak{I}_h}_{t_h}\psi_h(0) &= \sum_{j=0}^{+\infty}\frac{\bar{\mathcal{c}}^j_h\,\underline{\mathcal{t}}^j_{\star,h}\,\alpha_h j(\alpha_h j-1)\ldots(\alpha_h j-\mathfrak{I}_l+1)(0-a_h)^{\alpha_h j-\mathfrak{I}_h}}{\Gamma\left(\alpha_h j+1\right)}\\
&= \sum_{j=0}^{+\infty}\frac{\bar{\mathcal{c}}^j_h\,\underline{\mathcal{t}}^j_{\star,h}\,\alpha_h j(\alpha_h j-1)\ldots(\alpha_h j-\mathfrak{I}_h+1)\,\epsilon^{\alpha_h j-\mathfrak{I}_h}}{\Gamma\left(\alpha_h j+1\right)\,\underline{\mathcal{t}}_h^{\;\alpha_h j-\mathfrak{I}_h}}\\
&= \sum_{j=1}^{+\infty}\frac{\bar{\mathcal{c}}^j_h\,\underline{\mathcal{t}}^j_{\star,h}\,\alpha_h j(\alpha_h j-1)\ldots(\alpha_h j-\mathfrak{I}_h+1)\,\epsilon^{\alpha_h j-\mathfrak{I}_h}}{\Gamma\left(\alpha_h j+1\right)\,\underline{\mathcal{t}}_h^{\;\alpha_h j-\mathfrak{I}_h}}.
\end{aligned}
$$

Accordingly, recalling (3.145), we find that

$$\begin{aligned}\lim_{\epsilon\searrow 0}\epsilon^{\mathfrak{I}_h-\alpha_h}\partial_{t_h}^{\mathfrak{I}_h}\psi_h(0)&=\lim_{\epsilon\searrow 0}\sum_{j=1}^{+\infty}\frac{\overline{c}_h^{j}\,\underline{t}_{\star,h}^{j}\,\alpha_h j(\alpha_h j-1)\ldots(\alpha_h j-\mathfrak{I}_h+1)\,\epsilon^{\alpha_h(j-1)}}{\Gamma(\alpha_h j+1)\,\underline{t}_h^{\,\alpha_h j-\mathfrak{I}_h}}\\&=\frac{\overline{c}_h\,\underline{t}_{\star,h}\,\alpha_h(\alpha_h-1)\ldots(\alpha_h-\mathfrak{I}_h+1)}{\Gamma(\alpha_h+1)\,\underline{t}_h^{\,\alpha_h-\mathfrak{I}_h}}=\frac{\overline{c}_h\,\underline{t}_h^{\mathfrak{I}_h}\,\alpha_h(\alpha_h-1)\ldots(\alpha_h-\mathfrak{I}_h+1)}{\Gamma(\alpha_h+1)}.\end{aligned}\tag{3.210}$$

Also, recalling (3.128), we can write (3.209) as

$$0=\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\|I|\le|\overline{I}|}}\theta_{i,I,\mathfrak{I}}\,\partial_x^i\tau(0)\partial_{y_1}^{I_1}\phi_1(e_1+\epsilon Y_1)\ldots\partial_{y_M}^{I_M}\phi_M(e_M+\epsilon Y_M)\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\ldots\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0).\tag{3.211}$$

Moreover, we define

$$\Xi:=|\overline{I}|-\sum_{j=1}^{M}s_j+|\mathfrak{I}|-\sum_{h=1}^{l}\alpha_h.$$

Then, we multiply (3.211) by $\epsilon^{\Xi}\in(0,+\infty)$, and we send ϵ to zero. In this way, we obtain from (3.169), (3.206), (3.210) and (3.211) that

$$\begin{aligned}0&=\lim_{\epsilon\searrow 0}\epsilon^{\Xi}\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\|I|\le|\overline{I}|}}\theta_{i,I,\mathfrak{I}}\,\partial_x^i\tau(0)\partial_{y_1}^{I_1}\phi_1\left(e_1+\epsilon Y_1\right)\ldots\partial_{y_M}^{I_M}\phi_M\left(e_M+\epsilon Y_M\right)\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\ldots\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0)\\&=\lim_{\epsilon\searrow 0}\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\|I|\le|\overline{I}|}}\epsilon^{|\overline{I}|-|I|}\theta_{i,I,\mathfrak{I}}\,\partial_x^i\tau(0)\epsilon^{|I_1|-s_1}\partial_{y_1}^{I_1}\phi_1\left(e_1+\epsilon Y_1\right)\ldots\epsilon^{|I_M|-s_M}\partial_{y_M}^{I_M}\phi_M\left(e_M+\epsilon Y_M\right)\\&\qquad\times\epsilon^{\mathfrak{I}_1-\alpha_1}\partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\ldots\epsilon^{\mathfrak{I}_l-\alpha_l}\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0)\\&=\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\|I|=|\overline{I}|}}C_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\underline{x}_1^{i_1}\ldots\underline{x}_n^{i_n}\,e_1^{I_1}\ldots e_M^{I_M}\,(-e_1\cdot Y_1)_+^{s_1-|I_1|}\ldots(-e_M\cdot Y_M)_+^{s_M-|I_M|}\,\underline{t}_1^{\mathfrak{I}_1}\ldots\underline{t}_l^{\mathfrak{I}_l}\\&=(-e_1\cdot Y_1)_+^{s_1}\ldots(-e_M\cdot Y_M)_+^{s_M}\\&\qquad\times\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\|I|=|\overline{I}|}}C_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\underline{x}^i\,e^I\,(-e_1\cdot Y_1)_+^{-|I_1|}\ldots(-e_M\cdot Y_M)_+^{-|I_M|}\,\underline{t}^{\mathfrak{I}},\end{aligned}$$

for a suitable $C_{i,I,\mathfrak{I}}\neq 0$.

We observe that the latter equality is valid for any choice of the free parameters $(\underline{x},Y,\underline{t})$ in an open subset of $\mathbb{R}^{p_1+\ldots+p_n}\times\mathbb{R}^{m_1+\ldots+m_M}\times\mathbb{R}^l$, as prescribed in (3.161), (3.196) and (3.204).

Now, we take new free parameters $\underline{y}_j$ with $\underline{y}_j\in\mathbb{R}^{m_j}\setminus\{0\}$ for any $j=1,\ldots,M$, and perform in the latter identity the same change of variables done in (3.173), obtaining that

$$0=\sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\|I|=|\overline{I}|}}C_{i,I,\mathfrak{I}}\,\theta_{i,I,\mathfrak{I}}\,\underline{x}^i\underline{y}^I\underline{t}^{\mathfrak{I}},$$

for some $C_{i,I,\mathfrak{I}}\neq 0$. This completes the proof of (3.130) in case (3.190) is satisfied.

Hence, we now focus on the case in which (3.191) holds true. For any $j\in\{1,\ldots,M\}$, we consider the function $\psi\in H^{s_j}(\mathbb{R}^{m_j})\cap C_0^{s_j}(\mathbb{R}^{m_j})$ constructed in Lemma 3.19 and we call such function ϕ_j, to make it explicit its dependence on j in this case. We recall that

$$(-\Delta)_{y_j}^{s_j}\phi_j(y_j)=0\quad\text{ in }B_1^{m_j}.\tag{3.212}$$

Also, for every $j \in \{1, \dots, M\}$, we let e_j and Y_j be as in (3.160) and (3.161). Thanks to Lemma 3.19 and Remark 3.20, for any $I_j \in \mathbb{N}^{m_j}$, we know that

$$\lim_{\epsilon \searrow 0} \epsilon^{|I_j|-s_j} \partial_{y_j}^{I_j} \phi_j(e_j + \epsilon Y_j) = \kappa_{s_j} e_j^{I_j} (-e_j \cdot Y_j)_+^{s_j - |I_j|}, \tag{3.213}$$

for some $\kappa_{s_j} \neq 0$.

Moreover, for any $h = 1, \dots, l$, we define $\overline{\tau}_h(t_h)$ as

$$\overline{\tau}_h(t_h) := \begin{cases} e^{\underline{t}_h t_h} & \text{if} \quad t_h \in [-1, +\infty), \\ e^{-\underline{t}_h} \displaystyle\sum_{i=0}^{k_h - 1} \frac{\underline{t}_h^i}{i!} (t_h + 1)^i & \text{if} \quad t_h \in (-\infty, -1), \end{cases} \tag{3.214}$$

where $\underline{t} = (\underline{t}_1, \dots, \underline{t}_l) \in (1,2)^l$ are free parameters.

We notice that, for any $h \in \{1, \dots, l\}$ and $\mathfrak{I}_h \in \mathbb{N}$,

$$\partial_{t_h}^{\mathfrak{I}_h} \overline{\tau}_h(0) = \underline{t}_h^{\mathfrak{I}_h}. \tag{3.215}$$

Now, we define

$$w(x, y, t) := \tau(x) \phi_1(y_1 + e_1 + \epsilon Y_1) \dots \phi_M(y_M + e_M + \epsilon Y_M) \overline{\tau}_1(t_1) \dots \overline{\tau}_l(t_l), \tag{3.216}$$

where the setting of (3.142), (3.205) and (3.214) has been exploited. We have that $w \in \mathcal{A}$. Moreover, we point out that, since τ, $\phi_1, \dots, \phi_M$ are compactly supported, we have that $w \in C(\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-l})$, and, using Proposition 3.22, for any $j \in \{1, \dots, M\}$, it holds that $\phi_j \in C^\infty(\mathcal{N}_j)$ for some neighborhood $\mathcal{N}_j$ of the origin in $\mathbb{R}^{m_j}$. Hence $w \in C^\infty(\mathcal{N})$.

Furthermore, using (3.212), when y is in a neighborhood of the origin we have that

$$\begin{aligned} \Lambda_{-\infty} w(x, y, t) = \tau(x) \left(\mathscr{b}_1 (-\Delta)_{y_1}^{s_1} \phi_1(y_1 + e_1 + \epsilon Y_1) \right) \dots \phi_M(y_M + e_M + \epsilon Y_M) \overline{\tau}_1(t_1) \dots \overline{\tau}_l(t_l) \\ + \dots + \tau(x) \phi_1(y_1) \dots \left(\mathscr{b}_M (-\Delta)_{Y_M}^{s_M} \phi_M(y_M + e_M + \epsilon Y_M) \right) \overline{\tau}_1(t_1) \dots \overline{\tau}_l(t_l) = 0, \end{aligned}$$

which gives that $w \in \mathcal{H}$.

In addition, using (3.128), (3.206) and (3.215), we have that

$$\begin{aligned} 0 = \theta \cdot \partial^K w(0) = \sum_{|\iota| \le K} \theta_{i,I,\mathfrak{I}} \partial_x^i \partial_y^I \partial_t^{\mathfrak{I}} w(0) = \sum_{\substack{|\iota| \le K \\ |I| \le |\overline{I}|}} \theta_{i,I,\mathfrak{I}} \partial_x^i \partial_y^I \partial_t^{\mathfrak{I}} w(0) \\ = \sum_{\substack{|\iota| \le K \\ |I| \le |\overline{I}|}} \theta_{i,I,\mathfrak{I}} \, \underline{x}^i \partial_{y_1}^{I_1} \phi_1(e_1 + \epsilon Y_1) \dots \partial_{y_M}^{I_M} \phi_M(e_M + \epsilon Y_M) \underline{t}^{\mathfrak{I}}. \end{aligned}$$

Hence, we set

$$\Xi := |\overline{I}| - \sum_{j=1}^M s_j,$$

we multiply the latter identity by ϵ^Ξ and we exploit (3.213). In this way, we find that

$$0 \;=\; \lim_{\epsilon \searrow 0} \sum_{\substack{|\iota| \le K \\ |I| \le |\overline{I}|}} \epsilon^{|\overline{I}| - |I|} \theta_{i,I,\mathfrak{I}} \, \underline{x}^i \, \epsilon^{|I_1| - s_1} \partial_{y_1}^{I_1} \phi_1(e_1 + \epsilon Y_1) \dots \epsilon^{|I_M| - s_M} \partial_{y_M}^{I_M} \phi_M(e_M + \epsilon Y_M) \underline{t}^{\mathfrak{I}}$$

$$= \sum_{\substack{|\iota|\le K\\ |I|=|\overline{I}|}} \theta_{i,I,\mathfrak{I}}\,\kappa_{s_j}\,\underline{x}^i\,e^I\,(-e_1\cdot Y_1)_+^{s_1-|I_1|}\dots(-e_M\cdot Y_M)_+^{s_M-|I_M|}\,\underline{t}^{\mathfrak{I}}$$

$$= (-e_1\cdot Y_1)_+^{s_1}\dots(-e_M\cdot Y_M)_+^{s_M}\sum_{\substack{|\iota|\le K\\ |I|=|\overline{I}|}} \theta_{i,I,\mathfrak{I}}\,\kappa_{s_j}\,\underline{x}^i\,e^I\,(-e_1\cdot Y_1)_+^{-|I_1|}\dots(-e_M\cdot Y_M)_+^{-|I_M|}\,\underline{t}^{\mathfrak{I}},$$

and consequently

$$0 = \sum_{\substack{|\iota|\le K\\ |I|=|\overline{I}|}} \theta_{i,I,\mathfrak{I}}\,\kappa_{s_j}\,\underline{x}^i\,e^I\,(-e_1\cdot Y_1)_+^{-|I_1|}\dots(-e_M\cdot Y_M)_+^{-|I_M|}\,\underline{t}^{\mathfrak{I}}. \tag{3.217}$$

Now we take free parameters $\underline{y}\in\mathbb{R}^{m_1+\dots+m_M}\setminus\{0\}$ and we perform the same change of variables in (3.173). In this way, we deduce from (3.217) that

$$0 = \sum_{\substack{|i|+|I|+|\mathfrak{I}|\le K\\ |I|=|\overline{I}|}} C_{i,I,\mathfrak{I}}\theta_{i,I,\mathfrak{I}}\underline{x}^i\underline{y}^I\underline{t}^{\mathfrak{I}},$$

for some $C_{i,I,\mathfrak{I}}\neq 0$, and the first claim in (3.130) is proved in this case as well. □

Proof of (3.130), *case 4*. Notice that if there exists $j\in\{1,\dots,M\}$ such that $\mathfrak{b}_j\neq 0$, we are in the setting of case 3. Therefore, we assume that $\mathfrak{b}_j=0$ for every $j\in\{1,\dots,M\}$.

We let ψ be the function constructed in Lemma 3.4. For each $h\in\{1,\dots,l\}$, we let $\overline{\psi}_h(t_h):=\psi(t_h)$, to make the dependence on h clear and explicit. Then, by formulas (3.20) and (3.21), we know that

$$D^{\alpha_h}_{t_h,0}\overline{\psi}_h(t_h)=0 \quad \text{in } (1,+\infty) \tag{3.218}$$

and, for every $\ell\in\mathbb{N}$,

$$\lim_{\epsilon\searrow 0}\epsilon^{\ell-\alpha_h}\partial^\ell_{t_h}\overline{\psi}_h(1+\epsilon t_h)=\kappa_{h,\ell}\,t_h^{\alpha_h-\ell}, \tag{3.219}$$

in the sense of distribution, for some $\kappa_{h,\ell}\neq 0$.

Now, we introduce a set of auxiliary parameters $\underline{t}=(\underline{t}_1,\dots,\underline{t}_l)\in(1,2)^l$, and fix ϵ sufficiently small possibly depending on the parameters. Then, we define

$$a=(a_1,\dots,a_l):=\left(-\frac{\epsilon}{\underline{t}_1}-1,\dots,-\frac{\epsilon}{\underline{t}_l}-1\right)\in(-2,0)^l, \tag{3.220}$$

and

$$\psi_h(t_h):=\overline{\psi}_h(t_h-a_h). \tag{3.221}$$

With a simple computation we have that the function in (3.221) satisfies

$$D^{\alpha_h}_{t_h,a_h}\psi_h(t_h)=D^{\alpha_h}_{t_h,0}\overline{\psi}_h(t_h-a_h)=0 \quad \text{in } (1+a_h,+\infty)=\left(-\frac{\epsilon}{\underline{t}_h},+\infty\right), \tag{3.222}$$

thanks to (3.218). In addition, for every $\ell\in\mathbb{N}$, we have that $\partial^\ell_{t_h}\psi_h(t_h)=\partial^\ell_{t_h}\overline{\psi}_h(t_h-a_h)$, and therefore, in light of (3.219) and (3.220),

$$\epsilon^{\ell-\alpha_h}\partial^\ell_{t_h}\psi_h(0)=\epsilon^{\ell-\alpha_h}\partial^\ell_{t_h}\overline{\psi}_h(-a_h)=\epsilon^{\ell-\alpha_h}\partial^\ell_{t_h}\overline{\psi}_h\left(1+\frac{\epsilon}{\underline{t}_h}\right)\to\kappa_{h,\ell}\,\underline{t}_h^{\ell-\alpha_h}, \tag{3.223}$$

in the sense of distributions, as $\epsilon \searrow 0$.

Moreover, since for any $h = 1, \dots, l$, $\psi_h \in C^{k_h,\alpha_h}_{a_h}$, we can consider the extension

$$\psi_h^\star(t_h) := \begin{cases} \psi_h(t_h) & \text{if } t_h \in [a_h, +\infty), \\ \displaystyle\sum_{i=0}^{k_h-1} \frac{\psi_h^{(i)}(a_h)}{i!}(t_h - a_h)^i & \text{if } t_h \in (-\infty, a_h), \end{cases} \tag{3.224}$$

and, using Lemma A.3 in [CDV18] with $u := \psi_h$, $a := -\infty$, $b := a_h$ and $u_\star := \psi_h^\star$, we have that

$$\psi_h^\star \in C^{k_h,\alpha_h}_{-\infty} \quad \text{and} \quad D^{\alpha_h}_{t_h,-\infty}\psi_h^\star = D^{\alpha_h}_{t_h,a_h}\psi_h = 0 \quad \text{in every interval } I \Subset \left(-\frac{\epsilon}{\underline{t}_h}, +\infty\right). \tag{3.225}$$

Now, we fix a set of free parameters $\underline{y} = \left(\underline{y}_1, \dots, \underline{y}_M\right) \in \mathbb{R}^{m_1+\ldots+m_M}$, and consider $\overline{\tau} \in C^\infty(\mathbb{R}^{m_1+\ldots+m_M})$, such that

$$\overline{\tau}(y) := \begin{cases} \exp\left(\underline{y} \cdot y\right) & \text{if} \quad y \in B_1^{m_1+\ldots+m_M}, \\ 0 & \text{if} \quad y \in \mathbb{R}^{m_1+\ldots+m_M} \setminus B_2^{m_1+\ldots+m_M}, \end{cases} \tag{3.226}$$

where

$$\underline{y} \cdot y = \sum_{j=1}^{M} \underline{y}_j \cdot y_j,$$

denotes the standard scalar product.

We notice that, for any multi-index $I \in \mathbb{N}^{m_1+\ldots m_M}$,

$$\partial_y^I \overline{\tau}(0) = \underline{y}^I, \tag{3.227}$$

where the multi-index notation has been used.

Now, we define

$$w(x,y,t) := \tau(x)\overline{\tau}(y)\psi_1^\star(t_1)\ldots\psi_l^\star(t_l), \tag{3.228}$$

where the setting in (3.205), (3.224) and (3.226) has been exploited.

Using (3.225), we have that, for any (x,y) in a neighborhood of the origin and $t \in \left(-\frac{\epsilon}{2}, +\infty\right)^l$,

$$\begin{aligned} \Lambda_{-\infty} w(x,y,t) &= \tau(x)\overline{\tau}(y)\left(\mathfrak{c}_1 D^{\alpha_1}_{t_1,-\infty}\psi_1^\star(t_1)\right)\ldots\psi_l^\star(t_l) \\ &\quad + \ldots + \tau(x)\overline{\tau}(y)\psi_1^\star(t_1)\ldots\left(\mathfrak{c}_l D^{\alpha_l}_{t_l,-\infty}\psi_l^\star(t_l)\right) = 0. \end{aligned}$$

We have that $w \in \mathcal{A}$, and, since τ and $\overline{\tau}$ are compactly supported, we also have that $w \in C(\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-l})$. Also, from Lemma 3.4, for any $h \in \{1, \dots, l\}$, we know that $\overline{\psi}_h \in C^\infty((1,+\infty))$, hence $\psi_h \in C^\infty\left(\left(-\frac{\epsilon}{\underline{t}_h}, +\infty\right)\right)$. Thus, $w \in C^\infty(\mathcal{N})$, and consequently $w \in \mathcal{H}$.

Recalling (3.129), (3.206), and (3.227), we have that

$$\begin{aligned} 0 = \theta \cdot \partial^K w(0) &= \sum_{|\iota| \le K} \theta_{i,I,\mathfrak{I}} \partial_x^i \partial_y^I \partial_t^{\mathfrak{I}} w(0) = \sum_{\substack{|\iota| \le K \\ |\mathfrak{I}| \le |\overline{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}} \partial_x^i \partial_y^I \partial_t^{\mathfrak{I}} w(0) \\ &= \sum_{\substack{|\iota| \le K \\ |\mathfrak{I}| \le |\overline{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}}\, \underline{x}^i \underline{y}^I \partial_{t_1}^{\mathfrak{I}_1}\psi_1(0)\ldots\partial_{t_l}^{\mathfrak{I}_l}\psi_l(0). \end{aligned} \tag{3.229}$$

Hence, we set

$$\Xi := |\bar{\mathfrak{I}}| - \sum_{h=1}^{l} \alpha_h,$$

we multiply the identity in (3.229) by ϵ^{Ξ} and we exploit (3.223). In this way, we find that

$$\begin{aligned} 0 &= \lim_{\epsilon \searrow 0} \sum_{\substack{|\iota| \le K \\ |\mathfrak{I}| \le |\bar{\mathfrak{I}}|}} \epsilon^{|\bar{\mathfrak{I}}| - |\mathfrak{I}|} \theta_{i,I,\mathfrak{I}} \, \underline{x}^{i} \, \underline{y}^{I} \, \epsilon^{\mathfrak{I}_1 - \alpha_1} \partial_{t_1}^{\mathfrak{I}_1} \psi_1(0) \dots \epsilon^{\mathfrak{I}_l - \alpha_l} \partial_{t_l}^{\mathfrak{I}_l} \psi_l(0) \\ &= \sum_{\substack{|\iota| \le K \\ |\mathfrak{I}| = |\bar{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}} \, \kappa_{1,\mathfrak{I}_1} \dots \kappa_{l,\mathfrak{I}_l} \, \underline{x}^{i} \, \underline{y}^{I} \, \underline{t}_1^{\mathfrak{I}_1 - \alpha_1} \dots \underline{t}_l^{\mathfrak{I}_l - \alpha_l} \\ &= \underline{t}_1^{-\alpha_1} \dots \underline{t}_l^{-\alpha_l} \sum_{\substack{|\iota| \le K \\ |\mathfrak{I}| = |\bar{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}} \, \kappa_{1,\mathfrak{I}_1} \dots \kappa_{l,\mathfrak{I}_l} \, \underline{x}^{i} \, \underline{y}^{I} \, \underline{t}_1^{\mathfrak{I}_1} \dots \underline{t}_l^{\mathfrak{I}_l}, \end{aligned}$$

and consequently

$$0 = \sum_{\substack{|\iota| \le K \\ |\mathfrak{I}| = |\bar{\mathfrak{I}}|}} \theta_{i,I,\mathfrak{I}} \, \kappa_{1,\mathfrak{I}_1} \dots \kappa_{l,\mathfrak{I}_l} \, \underline{x}^{i} \, \underline{y}^{I} \, \underline{t}^{\mathfrak{I}},$$

and the second claim in (3.130) is proved in this case as well. $\square$

3.12 Every function is locally $\Lambda_{-\infty}$-harmonic up to a small error, and completion of the proof of Theorem 3.23

In this section we complete the proof of Theorem 3.23 (which in turn implies Theorem 3.1 via Lemma 3.24). By standard approximation arguments we can reduce to the case in which f is a polynomial, and hence, by the linearity of the operator $\Lambda_{-\infty}$, to the case in which is a monomial. The details of the proof are therefore the following:

3.12.1 Proof of Theorem 3.23 when f is a monomial

We prove Theorem 3.23 under the initial assumption that f is a monomial, that is

$$f(x,y,t) = \frac{x_1^{i_1} \dots x_n^{i_n} y_1^{I_1} \dots y_M^{I_M} t_1^{\mathfrak{I}_1} \dots t_l^{\mathfrak{I}_l}}{\iota!} = \frac{x^i y^I t^{\mathfrak{I}}}{\iota!} = \frac{(x,y,t)^{\iota}}{\iota!}, \tag{3.230}$$

where $\iota! := i_1! \dots i_n! I_1! \dots I_M! \mathfrak{I}_1! \dots \mathfrak{I}_l!$ and $I_\beta! := I_{\beta,1}! \dots I_{\beta,m_\beta}!$, $i_\chi! := i_{\chi,1}! \dots i_{\chi,p_\chi}!$ for all $\beta = 1, \dots M$. and $\chi = 1, \dots, n$. To this end, we argue as follows. We consider $\eta \in (0,1)$, to be taken sufficiently small with respect to the parameter $\epsilon > 0$ which has been fixed in the statement of Theorem 3.23, and we define

$$\mathcal{T}_\eta(x,y,t) := \left(\eta^{\frac{1}{r_1}} x_1, \dots, \eta^{\frac{1}{r_n}} x_n, \eta^{\frac{1}{2s_1}} y_1, \dots, \eta^{\frac{1}{2s_M}} y_M, \eta^{\frac{1}{\alpha_1}} t_1, \dots, \eta^{\frac{1}{\alpha_l}} t_l \right).$$

We also define

$$\gamma := \sum_{j=1}^{n} \frac{|i_j|}{r_j} + \sum_{j=1}^{M} \frac{|I_j|}{2s_j} + \sum_{j=1}^{l} \frac{\mathfrak{I}_j}{\alpha_j}, \tag{3.231}$$

and
$$\delta := \min\left\{\frac{1}{r_1},\dots,\frac{1}{r_n},\frac{1}{2s_1},\dots,\frac{1}{2s_M},\frac{1}{\alpha_1},\dots,\frac{1}{\alpha_l}\right\}. \tag{3.232}$$
We also take $K_0 \in \mathbb{N}$ such that
$$K_0 \geq \frac{\gamma+1}{\delta} \tag{3.233}$$
and we let
$$K := K_0 + |i| + |I| + |\mathfrak{I}| + \ell = K_0 + |\iota| + \ell, \tag{3.234}$$
where ℓ is the fixed integer given in the statement of Theorem 3.23.

