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Second variation techniques for stability in geometric inequalities

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Chapter 1

Introduction

1.1 Background

1.1.1 General statement and examples

Let I be a functional defined on a suitable class of sets $\mathcal{A}$. We are exploring the stability properties of the following minimization problem:

$$\min \{I(\Omega) : \Omega \in \mathcal{A}\}. \tag{1.1.1}$$

More precisely, we are interested in the following question. Let $\Omega^* \in \mathcal{A}$ be a stable critical point for I. Is it true that Ω^* is a strict local minimum? Can it be quantified?

A critical point is said to be strongly stable if for any $X \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ we have

$$\frac{d}{dt} I(\Phi_t(\Omega^*))|_{t=0} = 0, \qquad \frac{d^2}{dt^2} I(\Phi_t(\Omega^*))|_{t=0} > 0,$$

where $\Phi_t := Id + tX$. This implies that Ω^* is a local minimum for a certain class of deformations. This class of deformations is however very limited, and we would like to enlarge it. So we are wondering if we can upgrade infinitesimal stability to the following:

$$I(\Omega) - I(\Omega^*) \geq c\, \omega \left(\mathrm{dist}(\Omega, \Omega^*)\right), \tag{1.1.2}$$

where Ω is in a small neighborhood of Ω^*, 'dist' denotes some suitable notion of distance, and ω is some modulus of continuity. And if so, does (1.1.2) hold for every Ω, thus making Ω^* a global minimum? Moreover, we would like ω in the inequality (1.1.2) to be sharp, that is, we wish to have a sequence Ω_ε such that

$$I(\Omega_\varepsilon) \to I(\Omega^*) \text{ as } \epsilon \to 0, \qquad I(\Omega_\varepsilon) - I(\Omega^*) \sim \omega \left(\mathrm{dist}(\Omega_\varepsilon, \Omega^*)\right).$$

These questions were asked for various geometric inequalities and different types of sets. Not only such results have their own merit, but they have also been used recently for several probabilistic results. Such inequalities can be helpful in large deviation theory. See for example these works on the limits of certain discrete models: Berestycki and Cerf in [BC18] use quantitative Faber-Krahn inequality for a penalized random walk, and Cicalese and Leonardi in [CL19] apply quantitative Wulff inequality for fluctuations on certain lattices.

Probably the first result of this flavor goes back to the work of Bonnesen in [Bon24], where he proves (1.1.2) with I = perimeter and $\mathcal{A} = \{\text{convex sets in } \mathbb{R}^2\}$ (Ω^* in this case is a ball). Since then, inequalities of type (1.1.2) were proven for several functional such as: perimeter, the first eigenvalue of Laplace operator, Cheeger constant, etc., see [FMP08], [BDPV15], [FMP09], [FFM⁺15]. There were developed several approaches to this problem, including symmetrization, mass transportation, and second variation techniques (for the overview of these techniques for the isoperimetric inequality see [Fus15]). We will be focusing on the latter.

The question of choosing the topology in (1.1.2) is an important one. Local minimality is insured only in the topology which allows us to take derivatives. Consider the functional $F(u) := \int_0^1 u(x)^2 - u(x)^4\,dx$. In L^∞ topology the second derivative of F at $u_0 \equiv 0$ is positive and u_0 is indeed a local minimum among L^∞ functions. However, it is not a local minimum among L^4 functions, as $F(x^{-1/8}) < 0$. This example might seem artificial but we will see that problems of this sort arise also in real life when we present Lord Rayliegh's charged liquid drops model in Section 1.1.2. For a more thorough discussion on the topology see, for example, [GH04, Chapter 4].

1.1.2 Main results

Let us first state the main results of the thesis. All the original results of this thesis are contained in the following papers:

- *The sharp quantitative isocapacitary inequality*, 2019, joint with G. De Philippis, M.Marini, [DPMM19];
- *The sharp quantitative isocapacitary inequality (the case of p-capacity)*, 2020, [Muk20];
- *Minimality of the ball for a model of charged liquid droplets*, 2019, joint with G. Vescovo, [MV19].

Quantitative isocapacitary inequality

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be an open set. We define the *absolute capacity* of Ω as

$$\operatorname{cap}(\Omega) = \inf_{u \in C_c^\infty(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \geq 1 \text{ on } \Omega \right\}. \tag{1.1.3}$$

Moreover, for $\Omega \subset\subset B_R$ (B_R denotes the ball of radius R centered at the origin) we denote by $\operatorname{cap}_R(\Omega)$ the *relative capacity* of Ω with respect to B_R defined as

$$\operatorname{cap}_R(\Omega) = \inf_{u \in C_c^\infty(B_R)} \left\{ \int_{B_R} |\nabla u|^2 dx : u \geq 1 \text{ on } \Omega \right\}. \tag{1.1.4}$$

It is easy to see that for problem (1.1.3) (resp. (1.1.4)) there exists a unique function[1] $u \in D^{1,2}(\mathbb{R}^N)$ (resp. $u_R \in W_0^{1,2}(B_R)$) called *capacitary potential* of Ω such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \mathrm{cap}(\Omega) \qquad \Big(\text{resp. } \int_{B_R} |\nabla u_R|^2 = \mathrm{cap}_R(\Omega)\Big).$$

Moreover, they satisfy the Euler-Lagrange equations:

$$\begin{cases} \Delta u = 0 \text{ in } \overline{\Omega}^c \\ u = 1 \text{ on } \partial\Omega \\ u(x) \to 0 \text{ as } |x| \to \infty \end{cases} \qquad \begin{cases} \Delta u_R = 0 \text{ in } B_R \setminus \overline{\Omega} \\ u_R = 1 \text{ on } \partial\Omega \\ u_R = 0 \text{ on } \partial B_R. \end{cases}$$

The well-known *isocapacitary inequality* (resp. *relative isocapacitary inequality*) asserts that, among all sets with given volume, balls (resp. ball centered at the origin) have the smallest possible capacity, namely

$$\mathrm{cap}(\Omega) - \mathrm{cap}(B_r) \geq 0 \qquad (\text{resp. } \mathrm{cap}_{\mathrm{R}}(\Omega) - \mathrm{cap}_{\mathrm{R}}(B_r) \geq 0). \tag{1.1.5}$$

Here r is such that $|B_r| = |\Omega|$, where $|\cdot|$ denotes the Lebesgue measure.

So, we can ask if (1.1.2) holds for $I = \mathrm{cap}$ and $\Omega^* = B_r$. The answer is positive, and a good choice of distance is the so-called *Fraenkel asymmetry*.

Definition 1.1.1. Let Ω be an open set. The Fraenkel asymmetry of Ω, $\mathcal{A}(\Omega)$, is defined as:

$$\mathcal{A}(\Omega) = \inf\left\{\frac{|\Omega \Delta B|}{|B|} : B \text{ is a ball with the same volume as } \Omega\right\}.$$

To the best of our knowledge, the first results in this direction appeared in [HHW91] where they considered the case of planar sets [2] and of convex sets in general dimension. In the same paper the authors conjectured the validity of the following inequality:

Conjecture 1.1.2 ([HHW91]). *Let $N \geq 3$. There exists a constant $c = c(N)$ such that for any open set Ω such that $|\Omega| = |B_r|$ the following inequality holds:*

$$\frac{\mathrm{cap}(\Omega) - \mathrm{cap}(B_r)}{r^{N-2}} \geq c\mathcal{A}(\Omega)^2.$$

Note that by testing the inequality on ellipsoids with eccentricity ε one easily sees that the exponent 2 can not be replaced by any smaller number. Indeed, consider the family $\{\Omega_\varepsilon\}$ of ellipsoids defined as

$$\Omega_\varepsilon := \{(x', x_N) \in \mathbb{R}^N : |x'|^2 + (1+\varepsilon)x_N^2 \leq 1\}.$$

[1] Here and in the sequel, $D^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the homogeneous Sobolev norm:

$$\|u\|_{\dot{W}^{1,2}} := \|\nabla u\|_{L^2},$$

see [EG15, Section 4.7] and [LL97, Chapter 8]

[2] Note that for $N = 2$ the infimum (1.1.3) is 0 and one has to use the notion of logarithmic capacity.

Then $|\Omega_\varepsilon| = |B_{r_\varepsilon}|$ with $r_\varepsilon = 1 - \frac{1}{2N}\varepsilon + O(\varepsilon^2)$. One can easily show that $\mathcal{A}(\Omega_\varepsilon) \sim \varepsilon$. To see that $\operatorname{cap}(\Omega_\varepsilon) \leq 1 + \frac{1}{2N}\varepsilon + O(\varepsilon)$ we use $u = (|x'|^2 + (1+\varepsilon)x_N^2)^{-\frac{N-2}{2}}$ as a competitor in the definition of capacity. Thus we get

$$\frac{\operatorname{cap}(\Omega_\varepsilon) - \operatorname{cap}(B_{r_\varepsilon})}{{r_\varepsilon}^{N-2}} \leq C\varepsilon^2 \leq C\mathcal{A}(\Omega_\varepsilon)^2.$$

A positive answer to the above conjecture in dimension 2 has been given by Hansen and Nadirashvili in [HN92, Corollary 1]. For general dimension, the best known result is due to Fusco, Maggi, and Pratelli in [FMP09] where they prove the following:

Theorem 1.1.3 ([FMP09, Theorem 1.2])**.** *There exists a constant $c = c(N)$ such that for any open set Ω such that $|\Omega| = |B_r|$ the following inequality holds*

$$\frac{\operatorname{cap}(\Omega) - \operatorname{cap}(B_r)}{r^{N-2}} \geq c\mathcal{A}(\Omega)^4.$$

Remark 1.1.4. This theorem is more general, we will state the full version later.

With G. De Philippis and M. Marini in [DPMM19] we provide a positive answer to Conjecture 1.1.2 in every dimension.

Theorem 1.1.5 ([DPMM19, Theorem 1.4])**.** *Let Ω be an open set such that $|\Omega| = |B_1|$. Then*

(A) *if Ω is compactly contained in B_R, there exists a constant $c_1 = c_1(N, R)$ such that the following inequality holds:*

$$\operatorname{cap}_R(\Omega) - \operatorname{cap}_R(B_1) \geq c_1(N,R)|\Omega \Delta B_1|^2.$$

(B) *there exists a constant $c_2 = c_2(N)$ such that the following inequality holds:*

$$\operatorname{cap}(\Omega) - \operatorname{cap}(B_1) \geq c_2(N)\mathcal{A}(\Omega)^2.$$

Remark 1.1.6. By the scaling $\operatorname{cap}(\lambda\Omega) = \lambda^{N-2}\operatorname{cap}(\Omega)$, we can also get the analogous result for Ω with arbitrary volume.

Note that in the above theorem, in the case of the absolute capacity one bounds the distance of Ω from the set of balls, while in the case of the relative capacity one bounds the distance from the ball *centered at the origin* but the constant is R dependent. Indeed in the former case all balls have the same capacity (due to the translation invariance of the problem) and thus in order to obtain a quantitative improvement, one has to measure the distance from the set of *all minimizers*. On the contrary, for the relative capacity, the ball centered at the origin is the only minimizer. Since

$$\lim_{R\to+\infty} \operatorname{cap}_R(\Omega) = \operatorname{cap}(\Omega),$$

it is clear that the constant in (A) above needs to depend on R. Indeed, if we consider $\Omega = B_1(x)$ with $x \neq 0$, in the limit we have 0 on the left-hand side but the right-hand side is strictly positive. The dependence on R can also be inferred by the study of the linearized problem, see Section 4.1.2 below. We also remark that, as it will be clear from the proof, in the case of the relative capacity one can replace $|\Omega \Delta B_1|^2$ with the larger quantity $\alpha_R(\Omega)$ defined in Section 5.1 below.

Quantitative isocapacitary inequality - the case of general p

The second main result is a generalization of the part (B) of Theorem 1.1.5 to the case of p-capacity. First we introduce the following definition. Let $\Omega \subset \mathbb{R}^N$, be an open set. We define the *p-capacity* of Ω as

$$\mathrm{cap}_p(\Omega) = \inf_{u \in C_c^\infty(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p dx : u \geq 1 \text{ on } \Omega \right\} \tag{1.1.6}$$

for $1 < p < N$. Similar to the case $p = 2$, it is easy to see that for problem (1.1.6) there exists a unique function[3] $u \in D^{1,p}(\mathbb{R}^N)$ called *capacitary potential* of Ω such that

$$\int_{\mathbb{R}^N} |\nabla u|^p = \mathrm{cap}_p(\Omega).$$

Moreover, it satisfies the Euler-Lagrange equation:

$$\begin{cases} \mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 \text{ in } \overline{\Omega}^c, \\ u = 1 \text{ on } \partial\Omega, \\ u(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

In the same way as for standard capacity of the previous section, Pólya-Szegö principle yields the *isocapacitary inequality*, saying that, among all sets with given volume, balls have the smallest possible p-capacity, namely

$$\mathrm{cap}_p(\Omega) - \mathrm{cap}_p(B_r) \geq 0 \tag{1.1.7}$$

with r such that $|B_r| = |\Omega|$. Inequality (1.1.7) is rigid, that is, equality is attained only when Ω coincides with a ball, up to a set of zero p-capacity.

It is natural to wonder whether these inequalities are stable as it was in the case $p = 2$. That is indeed true and the first result to our knowledge is contained in the already mentioned paper by Fusco, Maggi, and Pratelli.

Theorem 1.1.7 ([FMP09, Theorem 1.2]). *There exists a constant $c = c(N, p)$ such that for any open set Ω such that $|\Omega| = |B_r|$ we have*

$$\frac{\mathrm{cap}_p(\Omega) - \mathrm{cap}_p(B_r)}{r^{N-p}} \geq c\mathcal{A}(\Omega)^{2+p}.$$

[3]Here $D^{1,p}$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the homogeneous Sobolev norm,

$$\|u\|_{D^{1,p}} := \|\nabla u\|_{L^p(\mathbb{R}^N)}$$

In [Muk20] we prove the sharp version of Thorem 1.1.7.

Theorem 1.1.8. *Let Ω be an open set such that $|\Omega| = |B_1|$. Then there exists a constant $c = c(N, p)$ such that the following inequality holds:*

$$\operatorname{cap}_p(\Omega) - \operatorname{cap}_p(B_1) \geq c\, \mathcal{A}(\Omega)^2.$$

By the scaling $\operatorname{cap}_p(\lambda\Omega) = \lambda^{N-p} \operatorname{cap}(\Omega)$, we can get the analogous result for Ω with arbitrary volume.

Charged liquid droplets

Here we will be considering a certain electrostatic energy in place of I, the exact definition requires some background.

In experiments one observes the following phenomenon: the shape of the liquid droplet is spherical in a small charge regime. Then, as soon as the value of the total charge increases, the droplet gradually deforms into an ellipsoid, it develops conical singularities, the so-called Taylor cones, [Tay64], and finally, the liquid starts emitting a thin jet ([DMV64], [DAM+03], [RPH89], [WT25]). The first experiments were conducted by Zeleny in 1914, [Zel17], but in a slightly different context.

Typically this behavior is modelled by defining a free energy composed by an attractive term, coming from surface tension forces, and a repulsive one, due to the electric forces generated by the interaction between charged particles. One may expect that for small values of the total charge the attractive part is predominant, forcing in this way the spherical shape of the minimizer.

The free energy in the classical model due to Lord Rayleigh is defined as follows:

$$P(E) + \frac{Q^2}{\mathcal{C}(E)}.$$

Here, $E \subset \mathbb{R}^3$ corresponds to the volume occupied by the droplet, $P(E)$ is its perimeter, Q is the total charge, and

$$\frac{1}{\mathcal{C}(E)} := \inf \left\{ \frac{1}{4\pi} \iint \frac{d\mu(x) d\mu(y)}{|x-y|} : \operatorname{spt} \mu \subset E, \mu(E) = 1 \right\}$$

takes into account the repulsive forces between charged particles. Note that μ can be thought as a (normalized) density of charges and that $\mathcal{C}(E)$ is the classical Newtonian capacity [4] of the set E. One assumes that the optimal shapes are given by the following variational problem:

$$\min_{|E|=V} \left\{ P(E) + \frac{Q^2}{\mathcal{C}(E)} \right\}.$$

[4] $\mathcal{C}(E)$ coincides with the capacity $\operatorname{cap}(E)$ defined above up to a constant, see [LL97, Section 11.15].

A difficulty is that contrary to the numerical and experimental observations this model is mathematically ill-posed, see [GNR15]. For a more exhaustive discussion we refer the reader to [MN16]. From a mathematical point of view, the issue with it is in line with the problem we mentioned above concerning the right topology in (1.1.2). For small values of the charge the ball is a minimizer but only if one restricts themselves to $C^{1,1}$-regular sets, while for wider classes of sets like sets with C^1 boundary or open sets the infimum is not attained.

As for the physical perspective, the main issue with the Rayleigh model comes from the tendency of charges to concentrate at the interface of the liquid. To restore the well-posedness one should consider a physical regularizing mechanism in the functional. With this purpose in mind, Muratov and Novaga in [MN16] integrate the entropic effects associated with the presence of free ions in the liquid. The advantage of this model is that the charges are now distributed inside of the droplet. They suggest to consider the following *Debye-Hückel-type free energy*

$$\mathcal{F}_{\beta,K,Q}(E,u,\rho) := P(E) + Q^2\left\{ \int_{\mathbb{R}^N} a_E|\nabla u|^2\,dx + K\int_E \rho^2\,dx \right\}. \tag{1.1.8}$$

Here $N \geq 3$, $E \subset \mathbb{R}^N$ represents the droplet, $P(E)$ is the De Giorgi perimeter, [Mag12, Chapter 12], the constant $Q > 0$ is the total charge enclosed in E, and

$$a_E(x) := \mathbf{1}_{E^c} + \beta\mathbf{1}_E,$$

where $\mathbf{1}_F$ is the characteristic function of a set F and $\beta > 1$ is the permittivity of the liquid.

The normalized density of charge $\rho \in L^2(\mathbb{R}^N)$ satisfies

$$\rho\mathbf{1}_{E^c} = 0 \qquad \text{and} \qquad \int \rho = 1 \tag{1.1.9}$$

and the electrostatic potential $u \in D^{1,2}(\mathbb{R}^N)$ is such that

$$-\operatorname{div}\big(a_E\,\nabla u\big) = \rho \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{1.1.10}$$

For a fixed set E we define the set of admissible pairs of functions u and ρ:

$$\mathcal{A}(E) := \big\{(u,\rho) \in D^{1,2}(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\colon\ u \text{ and } \rho \text{ satisfy (1.1.10) and (1.1.9)}\big\}.$$

The variational problem proposed in [MN16] is the following:

$$\min\big\{\mathcal{F}_{\beta,K,Q}(E,u,\rho) : |E| = V, E \subset B_R, (u,\rho) \in \mathcal{A}(E)\big\}. \tag{1.1.11}$$

By scaling (see the introduction of [DPHV19]), we can reduce the problem to the case $|E| = |B_1|$ and so we will work with the following problem:

$$\min\big\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|,\ E \subset B_R\big\}. \tag{$\mathcal{P}_{\beta,K,Q,R}$}$$

In [MV19] with G.Vescovo we obtain the following result.

Theorem 1.1.9. *Fix $K > 0$, $\beta > 1$. Then there exists $Q_0 = Q_0(\beta, K) > 0$ such that for all $Q < Q_0$ and any suitable R the only minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) are the balls of radius 1.*

The condition $E \subset B_R$ in the minimizing problem ($\mathcal{P}_{\beta,K,Q,R}$) is required to have existence of minimizers. However, thanks to Theorem 1.1.9 it can be dropped for small enough charges.

Corollary 1.1.10. *Fix $K > 0$, $\beta > 1$. Then there exists $Q_0 = Q_0(\beta, K) > 0$ such that for all $Q < Q_0$ the infimum in the problem*

$$\inf \left\{ \mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1| \right\} \tag{$\mathcal{P}_{\beta,K,Q}$}$$

is attained. Moreover, the only minimizers are the balls of radius 1.

To prove Theorem 1.1.9 we show that minimizers are close to the ball and regular if Q is small enough. We then use second variation techniques to prove stability of the ball with respect to smooth perturbations in the case of small charge.

The first step is to obtain $C^{2,\vartheta}$-regularity of minimizers. We improve the results of [DPHV19], where partial $C^{1,\vartheta}$-regularity is proven. In fact, we are able to prove C^∞-regularity of minimizers, a result that is interesting in itself.

Theorem 1.1.11 ($\boldsymbol{C^\infty}$-regularity)**.** *Given $N \geq 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exists $\varepsilon_{\text{reg}} = \varepsilon_{\text{reg}}(N, A, \vartheta) > 0$ such that if E is a minimizer of (1.1.11) with $Q + \beta + K + \frac{1}{K} \leq A$,*

$$x_0 \in \partial E \quad \textit{and} \quad r + \mathbf{e}_E(x_0, r) + Q^2\, D_E(x_0, r) \leq \varepsilon_{\text{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epi-graph of a C^∞-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a C^∞ $(N-1)$-dimensional manifold. Moreover [5]*,*

$$[f]_{C^{k,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C(N, A, k, \vartheta) \tag{1.1.12}$$

for every $k \in \mathbb{N}$ with $k \geq 2$.

We refer the reader to Notation 2.0.1 for the definition of $\mathbf{e}_E(x_0, r)$, $D_E(x_0, r)$ and $\mathbf{C}(x_0, r/2)$.

[5] Let $\Omega \subset \mathbb{R}^m$ be an open and bounded set, $f \in C(\overline{\Omega})$. Then

$$[f]_{C^{0,\vartheta}(\overline{\Omega})} := \sup_{x \neq y, x, y \in \overline{\Omega}} \frac{|f(x) - f(y)|}{|x - y|^\vartheta}.$$

Moreover, if $f \in C^k(\overline{\Omega})$ then

$$[f]_{C^{k,\vartheta}(\overline{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [D^\alpha f]_{C^{0,\vartheta}(\overline{\Omega})}.$$

1.2 Possible approaches

As was mentioned earlier, there are several approaches to proving inequalities of the type (1.1.2). We will go briefly through them and mostly focus on the second variation technique.

1.2.1 Symmetrization

Often minimizers of the problem (1.1.1) enjoy some symmetries. For example, balls are minimizers for a variety of problems of this sort - consider properly scaled perimeter, first eigenvalue of Laplacian, capacity, etc. One usually can prove that only balls are minimizers via appropriate symmetrization techniques and that proof can be quantified.

Let us illustrate the above idea by looking more closely at the isocapacitary inequality (1.1.7). Its proof is an easy combination of *Schwarz symmetrization* with *Pólya-Szegö principle*. Indeed, let Ω be an open set and let u be its capacitary potential. Schwarz symmetrization provides us with a radially symmetric function u^* such that, for every $t \in \mathbb{R}$,

$$|\{x : u(x) > t\}| = |\{x : u^*(x) > t\}| . \tag{1.2.1}$$

We use u^* as a test function for the set $\{x : u^*(x) = 1\} = B_r$ and we note that (1.2.1) yields that $|B_r| = |\Omega|$. Hence

$$\mathrm{cap}_p(B_r) \leq \int_{\mathbb{R}^N} |\nabla u^*|^p dx \leq \int_{\mathbb{R}^N} |\nabla u|^p dx = \mathrm{cap}_p(\Omega) \qquad |\Omega| = |B_r|,$$

where the second inequality follows by Pólya-Szegö principle, which in turn follows from isoperimetric inequality (for details see [Tal76, Section 1]).

Since the isocapacitary inequality is a consequence of the isoperimetric inequality, a reasonable strategy to obtain a quantitative improvement would be to rely on a quantitative isoperimetric inequality. This was indeed the strategy used in [FMP09] where they rely on the quantitative isoperimetric inequality established in [FMP08]. However, although the inequality proved in [FMP08] is sharp, in order to combine it with the Schwarz symmetrization procedure, it seems unavoidable to lose some exponent and to obtain a result as the one in [FMP09] (recall Theorem 1.1.7).

1.2.2 Mass transportation

Sometimes the minimizers are not symmetric. For example, that is the case for anisotropic perimeter, defined as follows. Suppose we have K - an open, bounded, convex set in $\mathbb{R}^N$ containing the origin. We define a weight function on directions as

$$\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}$$

for $\nu \in \mathbb{R}^N$ such that $|\nu| = 1$. Now for Ω - a sufficiently smooth set in $\mathbb{R}^N$ - we define its anisotropic perimeter as

$$P_K(\Omega) := \int_{\partial\Omega} \|\nu_\Omega(x)\|_* \, d\mathcal{H}^{N-1}.$$

Unlike the standard perimeter, P_K is not necessarily invariant under rotation and its unique minimizer (modulo translations) under a volume constraint is K itself, properly scaled. In other words, the following anisotropic isoperimetric inequality holds

$$\frac{P_K(\Omega)}{|\Omega|^{\frac{N-1}{N}}} \geq N|K|^{1/N}$$

with equality only for $\Omega = \lambda K + x$ for some $\lambda > 0$ and $x \in \mathbb{R}^N$. In [FMP10] Figalli, Maggi, and Pratelli show a quantitative version of this inequality. More precisely, they prove that

$$\frac{P_K(\Omega)}{|\Omega|^{\frac{N-1}{N}}} \geq N|K|^{1/N} \left(1 + \left(\frac{\mathcal{A}_K(\Omega)}{C(N)}\right)^2\right), \tag{1.2.2}$$

where $\mathcal{A}_K(\Omega)$ is asymmetry with respect to K defined as

$$\mathcal{A}_K(\Omega) = \inf\left\{\frac{|\Omega\Delta(x + rK)|}{|\Omega|} \,:\, x \in \mathbb{R}^N,\, r^N|K| = |\Omega|\right\}.$$

Moreover, they show that one can take

$$C(N) = \frac{181N^7}{\left(2 - 2^{\frac{N-1}{N}}\right)^{3/2}}$$

on the right hand side.

To explain the idea behind their proof we need to recall the following theorem.

Theorem 1.2.1 (Brenier's map, [Bre91, McC95])**.** *Let μ and ν be two probability measures in $\mathbb{R}^N$. Suppose that μ is absolutely continuous with respect to Lebesgue measure. Then there exists a convex function $\varphi : \mathbb{R}^N \to \mathbb{R}$ such that the map $T := \nabla\varphi$ transports μ onto ν, i.e. for every Borel function h the following holds*

$$\int_{\mathbb{R}^N} h(y)\, d\nu(y) = \int_{\mathbb{R}^N} h(T(x))\, d\mu(x).$$

An elegant proof of isoperimetric inequality due to Gromov in [MS86] goes as follows. Suppose T is a Brenier transport map (in the original proof instead of a Brenier map a less rigid Knothe one is used) between $\mu = \frac{1}{|\Omega|}\chi_\Omega\, dx$ and $\nu = \frac{1}{|K|}\chi_K\, dy$. Note that by change of variable $\det \nabla T(x) = |K|/|\Omega|$ for a.e. $x \in \Omega$. Then

$$\begin{aligned} P_K(\Omega) = \int_{\partial\Omega} \|\nu_\Omega(x)\|_* \, d\mathcal{H}^{N-1} &\geq \int_{\partial\Omega} \|T\|\|\nu_\Omega(x)\|_* \, d\mathcal{H}^{N-1} \geq \int_{\partial\Omega} T\cdot\nu_\Omega(x)\, d\mathcal{H}^{N-1} \\ &= \int_\Omega \operatorname{div} T\, dx \geq \int_\Omega N(\det\nabla T)^{1/N}\, dx = N|K|^{1/N}|\Omega|^{\frac{N-1}{N}}. \end{aligned}$$

In [FMP10] Figalli, Maggi, and Pratelli quantify this proof to get (1.2.2).

Unfortunately, for this proof the functional needs to have a particular structure which is not the case for problems we are concerned with in this thesis.

1.2.3 Second variation technique

The method, called Selection Principle, was introduced by Cicalese and Leonardi in [CL12] to give a new proof of the sharp quantitative isoperimetric inequality. It turned out to be an effective tool for such questions. See, for example, proofs of sharp quantitative Faber-Krahn inequality in [BDPV15] and sharp quantitative isoperimetric inequality for non-local perimeters in [FFM⁺15]. The idea is to get a sequence contradicting the inequality (1.1.2), then improve it to be a sequence of smooth sets. In the spirit of Ekeland's variational principle, the new sequence is selected as minimizers of penalized minimization problems. We also note that Acerbi, Fusco, and Morini in [AFM13] use a different penalization approach to prove stability of certain configurations for nonlocal isoperimetric problem.

To tackle the problem for smooth sets, we write an analog of Taylor expansion for the functional I. That is, we want to have a formula of the type

$$I(\Omega) = I(\Omega^*) + [\text{first order term}] + \frac{1}{2} I''(\Omega^*) \text{dist}(\Omega, \Omega^*)^2 + [\text{remainder term}]. \quad (1.2.3)$$

The first order term vanishes for a critical point. We would like to show that the second derivative $I''(\Omega^*)$ is positive and to get an appropriate bound on the remainder term. Such a computation for the perimeter was done first by Fuglede in [Fug89].

Thus, we have the following steps:

- Get a contradicting sequence.

  We argue by contradiction and for any $c > 0$ we get a sequence of sets Ω_h, such that
  $$I(\Omega_h) - I(\Omega^*) < c\,\text{dist}(\Omega_h, \Omega^*)^2$$
  and Ω_h converges to Ω^* in some (typically weak) topology.

- Improve a contradicting sequence.

  Now we want to have convergence in a stronger topology. As was mentioned, this is done by perturbing a sequence $\{\Omega_h\}$ to be a sequence of minimizers of some functionals. Then we will need to use regularity results.

- Prove (1.1.2) for smooth sets.

  To write the equality (1.2.3) we employ shape derivatives. The exact form of the distance and the remainder term may vary for different functionals. The bounds require certain regularity, which tells us the topology we should aim for in the previous step.

To use this approach we a priori don't need neither symmetry of the minimizer nor some specific structure of the functional. Note however that arguing by contradiction leaves us no chance to bound the constant c on the right hand side of (1.1.2).

1.3 Organization of the thesis

The rest of the thesis is organized as follows.

In Chapter 2 we collect the notions and conventions used throughout the thesis.

In Chapter 3 we define shape and material derivatives, introduce Hadamard's formula, and explain how to differentiate solutions of elliptic problems with respect to the domain.

Chapter 4 contains computations for the so-called spherical sets. We start by defining these sets and proving a technical lemma that allows us to deform the unit ball to an arbitrary nearly-spherical set in a smooth way. We then deal separately with 2-capacity, p-capacity and charged drops. In Section 4.1 we compute the first two derivatives of capacity near the ball and prove Theorem 1.1.5 for nearly-spherical sets. We make similar computations for p-capacity in Section 4.2 and prove Theorem 1.1.8 for nearly-spherical sets. Note that the computations become more technical as the equation for capacitary potential is degenerate in this case. Finally, in Section 4.3 in an analogous fashion we write Taylor expansion for the free energy defined by (1.1.8). Since this free energy contains perimeter we can be crude and only provide a bound for the second derivative of the repulsive term near the unit ball. We prove stability of the unit ball for the free energy $\mathcal{F}$ in the family of nearly-spherical sets, getting as a corollary Theorem 1.1.9 for nearly-spherical sets. For the sake of completeness we also provide the sharp bound of the second derivative of the repulsive term at the unit ball.

Chapter 5 (Chapter 6) concerns the proof of Selection Principle for 2-capacity (p-capacity). In both cases we first deal with bounded sets. In Section 5.1 we introduce a different notion of asymmetry that we will use in both chapters. We argue by contradiction and state Selection Principle in Section 5.2 (Section 6.1). We perturb the contradicting sequence, making it a sequence of minimizers of a certain functional in Section 5.3 (Section 6.2). We then prove that the new minimizing sequence consists of uniformly regular domains: we first get Lipschitz regularity and then use [AC81] ([DP05]) to get higher regularity. This is the content of Section 5.4 (Section 6.3). Finally, in Section 5.5 and Section 5.6 (Section 6.4 and Section 6.5) we reduce to the case of bounded sets and finish the proof of Theorem 1.1.5 (Theorem 1.1.8). The biggest differences of these two chapters lies in the part concerning regularity of the perturbed sequence, most of the other proofs can be repeated almost verbatim and we will omit some of them for the case of p-capacity.

In Chapter 7 we finish the proof of Theorem 1.1.9. As we proved it already for the case of nearly-spherical sets, it is enough to show that the minimizers are nearly-spherical for small enough charge. We start by collecting the regularity results of [DPHV19] that we will use in Section 7.1. In Section 7.2 we prove that minimizers are close to the unit ball in L^∞. We improve $C^{1,\vartheta}$-regularity of [DPHV19] to $C^{2,\vartheta}$ in Section 7.3 by looking at the minimizing pair (u, ρ) and utilizing Euler-Lagrange equation. We prove smooth regularity of minimizers in Section 7.4 via bootstrap procedure. Finally, we finish the proof of Theorem 1.1.9 and prove Corollary 1.1.10 in Section 7.5.

Chapter 2

Notation

Here we collect the notation and conventions we are going to use throughout this thesis.

Barycenter and direction.

For an open set Ω, x_Ω denotes the barycenter of Ω, namely

$$x_\Omega = \frac{1}{|\Omega|}\int_\Omega x dx.$$

For $x \in \mathbb{R}^n$ we denote

$$\theta := x/|x|.$$

Jacobians.

We denote by $J_{\Phi_t}(x)$ the jacobian of Φ_t at x:

$$J_{\Phi_t}(x) = \det \nabla\Phi_t(x)$$

and by $J^{\partial\Omega}_{\Phi_t}(x)$ the tangential jacobian of Φ_t at $x \in \partial\Omega$:

$$J^{\partial\Omega}_{\Phi_t}(x) = \det \nabla^\tau\Phi_t(x)$$

(see [Mag12, Section 11.1]).

Perimeter.

We are going to deal with sets of finite perimeter, for the definition and basic properties see [Mag12, Chapter 12]. For E - a set of finite perimeter we denote its perimeter by $P(E)$.

Harmonic extension and $H^{1/2}$ norm on the boundary.

Given a function $\varphi : \partial B_1 \to \mathbb{R}$ we define

(A) $H_R(\varphi) \in W^{1,2}_0(B_R)$ as the solution to

$$\begin{cases} \Delta H_R(\varphi) = 0 \text{ in } B_R\backslash B_1 \\ H_R(\varphi) = \varphi \text{ on } \partial B_1 \\ H_R(\varphi) = 0 \text{ on } \partial B_R \end{cases}$$

(B) $H(\varphi) \in D^{1,2}(\mathbb{R}^N)$ as the solution to

$$\begin{cases} \Delta H(\varphi) = 0 \text{ in } B_1^c \\ H(\varphi) = \varphi \text{ on } \partial B_1 \\ H(\varphi)(x) \to 0 \text{ as } |x| \to \infty \end{cases}$$

We are going to use the following norm:

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)} := \int_{\partial B_1} \varphi^2 d\mathcal{H}^{N-1} + \int_{B_1^c} |\nabla H(\varphi)|^2 dx$$

in the case of absolute capacity;

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)} := \int_{\partial B_1} \varphi^2 d\mathcal{H}^{N-1} + \int_{B_R \setminus B_1} |\nabla H_R(\varphi)|^2 dx$$

in the case of relative capacity.

Note that this norm is equivalent to the standard one, where the second integral is replaced by Gagliardo seminorm (see for example [Gri85, (1,3,3,3)]).

Normal vector and mean curvature.

When dealing with capacity we will denote by ν_Ω the *inward* normal to Ω. We will denote by $\mathscr{H}_{\partial\Omega}$ the mean curvature of $\partial\Omega$ with respect to the *inward* normal to Ω, that is $\mathscr{H}_{\partial\Omega} = \operatorname{div} \nu_\Omega$.

Note that when dealing with liquid drops we will denote by ν_Ω the *outward* normal to Ω.

Dealing with relative and absolute capacity simultaneously.

Since most of the argument will be similar for the relative and for the absolute capacity, we are going to use the following notational convention. Whenever possible, we will write α_*,cap_*, etc. instead of α/α_R, cap/cap_R or other notions that will come along. The convention is that $*$ denotes the same thing (R or the absence of it) throughout the equation or the computation where it appears.

Charged liquid drops model

Notation 2.0.1. Let $E \subset \mathbb{R}^N$ be a set of finite perimeter, $x \in \mathbb{R}^N$, $\nu \in \mathbb{S}^{N-1}$ and $r > 0$.

- We call $\mathbf{p}^\nu(x) := x - (x \cdot \nu)\, \nu$ and $\mathbf{q}^\nu(x) := (x \cdot \nu)\, \nu$, respectively, the *orthogonal projection* onto the plane $\nu^\perp$ and the *projection* on ν. For simplicity we write $\mathbf{p}(x) := \mathbf{p}^{e_N}(x)$ and $\mathbf{q}(x) := \mathbf{q}^{e_N}(x) = x_N$.
- We define the *cylinder* with center at $x_0 \in \mathbb{R}^N$ and radius $r > 0$ with respect to the direction $\nu \in \mathbb{S}^{N-1}$ as

$$\mathbf{C}(x_0, r, \nu) := \big\{x \in \mathbb{R}^N \,:\, |\mathbf{p}^\nu(x - x_0)| < r\,, |\mathbf{q}^\nu(x - x_0)| < r\big\},$$

  and write $\mathbf{C}_r := \mathbf{C}(0, r, e_N)$, $\mathbf{C} := \mathbf{C}_1$.

- We denote the $(N-1)$-dimensional *disk* centered at $y_0 \in \mathbb{R}^{N-1}$ and of radius r by
$$\mathbf{D}(y_0, r) := \{y \in \mathbb{R}^{N-1} : |y - y_0| < r\}.$$
We let $\mathbf{D}_r := \mathbf{D}(0, r)$ and $\mathbf{D} := \mathbf{D}(0, 1)$.

- We define
$$\mathbf{e}_E(x, r) := \inf_{\nu \in \mathbb{S}^{N-1}} \frac{1}{r^{N-1}} \int_{\partial^* E \cap B_r(x)} \frac{|\nu_E(y) - \nu|^2}{2} \, d\mathcal{H}^{N-1}(y).$$
We call $\mathbf{e}_E(x, r)$ the *spherical excess.* Note that from from the definition it follows that
$$\mathbf{e}_E(x, \lambda r) \leq \frac{1}{\lambda^{N-1}} \mathbf{e}_E(x, r)$$
for any $\lambda \in (0, 1)$.

- Let $(u, \rho) \in \mathcal{A}(E)$ be the minimizer of
$$\mathcal{G}_{\beta,K}(E) = \inf_{(u,\rho) \in \mathcal{A}(E)} \left\{ \int_{\mathbb{R}^N} a_E |\nabla u|^2 + K \int_E \rho^2 \right\}.$$
We define the *normalized Dirichlet energy* at x as
$$D_E(x, r) := \frac{1}{r^{N-1}} \int_{B_r(x)} |\nabla u|^2 \, dx.$$

Notation 2.0.2. Let $E \subset \mathbb{R}^N$ be such that $\partial E \cap \mathbf{C}(x_0, r)$ is described by the graph of a regular function f.

- If $x \in \mathbb{R}^N$, we write $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$.
- We denote by ν_E the outer-unit normal to ∂E. Moreover, we extend ν_E at every point in the following way
$$\nu_E(x', x_N) = \nu_E(x', f(x')) \quad \forall x = (x', x_N) \in \mathbf{C}(x_0, r).$$
- Let u be a solution of
$$-\mathrm{div}(a_E \nabla u) = \rho_E \quad \text{in } \mathcal{D}'\left(B_r(x_0)\right),$$
where
$$\rho_E \in L^\infty\left(B_r(x_0)\right) \quad \text{and} \quad a_E = \beta \mathbf{1}_E + \mathbf{1}_{E^c}.$$
We denote by
$$T_E u := \left(\partial_{\nu_E^\perp} u, (1 + (\beta - 1)\mathbf{1}_E) \partial_{\nu_E} u\right),$$
where
$$\partial_{\nu_E^\perp} u := \nabla u - (\nabla u \cdot \nu_E)\, \nu_E \quad \text{and} \quad \partial_{\nu_E} u := (\nabla u \cdot \nu_E)\, \nu_E.$$

- We denote by
$$[g]_{x,r} := \frac{1}{|B_r|}\int_{B_r(x)} g\,dx$$
the *mean value* of $g \in L^1(B_r(x))$. We simply write $[g]_r := [g]_{0,r}$.

- We denote the restrictions of a function v to E and E^c by v^+ and v^- respectively:
$$v^+ := v\,\mathbf{1}_E, \quad v^- := v\,\mathbf{1}_{E^c}.$$

Chapter 3

Shape derivative

3.1 Hadamard's formula

If the sets in the minimizing problem (1.1.1) are sufficiently smooth, one can try to deal with the problem in a classical way, i.e. look at the first and second variations. For a detailed overview of this approach see, for example [HP05, Chapter 5]. We are going to present briefly the tools we need.

Imagine you have a family of sufficiently smooth (for our needs $C^{2,\vartheta}$ boundary will be enough) sets Ω_t, $t \in [0,1]$. We want to learn how to take a derivative of $I(\Omega_t)$ with respect to t. Suppose that

$$\Omega_t = \Phi_t(\Omega_0),$$

where $\Phi_t(x) = Id + tX + o(t)$ with $X : \mathbb{R}^N \to \mathbb{R}^N$ a smooth vector field. Then the following lemma holds.

Lemma 3.1.1 (Hadamard's formula)**.** *If I has the form $I(\Omega_t) = \int_{\Omega_t} f_t(x)dx$ for $f(t,x) \in C^1([0,1] \times E)$ with $E \supset \Omega_t$ for any $t \in [0,1]$, then*

$$I'(\Omega_t) := \frac{d}{dt} I(\Omega_t) = \int_{\Omega_t} \frac{d}{dt} f_t(x)dx + \int_{\partial\Omega_t} f_t (X \cdot \nu) d\mathcal{H}^{N-1}.$$

Proof. We first use change of variables formula to move everything to the initial set:

$$I(\Omega_t) = \int_{\Omega_0} f_t(\Phi_t(x)) J_{\Phi_t}(x)dx.$$

Now the domain of integration is fixed and we have

$$\begin{aligned}
I'(\Omega_t) &= \int_{\Omega_0} \frac{d}{dt} \left(f_t(\Phi_t(x)) J_{\Phi_t}(x)\right)dx \\
&= \int_{\Omega_0} \frac{\partial}{\partial t} f(t, \Phi_t(x)) J_{\Phi_t}(x)dx + \int_{\Omega_0} \nabla f_t(\Phi_t(x)) \cdot X\, J_{\Phi_t}(x)dx + \int_{\Omega_0} f_t(\Phi_t(x)) \frac{d}{dt} J_{\Phi_t}(x)dx \\
&= \int_{\Omega_t} \frac{d}{dt} f_t(x)dx + \int_{\Omega_t} \nabla f_t(x) \cdot X\, dx + \int_{\Omega_t} f_t(x) \operatorname{div} X\, dx \\
&= \int_{\Omega_t} \frac{d}{dt} f_t(x)dx + \int_{\Omega_t} \operatorname{div}(f_t(x)X)\, dx = \int_{\Omega_t} \frac{d}{dt} f_t(x)dx + \int_{\partial\Omega_t} f_t(x) X \cdot \nu\, dx.
\end{aligned}$$

□

3.2 Derivatives of solutions of PDEs on changing domain

The problem is that even when the functionals we are interested in can be represented as in the statement of Hadamard's formula, usually the dependence of the function f on t is not direct. We are going to deal with solutions of Euler-Lagrange equation, which would be PDEs with the domain changing in time.

When talking about functions with changing domains, one can take derivatives in different ways. We are going to use mostly the shape derivative, defined as follows.

Definition 3.2.1. Suppose we have a family of functions $f_t : \Omega_t \to \mathbb{R}$. We define the *shape derivative* as

$$\dot{f}_t(x) := \frac{d}{dt} f_t(x) \text{ for } x \in \Omega_t.$$

Note that the shape derivative is a function defined on Ω_t.

Another notion we are going to need is that of material derivative.

Definition 3.2.2. Suppose we have a family of functions $f_t : \Omega_t \to \mathbb{R}$ and $\Omega_t = \Phi_t(\Omega)$. We define the *material derivative* as

$$\frac{d}{dt} f_t(x \circ \Phi_t) \text{ for } x \in \Omega.$$

Note that the material derivative is a function defined on Ω.

3.2.1 Dirichlet Laplacian

We state the following theorem for Dirichlet Laplacian with changing domain.

Proposition 3.2.3 ([SZ92, Proposition 3.1])**.** *Let Ω be a C^k domain in $\mathbb{R}^N$, $k \geq 2$. Suppose u_t is the solution in $H^1(\Omega_t)$ of*

$$\begin{cases} \Delta u_t = h_t \text{ in } \Omega_t, \\ u_t = z_t \text{ on } \partial\Omega_t \end{cases}$$

for some $h_t \in L^2(\Omega_t)$, $z_t \in H^{1/2}(\partial\Omega_t)$. Assume further that h_t and z_t have shape derivatives in $L^2(\Omega_t)$ and $H^{1/2}(\partial\Omega_t)$ respectively. Then there exists a shape derivative of h_t in $H^1(\Omega_t)$ and it is the solution of

$$\begin{cases} \Delta \dot{u}_t = \dot{h}_t \text{ in } \Omega_t, \\ \dot{u}_t = \dot{z}_t - (X \cdot \nu)\nabla u_t \cdot \nu \text{ on } \partial\Omega_t. \end{cases}$$

Remark 3.2.4. An analogous result holds if there is a uniformly elliptic operator instead of Laplacian. In particular, we are going to use it for an operator $u \mapsto \operatorname{div}\left((\kappa^2 + |\nabla u|^2)^{(p-2)/2} \nabla u\right)$ with $\kappa > 0$. The proof is similar to the one for Laplacian. For the scheme of the proof see Proposition 3.2.7.

3.2.2 Transmission problem

Let $\psi_t : \mathbb{R}^n \to \mathbb{R}$ be a solution in $D^{1,2}(\mathbb{R}^n)$ of

$$\begin{cases} -\beta\Delta\psi_t = -\frac{1}{K}\psi_t - f(t) \text{ in } \Omega_t, \\ \Delta\psi_t = 0 \text{ in } \Omega_t^c, \\ \psi_t^+ = \psi_t^- \text{ on } \partial\Omega_t, \\ \beta\nabla\psi_{\Omega_t}^+ \cdot \nu = \nabla\psi_{\Omega_t}^1 \cdot \nu \text{ on } \partial\Omega_t \end{cases} \tag{3.2.1}$$

or, in distributional form,

$$\int_{\mathbb{R}^N} a_{\Omega_t}\nabla\Psi\nabla\psi_t\, dx + \frac{1}{K}\int_{\Omega_t}\Psi\psi_t\, dx + \left(\int_{\Omega_t}\Psi\, dx\right) f(t) = 0$$

for any $\Psi \in D^{1,2}(\mathbb{R}^N)$.

First we notice that ψ_t is regular since it is a solution to a transmission problem. Indeed, $\psi_t \in C^{2,\vartheta'}(\overline{B}) \cap C^{2,\vartheta'}(\overline{B^c})$ by the following theorem.