By Lemma 3.25, there exist a neighborhood $\mathcal{N}$ of the origin and a function $w \in C\left(\mathbb{R}^N\right) \cap C_0\left(\mathbb{R}^{N-l}\right) \cap C^\infty\left(\mathcal{N}\right) \cap \mathcal{A}$ such that
$$\Lambda_{-\infty} w = 0 \text{ in } \mathcal{N}, \tag{3.235}$$
and such that
$$\begin{aligned}&\text{all the derivatives of } w \text{ in } 0 \text{ up to order } K \text{ vanish,}\\ &\text{with the exception of } \partial^\iota w\,(0) \text{ which equals } 1,\end{aligned} \tag{3.236}$$
being ι as in (3.230). Recalling the definition of $\mathcal{A}$ on page 61, we also know that
$$\partial^{k_h}_{t_h} w = 0 \text{ in } (-\infty, a_h), \tag{3.237}$$
for suitable $a_h \in (-2,0)$, for all $h \in \{1,\dots,l\}$.

In this way, setting
$$g := w - f, \tag{3.238}$$
we deduce from (3.236) that
$$\partial^\sigma g\,(0) = 0 \qquad \text{for any } \sigma \in \mathbb{N}^N \text{ with } |\sigma| \leq K.$$
Accordingly, in $\mathcal{N}$ we can write
$$g\,(x,y,t) = \sum_{|\tau|\geq K+1} x^{\tau_1} y^{\tau_2} t^{\tau_3} h_\tau\,(x,y,t), \tag{3.239}$$
for some h_τ smooth in $\mathcal{N}$, where the multi-index notation $\tau = (\tau_1,\tau_2,\tau_3)$ has been used.

Now, we define
$$u\,(x,y,t) := \frac{1}{\eta^\gamma} w\left(\mathcal{T}_\eta(x,y,t)\right). \tag{3.240}$$
In light of (3.237), we notice that $\partial^{k_h}_{t_h} u = 0$ in $(-\infty, a_h/\eta^{\frac{1}{\alpha_h}})$, for all $h \in \{1,\dots,l\}$, and therefore $u \in C\left(\mathbb{R}^N\right) \cap C_0(\mathbb{R}^{N-l}) \cap C^\infty\left(\mathcal{T}_\eta(\mathcal{N})\right) \cap \mathcal{A}$. We also claim that
$$\mathcal{T}_\eta([-1,1]^{N-l} \times (a_1,+\infty) \times \ldots \times (a_l,+\infty)) \subseteq \mathcal{N}. \tag{3.241}$$
To check this, let $(x,y,t) \in [-1,1]^{N-l} \times (a_1 + \infty) \times \ldots \times (a_l,+\infty)$ and $(X,Y,T) := \mathcal{T}_\eta(x,y,t)$. Then, we have that $|X_1| = \eta^{\frac{1}{r_1}}|x_1| \leq \eta^{\frac{1}{r_1}}$, $|Y_1| = \eta^{\frac{1}{2s_1}}|y_1| \leq \eta^{\frac{1}{2s_1}}$, $T_1 = \eta^{\frac{1}{\alpha_1}} t_1 > a_1 \eta^{\frac{1}{\alpha_1}} > -1$, provided η is small enough. Repeating this argument, we obtain that, for small η,
$$(X,Y,T) \text{ is as close to the origin as we wish.} \tag{3.242}$$

From (3.242) and the fact that $\mathcal{N}$ is an open set, we infer that $(X,Y,T)\in\mathcal{N}$, and this proves (3.241).

Thanks to (3.235) and (3.241), we have that, in $B_1^{N-l}\times(-1,+\infty)^l$,

$$\begin{aligned}&\eta^{\gamma-1}\,\Lambda_{-\infty}u\,(x,y,t)\\&=\sum_{j=1}^{n}\mathfrak{a}_j\partial_{x_j}^{r_j}w\,(\mathcal{T}_\eta(x,y,t))+\sum_{j=1}^{M}\mathfrak{b}_j(-\Delta)_{y_j}^{s_j}w\,(\mathcal{T}_\eta(x,y,t))+\sum_{j=1}^{l}\mathfrak{c}_jD_{t_h,-\infty}^{\alpha_h}w\,(\mathcal{T}_\eta(x,y,t))\\&=\Lambda_{-\infty}w\,(\mathcal{T}_\eta(x,y,t))\\&=0.\end{aligned}$$

These observations establish that u solves the equation in $B_1^{N-l}\times(-1+\infty)^l$ and u vanishes when $|(x,y)|\geq R$, for some $R>1$, and thus the claims in (3.118) and (3.119) are proved.

Now we prove that u approximates f, as claimed in (3.120). For this, using the monomial structure of f in (3.230) and the definition of γ in (3.231), we have, in a multi-index notation,

$$\frac{1}{\eta^\gamma}f\,(\mathcal{T}_\eta(x,y,t))=\frac{1}{\eta^\gamma\,\iota!}\,(\eta^{\frac{1}{r}}x)^i(\eta^{\frac{1}{2s}}y)^I\left(\eta^{\frac{1}{\alpha}}t\right)^{\mathfrak{I}}=\frac{1}{\iota!}x^iy^It^{\mathfrak{I}}=f(x,y,t).\tag{3.243}$$

Consequently, by (3.238), (3.239), (3.240) and (3.243),

$$\begin{aligned}u\,(x,y,t)-f\,(x,y,t)&=\frac{1}{\eta^\gamma}g\left(\eta^{\frac{1}{r_1}}x_1,\dots,\eta^{\frac{1}{r_n}}x_n,\eta^{\frac{1}{2s_1}}y_1,\dots,\eta^{\frac{1}{2s_M}}y_M,\eta^{\frac{1}{\alpha_1}}t_1,\dots,\eta^{\frac{1}{\alpha_l}}t_l\right)\\&=\sum_{|\tau|\geq K+1}\eta^{\left|\frac{\tau_1}{r}\right|+\left|\frac{\tau_2}{2s}\right|+\left|\frac{\tau_3}{\alpha}\right|-\gamma}x^{\tau_1}y^{\tau_2}t^{\tau_3}h_\tau\left(\eta^{\frac{1}{r}}x,\eta^{\frac{1}{2s}}y,\eta^{\frac{1}{\alpha}}t\right),\end{aligned}$$

where a multi-index notation has been used, e.g. we have written

$$\frac{\tau_1}{r}:=\left(\frac{\tau_{1,1}}{r_1},\dots,\frac{\tau_{1,n}}{r_n}\right)\in\mathbb{R}^n.$$

Therefore, for any multi-index $\beta=(\beta_1,\beta_2,\beta_3)$ with $|\beta|\leq\ell$,

$$\begin{aligned}&\partial^\beta\,(u\,(x,y,t)-f\,(x,y,t))\\&=\partial_x^{\beta_1}\partial_y^{\beta_2}\partial_t^{\beta_3}\,(u\,(x,y,t)-f\,(x,y,t))\\&=\sum_{\substack{|\beta_1'|+|\beta_1''|=|\beta_1|\\|\beta_2'|+|\beta_2''|=|\beta_2|\\|\beta_3'|+|\beta_3''|=|\beta_3|\\|\tau|\geq K+1}}c_{\tau,\beta}\,\eta^{\kappa_{\tau,\beta}}\,x^{\tau_1-\beta_1'}y^{\tau_2-\beta_2'}t^{\tau_3-\beta_3'}\partial_x^{\beta_1''}\partial_y^{\beta_2''}\partial_t^{\beta_3''}h_\tau\left(\eta^{\frac{1}{r}}x,\eta^{\frac{1}{2s}}y,\eta^{\frac{1}{\alpha}}t\right),\end{aligned}\tag{3.244}$$

where

$$\kappa_{\tau,\beta}:=\left|\frac{\tau_1}{r}\right|+\left|\frac{\tau_2}{2s}\right|+\left|\frac{\tau_3}{\alpha}\right|-\gamma+\left|\frac{\beta_1''}{r}\right|+\left|\frac{\beta_2''}{2s}\right|+\left|\frac{\beta_3''}{\alpha}\right|,$$

for suitable coefficients $c_{\tau,\beta}$. Thus, to complete the proof of (3.120), we need to show that this quantity is small if so is η. To this aim, we use (3.232), (3.233) and (3.234) to see that

$$\kappa_{\tau,\beta}\quad\geq\quad\left|\frac{\tau_1}{r}\right|+\left|\frac{\tau_2}{2s}\right|+\left|\frac{\tau_3}{\alpha}\right|-\gamma$$

$$\begin{aligned}
&\geq \delta\left(|\tau_1|+|\tau_2|+|\tau_3|\right)-\gamma\\
&\geq K\delta-\gamma\\
&\geq K_0\delta-\gamma\\
&\geq 1.
\end{aligned}$$

Consequently, we deduce from (3.244) that $\|u-f\|_{C^\ell\left(B_1^N\right)}\leq C\eta$ for some $C>0$. By choosing η sufficiently small with respect to ϵ, this implies the claim in (3.120). This completes the proof of Theorem 3.23 when f is a monomial.

3.12.2 Proof of Theorem 3.23 when f is a polynomial

Now, we consider the case in which f is a polynomial. In this case, we can write f as

$$f\left(x,y,t\right)=\sum_{j=1}^{J}c_j f_j\left(x,y,t\right),$$

where each f_j is a monomial, $J\in\mathbb{N}$ and $c_j\in\mathbb{R}$ for all $j=1,\ldots,J$.

Let

$$c:=\max_{j\in\{1,\ldots,J\}}c_j.$$

Then, by the work done in Subsection 3.12.1, we know that the claim in Theorem 3.23 holds true for each f_j, and so we can find $a_j\in(-\infty,0)^l$, $u_j\in C^\infty\left(B_1^N\right)\cap C\left(\mathbb{R}^N\right)\cap\mathcal{A}$ and $R_j>1$ such that $\Lambda_{-\infty}u_j=0$ in $B_1^{N-l}\times(-1,+\infty)^l$, $\|u_j-f_j\|_{C^\ell\left(B_1^N\right)}\leq\epsilon$ and $u_j=0$ if $|(x,y)|\geq R_j$.

Hence, we set

$$u\left(x,y,t\right):=\sum_{j=1}^{J}c_j u_j\left(x,y,t\right),$$

and we see that

$$\|u-f\|_{C^\ell\left(B_1^N\right)}\leq\sum_{j=1}^{J}|c_j|\,\|u_j-f_j\|_{C^\ell\left(B_1^N\right)}\leq cJ\epsilon. \tag{3.245}$$

Also, $\Lambda_{-\infty}u=0$ thanks to the linearity of $\Lambda_{-\infty}$ in $B_1^{N-l}\times(-1,+\infty)^l$. Finally, u is supported in B_R^{N-l} in the variables (x,y), being

$$R:=\max_{j\in\{1,\ldots,J\}}R_j.$$

This proves Theorem 3.23 when f is a polynomial (up to replacing ϵ with $cJ\epsilon$).

3.12.3 Proof of Theorem 3.23 for a general f

Now we deal with the case of a general f. To this end, we exploit Lemma 2 in [DSV17] and we see that there exists a polynomial $\tilde{f}$ such that

$$\|f-\tilde{f}\|_{C^\ell(B_1^N)}\leq\epsilon. \tag{3.246}$$

Then, applying the result already proven in Subsection 3.12.2 to the polynomial $\tilde{f}$, we can find $a\in(-\infty,0)^l$, $u\in C^\infty\left(B_1^N\right)\cap C\left(\mathbb{R}^N\right)\cap\mathcal{A}$ and $R>1$ such that

$$\Lambda_{-\infty}u=0 \quad \text{in } B_1^{N-l}\times(-1,+\infty)^l,$$

$$\begin{aligned} & u = 0 && \text{if } |(x,y)| \geq R, \\ & \partial^{k_h}_{t_h} u = 0 && \text{if } t_h \in (-\infty, a_h), \qquad \text{for all } h \in \{1, \dots, l\}, \\ \text{and} \qquad & \|u - \tilde{f}\|_{C^\ell(B_1^N)} \leq \epsilon. \end{aligned}$$

Then, recalling (3.246), we see that

$$\|u - f\|_{C^\ell(B_1^N)} \leq \|u - \tilde{f}\|_{C^\ell(B_1^N)} + \|f - \tilde{f}\|_{C^\ell(B_1^N)} \leq 2\epsilon.$$

Hence, the proof of Theorem 3.23 is complete. □

3.13 Applications

In this section we give some applications of the approximation results obtained and discussed in this chapter. These examples exploit particular cases of the operator Λ_a, namely, when $s \in (0,1)$ and Λ_a is the fractional Laplacian $(-\Delta)^s$, or the fractional heat operator $\partial_t + (-\Delta)^s$. Similar applications have been pointed out in [CDV17, AV19, RS17b].

Example 3.26 (The classical Harnack inequality fails for s-harmonic functions)**.** Harnack inequality, in its classical formulation, says that if u is a nontrivial and nonnegative harmonic function in B_1 then, for any $0 < r < 1$, there exists $0 < c = c(n,r)$ such that

$$\sup_{B_r} u \leq c \inf_{B_r} u. \tag{3.247}$$

The same result is not true for s-harmonic functions. To construct a counterexample, consider the smooth function $f(x) = |x|^2$, and, for a small $\epsilon > 0$, let $v = v_\epsilon$ be the function provided by Theorem 3.1, where we choose $\ell = 0$. Notice that, if $x \in B_1 \setminus B_{r/2}$,

$$v(x) \geq f(x) - \|v - f\|_{L^\infty(B_1)} \geq \frac{r^2}{4} - \epsilon > \frac{r^2}{8},$$

provided ϵ is small enough, while

$$v(0) \leq f(0) + \|v - f\|_{L^\infty(B_1)} \leq \epsilon < \frac{r^2}{8}.$$

Hence, we have that $v(0) < v(x)$ for any $x \in B_1 \setminus B_{r/2}$, and therefore the minimum of v in B_1 is attained in some point $\overline{x} \in \overline{B_{r/2}}$. Then, we define

$$u(x) := v(x) - v(\overline{x}).$$

Notice that u is s-harmonic in B_1 since so does v. Also, $u \geq 0$ in B_1 by construction, and $u > 0$ in $B_1 \setminus B_{r/2}$. On the other hand, since $\overline{x} \in B_r$

$$\inf_{B_r} u = u(\overline{x}) = 0,$$

which implies that u cannot satisfies an inequality such as (3.247).

As a matter of fact, in the fractional case, the analogue of the Harnack inequality requires u to be nonnegative in the whole of $\mathbb{R}^n$, hence a "global" condition is needed to obtain a "local" oscillation bound. See e.g. [Kas11] and the references therein for a complete discussion of nonlocal Harnack inequalities.

Example 3.27 (A logistic equation with nonlocal interactions)**.** We consider the logistic equation taken into account in [CDV17]

$$-(-\Delta)^s u + (\sigma - \mu u)u + \tau(J * u) = 0, \tag{3.248}$$

where $s \in (0,1]$, $\tau \in [0,+\infty)$ and σ, μ, J are nonnegative functions. The symbol $*$ denotes as usual the convolution product between J and u. Moreover, the convolution kernel J is assumed to be of unit mass and even, namely

$$\int_{\mathbb{R}^n} J(x)dx = 1$$

and

$$J(-x) = J(x) \quad \text{for any } x \in \mathbb{R}^n.$$

In this framework, the solution u denotes the density of a population living in some environment $\Omega \subseteq \mathbb{R}^n$, while the functions σ and μ model respectively the growing and dying effects on the population. The equation is driven by the fractional Laplacian that models a nonlocal dispersal strategy which has been observed experimentally in nature, and may be related to optimal hunting strategies and adaptation to the environment stimulated by natural selection.

We state here a result which translates the fact that a population with a nonlocal strategy can plan the distribution of resources in a strategic region better than populations with a local one.

Namely, fixed $\Omega = B_1$, one can find a solution of a slightly perturbed version of (3.248) in B_1, compactly supported in a larger ball B_{R_ϵ}, where $\epsilon \in (0,1)$ denotes the perturbation.

The strategic plan consists in properly adjusting the resources in $B_{R_\epsilon} \setminus B_1$ (that is, a bounded region in which the equation is not satisfied) in order to consume almost all the given resources in B_1.

The detailed statement goes as follows:

Theorem 3.28. *Let $s \in (0,1)$ and $\ell \in \mathbb{N}$, $\ell \geq 2$. Assume that $\sigma, \mu \in C^\ell(\overline{B_2})$, with*

$$\inf_{B_2} \mu > 0, \qquad \inf_{B_2} \sigma > 0.$$

Fixed $\epsilon \in (0,1)$, there exist a nonnegative function u_ϵ, $R_\epsilon > 2$ and $\sigma_\epsilon \in C^\ell(\overline{B_1})$ such that

$$(-\Delta)^s u_\epsilon = (\sigma_\epsilon - \mu u_\epsilon)u_\epsilon + \tau(J * u_\epsilon) \quad \textit{in } B_1,$$

$$u_\epsilon = 0 \quad \textit{in } \mathbb{R}^n \setminus B_{R_\epsilon},$$

$$\|\sigma_\epsilon - \sigma\|_{C^\ell(\overline{B_1})} \leq \epsilon,$$

$$u_\epsilon \geq \mu^{-1}\sigma_\epsilon \quad \textit{in } B_1.$$

Example 3.29. Higher order nonlocal equations also appear naturally in several contexts, see e.g. [CV13] for a nonlocal version of the Cahn-Hilliard phase coexistence model. Higher orders operators have also appeared in connection with logistic equations, see e.g. [Bha16]. In this spirit, we point out a version of Theorem 3.28 which is new and relies on Theorem 3.1. Its content is that nonlocal logistic equations (of any order and with nonlocality given in either time or space, or both) admits solutions which can arbitrarily well adapt to any given resource. The precise statement is the following:

Theorem 3.30. *Let $s \in (0,+\infty)$, $\alpha \in (0,+\infty)$ and $\ell \in \mathbb{N}$, $\ell \geq 2$. Assume that*

$$\text{either } s \notin \mathbb{N} \text{ or } \alpha \notin \mathbb{N}. \tag{3.249}$$

Let $\sigma, \mu \in C^\ell(\overline{B_2})$, with

$$\inf_{B_1} \mu > 0. \tag{3.250}$$

Fixed $\epsilon \in (0,1)$, there exist a nonnegative function u_ϵ, $R_\epsilon > 2$, $a_\epsilon < 0$, and $\sigma_\epsilon \in C^\ell(\overline{B_1})$ such that

$$\begin{gathered} D^\alpha_{t,a_\epsilon} u_\epsilon(x,t) + (-\Delta)^s u_\epsilon(x,t) = \Big(\sigma_\epsilon(x,t) - \mu(x,t) u_\epsilon(x,t)\Big) u_\epsilon(x,t) \\ \text{for all } (x,t) \in \mathbb{R}^p \times \mathbb{R} \text{ with } |(x,t)| < 1, \end{gathered} \tag{3.251}$$

$$u_\epsilon(x,t) = 0 \quad \text{if } |(x,t)| \geq R_\epsilon, \tag{3.252}$$

$$\|\sigma_\epsilon - \sigma\|_{C^\ell(\overline{B_1})} \leq \epsilon, \tag{3.253}$$

$$u_\epsilon = \mu^{-1}\sigma_\epsilon \geq \mu^{-1}\sigma - \epsilon \quad \text{in } B_1. \tag{3.254}$$

Proof. We use Theorem 3.1 in the case in which $\Lambda_a := D^\alpha_{t,a} + (-\Delta)^s$. Let $f := \sigma/\mu$. Then, by Theorem 3.1, which can be exploited here in view of (3.249), we obtain the existence of suitable u_ϵ, $R_\epsilon > 2$ and $a_\epsilon < 0$ satisfying (3.252),

$$\begin{gathered} D^\alpha_{t,a_\epsilon} u_\epsilon(x,t) + (-\Delta)^s u_\epsilon(x,t) = 0 \\ \text{for all } (x,t) \in \mathbb{R}^p \times \mathbb{R} \text{ with } |(x,t)| < 1, \end{gathered} \tag{3.255}$$

and

$$\|u_\epsilon - f\|_{C^\ell(\overline{B_1})} \leq \epsilon. \tag{3.256}$$

Then, we set $\sigma_\epsilon := \mu u_\epsilon$, and then, by (3.256),

$$\begin{aligned} \|\sigma_\epsilon - \sigma\|_{C^\ell(\overline{B_1})} &\leq C\, \|\mu\|_{C^\ell(\overline{B_1})} \left\| \frac{\sigma_\epsilon}{\mu} - \frac{\sigma}{\mu} \right\|_{C^\ell(\overline{B_1})} \\ &= C\, \|\mu\|_{C^\ell(\overline{B_1})}\, \|u_\epsilon - f\|_{C^\ell(\overline{B_1})} \\ &\leq C\, \|\mu\|_{C^\ell(\overline{B_1})}\, \epsilon, \end{aligned} \tag{3.257}$$

which gives (3.253), up to renaming ϵ.

Moreover, if $|(x,t)| < 1$,

$$(\sigma_\epsilon - \mu u_\epsilon) u_\epsilon = 0 = D^\alpha_{t,a_\epsilon} u_\epsilon + (-\Delta)^s u_\epsilon,$$

thanks to (3.255), and this proves (3.251).