Theorem 3.2.5 ([LU68, Theorem 16.2]). *Let Ω be a bounded set in $\mathbb{R}^N$. Denote by $\mathcal{B}_\nu$ the co-normal derivative and suppose u is a solution in $W^{1,2}(\Omega)$ of*

$$\begin{cases} \sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{i,j}(x)u_{x_j}) + \sum_{i=1}^N b_i(x)u_{x_i} + a(x)u = f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ [u] = 0 \text{ on } \partial E, \\ [p(x)\mathcal{B}_\nu u] = 0 \text{ on } \partial E, \end{cases}$$

where the coefficients satisfy the following conditions:

$$\nu|\xi|^2 \le \sum_{i,j=1}^N a_{i,j}\xi_i\xi_j \le \mu|\xi|^2, \quad \nu > 0, \quad 0 < p_0 \le p(x) \le p_1$$

and

$$a_{i,j}, \quad \frac{\partial a_{i,j}}{\partial x_k}, \quad p, \quad \frac{\partial p}{\partial x_k}, \quad b_i, \quad a, \quad f \in C^{\vartheta}(\overline{E}) \cap C^{\vartheta}(\overline{E^c}), \quad \partial E \in C^{2,\vartheta}.$$

Then $u \in C^{2,\vartheta'}(\overline{E}) \cap C^{2,\vartheta'}(\overline{E^c})$ for some $\vartheta' > 0$.

Remark 3.2.6. Note that in our case the equation is in the whole space $\mathbb{R}^N$ rather than in a bounded domain. However, the same proof applies.

Now we are ready to prove differentiability of ψ_t with respect to t.

Proposition 3.2.7. *Suppose f is bounded and Lipshitz with respect to t. Then the function $t \mapsto \psi_t$ is differentiable in t and its derivative $\dot{\psi}_t$ satisfies*

$$\begin{cases} -\beta\Delta\dot{\psi}_t = -\frac{1}{K}\dot{\psi}_t - f'(t) \text{ in } \Omega_t, \\ \Delta\dot{\psi}_t = 0 \text{ in } \Omega_t^c, \\ \dot{\psi}_t^+ - \dot{\psi}_t^- = -\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot\nu(X\cdot\nu) \text{ on } \partial\Omega_t, \\ \beta\nabla\dot{\psi}_t^+ \cdot \nu - \nabla\dot{\psi}_t^- \cdot \nu = -\left(\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)\cdot X\right)\cdot\nu \text{ on } \partial\Omega_t. \end{cases} \tag{3.2.2}$$

Proof. The proof is standard, see [HP05, Chapter 5] for the general strategy and [ADK07, Theorem 3.1] for a different kind of a transmission problem. We were unable to find a result covering our particular case in the literature, so we provide a proof here.

We first deal with material derivative of the function ψ, i.e. we shall look at the function $t \mapsto \tilde{\psi}_t := \psi_t(\Phi_t(x))$. The advantage is that its derivative in time is in H^1 as we will see. Note that the shape derivative of ψ_t is not in H^1 as it has a jump on $\partial\Omega_t$.

Step 1: moving everything to a fixed domain.

We introduce the following notation:

$$A_t(x) := D\Phi_t^{-1}(x) \left(D\Phi_t^{-1}\right)^t (x) J_{\Phi_t}(x).$$

Note that A_t is symmetric and positive definite and for t small enough it is elliptic with a constant independent of t.

Now let us write the equation for ψ_t in distributional form and perform a change of variables to get the equation for $\tilde{\psi}_t$:

$$\int_{\mathbb{R}^N} \nabla\Psi \left(a_B A_t \nabla\tilde{\psi}_t\right) dx + \frac{1}{K}\int_B \Psi\tilde{\psi}_t J_{\Phi_t}(x)dx + \left(\int_B \Psi J_{\Phi_t}(x)dx\right) f(t) = 0 \quad (3.2.3)$$

for any $\Psi \in D^{1,2}(\mathbb{R}^N)$.

Step 2: convergence of the material derivative.

We write the difference of equations (3.2.3) for $\tilde{\psi}_{t+h}$ and $\tilde{\psi}_t$ and divide it by h to get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\Psi \left(a_B \frac{A_{t+h}\nabla\tilde{\psi}_{t+h} - A_t\nabla\tilde{\psi}_t}{h}\right) dx + \frac{1}{K}\int_B \Psi \left(\frac{\tilde{\psi}_{t+h} - \tilde{\psi}_t}{h}\right) J_{\Phi_t}(x)dx \\
&+ \frac{1}{K}\int_B \Psi\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}\, dx + \left(\int_B \Psi J_{\Phi_t}(x)dx\right)\frac{f(t+h) - f(t)}{h} \\
&+ \left(\int_B \Psi \frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}\, dx\right) f(t+h) = 0
\end{aligned}$$

for any $\Psi \in D^{1,2}(\mathbb{R}^N)$.

Now, introducing $g_h(x) := \frac{\tilde{\psi}_{t+h} - \tilde{\psi}_t}{h}$ for convenience, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\Psi \left(a_B A_{t+h}\nabla g_h\right) dx + \frac{1}{K}\int_B \Psi g_h J_{\Phi_t}(x)dx + \int_{\mathbb{R}^N} \nabla\Psi\left(a_B \frac{A_{t+h} - A_t}{h}\nabla\tilde{\psi}_t\right) dx \\
&+ \frac{1}{K}\int_B \Psi\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}\, dx + \left(\int_B \Psi J_{\Phi_t}(x)dx\right)\frac{f(t+h) - f(t)}{h} \\
&+ \left(\int_B \Psi \frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}\, dx\right) f(t+h) = 0
\end{aligned} \quad (3.2.4)$$

for any $\Psi \in D^{1,2}(\mathbb{R}^N)$.

Now we want to get a uniform bound on g_h in $D^{1,2}(\mathbb{R}^N)$. We use g_h as a test function in (3.2.4) and get

$$\int_{\mathbb{R}^N} a_B \nabla g_h \cdot (A_{t+h}\nabla g_h)\, dx + \frac{1}{K}\int_B g_h^2 J_{\Phi_t}(x) dx + \int_{\mathbb{R}^N} a_B \nabla g_h \cdot \left(\frac{A_{t+h}-A_t}{h}\nabla\tilde{\psi}_t\right) dx$$
$$+ \frac{1}{K}\int_B g_h \tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}}-J_{\Phi_t}}{h}\, dx + \left(\int_B g_h J_{\Phi_t}(x)dx\right)\frac{f(t+h)-f(t)}{h}$$
$$+ \left(\int_B g_h \frac{J_{\Phi_{t+h}}(x)-J_{\Phi_t}(x)}{h}\, dx\right) f(t+h) = 0.$$

Since $\frac{A(t+h,x)-A(t,x)}{h}$ is bounded in L^∞ and A_t is uniformly elliptic we know that there exist some positive constant c independent of h such that

$$\int_{\mathbb{R}^N} a_B \nabla g_h \cdot (A_{t+h}\nabla g_h)\, dx + \int_{\mathbb{R}^N} a_B \nabla g_h \cdot \left(\frac{A_{t+h}-A_t}{h}\nabla\tilde{\psi}_t\right) dx$$
$$\geq c\int_{\mathbb{R}^N} |\nabla g_h|^2\, dx - C\int_{\mathbb{R}^N} |\nabla \psi_t|^2\, dx.$$

Thus, recalling the properties of Φ_t and the fact that f is bounded and Lipschitz, we have

$$c\int_{\mathbb{R}^N} |\nabla g_h|^2\, dx + \frac{1}{K}\int_B g_h^2 J_{\Phi_t}(x)dx \leq C\int_{\mathbb{R}^N} |\nabla \psi_t|^2\, dx + \frac{1}{K}\int_B \left| g_h\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}}-J_{\Phi_t}}{h}\right| dx$$
$$+ \left(\int_B |g_h J_{\Phi_t}(x)| dx\right)\left|\frac{f(t+h)-f(t)}{h}\right| + \left(\int_B \left| g_h\frac{J_{\Phi_{t+h}}(x)-J_{\Phi_t}(x)}{h}\right| dx\right)|f(t+h)|$$
$$\leq C\int_{\mathbb{R}^N} |\nabla \psi_t|^2\, dx + C\int_B \left| g_h\tilde{\psi}_{t+h}\right| dx + C\int_B |g_h|\, dx.$$

Recalling that ψ_t is in $D^{1,2}$ and L^∞, we further get

$$c\int_{\mathbb{R}^N} |\nabla g_h|^2\, dx + \frac{1}{K}\int_B g_h^2 J_{\Phi_t}(x)dx \leq C + C\int_B |g_h|\, dx.$$

Now, using Young's inequality and the fact that jacobian of Φ_t is close to 1 we finally get

$$\int_{\mathbb{R}^N} |\nabla g_h|^2\, dx + \frac{1}{K}\int_B g_h^2 J_{\Phi_t}(x)dx \leq C.$$

So, g_h is uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ and thus, up to a subsequence, there exists a weak limit g_0 as h goes to zero. Note that g_0 satisfies

$$\int_{\mathbb{R}^N} \nabla\Psi\,(a_B A_t \nabla g_0)\, dx + \int_{\mathbb{R}^N} \nabla\Psi\left(a_B \frac{d}{dt}A_t\nabla\tilde{\psi}_t\right) dx + \frac{1}{K}\int_B \Psi g_0 J_{\Phi_t}(x)dx$$
$$+ \frac{1}{K}\int_B \Psi\tilde{\psi}_t \frac{d}{dt}J_{\Phi_t}\, dx + \left(\int_B \Psi J_{\Phi_t}(x)dx\right) f'(t) + \left(\int_B \Psi\frac{d}{dt}J_{\Phi_t}\, dx\right) f(t) = 0$$

for any $\Psi \in D^{1,2}(\mathbb{R}^N)$, i.e. it is the solution of

$$\begin{cases} -\beta\,\mathrm{div}(A_t\nabla g_0) = -\frac{1}{K} g_0 J_{\Phi_t}(x) - \frac{1}{K}\tilde{\psi}_t \frac{d}{dt} J_{\Phi_t} - J_{\Phi_t}(x) f'(t) \\ \qquad\qquad -\frac{d}{dt} J_{\Phi_t} f(t) + \beta\,\mathrm{div}\left(\frac{d}{dt} A_t \nabla\tilde{\psi}_t\right) \text{ in } B_1, \\ \mathrm{div}(A_t\nabla g_0) = -\,\mathrm{div}\left(\frac{d}{dt} A_t\nabla\tilde{\psi}_t\right) \text{ in } B_1^c, \\ g_0^+ = g_0^1 \text{ on } \partial B_1, \\ \beta A_t \nabla g_0^+ \cdot \nu = A_t\nabla g_0^1 \cdot \nu \text{ on } \partial B_1. \end{cases}$$

So, the whole sequence g_h converges weakly to g_0 as h tends to 0.

To get the strong convergence of the material derivative, we observe that using g_h as a test function in its Euler-Lagrange equation, we get the convergence of the norm in H^1 to the norm of g_0. That, together with weak convergence, gives us strong convergence of g_h.

Step 3: existence of the shape derivative.

We want to show that

$$\dot{\psi}_t = \frac{d}{dt}\tilde{\psi}_t - X \cdot \nabla\psi_t$$

in $D^{1,2}(\Omega_t) \cap D^{1,2}(\Omega_t^c)$. Indeed, since $\psi_t(x) = \dot{\psi}_t(\Phi_t^{-1}(x))$, we have

$$\frac{\psi_{t+h}(x) - \psi_{t+h}(x)}{h} = \frac{\psi_{t+h}(\Phi_{t+h}^{-1}(x)) - \psi_t(\Phi_{t+h}^{-1}(x))}{h} + \frac{\psi_t(\Phi_{t+h}^{-1}(x)) - \psi_t(\Phi_t^{-1}(x))}{h}. \tag{3.2.5}$$

The first term on the right-hand side converges strongly to $\frac{d}{dt}\psi_t(\Phi_t^{-1}(x))$ as h goes to 0 by Step 2 and continuity of Φ_t. As for the second term, by the regularity of ψ_t and the definition of Φ, it converges to $-\nabla\psi_t(\Phi_t^{-1}(x)) \cdot X$ strongly in L^2.

Step 4: the equation for the shape derivative.

Now that we know that $t \mapsto \psi_t$ is differentiable, we can differentiate the Euler-Lagrange equation for ψ_t given by (3.2.1) and we get

$$\begin{cases} -\beta\Delta\dot{\psi}_t = -\frac{1}{K}\dot{\psi}_t - f'(t) \text{ in } \Omega_t, \\ \Delta\dot{\psi}_t = 0 \text{ in } \Omega_t^c, \\ \dot{\psi}_t^+ - \dot{\psi}_t^- = -\left(\nabla\psi_t^+ - \nabla\psi_t^-\right) \cdot X \text{ on } \partial\Omega_t, \\ \beta\nabla\dot{\psi}_t^+ \cdot \nu - \nabla\dot{\psi}_t^- \cdot \nu = -\left(\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right) \cdot X\right) \cdot \nu \text{ on } \partial\Omega_t. \end{cases}$$

Now we can use the boundary conditions in (3.2.1) to get rid of the tangential part in the right-hand side of the jump of $\dot{\psi}_t$ on the boundary. Indeed,

$$-\left(\nabla\psi_t^+ - \nabla\psi_t^-\right) \cdot X = -\left(\nabla^\tau\psi_t^+ - \nabla^\tau\psi_t^-\right) \cdot X^\tau - \left(\nabla\psi_t^+ - \nabla\psi_t^-\right) \cdot \nu(X \cdot \nu)$$

and $\nabla^\tau\psi_t^+ = \nabla^\tau\psi_t^-$ by differentiating the equality $\psi_t^+ = \psi_t^-$ on the boundary of Ω_t. □

Chapter 4

Fuglede's computation

In this chapter we are going to prove (1.1.2) int the cases $I = \operatorname{cap}_* / \operatorname{cap}_p / \mathcal{G}$ for sufficiently smooth sets Ω, that is, for nearly-spherical sets defined as follows.

Definition 4.0.1. An open bounded set $\Omega \subset \mathbb{R}^N$ is called nearly-spherical of class $C^{2,\vartheta}$ parametrized by φ, if there exists $\varphi \in C^{2,\vartheta}$ with $\|\varphi\|_{L^\infty} < \frac{1}{2}$ such that

$$\partial\Omega = \{(1 + \varphi(x))x : x \in \partial B_1\}.$$

The results we are going to get are analogues of the following theorem which we will need for liquid drops model.

Theorem 4.0.2 ([Fug89, Theorem 1.2])**.** *There exists a constant $c = c(N)$ such that for any Ω — nearly-spherical set parametrized by φ with $|\Omega| = |B_1|$, $x_\Omega = 0$, the following inequality holds*

$$P(\Omega) - P(B_1) \geq c\|\varphi\|^2_{H^1(\partial B_1)}.$$

We want to write a Taylor expansion for the energies we are dealing with. To that end we need to have a family of sets transforming B to Ω. We will use the following lemma.

Lemma 4.0.3. *Given $\vartheta \in (0, 1]$ there exists $\delta = \delta(N, \vartheta) > 0$, a modulus of continuity ω, and a constant $C > 0$ such that for every nearly-spherical set Ω parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|\Omega| = |B_1|$, we can find an autonomous vector field X_φ for which the following holds true:*

(i) $\operatorname{div} X_\varphi = 0$ *in a δ-neighborhood of ∂B_1;*

(ii) $X_\varphi = 0$ *outside a 2δ-neighborhood of ∂B_1;*

(iii) if $\Phi_t := \Phi(t, x)$ is the flow of X_φ, i.e.

$$\partial_t \Phi_t = X_\varphi(\Phi_t), \qquad \Phi_0(x) = x,$$

then $\Phi_1(\partial B_1) = \partial\Omega$ and $|\Phi_t(B_1)| = |B_1|$ for all $t \in [0, 1]$;

(iv) denote $\Omega_t := \Phi_t(B_1)$, then

$$\|\Phi_t - Id\|_{C^{2,\vartheta}} \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}) \textit{ for every } t \in [0,1], \tag{4.0.1}$$

$$|J_\Phi| \le C \textit{ in a neighborhood of } B_1, \tag{4.0.2}$$

$$\|\varphi - (X_\varphi \cdot \nu_{B_1})\|_{L^2(\partial B_1)} \le \omega(\|\varphi\|_{L^\infty(\partial B_1)})\|\varphi\|_{L^2(\partial B_1)}, \tag{4.0.3}$$

$$\|\varphi - (X_\varphi \cdot \nu_{B_1})\|_{H^{\frac{1}{2}}(\partial B_1)} \le \omega(\|\varphi\|_{L^\infty(\partial B_1)})\|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}, \tag{4.0.4}$$

$$\|X \cdot \nu\|_{H^1(\partial\Omega_t)} \le C\,\|\varphi\|_{H^1(\partial B_1)}\,, \tag{4.0.5}$$

$$(X \cdot x) \circ \Phi_t - X \cdot \nu_{B_1} = (X \cdot \nu_{B_1}) f_t,\ x \in \partial B_1, \tag{4.0.6}$$

where $\|f_t\|_{C^{2,\vartheta}(\partial B_1)} \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})$, and for the tangential part of X, defined as $X = X - (X \cdot \nu)\nu$, there holds

$$|X^\tau| \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})\,|X \cdot \nu| \textit{ on } \partial\Omega_t. \tag{4.0.7}$$

Proof. Let us construct a vector field satisfying all the properties except (ii) and then multiply it by a cut-off. Such a vector field can be constructed for any smooth set, see for example [Dam02]. However, for the ball one can write an explicit expression in a neighborhood of ∂B_1. The proof for the case of the ball can be found in [BDPV15, Lemma A.1]. For the convenience of the reader we provide the expression here, as well as a brief explanation of how to get the needed bounds. In polar coordinates, $\rho = |x|$, $\theta = x/|x|$ the field looks like this:

$$X_\varphi(\rho,\theta) = \frac{(1+\varphi(\theta))^N - 1}{N\rho^{N-1}}\theta,$$
$$\Phi_t(\rho,\theta) = \left(\rho^N + t\left((1+\varphi(\theta))^N - 1\right)\right)^{\frac{1}{N}}\theta$$

for $|\rho - 1| \ll 1$. Then we extend this vector field globally in order to satisfy (4.0.1). Notice that (4.0.2) is a direct consequence of (4.0.1).

By direct computation we get (4.0.6). Now we can get the bound (4.0.5). Indeed, (4.0.6) together with (4.0.2) gives us

$$\begin{aligned}\|X \cdot \nu\|_{H^1(\partial\Omega_t)} &\le |J_\Phi|\left(1 + \omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\|X \cdot \nu\|_{H^1(\partial B_1)} \\ &\le g\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\|X \cdot \nu\|_{H^1(\partial B_1)}.\end{aligned}$$

From the definition of X, on ∂B_1 we have

$$\varphi - X \cdot \nu = \frac{1}{N}\sum_{i=2}^{N}\binom{N}{i}\varphi^i,$$

yielding the inequalities (4.0.4) and (4.0.3).

To see (4.0.7) we use that by definition X is parallel to θ near ∂B_1. Thus,

$$\begin{aligned}|X^\tau \circ \Phi_t| &= |((X\cdot\theta)\,\theta)\circ\Phi_t - ((X\cdot\nu)\,\nu)\circ\Phi_t| \\ &= \Big|\,(X\cdot\nu_{\partial B_1})\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\nu_{\partial B_1}\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right) \\ &\quad - (X\cdot\nu_{\partial B_1})\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\nu_{\partial B_1}\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\Big| \\ &= \omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)|(X\cdot\nu)\circ\Phi_t|\,.\end{aligned}$$

□

In what follows we will sometimes omit the subscript φ for brevity.

4.1 Capacity: the case of $p = 2$

4.1.1 Second variation

We now compute the second order expansion of the capacity of a nearly-spherical set.

Lemma 4.1.1. *Given $\vartheta \in (0,1]$, there exists $\delta = \delta(N,\vartheta) > 0$ and a modulus of continuity ω such that for every nearly-spherical set Ω parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|\Omega| = |B_1|$, we have*

$$\operatorname{cap}_*(\Omega) \geq \operatorname{cap}_*(B_1) + \frac{1}{2}\partial^2 \operatorname{cap}_*(B_1)[\varphi,\varphi] - \omega(\|\varphi\|_{C^{2,\vartheta}})\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)},$$

where

(A)

$$\partial^2 \operatorname{cap}_R(B_1)[\varphi,\varphi] := 2\frac{(N-2)^2}{1-R^{-(N-2)}}\Big(\int_{B_R\setminus B_1}|\nabla H_R(\varphi)|^2dx \\ - (N-1)\int_{\partial B_1}\varphi^2 d\mathcal{H}^{N-1}\Big);$$

(B)

$$\partial^2 \operatorname{cap}(B_1)[\varphi,\varphi] := 2(N-2)^2\left(\int_{B_1^c}|\nabla H(\varphi)|^2dx - (N-1)\int_{\partial B_1}\varphi^2 d\mathcal{H}^{N-1}\right).$$

Proof. Now set $\Omega_t := \Phi_t(B_1)$ with Φ_t from Lemma 4.0.3 and let u_t be the capacitary potential of Ω_t. We define

$$c_*(t) := \operatorname{cap}_*(\Omega_t) = \begin{cases}\int_{B_R\setminus\Omega_t}|\nabla u_t|^2dx \text{ in the case of relative capacity;}\\ \int_{\Omega_t^c}|\nabla u_t|^2dx \text{ in the case of full capacity.}\end{cases}$$

By Proposition 3.2.3 $t\mapsto u_t$ is differentiable and its derivative $\dot u_t$ satisfies

(A)
$$\begin{cases} \Delta \dot{u}_t = 0 \text{ in } B_R \backslash \Omega_t, \\ \dot{u}_t = -\nabla u_t \cdot X_\varphi \text{ on } \partial\Omega_t, \\ \dot{u}_t = 0 \text{ on } \partial B_R; \end{cases}$$
(B)
$$\begin{cases} \Delta \dot{u}_t = 0 \text{ in } \Omega_t^c, \\ \dot{u}_t = -\nabla u_t \cdot X_\varphi \text{ on } \partial\Omega_t, \\ \dot{u}_t(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$
Using Hadamard formula, we compute:
$$\frac{1}{2}c_R'(t) = \int_{B_R\backslash\Omega_t} \nabla u_t \cdot \nabla \dot{u}_t dx + \frac{1}{2}\int_{\partial\Omega_t} |\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1},$$
where ν_{Ω_t} is the *inward* normal to $\partial\Omega_t$. Now we recall that u_t is harmonic in $B_R\backslash\Omega_t$ and we use the boundary conditions for $\dot{u}_t$ to get
$$\begin{aligned} \frac{1}{2}c_R'(t) &= \int_{B_R\backslash\Omega_t} \operatorname{div}(\dot{u}_t \nabla u_t) dx + \frac{1}{2}\int_{\partial\Omega_t} |\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} \\ &= \int_{\partial\Omega_t} \dot{u}_t \nabla u_t \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} + \frac{1}{2}\int_{\partial\Omega_t} |\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} \\ &= \int_{\partial\Omega_t} (-\nabla u_t \cdot X_\varphi)\nabla u_t \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} + \frac{1}{2}\int_{\partial\Omega_t} |\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1}. \end{aligned}$$
We know that u_t is identically 1 on $\partial\Omega_t$ and smaller than 1 outside, hence (recall that ν_{Ω_t} denotes the inner normal)
$$\nabla u_t = |\nabla u_t| \nu_{\partial\Omega_t} \text{ on } \partial\Omega_t. \tag{4.1.1}$$
Therefore,
$$\begin{aligned} \frac{1}{2}c_R'(t) &= \int_{\partial\Omega_t} -|\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} + \frac{1}{2}\int_{\partial\Omega_t} |\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} \\ &= -\frac{1}{2}\int_{\partial\Omega_t} |\nabla u_t|^2 X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} = -\frac{1}{2}\int_{B_R\backslash\Omega_t} \operatorname{div}(|\nabla u_t|^2 X_\varphi) dx. \end{aligned}$$
We proceed now with the second derivative, using again Hadamard's formula and recalling that X is autonomous and divergence-free in a neighborhood of ∂B_1 (hence, on $\partial\Omega_t$).
$$\begin{aligned} \frac{1}{2}c_R''(t) &= -\frac{1}{2}\int_{B_R\backslash\Omega_t} \operatorname{div}\Big(\frac{\partial}{\partial t}|\nabla u_t|^2 X_\varphi\Big) dx - \frac{1}{2}\int_{\partial\Omega_t} \operatorname{div}(|\nabla u_t|^2 X_\varphi)(X_\varphi \cdot \nu_{\Omega_t}) d\mathcal{H}^{N-1} \\ &= -\int_{\partial\Omega_t} (\nabla u_t \cdot \nabla \dot{u}_t) X_\varphi \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} - \frac{1}{2}\int_{\partial\Omega_t} (\nabla|\nabla u_t|^2 \cdot X_\varphi)(X_\varphi \cdot \nu_{\Omega_t}) d\mathcal{H}^{N-1} \\ &= \int_{\partial\Omega_t} \dot{u}_t \nabla \dot{u}_t \cdot \nu_{\Omega_t} d\mathcal{H}^{N-1} - \int_{\partial\Omega_t} (X_\varphi \cdot \nu_{\Omega_t})(\nabla^2 u_t[\nabla u_t] \cdot X_\varphi) d\mathcal{H}^{N-1} \\ &= \int_{B_R\backslash\Omega_t} |\nabla \dot{u}_t|^2 dx - \int_{\partial\Omega_t} (X_\varphi \cdot \nu_{\Omega_t})(\nabla^2 u_t[\nabla u_t] \cdot X_\varphi) d\mathcal{H}^{N-1} \end{aligned}$$

Note that in the second to last equality we have used (4.1.1) and the boundary condition for $\dot{u}_t$. Now since u_t is constant on $\partial\Omega_t$, we get

$$0 = \Delta u_t = |\nabla u_t|\mathscr{H}_{\partial\Omega_t} + \nabla^2[\nu_{\Omega_t}]\cdot\nu_{\Omega_t} \text{ on } \partial\Omega_t.$$

Taking this into account and denoting $X^\tau = X_\varphi - (X_\varphi\cdot\nu_{\Omega_t})\nu_{\Omega_t}$ on $\partial\Omega_t$, we get

$$\begin{aligned}\frac{1}{2}c_R''(t) &= \int_{B_R\setminus\Omega_t}|\nabla\dot{u}_t|^2dx \\ &\quad - \int_{\partial\Omega_t}(X_\varphi\cdot\nu_{\Omega_t})(\nabla^2u_t[|\nabla u_t|\nu_{\Omega_t}]\cdot((X_\varphi\cdot\nu_{\Omega_t})\nu + X^\tau))d\mathcal{H}^{N-1} \\ &= \int_{B_R\setminus\Omega_t}|\nabla\dot{u}_t|^2dx + \int_{\partial\Omega_t}(X_\varphi\cdot\nu_{\Omega_t})^2|\nabla u_t|^2\mathscr{H}_{\partial\Omega_t}d\mathcal{H}^{N-1} \\ &\quad - \int_{\partial\Omega_t}(X_\varphi\cdot\nu_{\Omega_t})(\nabla^2u_t[\nabla u_t]\cdot X^\tau)d\mathcal{H}^{N-1}.\end{aligned} \tag{4.1.2}$$

Now we wish to calculate $c_R''(0)$. We use that

- $\mathscr{H}_{\partial B_1} = -(N-1)$;
- $X^\tau = 0$ on ∂B_1;
- $u_0 = u_{B_1} = \frac{|x|^{-(N-2)}-R^{-(N-2)}}{1-R^{-(N-2)}}$ in $B_R\setminus B_1$;
- $\dot{u}_0 = H_R(-X_\varphi\cdot\nabla u_0)$.

$$\begin{aligned}\frac{1}{2}c_R''(0) &= \int_{B_R\setminus B_1}|\nabla H_R(-X_\varphi\cdot\nabla u_0)|^2dx - (N-1)\int_{\partial B_1}(X_\varphi\cdot\nu_{B_1})^2|\nabla u_0|^2d\mathcal{H}^{N-1} \\ &= \frac{(N-2)^2}{1-R^{-(N-2)}}\left(\int_{B_R\setminus B_1}|\nabla H_R(X_\varphi\cdot\nu_{B_1})|^2dx - (N-1)\int_{\partial B_1}(X_\varphi\cdot\nu_{B_1})^2d\mathcal{H}^{N-1}\right).\end{aligned}$$

As for the case of full capacity, the same computations apply with minor changes, obtaining

$$\frac{1}{2}c''(0) = (N-2)^2\left(\int_{B_1^c}|\nabla H_R(X_\varphi\cdot\nu_{B_1})|^2dx - (N-1)\int_{\partial B_1}(X_\varphi\cdot\nu_{B_1})^2d\mathcal{H}^{N-1}\right),$$

which formally corresponds to sending $R\to\infty$ in the formula for c_R''. Since balls minimize the capacity we also have that $c_*'(0) = 0$. Writing

$$\mathrm{cap}_*(\Omega) = c_*(1) = c_*(0) + \frac{1}{2}c_*''(0) + \int_0^1(1-t)(c_*''(t)-c_*''(0))dt\,,$$

one can now exploit Lemma 4.0.3 and perform the very same computations as in [BDPV15, Lemma A.2] to conclude. We put the computations here for the sake of completeness. We need to show the following bound:

$$|c_*''(t)-c_*''(0)| \le \omega(\|\varphi\|_{C^{2,\vartheta}})\|X\cdot\nu\|^2_{H^{\frac{1}{2}}(\partial B_1)}$$

for some modulus of continuity ω. Let us prove it for the relative capacity, the absolute capacity can be dealt with in a similar fashion. We recall (4.1.2) and pull it back on the unit ball:

$$\begin{aligned}\frac{1}{2}c_R''(t) &= \int_{B_R\setminus B_1} |\nabla \dot{u}_t|^2 \circ \Phi_t J_{\Phi_t} dx + \int_{\partial B_1} \left((X\cdot\nu_{\Omega_t})^2 |\nabla u_t|^2 \mathscr{H}_{\partial\Omega_t}\right)\circ\Phi_t J_{\Phi_t}^{\partial B_1} d\mathcal{H}^{N-1} \\ &\quad - \int_{\partial B_1} \left((X\cdot\nu_{\Omega_t})(\nabla^2 u_t[\nabla u_t]\cdot X^\tau)\right)\circ\Phi_t J_{\Phi_t}^{\partial B_1} d\mathcal{H}^{N-1} \\ &= I_1(t) + I_2(t) + I_3(t).\end{aligned}$$

By (4.0.1), we have

$$\|\mathscr{H}_{\partial\Omega_t}\circ\Phi_t - \mathscr{H}_{\partial B}\|_{L^\infty(\partial B)} + \|J_{\Phi_t}^{\partial B} - 1\|_{L^\infty(\partial B)} + \|J_{\Phi_t} - 1\|_{L^\infty(B)} \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right).$$

In addition, by Lemma 4.0.3, X is parallel to θ in a neighborhood of ∂B, so we have

$$|\left(X\cdot\nu_{\Omega_t}\right)\circ\Phi_t - X\cdot\nu_B| \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right)|X\cdot\nu_B|,$$

as well as

$$|X_\tau\circ\Phi_t| \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right)|X\cdot\nu_B|.$$

Considering the equation satisfied by $u_t\circ\Phi_t$ on B_1, by Schauder estimates we get

$$\|u - u_t\circ\Phi_t\|_{C^{2,\vartheta}(\overline{B_1})} \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right)$$

Thus, noticing that $I_3(0) = 0$, we get

$$|I_2(t) - I_2(0)| + |I_3(t) - I_3(0)| \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right)\|X\cdot\nu_B\|_{L^2(\partial B)}^2.$$

It remains to show that

$$|I_1(t) - I_1(0)| \le \omega(\|\varphi\|_{C^{2,\vartheta}})\|X\cdot\nu\|_{H^{1/2}(\partial B)}^2. \tag{4.1.3}$$

We define $w_t := \dot{u}_t\circ\Phi_t$. Notice that $\nabla w_t = (\nabla\Phi_t)^t\,\nabla\dot{u}_t\circ\Phi_t$, so by (4.0.1) to get (4.1.3) it is enough to prove that

$$\left|\int_{B_R\setminus B_1} |\nabla w_t|^2 - |\nabla\dot{u}_0|^2\,dx\right| \le \omega(\|\varphi\|_{C^{2,\vartheta}})\|X\cdot\nu\|_{H^{1/2}(\partial B)}^2.$$

We now move the equation for $\dot{u}_t$ onto the unit ball B and see that w_t satisfies

$$\begin{cases}\operatorname{div}\left(M_t\nabla w_t\right) = 0 \text{ in } B_R\setminus B_1,\\ w_t = -\left(\nabla u_t\cdot X\right)\circ\Phi_t \text{ on } \partial B_1,\\ w_t = 0 \text{ on } \partial B_R,\end{cases}$$

where $M_t = J_{\Phi_t}\left((\nabla\Phi_t)^{-1}\right)^t(\nabla\Phi_t)^{-1}$. Classical elliptic estimates together with (4.0.1) then give

$$\begin{aligned}&\|\nabla w_t - \nabla\dot{u}_0\|_{L^2(B_R\setminus B_1)} \\ &\quad\le C(N)\left(\|(M_t - Id)\nabla w_t\|_{L^2(B_R\setminus B_1)} + \|(\nabla u_t\cdot X)\circ\Phi_t - \nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)}\right) \\ &\quad\le \omega(\|\varphi\|_{C^{2,\vartheta}})\|\nabla w_t\|_{L^2(B_R\setminus B_1)} + C(N)\|(\nabla u_t\cdot X)\circ\Phi_t - \nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)}.\end{aligned}$$

Recalling that $\nabla u_0 = -|\nabla u_0|\theta$ on ∂B_1 and that $X = (X\cdot\theta)\theta$ near ∂B_1, we obtain

$$
\begin{aligned}
&\|(\nabla u_t \cdot X)\circ\Phi_t - \nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)} \\
&\quad\le \|((\nabla u_t\cdot\theta)\circ\Phi_t - \nabla u_0\cdot\theta)\,(X\cdot\theta)\circ\Phi_t\|_{H^{1/2}(\partial B_1)} \\
&\qquad + \||\nabla u_0|\,((X\cdot\theta)\circ\Phi_t - X\cdot\theta)\|_{H^{1/2}(\partial B_1)} \\
&\quad\le \|((\nabla u_t\circ\Phi_t - \nabla(u_t\circ\Phi_t))\,(\theta\circ\Phi_t))\,(X\cdot\theta)\circ\Phi_t\|_{H^{1/2}(\partial B_1)} \\
&\qquad + \|(\nabla(u_t\circ\Phi_t)\,(\theta\circ\Phi_t) - (\nabla u_0\cdot\theta))\,(X\cdot\theta)\circ\Phi_t\|_{H^{1/2}(\partial B_1)} \\
&\qquad + \||\nabla u_0|\,((X\cdot\theta)\circ\Phi_t - X\cdot\theta)\|_{H^{1/2}(\partial B_1)} \\
&\quad\le \omega(\|\varphi\|_{C^{2,\vartheta}})\|\nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\nabla w_t - \nabla\dot u_0\|_{L^2(B_R\setminus B_1)} &\le \omega(\|\varphi\|_{C^{2,\vartheta}})\left(\|\nabla w_t\|_{L^2(B_R\setminus B_1)} + \|\nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)}\right)\\
&\le \omega(\|\varphi\|_{C^{2,\vartheta}})\left(\|\nabla w_t - \dot u_0\|_{L^2(B_R\setminus B_1)} + \|\nabla\dot u_0\|_{L^2(B_R\setminus B_1)} + \|\nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)}\right)\\
&\le \omega(\|\varphi\|_{C^{2,\vartheta}})\left(\|\nabla w_t - \dot u_0\|_{L^2(B_R\setminus B_1)} + 2\|\nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)}\right),
\end{aligned}
$$

where in the last inequality we have used the equation for $\dot u_0$. We choose δ small enough so that $\omega(\|\varphi\|_{C^{2,\vartheta}}) \le 1/2$ and get

$$
\|\nabla w_t - \nabla\dot u_0\|_{L^2(B_R\setminus B_1)} \le \omega(\|\varphi\|_{C^{2,\vartheta}})\|\nabla u_0\cdot X\|_{H^{1/2}(\partial B_1)},
$$

Finally, we have

$$
\begin{aligned}
\left|\int_{B_R\setminus B_1} |\nabla w_t|^2 - |\nabla\dot u_0|^2\,dx\right| &\le \|\nabla w_t - \nabla\dot u_0\|_{L^2(B_R\setminus B_1)}\|\nabla w_t + \nabla\dot u_0\|_{L^2(B_R\setminus B_1)}\\
&\le 2\|\nabla\dot u_0\|_{L^2(B_R\setminus B_1)}\|\nabla w_t - \nabla\dot u_0\|_{L^2(B_R\setminus B_1)} + \|\nabla w_t - \nabla\dot u_0\|^2_{L^2(B_R\setminus B_1)}\\
&\le \omega(\|\varphi\|_{C^{2,\vartheta}})\|\nabla u_0\cdot X\|^2_{H^{1/2}(\partial B_1)},
\end{aligned}
$$

yielding (4.1.3) and hence finishing the proof. □

4.1.2 Inequality for nearly-spherical sets

We now establish a quantitative inequality for nearly-spherical sets in the spirit of those established by Fuglede in [Fug89], compare with [BDPV15, Section 3].

Theorem 4.1.2. *There exists $\delta = \delta(N), c = c(N,R)$ ($c = c(N)$ for the capacity in $\mathbb{R}^N$) such that if Ω is a nearly-spherical set of class $C^{2,\vartheta}$ parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}} \le \delta, |\Omega| = |B_1|$ (and $x_\Omega = 0$ for the case of the capacity in $\mathbb{R}^N$), then*

$$
\operatorname{cap}_*(\Omega) - \operatorname{cap}_*(B_1) \ge c\|\varphi\|^2_{H^{\frac12}(\partial B_1)},
$$

where

$$
\|\varphi\|^2_{H^{\frac12}(\partial B_1)} := \int_{\partial B_1}\varphi^2 d\mathcal{H}^{N-1} + \int_{B_1^c}|\nabla H_*(\varphi)|^2dx,
$$

where the second integral is intended on $B_R\setminus B_1$ if $* = R$.

Proof. We essentially repeat the proof of the Theorem 3.3 in [BDPV15]. First, we show that $\int_{\partial B_1} \varphi$ is small. Indeed, we know that

$$|B_1| = |\Omega| = \int_{\partial B_1} \frac{(1+\varphi(x))^N}{N} d\mathcal{H}^{N-1}$$
$$= |B_1| + \int_{\partial B_1} \varphi(x) d\mathcal{H}^{N-1} + \int_{\partial B_1} \sum_{i=2}^{N} \binom{N}{i} \frac{\varphi(x)^i}{N} d\mathcal{H}^{N-1}.$$

Hence,

$$\left| \int_{\partial B_1} \varphi(x) d\mathcal{H}^{N-1} \right| = \left| \int_{\partial B_1} \sum_{i=2}^{N} \binom{N}{i} \frac{\varphi(x)^i}{N} d\mathcal{H}^{N-1} \right|$$
$$\leq C(N) \int_{\partial B_1} \varphi(x)^2 d\mathcal{H}^{N-1} \leq C(N)\delta \|\varphi\|_{L^2}.$$

Moreover, for the case of the absolute capacity, also $\int_{\partial B_1} x_i \varphi$ is small. Indeed, using that the barycenter of Ω is at the origin, we get

$$\left| \int_{\partial B_1} x_i \varphi(x) d\mathcal{H}^{N-1} \right| \leq \int_{\partial B_1} \sum_{j=2}^{N} \binom{N}{j} \left| \frac{\varphi(x)^j}{N+1} \right| d\mathcal{H}^{N-1} \leq C(N)\delta \|\varphi\|_{L^2}.$$

Let us define

(A)

$$\mathcal{M}_\delta^R := \{\xi \in H^{\frac{1}{2}}(\partial B_1) : \left| \int_{\partial B_1} \xi d\mathcal{H}^{N-1} \right| \leq \delta \|\xi\|_{H^{1/2}}\};$$

(B)

$$\mathcal{M}_\delta := \left\{ \xi \in H^{\frac{1}{2}}(\partial B_1) : \left| \int_{\partial B_1} \xi d\mathcal{H}^{N-1} \right| + \left| \int_{\partial B_1} x \xi d\mathcal{H}^{N-1} \right| \leq \delta \|\xi\|_{H^{1/2}} \right\},$$

and note that, since $\|\xi\|_{L^2} \leq \|\xi\|_{H^{1/2}}$, we have just proved that φ belongs to $\mathcal{M}^*_{C\delta}$.

By Lemma 4.1.1 for δ small enough we have

$$\operatorname{cap}_*(\Omega) - \operatorname{cap}_*(B_1) \geq \frac{1}{2} \partial^2 \operatorname{cap}_*(B_1)[\varphi, \varphi] - \omega(\|\varphi\|_{C^{2,\vartheta}}) \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)}. \tag{4.1.4}$$

So, it is enough to check that

$$\partial^2 \operatorname{cap}_*(B_1)[\xi, \xi] \geq c \|\xi\|^2_{H^{\frac{1}{2}}(\partial B_1)}, \text{ for every } \xi \in \mathcal{M}^*_\delta$$

for small δ.

Step 1: linearized problem. First, we show that

$$\partial^2 \operatorname{cap}_*(B_1)[\xi, \xi] \geq c \|\xi\|^2_{H^{\frac{1}{2}}(\partial B_1)}, \text{ for every } \xi \in \mathcal{M}^*_0.$$

Note that

(A) $\mathcal{M}_0^R = \{\xi \in H^{\frac{1}{2}}(\partial B_1) : \int_{\partial B_1} \xi d\mathcal{H}^{N-1} = 0\}$;

(B) $\mathcal{M}_0 = \Big\{\xi \in H^{\frac{1}{2}}(\partial B_1) : \int_{\partial B_1} \xi d\mathcal{H}^{N-1} = \int_{\partial B_1} x_i \xi d\mathcal{H}^{N-1} = 0, i = 1, 2, \ldots, N\Big\}$.

We recall that

(A)

$$\partial^2 \operatorname{cap}_R(B_1)[\varphi, \varphi] := 2\frac{(N-2)^2}{1 - R^{-(N-2)}}\Big(\int_{B_R \setminus B_1} |\nabla H_R(\varphi)|^2 dx \\ - (N-1)\int_{\partial B_1} \varphi^2 d\mathcal{H}^{N-1}\Big);$$

(B)

$$\partial^2 \operatorname{cap}(B_1)[\varphi, \varphi] := 2(N-2)^2 \left(\int_{B_1^c} |\nabla H(\varphi)|^2 dx - (N-1)\int_{\partial B_1} \varphi^2 d\mathcal{H}^{N-1}\right).$$

We consider first the case of relative capacity. We need to estimate the quotient

$$\frac{\int_{B_R \setminus B_1} |\nabla H_R(\xi)|^2 dx}{\int_{\partial B_1} \xi^2 d\mathcal{H}^{N-1}}$$

from below for $\xi \in \mathcal{M}_0 \setminus \{0\}$. We note that it is the Rayleigh quotient for the operator $\xi \mapsto \nabla H_R(\xi) \cdot \nu$. Thus, we need to calculate its eigenvalues. We use spherical functions as a basis of $L^2(\partial B_1)$: $\xi = \sum_{m,n} a_{m,n} Y_{m,n}$. We now show that $H(Y_{m,n})$ can be written as $R_{m,n}(r) Y_{m,n}(\omega)$ for a suitable function $R_{m,n}(r)$. Indeed, by the equation defining $H(Y_{m.n})$ we have check that

$$\begin{cases} \Delta(R_{m,n}(r) Y_{m,n}(\omega)) = 0 \text{ in } B_R \setminus B_1 \\ R_{m,n}(1) Y_{m,n} = Y_{m,n} \\ R_{m,n}(R) Y_{m,n} = 0 \end{cases}$$

Since $\tilde{\Delta} Y_{m,n} = -m(m + N - 2) Y_{m,n}$, where $\tilde{\Delta}$ is the Laplace-Beltrami operator, one easily checks that

$$R_{m,n}(r) = -\frac{1}{R^{2m+N-2} - 1} r^m + \Big(1 + \frac{1}{R^{2m+N-2} - 1}\Big) r^{-(N+m-2)}$$

provides a solution. Hence, the first eigenvalue is zero and corresponds to constants, whereas the first non-zero one is $-R'_{1,n}(1) = (N-1) + \frac{1}{R^N - 1} N$.

For the case of the absolute capacity we estimate the quotient

$$\frac{\int_{B_1^c} |\nabla H(\xi)|^2 dx}{\int_{\partial B_1} \xi^2 d\mathcal{H}^{N-1}}$$

in an analogous way. The functions $R_{m,n}$ in this case is

$$R_{m,n}(r) = r^{-(N+m-2)}.$$

The first eigenvalue is zero and corresponds to constants, the second one is $N-1$ and corresponds to the coordinate functions, the next one is N.

Step 2: reducing to $\mathcal{M}_0^*$. We are going to apply Step 1 to the projection ξ_0 of ξ on $\mathcal{M}_0^*$ and show that the difference $|\partial^2 \operatorname{cap}_*(B_1)[\xi,\xi] - \partial^2 \operatorname{cap}_*(B_1)[\xi_0,\xi_0]|$ is small. Let ξ be in $\mathcal{M}_\delta^*$. Define

(A)

$$\xi_0 := \xi - \frac{1}{N|B_1|}\int_{\partial B_1} \xi d\mathcal{H}^{N-1};$$

(B)

$$\xi_0 := \xi - \frac{1}{N|B_1|}\int_{\partial B_1} \xi d\mathcal{H}^{N-1} - \frac{1}{|B_1|}\sum_{i=1}^{N} x_i \int_{\partial B_1} y_i \xi d\mathcal{H}^{N-1}.$$

It is immediate from the definition that ξ_0 belongs to $\mathcal{M}_0^*$. We denote $\zeta := \xi - \xi_0$. Since ξ belongs to $\mathcal{M}_{C\delta}^*$, we have

$$\|\zeta\|^2_{H^{\frac{1}{2}}(\partial B_1)} \leq C\|\zeta\|^2_{L^2(\partial B_1)} \leq C\delta^2\|\xi\|^2, \tag{4.1.5}$$

where we have used that since ζ belongs to an $N+1$ dimensional space, the $H^{1/2}$ and the L^2 are equivalent.