In addition, recalling (3.257) and (3.250),

$$u_\epsilon = \mu^{-1}\sigma_\epsilon \geq \mu^{-1}\sigma - \frac{1}{\inf_{\overline{B_1}} \mu} \|\sigma - \sigma_\epsilon\|_{L^\infty(B_1)} \geq \mu^{-1}\sigma - \frac{\|\mu\|_{C^\ell(\overline{B_1})}\, \epsilon}{\inf_{\overline{B_1}} \mu},$$

in B_1, which proves (3.254), up to renaming ϵ. $\square$

3.14 Open Problems

Is it Theorem 3.1 true with the same techniques even for nonlocal linear operators involving more general kernels? Using the notation in formulae (3.4), (3.6) we have in mind something like

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} (\delta_h u)(x,y)K(y)dy,$$

and

$$\mathcal{D}_{a+}u(t) := \int_a^t u^{(k)}(\tau)H(t-\tau)d\tau$$

for some $h, k \in \mathbb{N}$, where K satisfies the following assumption

$$K(y) = \frac{J(y)}{|y|^{n+2s}},$$

for some $s \in (h-1,h)$ and some function J measurable, even, bounded between two positive constants $\lambda < \Lambda$ and positively homogeneous of degree zero. The kernel H satisfies

$$\frac{c_1}{z^{\alpha-k+1}} \leq H(z) \leq \frac{c_2}{z^{\alpha-k+1}},$$

for some $0 < c_1 < c_2$ and some $\alpha \in (k-1,k)$. The main difficulty is the lack of explicit representation formulae of Green functions and Poisson kernels which does not allow us to prove results in the fashion of the ones given in Sections 3.7, 3.8; nevertheless, sharp asymptotic results are proved in [BFV18, Gru15] for solutions of more general nonlocal linear equations. We notice that in [RS17b, Theorem 4] a quantitative approximation result is given, and it involves as nonlocal linear operator the fractional power of an elliptic operator in divergence form. Actually, using Runge-type approximation techniques, the authors prove that the approximation property is guaranteed for solutions of nonlocal operators given by a finite sum of general local operators and a nonlocal operator that satisfies a weak unique continuation principle. Anyway, in that paper the authors approximate functions in the Sobolev space H^1_0 and not in C^k, and they do not take into account time-fractional derivatives as we do.

Chapter 4

A note on Riemann-Liouville fractional Sobolev spaces

4.1 Introduction and main results

The goal of this chapter is to analyze in detail the connection of some functional spaces defined through Riemann-Liouville fractional operator with classical Sobolev and BV spaces on an interval $I = (a.b)$ of the real line.

Fractional integrals and derivatives arise in many contests such as viscoelasticity, neurobiology, finance and so on, see for instance [ACV16], [ABM16], [Cap08], [DPPZ13], [DV18]. A new recent approach to the problem, suitable also for the n-dimensional case, has been investigated in [CS18, CS19, SSVS17, SSS15, SSS18, SS15, SS18, Šil19, Spe18, Spe19].

There are many examples of such operators in literature. Among these ones, Riemann-Liouville and Caputo fractional derivatives are the most exploited in the applications.

Here, we give answers to some questions posed in [BLNT17]; namely, the chapter is structered in the following way: after an introduction of the topics treated in the chapter, this section is devoted to preliminary notions and to introduce two of the main results of the chapter; namely the fact that $BV(I)$ and $W^{s,1}(I)$, continuously embed into $W^{s,1}_{RL,a+}(I)$. The continuity of these embeddings can be useful in many variational models involving this kind of fractional operators. Moreover, we extend [BLNT17, Theorem 4.1] from SBV to BV proving that $D^s_{a+}[u]\mathcal{L}^1 \rightharpoonup Du\mathcal{L}^1 + u(a+)\delta_a$ in $\mathcal{M}(\overline{I})$ as $s \to 1^-$. We notice that by proving 4.2 we recover the continuity given by 4.1, but the proofs of these two results exploit different techniques that, according to the authors, make both results interesting in their own way. We conclude the section with the proof of Theorem 4.1.

In the second section we extend with a density result the Marchaud representation formula for fractional derivatives to functions in the fractional Sobolev space $W^{s,1}(I)$; this fact, joint with a fractional Hardy-type inequality, allows us to prove Proposition 4.2. We conclude this section with a counterexample that deny Proposition 4.2 in the case of unbounded intervals.

In the third section we introduce the space $BV^s_{RL,a+}(I)$ given by the functions in $L^1(I)$ with $(1-s)$ fractional integral in $BV(I)$, and we analyze it in detail proving that it contains $W^{s,1}_{RL,a+}(I)$ and hence, thanks to Theorem 4.1 also $BV(I)$; moreover, we show through some examples that despite the regularization properties of the fractional integral, the measure D^s_{a+} it's not in general absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$, as one can expects for BV functions.

In the fourth section we study the continuity of the $(1-s)$ fractional integral in the Sobolev space $W^{1,p}(I)$ for $1 \leq p < \infty$. In the case $p = \infty$, we show through a simple example that if the function does not vanish in the initial point, its Riemann-Liouville fractional derivative cannot be essentially bounded, even if the function is locally analytic. We conclude obtaining as a corollary the well known result on the inclusion relations between Riemann-Liouville fractional Sobolev spaces.

In the fifth section we extend some results obtained in the previous sections taking into account higher order fractional derivatives; namely, continuity of the fractional integral between Sobolev spaces of greater integer order, and the inclusion of the space of functions with bounded Hessian in a higher order Riemann-Liouville space are proved.

We conclude the chapter with some open problems.

We point out that from the study of Riemann-Liouville fractional integrals done in the last century have been carried out many beautiful results about Riesz potential, which is a fundamental tool in linear potential Theory. In particular we mention the paper by Hardy and Littlewood, [HL28] in which the authors prove the continuity of the s-fractional integral from $L^p(I)$ into $L^q(I)$ where q is the critical exponent $q = p_s^* := \frac{p}{1-sp}$, provided $1 < p < 1/s$. This result holds even if one replace the bounded interval I with an half-line or the whole of $\mathbb{R}$. Indeed, It is worth noticing that for $f \in L^1(\mathbb{R})$, we may define the following "improper" fractional integrals:

$$\begin{aligned} I^s_{-\infty}[f](x) &:= \frac{1}{\Gamma(s)} \int_{-\infty}^{x} \frac{f(t)}{(x-t)^{1-s}}\, dt, \\ I^s_{+\infty}[f](x) &:= \frac{1}{\Gamma(s)} \int_{x}^{+\infty} \frac{f(t)}{(t-x)^{1-s}}\, dt. \end{aligned}$$

Up to a translation, it is immediate to check that

$$I^s_{-\infty}[f](x) = \frac{1}{\Gamma(s)} \int_{-\infty}^{0} \frac{f(x+t)}{|t|^{1-s}}\, dt, \tag{4.1}$$

$$I^s_{+\infty}[f](x) = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} \frac{f(x+t)}{t^{1-s}}\, dt. \tag{4.2}$$

It is well known that the function

$$u(x) := I^s_{-\infty}[f](x) + I^s_{+\infty}[f](x) = \frac{1}{\Gamma(s)} \int_{-\infty}^{+\infty} \frac{f(t)}{|x-t|^{1-s}} dt =: I^s[f](x), \tag{4.3}$$

is a distributional solution of

$$(-\Delta)^{s/2} u = f, \tag{4.4}$$

where, up to multiplicative constants, the operator I^s in (4.3) denotes the one-dimensional Riesz potential with parameter s, while the left hand side in (4.4) denotes the one-dimensional fractional Laplacian with parameter $s/2$, namely

$$\begin{aligned} (-\Delta)^{s/2} u(x) &:= \frac{2^{s-1}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+s}{2}\right)}{\left|\Gamma\left(-\frac{s}{2}\right)\right|} \int_{-\infty}^{+\infty} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{s+1}} dy \\ &= \frac{2^{s}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+s}{2}\right)}{\left|\Gamma\left(-\frac{s}{2}\right)\right|} \int_{-\infty}^{+\infty} \frac{u(x) - u(x+y)}{|y|^{s+1}} dy, \end{aligned}$$

where the last integral has to be intended in the Cauchy principal value sense. Indeed, by applying the Fourier transform $\mathcal{F}$ to equation (4.4), one has that

$$|\xi|^s \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi).$$

Multiplying by $|\xi|^{-s}$ and applying the inverse Fourier transform $\mathcal{F}^{-1}$, one has that

$$u(x) = \mathcal{F}^{-1}(|\cdot|^{-s}\mathcal{F}(f)(\cdot))(x) = \mathcal{F}^{-1}(|\cdot|^{-s})(x) * f(x),$$

where

$$\begin{aligned} \mathcal{F}^{-1}(|\cdot|^{-s})(x) &= \int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{|\xi|^s} d\xi = 2|x|^{s-1} \int_0^{+\infty} \frac{\cos(y)}{y^s} dy \\ &= 2\Gamma(1-s)\sin\left(\frac{s\pi}{2}\right)|x|^{s-1}. \end{aligned} \tag{4.5}$$

In addition, let us define the "improper" left and right Riemann-Liouville fractional derivatives of u at ∞ as

$$\begin{aligned} D^s_{-\infty}[u](x) &:= \frac{d}{dx} I^{1-s}_{-\infty}[u](x), \\ D^s_{+\infty}[u](x) &:= -\frac{d}{dx} I^{1-s}_{+\infty}[u](x). \end{aligned}$$

Then, if we consider the case $n = 1$ in the notion of fractional gradient

$$\begin{aligned} \nabla^s u(x) &:= \frac{2^s}{\sqrt{\pi}} \frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \int_{-\infty}^{+\infty} \frac{u(y)-u(x)}{|y-x|^{1+s}} \operatorname{sgn}(y-x)\, dy \\ &= \frac{2^{s-1}}{\sqrt{\pi}} \frac{s\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \int_{-\infty}^{+\infty} \frac{u'(y)}{|y-x|^s}\, dy, \end{aligned}$$

it is easy to see that

$$D^s_{-\infty}[u](x) - D^s_{+\infty}[u](x) = \frac{1}{\Gamma(1-s)} \int_{-\infty}^{+\infty} \frac{u'(x+y)}{|y|^s}\, dy = \mu_s \nabla^s u(x),$$

where

$$\mu_s := \frac{\sqrt{\pi}}{2^{s-1}\Gamma(1-s)} \frac{\Gamma\left(\frac{1-s}{2}\right)}{s\Gamma\left(\frac{s}{2}\right)}.$$

Moreover, using the equivalent Marchaud formulation one has that

$$\begin{aligned} D^s_{-\infty}[u](x) + D^s_{+\infty}[u](x) &= \frac{s}{\Gamma(1-s)} \int_0^{+\infty} \frac{2u(x)-u(x+y)-u(x-y)}{y^{s+1}} d\tau \\ &= \frac{s}{2\Gamma(1-s)} \int_{-\infty}^{+\infty} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{s+1}} dy \\ &= \frac{1}{2c_s} \frac{s}{\Gamma(1-s)} (-\Delta)^{s/2} u(x), \end{aligned}$$

where $c_s := \left(\int_{-\infty}^{+\infty} \frac{1-\cos(\omega)}{|\omega|^{2s+1}} d\omega \right)^{-1}$.

We conclude this introduction only noticing that the renormalized fractional gradient $\mu_s\nabla^s u(x)$ can be seen as the convolution between the weak first derivative u' and the tempered distribution $T := \frac{\text{P.V.}}{|\cdot|^s}$, where P.V. denotes the Cauchy principal value. Therefore, if we compute the Fourier transform of $\mu_s\nabla^s u(x)$ we have that for any $\xi \in \mathbb{C}$

$$\begin{aligned}\mu_s\mathcal{F}(\nabla^s u)(\xi) &= \frac{1}{\Gamma(1-s)}\mathcal{F}(u' * T)(\xi) = \frac{1}{\Gamma(1-s)}\mathcal{F}(u')(\xi)\mathcal{F}(T)(\xi)\\ &= \frac{1}{\Gamma(1-s)} i\xi\mathcal{F}(u)(\xi)\mathcal{F}(T)(\xi).\end{aligned}$$

Through analogous calculations as in formula (4.5), we get

$$\mathcal{F}(T)(\xi) = 2\Gamma(1-s)\sin(\frac{s\pi}{2})|\xi|^{s-1}.$$

Therefore

$$\mu_s\mathcal{F}(\nabla^s u)(\xi) = 2i\sin(\frac{s\pi}{2})|\xi|^{s-1}\xi\mathcal{F}(u)(\xi), \tag{4.6}$$

which means that $\nabla^s u$ is a Fourier multiplier with symbol $a(\xi) := 2\mu_s^{-1} i\sin(\frac{s\pi}{2})|\xi|^{s-1}\xi$.

It is also worth noticing that $|a(\xi)| \approx |\xi|^s$, which is exactly the symbol of the $s/2$-fractional Laplacian $(-\Delta)^{s/2}$. This fact is strictly related with the celebrated Kato's square root problem. See e.g. [AHL+02], where the problem is treated for a more general class of elliptic operators.

Now we are ready to claim the main statements of this chapter

Theorem 4.1. *Let $u \in BV(I)$. Then we have that $BV(I) \hookrightarrow W^{s,1}_{RL,a+}(I)$ for any $s \in (0,1)$, with*

$$\|u\|_{W^{s,1}_{RL,a+}(I)} \le \max\left\{1 + \frac{(b-a)^{-s}}{\Gamma(2-s)}, \frac{2(b-a)^{1-s}}{\Gamma(2-s)}\right\}\|u\|_{BV(I)}. \tag{4.7}$$

In addition

$$D^s_{a+}[u]\mathcal{L}^1 \rightharpoonup Du + u(a+)\delta_a \quad as \quad s \to 1^- \quad in \quad \mathcal{M}(\overline{I}). \tag{4.8}$$

Proposition 4.2. *For any $s \in (0,1)$ and any bounded open interval I, the embedding $W^{s,1}(I) \hookrightarrow W^{s,1}_{RL,a+}(I)$ is continuous.*

We start with a technical result concerning the action of the fractional integral on $\mathcal{M}(I)$.We notice that in the sequel with the notation $\int_a^x$ we refer to the integral on the open interval (a,x). However, thanks to the nonconcentration properties of Radon measures μ we have that $\mu(\{x\}) = 0$ for all but countably many $x \in (a,b)$. As a consequence there would be no ambiguity when integrating $I^s_{a+}[\mu]$ with respect $\mathcal{L}^1$.

Proposition 4.3. *Let $s \in (0,1)$. The map I^s_{a+} can be continuously extended to a map from $\mathcal{M}(I)$ into $L^1(I)$, by setting*

$$I^s_{a+}[\mu](x) := \frac{1}{\Gamma(s)}\int_a^x \frac{d\mu(t)}{(x-t)^{1-s}}.$$

Then, I^s_{a+} satisfies the following bound:

$$\left\|I^s_{a+}[\mu]\right\|_{L^1(I)} \le \frac{(b-a)^s}{\Gamma(1+s)}|\mu|(I), \tag{4.9}$$

for any $\mu \in \mathcal{M}(I)$.

Proof. Since the function $(x-t)^{s-1}$ is continuous in $t \in (a,x)$, for any fixed $x \in (a,b)$, we are allowed to integrate this function against any nonnegative $\mu \in \mathcal{M}(I)$. Hence,

$$I_{a+}^{s}[\mu](x) := \frac{1}{\Gamma(s)} \int_a^x \frac{d\mu(t)}{(x-t)^{1-s}}$$

is well defined for $\mu \geq 0$. Then, a simple computation similar to the one in Remark 2.14 shows that

$$\begin{aligned}
\left\| I_{a+}^{s}[\mu] \right\|_{L^1(I)} &= \int_a^b \left| I_{a+}^{s}[\mu](x) \right| dx = \frac{1}{\Gamma(s)} \int_a^b \left| \int_a^x \frac{d\mu(t)}{(x-t)^{1-s}} \right| dx = \frac{1}{\Gamma(s)} \int_a^b \int_a^x \frac{d\mu(t)}{(x-t)^{1-s}}\, dx \\
&= \frac{1}{\Gamma(s)} \int_a^b d\mu(t) \int_t^b \frac{dx}{(x-t)^{1-s}} = \frac{1}{s\Gamma(s)} \int_a^b (b-t)^s d\mu(t) \\
&\leq \frac{(b-a)^s}{\Gamma(1+s)} \int_a^b d\mu(t) = \frac{(b-a)^s}{\Gamma(1+s)} \mu(I).
\end{aligned}$$

In the general case of $\mu \in \mathcal{M}(I)$, we consider the Jordan decomposition $\mu = \mu^+ - \mu^-$ and we set

$$I_{a+}^{s}[\mu](x) := I_{a+}^{s}[\mu^+](x) - I_{a+}^{s}[\mu^-](x) = \frac{1}{\Gamma(s)} \int_a^x \frac{d\mu(t)}{(x-t)^{1-s}},$$

by the linearity of the integral. Therefore one has that, for any $\mu \in \mathcal{M}(I)$

$$\left\| I_{a+}^{s}[\mu] \right\|_{L^1(I)} \leq \frac{1}{\Gamma(s)} \int_a^b \int_a^x \frac{d|\mu|(t)}{(x-t)^{1-s}}\, dx \leq \frac{(b-a)^s}{\Gamma(1+s)} |\mu|(I),$$

which ends the proof. □

It is not difficult to see that 2.16 can be extended to couples of measures and essentially bounded functions.

Lemma 4.4. *Let $\mu \in \mathcal{M}(I)$, $\phi \in L^\infty(I)$ and $s \in (0,1)$. Then we have*

$$\int_a^b I_{a+}^{s}[\mu](x)\, \phi(x)\, dx = \int_a^b I_{b-}^{s}[\phi](x)\, d\mu(x). \tag{4.10}$$

Proof. Notice that, by Proposition 2.18, $I_{b-}^{s}[\phi] \in C^{0,s}(I)$, so that it is continuous and bounded, in particular. This implies that the integral in the right hand side of (4.10) is well defined. In addition, notice that

$$\begin{aligned}
\int_a^b \int_a^x \frac{|\phi(x)|}{(x-t)^{1-s}}\, d|\mu|(t)\, dx &\leq \|\phi\|_{L^\infty(I)} \int_a^b \int_t^b (x-t)^{s-1}\, dx\, d|\mu|(t) \\
&\leq \|\phi\|_{L^\infty(I)} \frac{(b-a)^s}{s} |\mu|(I) < \infty.
\end{aligned}$$

Therefore, we may apply Fubini's theorem, and we obtain

$$\begin{aligned}
\int_a^b I_{a+}^{s}[\mu](x)\, \phi(x)\, dx &= \frac{1}{\Gamma(s)} \int_a^b \int_a^x \phi(x) \frac{d\mu(t)}{(x-t)^{1-s}}\, dx = \frac{1}{\Gamma(s)} \int_a^b \int_t^b \frac{\phi(x)}{(x-t)^{1-s}}\, dx\, d\mu(t) \\
&= \int_a^b I_{b-}^{s}[\phi](t)\, d\mu(t).
\end{aligned}$$

□

Now, we focus on formula (2.15). We notice that, as a byproduct of the proof of [BLNT17, Theorem 3.3], this relation has been already extended to Sobolev functions. In view of this fact, our goal is to extend it to BV functions; by doing so, we also immediately prove the inclusion of $BV(I)$ in $W^{s,1}_{RL,a+}(I)$.

Proposition 4.5. *Let $u \in BV(I)$. Then, for any $s \in (0,1)$, we have that*

$$D^s_{a+}[u](x) = I^{1-s}_{a+}[Du](x) + \frac{1}{\Gamma(1-s)} \frac{u(a+)}{(x-a)^s}. \tag{4.11}$$

Proof. By Remark 2.14, we obtain immediately that $I^{1-s}_{a+} u \in L^1(I)$, since $u \in L^1(I)$. Let us now assume that $u \in AC(\overline{I})$. For any $x \in (a,b)$, formula (2.15) yields

$$\frac{d}{dx} I^{1-s}_{a+}[u](x) = I^{1-s}_{a+}[u'](x) + \frac{1}{\Gamma(1-s)} \frac{u(a)}{(x-a)^s}.$$

Now, let $u \in BV(I)$ and let $\rho \in C^\infty_c((-1,1))$ be a standard mollifier. It is well known that $\rho_\varepsilon * \tilde{u} \in C^\infty(I) \cap BV(I)$, so that $\rho_\varepsilon * \tilde{u} \in W^{1,1}(I) \subset AC(\overline{I})$, in particular. Then, for any $\phi \in C^1_c(I)$ we have

$$\int_a^b I^{1-s}_{a+}[\rho_\varepsilon * u]\, \phi' \, dx = -\int_a^b \left(I^{1-s}_{a+}[\rho_\varepsilon * Du] + \frac{1}{\Gamma(1-s)} \frac{(\rho_\varepsilon * u)(a)}{(x-a)^s} \right) \phi \, dx.$$

By (2.8), we get

$$\int_a^b I^{1-s}_{a+}[\rho_\varepsilon * Du]\, \phi \, dx = \int_a^b I^{1-s}_{b-}[\phi]\, (\rho_\varepsilon * Du) \, dx.$$

Then, since $I^{1-s}_{b-}[\phi]$ is continuous and bounded by Proposition 2.18, by [AFP00, Proposition 1.62] and (4.10), we get

$$\int_a^b I^{1-s}_{b-}[\phi]\, (\rho_\varepsilon * Du) \, dx \to \int_a^b I^{1-s}_{b-}[\phi] \, dDu = \int_a^b \phi \, I^{1-s}_{a+}[Du] \, dx.$$

On the other hand, we also obtain

$$\begin{aligned} \int_a^b I^{1-s}_{a+}[\rho_\varepsilon * u]\, \phi' \, dx &= \int_a^b (\rho_\varepsilon * u)\, I^{1-s}_{b-}[\phi'] \, dx \\ &\to \int_a^b u \, I^{1-s}_{b-}[\phi'] \, dx = \int_a^b I^{1-s}_{a+}[u]\, \phi' \, dx, \end{aligned}$$

by (2.8) and Lebesgue's dominated convergence theorem, since $I^{1-s}_{b-}[\phi'] \in L^1(I)$ and

$$|\rho_\varepsilon * u| \le \|u\|_{L^\infty(I)} \le C_{a,b} \|u\|_{BV(I)},$$

by (2.2). Now, since $(\rho_\varepsilon * u)(a) \to u(a+)$ by (2.3), and $\rho_\varepsilon * Du \rightharpoonup Du$ in $\mathcal{M}(I)$, we get

$$\begin{aligned} \int_a^b I^{1-s}_{a+}[u](x)\, \phi'(x) \, dx &= \lim_{\varepsilon \to 0} \int_a^b I^{1-s}_{a+}[\rho_\varepsilon * u](x)\, \phi'(x) \, dx \\ &= -\lim_{\varepsilon \to 0} \int_a^b I^{1-s}_{b-}[\phi](x)\, (\rho_\varepsilon * Du)(x) + \frac{1}{\Gamma(1-s)} \frac{(\rho_\varepsilon * u)(a)}{(x-a)^s} \phi(x) \, dx \\ &= -\int_a^b I^{1-s}_{b-}[\phi](x) \, dDu(x) - \frac{1}{\Gamma(1-s)} \int_a^b \frac{u(a+)}{(x-a)^s} \phi(x) \, dx \\ &= -\int_a^b \left(I^{1-s}_{a+}[Du](x) + \frac{1}{\Gamma(1-s)} \frac{u(a+)}{(x-a)^s} \right) \phi(x) \, dx, \end{aligned}$$

which yields (4.11). ∎

4.1.1 Proof of Theorem 4.1

Proof of Theorem 4.1. Thanks to Proposition 4.5, if $u \in BV(I)$ then $D^s_{a+}u \in L^1(I)$, and so $u \in W^{s,1}_{RL,a+}(I)$, since

$$\left\|D^s_{a+}u\right\|_{L^1(I)} \leq \frac{(b-a)^{1-s}}{\Gamma(2-s)}(|Du|(I) + |u(a+)|)$$

by (4.9). Then, it is clear that $|u(a+)| \leq \|u\|_{L^\infty(I)}$ and so, thanks to (2.2), we get

$$\begin{aligned}
\|u\|_{W^{s,1}_{RL,a+}(I)} &= \|u\|_{L^1(I)} + \left\|D^s_{a+}u\right\|_{L^1(I)} \leq \|u\|_{L^1(I)} + \frac{(b-a)^{1-s}}{\Gamma(2-s)}\left(|Du|(I) + |u^*(a)|\right) \\
&\leq \|u\|_{L^1(I)} + \frac{(b-a)^{1-s}}{\Gamma(2-s)}\left(|Du|(I) + \frac{1}{b-a}\|u\|_{L^1(I)} + |Du|(I)\right) \\
&= \left(1 + \frac{(b-a)^{-s}}{\Gamma(2-s)}\right)\|u\|_{L^1(I)} + \frac{2(b-a)^{1-s}}{\Gamma(2-s)}|Du|(I),
\end{aligned}$$

which easily implies (4.7) and the continuity of the embedding $BV(I) \hookrightarrow W^{s,1}_{RL,a+}(I)$. To prove the second part of the claim, we exploit (4.11) in order to obtain

$$\begin{aligned}
\int_a^b D^s_{a+}[u](x)\phi(x)dx &= \int_a^b I^{1-s}_{a+}[Du](x)\phi(x)dx + \frac{u(a+)}{\Gamma(1-s)}\int_a^b \frac{\phi(x)}{(x-a)^s}dx \\
&= \int_a^b I^{1-s}_{b-}[\phi](x)dDu(x) + \\
&\quad + \frac{u(a+)}{\Gamma(2-s)}\left(\phi(b)(b-a)^{1-s} - \int_a^b \phi'(x)(x-a)^{1-s}dx\right),
\end{aligned}$$

where $I^{1-s}_{b-}[\phi] \in C^{0,s}(I)$, by Proposition 2.18. Therefore, by Lemma 2.17 and Lebesgue's dominated convergence theorem, we get

$$\lim_{s \to 1^-}\int_a^b D^s_{a+}[u](x)\phi(x)dx = \int_a^b \phi(x)dDu(x) + u(a+)\phi(a).$$

Then, the claim plainly follows by the density of $C^1(\overline{I})$ in $C(\overline{I})$. □

Remark 4.6. *We notice that, thanks to* [BLNT17, Theorem 4.1]*, one can alternatively prove* (4.8) *by showing only that $D^s_{a+}[u_c] \rightharpoonup Du_c$ as $s \to 1^-$ in $\mathcal{M}(I)$, where u_c is the Cantor-type function such that $u = u_{ac} + u_j + u_c$. Indeed, if $\varphi \in C^1_c(I)$, we have that*

$$\begin{aligned}
&\int_a^b \varphi(x)D^s_{a+}[u_c](x)dx = \int_a^b \varphi(x)I^{1-s}_{a+}[Du_c](x)dx = \int_a^b I^{1-s}_{b-}[\varphi](x)dDu_c(x) = \\
&\int_a^b dDu_c(x)\left[\frac{1}{\Gamma(2-s)}\left(\varphi(b) - \int_x^b \varphi'(t)(t-x)^{1-s}dt\right)\right],
\end{aligned} \tag{4.12}$$

now, when $s \to 1^-$, the last integral approaches to

$$\int_a^b \varphi(x)dDu_c(x).$$

Eventually, we conclude using the density of $C^1_c(I)$ in $C_c(I)$.

4.2 Marchaud fractional derivative for functions in $W^{s,1}(I)$

Now, we prove that the Marchaud fractional derivative is well defined even if u is merely in the fractional Sobolev space $W^{s,1}(I)$; this is a key tool in the proof of Proposition 4.2

Lemma 4.7. *Let $s \in (0,1)$. If $u \in W^{s,1}(I)$, then the Marchaud fractional derivative ${}^M D^s_{a+}[u]$ is well defined and coincide a.e. with $D^s_{a+}[u]$.*

Proof. If $u \in W^{s,1}(I) \cap C^1(I)$, formula (2.17) holds true.

Otherwise, if $u \in W^{s,1}(I)$, we exploit the density of $C^1_c(I)$ in $W^{s,1}(I)$ (see Remark 2.11), which means that there exists a sequence u_n in $C^1_c(I)$ such that $\|u_n - u\|_{W^{s,1}(I)} \to 0$ as $n \to +\infty$. Now, we prove that, up to a subsequence,

$$D^s_{a+}[u_n](x) = \frac{1}{\Gamma(1-s)} \frac{u_n(x)}{(x-a)^s} + \frac{s}{\Gamma(1-s)} \int_a^x \frac{u_n(x) - u_n(t)}{(x-t)^{s+1}} dt$$

converges pointwise $\mathcal{L}^1$-a.e. in I to $D^s_{a+}[u](x)$.

For the second term in the right hand side, we proceed as follows: we set

$$f_n(x) := \int_a^x \frac{u_n(x) - u_n(t) - u(x) + u(t)}{(x-t)^{s+1}} dt.$$

The sequence f_n converges to 0 in $L^1(I)$. Indeed

$$\int_a^b |f_n(x)| dx \le [u_n - u]_{W^{s,1}(I)} \le \|u_n - u\|_{W^{s,1}(I)} \to 0 \quad \text{as} \quad n \to +\infty.$$

Therefore, up to a subsequence, f_n converges pointwise $\mathcal{L}^1$-a.e. to 0 in I, so that

$$\lim_{n\to+\infty} \int_a^x \frac{u_n(x) - u_n(t)}{(x-t)^{s+1}} dt = \int_a^x \frac{u(x) - u(t)}{(x-t)^{s+1}} dt$$

for $\mathcal{L}^1$-a.e. $x \in I$. Conversely, for the first term in the right hand side, up to a subsequence, we have convergence $\mathcal{L}^1$-a.e. in I thanks to the convergence of u_n to u in $W^{s,1}(I)$ and hence in $L^1(I)$, which implies pointwise convergence $\mathcal{L}^1$-a.e., up to a subsequence.

For the L^1 convergence, we argue as follows: employing the fractional Hardy inequality 2.9 with $n = p = 1$ and $\Omega = (a,b)$, we get

$$\int_a^b \frac{|u_n(x) - u(x)|}{(x-a)^s} dx \le \int_a^b \frac{|u_n(x) - u(x)|}{|\delta_I(x)|^s} dx \le C \|u_n - u\|_{W^{s,1}(I)} \to 0 \quad \text{as} \quad n \to +\infty,$$

where $|\delta_I(x)| = \min\{x-a, b-x\}$. To conclude, we notice that, for any $\phi \in C^1_c(I)$ it holds that

$$\begin{aligned}\int_a^b {}^M D^s_{a+}[u_n](x)\phi(x) dx &= \int_a^b D^s_{a+}[u_n](x)\phi(x) dx \\ &= -\int_a^b I^{1-s}_{a+}[u_n](x)\phi'(x) dx \to -\int_a^b I^{1-s}_{a+}[u](x)\phi'(x) dx,\end{aligned}$$

since $u_n \to u$ in $L^1(I)$ and I^{1-s}_{a+} is continuous from $L^1(I)$ to $L^1(I)$. On the other hand, we have just proved that ${}^M D^s_{a+}[u_n] \to {}^M D^s_{a+}[u]$ in $L^1(I)$, and so we conclude

$$\int_a^b {}^M D^s_{a+}[u](x)\phi(x) dx = -\int_a^b I^{1-s}_{a+}[u](x)\phi'(x) dx,$$

and this implies $u \in W^{s,1}_{RL,a+}(I)$ with ${}^M D^s_{a+}[u](x) = D^s_{a+}[u](x)$ for a.e. $x \in I$. □

Remark 4.8. *We notice that Hölder inequality does not work in the last computation, so that we need to employ the fractional Hardy inequality; indeed, since $u_n - u \in W^{s,1}(I)$, thanks to fractional Sobolev embedding Theorem (see e.g.*[DNPV12, Theorem 6.7.]*) we have that $u_n - u \in L^q(I)$ for any $q \in \left[1, \frac{1}{1-s}\right]$.*

Therefore, we get

$$\int_a^b \frac{|u_n(x) - u(x)|}{(x-a)^s} dx \leq \left(\int_a^b |u_n(x) - u(x)|^q dx\right)^{1/q} \left(\int_a^b \frac{dx}{(x-a)^{sq'}}\right)^{1/q'}.$$

Now, $q \leq \frac{1}{1-s}$, implies $sq' \geq 1$, and so

$$\int_a^b \frac{dx}{(x-a)^{sq'}} = +\infty,$$

and thus this estimate is not useful.

4.2.1 Proof of Proposition 4.2

Proof of Proposition 4.2. Since $u \in L^1(I)$, in particular, we have $I_{a+}^{1-s}[u] \in L^1(I)$ by (4.9) applied to $\mu = u\mathcal{L}^1$.

Thanks to Lemma 4.7, we have that the Riemann Liouville fractional derivative of u coincides with the Marchaud one, and so

$$D_{a+}^s[u](x) = \frac{1}{\Gamma(1-s)} \frac{u(x)}{(x-a)^s} + \frac{s}{\Gamma(1-s)} \int_a^x \frac{u(x) - u(t)}{(x-t)^{s+1}} dt$$

For the second right hand side term, it holds that

$$\int_a^b dx \left| \int_a^x \frac{u(x) - u(t)}{(x-t)^{s+1}} dt \right| \leq [u]_{W^{s,1}(I)}. \tag{4.13}$$

While for the first term, using Lemma 2.9 with $n = p = 1$, and $\Omega = (a, b)$ we have that

$$\int_a^b \frac{|u(x)|}{(x-a)^s} dx \leq \int_a^b \frac{|u(x)|}{|\delta_I(x)|^s} dx \leq C \|u\|_{W^{s,1}(I)}. \tag{4.14}$$

Eventually, using (4.9), (4.13) and (4.14), we obtain that there exists a positive constant $C = C(s, a, b)$ such that,

$$\|u\|_{W^{s,1}_{RL,a+}(I)} \leq C \|u\|_{W^{s,1}(I)}.$$

□

Actually, the inclusion of Proposition 4.2 is strict, as the following result shows

Proposition 4.9. *The space $W^{s,1}_{RL,a+}(I)$ strictly contains $W^{s,1}(I)$.*

Proof. Without loss of generality, let $I := (0, 1)$. We claim that the function $u(x) := x^{s-1} \in W^{s,1}_{RL,0+}(I) \setminus W^{s,1}(I)$. Clearly, $x^{s-1} \in L^1(I)$. Now we prove that $u \in W^{s,1}_{RL,0+}(I) \setminus W^{s,1}(I)$. Indeed, it is easy to check that

$$I_{0+}^{1-s}[u](x) = \Gamma(s),$$

and so $I_{0+}^{1-s}[u] \in W^{1,1}(I)$, which shows that $u \in W^{s,1}_{RL,0+}(I)$. Then, we need to prove that the Gagliardo-Slobodeckij seminorm of u is infinite. We see that

$$\begin{aligned}
[u]_{W^{s,1}(I)} &:= \int_0^1 \int_0^1 \frac{|x^{s-1} - y^{s-1}|}{|x-y|^{s+1}}\, dx\, dy = [x = yz] = \int_0^1 \int_0^{\frac{1}{y}} \frac{|z^{s-1} - 1| y^{s-1}}{|z-1|^{s+1} y^{s+1}} y\, dz\, dy \\
&= \int_0^\infty \int_0^{\min\left\{1, \frac{1}{z}\right\}} \frac{|z^{s-1} - 1|}{|z-1|^{s+1} y}\, dy\, dz \\
&= \int_0^1 \int_0^1 \frac{|z^{s-1} - 1|}{|z-1|^{s+1} y}\, dy\, dz + \int_1^\infty \int_0^{\frac{1}{z}} \frac{|z^{s-1} - 1|}{|z-1|^{s+1} y}\, dy\, dz = +\infty,
\end{aligned}$$

since $1/y \notin L^1((0,\delta))$, for any $\delta > 0$. □

Remark 4.10. *We notice that Proposition 4.2 does not hold for unbounded intervals; indeed, the functions $u(x) := \frac{1}{x^2}$ belongs to $W^{1,1}((1,+\infty))$, therefore $u \in W^{1/2,1}((1,+\infty))$, but we have that*

$$I_{1+}^{1/2}[u](x) = \frac{1}{\sqrt{\pi}} \left(\frac{\log(x) + 2\log\left(1 + \sqrt{\frac{x-1}{x}}\right)}{2x^{3/2}} + \frac{\sqrt{x-1}}{x} \right) \notin L^1((1,+\infty))$$

This example says also that the continuity of the fractional integral in $L^p(I)$ for $1 \le p \le 2$ fails if I is an unbounded interval.

Recalling that $W^{s,1}(I)$ coincides with the Besov space $B^s_{1,1}(I)$ (see Appendix A.2 for basic notions on real Interpolation Theory), we can extend Proposition 4.2 as follows

Corollary 4.11. *Let $0 < s < r < 1$, $1 \le p \le \infty$ and $1 \le q \le \infty$. It holds that the embedding $B^r_{p,q}(I) \hookrightarrow W^{s,1}_{RL;a+}(I)$ is continuous.*

Proof. It is sufficient to use Proposition 4.2 and Proposition A.6. □

Remark 4.12. *Unfortunately, Corollary 4.11 does not cover the case $r = s$ for any choice of p and q. Therefore, in the particular case $p = q$ we are unable to conclude that Proposition 4.2 extends to $W^{s,p}(I)$ for any $p > 1$. Indeed in [MS15] the authors prove that for $1 \le q < p \le \infty$ and $s > 0$, s not an integer, $W^{s,p}(I) \not\subseteq W^{s,q}(I)$.*

It is interesting to notice that some inequalities of Poincarè type hold true in the fractional context, for which we refer for instance to [Ana09, Chapter 17]. However, in general it is not possible to retrieve the classical Poincarè inequality by estimating the L^p norm of the difference between u and its average with the L^p norm of its Riemann-Liouville derivative. To this purpose, given a function $u \in L^1(I)$, we denote by u_I the mean of u on the interval I; i.e.

$$u_I := \frac{1}{b-a} \int_a^b u(x) dx.$$

Now consider for instance $u(x) := (x-a)^{s-1}$ for $x \in (a,b)$ and for some $s \in (0,1)$; we have that $u \in L^p(I)$ if and only if $p < \frac{1}{1-s}$, and, by the calculations in the proof of Proposition 4.9, $I_{a+}^{1-s}[u](x) = \Gamma(s)$ and $D^s_{a+}[u](x) = 0$ for any $x \in (a,b)$, so that $u \in W^{s,p}_{RL,a+}(I)$ for any $p \in [1, 1/(1-s))$. Thus, being u not constant, we cannot hope for any sort of Poincaré inequality.

4.3 The space $BV^s_{RL,a+}(I)$

In analogy with the previous definition of left Riemann-Liouville fractional Sobolev spaces, we introduce now the natural extension to the BV framework.

Definition 4.13. *Let $s \in (0,1)$. We define the space of* functions with left Riemann-Liouville fractional bounded variation *as*

$$BV^s_{RL,a+}(I) := \left\{u \in L^1(I),\ I^{1-s}_{a+}[u] \in BV(I)\right\}.$$

It is easy to see that u belongs to $BV^s_{RL,a+}(I)$ if and only if there exists a measure $\mu \in \mathcal{M}(I)$ satisfying

$$\int_a^b I^{1-s}_{a+}[u](x)\,\phi'(x)\,dx = -\int_a^b \phi(x)\,d\mu(x)$$

for any $\phi \in C^1_c(I)$, and we call $DI^{1-s}_{a+}[u] := \mu$ the weak left Riemann-Liouville s-fractional derivative.

It is not difficult to see that the space $BV^s_{RL,a+}(I)$, endowed with the norm

$$\|u\|_{BV^s_{RL,a+}(I)} := \|u\|_{L^1(I)} + \left\|I^{1-s}_{a+}[u]\right\|_{BV(I)},$$

is a Banach space.

Arguing analogously as in Lemma 2.29, we derive a duality relation between the left Riemann-Liouville weak s-fractional derivative and the right Caputo s-fractional derivative.

Corollary 4.14. *A function $u \in L^1(I)$ belongs to $BV^s_{RL,a+}(I)$ if and only if there exists $\mu \in \mathcal{M}(I)$ such that*

$$\int_a^b u(x)\,{}^C D^s_{b-}[\phi](x)\,dx = \int_a^b \phi(x)\,d\mu(x)$$

for every $\phi \in C^1_c(I)$. In that case, we set $DI^{1-s}_{a+}[u] := \mu$.

Remark 4.15. *Clearly, if $u \in W^{s,p}_{RL,a+}(I)$ for some $p \geq 1$, and $s \in (0,1)$, then $u \in BV^s_{RL,a+}(I)$, and $D^s_{a+}[u] = (D^s_{a+}[u])\mathcal{L}^1$.*

4.3.1 Fine properties of functions in $BV^s_{RL,a+}(I)$

Now, we focus on the decomposition of the measure $D^s_{a+}[u]$ for functions in $BV^s_{RL,a+}(I)$.

We start with the following

Proposition 4.16. *If $u \in BV(I)$, then $u \in BV^s_{RL,a+}(I)$ and*

$$D^s_{a+}[u] = (D^s_{a+}[u])_{ac}\mathcal{L}^1.$$

Proof. Using Theorem 4.1 and remark 4.15, if $u \in BV(I)$, then $u \in W^{s,1}_{RL,a+}(I)$ and so $I^{1-s}_{a+}[u] \in W^{1,1}(I) \subset BV(I)$, and this means that the measure $D^s_{a+}[u]$ is an absolutely continuous measure with respect to the Lebesgue measure $\mathcal{L}^1$, with $D^s_{a+}[u]$ as density. $\square$

In the spirit of Lemma 2.31, we can obtain a version of the Fundamental Theorem of Calculus for functions in $BV^s_{RL,a+}(I)$.

Lemma 4.17. *Let $s \in (0,1)$ and $u \in BV^s_{RL,a+}(I)$. Then, for $\mathcal{L}^1$-a.e. $x \in I$, we also have*

$$u(x) = D^s_{a+}[I^s_{a+}[u]](x) = I^s_{a+}[\mathcal{D}^s_{a+}[u]](x) + \frac{I^{1-s}_{a+}[u](a)}{\Gamma(s)}(x-a)^{s-1}. \tag{4.15}$$

In addition, if $u \in BV^s_{RL,a+}(I) \cap I^s_{a+}(L^1(I))$, then $u \in W^{s,1}_{RL,a+}(I) \cap I^s_{a+}(L^1(I))$, $I^{1-s}_{a+}[u](a) = 0$ and (2.28) holds.

Proof. The first equality in (4.15) follows immediately from (2.26). The second one can be proved as (2.27). Indeed, if $u \in BV^s_{RL,a+}(I)$, then $I^{1-s}_{a+}[u] \in BV(I)$ with weak derivative $\mathcal{D}^s_{a+}[u]$. Therefore, by [AFP00, Theorem 3.28], for $\mathcal{L}^1$-a.e. $x \in I$, we get

$$\begin{aligned} I^{1-s}_{a+}[u](x) &= \int_a^x d\mathcal{D}^s_{a+}[u](t) + I^{1-s}_{a+}[u](a+) \\ &= I^{1-s}_{a+}[I^s_{a+}[\mathcal{D}^s_{a+}[u]]](x) + I^{1-s}_{a+}\left[\frac{I^{1-s}_{a+}[u](a+)}{\Gamma(s)}(\cdot - a)^{s-1}\right](x) \end{aligned}$$

by (2.18). We notice that $\mathcal{D}^s_{a+}[u] \in \mathcal{M}(I)$, and so, by Proposition 4.3, $I^s_{a+}[\mathcal{D}^s_{a+}[u]] \in L^1(I)$. Thus, it is enough to apply D^{1-s}_{a+} to both sides of the equation and use (2.26) to obtain (4.15). Finally, if $u \in BV^s_{RL,a+}(I) \cap I^s_{a+}(L^1(I))$, then, by Lemma 2.30 with $p = 1$, we have that $u \in W^{s,1}_{RL,a+}(I), I^{1-s}_{a+}[u](a) = 0$, and so it satisfies the hypotheses for (2.28). This ends the proof. $\square$

Now, we show with a counterexample that the inclusion of $BV(I)$ into $BV^s_{RL,a+}(I)$ is strict. This fact suggests that, in general, if $u \in BV^s_{RL,a+}(I) \setminus BV(I)$, then the measure D^s_{a+} is not absolutely continuous with respect $\mathcal{L}^1$.

Example 4.18 ($BV^s_{RL,a+}(I)$ strictly contains $BV(I)$)**.** *Let $s \in (0,1)$, $J = (c,d)$ with $c, d \in \mathbb{R}$ such that $a < c < d < b$. We define the following function*

$$u(x) := \begin{cases} 0 \quad \text{if} \quad a < x \le c \\ \dfrac{(x-c)^{s-1}}{\Gamma(s)} \quad \text{if} \quad c < x \le d \\ 0 \quad \text{if} \quad d < x < b \end{cases} \tag{4.16}$$

It is plain to see that $u \notin BV(I)$, since $u \notin L^\infty(I)$. Now, we compute $I^{1-s}_{a+}[u](x)$. Clearly, when $x \in (a,c)$, $I^{1-s}_{a+}[u](x) = 0$, otherwise, for $x \in J$ we have that

$$I^{1-s}_{a+}[u](x) = \frac{1}{\Gamma(s)\Gamma(1-s)}\int_c^x (t-c)^{s-1}(x-t)^{-s}dt = \frac{1}{\Gamma(s)\Gamma(1-s)}\int_0^1 \sigma^{s-1}(1-\sigma)^{-s}d\sigma = 1.$$

Therefore, for any $x \in I$,

$$I^{1-s}_{a+}[u](x) = \begin{cases} 0 \quad \text{if} \quad x \in (a,c] \\ 1 \quad \text{if} \quad x \in (c,d] \\ \dfrac{1}{\Gamma(s)\Gamma(1-s)}\displaystyle\int_c^d (t-c)^{s-1}(x-t)^{-s}dt \quad \text{if} \quad x \in (d,b), \end{cases}$$

which coincides almost everywhere in I with the function $\chi_J(x) + f(x)\chi_{(d,b)}(x) \in BV(I)$, where

$$f(x) := \frac{1}{\Gamma(s)\Gamma(1-s)} \int_c^d (t-c)^{s-1}(x-t)^{-s} dt \in C([d,b)) \cap C^\infty((d,b)) \cap L^1((d,b)).$$

Therefore, $I_{a+}^{1-s}[u] \in BV(I)$, and hence $u \in BV_{RL,a+}^s(I) \setminus BV(I)$.

Remark 4.19. *From the previous example, we deduce that if $u \in BV_{RL,a+}^s(I) \setminus BV(I)$ then, the measure $D_{a+}^s[u]$ can have a jump part; indeed, for the function u given by (4.16), we have that*

$$D_{a+}^s[u] = \delta_c - \delta_d + f(d)\delta_d + f'(x)\chi_{(d,b)}\mathcal{L}^1 = \delta_c + f'(x)\chi_{(d,b)}\mathcal{L}^1,$$

where the second equality follows from the fact that $f(d) = \dfrac{1}{\Gamma(s)\Gamma(1-s)}\beta(s, 1-s) = 1.$

Now, we exhibit an example of $u \in BV_{RL,a+}^s(I)$ such that $I_{a+}^{1-s}[u] \in BV(I) \setminus SBV(I)$.

Example 4.20. *Consider the classical ternary Cantor function $C(x)$, and let $I = (0,1)$. It is well known that $C \in C^{0,\alpha_C}(\overline{I}) \cap BV(I)$, where $\alpha_C := \log_3 2$, and DC is a singular measure without atoms which means that $DC = (DC)_c$, in particular up to a multiplicative constant $DC = \mathcal{H}^{\alpha_C}$, see e.g.* [AFP00].

Now, since $C(0) = 0$ using Proposition 2.21, we have that C is representable as the $(1-s)$-fractional integral of a function in $C_0^{0,\alpha_C+s-1}(\overline{I})$, provided $s \in (1-\alpha_C, 1)$, but this implies that there exists $u \in C_0^{0,\alpha_C+s-1}(\overline{I})$ such that $I_{0+}^{1-s}[u](x) = C(x)$, and so $u \in BV_{RL,0+}^s(I)$, with $D_{0+}^s[u] = DC = (DC)_c$.

4.4 Action of the fractional integral on Sobolev functions

In this section we analyze the behaviour of the fractional integral when it acts on functions in the Sobolev space $W^{1,p}(I)$ for some $p \geq 1$. We start with the following statement

Proposition 4.21. *Let $1 \leq p < \infty$ and $s \in (0,1)$ such that $sp < 1$. If $u \in W^{1,p}(I)$, then $I_{a+}^{1-s}[u] \in W^{1,p}(I)$. Moreover, if $u(a+) = 0$, I_{a+}^{1-s} is a continuous operator from $W^{1,p}(I)$ into $W^{1,p}(I)$.*

Proof. Thanks to Proposition 2.21, $I_{a+}^{1-s}[u] \in L^p(I)$, and

$$\left\| I_{a+}^{1-s}[u] \right\|_{L^p(I)} \leq \frac{(b-a)^{1-s}}{\Gamma(2-s)} \left\| u \right\|_{L^p(I)}. \tag{4.17}$$

Now, we prove that $D_{a+}^s[u] \in L^p(I)$.

We notice that, $u \in W^{1,p}(I) \subset BV(I)$ for any $s \in (0,1)$; hence, using Theorem 4.1, we have that

$$D_{a+}^s[u](x) = \frac{u(a+)}{\Gamma(1-s)} \frac{1}{(x-a)^s} + \frac{1}{\Gamma(1-s)} \int_a^x \frac{u'(t)}{(x-t)^s} dt,$$

where u' denotes the weak derivative of u.