We now compare the norms of ξ and ξ_0:

$$\|\xi_0\|^2_{H^{\frac{1}{2}}(\partial B_1)} \geq \|\xi\|^2_{H^{\frac{1}{2}}(\partial B_1)} - \|\zeta\|^2_{H^{\frac{1}{2}}(\partial B_1)} \geq (1-C\delta^2)\|\xi\|^2_{H^{\frac{1}{2}}(\partial B_1)}. \tag{4.1.6}$$

Now we apply Step 1 to ξ_0 to get

$$\begin{aligned}\partial^2 \operatorname{cap}_*(B_1)[\xi,\xi] &= \partial^2 \operatorname{cap}_*(B_1)[\xi_0,\xi_0] + 2\partial^2 \operatorname{cap}_*(B_1)[\xi,\zeta] - \partial^2 \operatorname{cap}_*(B_1)[\zeta,\zeta] \\ &\geq c\|\xi_0\|^2_{H^{\frac{1}{2}}(\partial B_1)} - C\left(2\|\zeta\|_{H^{\frac{1}{2}}(\partial B_1)}\|\xi\|_{H^{\frac{1}{2}}(\partial B_1)} + \|\zeta\|^2_{H^{\frac{1}{2}}(\partial B_1)}\right)\end{aligned}$$

and thus , by (4.1.6) and (4.1.5),

$$\partial^2 \operatorname{cap}_*(B_1)[\xi,\xi] \geq c\|\xi_0\|^2_{H^{\frac{1}{2}}(\partial B_1)} - C\delta\|\xi\|^2 \geq \frac{c}{2}\|\xi\|^2_{H^{\frac{1}{2}}(\partial B_1)}$$

provided δ is chosen sufficiently small.

□

4.2 Capacity: the case of general p

In this section we are going to prove Theorem 1.1.8 for nearly-spherical sets. Some technical problems arise comparing to the case of standard capacity of the previous section. We are going to deal with them in a way similar to the one devised by Fusco and Zhang for proving analogous result for p-Faber-Krahn inequality in [FZ16] (note that we will be citing the preprint rather than the published version [FZ17] as it has more details).

First, we consider perturbed functionals to make the equation non-degenerate.

Definition 4.2.1. We define perturbed p-capacity as follows:

$$\operatorname{cap}_{p,\kappa}(\Omega) = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \left\{ \int_{\Omega^c} \left((\kappa^2 + |\nabla u|^2)^{\frac{p}{2}} - \kappa^p \right) dx : u = 1 \text{ on } \Omega \right\}.$$

Remark 4.2.2. Note that the infimum is achieved by the unique solution of the following equation

$$\begin{cases} \operatorname{div}((\kappa^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u) = 0 \text{ in } \Omega^c, \\ u = 1 \text{ on } \partial\Omega, \\ u(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

We will denote the minimizer by $u_{\kappa,\Omega}$.

Let Φ_t be the flow from Lemma 4.0.3 and define $\Omega_t = \Phi_t(B)$. For brevity we denote $u_{\kappa,t} := u_{\kappa,\Omega_t}$.

Remark 4.2.3. The function $u_{\kappa,t}$ satisfies the following equation

$$\begin{cases} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \nabla u_{\kappa,t}) = 0 \text{ in } \Omega_t^c, \\ u_{\kappa,t} = 1 \text{ on } \partial\Omega_t, \\ u_{\kappa,t}(x) \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{4.2.1}$$

We also note that $\nabla u_{\kappa,t} = |\nabla u_{\kappa,t}| \nu_{\partial\Omega_t}$ on $\partial\Omega_t$ since it is constant on the boundary and less than 1 outside of the set by maximum principle (here $\nu_{\partial\Omega_t}$ denotes *inner* normal).

We want to differentiate the perturbed p-capacity of Ω_t in t. We introduce the following notation

$$c_k(t) := \int_{\Omega_t^c} \left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p}{2}} - \kappa^p \right) dx.$$

Since for any $\kappa > 0$ equation (4.2.1) is elliptic, the following differentiability result holds (remember Remark 3.2.4).

Lemma 4.2.4 (Shape derivative of $\boldsymbol{u_{\kappa,t}}$)**.** *For any $\kappa > 0$ the derivative of $u_{\kappa,t}$ in t exists and it solves the following equation*

$$\begin{cases} \operatorname{div}\Big((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \nabla \dot{u}_{\kappa,t} \\ \qquad + (p-2)(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} (\nabla u_{\kappa,t} \cdot \nabla \dot{u}_{\kappa,t}) \nabla u_{\kappa,t} \Big) = 0 \textit{ in } \Omega_t^c, \\ \dot{u}_{\kappa,t} = -\nabla u_{\kappa,t} \cdot X \textit{ on } \partial\Omega_t. \end{cases} \tag{4.2.2}$$

We want to see what happens near the ball for the initial functional. To that end, we compute $u_0 := u_{0,0}$ and its gradient in B^c:

$$u_0 = |x|^{\frac{p-N}{p-1}}, \qquad \nabla u_0 = \frac{p-N}{p-1} |x|^{\frac{p-N}{p-1}-1} \theta.$$

Theorem 4.2.5 (convergence of $\boldsymbol{u_{\kappa,t}}$)**.** *Let $\kappa \in [0,1]$, $p > 1$, $\vartheta \in (0,1)$, $R > 1$. There exist $\tilde{\vartheta} \in (0,\vartheta)$ and a modulus of continuity $\omega = \omega(p,\vartheta,n)$ such that if Ω is a $C^{2,\vartheta}$ nearly-spherical set parametrized by φ and $\|\varphi\|_{C^{2,\vartheta}(\partial B)} < \delta$, then for all $t \in [0,1]$ and $\kappa \in [0,1]$ we have*

$$\|u_0 - u_{\kappa,t} \circ \Phi_t\|_{C^{1,\tilde{\vartheta}}(B_R \setminus B_1)} \leq \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B)} + \kappa).$$

Moreover, there exist $\delta' > 0$, $0 < \vartheta' < \vartheta$ and a modulus of continuity $\omega' = \omega'(p,\vartheta,n,\varepsilon)$, such that if $\|\varphi\|_{C^{2,\vartheta}(\partial B)} + \kappa < \delta'$, then for all $t \in [0,1]$

$$\|u_0 - u_{\kappa,t} \circ \Phi_t\|_{C^{2,\vartheta'}(B_R \setminus B_1)} \leq \omega'(\|\varphi\|_{C^{2,\vartheta}(\partial B)} + \kappa).$$

Proof. The proof goes in the same way as the one of [FZ16, Theorem 2.2]. We reproduce it here for the reader's convenience.

First, we notice that regularity for degenerate elliptic equations (see [Lie88, Theorem 1]) gives us

$$\|u_{\kappa,t}\|_{C^{1,\tilde{\vartheta}'}(B_R \setminus \Omega_t)} \leq C = C(p,\vartheta,n,\delta) \tag{4.2.3}$$

for some $\tilde{\vartheta}' \in (0,\vartheta)$, and every $\kappa \in [0,1]$, $t \in [0,1]$. Fix $\tilde{\vartheta} \in (0,\tilde{\vartheta}')$. To prove the first inequality we argue by contradiction. Suppose there exist sequences $\{\varphi_j\}$, $\{\kappa_j\}$, $\{t_j\}$ such that $\|\varphi_j\|_{C^{2,\vartheta}(\partial B)} + \kappa_j \to 0$,

$$\limsup_{j\to\infty} \|u_0 - u_{\kappa_j,t_j} \circ \Phi^j_{t_j}\|_{C^{1,\tilde{\vartheta}}(B_R \setminus B_1)} > 0, \tag{4.2.4}$$

where Φ^j is the flow associated with φ_j. Using (4.2.3), we extract a (non-relabelled) subsequence such that $\tilde{u}_j := u_{\kappa_j,t_j} \circ \Phi^j_{t_j}$ converges to a function u in $C^{1,\tilde{\vartheta}}$. Each function $\tilde{u}_j$ satisfies

$$\begin{cases} \operatorname{div}\left(\left(\kappa_j^2 + \left|\left(\left(\nabla\Phi^j_{t_j}\right)^{-1}\right)^t \nabla\tilde{u}_j\right|^2\right)^{\frac{p-2}{2}} M_j \nabla\tilde{u}_j\right) = 0 \text{ in } B^c, \\ \tilde{u}_j = 1 \text{ on } \partial B, \\ \tilde{u}_j(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $M_j = \det\nabla\Phi^j_{t_j}\left(\nabla\Phi^j_{t_j}\right)^{-1}\left(\left(\nabla\Phi^j_{t_j}\right)^{-1}\right)^t$. Thus, u, as a limit of $\tilde{u}_j$ in $C^{1,\tilde{\vartheta}}$, satisfies

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \text{ in } B^c, \\ u = 1 \text{ on } \partial B, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

meaning that u coincides with u_0, which contradicts (4.2.4).

To get convergence in $C^{2,\vartheta'}$, we notice that $0 < c \leq |\nabla u_0| \leq C$ in $B_R \setminus B_1$. The $C^{1,\tilde{\vartheta}'}$ converges gives us that the same is true for $\nabla u_{\kappa,t}$ if $\|\varphi\|_{C^{2,\vartheta}(\partial B)} + \kappa$ is small enough. From here equation for $u_{\kappa,t}$ and Shauder estimates give us

$$\|u_{\kappa,t}\|_{C^{2,\vartheta'}(B_R \setminus \Omega_t)} \leq C = C(p,\vartheta,n,\delta).$$

We now can argue in the same way as we did for $C^{1,\tilde{\vartheta}'}$ convergence. □

4.2.1 First derivative

Proposition 4.2.6. *For $\kappa > 0$, $t \in [0,1]$ the perturbed p-capacity is differentiable in t and the following formula holds*

$$\begin{aligned} c_\kappa'(t) &= -p\int_{\partial\Omega_t} |\nabla u_{\kappa,t}|^2(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(X\cdot\nu)d\mathcal{H}^{N-1} \\ &\quad + \int_{\partial\Omega_t}\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}-\kappa^p\right)(X\cdot\nu)d\mathcal{H}^{N-1} \\ &= -p\int_{\Omega_t^c}\operatorname{div}\left(|\nabla u_{\kappa,t}|^2(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}X\right) \\ &\quad + \int_{\Omega_t^c}\operatorname{div}\left(\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}-\kappa^p\right)X\right)dx. \end{aligned} \tag{4.2.5}$$

Moreover, for every $t \in [0,1]$ we also have

$$c_0'(t) = -(p-1)\int_{\partial\Omega_t}|\nabla u_{0,t}|^p(X\cdot\nu)d\mathcal{H}^{N-1}. \tag{4.2.6}$$

Proof. By Hadamard's formula,

$$\begin{aligned} c_\kappa'(t) &= \int_{\Omega_t^c} p\nabla u_{\kappa,t}\cdot\nabla\dot{u}_{\kappa,t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\,dx + \int_{\partial\Omega_t}\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}-\kappa^p\right)(X\cdot\nu)d\mathcal{H}^{N-1} \\ &= \int_{\partial\Omega_t}(-\nabla u_{\kappa,t}\cdot X)\,p(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla u_{\kappa,t}\cdot\nu\,d\mathcal{H}^{N-1} \\ &\quad + \int_{\partial\Omega_t}\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}-\kappa^p\right)(X\cdot\nu)d\mathcal{H}^{N-1}, \end{aligned}$$

where for the second equality we used the equations (4.2.1) and (4.2.2). It remains to notice that $\nabla u_{\kappa,t} = |\nabla u_{\kappa,t}|\nu$ on $\partial\Omega_t$ as noted in Remark 4.2.3. This gives us the first equality of (4.2.5), whereas the second equality of (4.2.5) follows from divergence theorem.

The convergence established in Theorem 4.2.5 gives us (4.2.6). □

4.2.2 Second derivative

To state the results for the second derivative we need to introduce the following weighted Sobolev space:

$$D^{1,2}(B^c,\mu) := \left\{u \in H^1_{loc}(B^c) : \int_{B^c}|\nabla u|^2\,d\mu < \infty\right\},$$

where $d\mu = |x|^{\left(\frac{p-n}{p-1}-1\right)(p-2)}dx$.

Proposition 4.2.7. *We define $X_\tau := X - (X\cdot\nu)\,\nu$. Then for $\kappa > 0, t\in[0,1]$ the perturbed p-capacity is twice differentiable and the following formula holds:*

$$
\begin{aligned}
\frac{1}{p}c_\kappa''(t) &= \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla\dot u_{\kappa,t}\cdot\nu)\dot u_{\kappa,t}d\mathcal{H}^{N-1}\\
&+(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla\dot u_{\kappa,t}\cdot\nabla u_{\kappa,t})(\nabla\dot u_{\kappa,t}\cdot\nu)\dot u_{\kappa,t}d\mathcal{H}^{N-1}\\
&-\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla^2u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&-(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2(\nabla^2u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\int_{\partial\Omega_t}|\nabla u_{\kappa,t}|^2(X\cdot\nu)^2(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\mathscr{H}_{\Omega_t^c}d\mathcal{H}^{N-1}.
\end{aligned}
\tag{4.2.7}
$$

Moreover,

$$
\begin{aligned}
\left(\frac{p-1}{N-p}\right)^{p-2}\frac{1}{p}c_0''(0) &= -(N-1)\int_{\partial B}\dot u_0^2 d\mathcal{H}^{N-1}\\
&\quad+\int_{B^c}|x|^{(p-2)\left(\frac{p-N}{p-1}-1\right)}\left(|\nabla\dot u_0|^2+(p-2)(\theta\cdot\nabla\dot u_0)^2\right)dx,
\end{aligned}
$$

where $\dot u_0$ solves

$$
\begin{cases}
\operatorname{div}\left(|x|^{(p-2)(\frac{p-N}{p-1})}\nabla\dot u_0+(p-2)|x|^{(p-2)(\frac{p-N}{p-1})}(\theta\cdot\nabla\dot u_0)\theta\right)=0 \text{ in } B^c,\\
\dot u_0=\frac{N-p}{p-1}\theta\cdot X \text{ on } \partial B
\end{cases}
\tag{4.2.8}
$$

in $W^{1,2}(B^c,d\mu)$.

Proof. **Computation.** First we use Hadamard's formula to differentiate the equality (4.2.5). We get

$$
\begin{aligned}
c_\kappa''(t) &= \int_{\partial\Omega_t} p\left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-2)/2}(\nabla u_{\kappa,t}\cdot\nabla\dot u_{\kappa,t})\,(X\cdot\nu)\,d\mathcal{H}^{N-1}\\
&+\int_{\partial\Omega_t}\operatorname{div}\left(\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}-\kappa^p\right)X\right)(X\cdot\nu)\,d\mathcal{H}^{N-1}\\
&-p\int_{\partial\Omega_t}\left((p-2)\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-4)/2}|\nabla u_{\kappa,t}|^2\,(\nabla u_{\kappa,t}\cdot\nabla\dot u_{\kappa,t})\,(X\cdot\nu)\,d\mathcal{H}^{N-1}\\
&-2p\int_{\partial\Omega_t}2\left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-2)/2}(\nabla u_{\kappa,t}\cdot\nabla\dot u_{\kappa,t})\,(X\cdot\nu)\,d\mathcal{H}^{N-1}\\
&-p\int_{\partial\Omega_t}\operatorname{div}\left(\left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-2)/2}|\nabla u_{\kappa,t}|^2X\right)(X\cdot\nu)\,d\mathcal{H}^{N-1}.
\end{aligned}
$$

Using (4.2.2), and the fact that X is divergence-free, we obtain

$$\begin{aligned}
c_\kappa''(t) &= p\int_{\partial\Omega_t} \left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-2)/2} \dot u_{\kappa,t}\left(\nabla \dot u_{\kappa,t}\cdot\nu\right) d\mathcal{H}^{N-1}\\
&-\int_{\partial\Omega_t} p\left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-2)/2}\left(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X\right)(X\cdot\nu)\, d\mathcal{H}^{N-1}\\
&+p\int_{\partial\Omega_t}\left((p-2)\left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-4)/2}|\nabla u_{\kappa,t}|^2\right)\dot u_{\kappa,t}\left(\nabla\dot u_{\kappa,t}\cdot\nu\right)\\
&-p(p-2)\int_{\partial\Omega_t}\left(\kappa^2+|\nabla u_{\kappa,t}|^2\right)^{(p-4)/2}|\nabla u_{\kappa,t}|^2\left(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X\right)(X\cdot\nu)\, d\mathcal{H}^{N-1}.
\end{aligned} \tag{4.2.9}$$

We now use Remark 4.2.3 to get the following equality on the boundary

$$\begin{aligned}
0 &= \operatorname{div}\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla u_{\kappa,t}\right)\\
&= (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|\mathscr{H}_{\Omega_t^c} + (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla^2 u_{\kappa,t}[\nu]\cdot\nu\\
&+(p-2)(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2\nabla^2 u_{\kappa,t}[\nu]\cdot\nu.
\end{aligned} \tag{4.2.10}$$

Now we plug (4.2.10) into (4.2.9) and get (4.2.7).

Convergence. Fix $R>1$. By Schauder estimates functions $u_{\kappa,t}\circ\Phi_t$ are equibounded in $C^{2,\vartheta}(B_R\backslash B)$ and $|\nabla u_{\kappa,t}|\in(c(R),C(R))$ for κ, t small. Thus, from (4.2.2), using classical elliptic estimates we get that $\dot u_{\kappa,t}$ are equibounded in $C^{1,\vartheta}(B_R\backslash B)$ and up to a subsequence converge to a function $\hat w\in C^1(B^c)$ uniformly on compacts.

Using $\dot u_{\kappa,t}$ as a test function in (4.2.2) and applying divergence theorem, we get

$$\begin{aligned}
&\int_{\Omega_t^c}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla\dot u_{\kappa,t}|^2dx+(p-2)\int_{\Omega_t^c}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla u_{\kappa,t}\cdot\nabla\dot u_{\kappa,t})^2dx\\
&=\int_{\partial\Omega_t^c}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\dot u_{\kappa,t}\nabla\dot u_{\kappa,t}\cdot\nu\\
&\quad+\int_{\partial\Omega_t^c}(p-2)(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla u_{\kappa,t}\cdot\nabla\dot u_{\kappa,t})\dot u_{\kappa,t}\nabla u_{\kappa,t}\cdot\nu\le C.
\end{aligned}$$

That means that $\hat w\in D^{1,2}(\Omega_t^c,\mu)$. Passing to the limit in (4.2.2) as $(\kappa,t)\to(0,0)$, we get

$$\int_{B^c}\nabla\hat w\cdot\nabla v+(p-2)(\theta\cdot\nabla\hat w)(\theta\cdot\nabla v)d\mu=0 \tag{4.2.11}$$

for any $v\in W^{1,2}(B_R\backslash\overline{B})$ with compact support in $B_R\backslash\overline{B}$.

It remains to show that the same identity holds for every v in $D_0^{1,2}(B^c;\mu)$ and that $\hat w$ is the unique solution of the equation (4.2.11) in $D^{1,2}(B^c;\mu)$ with a proper boundary condition. To that end, we fix $R>1$ and a cut-off function η_R such that $\eta_R\equiv 1$ in $B_R\backslash B$, $\eta_R\equiv 0$ in B_{2R}^c and $|\nabla\eta_R|\le C/R$. We also fix a constant c that we specify later, it will depend on R. For a function $v\in D_0^{1,2}(B^c;\mu)$ we plug $(v-c)\eta_R$

into (4.2.11) to get

$$\begin{aligned}
&\int_{B^c} \eta_R \nabla\hat{w}\cdot\nabla v + (p-2)\eta_R(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla v)d\mu \\
&= \int_{B^c} \eta_R \nabla\hat{w}\cdot\nabla(v-c) + (p-2)\eta_R(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla(v-c))d\mu \\
&= -\left(\int_{B^c} (v-c)\nabla\hat{w}\cdot\nabla\eta_R + (p-2)(v-c)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_R)d\mu\right).
\end{aligned}$$

Claim: $\int_{B^c}(v-c)\nabla\hat{w}\cdot\nabla\eta_R + (p-2)(v-c)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_R)d\mu \to 0$ as $R\to\infty$. Indeed, we have

$$\begin{aligned}
&\left|\int_{B^c} (v-c)\nabla\hat{w}\cdot\nabla\eta_R + (p-2)(v-c)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_R)d\mu\right| \\
&= \left|\int_{B_{2R}\setminus B_R} \left((v-c)\nabla\hat{w}\cdot\nabla\eta_R + (p-2)(v-c)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_R)\right)|x|^{\left(\frac{p-N}{p-1}-1\right)(p-2)}\,dx\right| \\
&\le (p-1)\frac{C}{R}\int_{B_{2R}\setminus B_R} |v-c||\nabla\hat{w}||x|^{\left(\frac{p-N}{p-1}-1\right)(p-2)}\,dx \\
&\le \frac{C}{R}\left(\int_{B_{2R}\setminus B_R}|\nabla\hat{w}|^2|x|^{\left(\frac{p-N}{p-1}-1\right)(p-2)}\,dx\right)^{1/2}\left(\int_{B_{2R}\setminus B_R}|v-c|^2|x|^{\left(\frac{p-N}{p-1}-1\right)(p-2)}\,dx\right)^{1/2}.
\end{aligned}$$

Since we know that $w\in D^{1,2}(B^c;\mu)$, if we manage to show that

$$\frac{1}{R}\left(\int_{B_{2R}\setminus B_R}|v-c|^2|x|^{\left(\frac{p-N}{p-1}-1\right)(p-2)}\,dx\right)^{1/2}\le C \tag{4.2.12}$$

for some $C=C(N,p)$, the claim will be proven. For convenience we denote $\gamma := \left(\frac{p-N}{p-1}-1\right)(p-2)$. We now choose $c = ⨍_{B_{2R}\setminus B_R} v$ and use Poincaré inequality to get

$$\begin{aligned}
\int_{B_{2R}\setminus B_R}|v-c|^2\,dx = \int_{B_{2R}\setminus B_R}\left|v - ⨍_{B_{2R}\setminus B_R} v\right|^2 dx \le CR^2\int_{B_{2R}\setminus B_R}|\nabla v|^2\,dx \\
= CR^{2-\gamma}\int_{B_{2R}\setminus B_R}|\nabla v|^2 R^\gamma\,dx \le CR^{2-\gamma},
\end{aligned} \tag{4.2.13}$$

where in the last inequality we used that $v\in D^{1,2}(B^c;\mu)$. This yields (4.2.12), since

$$\frac{1}{R}\left(\int_{B_{2R}\setminus B_R}|v-c|^2|x|^\gamma\,dx\right)^{1/2}\le CR^{-1+\gamma/2}\left(\int_{B_{2R}\setminus B_R}|v-c|^2\,dx\right)^{1/2}\le C,$$

where in the last inequality we applied (4.2.13). Thus we proved the claim and so (4.2.11) holds for every $v\in D_0^{1,2}(B^c;\mu)$. The boundary condition is the same as in (4.2.8) since $\dot{u}_{\kappa,t}\circ\Phi_t$ are converging to $\hat{w}$ regularly on ∂B. So, $\hat{w}$ solves the Dirichlet problem (4.2.8) and by uniqueness the whole sequence $\dot{u}_{\kappa,t}\circ\Phi_t$ converges to $\hat{w}$.

□

Lemma 4.2.8. *There exists a modulus of continuity ω such that*

$$|c''_\kappa(t) - c''_\kappa(0)| \le \omega(\|\varphi\|_{C^{2,\vartheta}} + \kappa)\|X\cdot\nu\|^2_{H^{1/2}(\partial B)}.$$

Proof. By divergence theorem and (4.2.2), using change of variables we can rewrite the second derivative of the energy in the following way:

$$\frac{1}{p}c''_\kappa(t) = I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) := \int_{B^c}\left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla \dot u_{\kappa,t}|^2\right)\circ\Phi_t\, J_{\Phi_t}dx$$
$$+ (p-2)\int_{B^c}\left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla \dot u_{\kappa,t}\cdot\nabla u_{\kappa,t})^2\right)\circ\Phi_t\, J_{\Phi_t}dx,$$
$$I_2(t) := -\int_{\partial B}\left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu)\right)\circ\Phi_t\, J^{\partial B}\Phi_t d\mathcal{H}^{N-1}$$
$$-(p-2)\int_{\partial B}\left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu)\right)\circ\Phi_t\, J^{\partial B}\Phi_t d\mathcal{H}^{N-1},$$
$$I_3(t) := \int_{\partial B}\left(|\nabla u_{\kappa,t}|^2(X\cdot\nu)^2(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\mathscr{H}_{\Omega_t^c}\right)\circ\Phi_t\, J^{\partial B}\Phi_t d\mathcal{H}^{N-1}.$$

By Lemma 4.0.3, we have

$$\|\mathscr{H}_{\partial\Omega_t}\circ\Phi_t - \mathscr{H}_{\partial B}\|_{L^\infty(\partial B)} + \|J^{\partial B}\Phi_t - 1\|_{L^\infty(\partial B)} \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right).$$

In addition, by Lemma 4.0.3, X is parallel to θ in a neighborhood of ∂B, so we have

$$|\,(X\cdot\nu_{\Omega_t})\circ\Phi_t - X\cdot\nu_B| \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right)|X\cdot\nu_B|,$$

as well as

$$|X_\tau\circ\Phi_t| \le \omega\left(\|\varphi\|_{C^{2,\vartheta}}\right)|X\cdot\nu_B|.$$

Thus, using Theorem 4.2.5 and noticing that $I_2(0) = 0$, we get

$$|I_2(t) - I_2(0)| + |I_3(t) - I_3(0)| \le \omega\left(\|\varphi\|_{C^{2,\vartheta}} + \kappa\right)\|X\cdot\nu_B\|^2_{L^2(\partial B)}.$$

It remains to show that

$$|I_1(t) - I_1(0)| \le \omega(\|\varphi\|_{C^{2,\vartheta}} + \kappa)\|X\cdot\nu\|^2_{H^{1/2}(\partial B)}. \tag{4.2.14}$$

We are going to sketch the proof of (4.2.14), for more details see the proof of [FZ16, Lemma 2.7]. We first move the equation for $\dot u_{\kappa,t}$ onto the unit ball B. To that end, we denote $w_{\kappa,t} := \dot u_{\kappa,t}\circ\Phi_t$, $\tilde u_{\kappa,t} := u_{\kappa,t}\circ\Phi_t$, and $N_t = (\nabla\Phi_t)^{-1}\left((\nabla\Phi_t)^{-1}\right)^t$. Then $w_{\kappa,t}$ satisfies

$$\begin{cases}\mathrm{div}\,\Big((\kappa^2 + |\left((\nabla\Phi_t)^{-1}\right)^t\nabla\tilde u_{\kappa,t}|^2)^{\frac{p-2}{2}}\det\nabla\Phi_t N_t\nabla w_{\kappa,t} \\ +(p-2)(\kappa^2 + |\left((\nabla\Phi_t)^{-1}\right)^t\nabla\tilde u_{\kappa,t}|^2)^{\frac{p-4}{2}}\det\nabla\Phi_t(N_t\nabla\tilde u_{\kappa,t}\cdot\nabla w_{\kappa,t})N_t\nabla w_{\kappa,t}\Big) = 0 \text{ in } B^c, \\ w_{\kappa,t} = -\left(\nabla u_{\kappa,t}\cdot X\right)\circ\Phi_t \text{ on } \partial B\end{cases}$$

and

$$I_1(t) := \int_B (\kappa^2 + |\left((\nabla\Phi_t)^{-1}\right)^t \nabla\tilde{u}_{\kappa,t}|^2)^{\frac{p-2}{2}} N_t \nabla w_{\kappa,t}\cdot\nabla w_{\kappa,t} \,\mathrm{det}\nabla\Phi_t dx$$
$$+ (p-2)\int_B (\kappa^2 + |\left((\nabla\Phi_t)^{-1}\right)^t \nabla\tilde{u}_{\kappa,t}|^2)^{\frac{p-4}{2}} (N_t\nabla\tilde{u}_{\kappa,t}\cdot\nabla w_{\kappa,t})^2 \,\mathrm{det}\nabla\Phi_t dx.$$

For convenience we define a bilinear form $L_{\kappa,t,\varphi}$ as

$$L_{\kappa,t,\varphi}(u,v) := \int_B (\kappa^2 + |\left((\nabla\Phi_{t,\varphi})^{-1}\right)^t \nabla\tilde{u}_{\kappa,t,\varphi}|^2)^{\frac{p-2}{2}} N_{t,\varphi}\nabla u\cdot\nabla v \,\mathrm{det}\nabla\Phi_t dx$$
$$+ (p-2)\int_B (\kappa^2 + |\left((\nabla\Phi_{t,\varphi})^{-1}\right)^t \nabla\tilde{u}_{\kappa,t,\varphi}|^2)^{\frac{p-4}{2}} (N_t\nabla\tilde{u}_{\kappa,t}\cdot\nabla u)(N_t\nabla\tilde{u}_{\kappa,t}\cdot\nabla v) \,\mathrm{det}\nabla\Phi_t dx,$$

so that proving (4.2.14) amounts to showing that

$$|L_{\kappa,t,\varphi}(w_{\kappa,t,\varphi,\kappa,t,\varphi}) - L_{\kappa,0,\varphi}(w_{\kappa,0,\varphi,\kappa,t,\varphi})| \le \omega(\|\varphi\|_{C^{2,\vartheta}} + \kappa)\|X\cdot\nu\|^2_{H^{1/2}(\partial B)}.$$

We argue by contradiction. Assume there exist sequences $\kappa_j \to 0$, $t_j \to t \in [0,1]$, $\varphi_j \to 0$ in $C^{2,\vartheta}(\partial B)$ such that

$$\lim_{j\to\infty} \frac{L_{\kappa_j,t_j,\varphi_j}(w_{\kappa_j,t_j,\varphi_j}, w_{\kappa_j,t_j,\varphi_j})}{\|X_j\cdot\nu_B\|^2_{H^{1/2}(\partial B)}} \ne \lim_{j\to\infty} \frac{L_{\kappa_j,0,\varphi_j}(w_{\kappa_j,0,\varphi_j}, w_{\kappa_j,0,\varphi_j})}{\|X_j\cdot\nu_B\|^2_{H^{1/2}(\partial B)}}. \tag{4.2.15}$$

Note that we can assume that both limits are finite. We define

$$\tilde{w}_j := \frac{w_{\kappa_j,t_j,\varphi_j}}{\|X_j\cdot\nu_B\|_{H^{1/2}(\partial B)}}, \qquad \tilde{w}_{0,j} := \frac{w_{\kappa_j,0,\varphi_j}}{\|X_j\cdot\nu_B\|_{H^{1/2}(\partial B)}}.$$

One can easily show that $\tilde{w}_j - \tilde{w}_{0,j} \to 0$ strongly in $H^{1/2}(\partial B)$. A bit more work is required to show that $\tilde{w}_j - \tilde{w}_{0,j} \to 0$ strongly in $W^{1,2}(B_R\setminus B)$ for every $r\in(0,1)$. To do that, one can prove first that both $\tilde{w}_j$ and $\tilde{w}_{0,j}$ converge weakly to the unique solution in $D^{1,2}(B^c,\mu)$ of

$$\begin{cases} \mathrm{div}\left(|x|^{(p-2)(\frac{p-N}{p-1})}\nabla w + (p-2)|x|^{(p-2)(\frac{p-N}{p-1})}(\theta\cdot\nabla w)\theta\right) = 0 \text{ in } B^c,\\ w = f \text{ on } \partial B,\end{cases}$$

where f is the weak limit in $H^{1/2}(\partial B)$ of the restriction of $\tilde{w}_j$ on ∂B (remember that the limit of restriction of $\tilde{w}_{0,j}$ is the same). To show the strong convergence consider z_j - the harmonic extension of $\tilde{w}_j - \tilde{w}_{0,j}$ from ∂B to B^c. Note that z_j converges strongly to zero in $D^{1,2}(B^c)$. Denote by $\zeta \in C_0^\infty(B_R)$ a cut-off function such that $\zeta\equiv 1$ on $B_R\setminus B$, $0\le\zeta\le 1$. By divergence theorem we get

$$L_{\kappa_j,t_j,\varphi_j}(\tilde{w}_j - \tilde{w}_{0,j}, (\tilde{w}_j - \tilde{w}_{0,j})\zeta) = L_{\kappa_j,t_j,\varphi_j}(\tilde{w}_j, z_j\zeta)$$
$$- (L_{\kappa_j,t_j,\varphi_j} - L_{\kappa_j,0,\varphi_j})(\tilde{w}_{0,j}, (\tilde{w}_j - \tilde{w}_{0,j})\zeta) - L_{\kappa_j,0,\varphi_j}(\tilde{w}_{0,j}, z_j\zeta) \to 0,$$

which yields strong convergence of $\tilde{w}_j - \tilde{w}_{0,j}$ to zero in $W^{1,2}(B_R\setminus B)$. Finally, one can now show that

$$\lim_{j\to\infty}\left(L_{\kappa_j,t_j,\varphi_j}(\tilde{w}_j,\tilde{w}_j) - L_{\kappa_j,0,\varphi_j}(\tilde{w}_{0,j},\tilde{w}_{0,j})\right) = 0,$$

contradicting (4.2.15). □

Lemma 4.2.9. *Given $\vartheta \in (0,1]$, there exists $\delta = \delta(N,p,\vartheta) > 0$ and a modulus of continuity ω such that for every nearly-spherical set Ω parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|\Omega| = |B_1|$, we have*

$$\operatorname{cap}_p(\Omega) \geq \operatorname{cap}_p(B_1) + \frac{1}{2}\partial^2 \operatorname{cap}_p(B_1)[\varphi,\varphi] - \omega(\|\varphi\|_{C^{2,\vartheta}})\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)},$$

where

$$\left(\frac{p-1}{N-p}\right)^{p-2} \frac{1}{p}\partial^2 \operatorname{cap}_p(B_1)[\varphi,\varphi] := -(N-1)\int_{\partial B}\left(\frac{N-p}{p-1}\varphi\right)^2 d\mathcal{H}^{N-1} + \int_{B^c} |x|^{(p-2)\left(\frac{p-N}{p-1}-1\right)}\left(|\nabla f(\varphi)|^2 + (p-2)\left(\theta\cdot\nabla f(\varphi)\right)^2\right) dx,$$

with $f(\varphi)$ satisfying

$$\begin{cases} \operatorname{div}\left(|x|^{(p-2)(\frac{p-N}{p-1})}\nabla f(\varphi) + (p-2)|x|^{(p-2)(\frac{p-N}{p-1})}(\theta\cdot\nabla f(\varphi))\theta\right) = 0 \ \textit{in } B^c, \\ f(\varphi) = \frac{N-p}{p-1}\varphi \ \textit{on } \partial B. \end{cases}$$

Proof. We write Taylor expansion for c_κ:

$$c_\kappa(1) = c_\kappa(0) + c_\kappa'(0) + \frac{1}{2}c_\kappa''(0) + \int_0^1 (1-t)(c_\kappa''(t) - c_\kappa''(0))\, dt.$$

From isocapacitary inequality we know that $c_0'(0) = 0$. So, we get the desired inequality using Lemma 4.2.8 and passing to the limit as $\kappa \to 0$. ☐

4.2.3 Inequality for nearly-spherical sets

We now establish a quantitative inequality for nearly-spherical sets, compare with [FZ16, Theorem 2.8].

Theorem 4.2.10. *There exists $\delta = \delta(N,p), c = c(N,p)$ such that if Ω is a nearly-spherical set of class $C^{2,\vartheta}$ parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}} \leq \delta, |\Omega| = |B_1|$ and $x_\Omega = 0$, then*

$$\operatorname{cap}_p(\Omega) - \operatorname{cap}_p(B_1) \geq c\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)}.$$

Proof. We introduce the following notation:

$$\dot{u}_0 = -\frac{p-N}{p-1}\hat{u},\ \ \Psi = \theta\cdot X.$$

Then $\hat{u}$ solves

$$\begin{cases} \operatorname{div}\left(|x|^{(p-2)(\frac{p-N}{p-1}-1)}\nabla\hat{u} + (p-2)|x|^{(p-2)(\frac{p-N}{p-1}-1)}(\theta\cdot\nabla\hat{u})\theta\right) = 0 \text{ in } B^c, \\ \hat{u} = \Psi \text{ on } \partial B \end{cases}$$

and

$$\left(\frac{p-1}{p-N}\right)^p \frac{1}{p} c_0''(0) = -(N-1)\int_{\partial B} \hat{u}^2 d\mathcal{H}^{N-1} + \int_{B^c} |x|^{(p-2)\left(\frac{p-N}{p-1}-1\right)} \left(|\nabla \hat{u}|^2 + (p-2)(\theta\cdot\nabla\hat{u})^2\right) dx.$$

We introduce the following notation:

$$Q[\Psi] = -(N-1)\int_{\partial B} \hat{u}^2 d\mathcal{H}^{N-1} + \int_{B^c} |x|^{(p-2)\left(\frac{p-N}{p-1}-1\right)} \left(|\nabla \hat{u}|^2 + (p-2)(\theta\cdot\nabla\hat{u})^2\right) dx.$$

We write Ψ in the basis of spherical harmonics, i.e.

$$\Psi = \sum_{k=0}^{\infty} \sum_{i=1}^{M(k,N)} a_{k,i} Y_{k,i},$$

where $Y_{k,i}$ for $i = 1, \ldots, M(k,N)$ are harmonic polynomials of degree k, normalized so that $\|Y_{k,i}\|_{L^2(\partial B)} = 1$. By (4.0.4) we have that $\|\Psi\|_{H^{1/2}(\partial B_1)} \geq c\|\varphi\|_{H^{1/2}(\partial B_1)}$ if δ is small enough. Thus, to prove the theorem, by Lemma 4.2.9 it is enough to show that $Q[\Psi] \geq c\|\Psi\|^2_{H^{1/2}(\partial B_1)}$, or, equivalently, that

$$Q[\Psi] = \sum_{k=0}^{\infty} \sum_{i=1}^{M(k,N)} kQ[Y_{k,i}] \geq c \sum_{k=0}^{\infty} \sum_{i=1}^{M(k,N)} (k+1)a_{k,i}^2. \tag{4.2.16}$$

We first note $\int_{\partial B} \Psi = 0$ as Φ_t conserves volume, and thus $a_0 = 0$. We then bound $\sum_{i=1^N} a_{1,i}^2$. We recall that $x_\Omega = 0$, hence

$$\int_{\partial B} x\left((1+\varphi)^{N+1} - 1\right) d\mathcal{H}^{N-1} = 0$$

and consequently, for any $\varepsilon > 0$ if δ is small enough we get

$$\left|\int_{\partial B} x\varphi \, d\mathcal{H}^{N-1}\right| \leq \varepsilon\|\varphi\|_{L^2(\partial B)}.$$

By (4.0.3) this in turn yields

$$\left|\int_{\partial B} x\Psi \, d\mathcal{H}^{N-1}\right| \leq 2\varepsilon\|\Psi\|_{L^2(\partial B)}$$

if δ is small enough. So we get

$$\sum_{i=1^N} a_{1,i}^2 \leq 2\sum_{k=2}^{\infty} \sum_{i=1}^{M(k,N)} (k+1)a_{k,i}^2$$

for δ small enough and to prove (4.2.16) it remains to show that

$$\sum_{k=0}^{\infty} \sum_{i=1}^{M(k,N)} kQ[Y_{k,i}] \geq c \sum_{k=2}^{\infty} \sum_{i=1}^{M(k,N)} (k+1)a_{k,i}^2. \tag{4.2.17}$$

We denote by $u_{k,i}$ the function $\hat{u}$ corresponding to $Y_{k,i}$ on the boundary. Then a straightforward computation tells us that

$$u_{k,i} = |x|^{\alpha_k} Y_{k,i},$$

where $\alpha_k < 0$ is the only negative solution of the following quadratic equation:

$$(p-1)\alpha_k^2 + (N-p)\,\alpha_k - k(k+N-2) = 0. \tag{4.2.18}$$

Recalling that

$$\int_{\partial B} |Y_{k,i}|^2 d\mathcal{H}^{N-1} = 1, \int_{\partial B} |\nabla_\tau Y_{k,i}|^2 d\mathcal{H}^{N-1} = k(k+N-2),$$

we get that

$$\begin{aligned} Q[Y_{k,i}] &= -(N-1) - \frac{\alpha_k^2(p-1) + k(k+N-2)}{(p-2)(\frac{p-N}{p-1} - 1) + 2(\alpha_k - 1) + N} \\ &= -(N-1) - \frac{(2k(k+N-2) - (N-p)\alpha_k)\,(p-1)}{(p-1)2\alpha_k + N - p}, \end{aligned}$$

where we used (4.2.18). Now, since by (4.2.18) we have

$$\alpha_k = -\frac{N - p + \sqrt{(N-p)^2 + 4(p-1)k(k+N-2)}}{2(p-1)},$$

we get after straightforward computations

$$Q[Y_{k,i}] = -(N-1) + \frac{N - p + \sqrt{(N-p)^2 + 4(p-1)k(k+N-2)}}{2}.$$

Notice that

$$Q[Y_{1,i}] = 0, \qquad Q[Y_{k,i}] \geq ck \text{ for } k \geq 2$$

for some $c = c(N,p) > 0$. This gives us (4.2.17) and hence (4.2.16) and so we conclude the proof of the theorem. □

4.3 Liquid drops

In this section we establish stability near the ball for the energy $\mathcal{F}$ defined in (1.1.8). For convenience we introduce the following notation:

$$\mathcal{G}_{\beta,K}(E) := \inf_{(u,\rho)\in\mathcal{A}(E)} \left\{ \int_{\mathbb{R}^N} a_E |\nabla u|^2 + K \int_E \rho^2 \right\}. \tag{4.3.1}$$

For $E \subset \mathbb{R}^N$ we set

$$\mathcal{F}_{\beta,K,Q}(E) := P(E) + Q^2 \mathcal{G}_{\beta,K}(E).$$

We are going to deal with $\mathcal{G}$ separately since the appropriate computation for the perimeter was done already by Fuglede in [Fug89].

4.3.1 Changing minimization problem

We first replace our problem with an equivalent one and write Euler-Lagrange equations for it. We do it to facilitate the computations of the first and second variation. For a fixed domain E we are solving the following minimization problem.

$$\mathcal{G}(E) = \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ \rho\mathbf{1}_{E^c}=0}} \left\{ \frac{1}{2}\int_{\mathbb{R}^N} \left(a_E|\nabla u|^2 + K\rho^2\right) dx : -\operatorname{div}(a_E\nabla u) = \rho, \int_{\mathbb{R}^N} \rho dx = 1 \right\}.$$

We want to get rid of the constraints and make it a minimization problem over single functions rather than over pairs. More precisely, we prove the following lemma.

Lemma 4.3.1.

$$\begin{aligned} \mathcal{G}(E) = \frac{K}{2|E|} - \inf_{\psi\in H^1(\mathbb{R}^N)} \Big(\frac{1}{2}\int_{\mathbb{R}^N} a_E|\nabla\psi|^2 dx + \frac{1}{|E|}\int_E \psi dx \\ - \frac{1}{2|E|K}\left(\int_E \psi dx\right)^2 + \frac{1}{2K}\int_E \psi^2 dx \Big). \end{aligned} \tag{4.3.2}$$

Proof. We use an “infinite dimension Lagrange multiplier":

$$\begin{aligned} \mathcal{G}(E) &= \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ \rho\mathbf{1}_{E^c}=0}} \Big\{ \frac{1}{2}\int_{\mathbb{R}^N} a_E|\nabla u|^2 dx + \frac{1}{2}\int_E K\rho^2 dx \\ &\quad + \sup_{\psi\in H^1(\mathbb{R}^N)} \left[\int_{\mathbb{R}^N} (a_E\nabla u\cdot\nabla\psi - \rho\psi) dx\right] : \int_E \rho dx = 1 \Big\} \\ &= \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ \rho\mathbf{1}_{E^c}=0}} \sup_{\psi\in H^1(\mathbb{R}^N)} \Big\{ \frac{1}{2}\int_{\mathbb{R}^N} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi) dx \\ &\quad + \frac{1}{2}\int_E (K\rho^2 - 2\rho\psi) dx : \int_E \rho dx = 1 \Big\}. \end{aligned}$$

The convexity of the problem allows us to use Sion minimax theorem ([Sio58, Corollary 3.3]) and interchange the infimum and the supremum:

$$\begin{aligned} \mathcal{G}(E) &= \sup_{\psi\in H^1(\mathbb{R}^N)} \inf_{\substack{u\in H^1(\mathbb{R}^N)\\ \rho\mathbf{1}_{E^c}=0}} \Big\{ \frac{1}{2}\int_{\mathbb{R}^N} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi) dx \\ &\quad + \frac{1}{2}\int_E (\rho^2 - 2\rho\psi) dx : \int_E \rho dx = 1 \Big\} \\ &= \sup_{\psi\in H^1(\mathbb{R}^N)} \Big\{ \inf_{u\in H^1(\mathbb{R}^N)} \frac{1}{2}\int_{\mathbb{R}^N} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi) dx \\ &\quad + \inf_{\substack{\rho\mathbf{1}_{E^c}=0\\ \int_E \rho dx = 1}} \frac{1}{2}\int_E (\rho^2 - 2\rho\psi) dx \Big\}. \end{aligned}$$

We denote the infimums inside by I and II, that is

$$\mathrm{I} := \inf_{u\in H^1(\mathbb{R}^N)} \left\{ \frac{1}{2}\int_{\mathbb{R}^N} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi)dx \right\};$$
$$\mathrm{II} := \inf_{\rho} \left\{ \frac{1}{2}\int_E (K\rho^2 - 2\rho\psi)dx : \int_E \rho dx = 1 \right\}.$$

We want to compute both I and II in terms of ψ.
For I the computation is immediate. Since a_E is positive we get that

$$\begin{aligned}
\mathrm{I} &= \inf_{u\in H^1(\mathbb{R}^N)} \left\{ \frac{1}{2}\int_{\mathbb{R}^N} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi)dx \right\} \\
&= \inf_{u\in H^1(\mathbb{R}^N)} \left\{ \frac{1}{2}\int_{\mathbb{R}^N} a_E\left(|\nabla u + \nabla\psi|^2 - |\nabla\psi|^2\right)dx \right\} \\
&= -\frac{1}{2}\int_{\mathbb{R}^N} a_E|\nabla\psi|^2 dx.
\end{aligned}$$

We note that the corresponding minimizing u equals to $-\psi$.
To compute II, note that

$$\begin{aligned}
\mathrm{II} &= \inf_{\rho} \left\{ \frac{1}{2}\int_E (K\rho^2 - 2\rho\psi)dx : \int_E \rho dx = 1 \right\} \\
&= \inf_{\rho} \left\{ \frac{1}{2}\int_E \left(\sqrt{K}\rho - \frac{\psi}{\sqrt{K}}\right)^2 dx : \int_E \rho dx = 1 \right\} - \frac{1}{2K}\int_E \psi^2 dx \\
&= \frac{\sqrt{K}}{2}\inf_f \left\{ \int_E \left(f - \left(\frac{\psi}{K} - \frac{1}{|E|}\right)\right)^2 dx : \int_E f dx = 0 \right\} - \frac{1}{2K}\int_E \psi^2 dx.
\end{aligned}$$

Then the minimizing function f^* is the projection in $L^2(E)$ of a function $\left(\frac{\psi}{K} - \frac{1}{|E|}\right)$ onto the linear space $\left\{f : \int_E f dx = 0\right\}$. Thus, $f^* = \left(\frac{\psi}{K} - \frac{1}{|E|}\right) - c$, where c is the constant such that $\int_E f^* = 0$, i.e. $c = \frac{\int_E\left(\frac{\psi}{K} - \frac{1}{|E|}\right)}{|E|}$. The corresponding minimizing ρ equals to $\mathbf{1}_E \frac{1}{K}\left(\psi + \frac{\left(1 - \frac{1}{K}\int_E \psi dx\right)K}{|E|}\right)$.