Therefore, we get

$$\int_a^b |D_{a+}^s[u](x)|^p dx \leq C_1 |u(a+)|^p \int_a^b \frac{dx}{(x-a)^{sp}} + C_2 \int_a^b dx \int_a^x \frac{|u'(t)|^p}{(x-t)^{sp}} dt.$$

Now, since $sp < 1$, the first term in the right hand side is finite. For the second term, we have that

$$\int_a^b dx \int_a^x \frac{|u'(t)|^p}{(x-t)^{sp}} dt = \int_a^b |u'(t)|^p dt \int_t^b \frac{dx}{(x-t)^{sp}}$$
$$= C_{s,p} \int_a^b |u'(t)|^p (b-t)^{1-sp} dt \le C_{s,p,a,b} \|u'\|_{L^p(I)}^p < \infty,$$

therefore

$$\left\| D_{a+}^s[u] \right\|_{L^p(I)} \le C \left(|u(a+)|^p + \|u'\|_{L^p(I)}^p \right)^{1/p}. \tag{4.18}$$

Moreover, if $u(a+) = 0$, summing up (4.17) and (4.18), we have that there exists $C > 0$ such that

$$\left\| I_{a+}^{1-s}[u] \right\|_{W^{1,p}(I)} \le C \|u\|_{W^{1,p}(I)}$$

for any $u \in W^{1,p}(I)$, and this concludes the proof. □

Corollary 4.22. *Let $1 \le p \le q < \infty$ and $r, s \in (0,1)$ such that $sp < 1$ and $r > s + \frac{1}{p'}$, where p' denotes the Hölder conjugate of p. Then we have that*

$$W_{RL,a+}^{r,q}(I) \subset W_{RL,a+}^{s,p}(I).$$

Proof. Since $W_{RL,a+}^{r,q}(I) \subseteq W_{RL,a+}^{r,p}(I)$, we are left to prove that $I_{a+}^{1-s}[u] \in W^{1,p}(I)$. We notice that

$$I_{a+}^{1-s}[u](x) = I_{a+}^{r-s}[I_{a+}^{1-r}[u]](x) = I_{a+}^{1-\gamma}[v](x),$$

where $v(x) := I_{a+}^{1-r}[u](x)$ and $\gamma := 1-r+s$. Thanks to Proposition 4.21, since $v \in W^{1,p}(I)$ we have that $I_{a+}^{1-\gamma}[v] \in W^{1,p}(I)$ provided $\gamma p < 1$, and this condition holds since $r > s + \frac{1}{p'}$. □

Remark 4.23. *Proposition 4.21 covers the case $p = \infty$ if and only if $u(a) = 0$; indeed, for $u \in W^{1,\infty}(I)$ with $u(a) = 0$ we have that*

$$D_{a+}^s[u](x) = \frac{1}{\Gamma(1-s)} \int_a^x \frac{u'(t)}{(x-t)^s} dt$$

and, for any $x \in \overline{I}$, we have that

$$|D_{a+}^s[u](x)| \le \frac{(b-a)^{1-s}}{\Gamma(2-s)} \|u'\|_{L^\infty(I)},$$

and so

$$\left\| I_{a+}^{1-s}[u] \right\|_{W^{1,\infty}(I)} \le \frac{(b-a)^{1-s}}{\Gamma(2-s)} \|u\|_{W^{1,\infty}(I)}.$$

If $u(a) \neq 0$ we have neither continuous embedding, nor inclusion; consider for instance $I := (0,1)$ and $u(x) := \cos(x) \in W^{1,\infty}(I)$. We have that

$$D_{0+}^s[u](x) = \frac{1}{\Gamma(1-s)} \left(\frac{1}{x^s} - \int_0^x \frac{\sin(t)}{(x-t)^s} dt \right).$$

Since the first term in the right hand side is not bounded when x is close to 0, clearly $D_{0+}^s[u] \notin L^\infty(I)$.

Remark 4.24. *We notice that the continuous embedding given by Corollary 4.22 can be obtained as a byproduct of* [IW13, Theorem 31]*, which attests that the embedding is compact.*

4.4.1 Regulatization properties of the fractional integral and a Sobolev-type embedding Theorem for functions in $W^{s,p}_{RL,a+}(I)$.

In this subsection, we want to prove that the fractional integral actually improve the (weak) differentiability of a Sobolev function. To this purpose, we start with a simple remark.

Remark 4.25. *Depending on the summability of a Sobolev function u we notice that I^{1-s}_{a+} enjoys different improvements in regularity. In particular we distinguish the case $p=1$ and the case $p>1$.*

1. *Case $p=1$*

   If $u \in W^{1,1}(I)$, using Sobolev Embedding Theorem $u \in L^q(I)$ for any $1 \le q \le \infty$, and so, thanks to Proposition 2.18, $I^{1-s}_{a+}[u] \in \bigcap\limits_{q>1/s} C^{0,s-\frac{1}{q}}(I)$,

2. *Case $p>1$*

   If $u \in W^{1,p}(I)$, again by Sobolev Embedding Theorem, we have that $u \in C^{0,1-\frac{1}{p}}(I)$.

   Using Proposition 2.21, for any $u \in C_0^{0,1-\frac{1}{p}}(I)$, we have that

   - $I^{1-s}_{a+}[u] \in C_0^{0,2-s-\frac{1}{p}}(I)$ *if* $s+\frac{1}{p}>1$,
   - $I^{1-s}_{a+}[u] \in H_0^{1,1}(I)$ *if* $s+\frac{1}{p}=1$,
   - $I^{1-s}_{a+}[u] \in C_0^{1,1-s-\frac{1}{p}}(I)$ *if* $s+\frac{1}{p}<1$.

   In the third case, follows that $D^s_{a+}[u] \in C^{0,1-s-\frac{1}{p}}(I)$.

Now, we are able to prove that when we apply the fractional integral I^{1-s}_{a+} to a function in $W_0^{1,p}(I)$ for some $p>1$, we gain more differentiability; this means that the function $I^{1-s}_{a+}[u]$ belongs to a higher order fractional Sobolev space as given by Remark 2.8.

The statement goes as follows

Proposition 4.26. *Let $p>1$ and $s \in \left(0, \min\left\{\frac{1}{p}, \frac{p-1}{2p}\right\}\right)$. For any $u \in W_0^{1,p}(I)$, we have that $I^{1-s}_{a+}[u] \in W^{s+1,p}(I)$.*

Proof. We notice that the conditions $sp<1$ and $s+\frac{1}{p}<1$ are satisfied, and so, by Proposition 4.21 and Remark 4.25, $I^{1-s}_{a+}[u] \in W_0^{1,p}(I) \cap C_0^{1,1-s-\frac{1}{p}}(\overline{I})$.

Now, we prove that $D^s_{a+}[u] \in W^{s,p}(I)$. Namely, we have to prove that

$$\int_a^b \int_a^b \frac{|D^s_{a+}[u](x) - D^s_{a+}[u](y)|^p}{|x-y|^{sp+1}} dxdy < \infty.$$

Now, we use the Hölder continuity of $D^s_{a+}[u]$ to say that

$$|D^s_{a+}[u](x) - D^s_{a+}[u](y)|^p \le C|x-y|^{p-sp-1},$$

for some $C>0$ and for any $x, y \in I$.

Therefore, we have that

$$\int_a^b \int_a^b \frac{|D^s_{a+}[u](x) - D^s_{a+}[u](y)|^p}{|x-y|^{sp+1}} dxdy \leq C \int_a^b \int_a^b \frac{1}{|x-y|^{2sp-p+2}} dxdy,$$

where the integral in the right hand side converges since $s < \frac{p-1}{2p}$. □

Corollary 4.27. *In the same hypoteses of Proposition 4.26, we have that*

$$u \in W^{s,r}_{RL,a+}(I) \quad \textit{for any} \quad r \in \left[1, \frac{p}{1-sp}\right]$$

Proof. Thanks to the fractional Sobolev Embedding, since $sp < 1$, we have that $D^s_{a+}[u] \in L^r(I)$ for any $r \in \left[1, \frac{p}{1-sp}\right]$, hence $I^{1-s}_{a+}[u]$ belongs to $W^{1,r}(I)$ in the same range for r, and this completely proves the claim. □

Now, we show an analogous of the Sobolev embedding Theorem for Riemann-Liouville fractional Sobolev spaces. To the knowledge of the authors, this is an original result in this setting. We refer the reader *e.g.* [AF03, Chapter 4] for the classical Sobolev embedding Theorem for Sobolev spaces of integer order or [DNPV12, Theorem 6.7.] for Sobolev spaces of fractional order.

Theorem 4.28 (Riemann-Liouville fractional Sobolev embedding)**.** *Let $s \in (0,1)$, $1 \leq p \leq \infty$ and $u \in W^{s,p}_{RL,a+}(I) \cap I^s_{a+}(L^1(I))$. Then, we have that*

(i) If $p = 1$, $u \in L^{\frac{1}{1-s},\infty}(I)$, in particular $u \in L^r(I)$ for any $r \in [1, \frac{1}{1-s})$,

(ii) If $1 < p < 1/s$, $u \in L^r(I)$ for any $r \in [1, \frac{p}{1-sp}]$,

(iii) If $sp = 1$, $u \in L^r(I)$ for any $r \in [1, +\infty)$,

(iv) If $sp > 1$, $u \in C^{0,\beta}(I)$ for any $\beta \in [0, s - \frac{1}{p}]$.

Proof. Since $u \in I^s_{a+}(L^1(I))$, thanks to Lemma 2.31 we have that

$$I^s_{a+}[D^s_{a+}[u]](x) = u(x),$$

for any $x \in I$, and $D^s_{a+}[u] \in L^p(I)$ since $u \in W^{s,p}_{RL,a+}(I)$. Now, using Proposition 2.18 we have distinguish among four cases:

- if $p = 1$, $u \in L^r(I)$ for any $r \in [1, 1/(1-s))$,
- if $1 < p < 1/s$, $u \in L^r(I)$ for any $r \in [1, \frac{p}{1-sp}]$,
- if $sp = 1$ $u \in L^r(I)$ for any $r \in [1, +\infty)$,
- if $sp > 1$, $u \in C^{0,\beta}(I)$ for any $\beta \in [0, s - \frac{1}{p}]$,

and this concludes the proof. □

Remark 4.29. *It is worth noticing that cases (i) and (iii) in Theorem 4.28 are sharp, as shown in Appendix A.4.*

Remark 4.30. *We notice that if we skip the L^1-representability hypotesis, it holds that for any $x \in I$*

$$u(x) = I^s_{a+}[D^s_{a+}[u]](x) + \frac{I^{1-s}_{a+}[u](a)}{\Gamma(s)}(x-a)^{s-1}.$$

Therefore, if $u \in W^{s,p}_{RL,a+}(I)$ for some $p \in [1, 1/s)$, we have that $u \in L^r(I)$ for any $r \in [1, \frac{1}{1-s})$. This means that we gain summability if and only if $s \in [\frac{1}{2}, 1)$. Indeed, if $s \in [1/2, 1)$, we have that $u \in L^r(I)$ even when $r \in \left(p, \frac{1}{1-s}\right) \supseteq (p, \frac{1}{s})$. In the cases $sp = 1$ and $sp > 1$ we have neither more regularity nor more summability for the presence of the second term in the right-hand side.

4.5 Higher order fractional derivatives

In this last section, we point out that some of the results presented in the chapter can be extended to higher order fractional derivatives

Definition 4.31. *Let $k \in \mathbb{N}$, $s \in (k-1, k)$ and u such that the fractional integral $I^{k-s}_{a+}[u]$ is sufficiently smooth; we define the Riemann Liouville fractional derivatives of u as*

$$D^s_{a+}[u](x) := \frac{d^k}{dx^k} I^{k-s}_{a+}[u](x).$$

$$D^s_{b-}[u](x) := (-1)^k \frac{d^k}{dx^k} I^{k-s}_{b-}[u](x).$$

From this definition, for u sufficiently smooth, we immediately obtain a definition for higher order Caputo fractional derivatives

$${}^C D^s_{a+}[u](x) = D^s_{a+}[u](x) - \sum_{j=0}^{k-1} \frac{u^{(j)}(a)}{\Gamma(j-s+1)}(x-a)^{j-s} = \frac{1}{\Gamma(k-s)} \int_a^x \frac{u^{(k)}(t)}{(x-t)^{s-k+1}} dt$$

and

$${}^C D^s_{b-}[u](x) = D^s_{b-}[u](x) - \sum_{j=0}^{k-1} (-1)^j \frac{u^{(j)}(b)}{\Gamma(j-s+1)}(b-x)^{j-s} = \frac{(-1)^k}{\Gamma(k-s)} \int_x^b \frac{u^{(k)}(t)}{(t-x)^{s-k+1}} dt.$$

These higher order fractional derivatives allow to define, for $p \geq 1$, $k \in \mathbb{N}$ and $s \in (k-1, k)$, higher order Riemann-Liouville fractional Sobolev spaces, which are given by functions $u \in W^{k-1,p}(I)$ such that $I^{k-s}_{a+}[u] \in W^{k,p}(I)$.

Proposition 4.32 (Continuity of the fractional integral in higher order Sobolev spaces)**.** *Let $k \geq 2$, $1 \leq p < \infty$ and $s \in (k-1, k-1+\frac{1}{p})$. Then, if $u \in W^{k,p}(I)$ and $u(a*) = u'(a*) = \ldots = u^{(k-2)}(a*) = 0$, then we have that $I^{k-s}_{a+}[u] \in W^{k,p}(I)$. Moreover, if in addition $u^{(k-1)}(a*) = 0$, the operator $I^{k-s}_{a+} : W^{k,p}(I) \to W^{k,p}(I)$ is continuous.*

Proof. Using the representation formula obtained via iterated integrations by parts

$$I^{k-s}_{a+}[u](x) = \frac{1}{\Gamma(k-s)} \left(c_{s,k,k} \int_a^x u^{(k)}(t)(x-t)^{2k-s-1} dt + \sum_{i=0}^{k-1} c_{s,k,i} u^{(i)}(a)(x-a)^{k-s+i} \right), \tag{4.19}$$

where

$$c_{s,k,h} := \begin{cases} 1 \quad \text{if} \quad h = 0 \\ \left(\prod_{l=0}^{h-1}(k-s+l)\right)^{-1} \quad \text{if} \quad h \geq 1, \end{cases},$$

it is an easy task to check that this function has all the derivatives up to order k in $L^p(I)$ if and only if u vanishes in a with all its derivatives up to order $k-2$. Furthermore, if $u^{(k-1)}(a) = 0$, the second term in the right-hand side of (4.19) completely vanishes and hence we have that

$$\left\| I_{a+}^{k-s}[u] \right\|_{W^{k,p}(I)} = \sum_{i=0}^{k} \left\| (I_{a+}^{k-s}[u])^{(i)} \right\|_{L^p(I)} \leq \sum_{i=0}^{k} C \left\| u^{(k)} \right\|_{L^p(I)} \leq C' \left\| u \right\|_{W^{k,p}(I)}$$

□

Remark 4.33. *The case $k = 1$ is covered by Proposition 4.21, where a homogeneous initial condition is not necessary to prove that $I_{a+}^{1-s}(W^{1,p}(I))$ is a vector subspace of $W^{1,p}(I)$, but only for the continuity of the $(1-s)$-fractional integral.*

Remark 4.34. *As stated in Remark 4.23 when $k = 1$, the same counterexample says us that in the case $p = \infty$ homogeneous conditions in the initial point for all the derivatives up to order $k-1$ are necessary also for the inclusion as vector subspace of $I_{a+}^{k-s}(W^{k,\infty}(I))$ into $W^{k,\infty}(I)$.*

The introduction of higher order Riemann-Liouville fractional Sobolev spaces allows us to prove the following proposition involving the space $BH(I) := \{u \in W^{1,1}(I) \,|\, u' \in BV(I)\}$, which is usually known as the space of functions with bounded Hessian in I. Originally introduced in [Dem84], BH is the natural setting for second order variational problems with linear growth (see e.g. [CLT04] for applications in image analysis).

Proposition 4.35. *Let $u \in BH_0(I)$, then $u \in W_{RL,a+}^{s,1}(I)$ for any $s \in (1,2)$.*

Proof. By definition $u \in W^{1,1}(I)$ and $u' \in BV(I)$; thanks to Theorem 4.1, we have that $u' \in W_{RL,a+}^{\sigma,1}(I)$ for any $\sigma \in (0,1)$, therefore $I_{a+}^{1-\sigma}[u'] = {}^C D_{a+}^{\sigma}[u] \in W^{1,1}(I)$. Now, since $u(a) = 0$, we have that ${}^C D_{a+}^{\sigma}[u](x) = D_{a+}^{\sigma}[u](x)$ for any $x \in I$, but this implies that $I_{a+}^{1-\sigma}[u] \in W^{2,1}(I)$ for any $\sigma \in (0,1)$. Now, if we set $\sigma := s-1$ for $s \in (1,2)$, the claim plainly follows. □

4.6 Open Problems

As remarked by Remark 4.12 we are not able to prove (or disprove) that for $s \in (0,1)$ and $p > 1$ the inclusion

$$W^{s,p}(I) \subseteq W_{RL,a+}^{s,p}(I),$$

holds. We conjecture that, if this were true, the condition $sp < 1$ should be imposed; indeed, if $sp < 1$ thanks to Remark 2.11 the set $C_c^1(I)$ is dense in $W^{s,p}(I)$; therefore, firstly one should be able to prove an analogous of Lemma 4.7 for functions in $W^{s,p}(I)$, and once proved that

$$D_{a+}^{s}[u](x) = \frac{1}{\Gamma(1-s)} \frac{u(x)}{(x-a)^s} + \frac{s}{\Gamma(1-s)} \int_a^x \frac{u(x)-u(t)}{(x-t)^{s+1}} dt,$$

one could be estimate the L^p norm of the first term in the right-hand side thanks to the fractional Hardy inequality 2.9, but we do not know how to handle the second term. Indeed, one has that

$$\int_a^b dx \left| \int_a^x \frac{u(x)-u(t)}{(x-t)^{s+1}} dt \right|^p \leq \int_a^b dx \int_a^x \frac{|u(x)-u(t)|^p}{|x-t|^{sp+p}} dt = J_1 + J_2,$$

where

$$J_1 := \int_a^b dx \int_{\max\{a,x+1\}}^{\min\{b,x-1\}} \frac{|u(x)-u(t)|^p}{|x-t|^{sp+p}} dt$$

and

$$J_2 := \int_a^b dx \int_{\max\{a,x-1\}}^{\min\{b,x+1\}} \frac{|u(x)-u(t)|^p}{|x-t|^{sp+p}} dt.$$

For J_1 we have that

$$J_1 \leq \int_a^b dx \int_{\max\{a,x+1\}}^{\min\{b,x-1\}} \frac{|u(x)-u(t)|^p}{|x-t|^{sp+1}} dt \leq [u]^p_{W^{s,p}(I)},$$

but we are not able to prove (or disprove) an estimate of the form

$$J_2 \leq C[u]^p_{W^{s,p}(I)},$$

or the weaker one

$$J_2 \leq C \left\| u \right\|^p_{W^{s,p}(I)},$$

for some $C > 0$.

Chapter 5

Local minimizers for nonlocal perimeters in Carnot Groups

5.1 Introduction and main results

In this chapter we study a minimization problem in a sub-Riemannian setting; in particular we will work in a Carnot Group.

We notice that variational problems in sub-Riemannian geometry are treated e.g. in [BF03, BLU07, CMS04, FS06].

In the first section, after an introduction of the framework, we present the main result of the paper; namely, the local minimality of halfspaces for nonlocal perimeters.

In the second section we introduce the notion of calibrations, which is analogous of the one given in the euclidean setting by [Cab19], [Pag19] and we proceed towards the proof of Theorem 5.2.

In the third section, we study the rescaled limit of our functional. Namely, following the results in [BP19] with appropriate modifications we prove that the horizontal perimeter with a given density bounds from below the $\Gamma - \liminf$ of the rescaled sequence $\frac{1}{\varepsilon}J_{K_\varepsilon}(E_\varepsilon, \Omega)$.

Γ-convergence of nonlocal perimeters in $\mathbb{R}^n$ has been treated e.g. in [ADPM11, BP19, Pag19]; in particular in the first work, the authors deal with the nonlocal perimeter of a measurable set in the whole of $\mathbb{R}^n$ obtaining a Γ-convergence result via a polyhedral approximation; in our setting this approximation could be made by an identification of the Group with the euclidean space via exponential coordinates, but in this way we would lose informations on the intrinsic geometry of the Group. See Section A.5 in the Appendix.

In the fourth section we prove that our main results hold even for functionals depending on the sub-Riemannian heat kernel.

We conclude with some open questions.

Now we are ready to introduce our framework.

Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q as defined in Chapter 2 and denote by $\|\cdot\|$ a symmetric and homogeneous norm on $\mathbb{G}$. Let $K\colon \mathbb{G} \to \mathbb{R}$ be such that

$$K \geq 0 \quad \text{in}\, \mathbb{G}, \tag{5.1}$$

$$K(\xi^{-1}) = K(\xi) \quad \text{for any}\, \xi \in \mathbb{G}, \tag{5.2}$$

$$\int_{\mathbb{G}} \min\{1, \|x\|\} K(x)\, dx < +\infty. \tag{5.3}$$

Define also for every measurable function $u\colon \mathbb{G} \to [0,+\infty]$ and every measurable set $\Omega \subseteq \mathbb{G}$ the functional

$$\begin{aligned} J_K(u;\Omega) &:= \frac{1}{2}\int_\Omega\int_\Omega K(y^{-1}x)|u(y)-u(x)|\, dydx + \int_\Omega\int_{\Omega^c} K(y^{-1}x)|u(y)-u(x)|\, dydx \\ &= \frac{1}{2}J^1_K(u;\Omega) + J^2_K(u;\Omega). \end{aligned} \tag{5.4}$$

Moreover, for A, B measurable sets, we define the interaction between A and B driven by the kernel K as

$$L_K(A,B) := \int_B\int_A K(y^{-1}x)dydx. \tag{5.5}$$

Notice that if $u = \chi_E$ for some measurable set E, we also set $P_K(E;\Omega) := J_K(\chi_E;\Omega)$, and we have that

$$P_K(E;\Omega) = L_K(E^c\cap\Omega, E\cap\Omega) + L_K(E^c\cap\Omega, E\cap\Omega^c) + L_K(E\cap\Omega, E^c\cap\Omega^c);$$

in particular, if $\Omega = \mathbb{G}$

$$P_K(E,\mathbb{G}) = L_K(E,E^c).$$

Remark 5.1. *For every measurable set $E \subseteq \mathbb{G}$ we notice that $P_K(E;\Omega)$ can also be written as*

$$P_K(E;\Omega) = \frac{1}{2}\int_{(\mathbb{G}\times\mathbb{G})\setminus(\Omega^c\times\Omega^c)} |\chi_E(y)-\chi_E(x)|K(y^{-1}x)\, dx\, dy. \tag{5.6}$$

Indeed we can write

$$\begin{aligned} \int_{(\mathbb{G}\times\mathbb{G})\setminus(\Omega^c\times\Omega^c)} |\chi_E(y)-\chi_E(x)|K(y^{-1}x)\, dx\, dy &= \int_{(\mathbb{G}\times\mathbb{G})\setminus(\Omega^c\times\Omega^c)} |\chi_E(y)-\chi_E(x)|^2K(y^{-1}x)\, dx\, dy \\ &= \int_{(\mathbb{G}\times\mathbb{G})\setminus(\Omega^c\times\Omega^c)} (\chi_E(y)-\chi_E(y)\chi_E(x))K(y^{-1}x)\, dx\, dy \\ &\quad + \int_{(\mathbb{G}\times\mathbb{G})\setminus(\Omega^c\times\Omega^c)} (\chi_E(x)-\chi_E(y)\chi_E(x))K(y^{-1}x)\, dx\, dy \\ &= 2\int_{(\mathbb{G}\times\mathbb{G})\setminus(\Omega^c\times\Omega^c)} \chi_E(x)\chi_{E^c}(y)K(y^{-1}x)\, dx\, dy \\ &= 2L_K(E^c\cap\Omega, E\cap\Omega) + 2L_K(E^c\cap\Omega, E\cap\Omega^c) + 2L_K(E\cap\Omega; E^c\cap\Omega^c) \\ &= 2P_K(E;\Omega). \end{aligned}$$

When $\mathbb{G}$ is the Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$, a typical example of radial kernel satisfying (5.3) is given by the fractional kernel $K(x) = |x|^{-n-\alpha}$, for some $\alpha \in (0,1)$. We refer e.g. to [Val13] where applications to phase transition problems are also treated.