Bringing it all together,

$$\begin{aligned}
\mathcal{G}(E) &= \frac{K}{2|E|} + \sup_{\psi\in H^1(\mathbb{R}^N)} \Big(-\frac{1}{2}\int_{\mathbb{R}^N} a_E|\nabla\psi|^2 dx - \frac{1}{|E|}\int_E \psi dx \\
&\qquad + \frac{1}{2|E|K}\left(\int_E \psi dx\right)^2 - \frac{1}{2K}\int_E \psi^2 dx \Big) \\
&= \frac{K}{2|E|} - \inf_{\psi\in H^1(\mathbb{R}^N)} \Big(\frac{1}{2}\int_{\mathbb{R}^N} a_E|\nabla\psi|^2 dx + \frac{1}{|E|}\int_E \psi dx \\
&\quad - \frac{1}{2|E|K}\left(\int_E \psi dx\right)^2 + \frac{1}{2K}\int_E \psi^2 dx \Big).
\end{aligned} \tag{4.3.3}$$

□

4.3.2 Euler-Lagrange equation

We now consider the following minimization problem:

$$\mathcal{J}(E) = \inf_{\psi \in H^1(\mathbb{R}^N)} \Big(\frac{1}{2} \int_{\mathbb{R}^N} a_E |\nabla \psi|^2 dx + \frac{1}{|E|} \int_E \psi dx - \frac{1}{2|E|K} \left(\int_E \psi dx \right)^2 + \frac{1}{2K} \int_E \psi^2 dx \Big). \tag{4.3.4}$$

Remark 4.3.2. Note that $\mathcal{J}(E) \leq 0$. By Lemma 4.3.1

$$\mathcal{G}(E) = \frac{K}{2|E|} - \mathcal{J}(E).$$

By the inequality (2.1) in [DPHV19], $\mathcal{G}(E) \leq C(N, K, \beta, |E|)$. This implies that

$$|\mathcal{J}(E)| \leq C(N, K, \beta, |E|). \tag{4.3.5}$$

A minimizer for this problem exists, and it is unique by convexity. Note that the minimizers in the definitions of $\mathcal{J}$ and $\mathcal{G}$ coincide since the set is fixed. We denote the minimizer by ψ_E. We would also need the interior and exterior restrictions of the function ψ_E, i.e.

$$\psi_E^+ := \psi_E|_E, \quad \psi_E^- := \psi_E|_{E^c}.$$

Proposition 4.3.3. *The following identities hold for ψ_E:*

(i) (Euler-Lagrange equation, integral form) for any $\Psi \in D^{1,2}(\mathbb{R}^N)$

$$\begin{aligned} &\int_{\mathbb{R}^N} a_E \nabla \psi_E \cdot \nabla \Psi dx + \frac{1}{K} \int_E \psi_E \Psi dx + \frac{1}{|E|} \left(\int_E \Psi dx \right) \left(1 - \frac{1}{K} \int_E \psi_E dx \right) \\ &= \int_{\mathbb{R}^N} \Psi \left(\frac{\mathbf{1}_E \psi_E}{K} - \operatorname{div}(a_E \nabla \psi_E) \right) dx + \int_{\partial E} \left(\beta \nabla \psi_E^+ - \nabla \psi_E^- \right) \cdot \nu \Psi \, d\mathcal{H}^{N-1} \\ &+ \frac{1}{|E|} \left(\int_E \Psi dx \right) \left(1 - \frac{1}{K} \int_E \psi_E dx \right) = 0. \end{aligned} \tag{4.3.6}$$

(ii) (Euler-Lagrange equation)

$$\begin{cases} -\beta \Delta \psi_E = -\frac{1}{K} \psi_E + \frac{2}{K} \mathcal{J}(E) - \frac{1}{|E|} \text{ in } E, \\ \Delta \psi_E = 0 \text{ in } E^c, \\ \psi_E^+ = \psi_E^- \text{ on } \partial E, \\ \beta \nabla \psi_E^+ \cdot \nu = \nabla \psi_E^- \cdot \nu \text{ on } \partial E. \end{cases} \tag{4.3.7}$$

(iii)

$$\mathcal{J}(E) = \frac{1}{2|E|} \int_E \psi_E dx. \tag{4.3.8}$$

(iv) There exists a constant $C = C(N, K, \beta, |E|)$ such that

$$\int_{\mathbb{R}^N} a_E |\nabla \psi_E|^2 dx \leq C. \tag{4.3.9}$$

Proof. To prove (4.3.8) we use ψ_E as a test function in (4.3.6).

To see (4.3.9), we use ψ_E as a test function in (4.3.6) and Cauchy-Schwarz inequality to get

$$\int_{\mathbb{R}^N} a_E |\nabla \psi_E|^2 dx \leq -\frac{1}{|E|}\left(\int_E \psi_E dx\right).$$

Now we apply (4.3.8) and (4.3.5) to obtain

$$\int_{\mathbb{R}^N} a_E |\nabla \psi_E|^2 dx \leq -2\mathcal{J}(E) \leq 2C(N, K, \beta, |E|).$$

□

Proposition 4.3.4. *Let ψ_0 be the minimizer for $\mathcal{J}(B_1)$. Then ψ_0 is radial.*

Proof. Let $W : \mathbb{R}^N \to \mathbb{R}^N$ be any rotation. Since $W(B_1) = B_1$, $\psi_0 \circ W$ is also a minimizer for $\mathcal{J}(B_1)$. But the minimizer is unique, so we got that $\psi_0 \circ W = \psi_0$ for any rotation W. This implies that ψ_0 is radial.

□

4.3.3 Inequality for nearly-spherical sets

First we show the following lemma that will allow us to prove Theorem 1.1.9 for nearly-spherical sets.

Lemma 4.3.5. *Given $\vartheta \in (0, 1]$, there exist $\delta = \delta(N, \vartheta) > 0$ and a constant $C > 0$ such that for every nearly-spherical set E parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|E| = |B_1|$, we have*

$$\mathcal{J}(E) \leq \mathcal{J}(B_1) + C\|\varphi\|^2_{H^1(\partial B_1)}.$$

First derivative

We want to compute $\frac{d}{dt}\mathcal{J}(\Omega_t)$.

Let ψ_t be the minimizer in the minimization problem (4.3.4) for Ω_t. Recall that by (4.3.7) it means that ψ_t satisfies

$$\begin{cases} -\beta\Delta\psi_t = -\frac{1}{K}\psi_t + \frac{2}{K}\mathcal{J}(\Omega_t) - \frac{1}{|B_1|} \text{ in } \Omega_t, \\ \Delta\psi_t = 0 \text{ in } \Omega_t^c, \\ \psi_t^+ = \psi_t^- \text{ on } \partial\Omega_t, \\ \beta\nabla\psi^+_{\Omega_t} \cdot \nu = \nabla\psi^1_{\Omega_t} \cdot \nu \text{ on } \partial\Omega_t. \end{cases} \tag{4.3.10}$$

First we notice that ψ_t is regular since it is a solution to a transmission problem. More precisely, by Lemma 7.4.2, the following holds.

Proposition 4.3.6. *There exists $\delta > 0$ such that if $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$, then*

$$\|\psi_t\|_{C^2(\overline{\Omega_t})} \le C \text{ for every } t \in [0,1]$$

for some constant $C = C(N, \delta)$.

Since we are going to use Hadamard's formula to compute the derivative of $\mathcal{J}(\Omega_t)$, we need the following proposition.

Proposition 4.3.7. *The function $t \mapsto \psi_t$ is differentiable in t and its derivative $\dot{\psi}_t$ satisfies*

$$\begin{cases} -\beta\Delta\dot{\psi}_t = -\frac{1}{K}\dot{\psi}_t + \frac{2}{K}\dot{\mathcal{J}}(\Omega_t) \ \text{in } \Omega_t, \\ \Delta\dot{\psi}_t = 0 \ \text{in } \Omega_t^c, \\ \dot{\psi}_t^+ - \dot{\psi}_t^- = -\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot\nu(X\cdot\nu) \ \text{on } \partial\Omega_t, \\ \beta\nabla\dot{\psi}_t^+\cdot\nu - \nabla\dot{\psi}_t^1\cdot\nu = -\left(\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)\cdot X\right)\cdot\nu \ \text{on } \partial\Omega_t. \end{cases} \tag{4.3.11}$$

Proof. Apply Proposition 3.2.7 to ψ_t with $f(t) = -\frac{2}{K}\mathcal{J}(\Omega_t) + \frac{1}{|B_1|}$. Note that f is Lipschitz due to [DPHV19, Lemma 3.2]. □

The following observation, which is a consequence of equation (4.3.11), will be useful for us.

Lemma 4.3.8. *There exists $f \in H^{3/2}(\Omega_t) \cap H^{3/2}(\Omega_t^c)$ such that*

$$f^\pm = \nabla\psi_t^\pm \cdot X \ \text{on } \partial\Omega_t, \quad \|f^\pm\|_{H^{3/2}} \le C\|\nabla\psi_t^\pm\cdot X\|_{H^1(\partial\Omega_t)}. \tag{4.3.12}$$

Consider a function $v := \dot{\psi}_t + f$, Then v satisfies the equation

$$\begin{cases} -\beta\Delta v = -\frac{1}{K}v + \frac{2}{K}\dot{\mathcal{J}}(\Omega_t) - \beta\Delta f + \frac{1}{K}f \ \text{in } \Omega_t, \\ \Delta v = \Delta f \ \text{in } \Omega_t^c, \\ v^+ - v^- = 0 \ \text{on } \partial\Omega_t, \\ \beta\nabla v^+\cdot\nu - \nabla v^-\cdot\nu = \left(-\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)\cdot X + \beta\nabla f^+ - \nabla f^-\right)\cdot\nu \ \text{on } \partial\Omega_t. \end{cases}$$

$$v = \dot{\psi}_t^\pm + \nabla\psi_t^\pm\cdot X \ \text{on } \partial\Omega_t. \tag{4.3.13}$$

Moreover, the following bounds hold:

$$\|v\|_{W^{1,2}(\Omega_t)} + \|v\|_{D^{1,2}(\Omega_t^c)} \le C\left(|\dot{\mathcal{J}}(\Omega_t)| + \|X\cdot\nu\|_{H^1(\partial\Omega_t)}\right); \tag{4.3.14}$$

$$\|u\|_{L^{2^*}(\mathbb{R}^N)} \le C\left(|\dot{\mathcal{J}}(\Omega_t)| + \|X\cdot\nu\|_{H^1(\partial\Omega_t)}\right). \tag{4.3.15}$$

Proof. The function f exists since $\nabla\psi_t^\pm\cdot X \in H^1(\partial\Omega_t)$. The equation for v follows from the equation for $\dot{\psi}_t$ and the definition of f. Using divergence theorem, we get

$$\int_{\Omega_t}\frac{1}{K}u^2dx + \int_{\Omega_t}\beta|\nabla u|^2dx + \int_{\Omega_t^c}|\nabla u|^2dx = \int_{\Omega_t}\left(\frac{2}{K}\dot{\mathcal{J}}(\Omega_t) - \beta\Delta f + \frac{1}{K}f\right)u\,dx$$
$$-\int_{\Omega_t^c}\Delta f\,u\,dx + \int_{\partial\Omega_t}\left(\left(-\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)\cdot X + \beta\nabla f^+ - \nabla f^-\right)\cdot\nu\right)u\,dx,$$

which by Young, Cauchy-Schwarz, and trace inequalities, recalling (4.3.12), implies that

$$\|u\|_{W^{1,2}(\Omega_t)} + \|u\|_{D^{1,2}(\Omega_t^c)} \leq C\left(|\dot{\mathcal{J}}(\Omega_t)| + \|\nabla\psi_t \cdot X\|_{H^1(\partial\Omega_t)}\right),$$

which in turn implies by Proposition 4.3.6 and (4.0.7)

$$\|u\|_{W^{1,2}(\Omega_t)} + \|u\|_{D^{1,2}(\Omega_t^c)} \leq C\left(|\dot{\mathcal{J}}(\Omega_t)| + \|X \cdot \nu\|_{H^1(\partial\Omega_t)}\right).$$

Moreover, we also can bound the L^{2^*} norm of v. Indeed, since v doesn't have a jump on the boundary of Ω_t, we know by (4.3.14) that it belongs to the space $D^{1,2}(\mathbb{R}^N)$. Thus, employing Gagliardo-Nirenberg-Sobolev inequality we get (4.3.15).

□

Proposition 4.3.9. *For any $t \in [0,1]$,*

$$\begin{aligned}\dot{\mathcal{J}}(\Omega_t) &= \left(1 - \frac{1}{K}\int_{\Omega_t}\psi_t dx\right)\frac{1}{|\Omega_t|}\int_{\partial\Omega_t}\psi_{\Omega_t}(X\cdot\nu)d\mathcal{H}^{N-1}\\ &+ \frac{1}{2}\int_{\partial\Omega_t}\left(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{N-1} + \frac{1}{2K}\int_{\partial\Omega_t}\psi_t^2(X\cdot\nu)d\mathcal{H}^{N-1}\\ &- \int_{\partial\Omega_t}\left(\nabla\psi_t^-\cdot\nu\right)\left(\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot\nu\right)(X\cdot\nu)d\mathcal{H}^{N-1}\\ &= \left(1 - \frac{1}{K}\int_{\Omega_t}\psi_t dx\right)\frac{1}{|\Omega_t|}\int_{\Omega_t}\operatorname{div}(\psi_t X)dx + \frac{1}{2}\int_{\mathbb{R}^N}\operatorname{div}(a_{\Omega_t}|\nabla\psi_t|^2 X)dx\\ &+ \frac{1}{2K}\int_{\Omega_t}\operatorname{div}(\psi_t^2 X)dx - \int_{\mathbb{R}^N}\operatorname{div}\left(a_{\Omega_t}\left(\nabla\psi_t\cdot\nu\right)^2 X\right)dx.\end{aligned}$$

In particular,

$$\dot{\mathcal{J}}(B_1) = 0.$$

Proof. We note that by (4.3.8)

$$\frac{d}{dt}\mathcal{J}(\Omega_t) = \frac{1}{2|\Omega_t|}\int_{\Omega_t}\dot{\psi}_t dx + \frac{1}{2|\Omega_t|}\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1}.$$

Now we use the definition of $\mathcal{J}$ to get

$$\begin{aligned}\dot{\mathcal{J}}(\Omega_t) &= \int_{\mathbb{R}^N}a_{\Omega_t}\nabla\psi_t\cdot\nabla\dot{\psi}_t dx + \frac{1}{2}\int_{\partial\Omega_t}\left(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{N-1}\\ &+ 2\dot{\mathcal{J}}(\Omega_t) - \frac{2}{K}|E|2\dot{\mathcal{J}}(\Omega_t)\mathcal{J}(\Omega_t) + \frac{1}{K}\int_{\Omega_t}\dot{\psi}_t\psi_t dx + \frac{1}{2K}\int_{\partial\Omega_t}\psi_t^2(X\cdot\nu)dx.\end{aligned}$$

Using (4.3.11), we obtain

$$\begin{aligned}
\dot{\mathcal{J}}(\Omega_t) &= -\frac{1}{|\Omega_t|}\left(\int_{\Omega_t}\dot{\psi}_t dx\right)\left(1-\frac{1}{K}\int_{\Omega_t}\psi_t dx\right) + 2\dot{\mathcal{J}}(\Omega_t)\left(1-\frac{1}{K}\int_{\Omega_t}\psi_t dx\right)\\
&+\frac{1}{2}\int_{\partial\Omega_t}\left(\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{N-1}+\frac{1}{2K}\int_{\partial\Omega_t}\psi_t^2(X\cdot\nu)dx\\
&+\int_{\partial\Omega_t}\left(\beta\dot{\psi}^+\nabla\psi_t^+\cdot\nu-\dot{\psi}^-\nabla\psi_t^-\cdot\nu\right)d\mathcal{H}^{N-1}\\
&=\left(1-\frac{1}{K}\int_{\Omega_t}\psi_t dx\right)\frac{1}{|\Omega_t|}\int_{\partial\Omega_t}\psi_{\Omega_t}(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{2}\int_{\partial\Omega_t}\left(\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{N-1}+\frac{1}{2K}\int_{\partial\Omega_t}\psi_t^2(X\cdot\nu)d\mathcal{H}^{N-1}\\
&-\int_{\partial\Omega_t}\left(\nabla\psi_t^-\cdot\nu\right)\left(\left(\nabla\psi_t^+-\nabla\psi_t^-\right)\cdot\nu\right)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&=\left(1-\frac{1}{K}\int_{\Omega_t}\psi_t dx\right)\frac{1}{|\Omega_t|}\int_{\Omega_t}\mathrm{div}(\psi_t X)dx+\frac{1}{2}\int_{\mathbb{R}^N}\mathrm{div}(a_{\Omega_t}|\nabla\psi_t|^2X)dx\\
&+\frac{1}{2K}\int_{\Omega_t}\mathrm{div}(\psi_t^2X)dx-\int_{\mathbb{R}^N}\mathrm{div}\left(a_{\Omega_t}\left(\nabla\psi_t\cdot\nu\right)^2X\right)dx.
\end{aligned}$$

Note that from the second to last expression it is easy to see that $\dot{\mathcal{J}}(B_1)=0$ as ψ_0 is radial by Proposition 4.3.4 and the volume of Ω_t is constant (hence $\int_{\partial B_1}(X\cdot\nu)d\mathcal{H}^{N-1}=0$).

□

Second derivative

Now we differentiate again to get

$$\begin{aligned}
\ddot{\mathcal{J}}(\Omega_t) &= -\frac{2}{K}\dot{\mathcal{J}}(\Omega_t)\int_{\Omega_t}\mathrm{div}(\psi_t X)dx\\
&+\frac{1-\frac{2}{K}|\Omega_t|\mathcal{J}(\Omega_t)}{|\Omega_t|}\left(\int_{\Omega_t}\mathrm{div}(\dot{\psi}_t X)dx+\int_{\partial\Omega_t}\mathrm{div}(\psi_t X)(X\cdot\nu)d\mathcal{H}^{N-1}\right)\\
&+\int_{\partial\Omega_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{2}\int_{\partial\Omega_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{K}\int_{\partial\Omega_t}\psi_t\dot{\psi}_t(X\cdot\nu)d\mathcal{H}^{N-1}+\frac{1}{K}\int_{\partial\Omega_t}\psi_t\nabla\psi_t\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}\\
&-2\int_{\partial\Omega_t}\left(\beta\left(\nabla\dot{\psi}_t^+\cdot\nu+\nabla\psi_t^+\cdot\dot{\nu}\right)\left(\nabla\psi_t^+\cdot\nu\right)-\left(\nabla\dot{\psi}_t^-\cdot\nu+\nabla\psi_t^-\cdot\dot{\nu}\right)\left(\nabla\psi_t^-\cdot\nu\right)\right)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&-\int_{\partial\Omega_t}\nabla\left[\beta(\nabla\psi_t^+\cdot\nu)^2-(\nabla\psi_t^-\cdot\nu)^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}.
\end{aligned}$$

Using that the vector field X is divergence-free in the neighborhood of ∂B_1 we get for t small enough

$$
\begin{aligned}
&\ddot{\mathcal{J}}(\Omega_t) = -\frac{2}{K}\dot{\mathcal{J}}(\Omega_t)\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1-\frac{2}{K}|\Omega_t|\mathcal{J}(\Omega_t)}{|\Omega_t|}\left(\int_{\partial\Omega_t}\dot{\psi}_t(X\cdot\nu)d\mathcal{H}^{N-1}+\int_{\partial\Omega_t}(\nabla\psi_t^+\cdot X)(X\cdot\nu)d\mathcal{H}^{N-1}\right)\\
&+\int_{\partial\Omega_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{2}\int_{\partial\Omega_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{K}\int_{\partial\Omega_t}\psi_t\dot{\psi}_t^+(X\cdot\nu)d\mathcal{H}^{N-1}+\frac{1}{K}\int_{\partial\Omega_t}\psi_t\nabla\psi_t^+\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}\\
&-2\int_{\partial\Omega_t}\left(\beta\left(\nabla\dot{\psi}_t^+\cdot\nu+\nabla\psi_t^+\cdot\dot{\nu}\right)(\nabla\psi_t^+\cdot\nu)-\left(\nabla\dot{\psi}_t^-\cdot\nu+\nabla\psi_t^-\cdot\dot{\nu}\right)(\nabla\psi_t^-\cdot\nu)\right)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&-\int_{\partial\Omega_t}\nabla\left[\beta(\nabla\psi_t^+\cdot\nu)^2-(\nabla\psi_t^-\cdot\nu)^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}.
\end{aligned}
$$

Now to prove Lemma 4.3.5 we only need the following bound on the second derivative.

Lemma 4.3.10. *There exist constants $\delta>0$ and $C=C(N,\delta)$ such that if $\|\varphi\|_{C^{2,\vartheta}}<\delta$, then*

$$\left|\ddot{\mathcal{J}}(\Omega_t)\right|\le C\|X\cdot\nu\|^2_{H^1(\partial B_1)}.$$

We will need the following proposition.

Proposition 4.3.11.

$$\|\dot{\psi}_t^+\|_{H^1(\partial\Omega_t)}+\|\dot{\psi}_t^-\|_{H^1(\partial\Omega_t)}\le C\left(\|X\cdot\nu\|_{H^1(\partial\Omega_t)}+\left|\dot{\mathcal{J}}(\Omega_t)\right|\right).$$

To prove the proposition we will use the following theorem concerning Sobolev bounds.

Theorem 4.3.12 ([McL02, Theorem 4.20])**.** *Let G_1 and G_2 be bounded open subsets of $\mathbb{R}^N$ such that $\overline{G}_1\Subset G_2$ and G_1 intersects an $(N-1)$-dimensional manifold Γ, and put*

$$\Omega_j^\pm=G_j\cap\Omega^\pm \text{ and } \Gamma_j=G_j\cap\Gamma \text{ for } j=1,2.$$

Suppose, for an integer $r\ge 0$, that Γ_2 is $C^{r+1,1}$, and consider two equations

$$\mathcal{P}u^\pm=f^\pm \text{ on } \Omega_2^\pm,$$

where $\mathcal{P}$ is strongly elliptic on G_2 with coefficients in $C^{r,1}(\overline{\Omega_2^\pm})$. If $u\in L^2(G_2)$ satisfies

$$u^\pm\in H^1(\Omega_2^\pm),\quad [u]_\Gamma\in H^{r+\frac{3}{2}}(\Gamma_2),\quad [\mathcal{B}_\nu u]_\Gamma\in H^{r+\frac{1}{2}}(\Gamma_2),$$

and if $f^\pm \in H^r(\Omega_2^\pm)$*, then* $u^\pm \in H^{r+2}(\Omega_1^\pm)$ *and*

$$
\begin{aligned}
\|u^+\|_{H^{r+2}(\Omega_1^+)} + \|u^+\|_{H^{r+2}(\Omega_1^-)} \le\ & C\left(\|u^+\|_{H^1(\Omega_2^+)} + \|u^-\|_{H^1(\Omega_2^-)}\right) \\
& + C\left(\|[u]_{\Gamma_2}\|_{H^{r+\frac{3}{2}}(\Gamma_2)} + \|[\mathcal{B}_\nu u]_{\Gamma_2}\|_{H^{r+\frac{1}{2}}(\Gamma_2)}\right) \\
& + C\left(\|f^+\|_{H^r(\Omega_2^+)} + \|f^-\|_{H^r(\Omega_2^-)}\right).
\end{aligned}
$$

We need an analogue of the above theorem for $r = -\frac{1}{2}$. To get it, we are going to interpolate between $r = 0$ and $r = -1$. We first prove the following lemma.

Lemma 4.3.13. *Let* E *be a set with the boundary in* $C^{1,1}$ *and let* $R > 0$ *be such that* $B_R \supset \overline{E}$*. Consider the equation*

$$
\begin{cases}
\beta\Delta u^+ = f^+ \text{ in } E, \\
\Delta u^- = f^- \text{ in } B_R \backslash E, \\
u^+ = u^- \text{ on } \partial E, \\
\beta\partial_n u^+ - \partial_n u^- = g \text{ on } \partial E, \\
u^- = 0 \text{ on } \partial B_R,
\end{cases}
\tag{4.3.16}
$$

where $f^+ \in H^{-1}(E), f^- \in H^{-1}(B_R\backslash E)$*, and* $g \in H^{-1/2}(\partial E)$ *are given. Then there exists* u *- the solution of* (4.3.16) *in* $W_0^{1,2}(B_R)$ *and it satisfies*

$$
\|u\|^2_{H^1(B_R)} \le C\left(\|f^+\|^2_{H^{-1}(E)} + \|f^-\|^2_{H^{-1}(B_R\backslash E)} + \|g\|^2_{H^{-1/2}(\partial E)}\right) \tag{4.3.17}
$$

with $C = C(N,R) > 0$*. Moreover, if* $f^+ \in H^{-1/2}(E), f^- \in H^{-1/2}(B_R\backslash E)$*, and* $g \in L^2(\partial E)$*, then*

$$
\|u\|^2_{H^{3/2}(B_R)} \le C\left(\|f^+\|^2_{H^{-1/2}(E)} + \|f^-\|^2_{H^{-1/2}(B_R\backslash E)} + \|g\|^2_{L^2(\partial E)}\right) \tag{4.3.18}
$$

with $C = C(N,R) > 0$*.*

Proof. First we observe that the solution in H^1 exists since it is a minimizer of the following convex functional:

$$
\int_{\Omega_t}\left(\beta\left|\nabla u^+\right|^2 - f_1 u^+\right) + \int_{\Omega_t^c}\left(\beta\left|\nabla u^-\right|^2 - f_2 u^-\right) + \int_{\partial\Omega_t} g(u^+ - u^-).
$$

Note that if we test the equation with the solution itself, we get

$$
\int_{\Omega_t}\beta\left|\nabla u^+\right|^2 dx + \int_{\Omega_t^c}\left|\nabla u^-\right|^2 dx = -\int_{\Omega_t} f_1 u^+ dx - \int_{\Omega_t} f_2 u^- dx + \int_{\partial\Omega_t} u_1 g\, d\mathcal{H}^{N-1}.
$$

By Poincaré, Cauchy-Schwarz, Young, and the trace inequality we obtain (4.3.17).

Now we consider an operator that takes the functions of the right-hand side and returns the solution of the corresponding transmission problem, i.e. we define

$T(f_1, f_2, g)$ for $f_1 \in H^r(\Omega_t)$, $f_2 \in H^r(\Omega_t^c)$, $g \in H^{r+\frac{1}{2}}(\partial\Omega_t)$ as the only H^1 solution of (4.3.16).

By (4.3.17), $T : H^r \times H^r \times H^{r+\frac{1}{2}} \to H^{r+2}$ for $r = -1$. Moreover, (4.3.17) together with Theorem 4.3.12 yields $T : H^r \times H^r \times H^{r+\frac{1}{2}} \to H^{r+2}$ for $r \geq 0$ - integer. Thus, interpolating between $r = 0$ and $r = -1$ we get that $T : H^{-\frac{1}{2}} \times H^{-\frac{1}{2}} \times L^2 \to H^{\frac{3}{2}}$, so (4.3.18) holds for appropriately regular right-hand side. □

Proof. (Proposition 4.3.11) Since we are interested only in the value of $\dot{\psi}_t$ on $\partial\Omega_t$, we multiply it by a cut-off function η. The function $\eta \in C_c^\infty(\mathbb{R}^N)$ is such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_2, \quad \eta \equiv 0 \text{ outside of } B_3, \quad |\nabla\eta| \leq 2, \quad |\Delta\eta| \leq 4.$$

We would also like to eliminate the jump on the boundary in order to use Lemma 4.3.13, so we consider a function $u := v\eta$, where v is as in Lemma 4.3.8 (we recall that $v = \dot{\psi}_t + f$, where f is a $H^{3/2}$ continuation of $\nabla\psi_t \cdot X$ from $\partial\Omega_t$ inside and outside). For δ small enough, all sets Ω_t lie inside of B_2, so

$$u = \dot{\psi}_t + \nabla\psi_t \cdot X \text{ on } \partial\Omega_t. \tag{4.3.19}$$

Note that u satisfies

$$\begin{cases} -\beta\Delta u = -\frac{1}{K}v + \frac{2}{K}\dot{\mathcal{J}}(\Omega_t) + \Delta f \text{ in } \Omega_t, \\ \Delta u = \nabla v \cdot \nabla\eta + \left(\dot{\psi}_t + f\right)\Delta\eta \text{ in } \Omega_t^c, \\ u^+ - u^- = 0 \text{ on } \partial\Omega_t, \\ \beta\nabla u^+ \cdot \nu - \nabla u^- \cdot \nu = \left(-\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right) \cdot X + \beta\nabla f^+ - \nabla f^-\right) \cdot \nu \text{ on } \partial\Omega_t, \\ u = 0 \text{ on } \partial B_3. \end{cases}$$

By Lemma 4.3.13,

$$\begin{aligned} \|u^+\|_{H^{\frac{3}{2}}(\Omega_t)} + \|u^-\|_{H^{\frac{3}{2}}(\Omega_t^c)} \leq{}& C\left(\left\|\left(\beta\nabla[\nabla\psi_t^+ \cdot X]\right) \cdot \nu\right\|_{L^2(\Gamma_2)} + \left\|\left(\nabla[\nabla\psi_t^- \cdot X]\right) \cdot \nu\right\|_{L^2(\Gamma_2)}\right) \\ &+ C\left(\left\|\left(\beta\nabla[\nabla\psi_t^+] \cdot X\right) \cdot \nu\right\|_{L^2(\Gamma_2)} + \left\|\left(\nabla[\nabla\psi_t^-] \cdot X\right) \cdot \nu\right\|_{L^2(\Gamma_2)}\right) \\ &+ C\left(\left\|\frac{1}{K}v\right\|_{H^{-\frac{1}{2}}(\Omega_t)} + \left\|\frac{2}{K}\dot{\mathcal{J}}(\Omega_t)\right\|_{H^{-\frac{1}{2}}(\Omega_t)} + \|\Delta f\|_{H^{-\frac{1}{2}}(\Omega_t)}\right) \\ &+ C\left(\|\nabla v \cdot \nabla\eta\|_{H^{-\frac{1}{2}}(\Omega_t^c)} + \|v\Delta\eta\|_{H^{-\frac{1}{2}}(\Omega_t^c)}\right). \end{aligned}$$

Now we employ Proposition 4.3.6, inequality (4.0.7), and the definition of f to get

$$\begin{aligned} \|u^+\|_{H^{\frac{3}{2}}(\Omega_t)} + \|u^-\|_{H^{\frac{3}{2}}(\Omega_t^c)} \leq{}& C\left(\|X \cdot \nu\|_{H^1(\partial\Omega_t)} + \left|\dot{\mathcal{J}}(\Omega_t)\right|\right) \\ &+ C\left(\|\nabla v \cdot \nabla\eta\|_{H^{-\frac{1}{2}}(\Omega_t^c)} + \|v\Delta\eta\|_{H^{-\frac{1}{2}}(\Omega_t^c)}\right). \end{aligned}$$

Remembering (4.3.19), using trace inequality and properties of η, we have

$$\begin{aligned}
\|\dot{\psi}_t^+\|_{H^1(\partial\Omega_t)} + \|\dot{\psi}_t^-\|_{H^1(\partial\Omega_t)} &\le C\left(\|X\cdot\nu\|_{H^1(\partial\Omega_t)} + \left|\dot{\mathcal{J}}(\Omega_t)\right|\right) \\
&\quad + C\left(\|\nabla v\cdot\nabla\eta\|_{H^{-\frac{1}{2}}(\Omega_t^c)} + \|v\Delta\eta\|_{H^{-\frac{1}{2}}(\Omega_t^c)}\right) \\
&\le C\left(\|X\cdot\nu\|_{H^1(\partial\Omega_t)} + \left|\dot{\mathcal{J}}(\Omega_t)\right|\right) + C\left(\|\nabla v\cdot\nabla\eta\|_{L^2(\Omega_t^c)} + \|v\Delta\eta\|_{L^2(\Omega_t^c)}\right) \\
&\le C\left(\|X\cdot\nu\|_{H^1(\partial\Omega_t)} + \left|\dot{\mathcal{J}}(\Omega_t)\right|\right) + C\left(\|\nabla v\|_{L^2(\Omega_t^c)} + \|v\|_{L^2(B_3\setminus B_2)}\right).
\end{aligned}$$

Now it remains to recall the bounds (4.3.14) and (4.3.15) and notice that

$$\|\cdot\|_{L^2(B_3\setminus B_2)} \le C\|\cdot\|_{L^{2^*}(B_3\setminus B_2)}.$$

□

Proof of Lemma 4.3.10. Let us first show that the lemma is implied by the following claim.

Claim: $\left|\ddot{\mathcal{J}}(\Omega_t)\right| \le C\left(\|X\cdot\nu\|^2_{H^1(\partial B_1)} + \dot{\mathcal{J}}(\Omega_t)\,\|X\cdot\nu\|_{H^1(\partial B_1)}\right).$

Indeed, suppose we proved the claim. Denote $\dot{\mathcal{J}}(\Omega_t)$ by $f(t)$. Then we know the following:

$$\begin{cases} |f'(t)| \le C\left(\|X\cdot\nu\|^2_{H^1(\partial B_1)} + f(t)\,\|X\cdot\nu\|_{H^1(\partial B_1)}\right), \\ f(0) = 0. \end{cases}$$

Let us show that

$$|f(t)| \le \|X\cdot\nu\|_{H^1(\partial B_1)}, \tag{4.3.20}$$

then the lemma will follow immediately. Suppose that there exists a time $t\in(0,1]$ such that the inequality (4.3.20) fails. We denote by t^* the first time when it happens, i.e.

$$t^* := \inf_{t\in[0,1]}\{t : (4.3.20)\text{ fails}\}.$$

Since inequality (4.3.20) is true for $t=0$, the following holds:

$$|f(t^*)| = \|X\cdot\nu\|_{H^1(\partial B_1)}, \quad |f(t)| \le \|X\cdot\nu\|_{H^1(\partial B_1)} \text{ for } t\in[0,t^*].$$

Now, as $f(0)=0$, we can write

$$f(t^*) = \int_0^{t^*} f'(t)dt$$

and thus

$$\begin{aligned}
\|X\cdot\nu\|_{H^1(\partial B_1)} = |f(t^*)| &\le \int_0^{t^*} |f'(t)|dt \\
&\le \int_0^{t^*} C\left(\|X\cdot\nu\|^2_{H^1(\partial B_1)} + f(t)\,\|X\cdot\nu\|_{H^1(\partial B_1)}\right)dt \le 2C\,\|X\cdot\nu\|^2_{H^1(\partial B_1)}.
\end{aligned}$$

However, that cannot hold for $\|X\cdot\nu\|_{H^1(\partial B_1)}$ small enough. That means that (4.3.20) holds for all times t.

Proof of the claim.

$$
\begin{aligned}
&\ddot{\mathcal{J}}(\Omega_t) = -\frac{2}{K}\dot{\mathcal{J}}(\Omega_t)\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{2}\int_{\partial\Omega_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\int_{\partial\Omega_t}\left(\frac{1-\frac{2}{K}|B_1|\mathcal{J}(\Omega_t)}{|B_1|}+\frac{1}{K}\psi_t\right)(\nabla\psi_t^+\cdot X)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\int_{\partial\Omega_t}\left(\left(\frac{1-\frac{2}{K}|B_1|\mathcal{J}(\Omega_t)}{|B_1|}\right)+\frac{1}{K}\psi_t\right)\dot{\psi}_t^+(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\int_{\partial\Omega_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)d\mathcal{H}^{N-1}\\
&=: I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t).
\end{aligned}
$$

We start with I_1.

$$
\begin{aligned}
-\frac{K}{2}I_1(t) =&\dot{\mathcal{J}}(\Omega_t)\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1} = \left(1-\frac{1}{K}\int_{\Omega_t}\psi_t dx\right)\frac{1}{|B_1|}\left(\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1}\right)^2\\
&+\frac{1}{2}\int_{\partial\Omega_t}\left(\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{N-1}\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1}\\
&+\frac{1}{2K}\int_{\partial\Omega_t}\psi_t^2(X\cdot\nu)d\mathcal{H}^{N-1}\int_{\partial\Omega_t}\psi_t(X\cdot\nu)d\mathcal{H}^{N-1}.
\end{aligned}
$$

To bound I_4 and I_5 we use Proposition 4.3.11 and Proposition 4.3.6. Let us show the inequality for I_5, I_4 can be treated in a similar way.

$$
\begin{aligned}
&\left|\int_{\partial\Omega_t}\left(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-\right)(X\cdot\nu)d\mathcal{H}^{N-1}\right|\\
&\le\int_{\partial\Omega_t}\left(\left|\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+\right|+\left|\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-\right|\right)|X\cdot\nu|d\mathcal{H}^{N-1}\\
&\le\left(\left(\int_{\partial\Omega_t}\left|\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+\right|^2\right)^{\frac{1}{2}}+\left(\int_{\partial\Omega_t}\left|\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-\right|^2\right)^{\frac{1}{2}}\right)\|X\cdot\nu\|_{L^2(\partial\Omega_t)} \qquad (4.3.21)\\
&\le g(\|\psi_t\|_{C^2(\overline{\Omega_t})})\left(\left(\int_{\partial\Omega_t}\left|\nabla\dot{\psi}_t^+\right|^2\right)^{\frac{1}{2}}+\left(\int_{\partial\Omega_t}\left|\nabla\dot{\psi}_t^-\right|^2\right)^{\frac{1}{2}}\right)\|X\cdot\nu\|_{L^2(\partial\Omega_t)}\\
&\le g(\|\psi_t\|_{C^2(\overline{\Omega_t})})\left(\|X\cdot\nu\|_{H^1(\partial\Omega_t)}+\left|\dot{\mathcal{J}}(\Omega_t)\right|\right)\|X\cdot\nu\|_{L^2(\partial\Omega_t)}
\end{aligned}
$$

□

Now we are ready to prove Lemma 4.3.5.

Proof of Lemma 4.3.5. By Taylor expansion for $g(t) = \mathcal{J}(E_t)$ at $t = 0$ we have

$$\mathcal{J}(E) = \mathcal{J}(B_1) + \dot{\mathcal{J}}(B_1) + \int_0^1 (1-s)\ddot{\mathcal{J}}(E_s)ds.$$

By Proposition 4.3.9 we know that $\dot{\mathcal{J}}(B_1) = 0$. Now use Lemma 4.3.10 to bound the integral. ☐

Theorem 1.1.9 for nearly-spherical sets

Finally, we can prove stability of the ball for $\mathcal{F}$ with small enough charge in the family of nearly-spherical sets.

Theorem 4.3.14. *Given $\vartheta \in (0,1]$, there exist $\delta = \delta(N,\vartheta) > 0$ and $Q_0 > 0$ such that for every nearly-spherical set E parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$, $x_E = 0$, and $|E| = |B_1|$, if $Q < Q_0$ we have*

$$\mathcal{F}(E) - \mathcal{F}(B_1) \geq c\|\varphi\|^2_{H^1(\partial B_1)}.$$

Proof. The proof is a combination of Lemma 4.3.5 and Theorem 4.0.2. Let δ be the one of Lemma 4.3.5. If Q is small enough, we have

$$\begin{aligned}\mathcal{F}(E) = P(E) + Q^2\mathcal{G}(E) &\geq P(B_1) + c\|\varphi\|^2_{H^1(\partial B_1)} + Q^2\left(\frac{K}{2|B_1|} - \mathcal{J}(E)\right)\\ &\geq P(B_1) + c\|\varphi\|^2_{H^1(\partial B_1)} + Q^2\left(\frac{K}{2|B_1|} - \mathcal{J}(B_1) - c'\|\varphi\|^2_{H^1(\partial B_1)}\right)\\ &\geq P(B_1) + Q^2\left(\frac{K}{2|B_1|} - \mathcal{J}(B_1)\right) + \frac{c}{2}\|\varphi\|^2_{H^1(\partial B_1)} = \mathcal{F}(B_1) + \frac{c}{2}\|\varphi\|^2_{H^1(\partial B_1)},\end{aligned}$$

yielding the desired result. ☐

4.3.4 Second derivative on the ball

We want to show that the second derivative of the energy which we know is bounded by $\|\varphi\|^2_{H^1}$ is actually bounded by a stronger $H^{1/2}$ norm on the ball. We don't need this for out main results but it is a sharp bound so we prove it for the sake of completeness. We have

$$\partial^2\mathcal{G}(B_1)[\varphi,\varphi] := \hat{c}_1\int_{\partial B_1}\varphi^2 d\mathcal{H}^{N-1} + \int_{\partial B_1}\hat{c}_2 H(\varphi) + \hat{c}_3(\nabla H(\varphi)^- \cdot \nu)\varphi d\mathcal{H}^{N-1}$$

with $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ are constants depending only on β, K and dimension n and $\tilde{H}(\varphi)$ is the unique solution of

$$\begin{cases}\beta\Delta u = \frac{1}{K}u \text{ in } B_1,\\ \Delta u = 0 \text{ in } B_1^c,\\ u^+ - u^1 = c_1\varphi \text{ on } \partial B_1,\\ \beta\nabla u^+\cdot\nu - \nabla u^1\cdot\nu = c_2\varphi \text{ on } \partial B_1,\end{cases}$$

where $c_1 = -(u'_-(1) - u'_2(1))$, $c_2 = -(\beta u''_-(1) - u''_2(1))$.

We are going to show

$$\partial^2 \mathcal{G}(B_1)[\varphi, \varphi] \geq -c\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)}.$$

Consider φ in the basis of spherical harmonics,

$$\varphi = \sum_{m=0}^{\infty} \sum_{i=1}^{N(m,n)} \alpha_{m,i} Y_{m,i}.$$

First, we would like to bound $\partial^2 \mathcal{G}(B_1)[Y_{m,i}, Y_{m,i}]$.

One can easily see that $H(Y_{m,i}) = R(r)Y_{m,i}$, where $R(r)$ is the only solution of the following system:

$$\begin{cases} R_1''(r) + \frac{N-1}{r} R_1'(r) + (-\frac{1}{\beta K} + \frac{\lambda_{m,n}}{r^2}) R_1(r) = 0 \text{ for } r \leq 1, \\ R_2''(r) + \frac{N-1}{r} R_2'(r) + \frac{\lambda_{m,i}}{r^2} R_2(r) = 0 \text{ for } r \geq 1, \\ R_1(1) - R_2(1) = c_1, \\ \beta R_1'(1) - R_2'(1) = c_2, \end{cases}$$

where $\lambda_{m,i} = -m(m+n-2)$.

A straightforward computation gives us that $R_2(r) = Ar^{-(m+n-2)}$ for some constant A.

Let us search for R_1 in the form $R_1(r) = \sum_{k=0}^{\infty} a_k r^k$. The equation for R_1 then will take the following form:

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)r^k + \frac{N-1}{r} \sum_{k=0}^{\infty} a_{k+1}(k+1)r^k + \left(-\frac{1}{\beta K} + \frac{\lambda_{m,n}}{r^2}\right) \sum_{k=0}^{\infty} a_k r^k = 0 \text{ for } r \leq 1$$

If $m \geq 2$, it means that

$$a_0 = 0, \quad a_1 = 0, \quad a_k\big(k(N+k-2) - m(N+m-2)\big) = \frac{1}{\beta K} a_{k-2} \text{ for } k \geq 2.$$

The recurrent condition can be rewritten as

$$a_k(k-m)(k+n+m-2) = -a_{k-2} \text{ for } k \geq 2.$$

Hence,

$$\begin{cases} a_m = C, \\ a_{m+2i} = \beta K a_{m+2(i-1)} \frac{1}{2i(2i+2m+N-2)} \text{ for } i \geq 1, \\ a_k = 0 \text{ for all other } k, \end{cases}$$

where C is a constant. So, the coefficients a_k decrease as $\frac{(\beta K)^k}{(k!)^2}$ and the series is absolutely converging. Note that $a_k = Cb_k$, where $\{b_k\}_k$ is the following fixed sequence:

$$\begin{cases} b_m = 1, \\ b_{m+2i} = (\beta K)^i \Pi_{j=1}^{i} \frac{1}{2j(2j+2m+N-2)} \text{ for } i \geq 1, \\ b_k = 0 \text{ for all other } k. \end{cases}$$

Our system for R then becomes

$$\begin{cases} R_1(r) = C\sum_{i=0}^{\infty} b_{m+2i} r^{m+2i} \text{ for } r \leq 1, \\ R_2(r) = Ar^{-(m+N-2)} \text{ for } r \geq 1, \\ C\sum_{i=0}^{\infty} b_{m+2i} - A = c_1, \\ \beta C\sum_{i=0}^{\infty}(m+2i)b_{m+2i} + A(m+N-2) = c_2 \end{cases}$$

with A and C unknowns. We are interested in the value of $|R_2'(1)|$:

$$|R_2'(1)| = |A(N+m-2)|$$

$$= \left| \frac{c_1(m+N-2) + c_2}{\sum_{i=0}^{\infty} b_{m+2i} + \beta\sum_{i=0}^{\infty}(m+2i)b_{m+2i}} \beta \sum_{i=0}^{\infty}(m+2i)b_{m+2i} - c_2 \right| \sim m.$$

Thus,

$$|\partial^2 \mathcal{G}(B_1)[Y_{m,i}, Y_{m,i}]| = |\hat{c}_1 + \hat{c}_2 A + \hat{c}_3 A(N+m-2)| \sim m. \tag{4.3.22}$$

Now recall that φ is such that $|\Omega| = |B_1|$ and $x_\Omega = 0$. It means that

$$|B_1| = |\Omega| = \int_{\partial B_1} \frac{(1+\varphi(x))^N}{N} d\mathcal{H}^{N-1};$$

$$0 = x_\Omega = \int_{\partial B_1} y \frac{(1+\varphi(x))^{N+1}}{N+1} d\mathcal{H}^{N-1}.$$

Hence,

$$\left| \int_{\partial B_1} \varphi(x) d\mathcal{H}^{N-1} \right| = \left| \int_{\partial B_1} \sum_{i=2}^{N} \binom{N}{i} \frac{\varphi(x)^i}{N} d\mathcal{H}^{N-1} \right|$$

$$\leq C(N) \int_{\partial B_1} \varphi(x)^2 d\mathcal{H}^{N-1} \leq C(N)\delta \|\varphi\|_{L^2}$$

and

$$\left| \int_{\partial B_1} x_i \varphi(x) d\mathcal{H}^{N-1} \right| \leq \int_{\partial B_1} \sum_{j=2}^{N} \binom{N}{j} \left| \frac{\varphi(x)^j}{N+1} \right| d\mathcal{H}^{N-1} \leq C(N)\delta \|\varphi\|_{L^2}.$$

Thus, for δ sufficiently small we have

$$a_0^2 + \sum_{i=1}^{N} a_{1,i}^2 \leq 2 \sum_{m=2}^{\infty} \sum_{i=1}^{N(m,n)} a_{m,i}^2,$$

which in turn implies

$$\partial^2 \mathcal{G}(B_1)[\varphi, \varphi] \geq -c\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)},$$

thanks to (4.3.22).

Chapter 5

Quantitative isocapacitary inequality

Now, as we have already the result for nearly-spherical sets, we can apply Selection Principle to the isocapacity inequality.

5.1 Another notion of asymmetry

As in [BDPV15], one of the key technical tools is to replace the Fraenkel asymmetry with a smoother (and stronger) version inspired by the distance among sets first used by Almgren, Taylor, and Wang in [ATW93] which resembles an L^2 type norm. Roughly speaking, while $\mathcal{A}(\Omega)$ represents an L^1 norm, $\alpha(\Omega)$ represents an L^2 norm, see (iii) in Lemma 5.1.2 below and the discussion in [BDPV15, Introduction].

Definition 5.1.1. Let Ω be an open set in $\mathbb{R}^N$. Then we define the asymmetry α in the following way:

(A)
$$\alpha_R(\Omega) = \int_{\Omega\Delta B_1} \big|1 - |x|\big| dx;$$

(B)
$$\alpha(\Omega) = \int_{\Omega\Delta B_1(x_\Omega)} \big|1 - |x - x_\Omega|\big| dx.$$

The next Lemma collects the main properties of α, its simple proof is identical to the one of [BDPV15, Lemma 4.2] but we put it here for the reader's convenience.

Lemma 5.1.2. *Let $\Omega \subset \mathbb{R}^N$, then*

(i) *There exists a constant $c = c(N)$ such that*

(A)
$$\alpha_R(\Omega) \geq c|\Omega\Delta B_1|^2$$

for any open set $\Omega \subset B_R$;

(B)
$$\alpha(\Omega) \geq c|\Omega\Delta B_1(x_\Omega)|^2$$

for any open set Ω.

(ii) *There exists a constant* $C = C(R)$ *such that*

$$|\alpha_*(\Omega_1) - \alpha_*(\Omega_2)| \le C|\Omega_1 \Delta \Omega_2|$$

for any $\Omega_1, \Omega_2 \subset B_R$. *In particular, if* $1_{\Omega_k} \to 1_\Omega$ *in* $L^1(B_R)$ *then* $\alpha_*(\Omega_k) \to \alpha_*(\Omega)$.

(iii) *There exist constants* $C = C(N)$, $\delta = \delta(N)$ *such that for every nearly-spherical set (see Definition 4.0.1 below)* Ω *with* $\|\phi\|_\infty \le \delta$ *(and* $x_\Omega = 0$ *in the case of* α*)*

$$\alpha_*(\Omega) \le C\|\phi\|^2_{L^2(\partial B_1)}.$$

Proof. First, we prove (i). We will prove the inequality for α_R. For α one can assume that $x_\Omega = 0$ and then proceed in the same way.

To that end, we use a simple rearranging argument. We notice that

$$\alpha_R(\Omega) = \int_{\Omega\setminus B_1} (|x| - 1)dx + \int_{B_1\setminus\Omega} (1 - |x|)dx. \tag{5.1.1}$$

Here in both summands we integrate monotone function of the modulus.

We introduce two annular regions T_{out} and T_{in} of the volumes $|\Omega\setminus B_1|$ and $|B_1\setminus\Omega|$:

$$\begin{aligned} T_{out} &:= \{x : 1 \le |x| \le R_{out}\}, R_{out} = \left(1 + \frac{|\Omega\setminus B_1|}{|B_1|}\right)^{\frac{1}{N}};\\ T_{in} &:= \{x : R_{in} \le |x| \le 1\}, R_{in} = \left(1 - \frac{|B_1\setminus\Omega|}{|B_1|}\right)^{\frac{1}{N}}. \end{aligned} \tag{5.1.2}$$

Now, by monotonicity,

$$\begin{aligned} \alpha_R(\Omega) &\ge \int_{T_{out}} (|x| - 1)dx + \int_{T_{in}} (1 - |x|)dx \\ &= \omega_n \left(\frac{R_{out}^{N+1} - 1}{N+1} - \frac{R_{out}^N - 1}{N} + \frac{R_{in}^{N+1} - 1}{N+1} - \frac{R_{in}^N - 1}{N}\right) \ge c|\Omega\Delta B_1|^2. \end{aligned} \tag{5.1.3}$$

Now we turn to the proof of (ii). Let us first prove the inequality for α_R. We observe that

$$\alpha_R(\Omega) = \int_{B_1} (1 - |x|)dx + \int_\Omega (|x| - 1)dx. \tag{5.1.4}$$

Thus,

$$|\alpha_R(\Omega_1) - \alpha_R(\Omega_2)| = \left|\int_{\Omega_1} (|x| - 1)dx - \int_{\Omega_2} (|x| - 1)dx\right| \le \int_{\Omega_1\Delta\Omega_2} \big|1 - |x|\big|dx. \tag{5.1.5}$$

Since both Ω_1 and Ω_2 lie inside B_R,

$$\int_{\Omega_1\Delta\Omega_2} |1 - |x||dx \le |\Omega_1\Delta\Omega_2|R. \tag{5.1.6}$$

As for α, notice that

$$\alpha(\Omega) = \int_{B_1} (1 - |x|)dx + \int_{\Omega} (|x - x_\Omega| - 1)dx. \tag{5.1.7}$$

Then

$$\begin{aligned}
&|\alpha_R(\Omega_1) - \alpha_R(\Omega_2)| = \left| \int_{\Omega_1} (|x - x_{\Omega_1}| - 1)dx - \int_{\Omega_2} (|x - x_{\Omega_2}| - 1)dx \right| \\
&\leq \left| \int_{\Omega_1} |x - x_{\Omega_1}|dx - \int_{\Omega_2} |x - x_{\Omega_2}|dx \right| + |\Omega_1 \Delta \Omega_2| \\
&\leq \int_{\Omega_1 \cap \Omega_2} |x_{\Omega_1} - x_{\Omega_2}|dx + \int_{\Omega_1 \setminus \Omega_2} |x - x_{\Omega_1}|dx + \int_{\Omega_2 \setminus \Omega_1} |x - x_{\Omega_2}|dx + |\Omega_1 \Delta \Omega_2| \\
&\leq |\Omega_1 \cap \Omega_2||x_{\Omega_1} - x_{\Omega_2}| + (2R+1)|\Omega_1 \Delta \Omega_2| \leq C(R)|\Omega_1 \Delta \Omega_2|.
\end{aligned} \tag{5.1.8}$$

Finally, we prove (iii). We prove the inequality for α_R as we have $\alpha_R = \alpha$ by assumption.

$$\begin{aligned}
\alpha_R(\Omega) = &\int_{\varphi \geq 0} \frac{(1+\varphi(x))^{N+1} - 1}{N+1} d\mathcal{H}^{N-1} - \int_{\varphi \geq 0} \frac{(1+\varphi(x))^{N} - 1}{N} d\mathcal{H}^{N-1} \\
+ &\int_{\varphi < 0} \frac{(1+\varphi(x))^{N+1} - 1}{N+1} d\mathcal{H}^{N-1} - \int_{\varphi < 0} \frac{(1+\varphi(x))^{N} - 1}{N} d\mathcal{H}^{N-1}.
\end{aligned} \tag{5.1.9}$$

But for $|t| \leq \frac{1}{2}$ there exists a constant $C = C(N)$ such that

$$\frac{(1+t)^{N+1} - 1}{N+1} \leq t + Ct^2, \; \frac{(1+t)^{N} - 1}{N} \geq t + \frac{1}{C}t^2.$$

This finishes the proof.

□

5.2 Stability for bounded sets with small asymmetry

We first want to prove the following theorem.

Theorem 5.2.1. *There exist constants $c = c(N,R)$, $\varepsilon_0 = \varepsilon_0(N,R)$ such that for any open set $\Omega \subset B_R$ with $|\Omega| = |B_1|$ and $\alpha_*(\Omega) \leq \varepsilon_0$ the following inequality holds:*

$$\operatorname{cap}_*(\Omega) - \operatorname{cap}_*(B_1) \geq c\alpha_*(\Omega).$$

We want to reduce our problem to nearly-spherical sets. To do that we argue by contradiction. Assume that there exists a sequence of domains $\tilde{\Omega}_j$ such that

$$|\tilde{\Omega}_j| = |B_1|, \;\; \alpha_*(\tilde{\Omega}_j) = \varepsilon_j \to 0, \;\; \operatorname{cap}_*(\tilde{\Omega}_j) - \operatorname{cap}_*(B_1) \leq \sigma^4 \varepsilon_j \tag{5.2.1}$$

for some σ small enough to be chosen later. We then prove the existence of a new contradicting sequence made of smooth sets via a selection principle.

Theorem 5.2.2 (Selection Principle). *There exists $\tilde{\sigma} = \tilde{\sigma}(N, R)$ such that if one has a contradicting sequence $\tilde{\Omega}_j$ as the one described above in (5.2.1) with $\sigma < \tilde{\sigma}$, then there exists a sequence of smooth open sets U_j such that*

(i) $|U_j| = |B_1|$,

(ii) $\partial U_j \to \partial B_1$ *in* C^k *for every* k,

(iii) $\limsup_{j\to\infty} \frac{\mathrm{cap}_*(U_j)-\mathrm{cap}_*(B_1)}{\alpha_*(\Omega_j)} \le C\sigma$ *for some* $C = C(N, R)$ *constant,*

(iv) *for the case of the capacity in* $\mathbb{R}^N$ *the barycenter of every* Ω_j *is in the origin.*

Proof of Theorem 5.2.1 assuming Selection Principle. Suppose Theorem 5.2.1 does not hold. Then for any $\sigma > 0$ we can find a contradicting sequence $\tilde{\Omega}_j$ as in (5.2.1). We apply Selection Principle to $\tilde{\Omega}_j$ to get a smooth contradicting sequence U_j.

By the properties of Ω_j, we have that for j big enough U_j is a nearly-spherical set. Thus, we can use Theorem 4.1.2 and get

$$c(N, R) \le \limsup_{j\to\infty} \frac{\mathrm{cap}_*(U_j) - \mathrm{cap}_*(B_1)}{\alpha_*(\Omega_j)} \le C(N, R)\sigma.$$

But this cannot happen for σ small enough depending only on N and R □

The proof of Theorem 5.2.2 is based on constructing the new sequence of sets by solving a variational problem. The existence of this new sequence is established in the next section while its regularity properties are studied in Section 5.4.

5.3 Proof of Theorem 5.2.2: Existence and first properties

5.3.1 Getting rid of the volume constraint

The first step consists in getting rid of the volume constraint in the isocapacitary inequality. Note that this has to be done locally since, by scaling, globally there exists no Lagrange multiplier. Furthermore, to apply the regularity theory for free boundary problems, it is crucial to introduce a *monotone* dependence on the volume. To this end, let us set

$$f_\eta(s) := \begin{cases} -\frac{1}{\eta}(s - \omega_N), & s \le \omega_N \\ -\eta(s - \omega_N), & s \ge \omega_N \end{cases}$$

and let us consider the new functional

$$\mathscr{C}^*_\eta(\Omega) = \mathrm{cap}_*(\Omega) + f_\eta(|\Omega|).$$

We now show that the above functional is uniquely minimized by balls. Note also that f_η satisfies

$$\eta(t - s) \le f_\eta(s) - f_\eta(t) \le \frac{(t - s)}{\eta} \qquad \text{for all } 0 \le s \le t. \tag{5.3.1}$$

Lemma 5.3.1 (Relative capacity)**.** *There exists an* $\hat{\eta} = \hat{\eta}(R) > 0$ *such that the only minimizer of* $\mathscr{C}_{\hat{\eta}}^R$ *in the class of sets contained in* B_R *is* B_1*, the unit ball centered at the origin.*

Moreover, there exists $c = c(R) > 0$ *such that for any ball* B_r *with* $0 < r < R$*, one has*

$$\mathscr{C}_{\hat{\eta}}^R(B_r) - \mathscr{C}_{\hat{\eta}}^R(B_1) \geq c|r-1|. \tag{5.3.2}$$

Lemma 5.3.2 (Absolute capacity)**.** *There exists an* $\hat{\eta} = \hat{\eta}(R) > 0$ *such that the only minimizer of* $\mathscr{C}_{\hat{\eta}}$ *in the class of sets contained in* B_R *is a translate of the unit unit ball* B_1*.*

Moreover, there exists $c = c(R) > 0$ *such that for any ball* B_r *with* $0 < r < R$*, one has*

$$\mathscr{C}_{\hat{\eta}}(B_r) - \mathscr{C}_{\hat{\eta}}(B_1) \geq c|r-1|. \tag{5.3.3}$$

Proof of Lemma 5.3.1. First of all, using symmetrization we get that any minimizer of $\mathscr{C}_{\eta}^R$ is a ball centered at zero. Thus, it is enough to show that for some $\eta > 0$

$$g(r) := \mathscr{C}_{\eta}^R(B_r)$$

attains its only minimum at $r = 1$ on the interval $(0, R)$. We recall that the (relative) capacitary potential of B_r in B_R is given by

$$u_R = \min\left\{ \frac{\left(|x|^{-(N-2)} - R^{-(N-2)}\right)_+}{r^{-(N-2)} - R^{-(N-2)}}, 1 \right\}$$

and thus

$$\mathrm{cap}_R(B_r) = \frac{(N-2)}{r^{-(N-2)} - R^{-(N-2)}},$$

hence

$$g(r) = \mathrm{cap}_R(B_r) + f_\eta(\omega_N r^N) = \frac{R^{N-2} - 1}{(\frac{R}{r})^{N-2} - 1} \mathrm{cap}_R(B_1) + f_\eta(\omega_N r^N).$$

For convenience let us denote

$$\varphi(r) := \mathrm{cap}_R(B_r) = c_1(R) \frac{R^{N-2} - 1}{R^{N-2} - r^{N-2}} r^{N-2},$$

and note that

$$\varphi'(r) = c_1(R)(N-2) \left(\frac{R^{N-2} - 1}{R^{N-2} - r^{N-2}} r^{N-3} + \frac{R^{N-2} - 1}{(R^{N-2} - r^{N-2})^2} r^{N-3} r^{N-2} \right).$$

Now we consider separately the two cases $0 < r \leq 1$ and $1 \leq r \leq R$.

- $0 < r \leq 1$

$$g'(r) = c_2(R)\left(\frac{R^{N-2}-1}{R^{N-2}-r^{N-2}}r^{N-3} + \frac{R^{N-2}-1}{(R^{N-2}-r^{N-2})^2}r^{N-3}r^{N-2}\right) - \frac{1}{\eta}\omega_N N r^{N-1}.$$

For $r \in (1/2, 1)$

$$g'(r) \leq c(R) - \frac{1}{\eta}\omega_N N\left(\frac{1}{2}\right)^{N-1}.$$

If we take $\eta < \eta(R) \ll 1$, then $g'(r) < -c_3(R)$ for $r \in (\frac{1}{2}, 1)$ and thus $g(r)$ attains its minimum at $r = 1$ on that interval.

Moreover for $r \in (0, 1/2)$

$$g(r) = \frac{R^{N-2}-1}{(\frac{R}{r})^{N-2}-1}\operatorname{cap}_R(B_1) + \frac{1}{\eta}\left(\omega_N\left(1-r^N\right)\right) \geq \frac{1}{\eta}\left(\omega_N\left(1-\left(\frac{1}{2}\right)^N\right)\right).$$

Since $g(1) = \operatorname{cap}_R(B_1) = c(R)$ we can take η small enough depending only on R to ensure that $g(r) \geq g(1)$ for all $r \in [0, 1/2)$.

- $1 \leq r < R$

$$\begin{aligned} g'(r) &= c(R)\left(\frac{R^{N-2}-1}{R^{N-2}-r^{N-2}}r^{N-3} + \frac{R^{N-2}-1}{(R^{N-2}-r^{N-2})^2}r^{N-3}r^{N-2}\right) - \eta\omega_N N r^{N-1} \\ &\geq c(R) - \eta\omega_N N R^{N-1}. \end{aligned}$$

Taking $\eta \ll 1$ depending only on r we get $g'(r) > c_4(R)$ for $r \in (1, R)$ and thus $g(r)$ attains its minimum at $r = 1$ also on this interval.

To prove the last claim just note that

$$\lim_{r\to 1^-} g'(r) \leq -c_3 \qquad \lim_{r\to 1^+} g'(r) \geq c_4.$$

□

Proof of Lemma 5.3.2. The proof works exactly as the one in the previous lemma, just using the equality

$$\operatorname{cap}(B_r) = \operatorname{cap}(B_1)r^{N-2}.$$

□

5.3.2 A penalized minimum problem

The sequence in Theorem 5.2.2 is obtained by solving the following minimum problem.

$$\min\{\mathscr{C}^*_{\hat{\eta},j}(\Omega) : \Omega \subset B_R\}, \tag{5.3.4}$$

where

$$\mathscr{C}^*_{\hat\eta,j}(\Omega) = \mathscr{C}^*_{\hat\eta}(\Omega)+\sqrt{\varepsilon_j^2+\sigma^2(\alpha_*(\Omega)-\varepsilon_j)^2} = \mathrm{cap}_*(\Omega)+f_{\hat\eta}(|\Omega|)+\sqrt{\varepsilon_j^2+\sigma^2(\alpha_*(\Omega)-\varepsilon_j)^2}.$$

We start proving the existence of minimizers. As in [BDPV15], in order to ensure the continuity of the asymmetry term, one needs to construct a minimizing sequence with equibounded perimeter. Recall also that a set is said to be quasi open if it is the zero level set of a $W^{1,2}$ function.

Lemma 5.3.3. *There exists $\sigma_0 = \sigma_0(N,R) > 0$ such that for every $\sigma < \sigma_0$ the minimum in (5.3.4) is attained by a quasi-open set Ω^*_j. Moreover, perimeters of Ω^*_j are bounded independently on j.*

Proof. We will focus on the capacity with respect to the ball. For the case of capacity in $\mathbb{R}^N$ one simply replaces $W^{1,2}_0(B_R)$ by $D^{1,2}(\mathbb{R}^N)$.

Step 1: finding minimizing sequence with bounded perimters. We consider $\{V_k\}_{k\in\mathbb{N}}$ – a minimizing sequence for $\mathscr{C}^R_{\hat\eta,j}$, satisfying

$$\mathscr{C}^R_{\hat\eta,j}(V_k) \le \inf \mathscr{C}^R_{\hat\eta,j} + \frac{1}{k}.$$

We denote by v_k the capacitary potentials of V_k, so $V_k = \{x\in B_R : v_k = 1\}$. We take as a variation the slightly enlarged set $\tilde V_k$:

$$\tilde V_k = \{x\in B_R : v_k > 1-t_k\},$$

where $t_k = \frac{1}{\sqrt{k}}$.

Note that the function $\tilde v_k = \frac{\min(v_k,1-t_k)}{1-t_k}$ is in $W^{1,2}_0(B_R)$ and $v_k = 1$ on $\tilde V_k$, so we can bound the capacity of $\tilde V_k$ by $\int_{B_R}|\nabla \tilde v_k|^2dx$. Since V_k is almost minimizing, we write

$$\begin{aligned}\int_{\{v_k<1\}} &|\nabla v_k|^2dx + f_{\hat\eta}(|\{v_k=1\}|) + \sqrt{\varepsilon_j^2+\sigma^2(\alpha(\{v_k=1\})-\varepsilon_j)^2}\\ &\le \int_{\{v_k<1-t_k\}}\left|\nabla\left(\frac{v_k}{1-t_k}\right)\right|^2dx + f_{\hat\eta}(|\{v_k\ge 1-t_k\}|)\\ &\qquad + \sqrt{\varepsilon_j^2+\sigma^2(\alpha(\{v_k\ge 1-t_k\})-\varepsilon_j)^2}+\frac{1}{k}.\end{aligned}$$

We use (5.3.1) and the fact that the function $t\mapsto \sqrt{\varepsilon_j^2+\sigma^2(t-\varepsilon_j)^2}$ is 1 Lipschitz to get

$$\begin{aligned}&\int_{\{1-t_k<v_k<1\}}|\nabla v_k|^2dx + \hat\eta|\{1-t_k<v_k<1\}|\\ &\le \sigma\big(|\alpha(\{v_k\ge 1-t_k\}-\alpha(\{v_k=1\})|\big)+\frac{1}{k}+\int_{\{v_k<1-t_k\}}\left(\left(\frac{1}{1-t_k}\right)^2-1\right)|\nabla v_k|^2dx\\ &\le C(R)\sigma|\{1-t_k<v_k\le 1\}|+\frac{1}{k}+\left(\left(\frac{1}{1-t_k}\right)^2-1\right)\mathrm{cap}_R(V_k)\\ &\le C(R)\sigma|\{1-t_k<v_k\le 1\}|+\frac{1}{k}+c(N,R)t_k,\end{aligned}$$

where in the second inequality we used Lemma 5.1.2, (ii). Taking $\sigma < \frac{\hat{\eta}}{2C(R)}$, we obtain

$$\int_{\{1-t_k<v_k<1\}} |\nabla v_k|^2 dx + \frac{\hat{\eta}}{2}(|\{1-t_k < v_k < 1\}|) \leq \frac{1}{k} + c(N,R)t_k.$$

We estimate the left-hand side from below, using the arithmetic-geometric mean inequality, the Cauchy-Schwarz inequality, and the co-area formula.

$$\begin{aligned}
&\int_{\{1-t_k<v_k<1\}} |\nabla v_k|^2 dx + \frac{\hat{\eta}}{2}(|\{1-t_k < v_k < 1\}|) \\
&\geq 2\left(\int_{1-t_k<v_k<1} |\nabla v_k|^2 dx\right)^{\frac{1}{2}} \left(\frac{\hat{\eta}}{2}(|\{1-t_k < v_k < 1\}|)\right)^{\frac{1}{2}} \\
&\geq \sqrt{2\hat{\eta}} \int_{1-t_k<v_k<1} |\nabla v_k| dx = \sqrt{2\hat{\eta}} \int_{1-t_k}^{1} P(v_k > s) ds.
\end{aligned}$$

where $P(E)$ denotes the De Giorgi perimeter of a set E. Hence, there exists a level $1-t_k < s_k < 1$ such that for $\hat{V}_k = \{v_k > s_k\}$

$$P(\hat{V}_k) \leq \frac{1}{t_k}\int_{1-t_k}^{1} P(\{v_k > s\}) ds \leq \frac{1}{t_k\sqrt{2\hat{\eta}}k} + c(N,R) = \frac{1}{\sqrt{2\hat{\eta}}k} + c(N,R).$$

where in the last equality we have used that $t_k = \frac{1}{\sqrt{k}}$. These $\hat{V}_k$ will give us the desired "good" minimizing sequence, indeed

$$\begin{aligned}
&\mathscr{C}^R_{\hat{\eta},j}(\hat{V}_k) \\
&\quad \leq \mathscr{C}^R_{\hat{\eta},j}(V_k) + f_{\hat{\eta}}(|\{v_k > s_k\}|) - f_{\hat{\eta}}(|\{v_k = 1\}|) + C\sigma|\{1-s_k < v_k < 1\}| \leq \mathscr{C}^R_{\hat{\eta},j}(V_k),
\end{aligned}$$

where in the first inequality we have used that $\hat{V}_k \subset V_k$ and in the second that, thanks to our choice of σ,

$$f_{\hat{\eta}}(|\{v_k > s_k\}|) - f_{\hat{\eta}}(|\{v_k = 1\}|) + C\sigma|\{1-s_k < v_k < 1\} \leq (C\sigma - \hat{\eta})|\{1-s_k < v_k < 1\} \leq 0.$$

Step 2: Existence of a minimizer. Since $\{\hat{V}_k\}_k$ is a sequence with equibounded perimeter,s there exists a Borel set $\hat{V}_\infty$ such that up to a (not relabelled) subsequence

$$1_{\hat{V}_k} \to 1_{\hat{V}_\infty} \text{ in } L_1(B_R) \text{ and a.e. in } B_R, \qquad P(\hat{V}_\infty) \leq C(N,R).$$

We want to show that $\hat{V}_\infty$ is a minimizer for $\mathcal{C}_{\eta,j}$. We set $\hat{v}_k = \frac{\min(v_k,s_k)}{s_k}$ and we note that they are the capacitary potentials of $\hat{V}_k$. Moreover the sequence $\{\hat{v}_k\}_k$ is bounded in $W^{1,2}_0(B_R)$. Thus, there exists a function $\hat{v} \in W^{1,2}_0(B_R)$ such that up to a (not relabelled) subsequence

$$\hat{v}_k \to \hat{v} \text{ strongly in } L^2(B_R) \text{ and a.e. in } B_R.$$

Let us define $\hat{V} = \{x : \hat{v} = 1\}$, we want to show that $\hat{V}$ is a minimizer. First, note that

$$1_{\hat{V}}(x) \geq \limsup 1_{\hat{V}_k}(x) = 1_{\hat{V}_\infty}(x) \qquad \text{for a.e. } x \in B_R,$$

hence $|\hat{V}_\infty \setminus \hat{V}| = 0$. Moreover, by the lower semicontinuity of Dirichlet integral, the monotonicity of $f_{\hat{\eta}}$ and the continuity of α with respect to the L^1 convergence, we have

$$\begin{aligned} \inf \mathscr{C}^R_{\hat{\eta},j} &= \lim_k \int |\nabla \hat{v}_k|^2 + f_{\hat{\eta}}(|\hat{V}_k|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\hat{V}_k) - \varepsilon_j)^2} \\ &\geq \operatorname{cap}_R(\hat{V}) + f_{\hat{\eta}}(|\hat{V}_\infty|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\hat{V}_\infty) - \varepsilon_j)^2}. \qquad \geq \operatorname{cap}_R(\hat{V}) + f_{\hat{\eta}}(|\hat{V}|) \end{aligned} \tag{5.3.5}$$

Hence

$$\begin{aligned} \operatorname{cap}_R(\hat{V}) + f_{\hat{\eta}}(|\hat{V}_\infty|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\hat{V}_\infty) - \varepsilon_j)^2} &\leq \inf \mathscr{C}^R_{\hat{\eta},j}(\Omega) \\ &\leq \operatorname{cap}_R(\hat{V}) + f_{\hat{\eta}}(|\hat{V}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\hat{V}) - \varepsilon_j)^2}. \end{aligned}$$

Using Lemma 5.1.2 (ii) we get

$$f_{\hat{\eta}}(|\hat{V}_\infty|) - f_{\hat{\eta}}(|\hat{V}|) \leq C\sigma |\hat{V} \Delta \hat{V}_\infty| = C\sigma |\hat{V} \setminus \hat{V}_\infty|.$$

Since $|\hat{V}| \geq |\hat{V}_\infty|$, (5.3.1) and our choice of σ yield

$$\hat{\eta} |\hat{V} \setminus \hat{V}_\infty| \leq f_{\hat{\eta}}(|\hat{V}_\infty|) - f_{\hat{\eta}}(|\hat{V}|) \leq C\sigma |\hat{V} \setminus \hat{V}_\infty| \leq \frac{\hat{\eta}}{2} |\hat{V} \setminus \hat{V}_\infty|,$$

from which we conclude that $|\hat{V} \Delta \hat{V}_\infty| = 0$ and thus, by (5.3.5) that $\hat{V}$ is the desired minimizer. □

5.3.3 First properties of the minimizers

Let us conclude by establishing some properties of the minimizers of (5.3.4).

Lemma 5.3.4. *Let $\{\Omega_j\}$ be a sequence of minimizers for (5.3.4). Then the following properties hold:*

(i) $|\alpha_*(\Omega_j) - \varepsilon_j| \leq 3\sigma\varepsilon_j$;

(ii) $\big||\Omega_j| - |B_1|\big| \leq C\sigma^4 \varepsilon_j$;

(iii) *(A) for the capacity in $\mathbb{R}^N$ up to translations $\Omega_j \to B_1$ in L^1,*
(B) for the relative capacity $\Omega_j \to B_1$ in L^1;

(iv) $0 \leq \mathscr{C}^*_{\hat{\eta}}(\Omega_j) - \mathscr{C}^*_{\hat{\eta}}(B_1) \leq \sigma^4 \varepsilon_j$.

Proof. Recall that the sequence $\{\Omega_j\}$ was obtained by a sequence $\{\tilde{\Omega}_j\}$ satisying

1. $|\tilde{\Omega}_j| = |B_1|$,
2. $\alpha_*(\tilde{\Omega}_j) = \varepsilon_j$,

3. $\operatorname{cap}_*(\tilde{\Omega}_j) - \operatorname{cap}_*(B_1) \leq \sigma^4 \varepsilon_j$.

We now use $\{\tilde{\Omega}_j\}$ as comparison domains for the functionals $\mathscr{C}^*_{\hat{\eta},j}$ to get

$$\mathscr{C}^*_{\hat{\eta}}(\Omega_j) + \varepsilon_j \leq \mathscr{C}^*_{\hat{\eta},j}(\Omega_j) \leq \mathscr{C}^*_{\hat{\eta},j}(\tilde{\Omega}_j) = \mathscr{C}^*_{\hat{\eta}}(\tilde{\Omega}_j) + \varepsilon_j \leq \mathscr{C}^*_{\hat{\eta}}(B_1) + \varepsilon_j(1+\sigma^4), \quad (5.3.6)$$

implying that

$$\mathscr{C}^*_{\hat{\eta}}(\Omega_j) - \mathscr{C}^*_{\hat{\eta}}(B_1) \leq \varepsilon_j \sigma^4,$$

which proves (iv). Note that we defined $f_{\hat{\eta}}$ in such a way that $\mathscr{C}^*_{\hat{\eta}}(\Omega_j) \geq \mathscr{C}^*_{\hat{\eta}}(B_1)$. Thus, using (5.3.6) we also deduce that

$$\sqrt{\varepsilon_j^2 + \sigma^2(\alpha_*(\Omega_j) - \varepsilon_j)^2} \leq \varepsilon_j(1+\sigma^4),$$

which gives (i). To estimate the volume of Ω_j, we use the classical isocapacitary inequality and properties of $f_{\hat{\eta}}$ and (5.3.2), (5.3.3). Indeed, let B^j be the ball centered in the origin such that $|B^j| = |\Omega_j|$.Then

$$\sigma^4 \varepsilon_j \geq \mathscr{C}^*_{\hat{\eta}}(\Omega_j) - \mathscr{C}^*_{\hat{\eta}}(B_1) \geq \mathscr{C}^*_{\hat{\eta}}(B^j) - \mathscr{C}^*_{\hat{\eta}}(B_1) \geq c(R)\big||\Omega_j| - |B_1|\big|,$$

where in the last inequality we have used (5.3.2), (5.3.3). This proves (ii). To prove (ii) we recall that the sets Ω_j have equibounded perimeter. Hence, the sequence $\{\Omega_j\}_j$ is precompact in $L^1(B_R)$. Since the asymmetry is continuous with respect to L^1 convergence any limit set has zero asymmetry. The only set with zero asymmetry is the unit ball (or a translated unit ball in the case of the absolute), proving (iii). □

5.4 Proof of Theorem 5.2.2: Regularity

In this section, we show that the sequence of minimizers of (5.3.4) converges smoothly to the unit ball. This will be done by relying on the regularity theory for free boundary problems established in [AC81].

5.4.1 Linear growth away from the free boundary

Let u_j be the capacitary potential for Ω_j, a minimizer of (5.3.4). Let us set $v_j := 1 - u_j$, so that $\Omega_j = \{v_j = 0\}$, $v_j = 1$ on ∂B_R, following [AC81] we are going to show that

$$v_j(x) \sim \operatorname{dist}(x, \Omega_j).$$

where the implicit constant depends only on R. The above estimate is obtained by suitable comparison estimates. In order to be able to perform them with constants which depend only on R, we need to know that $\{u_j = 1\}$ is uniformly far from ∂B_R. This will be achieved by first establishing (uniform in j) Hölder continuity of u_j.

Hölder continuity

The proof of Hölder continuity is quite standard and it is based on establishing a decay estimate for the integral oscillation of u_j. Since, thanks to the minimizing property, u_j is close to the harmonic function in $B_r(x_0) \cap B_R$ with the same boundary value, we start by recalling the decay of the harmonic functions both in the interior and at the boundary. The following is well known, see for instance [GM12, Proposition 5.8].

Lemma 5.4.1. *Suppose $w \in W^{1,2}(\Omega)$ is harmonic, $x_0 \in \Omega$. Then there exists a constant $c = c(N)$ such that for any balls $B_{r_1}(x_0) \subset B_{r_2}(x_0) \Subset \Omega$*

$$\fint_{B_{r_1}(x_0)} \left(w - \fint_{B_{r_1}(x_0)} w \right)^2 \leq c \left(\frac{r_1}{r_2} \right)^2 \fint_{B_{r_2}(x_0)} \left(w - \fint_{B_{r_2}(x_0)} w \right)^2. \tag{5.4.1}$$

Next lemma studies the decay at the boundary, the result is well known. Since we have not been able to find a precise reference for this statement, we report its simple proof.

Lemma 5.4.2. *Let Ω be an open set such that $0 \in \partial\Omega$ and let $w \in W^{1,2}(B_r)$ be harmonic in $\Omega \cap B_r$, $w \equiv 0$ on $B_r \setminus \Omega$. Assume that there exists $\delta > 0$ such that for $\rho \leq r$*

$$\frac{|\Omega^c \cap B_\rho|}{|B_\rho|} \geq \delta.$$

Then there exist a constant $c = c(\delta)$ and an exponent $\gamma = \gamma(\delta) > 0$ such that for any $0 < r_1 < r_2 < r$ we have

$$\fint_{B_{r_1}} w^2 \leq c \left(\frac{r_1}{r_2} \right)^\gamma \fint_{B_{r_2}} w^2.$$

Remark 5.4.3. Note that as w is harmonic in $\Omega \cap B_r$ and 0 on $B_r \setminus \Omega$, w^2 is subharmonic in B_r, thus its means over balls increase with the radius. In particular,

$$\sup_{B_r} w^2 \leq c(N) \fint_{B_{3r}} w^2. \tag{5.4.2}$$

Proof of Lemma 5.4.2. For convenience, we assume that $r > 1$ (we can reduce to this case by scaling). First, we note that it is enough to show the result for radii with the ratio equal to a positive power of $\frac{1}{4}$. Indeed, take $k \in \mathbb{Z}_+$ such that $\frac{1}{4^{k+1}} \leq \frac{r_1}{r_2} < \frac{1}{4^k}$. Then

$$\fint_{B_{r_1}} w^2 \leq C 4^{-\gamma k} \fint_{B_{r_1 4^k}} w^2 \leq C 4^{-\gamma k} \fint_{B_{r_2}} w^2 \leq C 4^\gamma \left(\frac{r_1}{r_2} \right)^{-\gamma} \fint_{B_{r_2}} w^2.$$

We work with powers of $\frac{1}{4}$. We start by showing

$$\sup_{B_{\frac{1}{4}}} w \leq (1 - c) \sup_{B_1} w. \tag{5.4.3}$$

For any $\varepsilon > 0$ there exists some $x_0 \in B_{\frac{1}{4}}$ such that $\sup_{B_{\frac{1}{4}}} w \leq w(x_0) + \varepsilon$, so we can write

$$\begin{aligned}
\sup_{B_{\frac{1}{4}}} w - \varepsilon \leq w(x_0) &\leq \fint_{B_{\frac{3}{4}}(x_0)} w \leq \frac{|\Omega \cap B_{\frac{3}{4}}(x_0)|}{|B_{\frac{3}{4}}(x_0)|} \sup_{B_1} w \\
&= \left(1 - \frac{|\Omega^c \cap B_{\frac{3}{4}}(x_0)|}{|B_{\frac{3}{4}}(x_0)|}\right) \sup_{B_1} w \leq \left(1 - \frac{|\Omega^c \cap B_{\frac{1}{4}}|}{|B_{\frac{3}{4}}|}\right) \sup_{B_1} w \\
&\leq \left(1 - \delta \frac{|B_{\frac{1}{4}}|}{|B_{\frac{3}{4}}|}\right) \sup_{B_1} w,
\end{aligned}$$

which proves (5.4.3) since ε is arbitrary. Using induction and scaling we can extend this result to all powers of $\frac{1}{4}$. Indeed $\tilde{w}(x) = w(x/4)$ satisfies the hypothesis of the theorem. Hence,

$$\sup_{B_{\frac{1}{16}}} w = \sup_{B_{\frac{1}{4}}} \tilde{w} \leq (1-c) \sup_{B_1} \tilde{w} = (1-c) \sup_{B_{\frac{1}{4}}} w \leq (1-c)^2 \sup_{B_1} w,$$

and thus

$$\sup_{B_{\frac{1}{4^k}}} w \leq (1-c)^k \sup_{B_1} w.$$

In the same way

$$\sup_{B_{\frac{1}{4^k} r}} w \leq (1-c)^k \sup_{B_r} w.$$

Now

$$\begin{aligned}
\fint_{B_{\frac{1}{4^k}}} w^2 &\leq (\sup_{B_{\frac{1}{4^k}}} w)^2 \leq (1-c)^{2(k-1)} (\sup_{B_{\frac{1}{4}}} w)^2 \\
&\leq (1-c)^{2(k-1)} \left(\fint_{B_{\frac{3}{4}}(x_0)} w^2\right) \leq (1-c)^{2(k-1)} \left(c' \fint_{B_1} w^2\right),
\end{aligned}$$

where we have used (5.4.2). We get from powers of $\frac{1}{4}$ to other radii again by scaling. This concludes the proof with $\gamma = -\log_4(1-c)$. $\square$

Corollary 5.4.4. *Let w be as in the statement of Lemma 5.4.2, then*

$$\fint_{B_{r_1}} \left(w - \fint_{B_{r_1}} w\right)^2 \leq C \left(\frac{r_1}{r_2}\right)^\gamma \fint_{B_{r_2}} \left(w - \fint_{B_{r_2}} w\right)^2$$

for any $0 < r_1 < r_2 < r$ with C a constant depending only on δ.

Proof. The proof follows from Lemma 5.4.2 and the simple observation that for a function w vanishing on a fixed fraction of B_ρ, the L^2 norm and the variance are comparable. Namely there exists a constant $c = c(\delta)$ such that

$$\frac{1}{c} \int_{B_\rho} \left(w - \fint_{B_\rho} w\right)^2 \leq \int_{B_\rho} w^2 \leq c \int_{B_\rho} \left(w - \fint_{B_\rho} w\right)^2.$$

Indeed, the first inequality is true for every w with $c = 1$. For the second one note that

$$\int_{B_\rho}\left(w - \fint_{B_\rho} w\right)^2 = \int_{B_\rho} w^2 - |B_\rho|\left(\fint_{B_\rho} w\right)^2.$$

Hence we need to estimate $\left(\fint_{B_\rho} w\right)^2$ in terms of $\int_{B_\rho} w^2$. Since w is non-zero only inside Ω, using Hölder inequality, we obtain

$$\left(\fint_{B_\rho} w\right)^2 \leq \left(\frac{|\Omega \cap B_\rho|}{|B_\rho|}\right)\fint_{B_\rho} w^2 \leq (1-\delta)\fint_{B_\rho} w^2,$$

hence

$$\int_{B_\rho}\left(w - \fint_{B_\rho} w\right)^2 \geq \delta \int_{B_\rho} w^2,$$

concluding the proof. □

To prove Hölder continuity of u_j we will use several times the following comparison estimates.

Lemma 5.4.5. *Let u_j be the capacitary potential of a minimizer for (5.3.4). Let $A \subset B_R$ be an open set with Lipschitz boundary and let $w \in W^{1,2}(\mathbb{R}^N)$ coincide with u_j on the boundary of A in the sense of traces.*

Then

$$\int_A |\nabla u_j|^2 dx - \int_A |\nabla w|^2 dx \leq \left(\frac{1}{\hat\eta} + C\sigma\right)\big|A \cap (\{u=1\}\Delta\{w=1\})\big|.$$

Moreover, if $u_j \leq w \leq 1$ in A, then

$$\int_A |\nabla u_j|^2 dx + \frac{\hat\eta}{2}\big|A \cap (\{u=1\}\Delta\{w=1\})\big| \leq \int_A |\nabla w|^2 dx,$$

provided $\sigma \leq \sigma(R)$.

Proof. We prove the result for the relative capacity. The case of the capacity in $\mathbb{R}^N$ can be treated in the same way. Since u_j is fixed we drop the subscript j. Consider $\tilde u$ defined as

$$\begin{cases} \tilde u = w & \text{in } A \\ \tilde u = u & \text{else.} \end{cases}$$

Take $\tilde\Omega = \{\tilde u = 1\}$ as a comparison domain. Since Ω is minimizing, we can write

$$\int_{B_R} |\nabla u|^2 dx + f_{\hat\eta}(\Omega) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha_R(\Omega) - \varepsilon_j)^2} = \mathscr{C}^R_{\hat\eta,j}(\Omega)$$
$$\leq \mathscr{C}^R_{\hat\eta,j}(\tilde\Omega) \leq \int_{B_R} |\nabla \tilde u|^2 dx + f_{\hat\eta}(\tilde\Omega) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\tilde\Omega) - \varepsilon_j)^2}.$$

Hence, by Lemma 5.1.2, (ii) and (5.3.1).

$$\int_A |\nabla u|^2 dx - \int_A |\nabla w|^2 dx \le |f_{\hat\eta}(\Omega) - f_{\hat\eta}(\tilde\Omega)| + C\sigma|\Omega\Delta\tilde\Omega| \le (\frac{1}{\hat\eta} + C\sigma)|\Omega\Delta\tilde\Omega|.$$

To prove the second inequality we observe that $u \le w \le 1$ implies $\{u = 1\} \subset \{\tilde u = 1\}$, i.e. $\Omega \subset \tilde\Omega$. Hence, by (5.3.1):

$$\int_A |\nabla u|^2 dx - \int_A |\nabla w|^2 dx \le -f_{\hat\eta}(\Omega) + f_{\hat\eta}(\tilde\Omega) + C\sigma|\Omega\Delta\tilde\Omega| \le -\hat\eta|\tilde\Omega \setminus \Omega| + C\sigma|\tilde\Omega \setminus \Omega|,$$

from which the inequality follows choosing σ small enough. □

Remark 5.4.6. Note that if w is harmonic in A, then

$$\int_A |\nabla u|^2 dx - \int_A |\nabla w|^2 dx = \int_A |\nabla(u-w)|^2 dx,$$

meaning that the first inequality from the lemma becomes

$$\int_A |\nabla(u-w)|^2 dx \le \left(\frac{1}{\hat\eta} + C\sigma\right) \big|A \cap \{u=1\}\Delta\{w=1\}\big|. \tag{5.4.4}$$

Let us also recall the following technical result

Lemma 5.4.7 ([Lemma 5.13 in [GM12]). *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function satisfying*

$$\phi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^\alpha + \varepsilon\right]\phi(R) + BR^\beta,$$

for some $A, \alpha, \beta > 0$, with $\alpha > \beta$ and for all $0 < \rho \le R \le R_0$, where $R_0 > 0$ is given. Then there exist constants $\varepsilon_0 = \varepsilon_0(A,\alpha,\beta)$ and $c = c(A,\alpha,\beta)$ such that if $\varepsilon \le \varepsilon_0$, we have

$$\phi(\rho) \le c\left[\frac{\phi(R)}{R^\beta} + B\right]\rho^\beta$$

for all $0 \le \rho \le R \le R_0$.

Lemma 5.4.8. *There exists $\alpha \in (0, 1/2)$ such that every minimizer of (5.3.4) staisfies $u_j \in C^{0,\alpha}(\overline{B_R})$. Moreover, the Hölder norm is bounded by a constant independent on j.*

Proof. Let us extend u_j by 0 outside of B_R. As usual, we drop the subscript j. By Camapanato's criterion it is enough to show that

$$\phi(r) := \int_{B_r(x_0)} \left(u - \fint_{B_r(x_0)} u\right)^2 \le Cr^{2\alpha}$$

for all r small enough (say less that 1/2).

Step 1: estimates on the boundary. Let $x_0 \in \partial B_R$. Let w be the harmonic extension of u in $B_{r'}(x_0) \cap B_R$. By Corollary 5.4.4 we know that

$$\int_{B_r(x_0)} \left(w - \fint_{B_r(x_0)} w\right)^2 \leq C\left(\frac{r}{r'}\right)^{N+\gamma} \int_{B_{r'}(x_0)} \left(w - \fint_{B_{r'}(x_0)} w\right)^2$$

for some $\gamma > 0$. Let $g := u - w$. Then

$$\begin{aligned}\int_{B_r(x_0)} \left(u - \fint_{B_r(x_0)} u\right)^2 &\leq 2\int_{B_r(x_0)} \left(w(x) - \fint_{B_r(x_0)} w\right)^2 + 2\int_{B_r(x_0)} \left(g - \fint_{B_r(x_0)} g\right)^2 \\ &\leq 2C\left(\frac{r}{r'}\right)^{N+\gamma} \int_{B_{r'}(x_0)} \left(w - \fint_{B_r(x_0)} w\right)^2 + 2\int_{B_{r'}(x_0)} g^2 \\ &\leq C\left(\frac{r}{r'}\right)^{N+\gamma} \int_{B_{r'}(x_0)} \left(u - \fint_{B_r(x_0)} u\right)^2 + C\int_{B_{r'}(x_0)} g^2.\end{aligned}$$

To estimate $\int_{B_{r'}(x_0)} g^2$ we recall that $g \in W_0^{1,2}(B_{r'}(x_0))$ and vanishes outside $B_{r'}(x_0) \cap B_R$, hence by Poincaré's inequality and (5.4.4)

$$\int_{B_{r'}(x_0)} g^2 \leq C(r')^2 \int_{B_{r'}(x_0)\cap B_R} |\nabla g|^2 \leq C(r')^{N+2}.$$

Combining the last two inequalities, we get

$$\phi(r) \leq c\left(\frac{r}{r'}\right)^{N+\gamma} \phi(r') + C(r')^{N+2}.$$

Using Lemma 5.4.7 we obtain

$$\phi(r) \leq c\left(\left(\frac{r}{r'}\right)^{N+\gamma} \phi(r') + Cr^{N+\gamma}\right)$$

for any $r < r' < 1$. In particular,

$$\phi(r) \leq c\left(\|u\|^2_{L^2(\mathbb{R}^N)} + C\right) r^{N+\gamma}.$$

Step 2: estimates at the interior. Assume that $x_0 \in B_R$, $r < r' < \operatorname{dist}(x_0, \partial B_R)$, so that $B_r(x_0) \subset B_{r'}(x_0) \subset B_R$. Then one can proceed in the same way as in the previous step using Lemma 5.4.1 instead of Corollary 5.4.4. Hence

$$\phi(r) \leq C\left(\left(\frac{r}{r'}\right)^{N+\gamma} \phi(r') + Cr^{N+\gamma}\right)$$

for $r < r' < \operatorname{dist}(x_0, \partial B_R)$ and, in particular,

$$\phi(r) \leq c\left(\left(\frac{1}{\operatorname{dist}(x_0, \partial B_R)}\right)^{N+\gamma} \|u\|^2_{L^2(\mathbb{R}^N)} + C\right) r^{N+\gamma}. \tag{5.4.5}$$

Step 3: global estimates. We now combine the previous steps, distinguishing several cases:

- $\operatorname{dist}(x_0, \partial B_R) > 1/2$. By Step 2
$$\phi(r) \le C\left(\|u\|^2_{L^2(\mathbb{R}^N)} + C\right) r^{N+\gamma}.$$
- $r \le \rho := \operatorname{dist}(x_0, \partial B_R) \le 1/2$. Let $y_0 = R\frac{x_0}{|x_0|}$ be the radial projection of x_0 on ∂B_R. Then, using Step 2 and Step 1, we have
$$\begin{aligned}
\phi(r) &\le C\left(\left(\frac{r}{\rho}\right)^{N+\gamma}\phi(\rho) + Cr^{N+\gamma}\right)\\
&= C\left(\left(\frac{r}{\rho}\right)^{N+\gamma}\int_{B_\rho(x_0)}\left(u - \fint_{B_\rho(x_0)} u\right)^2 + Cr^{N+\gamma}\right)\\
&\le C\left(\left(\frac{r}{\rho}\right)^{N+\gamma}\int_{B_{2\rho}(y_0)}\left(u - \fint_{B_{2\rho}(y_0)} u\right)^2 + Cr^{N+\gamma}\right)\\
&\le C\left(\left(\frac{r}{\rho}\right)^{N+\gamma}(2\rho)^{N+\gamma}\int_{B_1(y_0)}\left(u - \fint_{B_1(y_0)} u\right)^2 + Cr^{N+\gamma}\right)\\
&\le C\left(\|u\|^2_{L^2(\mathbb{R}^N)} + C\right) r^{N+\gamma}.
\end{aligned}$$
- $\rho := \operatorname{dist}(x_0, \partial B_R) \le r \le 1/2$. Again we set y_0 to be the radial projection of x_0 onto ∂B_R. We use Step 1 and get
$$\begin{aligned}
\phi(r) = \int_{B_r(x_0)}\left(u - \fint_{B_r(x_0)} u\right)^2 &\le \int_{B_{2r}(y_0)}\left(u - \fint_{B_{2r}(y_0)} u\right)^2\\
&\le C\left(r^{N+\gamma}\int_{B_1(y_0)}\left(u - \fint_{B_1(y_0)} u\right)^2 + Cr^{N+\gamma}\right)\\
&\le C\left(\|u\|^2_{L^2(\mathbb{R}^N)} + C\right) r^{N+\gamma}.
\end{aligned}$$

In conclusion,
$$\phi(r) \le C\left(\|u\|^2_{L^2(\mathbb{R}^N)} + C\right) r^{N+\gamma},$$
which by Campanato criterion implies that $u \in C^{\frac{\gamma}{2}}$. Note furthermore that the dependence on j is realized only by the L^2 norm of u_j which is uniformly bounded by $\sqrt{|B_R|}$. □

Lipschitz continuity and density estimates on the boundary

We now prove two lemmas similar to those in Section 3 of [AC81]. These are obtained by adding or removing a small ball from an optimizer of (5.3.4). Since our competitors are constrained to lie in B_R removing a ball is not a problem. On the other hand adding might lead to a non admissible competitor. For the case of the relative capacity, we use the Hölder estimate of the previous section. Indeed it implies that there exists $\rho_0 = \rho_0(R) > 0$ such that

$$\Omega_j \subset B_{R-\rho_0}. \tag{5.4.6}$$

Lemma 5.4.9. *For $\kappa < 1$ there is a constant $c = c(N, \kappa, R)$ such that if u_j is a capacitary potential for a minimizer of (5.3.4) and $v_j = 1 - u_j$ satisfies*

$$⨍_{\partial B_r(x_0)} v_j \le cr \tag{5.4.7}$$

for some $x_0 \in B_R$, then $v_j = 0$ in $B_{\kappa r}(x_0)$. In the case of the relative capacity we assume $r \le \rho_0$ where ρ_0 is as in (5.4.6).