For any $\varepsilon > 0$, we define the rescaled kernel K_ε as

$$K_\varepsilon(x) := \frac{1}{\varepsilon^Q}K(\delta_{1/\varepsilon}x),$$

and we introduce the functionals J_ε and L_ε accordingly as J and L by replacing K with K_ε.

Our main Theorem goes as follows

Theorem 5.2. *Let H be a vertical halfspace and denote by $B := B(0,1)$. Then*

$$J_K(\chi_H;B) \le J_K(v;B),$$

for every measurable $v\colon \mathbb{G} \to [0,1]$ such that $v = \chi_H$ almost everywhere on B^c. Moreover, if $u\colon \mathbb{G} \to [0,1]$ is such that $u = \chi_H$ almost everywhere on B^c and $J_K(u;B) \le J_K(\chi_H;B)$, then $u = \chi_H$ almost everywhere on $\mathbb{G}$.

5.2 Calibrations

In order to prove Theorem 5.2 we adapt the notion of nonlocal calibration given in [Pag19] in the euclidean setting. We notice that in [Cab19] a notion of calibration for nonlocal functionals is also given; in particular, the author proves that in presence of a foliation[1] made of sub and super solution of the nonlocal mean curvature flow

$$H_K[E] = 0,$$

where the nonlocal mean curvature is given by

$$H_K[E](x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} (\chi_{E^c}(y) - \chi_E(y)) K(x-y) dy, \tag{5.7}$$

a calibration for the nonlocal perimeter is available and, as in the local case, we can prove the minimality of each leaf of the foliation for its own boundary datum.

Definition 5.3. *Let $u\colon \mathbb{G} \to [0,1]$ and $\zeta\colon \mathbb{G}\times\mathbb{G} \to [-1,1]$ be measurable functions. We say that ζ is a* calibration *for u if the following two facts hold.*

(i) The map $F_\varepsilon(p) = \int_{\mathbb{G}\setminus B(p,\varepsilon)} K(y^{-1}p)(\zeta(y,p) - \zeta(p,y))\, dy$ is such that

$$\lim_{\varepsilon\to 0} \|F_\varepsilon\|_{L^1(\Omega)} = 0. \tag{5.8}$$

(ii) for almost every $(p,q) \in \mathbb{G}\times\mathbb{G}$ such that $u(p) \neq u(q)$ one has

$$\zeta(p,q)(u(q) - u(p)) = |u(q) - u(p)|. \tag{5.9}$$

Remark 5.4. *If $\zeta\colon \mathbb{G}\times\mathbb{G} \to [-1,1]$ is a calibration for $u\colon \mathbb{G}\to[0,1]$, then also the anti-symmetric function $\widehat{\zeta}(p,q) := \frac{1}{2}(\zeta(p,q) - \zeta(q,p))$ is a calibration for u.*

The proof of the following theorem follows closely the proof of [Pag19, Theorem 2.3].

Theorem 5.5. *Let $E_0 \subseteq \mathbb{G}$ be a measurable set such that $J_K(\chi_{E_0};\Omega) < +\infty$ and define*

$$\mathcal{F} := \{v : \mathbb{G} \to [0,1] \text{ measurable} \mid v = \chi_{E_0} \text{ on } \Omega^c\}.$$

Let $u \in \mathcal{F}$ and let $\zeta\colon \mathbb{G}\times\mathbb{G} \to [-1,1]$ be a calibration for u. Then

$$J_K(u;\Omega) \le J_K(v;\Omega),$$

for every $v \in \mathcal{F}$. Moreover, if $\widetilde{u} \in \mathcal{F}$ is such that $J_K(\widetilde{u};\Omega) \le J_K(u;\Omega)$, then ζ is a calibration for $\widetilde{u}$.

---

[1] If $\Omega \subset \mathbb{R}^n$ is a bounded open set and E is a measurable set, we say that Ω is foliated by sub and super solutions adapted to E whenever there exists a measurable function $\phi_E : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $E = \{\phi_E(x) > 0\}$ up to $\mathcal{L}^n$-negligible sets,

(ii) The limit in (5.7) exists for a.e. $x \in \Omega$ and the sequence indiced by ε given by the integrals in the right-hand side of (5.7) converge in $L^1(\Omega)$ to $H_K[\phi_E]$, as $\varepsilon \to 0^+$,

(iii) $H_K[\phi_E](x) \le 0$ for a.e. $x \in \Omega \cap E$ and $H_K[\phi_E](x) \ge 0$ for a.e. $x \in \Omega \setminus E$.

Proof. We can assume without loss of generality that $J_K(v;\Omega) < +\infty$ for every $v \in \mathcal{F}$. Since $|v(y) - v(x)| \geq \zeta(x,y)(v(y) - v(x))$ we can write for any $v \in \mathcal{F}$

$$J_K(v;\Omega) \geq a(v) - b_1(v) + b_0,$$

where a, b_1 and b_0 are respectively defined by

$$\begin{aligned}
a(v) &:= \frac{1}{2}\int_\Omega \int_\Omega K(y^{-1}x)\zeta(x,y)(v(y) - v(x))\, dydx,\\
b_1(v) &:= \int_\Omega \int_{\Omega^c} K(y^{-1}x)\zeta(x,y)v(x)\, dydx,\\
b_0 &:= \int_\Omega \int_{\Omega^c} K(y^{-1}x)\zeta(x,y)\chi_{E_0}(y)\, dydx.
\end{aligned}$$

By (5.9), we notice that $J_K(u;\Omega) = a(u) - b_1(u) + b_0$. It is then enough to prove that, for every $v \in \mathcal{F}$, one has $a(v) = b_1(v)$. By Remark 5.4, we can assume that ζ is antisymmetric. Combining this with the fact that $K(\xi^{-1}) = K(\xi)$, we easily get

$$a(v) = -\int_\Omega \int_\Omega K(y^{-1}x)\zeta(x,y)v(x)\, dydx. \tag{5.10}$$

By (5.8), for almost every $x \in \Omega$, we have

$$\begin{aligned}
&\lim_{r\to 0} \int_{B(x,r)^c} K(y^{-1}x)\zeta(x,y)\, dy =\\
&\qquad \lim_{r\to 0} \int_{B(x,r)^c\cap\Omega} K(y^{-1}x)\zeta(x,y)\, dy + \int_{\Omega^c} K(y^{-1}x)\zeta(x,y)\, dydx = 0.
\end{aligned}$$

Implementing this identity in (5.10) and using the dominated convergence theorem, we get

$$\begin{aligned}
a(v) &= -\int_\Omega \int_\Omega K(y^{-1}x)\zeta(x,y)v(x)\, dydx\\
&= -\lim_{r\to 0} \int_\Omega \int_{B(x,r)^c\cap\Omega} K(y^{-1}x)\zeta(x,y)v(x)\, dydx\\
&= \int_\Omega \int_{\Omega^c} K(y^{-1}x)\zeta(x,y)v(x)\, dydx = b_1(v).
\end{aligned}$$

We are left to prove that if $\widetilde{u} \in \mathcal{F}$ is such that $J_K(\widetilde{u};\Omega) \leq J_K(u;\Omega)$, then ζ is a calibration for $\widetilde{u}$. Since $u = \widetilde{u}$ on Ω^c we get

$$\zeta(x,y)(\widetilde{u}(y) - \widetilde{u}(x)) = |\widetilde{u}(y) - \widetilde{u}(x)|, \tag{5.11}$$

for almost every $(x,y) \in \Omega^c \times \Omega^c$ satisfying $u(x) \neq u(y)$. Since $J_K(\widetilde{u};\Omega) = b_0$, we also have that $J_K(\widetilde{u};\Omega) = a(\widetilde{u}) - b_1(\widetilde{u}) + b_0$. This implies that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \int_\Omega K(y^{-1}x)\left(|\widetilde{u}(y) - \widetilde{u}(x)| - \zeta(x,y)(\widetilde{u}(y) - \widetilde{u}(x))\right) dydx\\
&+ \int_\Omega \int_{\Omega^c} K(y^{-1}x)\left(|\widetilde{u}(y) - \widetilde{u}(x)| - \zeta(x,y)(\widetilde{u}(y) - \widetilde{u}(x))\right) dydx = 0.
\end{aligned}$$

Since both integrands are positive, we get that (5.11) holds true for almost every $(x,y) \in \Omega \times \mathbb{G}$ with $\widetilde{u}(x) \neq \widetilde{u}(y)$. To get (5.11) for almost every $(x,y) \in \Omega^c \times \Omega$ it is enough to use the antisymmetry of ζ. $\square$

Proposition 5.6. *For any $\nu \in \mathfrak{g}_1 \setminus \{0\}$, the map $\zeta_\nu\colon \mathbb{G}\times\mathbb{G} \to [0,1]$ defined by*

$$\zeta_\nu(x,y) \coloneqq \operatorname{sign}\left(\langle \pi_1 \log(x^{-1}y), \nu\rangle\right),$$

is a calibration for χ_{H_ν}.

Proof. Denote for shortness $H = H_\nu$ and $\zeta = \zeta_\nu$. Let us first prove property (ii) of Definition 5.3, namely that for almost every $(x,y) \in \mathbb{G}\times\mathbb{G}$ with $\chi_H(x) \neq \chi_H(y)$ one has

$$\zeta(x,y)(\chi_H(y) - \chi_H(x)) = |\chi_H(y) - \chi_H(x)|.$$

It is not restrictive to assume that $x \in H$ and $y \in H^c$. Then we just observe that

$$\langle \pi_1 \log(x^{-1}y), \nu\rangle = -\langle \pi_1 \log x, \nu\rangle + \langle \pi_1 \log y, \nu\rangle < 0.$$

Concerning property (i) of Definition 5.3 we observe that for every $r > 0$ and every $x \in \mathbb{G}$ one has

$$\begin{aligned}
&\int_{\mathbb{G}\setminus B(x,r)} K(y^{-1}x)\left(\operatorname{sign}(\langle \pi_1 \log(x^{-1}y),\nu\rangle) - \operatorname{sign}(\langle \pi_1 \log(y^{-1}x),\nu\rangle)\right) dy \\
&\qquad = 2\int_{\mathbb{G}\setminus B(x,r)\cap xH} K(y^{-1}x)\, dy - 2\int_{\mathbb{G}\setminus B(x,r)\cap xH^c} K(y^{-1}x)\, dy \\
&\qquad = 2\int_{\mathbb{G}\setminus B(0,r)\cap H} K(z)\, dz - 2\int_{\mathbb{G}\setminus B(0,r)\cap H^c} K(z)\, dz = 0.
\end{aligned}$$

The last identity comes from the fact that $\mathcal{H}^Q(\{x \in \mathbb{G} : \langle \pi_1 \log x, \nu\rangle = 0\}) = 0$, $K(x^{-1}) = K(x)$ and the inversion $\xi \mapsto \xi^{-1}$ preserves the volume and maps H onto H^c (up to sets of measure zero). $\square$

5.2.1 Proof of Theorem 5.2

Proof. By Proposition 5.6 and Theorem 5.5 we only have to show that minimizers are unique (up to sets of measure zero). Let $\nu \in \mathfrak{g}_1 \setminus \{0\}$ be such that $H = H_\nu$ and let $u\colon \mathbb{G} \to [0,1]$ be such that $u = \chi_H$ almost everywhere on B^c and $J_K(u;B) \le J_K(\chi_H;B)$. Consider the map $\zeta(x,y) = \operatorname{sign}(\langle \pi_1 \log(x^{-1}y),\nu\rangle)$ which is a calibration of χ_H. By Theorem 5.5, ζ is also a calibration for u. Let $N \subseteq \mathbb{G}\times\mathbb{G}$ be a set of $\mathcal{H}^Q \otimes \mathcal{H}^Q$-measure zero such that

$$\operatorname{sign}(\langle \pi_1 \log(x^{-1}y),\nu\rangle)(u(y) - u(x)) = |u(y) - u(x)|, \quad \text{for every } (x,y) \in N^c. \tag{5.12}$$

We now prove that (5.12) holds indeed for every $(x,y) \in \mathbb{G}\times\mathbb{G}$. Consider a radial function $\rho : \mathbb{G} \to [0,+\infty)$ with compact support in $B(0,1)$ and such that $\int_{B(0,1)} \rho\, d\mathcal{H}^Q = 1$. For every $\varepsilon > 0$, consider the family $\rho_\varepsilon(x) = \frac{1}{\varepsilon^Q}\rho(\delta_{1/\varepsilon}x)$ and define

$$u_\varepsilon(x) = u * \rho_\varepsilon(x) = \int_{\mathbb{G}} u(\xi^{-1}x)\rho_\varepsilon(\xi)\, d\xi.$$

Then, for every $(x,y) \in \mathbb{G}\times\mathbb{G}$ one has

$$\begin{aligned}
u_\varepsilon(y) - u_\varepsilon(x) &= \int_{\mathbb{G}}\int_{\mathbb{G}} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\left(u(\eta^{-1}y) - u(\xi^{-1}x)\right) d\eta d\xi \\
&= \int_{B(0,\varepsilon)\times B(0,\varepsilon)} \rho_\varepsilon(\xi)\rho_\varepsilon(\eta)\left(u(\eta^{-1}y) - u(\xi^{-1}x)\right) d\eta d\xi.
\end{aligned}$$

Assume without loss of generality that $\langle \pi_1 \log(x^{-1}y), \nu\rangle > 0$. Then, for $\varepsilon > 0$ small enough, we also have that

$$\langle \pi_1 \log(x^{-1}\xi\eta^{-1}y), \nu\rangle > 0$$

for almost every $\xi, \eta \in B(0, \varepsilon)$. By (5.12), we therefore obtain that $u(\eta^{-1}y) - u(\xi^{-1}x) > 0$, for almost every $\xi, \eta \in B(0, \varepsilon)$, and this implies $u_\varepsilon(y) - u_\varepsilon(x) > 0$. Letting $\varepsilon \to 0$, we obtain the implication

$$\langle \pi_1 \log(x^{-1}y), \nu\rangle > 0 \Rightarrow u(y) \geq u(x)$$

for every $(x, y) \in \mathbb{G} \times \mathbb{G}$. For every $t \in (0, 1)$, define the set $E_t \coloneqq \{\xi \in \mathbb{G} : u(\xi) > t\}$. For every $(x, y) \in E_t \times E_t^c$ one has $u(x) > u(y)$ and therefore $\langle \pi_1 \log x, \nu\rangle \geq \langle \pi_1 \log y, \nu\rangle$. By Dedekind's Theorem, for every $t \in (0, 1)$, there exists $\lambda_t \in \mathbb{R}$ such that $E_t \subseteq \{\xi \in \mathbb{G} : \langle \pi_1 \log \xi, \nu\rangle \geq \lambda_t\}$ and $E_t^c \subseteq \{\xi \in \mathbb{G} : \langle \pi_1 \log \xi, \nu\rangle \leq \lambda_t\}$.
This implies that for all $t \in (0, 1)$ one has

$$\mathcal{H}^Q(E_t \Delta \{\xi \in \mathbb{G} : \langle \pi_1 \log \xi, \nu\rangle \geq \lambda_t\}) = 0.$$

Combining this with the fact that $u = \chi_H$ almost everywhere on B^c, we get that $\lambda_t = 0$ for every $t \in (0, 1)$, and therefore

$$\mathcal{H}^Q(E_t \Delta H) = 0,$$

for every $t \in (0, 1)$. Consider now a sequence (t_j) in $(0, 1)$ that converges to 0 as $j \to +\infty$. Since u has values in $[0, 1]$, we get

$$\{\xi \in \mathbb{G} : u(\xi) \leq 0\} = \{\xi \in \mathbb{G} : u(\xi) = 0\} = \bigcap_{j \in \mathbb{N}} E_{t_j}^c,$$

and similarly

$$\{\xi \in \mathbb{G} : u(\xi) = 1\} = \bigcap_{j \in \mathbb{N}} E_{1-t_j}.$$

The proof is completed by observing that the identities $\mathcal{H}^Q(\{\xi \in \mathbb{G} : u(\xi) = 0\}\Delta H^c) = 0$ and $\mathcal{H}^Q(\{\xi \in \mathbb{G} : u(\xi) = 1\}\Delta H) = 0$ hold. □

Proposition 5.7. *Let Ω be an open set and let $u \in BV_{\mathbb{G}}(\Omega)$. Let $p \in \Omega$, let $r > 0$ be such that $\overline{B(p, 2r)} \subseteq \Omega$ and let $g \in B(0, r)$. Then*

$$\int_{B(p,r)} |u(x \cdot g) - u(x)|\, dx \leq d(0, g)|D_X u|(\Omega).$$

Proof. Fix a basis $(X_1, \ldots, X_m)$ and assume without loss of generality that $u \in C^\infty(\Omega)$. Let $\varepsilon > 0$ and let $\gamma\colon [0, 1] \to \mathbb{G}$ be a horizontal curve satisfying

$$\gamma(0) = 0, \quad \gamma(1) = g \quad \text{and} \quad \dot\gamma(t) = \sum_{i=1}^m h_i(t) X_i(\gamma(t)) \quad \text{for a. e. } t \in [0, 1],$$

where $(h_1, \ldots, h_m) \in L^\infty([0, 1]; \mathbb{R}^m)$ with $\|(h_1, \ldots, h_m)\|_\infty \leq d(g, 0) + \varepsilon$. Notice that, for every $x \in \mathbb{G}$, the curve $\gamma_x\colon [0, 1] \to \mathbb{G}$ defined by $\gamma_x(t) = x \cdot \gamma(t)$ is horizontal, joins x and $x \cdot g$, and $\|\dot\gamma_x\|_\infty = \|(h_1, \ldots, h_m)\|_\infty$. Therefore, for any $x \in B(p, r)$, one has

$$|u(x \cdot g) - u(x)| = \left| \int_0^1 \frac{d}{dt} u(\gamma_x(t))\, dt \right| \leq (d(g, 0) + \varepsilon) \int_0^1 \|\nabla_X u(\gamma_x(t))\|\, dt.$$

Integrating both sides on $B(p,r)$ we get

$$\int_{B(p,r)} |u(x\cdot g)-u(x)|\,dx \le (d(g,0)+\varepsilon)\int_{B(p,r)}\int_0^1 \|\nabla_X u\,(x\cdot\gamma(t))\,\|\,dt\,dx,$$

and exchanging the order of integration we get

$$\int_{B(p,r)} |u(x\cdot g)-u(x)|\,dx \le (d(g,0)+\varepsilon)\int_0^1\int_{B(p,2r)} \|\nabla_X u(\xi)\|\,d\xi\,dt.$$

Notice that in the last inequality we have used that $d(x\cdot\gamma(t),p)\le 2r$, for almost every $t\in[0,1]$. Indeed, by the triangular inequality and the assumption on g, one has

$$d(x\cdot\gamma(t),p)\le d(x\cdot\gamma(t),x)+d(x,p)=d(\gamma(t),0)+d(x,p)\le r+r=2r.$$

Finally, since $\overline{B(p,2r)}\subseteq\Omega$ one gets

$$\begin{aligned}\int_{B(p,r)} |u(x\cdot g)-u(x)|\,dx &\le (d(g,0)+\varepsilon)\int_0^1 |D_X u|(B(p,2r))\,dt\\ &\le (d(g,0)+\varepsilon)\,|D_X u|(\Omega).\end{aligned}$$

By the arbitrariness of ε, the proof is complete. □

Before proving the following proposition we introduce the notation

$$C(K):=\int_{\mathbb{G}} K(\xi)d(\xi,0)\,d\xi.$$

Proposition 5.8. *Let $E,F\subseteq\mathbb{G}$ be measurable sets. Then the following facts hold.*

(i) If $N\subseteq\mathbb{G}$ is a set of finite perimeter in $\mathbb{G}$ such that $E\subseteq N$ and $F\subseteq N^c$, then

$$\limsup_{\varepsilon\to0}\frac{1}{\varepsilon}L_\varepsilon(E,F)\le\frac{C(K)}{2}P(N;\mathbb{G}).$$

(ii) If $d(E,F)>0$ and $C(K)<+\infty$ then,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}L_\varepsilon(E,F)=0.$$

(iii) If $\mu(F)<+\infty$ and $d(E;F)>0$, then

$$\lim_{\varepsilon\to0}L_\varepsilon(E,F)=0.$$

Proof. (i) By a change of variable formula and Proposition 5.7 we have

$$\begin{aligned}\frac{1}{\varepsilon}L_\varepsilon(E,F)&\le\frac{1}{\varepsilon}\int_N\int_{N^c}\frac{1}{\varepsilon^Q}K(\delta_{1/\varepsilon}(y^{-1}x))\,dydx\\ &=\frac{1}{2\varepsilon}\int_{\mathbb{G}}\int_{\mathbb{G}}K(g)|\chi_N(x\delta_\varepsilon g)-\chi_N(x)|\,dgdx\\ &\le\frac{1}{2}P(N;\mathbb{G})\int_{\mathbb{G}}K(g)d(g,0)\,dg.\end{aligned}$$

(ii) Denote by $\eta := \min\{1, d(E,F)\} > 0$. Again by a change of variable formula, we can write

$$\begin{aligned}\frac{1}{\varepsilon}L_\varepsilon(E,F) &\leq \frac{1}{\eta\varepsilon}\int_E\int_F K_\varepsilon(y^{-1}x)\min\{1, d(y,x)\}\,dxdy \\ &= \frac{1}{\eta}\int_E\int_{\mathbb{G}} K(g)\min\{1, d(g,0)\}\chi_F(y\delta_\varepsilon g)\,dgdy.\end{aligned}$$

By noticing that $\chi_F(y\delta_\varepsilon g)$ converges to 0 as $\varepsilon \to 0$, for almost every $y \in E$, we conclude the proof by means of the Dominated Convergence Theorem.
(iii). By definition of L_ε and by a change of variable formula, we have

$$L_\varepsilon(E,F) = \frac{1}{\varepsilon^Q}\int_E\int_F K(\delta_{1/\varepsilon}(y^{-1}x))\,dydx = \varepsilon^Q\int_{\delta_{1/\varepsilon}E}\int_{\delta_{1/\varepsilon}F} K(y^{-1}x)\,dydx$$

Denoting by $\eta := d(E,F) > 0$ and by $F^r := \{x \in \mathbb{G} : d(x,F) \geq r\}$, for any positive r, we notice that $\delta_{1/\varepsilon}E \subseteq \delta_{1/\varepsilon}F^{\eta/\varepsilon}$ and therefore

$$\begin{aligned}L_\varepsilon(E,F) &\leq \varepsilon^Q\int_{\delta_{1/\varepsilon}F^{\eta/\varepsilon}}\int_{\delta_{1/\varepsilon}F} K(y^{-1}x)\,dydx = \varepsilon^Q\mu(\delta_{1/\varepsilon}F)\int_{B(0,\varepsilon^{-1}\eta)^c} K(\xi)\,d\xi \\ &= \mu(F)\int_{B(0,\varepsilon^{-1}\eta)^c} K(\xi)\,d\xi.\end{aligned}$$

The thesis then follows by (5.3). □

The following Theorem provides a compactness criterion in $L^1(\Omega)$ for our functional with a geometrical prescription on the domain Ω; namely we require that Ω is a John domain, a condition that generalize the cone condition treated e.g. in [AF03]. We put off the reader to definition A.12. Before we state it, we remark the validity of the following fact, whose proof is an immediate calculation.

Lemma 5.9. *Let $G \in L^1(\mathbb{G})$ be a positive function. Then, for any $u \in L^\infty(\mathbb{G})$ it holds that*

$$\int_{\mathbb{G}\times\mathbb{G}} (G * G)(y)|u(x\cdot y) - u(x)|\,dydx \leq 2\,\|G\|_{L^1(\mathbb{G})}\,J_G(u,\mathbb{G}).$$

In particular, if we choose $u = \chi_E$ we have

$$\int_{\mathbb{G}\times\mathbb{G}} (G * G)(y)|\chi_E(x\cdot y) - \chi_E(x)|\,dydx \leq 4\,\|G\|_{L^1(\mathbb{G})}\,P_G(E,\mathbb{G}).$$

Theorem 5.10. *Let $\Omega \subseteq \mathbb{G}$ be an open John domain with finite measure, let (ε_n) be an infinitesimal sequence of positive numbers and let (E_n) be a sequence of measurable sets in Ω. Assume that Ω is a John domain and that there exists $C > 0$ such that*

$$\frac{1}{\varepsilon_n}J_{\varepsilon_n}(E_n,\Omega) \leq C, \quad \forall n \in \mathbb{N}. \tag{5.13}$$

Then, there exist a subsequence (E_{n_k}) of (E_n) and a set E of finite perimeter in Ω such that (E_{n_k}) converges to E in $L^1(\Omega)$.

Proof. We write E_ε in place of E_n, to avoid inconvenient notation. Let φ be a positive function in $C_c^\infty(\mathbb{G}) \setminus \{0\}$ and define for every $\varepsilon > 0$ the map

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^Q \int_{\mathbb{G}} \varphi(\xi)\, d\xi} \varphi(\delta_{1/\varepsilon} x),$$

and consequently set $v_\varepsilon := \varphi_\varepsilon * \chi_{E_\varepsilon}$. We can therefore estimate

$$\begin{aligned} \int_{\mathbb{G}} |v_\varepsilon(\xi) - \chi_{E_\varepsilon}(\xi)|\, d\xi &\leq \int_{\mathbb{G}} \int_{\mathbb{G}} \varphi_\varepsilon(\eta^{-1}\xi) |\chi_{E_\varepsilon}(\eta) - \chi_{E_\varepsilon}(\xi)|\, d\eta d\xi \\ &= \int_{\mathbb{G}} \int_{\mathbb{G}} \varphi_\varepsilon(\xi) |\chi_{E_\varepsilon}(\eta\xi) - \chi_{E_\varepsilon}(\eta)|\, d\eta d\xi. \end{aligned} \tag{5.14}$$

Reasoning in a similar way on the horizontal gradient of v_ε we get

$$\begin{aligned} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} v_\varepsilon(\xi)|\, d\xi &= \int_{\mathbb{G}} \left| \int_{\mathbb{G}} \nabla_{\mathbb{G}} \varphi_\varepsilon(\eta^{-1}\xi) \chi_{E_\varepsilon}(\eta)\, d\eta \right| d\xi \\ &\leq \int_{\mathbb{G}} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} \varphi_\varepsilon(\eta^{-1}\xi)| |\chi_{E_\varepsilon}(\eta) - \chi_{E_\varepsilon}(\xi)|\, d\eta d\xi \\ &\quad + \int_{\mathbb{G}} \chi_{E_\varepsilon}(\xi) \left| \int_{G} \nabla_{\mathbb{G}} \varphi_\varepsilon(\eta^{-1}\xi)\, d\eta \right| d\xi \\ &= \int_{\mathbb{G}} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} \varphi_\varepsilon(\xi)| |\chi_{E_\varepsilon}(\eta\xi) - \chi_{E_\varepsilon}(\eta)|\, d\eta d\xi. \end{aligned} \tag{5.15}$$

Define the map

$$T(s) := \begin{cases} s & \text{if } |s| \leq 1, \\ 1 & \text{otherwise}, \end{cases}$$

and consider the truncated kernel $G := T \circ \min\{1, d(\cdot, 0)\} K$. We notice that $G \geq 0$ and, since $G \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, the map $G * G$ is continuous. This is a consequence of the estimate

$$|(G * G)(p) - (G * G)(q)| \leq \|G\|_\infty \|\tau_{p^{-1}q} G - G\|_1,$$

and the Dominated Convergence Theorem. We now choose a positive $\varphi \in C_c^\infty(\mathbb{G}) \setminus \{0\}$ such that

$$\varphi \leq G * G \quad \text{and} \quad |\nabla_{\mathbb{G}} \varphi| \leq G * G.$$

Setting $G_\varepsilon(\xi) := \varepsilon^{-Q} G(\delta_{1/\varepsilon}\xi)$, and taking (5.14) and (5.15) into account we obtain

$$\int_{\mathbb{G}} |v_\varepsilon(\xi) - \chi_{E_\varepsilon}(\xi)|\, d\xi \leq \int_{\mathbb{G}} \int_{\mathbb{G}} (G_\varepsilon * G_\varepsilon)(\xi) |\chi_{E_\varepsilon}(\eta\xi) - \chi_{E_\varepsilon}(\eta)|\, d\eta d\xi, \tag{5.16}$$

and

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} v_\varepsilon(\xi)|\, d\xi \leq \frac{1}{\varepsilon} \int_{\mathbb{G}} \int_{\mathbb{G}} (G_\varepsilon * G_\varepsilon)(\xi) |\chi_{E_\varepsilon}(\eta\xi) - \chi_{E_\varepsilon}(\eta)|\, d\eta d\xi, \tag{5.17}$$

where the last inequality comes from the fact that

$$(\nabla_{\mathbb{G}} \varphi_\varepsilon)(\xi) = \frac{1}{\varepsilon^{Q+1}} (\nabla_{\mathbb{G}} \varphi)(\delta_{1/\varepsilon}\xi),$$

and that

$$(G_\varepsilon * G_\varepsilon)(\xi) = \frac{1}{\varepsilon^Q} (G * G)(\delta_{1/\varepsilon}\xi).$$

By applying Lemma 5.9, we then have

$$\begin{aligned}\int_{\mathbb{G}}\int_{\mathbb{G}}(G_\varepsilon * G_\varepsilon)(\xi)|\chi_{E_\varepsilon}(\eta\xi)-\chi_{E_\varepsilon}(\eta)|\,d\eta d\xi &\le 4\|G\|_1 P_{G_\varepsilon}(E_\varepsilon)\\ &\le 4\|G\|_1 P_{K_\varepsilon}(E_\varepsilon) = 4\|G\|_1\left(\frac{1}{2}J^1_\varepsilon(E_\varepsilon,\Omega)+J^2_\varepsilon(E_\varepsilon,\Omega)\right)\\ &= 4\|G\|_1 J_\varepsilon(E_\varepsilon;\Omega).\end{aligned}$$

Condition (5.13) then gives $M>0$ such that

$$\frac{1}{\varepsilon}\int_{\mathbb{G}}\int_{\mathbb{G}}(G_\varepsilon * G_\varepsilon)(\xi)|\chi_{E_\varepsilon}(\eta\xi)-\chi_{E_\varepsilon}(\eta)|\,d\eta d\xi \le M\|G\|_1.$$

Since Ω has finite measure, the estimates (5.16) and (5.17) imply that (v_ε) is equibounded in $W^{1,1}_{\mathbb{G}}(\Omega)$ and therefore, since Ω is a John domain, by Theorem A.13, up to subsequences, v_ε converges in $L^1(\Omega)$ to some w. We moreover observe that (5.16) also tells us that $w=\chi_E$ for some E with finite measure in Ω. Inequality (5.17) together with the lower semicontinuity of the total variation implies that E has finite perimeter in Ω. □

Remark 5.11. *In case Ω has finite perimeter and the stronger integrability condition*

$$\int_{\mathbb{G}} K(x)d(x,0)\,dx < +\infty \tag{5.18}$$

is satisfied, Theorem 5.10 can be strengthened replacing condition (5.13) with the weaker

$$\frac{1}{\varepsilon_n}J^1_{\varepsilon_n}(E_n,\Omega)\le C, \quad \forall n\in\mathbb{N}.$$

Indeed, applying (i) of Proposition 5.8 with $N=\Omega$ one some $C_2>0$ such that

$$\frac{1}{\varepsilon_n}J^2_{\varepsilon_n}(E_{\varepsilon_n},\Omega)=\frac{1}{\varepsilon_n}L_{\varepsilon_n}(\Omega\cap E_{\varepsilon_n},\Omega^c\cap E^c_{\varepsilon_n})\le\frac{1}{2}P(\Omega;\mathbb{G})\int_{\mathbb{G}}K(x)d(x,0)\,dx\le C_2, \quad \forall n\in\mathbb{N}.$$

Notice however that condition (5.18) is in contrast with (5.20) below, that will be used in Theorem 5.14.

5.3 A Γ-liminf inequality

Denote for shortness $B:=B(0,1)$. For every halfspace $H\subseteq\mathbb{G}$ we set

$$b(H):=\inf\left\{\liminf_{\varepsilon\to 0}\frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,B(0,1)) : E_\varepsilon\to H \text{ in } L^1(B(0,1))\right\}. \tag{5.19}$$

Proposition 5.12. *The following facts hold*

(i) Assume that

$$\inf_{r>1} K(r)r^{Q+1}>0. \tag{5.20}$$

Then

$$\inf\{b(H) : H \textit{ is a vertical halfspace}\}>0.$$

(ii) If $\mathbb{G}$ is a free group and $H_1, H_2 \subseteq \mathbb{G}$ are vertical halfspaces in $\mathbb{G}$, then $b(H_1) = b(H_2)$.

Proof. (i). Fix a halfspace H. We first prove that $b(H) > 0$. By definition of $b(H)$ and a diagonal argument, there exists a sequence E_ε that converges to χ_H in $L^1(B)$ as $\varepsilon \to 0$ such that

$$\liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} J^1_\varepsilon(E_\varepsilon, B) = b(H).$$

Thanks to Severini-Egorov's Theorem there exists an open set $A \subseteq B$ such that

$$\mathcal{H}^Q(B \setminus A) < \frac{\mathcal{H}^Q(H \cap B)}{2} \tag{5.21}$$

and χ_{E_ε} converges to χ_H uniformly on A, as $\varepsilon \to 0$. We therefore find ε_0 such that

$$\sup_{x \in A} |\chi_{E_\varepsilon}(x) - \chi_H(x)| < 1, \quad \forall \varepsilon \leq \varepsilon_0,$$

and hence, for every $\varepsilon \leq \varepsilon_0$ we have $E_\varepsilon \cap A = H \cap A =: C^+$. By reasoning in the same way on E^c_ε, we may assume without loss of generality that, for every $\varepsilon \leq \varepsilon_0$, we also have $E^c_\varepsilon \cap A = H^c \cap A =: C^-$. Notice that, by (5.21), we have

$$\min\{\mathcal{H}^Q(C^+), \mathcal{H}^Q(C^-)\} > 0. \tag{5.22}$$

For every $\varepsilon \leq \varepsilon_0$, we have

$$\begin{aligned}
\frac{1}{2\varepsilon} J^1_\varepsilon(E_\varepsilon, B) &= \frac{1}{2\varepsilon} \int_{E_\varepsilon} \int_{E^c_\varepsilon \cap B} K_\varepsilon(y^{-1}x)\, dydx \geq \frac{\varepsilon^{Q-1}}{2} \int_{\delta 1/\varepsilon C^+} \int_{\delta 1/\varepsilon C^-} K(y^{-1}x)\, dydx \\
&\geq \frac{\varepsilon^{Q-1}}{2} K(\operatorname{diam}(\delta_{1/\varepsilon} C^+ \cup \delta_{1/\varepsilon} C^-)) \mathcal{H}^Q(\delta_{1/\varepsilon} C^+) \mathcal{H}^Q(\delta_{1/\varepsilon} C^-) \\
&= \frac{1}{2\varepsilon^{Q+1}} K\left(\frac{\operatorname{diam}(C^+ \cup C^-)}{\varepsilon}\right) \mathcal{H}^Q(C^+) \mathcal{H}^Q(C^-),
\end{aligned}$$

which, by (5.20) and (5.22), is a positive lower bound independent of ε.

To conclude the proof of (i), it is enough to check that b is lower-semicontinuous. In fact, if this were true, by the compactness of the sphere $\mathbb{S}^{m-1}$, we would have that b admits a minimum, that, by the previous step would be strictly positive.
Let H_η be a family of vertical halfspaces that converges to H in $L^1(B)$, as $\eta \to 0$. Fix $\sigma > 0$. For every $\eta > 0$ we can find F^η_ε converging to H_η in $L^1(B)$, as $\varepsilon \to 0$ such that

$$\liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} J^1_\varepsilon(F^\eta_\varepsilon, B) \leq b(H_\eta) + \sigma.$$

Considering $E_\varepsilon := F^\varepsilon_\varepsilon$, we easily find that $E_\varepsilon \to H$ in $L^1(B)$, as $\varepsilon \to 0$ and hence

$$b(H) \leq \liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} J^1_\varepsilon(E_\varepsilon, B) \leq \liminf_{\varepsilon \to 0} b(H_\varepsilon) + \sigma.$$

The thesis follows by the arbitrariness of σ.

(ii). Let $\nu_1, \nu_2 \in \mathfrak{g}_1 \setminus \{0\}$ such that $H_1 = H_{\nu_1}$ and $H_2 = H_{\nu_2}$. It is enough to show that $b(H_1) \leq b(H_2)$. Let E^2_ε be a family of measurable set in B such that $E^2_\varepsilon \to H_{\nu_2}$ in $L^1(B)$ as $\varepsilon \to 0$. Now consider an orthogonal isomorphism $T : \mathfrak{g}_1 \to \mathfrak{g}_1$ such that $T(\nu_2) = \nu_1$. Since

$\mathbb{G}$ is free, the map T extends in a unique way to a Lie algebra isomorphism $T : \mathfrak{g} \to \mathfrak{g}$ that induces an isometry $I : \mathbb{G} \to \mathbb{G}$ defined by

$$I := \exp \circ T \circ \log .$$

We claim that $I(H_2) = H_1$. Indeed, for every $\xi \in \mathbb{G}$, one has

$$\begin{aligned}\langle \pi_1 \log \xi, \nu_1 \rangle =& \langle \pi_1 \log \xi, T(\nu_2) \rangle = \langle T(\pi_1 \log \xi), \nu_2 \rangle \\ =& \langle \pi_1 T(\log \xi), \nu_2 \rangle = \langle \pi_1 \log I(\xi), \nu_2 \rangle .\end{aligned}$$

Since K is radial and I is an isometry, it is easy to see that $J^1(A, B) = J^1(I(A), I(B))$. By noticing that $I(B) = B$ and that $I(E^2_\varepsilon) \to H_1$ in $L^1(B)$ as $\varepsilon \to 0$, we have that

$$b(H_1) \leq \liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} J^1_\varepsilon(I(E^2_\varepsilon), B) = \liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} J^1_\varepsilon(E^2_\varepsilon, B),$$

whence $b(H_1) \leq b(H_2)$. $\square$

Remark 5.13. *Let $\mathbb{G}$ be a Carnot group satisfying property $\mathcal{R}$ and let E be a set of locally finite perimeter in some open set $\Omega \subseteq \mathbb{G}$. Then, by* [FSSC03, Lemma 3.8]*, if $\mathbb{G}$ satisfies property $\mathcal{R}$, for every $p \in \mathcal{F}E$, one has*

$$\lim_{r \to 0} \frac{P_{\mathbb{G}}(E; B(p, r))}{r^{Q-1}} = P_{\mathbb{G}}(H_{\nu_E(p)}; B(0, 1)) =: \vartheta(\nu_E(p)). \tag{5.23}$$

Notice also that, since H_ν has smooth boundary, for any $\nu \in \mathfrak{g}$, its perimeter can be explicitly computed (up to identification of $\mathbb{G}$ with $\mathbb{R}^n$ by means of exponential coordinates) getting

$$\vartheta(\nu) = \mathcal{H}^{n-1}_e(\partial H_\nu \cap B(0, 1)),$$

where $\mathcal{H}_e$ denotes the Hausdorff measure with respect to the Euclidean metric (see e.g. [Mon01, Theorem 5.1.3.] *or* [FSSC03, Proposition 2.22]*).*

Theorem 5.14. *Let $\mathbb{G}$ be a Carnot group satisfying property $\mathcal{R}$, let Ω be an open and bounded John domain in $\mathbb{G}$ and assume $K\colon \mathbb{G} \to [0, +\infty)$ satisfies (5.20). Then, there exists a positive density $\rho\colon \mathfrak{g}_1 \to (0, +\infty)$ such that, for every family (E_ε) of measurable sets converging in $L^1(\Omega)$ to $E \subseteq \Omega$, one has*

$$\int_\Omega \rho(\nu_E)\, dP_{\mathbb{G}}(E; \cdot) \leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} J_\varepsilon(E_\varepsilon, \Omega). \tag{5.24}$$

More precisely, for every $\nu \in \mathfrak{g}_1$, one has

$$\rho(\nu) = \frac{b(H_\nu)}{\vartheta(\nu)}.$$

Proof. Define, for every $\varepsilon > 0$, the function

$$f_\varepsilon(\xi) := \begin{cases} \dfrac{1}{2\varepsilon} \displaystyle\int_{E^c_\varepsilon \cap \Omega} K_\varepsilon(\eta^{-1}\xi)\, d\eta + \frac{1}{\varepsilon} \int_{\Omega^c \cap E^c_\varepsilon} K(\eta^{-1}\xi)\, d\eta, & \text{if } \xi \in E_\varepsilon \\ \dfrac{1}{2\varepsilon} \displaystyle\int_{E_\varepsilon \cap \Omega} K_\varepsilon(\eta^{-1}\xi)\, d\eta, & \text{if } \xi \in E^c_\varepsilon, \end{cases}$$

and set $\nu_\varepsilon := f_\varepsilon \mathcal{H}^Q_{|\Omega}$. Notice that

$$\|\nu_\varepsilon\| = |\nu_\varepsilon|(\Omega) = \frac{1}{2\varepsilon} J_\varepsilon(E_\varepsilon, \Omega).$$

Without loss of generality we can assume that there exists $M > 0$ such that

$$\frac{1}{\varepsilon} J_\varepsilon(E_\varepsilon, \Omega) \leq M, \quad \forall \varepsilon > 0.$$

By this uniform bound and the assumptions on Ω, we get that, by Theorem 5.10, E has finite perimeter in Ω. Moreover, by a standard argument of Measure Theory (see e.g. [AFP00]), we find a positive measure ν such that $\nu_\varepsilon \rightharpoonup^* \nu$ up to subsequences as $\varepsilon \to 0$, and hence

$$\|\nu\| \leq \liminf_{\varepsilon \to 0} \|\nu_\varepsilon\|.$$

To prove (5.24), it is enough to show that

$$\|\nu\| \geq \int_\Omega \rho(\nu_E)\, dP_{\mathbb{G}}(E;\cdot),$$

for some ρ that will be determined in the sequel. Letting $P_E := P_K(E;\cdot)$, we aim to prove that

$$\frac{d\nu}{dP_E}(p) \geq \rho(\nu_E(p)), \quad \text{for } P_E\text{-a.e. } p \in \Omega,$$

where $\frac{d\nu}{dP_E}(p)$ denotes the Radon-Nikodym derivative of ν with respect to P_E. Fix $p \in \mathcal{F}E \cap \Omega$. Since $\mathbb{G}$ satisfies property $\mathcal{R}$, by (5.23) we have

$$\frac{d\nu}{dP_E}(p) = \frac{1}{\vartheta(\nu_E(p))} \lim_{r \to 0} \frac{\nu(B(p,r))}{r^{Q-1}}.$$

Since ν_ε weakly* converges to ν as $\varepsilon \to 0$, we have that $\nu_\varepsilon(B(p,r))$ converges to $\nu(B(p,r))$ for every $r > 0$ outside a countable subset $Z \subseteq (0, +\infty)$ of radii. We therefore have

$$\frac{d\nu}{dP_E}(p) = \frac{1}{\vartheta(\nu_E(p))} \lim_{r \to 0, r \notin Z} \left(\lim_{\varepsilon \to 0} \frac{\nu_\varepsilon(B(p,r))}{r^{Q-1}} \right).$$

By a diagonal argument, we may choose two infinitesimal sequences (ε_j) and (r_j) such that

$$\lim_j \frac{\varepsilon_j}{r_j} = 0,$$

and so that

$$\frac{d\nu}{dP_E}(p) = \frac{1}{\vartheta(\nu_E(p))} \lim_j \frac{\nu_{\varepsilon_j}(B(p,r_j))}{r_j^{Q-1}}.$$

By making the computation explicit, we can write

$$\begin{aligned}\frac{d\nu}{dP_E}(p) = \frac{1}{\vartheta(\nu_E(p))} \lim_j \frac{1}{\varepsilon_j r_j^{Q-1}} \Bigg(&\frac{1}{2} \int_{E_{\varepsilon_j} \cap \Omega \cap B(p,r_j)} \int_{E^c_{\varepsilon_j} \cap \Omega} K_{\varepsilon_j}(y^{-1}x)\, dy dx \\ &+ \frac{1}{2} \int_{E^c_{\varepsilon_j} \cap \Omega \cap B(p,r_j)} \int_{E_{\varepsilon_j} \cap \Omega} K_{\varepsilon_j}(y^{-1}x)\, dy dx \\ &+ \int_{E_{\varepsilon_j} \cap \Omega \cap B(p,r_j)} \int_{\Omega^c \cap E^c_\varepsilon} K_{\varepsilon_j}(y^{-1}x)\, dy dx \Bigg),\end{aligned}$$

and hence, since $J_\varepsilon \geq J^1_\varepsilon$ and since, for j sufficiently large, one has $B(p, r_j) \subseteq \Omega$, we get

$$\begin{aligned}\frac{d\nu}{dP_E}(p) &\geq \frac{1}{\vartheta(\nu_E(p))} \limsup_j \frac{1}{2\varepsilon_j r_j^{Q-1}} J^1_{\varepsilon_j}(E_{\varepsilon_j}, B(p, r_j) \cap \Omega)\\ &= \frac{1}{\vartheta(\nu_E(p))} \limsup_j \frac{1}{2\varepsilon_j r_j^{Q-1}} J^1_{\varepsilon_j}(E_{\varepsilon_j}, B(p, r_j)).\end{aligned}$$

By a change of variable, since J^1 is left unchanged by isometries, we have

$$J^1_{\varepsilon_j}(E_{\varepsilon_j}, B(p, r_j)) = r_j^Q J^1_{\varepsilon_j/r_j}\left(\delta_{1/r_j} p^{-1} E_{\varepsilon_j}, B\right).$$

This implies that

$$\frac{d\nu}{dP_E}(p) \geq \frac{1}{\vartheta(\nu_E(p))} \limsup_j \frac{r_j}{2\varepsilon_j} J^1_{\varepsilon_j r_j}\left(\delta_{1/r_j} p^{-1} E_{\varepsilon_j}, B\right).$$

Since, by property $\mathcal{R}$, the sequence $\delta_{1/\varepsilon_j} p^{-1} E_{\varepsilon_j}$ is converging to H in $L^1(B)$ as $j \to \infty$ we get

$$\frac{d\nu}{dP_E}(p) \geq \frac{1}{\vartheta(\nu_E(p))} b(H_{\nu_E(p)}). \qquad \square$$

5.4 Applications

In this section we want to observe that Theorems 5.2 and 5.14 hold even for a particular kernel induced by the sub-Riemannian heat kernel; the connection between the fractional perimeter and the asymptotic behaviour of the fractional heat semigroup in Carnot Groups has been analyzed in [FMP+18].

Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q, $\alpha \in (0,1)$ and let $\widetilde{R}_\alpha \colon \mathbb{G} \to [0, +\infty)$ be defined as

$$\widetilde{R}_\alpha(x) := -\frac{\alpha}{2\Gamma(-\alpha/2)} \int_0^{+\infty} t^{-\frac{\alpha}{2}-1} h(t, x) dt,$$

where $h \colon [0, +\infty) \times \mathbb{G} \to \mathbb{R}$ is the fundamental solution of the sub-Riemannian heat operator

$$\mathcal{H} := \partial_t + \mathcal{L},$$

where

$$\mathcal{L} := \sum_{i=1}^m X_i^2$$

is a positive sub-Laplacian associated with a basis $(X_1, \dots, X_m)$ of the horizontal layer $\mathfrak{g}_1$ of $\mathbb{G}$. Notice that $\widetilde{R}_\alpha(x^{-1}) = \widetilde{R}_\alpha(x)$ and $\widetilde{R}_\alpha(\delta_\lambda x) = \lambda^{-\alpha-Q} \widetilde{R}_\alpha(x)$ for any $x \in \mathbb{G}$ and $\lambda \geq 0$. The quantity

$$\|x\|_\alpha := \left(\widetilde{R}_\alpha(x)\right)^{-\frac{1}{\alpha+Q}},$$

defines a homogeneous symmetric norm on $\mathbb{G}$ and therefore it is equivalent to the norm induced by the Carnot-Carathéodory distance. In particular, K_α satisfies conditions (5.1), (5.2), (5.3) and (5.20). All the results obtained in this paper therefore apply to the special case $K = K_\alpha$.