Proof. We drop the subscript j for simplicity. We first check that $B_{\kappa r}(x_0) \subset B_R$. By our restriction on r this is clear in the case of the relative capacity. Let us show that this is the case also for the absolute capacity provided we choose c small enough (depending only on R and N, κ). To prove this we use that v cannot be too small outside of B_R. More precisely, by comparison principle we know that

$$v(x) \ge v_{B_R}(x) = 1 - \frac{R^{N-2}}{|x|^{N-2}},$$

where v_{B_R} is the corresponding function for B_R. Suppose that $B_{\kappa r}(x_0) \backslash B_R \neq \emptyset$. Then the part of $\partial B_r(x_0) \backslash B_R$ with the distance at least $\frac{1-\kappa}{2} r$ from the boundary of the ball B_R has measure at least $c(\kappa) r^{N-1}$. Then

$$⨍_{\partial B_r(x_0)} v \ge c(\kappa) \left(1 - \frac{R^{N-2}}{\left(R + \frac{1-\kappa}{2} r\right)^{N-2}} \right) \ge c(N, \kappa, R) r,$$

in contradiction with (5.4.7) if c is small enough depending on κ, N, R.

Now we turn to the proof of the lemma for both cases. Since x_0 is fixed we simply write B_r for $B_r(x_0)$. The idea is to take as a variation a domain, defined by a function coinciding with v everywhere outside $B_{\sqrt{\kappa} r}$ and being zero inside $B_{\kappa r}$. More precisely, define w in $B_{\sqrt{\kappa} r}$ as the solution of

$$\begin{cases} \Delta w = 0 \text{ in } B_{\sqrt{\kappa} r} \backslash B_{\kappa r} \\ w = 0 \text{ in } B_{\kappa r} \\ w = \overline{v} \text{ on } \partial B_{\sqrt{\kappa} r} \end{cases},$$

where $\overline{v} = \sup_{B_{\sqrt{\kappa} r}} v$. Note that since v is subharmonic, $\overline{v} \le c(N, \kappa) ⨍_{\partial B_r} v$. Moreover, one easily estimates

$$\left| \frac{\partial w}{\partial \nu} \right| \le C(N, \kappa) \frac{\overline{v}}{r} \qquad \text{on } \partial B_{\kappa r}. \tag{5.4.8}$$

Using the second inequality in Lemma 5.4.5 with $A = B_{\sqrt{\kappa r}}$ and $\max(u, 1-w) = 1 - \min(v, w)$ in the place of w, we get

$$\int_{B_{\sqrt{\kappa r}}} |\nabla v|^2 dx + \frac{\hat{\eta}}{2} |B_{\sqrt{\kappa r}} \cap \{v > 0, w = 0\}| \leq \int_{B_{\sqrt{\kappa r}}} |\nabla \min(v, w)|^2 dx.$$

Using that $|a|^2 + |b|^2 \geq 2a \cdot b$, we obtain

$$\begin{aligned}
\int_{B_{\kappa r}} \left(|\nabla v|^2 + \frac{\hat{\eta}}{2} 1_{\{v>0\}} \right) dx &\leq \int_{B_{\sqrt{\kappa r}} \setminus B_{\kappa r}} \left(|\nabla \min(v, w)|^2 - |\nabla v|^2 \right) dx \\
&\leq 2 \int_{(B_{\sqrt{\kappa r}} \setminus B_{\kappa r}) \cap \{v > w\}} \left(|\nabla w|^2 - \nabla v \cdot \nabla w \right) dx = -2 \int_{B_{\sqrt{\kappa r}} \setminus B_{\kappa r}} \nabla \max(v - w, 0) \nabla w dx \\
&= 2 \int_{\partial B_{\kappa r}} v \frac{\partial w}{\partial \nu} d\mathcal{H}^{N-1} \leq c(N, \kappa) \frac{\overline{v}}{r} \int_{\partial B_{\kappa r}} v d\mathcal{H}^{N-1}.
\end{aligned}$$

where we have used (5.4.8). We will now bound $\int_{\partial B_{\kappa r}} v d\mathcal{H}^{N-1}$ from above by a constant times the left-hand side. Since $\frac{\overline{v}}{r}$ can be made as small as we wish, this will conclude the proof. In order to do that we use first the trace inequality, then AM-GM to get

$$\begin{aligned}
\int_{\partial B_{\kappa r}} v d\mathcal{H}^{N-1} &\leq c(N, \kappa) \left(\frac{1}{r} \int_{B_{\kappa r}} v dx + \int_{B_{\kappa r}} |\nabla v| dx \right) \\
&\leq c(N, \kappa) \left(\frac{\overline{v}}{r} \frac{2}{\hat{\eta}} \int_{B_{\kappa r}} \frac{\hat{\eta}}{2} 1_{\{v>0\}} dx + \frac{1}{2} \int_{B_{\kappa r}} \left(|\nabla v|^2 + 1_{\{v>0\}} \right) dx \right) \\
&\leq c(N, \kappa, R) \int_{B_{\kappa r}} \left(|\nabla v|^2 + \frac{\hat{\eta}}{2} 1_{\{v>0\}} \right) dx.
\end{aligned}$$

□

Lemma 5.4.10. *There exists $M = M(N, R)$ such that if u_j is a minimizer for (5.3.4) and $v_j = 1 - u_j$ satisfies*

$$\fint_{\partial B_r(x_0)} v_j d\mathcal{H}^{N-1} \geq Mr,$$

then $v_j > 0$ in $B_r(x_0)$.

Proof. Let us drop the subscript j as usual. As a comparison domain here we consider $\Omega \setminus B_r(x_0)$, note that it is a subset of B_R.More precisely, we define w as the solution of

$$\begin{cases} \Delta w = 0 \text{ in } B_r(x_0) \\ w = v \text{ on } \mathbb{R}^N \setminus B_r(x_0). \end{cases}$$

We use Lemma 5.4.5 and Remark 5.4.6 with $A = B_r$, $1 - w$ as w to deduce

$$\int_{B_r(x_0)} |\nabla (v - w)|^2 dx \leq \left(\frac{1}{\hat{\eta}} + C\sigma \right) |\{v = 0\} \cap B_r(x_0)|. \tag{5.4.9}$$

We now estimate $|\{v = 0\} \cap B_r|$ by the left-hand side. This can be done by arguing as in [AC81, Lemma 3.2]. Here we present a slightly different proof [1]. First we

[1] We warmly thank Jonas Hirsch for suggesting this proof.

change coordinates so that $x_0 = 0$. Then by the representation formula

$$w(x) \geq c(N)\frac{r-|x|}{r} \fint_{\partial B_r} v \geq c(N)(r-|x|)M. \tag{5.4.10}$$

If we now apply Hardy inequality,

$$\int_{B_r} \frac{g^2}{(r-|x|)^2} \leq C(N) \int_{B_r} |\nabla g|^2 \qquad g \in W_0^{1,2}(B_r),$$

to the function $g = v - w$ and we take into account (5.4.10) and (5.4.9), we get

$$\begin{aligned} c(N)M^2|\{v=0\} \cap B_r| &\leq \int_{\{v=0\}\cap B_r} \frac{w^2}{(r-|x|)^2} \leq \int_{B_r} \frac{(w-v)^2}{(r-|x|)^2} \\ &\leq c(N) \int_{B_r} |\nabla(v-w)|^2 \leq C(N,R)|\{v=0\} \cap B_r|, \end{aligned}$$

which is impossible if M is large enough depending in N, R unless $v > 0$ almost everywhere in B_r. □

As in Section 3 of [AC81] these two lemmas imply Lipschitz continuity of minimizers and density estimates on the boundary of minimizing domains. Note that we use here Lemma 5.4.8 as we need to apply the lemmas for the balls of all radii less or equal to some ρ_0, see (5.4.6).

Lemma 5.4.11. *Let v_j be as above, $\Omega_j = \{v_j = 0\}$. Then Ω_j is open and there exist constants $C = C(N,R)$, $\rho_0 = \rho_0(N,R) > 0$ such that*

(i) *for every $x \in B_R$*

$$\frac{1}{C}\operatorname{dist}(x,\Omega_j) \leq v_j \leq C\operatorname{dist}(x,\Omega_j);$$

(ii) *v_j are equi-Lipschitz;*

(iii) *for every $x \in \partial\Omega_j$ and $r \leq \rho_0$*

$$\frac{1}{C} \leq \frac{|\Omega_j \cap B_r(x)|}{|B_r(x)|} \leq \left(1 - \frac{1}{C}\right).$$

Applying [AC81, Theorem 4.5] to $v_j = (1 - u_j)$ we also have the following

Lemma 5.4.12. *Let u_j be as above, then there exists a Borel function q_{u_j} such that*

$$\Delta u_j = q_{u_j}\mathcal{H}^{N-1} \llcorner \partial^*\Omega_j. \tag{5.4.11}$$

Moreover, $0 < c \leq -q_{u_j} \leq C$, $c = c(N,R)$, $C = C(N,R)$ and $\mathcal{H}^{N-1}(\partial\Omega_j \setminus \partial^*\Omega_j) = 0$.

Since Ω_j converge to B_1 in L^1 by Lemma 5.3.4, the density estimates also give us the following convergence of boundaries.

Lemma 5.4.13. *Let Ω_j be minimizers of* (5.3.4)*. Then:*

(A) *For the capacity with respect to the ball B_R*

$$\partial\Omega_j \xrightarrow[j\to\infty]{} \partial B_1,$$

in the Kuratowski sense.

(B) *For the capacity in $\mathbb{R}^N$ every limit point of Ω_j with respect to L^1 convergence is the unit ball centered at some $x_\infty \in B_R$. Moreover, the convergence holds also in the Kuratowski sense.*

Corollary 5.4.14. *In the setting of Lemma 5.4.13, for every $\delta > 0$ there exists j_δ such that for $j \geq j_\delta$*

(A) *$B_{1-\delta} \subset \Omega_j \subset B_{1+\delta}$ in the case of the relative capacity;*

(B) *$B_{1-\delta}(x_j) \subset \Omega_j \subset B_{1+\delta}(x_j)$ for some $x_j \in B_R$ in the case of the capacity in $\mathbb{R}^N$.*

5.4.2 Higher regularity of the free boundary

In order to address the higher regularity of $\partial\Omega_j$, we need to prove that q_{u_j} is smooth. This will be done by using the Euler-Lagrange equations for our minimizing problem. We defined Ω_j in such a way that the following minimizing property holds

(A)

$$\begin{aligned}&\int_{B_R} |\nabla u_j|^2 dx + f_{\hat{\eta}}(|\{u_j = 1\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha_R(\{u_j = 1\}) - \varepsilon_j)^2}\\ &\leq \int_{B_R} |\nabla u|^2 dx + f_{\hat{\eta}}(|\{u = 1\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha_R(\{u = 1\}) - \varepsilon_j)^2}\end{aligned} \tag{5.4.12}$$

for any $u \in W_0^{1,2}(B_R)$ such that $0 \leq u \leq 1$.

(B)

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u_j|^2 dx + f_{\hat{\eta}}(|\{u_j = 1\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\{u_j = 1\}) - \varepsilon_j)^2}\\ &\leq \int_{\mathbb{R}^N} |\nabla u|^2 dx + f_{\hat{\eta}}(|\{u = 1\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\{u = 1\}) - \varepsilon_j)^2}\end{aligned} \tag{5.4.13}$$

for any $u \in W^{1,2}(\mathbb{R}^N)$ such that $0 \leq u \leq 1$, $\{u = 1\} \subset B_R$.

To write Euler-Lagrange equations for u_j, we need to have (5.4.12) or (5.4.13))respectively for $u_j \circ \Phi$ where Φ is a diffeomorphism of $\mathbb{R}^N$ close to the identity. Note that to make sure that $\{u_j \circ \Phi = 1\}$ is contained in B_R one needs to know that $\text{dist}(u_j, \partial B_r) > 0$. This follows from Corollary 5.4.14, up translate Ω_j in the case of the absolute capacity (note that in this case the problem is invariant by translation). More precisely we will get the following optimality condition

(A)

$$q_{u_j}^2 - \frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x| = \Lambda_j;$$

(B)

$$q_{u_j}^2 - \frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x-x_{\Omega_j}| - \left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x\right) = \Lambda_j$$

for some constant $\Lambda_j > 0$. These equations are an immediate consequence of the following lemma whose proof is almost the same as [BDPV15, Lemma 4.15] (which in turn is based on [AAC86]). For this reason we only highlight the most relevant changes, referring the reader to [BDPV15, Lemma 4.15] for more details.

Lemma 5.4.15. *There exists j_0 such that for any $j \geq j_0$ and any two points x_1 and x_2 in the reduced boundary of Ω_j the following equality holds:*

(A)

$$q_{u_j}^2(x_1) - \frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x_1| = q_{u_j}^2(x_2) - \frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x_2|;$$

(B)

$$\begin{aligned} q_{u_j}^2(x_1) &- \frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x_1-x_{\Omega_j}| - \left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x_1\right) \\ &= q_{u_j}^2(x_2) - \frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x_2-x_{\Omega_j}| - \left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x_2\right). \end{aligned}$$

Proof. We argue by contradiction. Assume there exist $x_1, x_2 \in \partial^*\{u_j = 1\}$ such that

(A)

$$q_{u_j}^2(x_1) - \frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x_1| < q_{u_j}^2(x_2) - \frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x_2|; \tag{5.4.14}$$

(B)

$$\begin{aligned} q_{u_j}^2(x_1) &- \frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x_1-x_{\Omega_j}| - \left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x_1\right) \\ &< q_{u_j}^2(x_2) - \frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x_2-x_{\Omega_j}| - \left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x_2\right). \end{aligned} \tag{5.4.15}$$

Using this inequalities, we are going to construct a variation contradicting (5.4.12). We take a smooth radial symmetric function $\phi(x) = \phi(|x|)$ supported in B_1 and define the following diffeomorphism for small τ and ρ:

$$\Phi_\tau^\rho(x) = \begin{cases} x + \tau\rho\phi(|\frac{x-x_1}{\rho}|)\nu(x_1), & x \in B_\rho(x_1), \\ x - \tau\rho\phi(|\frac{x-x_2}{\rho}|)\nu(x_2), & x \in B_\rho(x_2), \\ x, & \text{otherwise.} \end{cases}$$

We define the function

$$u_\tau^\rho := u \circ (\Phi_\tau^\rho)^{-1}$$

and we define a competitor domain Ω_τ^ρ as the domain with u_τ^ρ for capacitary potential, i.e.

$$\Omega_\tau^\rho := \{u_\tau^\rho = 1\}.$$

Now we are going to show that for τ and ρ small enough $\mathscr{C}_{\hat\eta}^*(\Omega_\tau^\rho) < \mathscr{C}_{\hat\eta}^*(\Omega)$. To do that, we first compute the variation of all the terms involved in $\mathscr{C}_{\hat\eta}^*$.

Volume. By arguing as in [BDPV15, Lemma 4.15] one gets

$$\begin{aligned} |\Omega_\tau^\rho| - |\Omega| &= \tau\rho^N \left(\int_{\{y\cdot\nu(x_1)=0\}\cap B_1} \phi\left(|y|\right) - \int_{\{y\cdot\nu(x_2)=0\}\cap B_1} \phi\left(|y|\right) \right) + o(\tau)\rho^N + o_\tau(\rho^N) \\ &= o(\tau)\rho^N + o_\tau(\rho^N), \end{aligned}$$

where $o_\tau(\rho^N)\rho^{-N}$ goes to zero as $\rho \to 0$ and $o(\tau)$ is independent on ρ.

Barycenter.(for the case of the capacity in $\mathbb{R}^N$). Assume that that $x_\Omega = 0$, as in [BDPV15, Lemma 4.15] one gets,

$$x_{\Omega_\tau^\rho} = -\rho^N\tau\frac{x_1 - x_2}{|\Omega|}\left(\int_{\{y_1=0\}\cap B_1} \phi(|y|)\right) + \rho^N o(\tau) + o_\tau(\rho^N).$$

Asymmetry. Again by the very same computations as in [BDPV15, Lemma 4.15] one gets

$$\alpha_R(\Omega_\tau^\rho) - \alpha_R(\Omega) = -\rho^N\tau\left(\int_{\{y_1=0\}\cap B_1} \phi(|y|)\right)(|x_1| - |x_2|) + o(\tau)\rho^N + o_\tau(\rho^N).$$

In the case of asymmetry $\alpha(\Omega)$ we get an additional term:

$$\begin{aligned} \alpha(\Omega_\tau^\rho) - \alpha(\Omega) = -\rho^N\tau\left(\int_{\{y_1=0\}\cap B_1} \phi(|y|)\right)\left(|x_1| - |x_2| + \left(\fint_\Omega \frac{y}{|y|}dy\right)\cdot(x_1 - x_2)\right) \\ + o(\tau)\rho^N + o_\tau(\rho^N). \end{aligned}$$

.

Dirichlet energy. Again one can argue as in [BDPV15, Lemma 4.15] to get

$$\mathrm{cap}_*(\Omega_\tau^\rho) - \mathrm{cap}_*(\Omega) \le \tau\rho^N\left(|q(x_1)|^2 - |q(x_2)|^2\right)\int_{B_1\cap\{y_1=0\}} \phi(|y|)dy + o(\tau)\rho^N + o_\tau(\rho^N).$$

Combining the above estimates one gets

(A)

$$\left(\int_{B_1\cap\{y_1=0\}}\phi(|y|)dy\right)^{-1}\frac{\mathscr{C}^R_{\hat\eta,j}(\Omega^\rho_\tau)-\mathscr{C}^R_{\hat\eta,j}(\Omega)}{\rho^N}$$
$$=\tau\left(|q(x_1)|^2-|q(x_2)|^2-\frac{\sigma^2(\alpha_R(\Omega)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega)-\varepsilon_j)^2}}(|x_1|-|x_2|)\right)+o(\tau)+o_\tau(1);$$

(B)

$$\left(\int_{B_1\cap\{y_1=0\}}\phi(|y|)dy\right)^{-1}\frac{\mathscr{C}_{\hat\eta,j}(\Omega^\rho_\tau)-\mathscr{C}_{\hat\eta,j}(\Omega)}{\rho^N}$$
$$=\tau\Bigg(|q(x_1)|^2-|q(x_2)|^2$$
$$-\frac{\sigma^2(\alpha(\Omega)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega)-\varepsilon_j)^2}}\left(|x_1|-|x_2|+\left(\fint_\Omega\frac{y}{|y|}dy\right)\cdot(x_1-x_2)\right)\Bigg)$$
$$+o(\tau)+o_\tau(1).$$

According to (5.4.14) and (5.4.15) the quantity in parentheses is strictly negative. Thus, we get a contradiction with the minimality of Ω for ρ and τ small enough. □

Lemma 5.4.16 (Smoothness of $\boldsymbol{q_u}$)**.** *There exist constants $\delta=\delta(N,R)>0$, $j_0=j_0(N,R)$, $\sigma_0=\sigma_0(N,R)>0$ such that for every $j\geq j_0$, $\sigma\leq\sigma_0$ the functions q_{u_j} belong to $C^\infty(\mathcal{N}_\delta(\partial\Omega_j))$.*

Moreover, for every k there exists a constant $C=C(k,N,R)$ such that

$$\|q_{u_j}\|_{C^k(\mathcal{N}_\delta(\partial\Omega_j))}\leq C$$

for every $j\geq j_0$.

Proof. We would like to write an explicit formula for q_{u_j} using Euler-Lagrange equations, namely

(A)

$$q_{u_j}=-\left(\frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x|+\Lambda_j\right)^{\frac12};\tag{5.4.16}$$

(B)

$$q_{u_j}=-\left(\frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x-x_{\Omega_j}|-\left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x\right)+\Lambda_j\right)^{\frac12}.\tag{5.4.17}$$

To do that, we need to show that the quantity in the parenthesis is bounded away from zero. Indeed, q_{u_j} is bounded from above and below independently of j and

(A)

$$\left|\frac{\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha_R(\Omega_j)-\varepsilon_j)^2}}|x|\right| \le C(N,R)\sigma; \tag{5.4.18}$$

(B)

$$\left|\frac{\sigma^2(\alpha(\Omega_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega_j)-\varepsilon_j)^2}}\left(|x-x_{\Omega_j}|-\left(\fint_{\Omega_j}\frac{y-x_{\Omega_j}}{|y-x_{\Omega_j}|}dy\right)\cdot x\right)\right| \le C(N,R)\sigma. \tag{5.4.19}$$

Then it follows from the Euler-Lagrange equations that also Λ_j is bounded from above and below independently of j. Thus, for σ small enough we can write the above-mentioned explicit formula for q_{u_j} and get the conclusion of the lemma. □

Now we are ready to apply the results of [AC81]. Indeed thanks to Lemma 5.4.15, $v_j = (1-u_j)$ is a weak solution of the free boundary problem First, we need to recall the definition of flatness for the free boundary, see [AC81, Definition 7.1] (here it is applied to $u = (1-v)$).

Definition 5.4.17. Let $\mu_-, \mu_+ \in (0,1]$. A weak solution u of (5.4.11) is said to be of class $F(\mu_-, \mu_+, \infty)$ in $B_\rho(x_0)$ in a direction $\nu \in S^{N-1}$ if $x_0 \in \partial\{u=1\}$ and

$$\begin{cases} u(x) = 1 & \text{for } (x-x_0)\cdot\nu \le -\mu_-\rho, \\ 1-u(x) \ge q_u(x_0)((x-x_0)\cdot\nu - \mu_+\rho) & \text{for } (x-x_0)\cdot\nu \ge \mu_+\rho, \end{cases}$$

We are going to use that flat free boundaries are smooth (again we apply [AC81, Theorem 8.1] to $v = (1-u)$)

Theorem 5.4.18 (Theorem 8.1 in [AC81])**.** *Let u be a weak solution of* (5.4.11)*) and assume that q_u is Lipschitz continuous. There are constants γ, μ_0, κ, C such that if u is of class $F(\mu, 1, \infty)$ in $B_{4\rho}(x_0)$ in some direction $\nu \in S^{N-1}$ with $\mu \le \mu_0$ and $\rho \le \kappa\mu^2$, then there exists a $C^{1,\gamma}$ function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ with $\|f\|_{C^{1,\gamma}} \le C\mu$ such that*

$$\partial\{u=1\} \cap B_\rho(x_0) = (x_0 + \mathrm{graph}_\nu f) \cap B_\rho(x_0), \tag{5.4.20}$$

where $\mathrm{graph}_\nu f = \{x \in \mathbb{R}^N : x\cdot\nu = f(x - x\cdot\nu)\nu)\}$*. Moreover if $q_u \in C^{k,\gamma}$ in some neighborhood of $\{u_j = 1\}$, then $f \in C^{k+1,\gamma}$ and $\|f\|_{C^{k+1,\gamma}} \le C(N, R, \|q_u\|_{C^{k,\gamma}})$.*

We are now ready to prove Theorem 5.2.2, cp. [BDPV15, Proposition 4.4].

Proof of Theorem 5.2.2. We define Ω_j as minimizers of (5.3.4). The desired sequence of Selection Principle will be properly rescaled $\{\Omega_j\}$. We need to show that $\{\Omega_j\}$ converges smoothly to the ball B_1. Indeed one then define

$$U_j = \lambda_j(\Omega_j - x_*),$$

where $\lambda_j = \left(\frac{|B_1|}{|\Omega_j|}\right)^{1/n}$, $x_* = 0$ in the case of the relative capacity and $x_* = x_{\Omega_j}$ in the case of the absolute capacity. Lemma 5.3.4 then implies all the desired properties of U_j, compare with [BDPV15, Proof of Proposition 4.4].

Let μ_0, κ be as in Theorem 5.4.18 and $\mu < \mu_0$ to be fixed later. Let $\overline{x}$ be some point on the boundary of B_1. As ∂B_1 is smooth, it lies inside a narrow strip in the neighborhood of $\overline{x}$. More precisely, there exists $\rho_0 = \rho_0(\mu) \leq \kappa\mu^2$ such that for every $\rho < \rho_0$ and every $\overline{x} \in \partial B_1$

$$\partial B_1 \cap B_{5\rho}(\overline{x}) \subset \{x : |(x - \overline{x}) \cdot \nu_{\overline{x}}| \leq \mu\rho\}.$$

We know that $\partial\Omega_j$ are converging to ∂B_1 in the sense of Kuratowski. Thus, there exists a point $x_0 \in \partial\Omega_j \cap B_{\mu\rho_0}(\overline{x})$ such that

$$\partial\Omega_j \cap B_{4\rho_0}(x_0) \subset \{x : |(x - x_0) \cdot \nu_{\overline{x}}| \leq 4\mu\rho_0\}.$$

So, u_j is of class $F(\mu, 1, \infty)$ in $B_{4\rho_0}(x_0)$ with respect to the direction $\nu_{\overline{x}}$ and by Theorem 5.4.18, $\partial\Omega_j \cap B_{\rho_0}(x_0)$ is the graph of a smooth function with respect to $\nu_{\overline{x}}$. More precisely, for μ small enough there exists a family of smooth functions $g_j^{\overline{x}}$ with uniformly bounded C^k norms such that

$$\partial\Omega_j \cap B_{\rho_0}(\overline{x}) = \{x + g_j^{\overline{x}}(x)x : x \in \partial B_1\} \cap B_{\rho_0}(\overline{x}).$$

By a covering argument this gives a family of smooth functions g_j with uniformly bounded C^k norms such that

$$\partial\Omega_j = \{x + g_j(x)x : x \in \partial B_1\}.$$

By Ascoli-Arzelà and convergence to ∂B_1 in the sense of Kuratowski, we get that $g_j \to 0$ in $C^{k-1}(\partial B_1)$, hence the smooth convergence of $\partial\Omega_j$. □

5.5 Reduction to bounded sets

To complete the proof of Theorem 1.1.5 one needs to show that in the case of the full capacity one can just consider sets with uniformly bounded diameter. To this end let us introduce the following

Definition 5.5.1. Let Ω be an open set in $\mathbb{R}^N$ with $|\Omega| = |B_1|$. Then we define the deficit of Ω as the difference between its capacity and the capacity of the unit ball:

$$D(\Omega) = \mathrm{cap}(\Omega) - \mathrm{cap}(B_1).$$

Here is the key lemma for reducing Theorem 1.1.5 to Theorem 5.2.1.

Lemma 5.5.2. *There exist constants $C = C(N)$, $\delta = \delta(N) > 0$ and $d = d(N)$ such that for any $\Omega \subset \mathbb{R}^N$ open with $|\Omega| = |B_1|$ and $D(\Omega) \leq \delta$, we can find a new set $\tilde{\Omega}$ enjoying the following properties*

1. $\operatorname{diam}(\tilde{\Omega}) \leq d$,
2. $|\tilde{\Omega}| = |B_1|$,
3. $D(\tilde{\Omega}) \leq CD(\Omega)$,
4. $\mathcal{A}(\tilde{\Omega}) \geq \mathcal{A}(\Omega) - CD(\Omega)$.

We are going to define $\tilde{\Omega}$ as a suitable dilation of $\Omega \cap B_S$ for some large S. Hence, we first show the following estimates on the capacity of $\Omega \cap B_S$.

Lemma 5.5.3. *Let $S' > S$. Then there exists a constant $c = c(S')$ such that for any open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| = |B_1|$ the following inequalities hold:*

$$\operatorname{cap}(B_1)\left(1 - \frac{|\Omega \setminus B_S|}{|B_1|}\right)^{\frac{N-2}{N}} \leq \operatorname{cap}(\Omega \cap B_S) \leq \operatorname{cap}(\Omega) - c\left(1 - \frac{S}{S'}\right)^2 |\Omega \setminus B_{S'}|^{\frac{N-2}{N}}.$$

Proof. The first inequality is a direct consequence of the classical isocapacitary inequality. To prove the second one we are going to use the estimates for the capacitary potential of B_S for which the exact formula can be written. Denote by u_Ω and u_S the capacitary potentials of Ω and $\Omega \cap B_S$ respectively. We first compute

$$\begin{aligned}
&\operatorname{cap}(\Omega \cap B_S) = \operatorname{cap}(\Omega) + \int_{\mathbb{R}^N} |\nabla u_S|^2 - |\nabla u_\Omega|^2 \\
&= \operatorname{cap}(\Omega) - \int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^2 + 2\int_{(\Omega\cap B_S)^c} \nabla u_S \cdot \nabla(u_S - u_\Omega) \\
&= \operatorname{cap}(\Omega) - \int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^2 - 2\int_{(\Omega\cap B_S)^c} (\Delta u_S)(u_S - u_\Omega) \\
&\qquad + 2\int_{\partial(\Omega\cap B_S)} (u_S - u_\Omega)\nabla u_S \cdot \nu d\mathcal{H}^{N-1} \\
&= \operatorname{cap}(\Omega) - \int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^2
\end{aligned}$$

since $u_S = u_\Omega = 1$ on $\partial(\Omega \cap B_S)$. We would like to show that $\int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^2$ cannot be too small. To this end let us set $v_\Omega = 1 - u_\Omega$ and similarly for v_S. By Sobolev's embedding we get

$$\begin{aligned}
&\int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^2 = \int_{(\Omega\cap B_S)^c} |\nabla(v_\Omega - v_S)|^2 \\
&\geq c(N)\left(\int_{(\Omega\cap B_S)^c} |v_\Omega - v_S|^{2^*}\right)^{\frac{2}{2^*}} \geq c\left(\int_{\Omega\setminus B_S} |v_S|^{2^*}\right)^{\frac{2}{2^*}},
\end{aligned}$$

where 2^* is the Sobolev exponent and in the last inequality we used that $v_\Omega \equiv 0$ on Ω. Let us also set

$$z_S = \left(1 - \frac{S^{N-2}}{|x|^{N-2}}\right)_+.$$

By the maximum principle, $v_S \geq z_S$, hence

$$\int_{\Omega\setminus B_S} |v_S|^{2^*} \geq \int_{\Omega\setminus B_S} |z_S|^{2^*}$$
$$\geq \int_{\Omega\setminus B_{S'}} |z_S|^{2^*} \geq \left(1-\left(\frac{S}{S'}\right)^{N-2}\right)^{2^*} |\Omega \setminus B_{S'}|.$$

Hence

$$\operatorname{cap}(\Omega\cap B_S) \leq \operatorname{cap}(\Omega) - c(N)\left(1-\left(\frac{S}{S'}\right)^{N-2}\right)^{2} |\Omega \setminus B_{S'}|^{\frac{N-2}{N}}$$
$$\leq \operatorname{cap}(\Omega) - c\left(1-\frac{S}{S'}\right)^{2} |\Omega \setminus B_{S'}|^{\frac{N-2}{N}},$$

concluding the proof. □

We can now prove Lemma 5.5.2.

Proof of Lemma 5.5.2. Let us assume without loss of generality that the ball achieving the asymmetry of Ω is B_1. As was already mentioned, we are going to show that there exists an $\tilde{\Omega}$ of the form $\lambda(\Omega\cap B_S)$ for suitable S and λ satisfying all the desired properties. Let us set

$$b_k := \frac{|\Omega\setminus B_{2-2^{-k}}|}{|B_1|} \leq 1.$$

Note that by Theorem 1.1.7 we can assume that $b_1 \leq 2\mathcal{A}(\Omega)$ is as small as we wish (independently on Ω up to choose δ sufficiently small). Lemma 5.5.3 gives

$$\operatorname{cap}(\Omega) - c\left(\frac{2^{-(k+1)}}{2-2^{-(k+1)}}\right)^{2} b_{k+1}^{\frac{N-2}{N}} \geq \operatorname{cap}(B_1)(1-b_k)^{\frac{N-2}{N}} \geq \operatorname{cap}(B_1) - \operatorname{cap}(B_1)b_k,$$

which implies

$$cb_{k+1} \leq \left(4^{N/(N-2)}\right)^{k} (D(\Omega)+Cb_k)^{\frac{N}{N-2}}. \tag{5.5.1}$$

We now claim that there exists $\bar{k}$ such that

$$b_{\bar{k}} \leq D(\Omega).$$

Indeed, otherwise by (5.5.1) we would get

$$b_{k+1} \leq C\left(4^{N/(N-2)}\right)^{k} (D(\Omega)+Cb_k)^{\frac{N}{N-2}} \leq \left(4^{N/(N-2)}\right)^{k} C' b_k^{\frac{N}{N-2}} \leq M^k b_k^{\frac{N}{N-2}}$$

for all $k \in \mathbb{N}$, where $M = M(N)$. Iterating the last inequality, we obtain

$$b_{k+1} \leq M^{\frac{N-2}{2}k}(Mb_1)^{(\frac{N}{N-2})^k} \xrightarrow[k\to\infty]{} 0$$

if $b_1 < \min(1, M^{-2})$, which by Theorem 1.1.7 we can assume up to choosing $\delta = \delta(N) \ll 1$.

We define $\tilde{\Omega}$ as a properly rescaled intersection of Ω with a ball. Let $\bar{k}$ be such that $b_{\bar{k}} \leq D(\Omega)$

$$\tilde{\Omega} := \left(\frac{|B_1|}{|\Omega \cap B_R|}\right)^{\frac{1}{N}} (\Omega \cap B_R) = (1 - b_{\bar{k}})^{-\frac{1}{N}} (\Omega \cap B_S),$$

where $S := 2 - 2^{-\bar{k}} \leq 2$. Note that $|\tilde{\Omega}| = |B_1|$. We now check all the remaining properties:

- *Bound on the diameter*:

$$\operatorname{diam}(\tilde{\Omega}) \leq 2 \cdot 2(1 - D(\Omega))^{-\frac{1}{N}} \leq 4(1-\delta)^{-\frac{1}{N}} \leq 4.$$

  up to choose $\delta = \delta(N) \ll 1$.
- *Bound on the deficit*:

$$\begin{aligned} D(\tilde{\Omega}) &= \operatorname{cap}(\tilde{\Omega}) - \operatorname{cap}(B_1) = \operatorname{cap}(\Omega \cap B_S)(1 - b_{\bar{K}})^{-\frac{N-2}{N}} - \operatorname{cap}(B_1) \\ &\leq \operatorname{cap}(\Omega)(1 - b_{\bar{k}})^{-\frac{N-2}{N}} - \operatorname{cap}(B_1) \\ &\leq \operatorname{cap}(\Omega) - \operatorname{cap}(B_1) + \frac{2(N-2)\operatorname{cap}(\Omega)}{N} b_{\bar{k}} \leq C(N) D(\Omega). \end{aligned}$$

  since $b_{\bar{k}} \leq D(\Omega) \ll 1$ and, in particular, $\operatorname{cap}(\Omega) \leq 2\operatorname{cap}(B_1)$.
- *Bound on the asymmetry*: Let $r := (1 - b_{\bar{k}})^{-1} \in (1, 2)$, that is r is such that $\tilde{\Omega} = r^N(\Omega \cap B_S)$ with $S = 2 - 2^{-\bar{k}} \leq 2$. Let x_0 be such that $B_1(x_0)$ is a minimizing ball for $\mathcal{A}(\tilde{\Omega})$. Then, recalling that $b_{\bar{k}} = |B_1|^{-1}|\Omega \setminus B_S| \leq C(N) D(\Omega)$,

$$\begin{aligned} |B_1|\mathcal{A}(\Omega) &\leq |\Omega \Delta B_1\left(\frac{x_0}{r}\right)| \leq |\Omega \setminus B_S| + \left|(\Omega \cap B_S)\Delta B_1\left(\frac{x_0}{r}\right)\right| \\ &\leq CD(\Omega) + \left|(\Omega \cap B_S)\Delta B_{\frac{1}{r}}\left(\frac{x_0}{r}\right)\right| \\ &\quad + \left|B_{\frac{1}{r}}\left(\frac{x_0}{r}\right)\Delta B_1\left(\frac{x_0}{r}\right)\right| \\ &\leq CD(\Omega) + \frac{|B_1|}{r^N}\mathcal{A}(\tilde{\Omega}) + |B_1|\left(1 - \frac{1}{r^N}\right) \\ &\leq CD(\Omega) + |B_1|\mathcal{A}(\tilde{\Omega}) + C(N) b_{\bar{k}} \\ &\leq CD(\Omega) + |B_1|\mathcal{A}(\tilde{\Omega}). \end{aligned}$$

□

5.6 Proof of Theorem 1.1.5

In order to reduce it to Theorem 5.2.1, we need to start with a set which is already close to a ball. In the case of the absolute capacity, thanks to Theorem 1.1.7, this can be achieved by assuming the deficit sufficiently small (the quantitative inequality being trivial in the other regime). The next lemma contains the same "qualitative" result in the case of the relative capacity.

Lemma 5.6.1. *For all* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, R) > 0$ *such that if* $\Omega \subset B_R$ *is an open set with* $|\Omega| = 1$ *and*

$$\mathrm{cap}_R(\Omega) \le \mathrm{cap}_R(B_1) + \delta$$

then

$$\alpha_R(\Omega) \le \varepsilon.$$

Proof. We argue by contradiction. Suppose there exists an $\varepsilon_0 > 0$ and a sequence of open sets $\Omega_j \subset B_R$ with $|\Omega_j| = |B_1|$ such that $\alpha_R(\Omega_j) \ge \varepsilon_0$ but

$$\mathrm{cap}_R(B_1) \le \mathrm{cap}_R(\Omega_j) \le \mathrm{cap}_R(B_1) + 1/j.$$

We denote by $u_j \in W_0^{1,2}(B_R)$ the capacitary potential of Ω_j. The above inequality grants that

$$\int_{B_R} |\nabla u_j|^2 dx \to \mathrm{cap}_R(B_1).$$

Thus, up to a not-relabelled subsequence, there exists a function u in $W_0^{1,2}(B_R)$ such that $u_j \rightharpoonup u$ in $W_0^{1,2}(B_R)$, $u_j \to u$ in $L^2(B_R)$ and almost everywhere in B_R. We define Ω as $\{u = 1\}$. From the lower semi-continuity of Dirichlet integral we have that

$$\mathrm{cap}_R(\Omega) \le \int_{B_R} |\nabla u|^2 dx \le \liminf \int_{B_R} |\nabla u_j|^2 dx = \mathrm{cap}_R(B_1).$$

On the other hand, we have $1_\Omega \ge \limsup 1_{\Omega_j}$, meaning that $|\Omega_j \setminus \Omega| \to 0$ and $|\Omega| \ge |\Omega_j| = |B_1|$. The isocapacitary inequality then implies that $\Omega = B_1$. In particular, $|\Omega_j| = |\Omega|$ for all j and

$$|\Omega \setminus \Omega_j| = |\Omega_j \setminus \Omega| \to 0,$$

and thus $1_{\Omega_j} \to 1_\Omega = 1_{B_1}$ in $L^1(B_R)$. Hence by Lemma 5.1.2, (ii), $\alpha_R(\Omega_j) \to 0$, a contradiction. □

We have now all the ingredients to prove Theorem 1.1.5.

Proof of Theorem 1.1.5. We will consider separately the cases of the absolute and relative capacity.

Absolute capacity. First note that if $D(\Omega) \ge \delta_0$ then, since $\mathcal{A}(\Omega) \ge 2$,

$$D(\Omega) \ge 4\frac{\delta_0}{4} \ge \frac{\delta_0}{4}\mathcal{A}(\Omega)^2.$$

Hence we can assume that $D(\Omega)$ is as small as we wish as long as the smallness depends only on N. We now δ_0 smaller than the constant δ in Lemma 5.5.2 and, assuming that $D(\Omega) \le \delta_0$, we use Lemma 5.5.2 to find a set $\tilde{\Omega}$ with $\mathrm{diam}(\tilde{\Omega}) \le d = d(N)$ and satisfying all the properties there. In particular, up to a translation we can assume that $\tilde{\Omega} \subset B_d$. Up to choosing δ_0 smaller we can apply Theorem 1.1.7 and Lemma 5.1.2 (ii) to ensure that $\alpha(\tilde{\Omega}) \le \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(N, d) = \varepsilon_0(N)$ is the constant appearing in the statement of Theorem 5.2.1. This, together with Lemma 5.1.2, (i), grants that

$$D(\tilde{\Omega}) \geq c(N)\alpha(\tilde{\Omega}) \geq c(N)\mathcal{A}(\tilde{\Omega})^2.$$

Hence, by Lemma 5.5.2 and assuming that $\mathcal{A}(\Omega) \geq CD(\Omega)$ (since otherwise there is nothing to prove),

$$D(\Omega) \geq cD(\tilde{\Omega}) \geq cA(\tilde{\Omega})^2 \geq c\mathcal{A}(\tilde{\Omega})^2 \geq c\mathcal{A}(\Omega)^2 - CD(\Omega)^2$$

from which the conclusion easily follows since $D(\Omega) \leq \delta_0 \ll 1$.

Relative capacity. Since $\alpha_R(\Omega) \leq C(R, N)$ by arguing as in the previous case, we can assume that $\mathrm{cap}_R(\Omega) - \mathrm{cap}_R(B_1) \leq \delta_1(N, R) \ll 1$. By Lemma 5.6.1 we can assume that $\alpha_R(\Omega) \leq \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(N, R)$ is the constant in Theorem 5.2.1. Hence

$$\mathrm{cap}_R(\Omega) - \mathrm{cap}_R(B_1) \geq c(N, R)\alpha_R(\Omega) \geq c(N, R)|\Omega \Delta B_1|^2.$$

□

Chapter 6

Quantitative isocapacitary inequality: case of the general p

The first steps in the proof of quantitative isocapacitary inequality in the case of general p repeat those for the case $p = 2$. We go through them briefly indicating the differences.

6.1 Stability for bounded sets with small asymmetry

Our aim is to prove the following theorem, and then reduce to the general case. Note that also here instead of Fraenkel asymmetry we are using another notion defined in Section 5.1.

Theorem 6.1.1. *There exist constants $c = c(N, p, R)$, $\varepsilon_0 = \varepsilon_0(N, p, R)$ such that for any open set $\Omega \subset B_R$ with $|\Omega| = |B_1|$ and $\alpha(\Omega) \leq \varepsilon_0$ the following inequality holds:*

$$\mathrm{cap}_p(\Omega) - \mathrm{cap}_p(B_1) \geq c\,\alpha(\Omega).$$

To prove Theorem 6.1.1 we are going to argue by contradiction. Suppose that the theorem doesn't hold. Then there exists a sequence of open sets $\tilde{\Omega}_j \subset B_R$ such that

$$|\tilde{\Omega}_j| = |B_1|, \quad \alpha(\tilde{\Omega}_j) = \varepsilon_j \to 0, \quad \frac{\mathrm{cap}_p(\tilde{\Omega}_j) - \mathrm{cap}_p(B_1)}{\varepsilon_j} \leq \sigma^4 \tag{6.1.1}$$

for some small σ to be chosen later. We then perturb the sequence $\tilde{\Omega}_j$ so that it converges to B_1 in a smooth way. More precisely, we are going to show the following.

Theorem 6.1.2 (Selection Principle)**.** *There exists $\tilde{\sigma} = \tilde{\sigma}(N, p, R)$ such that if one has a contradicting sequence $\tilde{\Omega}_j$ as the one described above in (6.1.1) with $\sigma < \tilde{\sigma}$, then there exists a sequence of smooth open sets U_j such that*

(i) $|U_j| = |B_1|$,

(ii) $\partial U_j \to \partial B_1$ *in* C^k *for every* k,

(iii) $\limsup_{j\to\infty} \frac{\mathrm{cap}_p(U_j)-\mathrm{cap}_p(B_1)}{\alpha(\Omega_j)} \leq C\sigma$ *for some* $C = C(N, p, R)$ *constant,*

(iv) *the barycenter of every* Ω_j *is in the origin.*

6.2 Proof of Theorem 6.1.2: Existence and first properties

6.2.1 Getting rid of the volume constraint

The first step consists in getting rid of the volume constraint in the isocapacitary inequality. We again use the following function

$$f_\eta(s) := \begin{cases} -\frac{1}{\eta}(s-\omega_N), & s \leq \omega_N \\ -\eta(s-\omega_N), & s \geq \omega_N \end{cases}$$

and consider the new functional

$$\mathscr{C}_\eta(\Omega) = \mathrm{cap}_p(\Omega) + f_\eta(|\Omega|).$$

Remember that f_η satisfies

$$\eta(t-s) \leq f_\eta(s) - f_\eta(t) \leq \frac{(t-s)}{\eta} \qquad \text{for all } 0 \leq s \leq t.$$

Analogously to Lemma 5.3.2, we have the following.