5.5 Open problems

We are unable to prove that the horizontal perimeter is actually a Γ-limit for the sequence of rescaled nonlocal perimeters; in fact to do this, we should prove the Γ-limsup inequality, namely that for every set of finite perimeter E in Ω, there exists a sequence E_ε of measurable sets converging in L^1 to E and such that

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} J_\varepsilon(E_\varepsilon, \Omega) \leq \int_\Omega \rho(\nu_E) dP_{\mathbb{G}}(E; \cdot),$$

but in the euclidean setting (see [BP19, Proposition 3.6.])the following result is exploited

Theorem 5.15. [Dáv02, Theorem 1] *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary, $u \in BV(\Omega)$, and consider a sequence $(\rho_j)_{j\in\mathbb{N}}$ of positive radial mollifiers. Then*

$$\lim_{j\to+\infty} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|} \rho_j(x-y) dx dy = C_{1,n} |Du|(\Omega),$$

where

$$C_{1,n} := \frac{1}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} |v \cdot e| d\mathcal{H}^{n-1}(v),$$

for some $e \in \mathbb{S}^{n-1}$.

Unfortunately, to the knowledge of the authors, an analogue of Davila's result is not known in the framework of Carnot Groups.

We propose to investigate some asymptotic results in a future work; namely, if the kernel K is the fractional kernel $K(p) := d(p, 0)^{-Q-\alpha}$ for some $\alpha \in (0, 1)$, it would be interesting to study the following limits

$$\lim_{\alpha\to 0^+} \alpha P_K(E, \Omega) \tag{5.25}$$

and

$$\lim_{\alpha\to 1^-} (1-\alpha) P_K(E, \Omega). \tag{5.26}$$

In Carnot Groups some asymptotic results are obtained in [MP19, Section 5.2.] when $\Omega = \mathbb{G}$. In the euclidean setting we refer to [DFPV13], where the authors provide necessary and sufficient conditions so that the limit in (5.25) exist and coincide with the Lebesgue measure of E, up to multiplicative costants, while concerning the limit in (5.26) we refer e.g. to [ADPM11, CV11] where the convergence to the classical perimeter is proved.

Appendices

Appendix A

Appendix

In this appendix, we give some details on technical tools used in this thesis.

A.1 Well-posedness and polynomial extension for Caputo fractional derivatives

Following [CDV18], we remark that Caputo-stationary functions with initial point $-\infty$ that have vanishing kth derivative near $-\infty$ are also Caputo-stationary for a fixed point beyond its constancy interval. To do this, we introduce the natural setting in which Caputo fractional derivatives are defined.

If $I \subseteq \mathbb{R}$ is an interval, we define the space

$$AC^{k-1}(I) := \left\{ f \in C^{k-1}(I) \text{ s.t. } f, f', \dots, f^{(k-1)} \in AC(I) \right\},$$

where $C^{k-1}(I)$ denotes the space of $(k-1)$-times continuously differentiable functions on I, and $AC(I)$ denotes the space of absolutely continuous functions on I.

Given $t > a$, $k \in \mathbb{N}$, $\beta > 0$, and $f : [a, +\infty) \to \mathbb{R}$, we also define the function

$$(a,t) \ni \tau \mapsto \Theta_{k,\beta,f,t}(\tau) := f^{(k)}(\tau)(t-\tau)^{k-\beta-1} \tag{A.1}$$

and we set

$$\begin{aligned} C^{k,\beta}_{a+} := \Big\{ f : \overline{(a,+\infty)} \to \mathbb{R} \ \text{ s.t. } \ & f \in AC^{k-1}\big(\overline{(a,t)}\big) \\ \text{and } \ & \Theta_{k,\beta,f,t} \in L^1\big((a,t)\big), \ \text{ for all } t > a \Big\}. \end{aligned} \tag{A.2}$$

We observe that the Caputo derivative in (3.6) is well defined for all u belonging to $C^{k,\alpha}_{a+}$.

Analogously, for $t < b$, $f : (-\infty, b] \to \mathbb{R}$, and

$$(t,b) \ni \tau \mapsto \Psi_{k,\beta,f,t}(\tau) := f^{(k)}(\tau)(\tau - t)^{k-\beta-1} \tag{A.3}$$

one can define

$$\begin{aligned} C^{k,\beta}_{b-} := \Big\{ f : \overline{(-\infty,b)} \to \mathbb{R} \ \text{ s.t. } \ & f \in AC^{k-1}\big(\overline{(t,b)}\big) \\ \text{and } \ & \Psi_{k,\beta,f,t} \in L^1\big((t,b)\big), \ \text{ for all } t < b \Big\}. \end{aligned} \tag{A.4}$$

From now on, we will argue only on left derivatives, but the following computations repeat straighforwardly for right derivatives.

A.1.1 Caputo-stationary functions with vanishing kth derivatives near $-\infty$

Lemma A.1. *Let $a \in \mathbb{R}$. Let $I \Subset (a,+\infty)$ be an interval. Let $k \in \mathbb{N}$ and $\alpha \in (k-1,k)$, and assume that $u \in C^{k,\alpha}_{-\infty}$, and that $u^{(k)} = 0$ in $(-\infty, a)$.*

Then,

$$u \in C^{k,\alpha}_{a+} \tag{A.5}$$
$$\text{and} \qquad D^{\alpha}_{a+}[u] = D^{\alpha}_{-\infty}[u] \quad \text{in } I. \tag{A.6}$$

Proof. By (A.2), we see that if $c \in (-\infty, a] \cup \{-\infty\}$, then $C^{k,\alpha}_{c+} \subseteq C^{k,\alpha}_{a+}$, and so (A.5) plainly follows. Furthermore, $u^{(k)}$ vanishes in $(-\infty, a)$, and consequently, for any $t \in I$,

$$0 = \int_{-\infty}^{t} \frac{u^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}}\,d\tau = \int_{a}^{t} \frac{u^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}}\,d\tau,$$

which proves (A.6). $\square$

A counterpart of Lemma A.1 allows us to extend a function with its Taylor polynomial maintaining its Caputo derivative. For this, we first point out that this operation is compatible with the functional setting in (A.2):

Lemma A.2. *Let $a \in \mathbb{R} \cup \{-\infty\}$ and $c \in (a,+\infty)$. Let $k \in \mathbb{N}$ and $\alpha \in (k-1,k)$. Let $f \in C^{k,\alpha}_{a+}$, $g \in C^{k,\alpha}_{c+}$ and assume that*

$$f^{(j)}(c) = g^{(j)}(c) \qquad \text{for all } j \in \{0,\dots,k-1\}. \tag{A.7}$$

Let

$$\overline{(a,+\infty)} \ni t \mapsto h(t) := \begin{cases} f(t) & \text{if } t \in \overline{(a,c)}, \\ g(t) & \text{if } t \in (c,+\infty). \end{cases}$$

Then $h \in C^{k,\alpha}_{a+}$.

Proof. Since $f \in C^{k-1}\big(\overline{(a,+\infty)}\big)$ and $g \in C^{k-1}([c,+\infty))$, we obtain from (A.7) that $h \in C^{k-1}\big(\overline{(a,+\infty)}\big)$, and, for every $t \in \overline{(a,+\infty)}$ and $j \in \{0,\dots,k-1\}$,

$$h^{(j)}(t) = \begin{cases} f^{(j)}(t) & \text{if } t \in \overline{(a,c)}, \\ g^{(j)}(t) & \text{if } t \in (c,+\infty). \end{cases}$$

In particular, we see from (A.7) that

$$h^{(j)}(c) = f^{(j)}(c) = g^{(j)}(c), \qquad \text{for all } j \in \{0,\dots,k-1\}. \tag{A.8}$$

Using that $f^{(j)} \in AC\big(\overline{(a,c)}\big)$ for each $j \in \{0,\dots,k-1\}$, we can write that, for every $t_1, t_2 \in \overline{(a,b)}$,

$$f^{(j)}(t_2) - f^{(j)}(t_1) = \int_{t_1}^{t_2} F_j(\tau)\,d\tau,$$

for a suitable Lebesgue integrable function F_j.

Similarly, if $T > c$, since $g^{(j)} \in AC([c,T])$, we have that for every $t_1, t_2 \in [c,T]$,

$$g^{(j)}(t_2) - g^{(j)}(t_1) = \int_{t_1}^{t_2} G_j(\tau)\, d\tau,$$

for a suitable Lebesgue integrable function G_j.

Then, given $T > c$, we define

$$H_j(t) = \begin{cases} F_j(t) & \text{if } t \in \overline{(a,c)}, \\ G_j(t) & \text{if } t \in (c,T]. \end{cases} \tag{A.9}$$

We have that H_j is Lebesgue integrable and, if $t_1 \in \overline{(a,c)}$ and $t_2 \in (c,T]$, recalling (A.8) we see that

$$\begin{aligned} h^{(j)}(t_2) - h^{(j)}(t_1) &= g^{(j)}(t_2) - f^{(j)}(t_1) \\ &= g^{(j)}(t_2) - g^{(j)}(b) + f^{(j)}(b) - f^{(j)}(t_1) \\ &= \int_b^{t_2} G_j(\tau)\, d\tau + \int_{t_1}^{b} F_j(\tau)\, d\tau \\ &= \int_{t_1}^{t_2} H_j(\tau)\, d\tau. \end{aligned}$$

From this, we conclude that

$$h^{(j)} \in AC\big(\overline{(a,T)}\big) \qquad \text{for all } j \in \{0, \dots, k-1\}. \tag{A.10}$$

Hence, in view of (A.2), to complete the proof of the desired result it remains to check that $\Theta_{k,\alpha,h,T} \in L^1\big((a,T)\big)$, for every $T > a$, namely that

$$\int_a^T \frac{|h^{(k)}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau < +\infty. \tag{A.11}$$

We remark that here $h^{(k)}$ is intended in the Lebesgue sense, being $h^{(k-1)} \in AC\big(\overline{(a,T)}\big)$, due to (A.10). Hence, in the setting of (A.9), we have that $h^{(k)} = H_{k-1}$ and therefore

$$\int_a^T \frac{|h^{(k)}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau = \int_a^T \frac{|H_{k-1}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau. \tag{A.12}$$

Consequently, if $T \le c$ we have that

$$\int_a^T \frac{|h^{(k)}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau = \int_a^T \frac{|F_{k-1}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau = \|\Theta_{k,\alpha,f,T}\|_{L^1(a,T)},$$

which is finite since $f \in C^{k,\alpha}_{a+}$.

If instead $T > c$, we have that

$$\begin{aligned} \int_a^T \frac{|h^{(k)}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau &= \int_a^c \frac{|F_{k-1}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau + \int_c^T \frac{|G_{k-1}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau \\ &\le \int_a^c \frac{|F_{k-1}(\tau)|}{(b-\tau)^{\alpha-k+1}}\, d\tau + \int_c^T \frac{|G_{k-1}(\tau)|}{(T-\tau)^{\alpha-k+1}}\, d\tau \\ &= \|\Theta_{k,\alpha,f,b}\|_{L^1(a,c)} + \|\Theta_{k,\alpha,g,T}\|_{L^1(c,T)}, \end{aligned}$$

which are finite since $f \in C^{k,\alpha}_{a+}$ and $g \in C^{k,\alpha}_{c+}$. This completes the proof of (A.11) and of the desired result. $\square$

With this, we can obtain a counterpart of Lemma A.1 (which is not explicitly used here, but that can be useful for further investigations), as follows:

Lemma A.3. *Let $a \in \mathbb{R} \cup \{-\infty\}$ and $c \in (a, +\infty)$. Let $I \Subset (c, +\infty)$ be an interval. Let $k \in \mathbb{N}$ and $\alpha \in (k-1, k)$, and assume that $u \in C^{k,\alpha}_{c+}$.*

Let also

$$u_\star(t) := \begin{cases} u(t) & \text{if } t \in [c, +\infty), \\ \displaystyle\sum_{j=0}^{k-1} \frac{u^{(j)}(c)}{j!}(t-c)^j & \text{if } t \in (-\infty, c). \end{cases}$$

Then, $u_\star \in C^{k,\alpha}_{a+}$ and $D^\alpha_{a+}[u_\star] = D^\alpha_{c+}[u]$ in I.

Proof. We apply Lemma A.2 with

$$f(t) := \sum_{j=0}^{k-1} \frac{u^{(j)}(c)}{j!}(t-c)^j,$$

$g(t) := u(t)$, and $h(t) := u_\star(t)$. Notice that, in this setting, for each $j \in \{0, \dots, k-1\}$, we have that $f^{(j)}(c) = u^{(j)}(c) = g^{(j)}(c)$, and therefore condition (A.7) is fulfilled. Hence, the use of Lemma A.2 gives that $u_\star \in C^{k,\alpha}_{a+}$, as desired. In addition, we have that $u_\star^{(k)} = 0$ in $(-\infty, c)$ and therefore, if $t \in I$,

$$\int_a^t \frac{u_\star^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau = \int_c^t \frac{u_\star^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau = \int_c^t \frac{u^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau,$$

which says that $D^\alpha_{a+}[u_\star](t) = D^\alpha_{c+}[u](t)$. $\square$

A.2 Some tools from Interpolation Theory

In this section we collect some basic results from the real Interpolation Theory. We will refer essentially on the monography [Lun18].

Let X, Y be Banach spaces. We say that (X, Y) is an *interpolation couple* if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$; it is well known that this topological structure make both $X \cap Y$ and $X + Y$ two Banach spaces.

An *intermediate space* is any Banach space E such that

$$X \cap Y \subset E \subset X + Y.$$

An *interpolation space* is any intermediate space such that for any operator $T \in \mathcal{L}(X) \cap \mathcal{L}(Y)$, the restriction of T to E belongs to $\mathcal{L}(E)$.

Definition A.4. *Let X, Y be Banach spaces and $\theta \in (0,1)$, $1 \le q \le \infty$. We define the real interpolation space $(X, Y)_{\theta,q}$ as*

$$(X, Y)_{\theta,q} := \left\{ z \in X + Y : (0, +\infty) \ni t \to t^{-\theta} K(t, z) \in L^q((0, +\infty), \frac{dt}{t}) \right\},$$

where $K(t, z)$ denotes the Peetre interpolation functional defined by

$$K(t, z) = K(t, z, X, Y) := \inf_{z = x + y \in X + Y} (\|x\|_X + t\,\|y\|_Y).$$

In particular, choosing $X = L^p(I)$ and $Y = W^{1,p}(I)$ for some $1 \le p < \infty$ and some interval I, we obtain the Besov space

$$\big(L^p(I), W^{1,p}(I)\big)_{\theta,q} := B^\theta_{p,q}(I),$$

and, if $q = p$

$$\big(L^p(I), W^{1,p}(I)\big)_{\theta,p} := B^\theta_{p,p}(I) = W^{\theta,p}(I).$$

while with the choices $X = L^1(I)$ and $Y = L^\infty(I)$ we obtain the Marcinkiewicz space

$$\big(L^1(I), L^\infty(I)\big)_{\theta,\infty} := L^{\frac{1}{1-\theta},\infty}(I).$$

Remark A.5. *From definition A.4 immediately follows that if X_1 and Y_1 are two Banach spaces continuously embedded in X_0 and Y_0 respectively, we have that, for any $\theta \in (0,1)$ and any $1 \le p \le \infty$*

$$(X_1, Y_1)_{\theta,p} \hookrightarrow (X_0, Y_0)_{\theta,p}.$$

Now, we recall a fundamental result on the inclusion between real interpolation spaces

Proposition A.6 (Prop. 1.4 in [Lun18])**.** *Let X, Y be Banach spaces such that $Y \subset X$ and (X, Y) be an interpolation couple.*

For any $1 \le p, q \le \infty$ and $0 < s < r < 1$, we have that

$$(X, Y)_{r,p} \subset (X, Y)_{s,q}.$$

Corollary A.7. *Let $1 \le q < p < \infty$, $0 < s < r < 1$ and Ω be an open bounded domain with the extension property. We have that*

$$W^{r,p}(\Omega) \subset B^s_{p,q}(\Omega) \subset W^{s,q}(\Omega)$$

Proof. For the first inclusion it is sufficient to apply A.6 with $X = L^p(\Omega)$ and $Y = W^{1,p}(\Omega)$, while for the second we apply Remark A.5 with $X_1 = L^p(\Omega)$, $Y_1 = W^{1,p}(\Omega)$, $X_0 = L^q(\Omega)$ and $Y_0 = W^{1,q}(\Omega)$. □

A.3 Addendum to Proposition 2.21

According to Proposition 2.21, if $u \in C^{0,s}_0(\overline{I})$ then its $(1-s)$-fractional integral is not Lipschitz continuous in $\overline{I}$ but merely log-Lipschitz continuous.

For the sake of completeness, and for the absence of an explicit example at least in the works mentioned in the bibliography, we want to show an explicit function in $C^{0,s}_0(\overline{I})$ with $I^{1-s}_{a+}[u] \notin C^{0,1}_0(\overline{I})$.

Example A.8. *Let $s \in (0,1)$, $I := (0,1)$ and*

$$u(x) = \begin{cases} x^s & \text{if} \quad 0 \le x \le \frac{1}{2} \\ \frac{1}{2^s} & \text{if} \quad \frac{1}{2} \le x \le 1. \end{cases}$$

We firstly notice that $u(0) = 0$ and u is continuous in $x = 1/2$. Concerning to the Hölder regularity, if x, y are both in $[0, \frac{1}{2}]$ or in $[\frac{1}{2}, 1]$ the claim is straightforward, while for $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$ we have that

$$|u(x) - u(y)| = \left| x^s - \frac{1}{2^s} \right| \le C \left| x - \frac{1}{2} \right|^s = C \left(\frac{1}{2} - x \right)^s \le C(y-x)^s = C|x-y|^s,$$

and the same computation holds if we interchange x and y. Therefore, $u \in C_0^{0,s}(\overline{I})$.

Eventually, the computation of the $(1-s)$ fractional integral gives us

$$I_{0+}^{1-s}[u](x) = \begin{cases} \Gamma(s+1)x & \text{if} \quad 0 \le x \le \frac{1}{2} \\ \frac{x}{\Gamma(1-s)} \int_0^{\frac{1}{2x}} t^s (1-t)^{-s} dt + \frac{1}{\Gamma(2-s)} \frac{(2x-1)^{1-s}}{2} & \text{if} \quad \frac{1}{2} \le x \le 1, \end{cases}$$

which clearly is not Lipschitz continuous in $\overline{I}$ having unbounded first derivative in $x = \frac{1}{2}$.

Remark A.9. *Since $C^{0,s}(\overline{I}) = W^{s,\infty}(I)$ (see e.g.* [DNPV12, Section 8]*), we are immediately able to conclude that $I_{a+}^{1-s}(W_0^{s,\infty}(I)) \not\subset W_0^{1,\infty}(I)$.*

A.4 Addendum to Theorem 4.28

We notice here that the embedding in Theorem 4.28 is sharp. The continuity of the fractional integral I_{a+}^s from $L^p(I)$ into $L^r(I)$, with $1 < p < \frac{1}{s}$ and $1 \le r \le \frac{p}{1-sp}$ has been proved by Hardy and Littlewood in [HL28, Theorem 4], but in the limiting cases $p = 1$ and $p = 1/s$ the continuity fails, as shown by the following examples

Example A.10. *Let $s \in (0,1)$, $1 < \beta \le 2-s$, $I = (0,1)$ and*

$$f(x) = \begin{cases} \dfrac{1}{x|\log(x)|^\beta} & \text{if} \quad 0 < x \le 1/2 \\ 0 \quad \text{if} \quad 1/2 < x < 1. \end{cases}$$

Now, let $u := I_{0+}^s[f]$. Clearly $u \in I_{0+}^s(L^1(I)) = I_{0+}^s(L^1(I)) \cap W_{RL,0+}^{s,1}(I)$ since $f \in L^1(I)$, but

$$u(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{dt}{t|\log(t)|^\beta (x-t)^{1-s}} > \frac{x^{s-1}}{\Gamma(s)} \int_0^x \frac{dt}{t|\log(t)|^\beta} = \frac{1}{\Gamma(s)(\beta-1)} x^{s-1} |\log(x)|^{1-\beta},$$

therefore

$$\int_0^1 |u(x)|^{\frac{1}{1-s}} dx \ge \int_0^{1/2} |u(x)|^{\frac{1}{1-s}} dx > K \int_0^{1/2} \frac{dx}{x|\log(x)|^{\frac{\beta-1}{1-s}}} = +\infty,$$

since $\dfrac{\beta-1}{1-s} \le 1$, and so $u \notin L^{\frac{1}{1-s}}(I)$.

Example A.11. *Let $s \in (0,1)$, $I := (0,1)$ and*

$$f(x) = \begin{cases} 0 \quad \text{if} \quad 0 < x < 1/2 \\ \dfrac{1}{(1-x)^s |\log(1-x)|} & \text{if} \quad 1/2 \le x < 1 \end{cases}.$$

Now let $u := I_{0+}^s[f]$; since $f \in L^{1/s}(I)$, by Lemma 2.27 we have that $u \in I_{0+}^s(L^{1/s}(I)) \subset W_{RL,0+}^{s,1/s}(I) \cap I_{0+}^s(L^1(I))$. Now, we notice that

$$\lim_{x \to 1^-} u(x) = \frac{1}{\Gamma(s)} \int_{1/2}^1 \frac{dt}{(1-t)|\log(1-t)|} = +\infty,$$

which implies that $u \notin L^\infty(I)$.

A.5 Some basic notions from sub-Riemannian geometry

Following the notation used in Section 2.5, we want to notice here that each Carnot Group $\mathbb{G}$ can be identified with the euclidean space. Namely, if we choose a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i.e., such that $e_{h_{j-1}+1}, \dots, e_{h_j}$ is a basis of V_j for each $j = 1, \dots, k$ we can define a family $X := \{X_1, \dots, X_n\}$ of left invariant vector fields such that $X_i(0) = e_i$, $i = 1, \dots, n$.

The sub-bundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \dots, X_m$ plays a particularly important role in sub-Riemannian geometry and it is called the *horizontal bundle* $H\mathbb{G}$. The fibers of $H\mathbb{G}$ are

$$H_x\mathbb{G} = \operatorname{span}\{X_1(x), \dots, X_m(x)\}, \qquad x \in \mathbb{G}.$$

We notice that each fiber of $H\mathbb{G}$ can be endowed with an inner product $\langle \cdot, \cdot \rangle$ that makes the basis $X_1(x), \dots, X_m(x)$ an orthonormal basis. The sections of $H\mathbb{G}$ are called *horizontal sections* and the elements of $H_x\mathbb{G}$ are called *horizontal vectors*. Each horizontal section is identified by its canonical coordinates with respect to this moving frame $X_1(x), \dots, X_m(x)$; in this way, a horizontal section ϕ is identified with a function $\phi = (\phi_1, \dots, \phi_m) : \mathbb{R}^n \to \mathbb{R}^m$. Since the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism, for any adapted basis $(X_1, \dots, X_n)$ of $\mathfrak{g}$ and any $x \in \mathbb{G}$, there exists a unique $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that

$$x = \exp(x_1X_1 + \ldots + x_nX_n).$$

We therefore identify x with $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the group operation on $\mathbb{R}^n$ is determined by the Baker-Campbell-Hausdorff formula on $\mathfrak{g}$ (see [BLU07, Chapter 14, Section 2]). The coordinates $x_1, \dots, x_n \in \mathbb{R}$, defined as above, are often referred to as *exponential coordinates of the first kind*. Although this identification allow to do explicit computations with the Group operation, it has the drawback of losing informations about the intrinsic structure of the Group.

A.6 Rellich-Kondrachov Theorem in Metric Measure Spaces

We introduce the class of John domains, which play a fundamental role in the proof of Theorem 5.10.

Definition A.12. *Let $\mathbb{G}$ be a Carnot group, and $\Omega \subset \mathbb{G}$ a bounded, open set. We say that Ω is a John domain if there exist $p \in \Omega$ and $C > 0$ such that, for every $q \in \Omega$, there is $T > 0$ and a continuous and rectifiable curve $\gamma : [0, T] \to \Omega$ parametrized by arclength such that $\gamma(0) = p$, $\gamma(T) = q$ and*

$$d(\gamma(t), \Omega^c) \geq Ct,$$

for any $t \in [0, T]$.

It was proved in [HK00] that a Rellich-Kondrachov-type Theorem holds for John domains in metric measure spaces with doubling property and Poincaré inequality. In the setting of Carnot groups, this result reads as follows.

Theorem A.13. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q and $\Omega \subseteq \mathbb{G}$ a John domain. Then the following facts hold.*

(i) If $1 \leq p < Q$ and $1 \leq q < p^*$, the embedding $W^{1,p}_{\mathbb{G}}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

(ii) If $p \geq Q$ and $q \geq 1$, the embedding $W^{1,p}_{\mathbb{G}}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

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