Lemma 6.2.1. *There exists an* $\hat{\eta} = \hat{\eta}(N, p, R) > 0$ *such that the only minimizer of* $\mathscr{C}_{\hat{\eta}}$ *in the class of sets contained in* B_R *is a translate of the unit unit ball* B_1.

Moreover, there exists $c = c(N, p, R) > 0$ *such that for any ball* B_r *with* $0 < r < R$*, one has*

$$\mathscr{C}_{\hat{\eta}}(B_r) - \mathscr{C}_{\hat{\eta}}(B_1) \geq c|r-1|.$$

Proof. The proof is similar to the proofs of Lemma 5.3.1 and Lemma 5.3.2. The computation can be repeated almost verbatim, the only difference being scaling of capacity:

$$\mathrm{cap}_p(B_r) = r^{N-p}\,\mathrm{cap}(B_1).$$

□

6.2.2 A penalized minimum problem

The sequence in Theorem 6.1.2 is obtained by solving the following minimum problem.

$$\min\{\mathscr{C}^p_{\hat{\eta},j}(\Omega) : \Omega \subset B_R\}, \tag{6.2.1}$$

where

$$\mathscr{C}^p_{\hat\eta,j}(\Omega) = \mathscr{C}^p_{\hat\eta}(\Omega)+\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega)-\varepsilon_j)^2} = \mathrm{cap}_p(\Omega)+f_{\hat\eta}(|\Omega|)+\sqrt{\varepsilon_j^2+\sigma^2(\alpha(\Omega)-\varepsilon_j)^2}.$$

As in the case $p = 2$, we construct a minimizing sequence with equibounded perimeter. Recall also that a set is said to be p-quasi-open if it is the zero level set of a $W^{1,p}$ function.

Lemma 6.2.2. *There exists $\sigma_0 = \sigma_0(N, p, R) > 0$ such that for every $\sigma < \sigma_0$ the minimum in (6.2.1) is attained by a p-quasi-open set Ω_j. Moreover, perimeters of Ω_j are bounded independently on j.*

Proof. The proof is identical to the proof of Lemma 5.3.3 in case of absolute capacity up to changing 2 to p. □

6.2.3 First properties of the minimizers

We finish the section by stating some properties of minimizers.

Lemma 6.2.3. *Let $\{\Omega_j\}$ be a sequence of minimizers for (6.2.1). Then the following properties hold:*

(i) $|\alpha(\Omega_j) - \varepsilon_j| \le 3\sigma\varepsilon_j$;

(ii) $\big||\Omega_j| - |B_1|\big| \le C\sigma^4\varepsilon_j$;

(iii) *up to translations* $\Omega_j \to B_1$ *in* L^1,

(iv) $0 \le \mathscr{C}_{\hat\eta}(\Omega_j) - \mathscr{C}_{\hat\eta}(B_1) \le \sigma^4\varepsilon_j$.

Proof. The lemma follows easily from Lemma 6.2.1. To prove (iii) we need to recall that the sets Ω_j have bounded perimeter. For more details see the proof of Lemma 5.3.4. □

6.3 Proof of Theorem 6.1.2: Regularity

In this section, we show that the sequence of minimizers of (6.2.1) converges smoothly to the unit ball. This will be done by relying on the regularity theory for free boundary problems established in [DP05].

6.3.1 Linear growth away from the free boundary

Let u_j be the capacitary potential for Ω_j, a minimizer of (6.2.1). Let us also introduce $v_j := 1 - u_j$, so that $\Omega_j = \{v_j = 0\}$. Following [DP05] we are going to show that

$$v_j(x) \sim \mathrm{dist}(x, \Omega_j).$$

where the implicit constant depends only on R. The above estimate is obtained by suitable comparison estimates. We will need to have some compactness properties, so we first prove Hölder continuity, also with the constant depending only on R.

Hölder continuity

The proof is based on establishing a decay estimate for the integral oscillation of u_j and it is almost identical to the case of 2-capacity.

We are going to use the following growth result for p-harmonic functions. The proof can be found, for example, in [Giu03, Theorem 7.7].

Lemma 6.3.1. *Suppose $w \in W^{1,2}(\Omega)$ is p-harmonic, $x_0 \in \Omega$. Then there exists a constant $c = c(N,p)$, $\beta > 0$ such that for any balls $B_{r_1}(x_0) \subset B_{r_2}(x_0) \Subset \Omega$*

$$\fint_{B_{r_1}(x_0)} |\nabla w|^p \le c \left(\frac{r_1}{r_2}\right)^{p\beta - p} \fint_{B_{r_2}(x_0)} |\nabla w|^p.$$

Remark 6.3.2. In [Giu03] the result is proven for the functions in De Giorgi class. One can prove that in the case of p-harmonic functions the inequality holds for $\beta = 1$, but we are not going to need that.

To prove Hölder continuity of u_j we will use several times the following comparison estimates.

Lemma 6.3.3. *Let u_j be the capacitary potential of a minimizer for (6.2.1). Let $A \subset B_R$ be an open set with Lipschitz boundary and let $w \in W^{1,p}(\mathbb{R}^N)$ coincide with u_j on the boundary of A in the sense of traces.*

Then

$$\int_A |\nabla u_j|^p dx - \int_A |\nabla w|^p dx \le \left(\frac{1}{\hat{\eta}} + C\sigma\right) \left|A \cap (\{u = 1\} \Delta \{w = 1\})\right|.$$

Moreover, if $u_j \le w \le 1$ in A, then

$$\int_A |\nabla u_j|^p dx + \frac{\hat{\eta}}{2} \left|A \cap (\{u = 1\} \Delta \{w = 1\})\right| \le \int_A |\nabla w|^p dx,$$

provided $\sigma \le \sigma(R)$.

Proof. The proof is the same as the proof of Lemma 5.4.5, modulo changing exponents from 2 to p. The idea is to consider $\tilde{u}$ defined as

$$\begin{cases} \tilde{u} = w & \text{in } A \\ \tilde{u} = u & \text{else} \end{cases}$$

and take $\tilde{\Omega} = \{\tilde{u} = 1\}$ as a comparison domain. □

Remark 6.3.4. Note that if w is p-harmonic in A, then by Lemma B.0.2

- if $p \ge 2$, we have

$$\int_A |\nabla u|^p dx - \int_A |\nabla w|^p dx \ge c \int_A |\nabla (u - w)|^p dx;$$

- if $1 < p < 2$, then

$$\int_A |\nabla u|^p dx - \int_A |\nabla w|^p dx \geq c \int_A |\nabla(u-w)|^2 \left(|\nabla w|^2 + |\nabla(u-w)|^2\right)^{\frac{p-2}{2}} dx.$$

Hence the first inequality from the lemma becomes

- for $p \geq 2$

$$\int_A |\nabla(u-w)|^p dx \leq C(p)\left(\frac{1}{\hat{\eta}} + C\sigma\right)\big|A \cap (\{u=1\}\Delta\{w=1\})\big|; \tag{6.3.1}$$

- for $1 < p < 2$

$$\begin{aligned}\int_A |\nabla(u-w)|^2 \left(|\nabla w|^2 + |\nabla(u-w)|^2\right)^{\frac{p-2}{2}} dx \\ \leq C(p)\left(\frac{1}{\hat{\eta}} + C\sigma\right)\big|A \cap (\{u=1\}\Delta\{w=1\})\big|.\end{aligned} \tag{6.3.2}$$

Lemma 6.3.5. *There exists $\alpha \in (0, 1/2)$ such that every minimizer of (6.2.1) satisfies $u_j \in C^{0,\alpha}(\overline{B_R})$. Moreover, the Hölder norm is bounded by a constant independent on j.*

Proof. The proof is similar to the proof of Lemma 5.4.8. As usual, we drop the subscript j. By Campanato's criterion it is enough to show that

$$\int_{B_r(x_0)} \left|u - \fint_{B_r(x_0)} u\right|^p \leq Cr^{N+2\alpha}$$

for all r small enough (say less that $1/2$). We are going to show instead that

$$\phi(r) := \int_{B_r(x_0)} |\nabla u|^p \leq Cr^{N+2\alpha-p}$$

which yields the previous inequality by Poincaré.
Let $x_0 \in B_R$. Let w be the p-harmonic extension of u in $B_{r'}(x_0)$. By Lemma 6.3.1 we know that

$$\int_{B_r(x_0)} |\nabla w|^p \leq C\left(\frac{r}{r'}\right)^{N+p\beta-p} \int_{B_{r'}(x_0)} |\nabla w|^p.$$

Let $g := u - w$. Then

$$\begin{aligned}\int_{B_r(x_0)} |\nabla u|^p dx &\leq C\int_{B_r(x_0)} |\nabla w|^p dx + C\int_{B_r(x_0)} |\nabla g|^p \\ &\leq C\left(\frac{r}{r'}\right)^{N+p\beta-p} \int_{B_{r'}(x_0)} |\nabla w|^p dx + C\int_{B_{r'}(x_0)} |\nabla g|^p.\end{aligned}$$

We want to show the following bound:

$$\int_{B_{r'}(x_0)} |\nabla g|^p \leq C_\varepsilon (r')^N + C\varepsilon \int_{B_{r'}(x_0)} |\nabla w|^p \tag{6.3.3}$$

for $\varepsilon < \varepsilon_0 = \varepsilon_0(N,p)$. By (6.3.1) it is immediate for $p \geq 2$ (even without the second summand on the right hand side). For $1 < p < 2$ we use Young inequality to get

$$\int_{B_{r'}(x_0)} |\nabla g|^p \leq C_\varepsilon \int_{B_{r'}(x_0)} |\nabla g|^2 \left(|\nabla w|^2 + |\nabla g|^2\right)^{(p-2)/2} + \varepsilon \int_{B_{r'}(x_0)} \left(|\nabla w|^2 + |\nabla g|^2\right)^{p/2}$$
$$\leq C_\varepsilon (r')^N + C\varepsilon \int_{B_{r'}(x_0)} (|\nabla w|^p + |\nabla g|^p),$$

yielding (6.3.3) for ε small enough. Note that in the last inequality we used (6.3.2). So we have

$$\begin{aligned}\int_{B_r(x_0)} |\nabla u|^p dx &\leq C\left(\left(\frac{r}{r'}\right)^{N+p\beta-p} + \varepsilon\right) \int_{B_{r'}(x_0)} |\nabla w|^p dx + C_\varepsilon (r')^N \\ &\leq C\left(\left(\frac{r}{r'}\right)^{N+p\beta-p} + \varepsilon\right) \int_{B_{r'}(x_0)} |\nabla u|^p dx + C_\varepsilon (r')^N \\ &\quad + C\left(\left(\frac{r}{r'}\right)^{N+p\beta-p} + \varepsilon\right) \int_{B_{r'}(x_0)} |\nabla g|^p dx \\ &\leq C\left(\left(\frac{r}{r'}\right)^{N+p\beta-p} + \varepsilon\right) \int_{B_{r'}(x_0)} |\nabla u|^p dx + C_\varepsilon (r')^N,\end{aligned}$$

which gives us

$$\phi(r) \leq c\left(\left(\frac{r}{r'}\right)^{N+p\beta-p} + \varepsilon\right) \phi(r') + C_\varepsilon (r')^N.$$

Using Lemma A.0.2 we obtain

$$\phi(r) \leq c\left(\left(\frac{r}{r'}\right)^{N+p\beta-p} \phi(r') + C r^N\right)$$

for any $r < r' < 1$. In particular,

$$\phi(r) \leq c\left(\|u\|^p_{L^p(\mathbb{R}^N)} + C\right) r^{N+p\beta-p}.$$

□

Lipschitz continuity and density estimates on the boundary

The following lemma is an analogue of [DP05, Lemma 3.2] and it will give us uniform Lipschitz continuity.

Lemma 6.3.6. *There exists $M = M(N,p,R)$ such that if u_j is a minimizer for (6.2.1) and $v_j = 1 - u_j$ satisfies $v_j(x_0) = 0$, then*

$$\sup_{B_{r/4}(x_0)} v_j \leq Mr.$$

Proof. **Step 1.** We argue by contradiction and get a sequence v_{j_k}, $B_{r_k}(y_k) \subset B_R$ such that $v_{j_k}(y_k) = 0$, $\sup_{B_{r_k/4}(y_k)} v_{j_k} \geq kr_k$. We now consider blow-ups around y_k, that is, we define

$$\tilde{v}_k(x) := \frac{v_{j_k}(y_k + r_k x)}{r_k}.$$

Note that $\tilde{v}_k$ minimizes

$$\int_{\mathbb{R}^N} |\nabla v|^p dx + r_k^{-n} f_{\hat{\eta}}(r_k^N \{v = 0\}) + r_k^{-n} \sqrt{\varepsilon_{j_k}^2 + \sigma^2(\alpha(\Phi_k(\{v = 0\})) - \varepsilon_{j_k})^2}$$

among functions such that $\Phi_k(\{v = 0\}) \subset B_R$, where $\Phi_k(x) = y_k + r_k x$. Additionally, we have

$$\tilde{v}_k(0) = 0, \ \sup_{B_{1/4}} \tilde{v}_k \geq k.$$

We define a function

$$d_k(x) := \operatorname{dist}(x, \{\tilde{v}_k = 0\})$$

and a set

$$V_k := \left\{ x \in B : d_k(x) \leq \frac{1 - |x|}{3} \right\}.$$

The following properties hold for V_k:

- $B_{1/4} \subset V_k$. This is due to the fact that $\tilde{v}_k(0) = 0$ and thus $d_k(x) \leq |x|$.
- $m_k := \sup_{x \in V_k} (1 - |x|)\tilde{v}_k(x) \geq \frac{3k}{4}$. This follows from the previous property.

Since $\tilde{v}_k$ is continuous and $(1 - |x|)\tilde{v}_k(x) = 0$ on ∂B, m_k is obtained at some point $x_k \in V_k$. We notice that the following holds for x_k:

$$\tilde{v}_k(x_k) = \frac{m_k}{1 - |x_k|} \geq m_k \geq \frac{3k}{4}; \quad \delta_k := d_k(x_k) \leq \frac{1 - |x_k|}{3}.$$

We now take projections of x_k onto $\{\tilde{v}_k = 0\}$, that is, we consider a sequence z_k such that $z_k \in \{\tilde{v}_k = 0\} \cap B$, $|z_k - x_k| = \delta_k$. Note that $B_{2\delta_k}(z_k) \subset B$. Moreover, $B_{\delta_k/2}(z_k) \subset V_k$ since for any $x \in B_{\delta_k/2}(z_k)$ we have

$$1 - |x| \geq 1 - |x_k| - |x_k - x| \geq 1 - |x_k| - \frac{3}{2}\delta_k \geq \frac{1 - |x_k|}{2}.$$

Now let us show that $\sup_{B_{\delta_k/4}} \tilde{v}_k \sim \tilde{v}_k(x_k)$. Indeed, for the upper bound it is enough to notice that

$$\sup_{B_{\delta_k/2}(z_k)} \tilde{v}_k \leq \tilde{v}_k(x_k)(1 - |x_k|) \sup_{B_{\delta_k/2}(z_k)} \frac{1}{1 - |x|} \leq 2\tilde{v}_k(x_k).$$

On the other hand, since $B_{\delta_k}(x_k) \subset \{\tilde{v}_k > 0\}$, $\tilde{v}_k$ is p-harmonic in $B_{\delta_k}(x_k)$ and thus, by Harnack inequality (see, for example, [Lin06, Theorem 2.20]), we get

$$\sup_{B_{\delta_k/4}(z_k)} \tilde{v}_k \geq \inf_{B_{4\delta_k/5}(x_k)} \tilde{v}_k \geq c_0 \sup_{B_{4\delta_k/5}(x_k)} \tilde{v}_k - C \geq \frac{c_0}{2} \tilde{v}_k(x_k),$$

where the last inequality holds for k big enough.

Step 2. We now consider blow-ups around z_k. We define

$$\hat{v}_k(x) := \frac{\tilde{v}_k(z_k + \frac{\delta_k}{2}x)}{\tilde{v}_k(x_k)}.$$

We note that

$$\sup_B \hat{v}_k \leq 2, \quad \sup_{B_{1/2}} \hat{v}_k \geq c_0/2, \quad \hat{v}_k(0) = 0$$

and $\hat{v}_k$ is a minimizer of

$$\int_{\mathbb{R}^N} |\nabla v|^p dx + (\delta_k/2)^{p-n}\, \tilde{v}_k(x_k)^{-p} r_k^{-n} f_{\hat{\eta}}((\delta_k/2)^N r_k^N \{v=0\}) + \\ (\delta_k/2)^{p-n}\, \tilde{v}_k(x_k)^{-p} r_k^{-n} \sqrt{\varepsilon_{j_k}^2 + \sigma^2(\alpha(\Psi_k(\{v=0\})) - \varepsilon_{j_k})^2} \tag{6.3.4}$$

among functions such that $\Psi_k(\{v=0\}) \subset B_R$, where $\Psi_k(x) = y_k + r_k z_k + \frac{r_k \delta_k x}{2}$.

We introduce w_k - a p-harmonic continuation of $\hat{v}_k$ in $B_{3/4}$:

$$\begin{cases} \operatorname{div}(|w_k|^{p-2}\nabla w_k) = 0 \text{ in } B_{3/4}, \\ w_k = \hat{v}_k \text{ in } B_{3/4}^c. \end{cases}$$

By maximum principle (see, for example, [Lin06, Corollary 2.21]) $w_k > 0$ in $B_{3/4}$ and thus

$$\{\hat{v}_k = 0\}\Delta\{w_k = 0\} = \{\hat{v}_k = 0\} \cap B_{3/4}.$$

So now, remembering that $\hat{v}_k$ is a minimizer for (6.3.4) and using w_k as a comparison function we obtain

$$\int_{B_{3/4}} |\nabla \hat{v}_k|^p\, dx \leq \int_{B_{3/4}} |\nabla w_k|^p\, dx + \frac{C}{k^p}.$$

From this we can infer the convergence of $v_k - w_k$ to zero. In order to do that, we define

$$v_k^s = s\hat{v}_k + (1-s)w_k.$$

Now, we write

$$\frac{C}{k^p} \geq \int_{B_{3/4}} |\nabla \hat{v}_k|^p\, dx - \int_{B_{3/4}} |\nabla w_k|^p\, dx = p\int_0^1 \frac{1}{s}\, ds \int_{B_{3/4}} |\nabla v_k^s|^{p-2}\nabla v_k^s \cdot \nabla(v_k^s - w_k)\, dx \\ = p\int_0^1 \frac{1}{s}\, ds \int_{B_{3/4}} \left(|\nabla v_k^s|^{p-2}\nabla v_k^s - |\nabla w_k|^{p-2}\nabla w_k\right) \cdot \nabla(v_k^s - w_k)\, dx.$$

We want to show that the convergence of $v_k - w_k$ is strong. We use Lemma B.0.1 for that. We need to consider two cases. For $p \geq 2$ by the inequality (B.0.1) we get

$$\int_{B_{3/4}} |\nabla \hat{v}_k - \nabla w_k|^p\, dx \leq \frac{C}{k^p},$$

yielding the strong convergence of $\hat{v}_k - w_k$ to zero in $W^{1,p}(B_{3/4})$ as $k \to \infty$. To deal with the case $1 < p < 2$, we observe that $\hat{v}_k$ is bounded in $D^{1,p}(B_{3/4})$. We infer that w_k is bounded in $D^{1,p}$ too and hence, by the inequality (B.0.2) we also have the strong convergence of $\hat{v}_k - w_k$ to zero in $W^{1,p}(B_{3/4})$ as $k \to \infty$.

We recall now that $\hat{v}_k$ is equibounded in $C^{0,\alpha}(B_{3/4})$ and hence, up to a non-relabelled subsequence we have that $\hat{v}_k$ converges to some continuous function v_∞ locally uniformly and weakly in $W^{1,p}$. This means that also w_k converges to v_∞ weakly in $W^{1,p}$. Elliptic regularity for w_k tell us that w_k is locally bounded in $C^{1,\beta}(B_{3/4})$ and so up to a subsequence w_k converges to v_∞ strongly in $W^{1,p}$. But then $v_\infty \geq 0$ is p-harmonic with $v_\infty(0) = 0$, $\sup_{B_{1/2}} v_\infty \geq c_0/2$. This contradicts the maximum principle. □

The following lemma is an analogue of [DP05, Lemma 4.2] and the proof is almost identical.

Lemma 6.3.7 (non-degeneracy). *For $\kappa < 1$, $\gamma > p-1$ there exists a constant $c_{nd} = c_{nd}(N, \kappa, \gamma, R)$ such that if u_j is a minimizer for (6.2.1) and $v_j = 1 - u_j$ satisfies*

$$\left(\fint_{\partial B_r(x_0)} v_j^\gamma \right)^{\frac{1}{\gamma}} \leq cr, \tag{6.3.5}$$

then $v_j = 0$ in $B_{\kappa r}(x_0)$.

Proof. We will omit the subscript j for convenience and write v instead of v_j. None of the bounds will depend on j.

First, we want to show that if c is small enough (depending only on N,κ,γ, and R), then the inequality(6.3.5) yields $B_{\kappa r} \subset B_R$. The idea is that v is sufficiently big outside of B_R. Indeed, by maximum principle

$$v(x) \geq 1 - u_{B_R}(x) = 1 - \frac{R^{\frac{N-p}{p-1}}}{|x|^{\frac{N-p}{p-1}}}. \tag{6.3.6}$$

If $B_{\kappa r}(x_0) \setminus B_R \neq \emptyset$, then $|B_r(x_0) \setminus B_{R+\frac{1-\kappa}{2}r}| \geq c(\kappa)|B_r|$ and, using (6.3.6), we get

$$\left(\fint_{\partial B_r(x_0)} v_j^\gamma \right)^{\frac{1}{\gamma}} \geq c(\kappa) \left(1 - \frac{R^{\frac{N-p}{p-1}}}{|R + \frac{1-\kappa}{2} r|^{\frac{N-p}{p-1}}} \right) \geq c(\kappa, N, R) r,$$

contradicting (6.3.5) for c_{nd} small enough.

Now we define

$$\varepsilon := \frac{1}{\sqrt{\kappa r}} \sup_{B_{\sqrt{\kappa r}}} v \leq C \frac{1}{r} \left(\fint_{\partial B_r(x_0)} v^\gamma \right)^{\frac{1}{\gamma}} \leq C\, c_{nd},$$

where we used Harnack inequality for p-subharmonic functions (see [Tru67, Theorem 1.3]). We set $\varphi(x) = \varphi(|x|)$ to be the solution of

$$\begin{cases} \Delta_p \varphi = 0 & \text{in } B_{\sqrt{\kappa r}} \setminus B_{\kappa r}, \\ \varphi = 0 & \text{on } \partial B_{\kappa r}, \\ \varphi = 1 & \text{on } \partial B_{\sqrt{\kappa r}}, \end{cases}$$

defined as 0 in $B_{\kappa r}$. Now we define

$$v' := \varepsilon\sqrt{\kappa r}\varphi.$$

Note that $v' \geq v$ on $\partial B_{\sqrt{\kappa r}}$. Finally, we define

$$w := \min(v, v') \text{ in } B_{\sqrt{\kappa r}}, \quad w := v \text{ in } B^c_{\sqrt{\kappa r}},$$

and we use w as a comparison function in (6.2.1). We notice that $\{w = 0\} \supset \{v = 0\}$ and so from minimality of v we conclude

$$\int_{B_{\sqrt{\kappa r}}} |\nabla v|^p \, dx + \frac{\hat{\eta}}{2} |\{w = 0\} \setminus \{v = 0\}| \leq \int_{B_{\sqrt{\kappa r}}} |\nabla w|^p \, dx = \int_{B_{\sqrt{\kappa r}} \setminus B_{\kappa r}} |\nabla w|^p \, dx.$$

Now we use the definition of w, positivity of v' in $B_{\sqrt{\kappa r}} \setminus B_{\kappa r}$, and convexity of $t \to t^p$ to get

$$\int_{B_{\kappa r}} |\nabla v|^p \, dx + \frac{\hat{\eta}}{2} |B_{\kappa r} \cap \{v > 0\}| \leq \int_{B_{\sqrt{\kappa r}} \setminus B_{\kappa r}} (|\nabla w|^p - |\nabla v|^p) \, dx$$
$$\leq p \int_{B_{\sqrt{\kappa r}} \setminus B_{\kappa r}} |\nabla w|^{p-2} \nabla w \cdot \nabla(w - v) \, dx = p \int_{\partial B_{\kappa r}} |\nabla v'|^{p-2} v \nabla v' \cdot \nu.$$

From the definition of v' we have

$$|\nabla v'| \leq C \frac{\varepsilon\sqrt{\kappa r}}{\kappa r - \sqrt{\kappa r}} \leq C\varepsilon.$$

So we have

$$\int_{B_{\kappa r}} |\nabla v|^p \, dx + \frac{\hat{\eta}}{2} |B_{\kappa r} \cap \{v > 0\}| \leq C\varepsilon^{p-1} \int_{\partial B_{\kappa r}} v.$$

On the other hand, by trace inequality and Young inequality, and remembering the definition of ε, we can get

$$\int_{\partial B_{\kappa r}} v \leq C \left(\frac{1}{r} \int_{B_{\kappa r}} v \, dx + \int_{B_{\kappa r}} |\nabla v| \, dx \right)$$
$$\leq C \left(\sqrt{\kappa}\varepsilon |B_{\kappa r} \cap \{v > 0\}| + \frac{1}{p} \int_{B_{\kappa r}} |\nabla v|^p \, dx + \frac{p-1}{p} |B_{\kappa r} \cap \{v > 0\}| \right)$$
$$\leq C(1 + \varepsilon) \left(\int_{B_{\kappa r}} |\nabla v|^p \, dx + |B_{\kappa r} \cap \{v > 0\}| \right).$$

Bringing it all together, we get

$$\int_{B_{\kappa r}} |\nabla v|^p \, dx + \frac{\hat{\eta}}{2} |B_{\kappa r} \cap \{v > 0\}| \leq C\varepsilon^{p-1}(1+\varepsilon) \left(\int_{B_{\kappa r}} |\nabla v|^p \, dx + |B_{\kappa r} \cap \{v > 0\}| \right).$$

It remains to choose c from the statement of the lemma small enough for $C\varepsilon^{p-1}(1+\varepsilon)$ to be smaller that $\min\{\frac{1}{2}, \frac{\hat{\eta}}{4}\}$.

□

As in Section 4 of [DP05] these two lemmas imply Lipschitz continuity of minimizers and density estimates on the boundary of minimizing domains.

Lemma 6.3.8. *Let v_j be as above, $\Omega_j = \{v_j = 0\}$. Then Ω_j is open and there exist constants $C = C(N, p, R)$, $\rho_0 = \rho_0(N, p, R) > 0$ such that*

(i) *for every $x \in B_R$*
$$\frac{1}{C}\operatorname{dist}(x, \Omega_j) \leq v_j \leq C \operatorname{dist}(x, \Omega_j);$$

(ii) *v_j are equi-Lipschitz;*

(iii) *for every $x \in \partial\Omega_j$ and $r \leq \rho_0$*
$$\frac{1}{C} \leq \frac{|\Omega_j \cap B_r(x)|}{|B_r(x)|} \leq \left(1 - \frac{1}{C}\right).$$

Applying [DP05, Theorem 5.1] to v_j (for more details on the proof see [AC81, Theorem 4.5]) we also have the following

Lemma 6.3.9. *Let v_j be as above, then there exists a Borel function q_{u_j} such that*

$$\operatorname{div}(|\nabla v_j|^{p-2}\nabla v_j) = q_{v_j}\mathcal{H}^{N-1} \llcorner \partial^*\Omega_j. \tag{6.3.7}$$

Moreover, $0 < c \leq -q_{v_j} \leq C$, $c = c(N, p, R)$, $C = C(N, p, R)$ and $\mathcal{H}^{N-1}(\partial\Omega_j \backslash \partial^*\Omega_j) = 0$.

Since Ω_j converge to B_1 in L^1 by Lemma 6.2.3, the density estimates also give us the following convergence of boundaries.

Lemma 6.3.10. *Let Ω_j be minimizers of (6.2.1). Then every limit point of Ω_j with respect to L^1 convergence is the unit ball centered at some $x_\infty \in B_R$. Moreover, the convergence holds also in the Kuratowski sense.*

Corollary 6.3.11. *In the setting of Lemma 6.3.10, for every $\delta > 0$ there exists j_δ such that for $j \geq j_\delta$*
$$B_{1-\delta}(x_j) \subset \Omega_j \subset B_{1+\delta}(x_j)$$
for some $x_j \in B_R$.

6.3.2 Higher regularity of the free boundary

In order to address the higher regularity of $\partial\Omega_j$, we need to prove that q_{v_j} is smooth. This will be done by using the Euler-Lagrange equations for our minimizing problem. We defined Ω_j in such a way that the following minimizing property holds

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_j|^p dx + f_{\hat{\eta}}(|\{v_j = 0\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\{v_j = 0\}) - \varepsilon_j)^2} \\ &\leq \int_{\mathbb{R}^N} |\nabla v|^p dx + f_{\hat{\eta}}(|\{v = 0\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\{v = 0\}) - \varepsilon_j)^2} \end{aligned} \tag{6.3.8}$$

for any $v \in W^{1,2}(\mathbb{R}^N)$ such that $0 \leq v \leq 1$, $\{v = 0\} \subset B_R$.

To write Euler-Lagrange equations for v_j, we need to have (6.3.8) for $v_j \circ \Phi$ where Φ is a diffeomorphism of $\mathbb{R}^N$ close to the identity. Note that to make sure that $\{v_j \circ \Phi = 0\}$ is contained in B_R one needs to know that $\operatorname{dist}(\{v_j = 0\}, \partial B_R) > 0$. This follows from Corollary 6.3.11, up to translating Ω_j. More precisely we will get the following optimality condition

$$(p-1)q_{v_j}^p - \frac{\sigma^2(\alpha(\Omega_j) - \varepsilon_j)}{\sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\Omega_j) - \varepsilon_j)^2}} \left(|x - x_{\Omega_j}| - \left(\fint_{\Omega_j} \frac{y - x_{\Omega_j}}{|y - x_{\Omega_j}|} dy \right) \cdot x \right) = \Lambda_j$$

for some constant $\Lambda_j > 0$. As in the case $p = 2$, these equations are an immediate consequence of the following lemma which is analogous to Lemma 5.4.15.

Lemma 6.3.12. *There exists j_0 such that for any $j \geq j_0$ and any two points x_1 and x_2 in the reduced boundary of Ω_j the following equality holds:*

$$\begin{aligned}(p-1)q_{v_j}^p(x_1) &- \frac{\sigma^2(\alpha(\Omega_j) - \varepsilon_j)}{\sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\Omega_j) - \varepsilon_j)^2}} \left(|x_1 - x_{\Omega_j}| - \left(\fint_{\Omega_j} \frac{y - x_{\Omega_j}}{|y - x_{\Omega_j}|} dy \right) \cdot x_1 \right) \\ = (p-1)q_{v_j}^p(x_2) &- \frac{\sigma^2(\alpha(\Omega_j) - \varepsilon_j)}{\sqrt{\varepsilon_j^2 + \sigma^2(\alpha(\Omega_j) - \varepsilon_j)^2}} \left(|x_2 - x_{\Omega_j}| - \left(\fint_{\Omega_j} \frac{y - x_{\Omega_j}}{|y - x_{\Omega_j}|} dy \right) \cdot x_2 \right).\end{aligned}$$

Proof. The proof repeats the proof of Lemma 5.4.15. The only computation which is different is the perturbation of p-capacity. One can argue as in the proof of [FZ16, Lemma 3.19] to get

$$\begin{aligned}\operatorname{cap}_p(\Omega_\tau^\rho) - \operatorname{cap}_p(\Omega) \leq \tau\rho^N(p-1)\left(|q(x_1)|^p - |q(x_2)|^p\right) \int_{B_1 \cap \{y_1 = 0\}} \phi(|y|)dy \\ + o(\tau)\rho^N + o_\tau(\rho^N),\end{aligned}$$

where Ω_τ^ρ is defined as in the proof of Lemma 5.4.15. □

Lemma 6.3.13 (Smoothness of $\boldsymbol{q_v}$)**.** *There exist constants $\delta = \delta(N,p,R) > 0$, $j_0 = j_0(N,p,R)$, $\sigma_0 = \sigma_0(N,p,R) > 0$ such that for every $j \geq j_0$, $\sigma \leq \sigma_0$ the functions q_{v_j} belong to $C^\infty(\mathcal{N}_\delta(\partial\Omega_j))$.*

Moreover, for every k there exists a constant $C = C(k,N,p,R)$ such that

$$\|q_{v_j}\|_{C^k(\mathcal{N}_\delta(\partial\Omega_j))} \leq C$$

for every $j \geq j_0$.

Proof. The proof is identical to the proof of Lemma 5.4.16 since we have a similar Euler-Lagrange equation for q. □

Now we want to apply the results of [DP05]. We can't apply them directly, since the equation there is slightly different. More precisely, in [DP05] the authors are considering solutions of the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \mathcal{H}^{N-1} \llcorner \partial\{u > 0\},$$

whereas v_j satisfies

$$\operatorname{div}\left(|\nabla v_j|^{p-2}\nabla v_j\right) = q_{v_j}\mathcal{H}^{N-1} \llcorner \partial\{v_j > 0\}.$$

However, since q_{v_j} is smooth, the proof works in exactly the same way (see also Appendix of [FZ16] for the same result for a slightly different equation, the proof becomes more involved in that case). The idea is that flatness improves in smaller balls if the free boundary is sufficiently flat in some ball.

First, we need to recall the definition of flatness for the free boundary, see [AC81, Definition 7.1] (here it is applied to v).

Definition 6.3.14. Let $\mu_-, \mu_+ \in (0,1]$. A weak solution u of (6.3.7) is said to be of class $F(\mu_-, \mu_+, \infty)$ in $B_\rho(x_0)$ in a direction $\nu \in S^{N-1}$ if $x_0 \in \partial\{v = 0\}$ and

$$\begin{cases} v(x) = 0 & \text{for } (x - x_0)\cdot\nu \le -\mu_-\rho, \\ v(x) \ge q_v(x_0)((x - x_0)\cdot\nu - \mu_+\rho) & \text{for } (x - x_0)\cdot\nu \ge \mu_+\rho. \end{cases}$$

We are going to use that flat free boundaries are smooth. The following theorem is a slight generalization of [DP05, Theorem 9.1] and we omit the proof since it is almost identical.

Theorem 6.3.15. *Let u be a weak solution of (6.3.7) and assume that q_v is Lipschitz continuous. There are constants γ, μ_0, κ, C such that if v is of class $F(\mu, 1, \infty)$ in $B_{4\rho}(x_0)$ in some direction $\nu \in S^{N-1}$ with $\mu \le \mu_0$ and $\rho \le \kappa\mu^2$, then there exists a $C^{1,\gamma}$ function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ with $\|f\|_{C^{1,\gamma}} \le C\mu$ such that*

$$\partial\{v = 0\} \cap B_\rho(x_0) = (x_0 + \operatorname{graph}_\nu f) \cap B_\rho(x_0), \tag{6.3.9}$$

where $\operatorname{graph}_\nu f = \{x \in \mathbb{R}^N : x\cdot\nu = f(x - x\cdot\nu)\nu)\}$.

Moreover if $q_v \in C^{k,\gamma}$ in some neighborhood of $\{u_j = 1\}$, then $f \in C^{k+1,\gamma}$ and $\|f\|_{C^{k+1,\gamma}} \le C(N, R, \|q_v\|_{C^{k,\gamma}})$.

Proof of Theorem 6.1.2. The proof goes exactly as the proof of Theorem 5.2.2 using Lemma 6.2.3 and Theorem 6.3.15. □

6.4 Reduction to bounded sets

To complete the proof of Theorem 1.1.8 one needs to show that one can consider only sets with uniformly bounded diameter. To this end let us introduce the following.

Definition 6.4.1. Let Ω be an open set in $\mathbb{R}^N$ with $|\Omega| = |B_1|$. Then we define the deficit of Ω as the difference between its p-capacity and the p-capacity of the unit ball:

$$D(\Omega) = \operatorname{cap}_p(\Omega) - \operatorname{cap}_p(B_1).$$

Here is the key lemma for reducing Theorem 1.1.8 to Theorem 6.1.1.

Lemma 6.4.2. *There exist constants $C = C(N,p)$, $\delta = \delta(N,p) > 0$ and $d = d(N,p)$ such that for any $\Omega \subset \mathbb{R}^N$ open with $|\Omega| = |B_1|$ and $D(\Omega) \leq \delta$, we can find a new set $\tilde{\Omega}$ enjoying the following properties*

1. $\operatorname{diam}(\tilde{\Omega}) \leq d$,
2. $|\tilde{\Omega}| = |B_1|$,
3. $D(\tilde{\Omega}) \leq CD(\Omega)$,
4. $\mathcal{A}(\tilde{\Omega}) \geq \mathcal{A}(\Omega) - CD(\Omega)$.

We are going to define $\tilde{\Omega}$ as a suitable dilation of $\Omega \cap B_S$ for some large S. Hence, we first show the following estimates on the p-capacity of $\Omega \cap B_S$.

Lemma 6.4.3. *Let $S' > S$. Then there exists a constant $c = c(S', N, p)$ such that for any open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| = |B_1|$ the following inequalities hold:*

$$\operatorname{cap}_p(B_1)\left(1 - \frac{|\Omega \setminus B_S|}{|B_1|}\right)^{\frac{N-p}{N}} \leq \operatorname{cap}_p(\Omega \cap B_S) \leq \operatorname{cap}(\Omega) - c\left(1 - \frac{S}{S'}\right)^p |\Omega \setminus B_{S'}|^{\frac{N-p}{N}}.$$

Proof. The first inequality is a direct consequence of the classical isocapacitary inequality. To prove the second one we are going to use the estimates for the capacitary potential of B_S for which the exact formula can be written. Denote by u_Ω and u_S the capacitary potentials of Ω and $\Omega \cap B_S$ respectively. We first write

$$\operatorname{cap}_p(\Omega \cap B_S) = \operatorname{cap}_p(\Omega) + \int_{\mathbb{R}^N} |\nabla u_S|^p - |\nabla u_\Omega|^p = \operatorname{cap}_p(\Omega) - \int_{(\Omega \cap B_S)^c} (|\nabla u_\Omega|^p - |\nabla u_S|^p).$$

Let us show that

$$\int_{(\Omega \cap B_S)^c} |\nabla u_\Omega|^p - |\nabla u_S|^p \geq c(p) \int_{\Omega \setminus B_S} |\nabla u_S|^p.$$

To that end we need to consider two cases. For both we will be using an inequality of Lemma B.0.2. For $p \geq 2$ we have

$$\begin{aligned}\int_{(\Omega\cap B_S)^c} |\nabla u_\Omega|^p - |\nabla u_S|^p &\geq c(p)\int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^p \\ &+ p\int_{(\Omega\cap B_S)^c} |\nabla u_S|^{p-2}\nabla u_S\cdot\nabla(u_\Omega - u_S) \\ &= c(p)\int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^p - p\int_{(\Omega\cap B_S)^c} \operatorname{div}(|\nabla u_S|^{p-2}\nabla u_S)(u_\Omega - u_S) \\ &\quad + p\int_{\partial(\Omega\cap B_S)} (u_\Omega - u_S)\nabla u_S\cdot\nu d\mathcal{H}^{N-1} \\ &= c(p)\int_{(\Omega\cap B_S)^c} |\nabla(u_\Omega - u_S)|^p \geq c(p)\int_{\Omega\setminus B_S} |\nabla(u_\Omega - u_S)|^p \\ &= c(p)\int_{\Omega\setminus B_S} |\nabla u_S|^p\end{aligned}$$

As for the case $1 < p < 2$, we have

$$\begin{aligned}\int_{(\Omega\cap B_S)^c} |\nabla u_\Omega|^p - |\nabla u_S|^p &\geq c(p)\int_{(\Omega\cap B_S)^c} \left(|\nabla u_\Omega|^2 + |\nabla u_S|^2\right)^{\frac{p-2}{2}} |\nabla(u_\Omega - u_S)|^2 \\ &\quad + p\int_{(\Omega\cap B_S)^c} |\nabla u_S|^{p-2}\nabla u_S\cdot\nabla(u_\Omega - u_S) \\ &= c(p)\int_{(\Omega\cap B_S)^c} \left(|\nabla u_\Omega|^2 + |\nabla u_S|^2\right)^{\frac{p-2}{2}} |\nabla(u_\Omega - u_S)|^2 \\ &\geq c(p)\int_{\Omega\setminus B_S} \left(|\nabla u_\Omega|^2 + |\nabla u_S|^2\right)^{\frac{p-2}{2}} |\nabla(u_\Omega - u_S)|^2 \\ &= c(p)\int_{\Omega\setminus B_S} |\nabla u_S|^p,\end{aligned}$$

where in the last equality we used that $u_\Omega \equiv 1$ in Ω. We would like to show that $\int_{\Omega\setminus B_S} |\nabla u_S|^p$ cannot be too small. To this end let us set $v_S = 1 - u_S$. By Sobolev's embedding we get

$$\int_{\Omega\setminus B_S} |\nabla u_S|^p = \int_{\Omega\setminus B_S} |\nabla v_S|^p \geq c(N)\left(\int_{\Omega\setminus B_S} |v_S|^{p^*}\right)^{\frac{p}{p^*}},$$

where p^* is the Sobolev exponent. Let us denote by z_S the capacitary potential of B_S:

$$z_S = \left(1 - \frac{S^{\frac{N-p}{p-1}}}{|x|^{\frac{N-p}{p-1}}}\right)_+.$$

By the maximum principle, $v_S \geq z_S$, hence

$$\begin{aligned}\int_{\Omega\setminus B_S} |v_S|^{p^*} &\geq \int_{\Omega\setminus B_S} |z_S|^{p^*} \\ &\geq \int_{\Omega\setminus B_{S'}} |z_S|^{p^*} \geq \left(1 - \left(\frac{S}{S'}\right)^{\frac{N-p}{p-1}}\right)^{p^*} |\Omega\setminus B_{S'}|.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{cap}_p(\Omega \cap B_S) &\leq \operatorname{cap}_p(\Omega) - c(N)\left(1 - \left(\frac{S}{S'}\right)^{\frac{N-p}{p-1}}\right)^p |\Omega \setminus B_{S'}|^{\frac{N-p}{N}} \\ &\leq \operatorname{cap}_p(\Omega) - c\left(1 - \frac{S}{S'}\right)^p |\Omega \setminus B_{S'}|^{\frac{N-p}{N}},\end{aligned}$$

concluding the proof. □

We can now prove Lemma 6.4.2.

Proof of Lemma 6.4.2. The proof is almost identical to the proof of Lemma 5.5.2, using Lemma 6.4.3 in the place of Lemma 5.5.3. □

6.5 Proof of Theorem 1.1.8

In order to reduce it to Theorem 6.1.1, we need to start with a set which is already close to a ball. Thanks to Theorem 1.1.7, this can be achieved by assuming the deficit sufficiently small (the quantitative inequality being trivial in the other regime).

We have now all the ingredients to prove Theorem 1.1.8.

Proof of Theorem 1.1.8. The proof is identical to the proof of Theorem 1.1.9 using Lemma 6.4.2 and Theorem 6.1.1 in place of Lemma 5.5.2 and Theorem 5.2.1. □

Chapter 7

Liquid drops

In this chapter we apply Selection Principle to prove Theorem 1.1.9.

7.1 Preliminary results

In this section we collect some results obtained in [DPHV19] which will be useful in the proof of regularity.

Convention 7.1.1 (Universal constants)**.** Let $A > 0$ be a positive constant. We say that

- the parameters β, K, Q with $\beta \geq 1$ are *controlled by* A if

$$\beta + K + \frac{1}{K} + Q \leq A;$$

- a constant is *universal* if it depends only on the dimension N and on A.

Note that in particular universal constants *do not depend* on the size of the container where the minimization problem is set.

In the following theorem we collect some properties of minimizers. For the proofs we refer the reader to [DPHV19].

Theorem 7.1.2. *Let $E \subset \mathbb{R}^N$ be a set of finite measure. Then*

(i) *there exists a unique pair $(u_E, \rho_E) \in \mathcal{A}(E)$ minimizing $\mathcal{G}_{\beta,K}(E)$. Moreover,*

$$u_E + K\rho_E = \mathcal{G}_{\beta,K}(E) \qquad \text{in } E,$$

and

$$0 \leq u_E \leq \mathcal{G}_{\beta,K}(E), \qquad 0 \leq K\rho_E \leq \mathcal{G}_{\beta,K}(E)\mathbf{1}_E.$$

In particular, $\rho_E \in L^p$ for all $p \in [1, \infty]$ with

$$\|\rho_E\|_p \leq C(N, \beta, K, 1/|E|).$$

(ii) (Euler-Lagrange equation) *If E is a minimizer of* ($\mathcal{P}_{\beta,K,Q,R}$), *then*

$$\int_{\partial^* E} \mathrm{div}_E \eta \, d\mathcal{H}^{n-1} - Q^2 \int_{\mathbb{R}^n} a_E \Big(|\nabla u_E|^2 \mathrm{div}\eta - 2\nabla u_E \cdot (\nabla \eta \, \nabla u_E) \Big) \, dx$$
$$- Q^2 \, K \int_{\mathbb{R}^n} \rho_E^2 \mathrm{div}\eta \, dx = 0$$

for all $\eta \in C^1_c(B_R;\mathbb{R}^n)$ with $\int_E \mathrm{div}\eta \, dx = 0$.

(iii) (Compactness) *Let $K_h, Q_h \in \mathbb{R}$, $\beta_h \geq 1$ and $R_h \geq 1$ be such that*

$$K_h \to K > 0\,, \quad \beta_h \to \beta \geq 1\,, \quad R_h \to R \geq 1\,, \quad Q_h \to Q \geq 0,$$

when $h \to \infty$. For every $h \in \mathbb{N}$ let E_h be a minimizer of ($\mathcal{P}_{\beta_h,K_h,Q_h,R_h}$). Then, up to a non relabelled subsequence, there exists a set of finite perimeter E such that

$$\lim_{h\to\infty} |E \Delta E_h| = 0.$$

Moreover, E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) *and*

$$\mathcal{F}_{\beta,K,Q}(E) = \lim_{h\to\infty} \mathcal{F}_{\beta_h,K_h,Q_h}(E_h), \qquad \lim_{h\to\infty} P(E_h) = P(E).$$

Let $A > 0$. For the following properties we require that β, K and Q are controlled by A.

(iv) (Boundedness of the normalized Dirichlet) *There exists a universal constant $C_{\mathrm{e}} > 0$ such that, if E is a minimizer of* ($\mathcal{P}_{\beta,K,Q,R}$), *then for all $x \in \overline{B_R}$,*

$$Q^2 D_E(x,r) = \frac{Q^2}{r^{N-1}} \int_{B_r(x)} |\nabla u|^2 \, dx \leq C_{\mathrm{e}}.$$

(v) (Density estimates) *There exist universal constants C_{o}, $C_{\mathrm{i}} > 0$ and $\bar{r} > 0$ such that, if E is a minimizer of* ($\mathcal{P}_{\beta,K,Q,R}$), *then*[1]

$$\frac{1}{C_{\mathrm{i}}} r^{N-1} \leq P(E, B_r(x)) \leq C_{\mathrm{o}} r^{N-1} \qquad \text{for all } x \in \partial E \text{ and } r \in (0,\bar{r}),$$

and

$$\frac{1}{C_{\mathrm{i}}} \leq \frac{|B_r(x) \cap E|}{|B_r(x)|} \leq C_{\mathrm{o}} \qquad \text{for all } x \in E \text{ and } r \in (0,\bar{r}).$$

(vi) (Excess improvement) *There exists a universal constant $C_{\mathrm{dec}} > 0$ such that for all $\lambda \in (0,1/4)$ there exists $\varepsilon_{\mathrm{dec}} = \varepsilon_{\mathrm{dec}}(N,A,\lambda) > 0$ satisfying the following: if E is a minimizer of* ($\mathcal{P}_{\beta,K,Q,R}$) *and*

$$x \in \partial E, \quad r + Q^2 D_E(x,r) + \mathbf{e}_E(x,r) \leq \varepsilon_{\mathrm{dec}},$$

[1] Here and in the sequel we will always work with the representative of E such that

$$\partial E = \left\{ x : \frac{|B_r(x) \setminus E|}{|B_r(x)|} \cdot \frac{|B_r(x) \cap E|}{|B_r(x)|} > 0 \quad \text{for all } r > 0 \right\},$$

see [Mag12, Proposition 12.19].

then

$$Q^2 D_E(x,\lambda r) + \mathbf{e}_E(x,\lambda r) \le C_{\mathrm{dec}}\lambda\Big(\mathbf{e}_E(x,r) + Q^2 D_E(x,r) + r\Big).$$

(vii) (Decay of the Dirichlet energy) *There exists a universal constant $C_{\mathrm{dir}} > 0$ such that for all $\lambda \in (0,1/2)$ there exists $\varepsilon_{\mathrm{dir}} = \varepsilon_{\mathrm{dir}}(N,A,\lambda)$ satisfying the following: if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$ and*

$$r + \mathbf{e}_E(x,r,e_N) \le \varepsilon_{\mathrm{dir}},$$

then

$$D_E(x,\lambda r) \le C_{\mathrm{dir}}\lambda\Big(D_E(x,r) + r\Big).$$

Proof. The proofs of (i), (iii), (iv), (v), (vi) and (vii) can be found respectively in [DPHV19, Proposition 2.1, Proposition 5.1, Lemma 6.5, Proposition 6.4, Proposition 6.6, Theorem 7.1, Proposition 7.6]. There is no detailed proof of (ii) in [DPHV19]. Moreover, we believe that the formula given in [DPHV19, Corollary 3.3] is slightly wrong. For those reasons we give a proof of (ii) here.

We start by showing the following identity for any $\rho \in L^2(\mathbb{R}^n)$:

$$\begin{aligned}\inf&\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 : u \in D^1(\mathbb{R}^n), -\operatorname{div}(a_E\nabla u) = \rho\right\}\\ &= \inf\left\{\int_{\mathbb{R}^n} \frac{|V|^2}{a_E} : V \in L^2(\mathbb{R}^n;\mathbb{R}^n), -\operatorname{div}(V) = \rho\right\}.\end{aligned} \tag{7.1.1}$$

Right-hand side is trivially not larger than the left-hand side, as we can take $V = a_E\nabla u$ as a competitor for the right-hand side. So we only need to show that

$$\begin{aligned}\inf&\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 : u \in D^1(\mathbb{R}^n), -\operatorname{div}(a_E\nabla u) = \rho\right\}\\ &\le \inf\left\{\int_{\mathbb{R}^n} \frac{|V|^2}{a_E} : V \in L^2(\mathbb{R}^n;\mathbb{R}^n), -\operatorname{div}(V) = \rho\right\}.\end{aligned}$$

We use that the infimum is achieved in both cases by convexity. Hence, the right-hand side has a minimizer V_0 and it satisfies the corresponding Euler-Langrange equation, that is

$$\frac{V_0}{a_E}\cdot X = 0 \qquad \text{for any } X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ such that } \operatorname{div} X = 0.$$

But that gives us that $\frac{V_0}{a_E} = \nabla u_0$ for some $u_0 \in D^1(\mathbb{R}^n)$ and u_0. Since $\operatorname{div}(V_0) = \rho$, we get $-\operatorname{div}(a_E\nabla u_0) = \rho$ and thus

$$\begin{aligned}\inf&\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 : u \in D^1(\mathbb{R}^n), -\operatorname{div}(a_E\nabla u) = \rho\right\} \le \int_{\mathbb{R}^n} a_E|\nabla u_0|^2\\ &\le \int_{\mathbb{R}^n}\frac{|V_0|^2}{a_E} = \inf\left\{\int_{\mathbb{R}^n} \frac{|V|^2}{a_E} : V \in L^2(\mathbb{R}^n;\mathbb{R}^n), -\operatorname{div}(V) = \rho\right\},\end{aligned}$$

which finishes the proof of (7.1.1).

Suppose now that E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and $(u,\rho)\in\mathcal{A}$ is the pair minimizing $\mathcal{G}(E)$. We fix a vector field $\eta\in C_c^\infty(B_R;\mathbb{R}^n)$ with $\int_E \mathrm{div}\eta\,dx=0$ and, following [DPHV19, Lemma 3.1], we define

$$\varphi_t(x):=x+t\eta,\qquad u_t:=u\circ\varphi_t^{-1},\qquad \tilde\rho_t:=\det(\nabla\varphi_t^{-1})\rho\circ\varphi_t^{-1}.$$

By [DPHV19, Lemma 3.1] we have

$$-\,\mathrm{div}(a_{E_t}A_t\nabla u_t)=\rho_t, \tag{7.1.2}$$

where $E_t=\varphi_t(E)$, $A_t=\det(\nabla\varphi_t^{-1})(\nabla\varphi_t^{-1})^{-t}(\nabla\varphi_t^{-1})^{-1}$. Note that $|E_t|=|E|=|B_1|$ since $\int_E \mathrm{div}\eta\,dx=0$.

Now we recall that E is a minimizer and we use (7.1.1) to get that

$$\begin{aligned}
&P(E)+Q^2\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx+K\int_E\rho^2\,dx\right\}\\
&=\min\left\{P(E)+Q^2\inf_{(u,\rho)\in\mathcal{A}(E)}\left\{\int_{\mathbb{R}^n}a_E|\nabla u|^2+K\int_E\rho^2\right\}:|E|=|B_1|,E\subset B_R\right\}\\
&=\min\left\{P(E)+Q^2\inf_{\rho\in L^2(\mathbb{R}^n)}\left\{\inf_{-\mathrm{div}(a_E\nabla u)=\rho}\int_{\mathbb{R}^n}a_E|\nabla u|^2+K\int_E\rho^2\right\}:|E|=|B_1|,E\subset B_R\right\}\\
&=\min\left\{P(E)+Q^2\inf_{\rho\in L^2(\mathbb{R}^n)}\left\{\inf_{-\mathrm{div}(V)=\rho}\int_{\mathbb{R}^n}\frac{|V|^2}{a_E}+K\int_E\rho^2\right\}:|E|=|B_1|,E\subset B_R\right\}\\
&\le P(E_t)+Q^2\left\{\int_{\mathbb{R}^n}a_{E_t}|A_t\nabla u_t|^2+K\int_{E_t}\rho_t^2\right\},
\end{aligned}$$

where for the last inequality we used (7.1.2) and the fact that $|E_t|=|B_1|$ by the choice of η. To get Euler-Lagrange equation it remains to expand the last quantity in terms of t. It is well known (see, for example, [Mag12, Theorem 17.8]) that

$$P(E_t)=P(E)+t\int_{\partial^*E}\mathrm{div}_E\,\eta+o(t), \tag{7.1.3}$$

where $\mathrm{div}_E\,\eta=\mathrm{div}\,\eta-\nu_E\cdot(\nabla\eta[\nu_E])$. As for the other part of the energy, using change of variables, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} a_{E_t}|A_t\nabla u_t|^2 + K\int_{E_t}\rho_t^2 &= \int_{\mathbb{R}^n} a_E|\det(\nabla\varphi_t^{-1})(\nabla\varphi_t)^t(\nabla\varphi_t)(\nabla\varphi_t)^{-t}\nabla u|^2\det(\nabla\varphi_t)\\
&\quad + K\int_E \rho^2\det{}^2(\nabla\varphi_t^{-1})\det(\nabla\varphi_t)\\
&= \int_{\mathbb{R}^n} a_E|(1-t\,\mathrm{div}\,\eta)(\mathrm{Id}+t(\nabla\eta)^t)(\mathrm{Id}+t\nabla\eta)(\mathrm{Id}-t(\nabla\eta)^t)\nabla u|^2(1+t\,\mathrm{div}\,\eta)\\
&\quad + K\int_E\rho^2(1-t\,\mathrm{div}\,\eta)+o(t)\\
&= \int_{\mathbb{R}^n} a_E|(1-t\,\mathrm{div}\,\eta)(\mathrm{Id}+t\nabla\eta)\nabla u|^2(1+t\,\mathrm{div}\,\eta)+K\int_E\rho^2(1-t\,\mathrm{div}\,\eta)+o(t)\\
&= \int_{\mathbb{R}^n} a_E|(1+t(-\mathrm{div}\,\eta\,\mathrm{Id}+\nabla\eta))\nabla u|^2(1+t\,\mathrm{div}\,\eta)+K\int_E\rho^2(1-t\,\mathrm{div}\,\eta)+o(t)\\
&= \int_{\mathbb{R}^n} a_E\left(|\nabla u|^2+t\left(-2\,\mathrm{div}\,\eta|\nabla u|^2+2\nabla u\cdot(\nabla\eta\nabla u)\right)\right)(1+t\,\mathrm{div}\,\eta)\\
&\quad + K\int_E\rho^2(1-t\,\mathrm{div}\,\eta)+o(t)\\
&= \int_{\mathbb{R}^n} a_E|\nabla u|^2+K\int_E\rho^2+t\left(\int_{\mathbb{R}^n} a_E\left(-\mathrm{div}\,\eta|\nabla u|^2+2\nabla u\cdot(\nabla\eta\nabla u)\right)-K\int_E\rho^2\,\mathrm{div}\,\eta\right)\\
&\qquad + o(t),
\end{aligned}$$

where for the first equality we used that $\nabla\varphi_t = \left(\nabla\varphi_t^{-1}\right)^{-1}\circ\varphi_t$. Bringing it all together we get that for any $\eta\in C_c^\infty(B_R;\mathbb{R}^n)$ with $\int_E \mathrm{div}\eta\,dx = 0$ for any t we have

$$\begin{aligned}
&P(E)+Q^2\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx+K\int_E\rho^2\,dx\right\} \le P(E)+Q^2\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2+K\int_E\rho^2\right\}\\
&+t\left(\int_{\partial^*E}\mathrm{div}_E\,\eta+Q^2\left(\int_{\mathbb{R}^n} a_E\left(-\mathrm{div}\,\eta|\nabla u|^2+2\nabla u\cdot(\nabla\eta\nabla u)\right)-K\int_E\rho^2\,\mathrm{div}\,\eta\right)\right)+o(t),
\end{aligned}$$

which gives us (ii). □

We state now the ε-regularity theorem.

Theorem 7.1.3 ([DPHV19, Theorem 1.2]). *Given $N\ge 3$, $A>0$ and $\vartheta\in(0,1/2)$, there exists $\varepsilon_{\mathrm{reg}}=\varepsilon_{\mathrm{reg}}(N,A,\vartheta)>0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) with $Q+\beta+K+\frac{1}{K}\le A$, $x\in\partial E$ and*

$$r+\mathbf{e}_E(x,r)+Q^2\,D_E(x,r)\le\varepsilon_{\mathrm{reg}},$$

then $E\cap\mathbf{C}(x,r/2)$ coincides with the epi-graph of a $C^{1,\vartheta}$ function. In particular, $\partial E\cap\mathbf{C}(x,r/2)$ is a $C^{1,\vartheta}$ $(N-1)$-dimensional manifold.

7.2 Closeness to the ball

In this section we deduce the L^∞-closeness of minimizers to the unitary ball in the small charge regime. Let us start with the following proposition.

Proposition 7.2.1 (**L^1**-closeness to the ball)**.** *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then, up to translations, $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$.*

Proof. By the quantitative isoperimetric inequality, [FMP08, Theorem 1.1], for every $h \in \mathbb{N}$ there exists a point $x_h \in \mathbb{R}^N$ such that

$$|E_h \Delta B_1(x_h)|^2 \leq C\left(P(E_h) - P(B_1)\right)$$

for some constant $C = C(N) > 0$ which depends only on n. By translating each set E_h we can assume without loss of generality that the following inequality holds:

$$|E_h \Delta B_1|^2 \leq C\left(P(E_h) - P(B_1)\right). \tag{7.2.1}$$

By the minimality of E_h we have

$$\begin{aligned}\mathcal{F}_{\beta,K,Q_h,R}(E_h) &= P(E_h) + Q_h^2\,\mathcal{G}_{\beta,K}(E_h)\\ &\leq P(B_1) + Q_h^2\,\mathcal{G}_{\beta,K}(B_1) = \mathcal{F}_{\beta,K,Q_h,R}(B_1), \quad \forall\, h \in \mathbb{N}.\end{aligned}$$

Hence, (7.2.1) yields

$$|E_h \Delta B_1|^2 \leq C\left(P(E_h) - P(B_1)\right) \leq C\,Q_h^2\,\mathcal{G}_{\beta,K}(B_1) \quad \forall\, h \in \mathbb{N},$$

for some constant $C = C(N) > 0$ which depends only on the dimension n.

Then $Q_h \to 0$ implies $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$. □

Thanks to the density estimates (see Theorem 7.1.2 (v)), we can improve the convergence of Proposition 7.2.1.

Proposition 7.2.2 (**L^∞**-closeness to the ball)**.** *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then, up to translations, $E_h \to \overline{B}_1$ and $\partial E_h \to \partial B_1$ in the Kuratowski sense.*

7.3 Higher regularity

In this section we improve Theorem 7.1.3. To be more precise, we deduce the partial $C^{2,\vartheta}$ regularity of minimizers. The first step is to obtain better regularity for a couple $(u,\rho) \in \mathcal{A}(E)$, where $E \subset \mathbb{R}^N$ is a minimizer of the problem $(\mathcal{P}_{\beta,K,Q,R})$: we prove that u is $C^{1,\eta}$-regular up to the boundary of E.

Lemma 7.3.1. *Given a minimizer E of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho)\in\mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$. Assume that $\partial E\cap\mathbf{C}(x_0,r)$ is a $C^{1,\vartheta}$-manifold. Then for every $\gamma\in(0,1)$ there exist $0<\bar r\le r$ and $C>0$ such that the following inequality holds true*

$$Q^2\int_{B_{\tilde r}(x_0)}|\nabla u|^2\,dx\le C\,\tilde r^{N-\gamma}$$

for every $\tilde r\le\bar r$.

Proof. Fix $\gamma\in(0,1)$. Choose $\lambda\in(0,1/4)$ such that

$$(1+C_{\text{dec}})\,\lambda\le\lambda^{1-\gamma},$$

where C_{dec} is as in Theorem 7.1.2 (vi). Let $s=s(\lambda)<\frac{1}{2}$ be such that

$$C_{\text{dir}}(C_{\text{e}}+1)\,s(\lambda)\le\frac{\varepsilon_{\text{dec}}(\lambda)}{2},\tag{7.3.1}$$

where ε_{dec} , C_{dir} and C_{e} are as in Theorem 7.1.3 and Theorem 7.1.2 (vii), (iv). Define

$$\varepsilon(\lambda):=\min\left\{s^{N-1}\frac{\varepsilon_{\text{dec}}(\lambda)}{2},\varepsilon_{\text{dir}}(\lambda)\right\}.$$

Since $\partial E\cap\mathbf{C}(x_0,r)$ is regular, we can take a radius $0<\bar r<\min\left(r,1,\frac{1}{Q^2}\right)$ such that

$$\bar r+\mathbf{e}_E(x_0,\bar r)\le\varepsilon(\lambda).$$

Then, thanks to the definition of $\varepsilon(\lambda)$, Theorem 7.1.2 (vii), (iv), and (7.3.1) we have

$$Q^2\,D_E(x_0,s\bar r)\le C_{\text{dir}}s\,(Q^2\,D_E(x_0,\bar r)+Q^2\bar r)\le C_{\text{dir}}(C_{\text{e}}+1)s\le\frac{\varepsilon_{\text{dec}}(\lambda)}{2}.\tag{7.3.2}$$

Furthermore, notice that

$$s\bar r+\mathbf{e}_E(x_0,s\bar r)\le\bar r+\frac{1}{s^{N-1}}\mathbf{e}_E(x_0,\bar r)\le\frac{\varepsilon_{\text{dec}}(\lambda)}{2}.\tag{7.3.3}$$

Combining (7.3.2) and (7.3.3), we have

$$s\bar r+Q^2\,D_E(x_0,s\bar r)+\mathbf{e}_E(x_0,s\bar r)\le\varepsilon_{\text{dec}}(\lambda).$$

The hypothesis of Theorem 7.1.2 (vi) is satisfied, hence (recall that $\lambda s\bar r\le\varepsilon_{\text{dec}}(\lambda)$)

$$\begin{aligned}&Q^2\,D_E(x_0,\lambda s\bar r)+\mathbf{e}_E(x_0,\lambda s\bar r)+\lambda s\bar r\le\lambda^{1-\gamma}\,\left(\mathbf{e}_E(x_0,s\bar r)+Q^2\,D_E(x_0,s\bar r)+s\bar r\right)\\&\le\lambda^{1-\gamma}\varepsilon_{\text{dec}}(\lambda)\le\varepsilon_{\text{dec}}(\lambda).\end{aligned}$$

Exploiting again Theorem 7.1.2, we obtain

$$\begin{aligned}Q^2\,D_E(x_0,\lambda^2 s\bar r)+\mathbf{e}_E(x_0,\lambda^2 s\bar r)+\lambda^2 s\bar r&\le\lambda^{(1-\gamma)}\,\left(\mathbf{e}_E(x_0,\lambda s\bar r)+Q^2\,D_E(x_0,\lambda s\bar r)+\lambda s\bar r\right)\\&\le\lambda^{2(1-\gamma)}\,\left(\mathbf{e}_E(x_0,s\bar r)+Q^2\,D_E(x_0,s\bar r)+s\bar r\right)\\&\le\lambda^{2(1-\gamma)}\varepsilon_{\text{dec}}(\lambda)\le\varepsilon_{\text{dec}}(\lambda).\end{aligned}$$

Iterating this argument k times, we conclude that

$$Q^2\, D_E(x_0, \lambda^k s\bar{r}) + \mathbf{e}_E(x_0, \lambda^k s\bar{r}) + \lambda^k s\bar{r} \le \lambda^{k(1-\gamma)}\varepsilon_{\rm dec}(\lambda), \quad \forall k \in \mathbb{N}.$$

In particular, the inequality above yields

$$Q^2\, D_E(x_0, \lambda^k s\bar{r}) \le \lambda^{k(1-\gamma)}\varepsilon_{\rm dec}(\lambda), \quad \forall k \in \mathbb{N}.$$

Therefore,

$$Q^2 \int_{B_{\lambda^k s\bar{r}}(x_0)} |\nabla u|^2\, dx \le C\, (\lambda^k s\bar{r})^{(N-\gamma)}, \quad \forall k \in \mathbb{N}$$

for some constant $C > 0$. Now if we take any $\tilde{r} \le \lambda s\bar{r}$, there exists an integer $k > 0$ such that $\lambda^{k+1} s\bar{r} < \tilde{r} \le \lambda^k s\bar{r}$, hence

$$Q^2 \int_{B_{\tilde{r}}(x_0)} |\nabla u|^2\, dx \le Q^2 \int_{B_{\lambda^k s\bar{r}}(x_0)} |\nabla u|^2\, dx \le C\, (\lambda^k s\bar{r})^{(N-\gamma)} \le \frac{C}{\lambda^{N-\gamma}}\, \tilde{r}^{(N-\gamma)}.$$

□

Proposition 7.3.2. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$, $x_0 \in \partial E$, and $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$. Suppose that $Q \le 1$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_N) \in \mathbf{D}(x_0', r) \times \mathbb{R} \,:\, x_N < f(x')\} \cap \mathbf{C}(x_0, r),$$

for some $0 < r \le \min\{\bar{r}, 1\}$, where $\bar{r}$ is as in Lemma 7.3.1. Then there exist $\alpha = \alpha(\vartheta) \in (0,1)$ and a constant $C = C(N, \beta, \vartheta, \|\rho\|_\infty) > 0$ such that

$$Q^2 \int_{B_{\lambda r}(x_0)} |T_E u - [T_E u]_{x_0,\lambda r}|^2\, dx \le C\, Q^2 \lambda^{N+2\alpha} \int_{B_r(x_0)} |T_E u - [T_E u]_{x_0,r}|^2\, dx + C\, r^{N+\alpha}. \tag{7.3.4}$$

Proof. Without loss of generality assume $0 \in \partial E$, $x_0 = 0$. Let $\lambda \in (0, 1/2)$ be given and let v be the solution of

$$\begin{cases} -\mathrm{div}(a_H \nabla v) = \rho & \text{in } B_{r/2} \\ v = u & \text{on } \partial B_{r/2} \end{cases}$$

where H is the half-space $\{x = (x', x_N) \,:\, x_N < 0\}$. In particular, $w = v - u \in W_0^{1,2}(B_{r/2})$ and

$$-\,\mathrm{div}(a_H \nabla w) = -\mathrm{div}\left((a_E - a_H)\nabla u\right). \tag{7.3.5}$$

Since $[T_E g]_s$ minimizes the functional $m \mapsto \int_{B_s} |T_E g - m|^2\, dx$, we have

$$\begin{aligned}\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\, dx &\le \int_{B_{\lambda r}} |T_E u - [T_H u]_{\lambda r}|^2\, dx \\ &\le 2\left(\int_{B_{\lambda r}} |T_H u - [T_H u]_{\lambda r}|^2\, dx + \int_{B_{\lambda r}} |T_E u - T_H u|^2\, dx \right).\end{aligned} \tag{7.3.6}$$

We want now to estimate the first term in the right hand side of (7.3.6). Notice that, since $u = v - w$, by linearity of T_H we have

$$|T_H u - [T_H u]_{\lambda r}|^2 \le 2\left(|T_H v - [T_H v]_{\lambda r}|^2 + |T_H w - [T_H w]_{\lambda r}|^2\right).$$

Hence, integrating the above inequality on $B_{\lambda r}$ we obtain

$$\begin{aligned}\int_{B_{\lambda r}} |T_H u - [T_H u]_{\lambda r}|^2\,dx &\le 2\left(\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx + \int_{B_{\lambda r}} |T_H w - [T_H w]_{\lambda r}|^2\,dx\right)\\ &\le 2\left(\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx + \int_{B_{\lambda r}} |T_H w|^2\,dx\right)\\ &\le C\left(\int_{B_{\lambda r}} |\nabla w|^2\,dx + \int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx\right).\end{aligned} \tag{7.3.7}$$

To estimate the second term in the right hand side of (7.3.6), recall the Notation 2.0.2

$$\partial_{\nu_E^\perp} u = \nabla u - (\nabla u\cdot\nu_E)\,\nu_E \quad \text{and} \quad \partial_{e_N^\perp} u = \nabla u - (\nabla u\cdot e_N).$$

Hence,

$$|T_E u - T_H u| = |(\nabla u\cdot\nu_E)\,\nu_E - (\nabla u\cdot e_N)\,e_N| \le 2|\nabla u|\,|\nu_E - e_N|.$$

Therefore,

$$\int_{B_{\lambda r}} |T_E u - T_H u|^2\,dx \le 4\int_{B_{\lambda r}} |\nabla u|^2\,|\nu_E - e_N|^2\,dx. \tag{7.3.8}$$

Combining (7.3.6), (7.3.7) and (7.3.8) we obtain

$$\begin{aligned}\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\,dx &\le C\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx\\ &\quad + C\int_{B_{r/2}} |\nabla w|^2\,dx + C\int_{B_{\lambda r}} |\nabla u|^2\,|\nu_E - e_N|^2\,dx.\end{aligned}$$

By Lemma A.0.3 we have

$$\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx \le C\,\lambda^{N+2\gamma}\int_{B_{r/2}} |T_H v - [T_H v]_{r/2}|^2\,dx + C\,r^{N+1}. \tag{7.3.9}$$

By arguing as above one can easily see that

$$\begin{aligned}\int_{B_{r/2}} |T_H v - [T_H v]_{r/2}|^2\,dx &\le C\int_{B_{r/2}} |T_E u - [T_E u]_{r/2}|^2\,dx\\ &\quad + C\int_{B_{r/2}} |\nabla w|^2\,dx + C\int_{B_{r/2}} |\nabla u|^2\,|\nu_E - e_N|^2\,dx.\end{aligned}$$

Hence,

$$\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\, dx \le C\,\lambda^{N+2\gamma} \int_{B_{r/2}} |T_E u - [T_E u]_{r/2}|^2\, dx$$
$$+ C \int_{B_{r/2}} |\nabla u|^2\, |\nu_E - e_N|^2\, dx + C \int_{B_{r/2}} |\nabla w|^2\, dx. \tag{7.3.10}$$

We need to estimate the last two terms in the right hand side of the above inequality. Since E is parametrised by $f \in C^{1,\vartheta}(\mathbf{D}_r)$ in the cylinder $\mathbf{C}(x_0, r)$, there exists a constant $C > 0$ such that

$$\frac{|(E\Delta H)\cap B_r|}{|B_r|} \le C\, r^{\vartheta}. \tag{7.3.11}$$

By testing (7.3.5) with w we deduce

$$\int_{B_{r/2}} |\nabla w|^2\, dx \le \int_{B_{r/2}} a_H\, |\nabla w|^2\, dx = \int_{B_{r/2}} (a_E - a_H)\, \nabla u \cdot \nabla w\, dx. \tag{7.3.12}$$

By applying Hölder inequality in (7.3.12) we obtain

$$\int_{B_{r/2}} |\nabla w|^2\, dx \le \int_{B_{r/2}} (a_E - a_H)^2\, |\nabla u|^2\, dx \le C(\beta) \int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\, dx. \tag{7.3.13}$$

By the higher integrability [DPHV19, Lemma 6.1], there exists $p > 1$ such that

$$\left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}} |\nabla u|^{2p}\, dx\right)^{\frac{1}{p}} \le C\, \frac{1}{|B_r|} \int_{B_r} |\nabla u|^2\, dx + C\, r^{N+2}\, \|\rho\|_\infty^2. \tag{7.3.14}$$

Hence by exploiting Hölder inequality, (7.3.11), and (7.3.14) we have

$$\int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\, dx \le \left|(E\Delta H)\cap B_{r/2}\right|^{1-\frac{1}{p}} \left(\int_{B_{r/2}} |\nabla u|^{2p}\, dx\right)^{\frac{1}{p}}$$
$$\le C\,|B_r| \left(\frac{|(E\Delta H)\cap B_r|}{|B_r|}\right)^{1-\frac{1}{p}} \left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}} |\nabla u|^{2p}\, dx\right)^{\frac{1}{p}}$$
$$\le C\, r^{\vartheta\,(1-\frac{1}{p})} \left\{\int_{B_r} |\nabla u|^2\, dx + r^{N+2}\, \|\rho\|_\infty^2\right\}. \tag{7.3.15}$$

Therefore, (7.3.13) together with (7.3.15) (recall $r < 1$) yield

$$\int_{B_{r/2}} |\nabla w|^2\, dx \le C \left\{ r^{\vartheta(1-\frac{1}{p})} \int_{B_r} |\nabla u|^2 + r^{N+2} \|\rho\|_\infty^2 \right\}. \tag{7.3.16}$$

On the other hand, by Lemma 7.3.1 we have

$$Q^2 \int_{B_s} |\nabla u|^2\, dx \le C\, s^{N-\gamma} \quad \forall\, s < \bar{r}. \tag{7.3.17}$$

Hence, combining (7.3.16) and (7.3.17), we obtain

$$Q^2 \int_{B_r} |\nabla w|^2 \, dx \leq C \left\{ r^{\vartheta(1-\frac{1}{p})+n-\gamma} + r^{N+2} \|\rho\|_\infty^2 \right\}.$$

Finally, we estimate the second term in (7.3.10). Notice that

$$\begin{aligned} \int_{B_{r/2}} |\nabla u|^2 \, |\nu_E - e_N|^2 \, dx &= \int_{B_{r/2}} |\nabla u(x', x_N)|^2 \, |\nu_E(x', x_N) - e_N|^2 \, dx \\ &= \int_{B_{r/2}} |\nabla u|^2 \, |\nu_E(x', f(x')) - e_N|^2 \, dx. \end{aligned}$$

Since $\sqrt{1+t} \leq 1 + \frac{t}{2}$ for every $t > 0$,

$$|\nu_E(x', f(x')) - e_N|^2 = 2 - \frac{2}{\sqrt{1 + |\nabla f(x')|^2}} \leq 2 \left(\frac{\sqrt{1+|\nabla f(x')|^2} - 1}{\sqrt{1+|\nabla f(x')|^2}} \right) \leq |\nabla f(x')|^2. \tag{7.3.18}$$

Thanks to (7.3.17) and (7.3.18), and using that ∇f is ϑ-Hölder, we deduce

$$Q^2 \int_{B_{r/2}} |\nabla u|^2 \, |\nu_E - e_N|^2 \, dx \leq C \, r^{N+2\vartheta-\gamma}. \tag{7.3.19}$$

Let

$$\alpha := \min \left\{ \gamma, \vartheta \left(1 - 1/p\right) - \gamma, 2\vartheta - \gamma \right\}.$$

Therefore, by multiplying (7.3.10) and (7.3.16) with Q^2 and by recalling that $Q < 1$ we have that (7.3.19) implies (7.3.4). □

We are now ready to prove that u is regular up to the boundary. Recall that $u^+ = u \, \mathbf{1}_E$ and $u^- = u \, \mathbf{1}_{E^c}$.

Theorem 7.3.3. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u, \rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$ and $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$. Suppose $Q \leq 1$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_N) \in \mathbf{D}(x_0', r) \times \mathbb{R} \, : \, x_N < f(x')\} \cap \mathbf{C}(x_0, r)$$

for some $0 < r \leq \min\{\bar{r}, 1\}$, where $\bar{r}$ is as in Lemma 7.3.1. Then there exists $\eta = \eta(\vartheta) \in (0,1)$ such that $u^+ \in C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u^- \in C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))$. Furthermore, let $A > 0$ and let β, K, Q be controlled by A and $R \geq 1$. Then there exists a universal constant $C = C(N, A) > 0$ such that

$$\|Q \, u^+\|_{C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))} \leq C \quad \text{and} \quad \|Q \, u^-\|_{C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))} \leq C. \tag{7.3.20}$$

Proof. Let $u_Q := Q \, u$. By Proposition 7.3.2 there exists $C = C(N, \beta, \gamma, \|\rho\|_\infty) > 0$ such that

$$\int_{B_{\lambda r}(x_0)} |T_E u_Q - [T_E u_Q]_{x_0, \lambda r}|^2 \, dx \leq C \lambda^{N+2\alpha} \int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x_0, r}|^2 \, dx + C \, r^{N+\alpha}, \tag{7.3.21}$$

where $\alpha \in (0,1)$ is as in Proposition 7.3.2. Therefore, Lemma A.0.2 implies that there exists a universal constant $C = C(N, A) > 0$ such that

$$\frac{1}{|B_r|}\int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x,r}|^2\, dy \le C\left(\frac{r}{R}\right)^{2\eta}, \quad \forall B_r(x_0) \subset B_R. \tag{7.3.22}$$

for some $\eta = \eta(\vartheta) \in (0,1)$. Hence, by Lemma A.0.1, recalling the definition of T_E, we get $u_Q \mathbf{1}_E \in C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u_Q \mathbf{1}_{E^c} \in C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))$ and (7.3.20). ☐

In the next proposition we rewrite the Euler-Lagrange equation (see Theorem 7.1.2 (ii)) in a more convenient form by exploiting the regularity of ∂E.

Proposition 7.3.4 (Euler-Lagrange equation)**.** *Let E be a minimizer for $(\mathcal{P}_{\beta,K,Q,R})$ and $(u,\rho) \in \mathcal{A}(E)$. Assume that $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} \,:\, x_n < f(x')\} \cap \mathbf{C}(x_0, r).$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned} -\mathrm{div}\left(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\right) &= Q^2\left(\beta|\nabla u^+|^2 - |\nabla u^-|^2 + K\rho^2\right)(x', f(x')) \\ &\quad - 2Q^2\left(\beta \partial_n u^+ \nabla u^+ - \partial_n u^- \nabla u^-\right)(x', f(x')) \cdot (-\nabla f(x'), 1) + C \end{aligned} \tag{7.3.23}$$

weakly in $\mathbf{D}(x_0', r)$.

Proof. Let $E \subset \mathbb{R}^n$ be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and let $(u,\rho) \in \mathcal{A}(E)$.

Notice that $E \cap \mathbf{C}(x_0, r)$ is an open set of $\mathbb{R}^n$. Moreover, by an approximation argument, we can integrate over $E \cap \mathbf{C}(x_0, r)$ the following identity,

$$\begin{aligned} |\nabla u^+|^2\, \mathrm{div}\, \eta &= \mathrm{div}(|\nabla u^+|^2 \eta) - \nabla |\nabla u^+|^2 \cdot \eta \\ &= \mathrm{div}(|\nabla u^+|^2 \eta) - 2\, \mathrm{div}(\nabla u^+ (\nabla u^+ \cdot \eta)) + 2\Delta u^+\, \nabla u^+ \cdot \eta + 2 \nabla u^+ \cdot (\nabla \eta \nabla u^+) \end{aligned}$$

for every $\eta \in C_c^\infty(\mathbf{C}(x_0, r), \mathbb{R}^n)$. Therefore,

$$\begin{aligned} \int_{E \cap \mathbf{C}(x_0,r)} \left(|\nabla u^+|^2\, \mathrm{div}\, \eta - 2\nabla u^+ \cdot (\nabla \eta \nabla u^+)\right) dx &= \int_{E \cap \mathbf{C}(x_0,r)} \mathrm{div}(|\nabla u^+|^2 \eta)\, dx \\ &\quad - \int_{E \cap \mathbf{C}(x_0,r)} 2\, \mathrm{div}(\nabla u^+ (\nabla u^+ \cdot \eta))\, dx \\ &\quad + \int_{E \cap \mathbf{C}(x_0,r)} 2\Delta u^+\, \nabla u^+ \cdot \eta\, dx. \end{aligned} \tag{7.3.24}$$

On the other hand, since $(u,\rho) \in \mathcal{A}(E)$, we have

$$-\beta \Delta u^+ = \rho \quad \text{in} \quad \mathcal{D}'(E \cap \mathbf{C}(x_0, r)).$$

Moreover, by Theorem 7.1.2 (i) we deduce

$$\nabla u^+ = -K\,\nabla\rho \quad \text{in} \quad E\cap \mathbf{C}(x_0,r).$$

Then, by multiplying equation (7.3.24) by β, we have

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)} \beta\left(|\nabla u^+|^2 \operatorname{div}\eta - 2\nabla u^+\cdot(\nabla\eta\nabla u^+)\right)\,dx &= \int_{E\cap\mathbf{C}(x_0,r)} \beta\operatorname{div}(|\nabla u^+|^2\eta)\,dx\\ &\quad - \int_{E\cap\mathbf{C}(x_0,r)} 2\beta\operatorname{div}(\nabla u^+(\nabla u^+\cdot\eta))\,dx\\ &\quad + K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\cdot\eta\,dx.\end{aligned} \tag{7.3.25}$$

Integrating by parts the first and the second term in the right hand side of (7.3.25), we can write

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)} \beta\left(|\nabla u^+|^2 \operatorname{div}\eta - 2\nabla u^+\cdot(\nabla\eta\nabla u^+)\right)\,dx &= \int_{\partial E\cap\mathbf{C}(x_0,r)} \beta|\nabla u^+|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad - \int_{\partial E\cap\mathbf{C}(x_0,r)} 2\beta\,(\nabla u^+\cdot\eta)(\nabla u^+\cdot\nu_E)\,d\mathcal{H}^{n-1}\\ &\quad + K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\cdot\eta\,dx.\end{aligned} \tag{7.3.26}$$

By arguing similarly as above, one can also prove

$$\begin{aligned}\int_{E^c\cap\mathbf{C}(x_0,r)} \left(|\nabla u^-|^2 \operatorname{div}\eta - 2\nabla u^-\cdot(\nabla\eta\nabla u^-)\right)\,dx &= \int_{E^c\cap\mathbf{C}(x_0,r)} \operatorname{div}(|\nabla u^-|^2\eta)\,dx\\ &\quad - \int_{E^c\cap\mathbf{C}(x_0,r)} 2\operatorname{div}(\nabla u^-(\nabla u^-\cdot\eta))\,dx.\end{aligned} \tag{7.3.27}$$

Integrating by parts the right hand side of (7.3.27), we can write

$$\begin{aligned}\int_{E^c\cap\mathbf{C}(x_0,r)} \left(|\nabla u^-|^2 \operatorname{div}\eta - 2\nabla u^-\cdot(\nabla\eta\nabla u^-)\right)\,dx &= -\int_{\partial E\cap\mathbf{C}(x_0,r)} |\nabla u^-|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad + \int_{\partial E\cap\mathbf{C}(x_0,r)} 2\,(\nabla u^-\cdot\eta)\,(\nabla u^-\cdot\nu_E)\,d\mathcal{H}^{n-1}.\end{aligned} \tag{7.3.28}$$

Therefore, combining (7.3.26) and (7.3.28), we get

$$\begin{aligned}\int_{\mathbb{R}^n} a_E \left(\mathrm{div}\eta\, |\nabla u|^2 - 2\nabla u\cdot(\nabla\eta\nabla u)\right) dx &= \int_{\partial E} \left(\beta|\nabla u^+|^2 - |\nabla u^-|^2\right) \eta\cdot\nu_E \, d\mathcal{H}^{n-1}\\ &- \int_{\partial E\cap \mathbf{C}(x_0,r)} 2\big(\beta\,(\nabla u^+\cdot\eta)(\nabla u^+\cdot\nu_E) - (\nabla u^-\cdot\eta)\,(\nabla u^-\cdot\nu_E)\big)\, d\mathcal{H}^{n-1}\\ &+ K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\cdot\eta\, dx.\end{aligned} \tag{7.3.29}$$

Notice that the following identity holds true

$$\begin{aligned}K\int_{\mathbb{R}^n}\rho^2 \mathrm{div}\eta &= K\int_{E\cap\mathbf{C}(x_0,r)} \mathrm{div}(\rho^2\eta)\, dx - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\,\cdot\eta\, dx\\ &= K\int_{\partial E\cap\mathbf{C}(x_0,r)} \rho^2\eta\cdot\nu_E\, d\mathcal{H}^{n-1} - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\,\cdot\eta\, dx.\end{aligned} \tag{7.3.30}$$

Combining the Euler-Lagrange equation of Theorem 7.1.2 (ii), (7.3.29) and (7.3.30), we find

$$\begin{aligned}\int_{\partial E}\mathrm{div}_E\eta\, d\mathcal{H}^{n-1} &= Q^2\int_{\partial E}\left(\beta|\nabla u^+|^2 - |\nabla u^-|^2 + K\,\rho^2\right)\,\eta\cdot\nu_E\, d\mathcal{H}^{n-1}\\ &- 2Q^2\int_{\partial E}\beta(\eta\cdot\nabla u^+)\,(\nabla u^+\cdot\nu_E) - (\eta\cdot\nabla u^-)\,(\nabla u^-\cdot\nu_E)\, d\mathcal{H}^{n-1}\end{aligned} \tag{7.3.31}$$

for every $\eta\in C^1_c(B_r(x_0),\mathbb{R}^n)$ with $\int_E \mathrm{div}\,\eta\, dx = 0$.

Now we are ready to prove (7.3.23). The tangential divergence of η on ∂E is

$$\mathrm{div}_E\eta := \mathrm{div}\eta - \sum_{i,j=1}^{n}(\nu_E)_i\,(\nu_E)_j\,\partial_j\eta_i \quad \text{on } \partial E, \tag{7.3.32}$$

where $\nu_E : \partial E\to\mathbb{S}^{n-1}$ is the normal vector to ∂E:

$$\nu_E := \frac{1}{\sqrt{1+|\nabla f|^2}}(-\nabla f, 1).$$

Let $\eta := (0,\dots,0,\eta_n)$, then by (7.3.32) we have

$$\mathrm{div}_E\eta := \partial_n\eta_n + \frac{1}{1+|\nabla f|^2}\left\{\sum_{j=1}^{n-1}\partial_j\eta_n\,\partial_j f - \partial_n\eta_n\right\} \quad \text{on } \partial E. \tag{7.3.33}$$

Choose $\eta_n(x) := \varphi(\mathbf{p}x)\,s(x_n)$, where $\varphi\in C^1_c(\mathbf{D}(x_0',r))$ is such that $\int_{\mathbf{D}(x_0',r)}\varphi = 0$ and $s:(-1,1)\to\mathbb{R}^n$ is such that $s(t)=1$ for every $|t|\le\|f\|_\infty$. Since now η_n does not depend on the n-th component on ∂E, we have

$$\eta\cdot\nu_E = \frac{\varphi(\mathbf{p}x)}{\sqrt{1+|\nabla f|^2}} \quad \text{on } \partial E\cap\mathbf{C}(x_0,r), \tag{7.3.34}$$

and the above equation (7.3.33) reads as

$$\mathrm{div}_E\eta := \frac{1}{1+|\nabla f|^2}\nabla\varphi\cdot\nabla f \quad \text{on } \partial E\cap \mathbf{C}(x_0,r). \tag{7.3.35}$$

Moreover,

$$\begin{aligned}\int_E \mathrm{div}\eta\,dx &= \int_{\partial E}(\eta\cdot\nu_E)\,d\mathcal{H}^{n-1} = \int_{\partial E\cap\mathbf{C}(x_0,r)}\eta_n(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\\ &= \int_{\partial E\cap\mathbf{C}(x_0,r)}\varphi(\mathbf{p}x)s(f(x))\,(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\\ &= \int_{\partial E\cap\mathbf{C}(x_0,r)}\frac{\varphi(\mathbf{p}x)}{\sqrt{1+|\nabla f(\mathbf{p}x)|}}\,d\mathcal{H}^{n-1} = \int_{\mathbf{p}(\partial E\cap\mathbf{C}(x_0,r))}\varphi\,dx = 0.\end{aligned}$$

This implies that η is admissible in (7.3.31). Hence by using η as a test function in (7.3.31), by combining (7.3.34) and (7.3.35), we have

$$\begin{aligned}\int_{\mathbf{D}(x_0',r)}\frac{\nabla f}{1+|\nabla f|^2}\cdot\nabla\varphi\,dx' &= Q^2\int_{\mathbf{D}(x_0',r}\left(\beta|\nabla u^+|^2-|\nabla u^-|^2+K\rho^2\right)(x',f(x'))\,\frac{\varphi(x')}{\sqrt{1+|\nabla f|^2}}\,dx'\\ &\quad -2Q^2\int_{\mathbf{D}(x_0',r}\left(\beta\partial_n u^+\nabla u^+-\partial_n u^-\nabla u^-\right)(x',f(x'))\cdot\frac{(-\nabla f,1)}{\sqrt{1+|\nabla f|^2}}\,\varphi(x')\,dx'\end{aligned}$$

for any $\varphi\in C_c^1(\mathbf{D}(x_0',r))$ with $\int_{\mathbf{D}(x_0',r)}\varphi=0$. It remains to multiply this equality by $\sqrt{1+|\nabla f|^2}$ and use divergence theorem on the left-hand side. □

Corollary 7.3.5. *Let E be a minimizer for ($\mathcal{P}_{\beta,K,Q,R}$) and $(u,\rho)\in\mathcal{A}(E)$. Assume that $f\in C^{1,\vartheta}(\mathbf{D}(x_0',r))$ and*

$$E\cap\mathbf{C}(x_0,r) = \{x=(x',x_N)\in\mathbf{D}(x_0',r)\times\mathbb{R} \,:\, x_N<f(x')\}\cap\mathbf{C}(x_0,r).$$

Then there exists a function M such that the matrix $\nabla M(\nabla f)$ is uniformly elliptic and a Hölder continuous function G such that

$$-\mathrm{div}\,(\nabla M(\nabla f)\,\nabla\partial_i f) = \partial_i G \quad \text{a.e. on } \partial E\cap\mathbf{C}(x_0,r/2)$$

for every $i=1,\dots,n$.

Proof. Exploiting Proposition 7.3.4, we have

$$-\,\mathrm{div}\left(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\right) = G(x',f(x')) \quad \text{for a.e. } x'\in\mathbf{D}(x_0',r/2), \tag{7.3.36}$$

where

$$\begin{aligned}G(x',f(x')) &= Q^2\left(\beta|\nabla u^+|^2-|\nabla u^-|^2+K\rho^2\right)(x',f(x'))\\ &\quad -2Q^2\left(\beta\partial_n u^+\,\nabla u^+-\partial_n u^-\,\nabla u^-\right)(x',f(x'))\cdot(-\nabla f(x'),1)+C, \quad x'\in\mathbf{D}(x_0',r/2).\end{aligned}$$

for $x' \in \mathbf{D}(x_0', r/2)$. Hence, (7.3.36) is equivalent to

$$-\operatorname{div}(M(\nabla f)) = G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2), \tag{7.3.37}$$

where

$$M(\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}, \quad \forall \xi \in \mathbb{R}^N.$$

By [Mag12, Theorem 27.1] we can take the derivatives of (7.3.37). Then,

$$-\operatorname{div}(\nabla M(\nabla f)\, \nabla \partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2)$$

for every $i = 1, \ldots, n$. Notice that

$$\nabla M(\xi) = \frac{1}{\sqrt{1+|\xi|^2}} \left(\operatorname{Id} - \frac{\xi \otimes \xi}{1+|\xi|^2} \right) \quad \forall \xi \in \mathbb{R}^N,$$

meaning that the matrix $\nabla M(\nabla f)$ is uniformly elliptic, more precisely

$$|\eta|^2 \geq \nabla M(\nabla f)\eta \cdot \eta \geq (1 + \|\nabla f\|_\infty)^{-3/2} |\eta|^2 \quad \forall \eta \in \mathbb{R}^N.$$

It follows from Theorem 7.3.3 that G is Hölder continuous. By the definition of M and by the regularity of f we also have that $\nabla M(\nabla f)$ is Hölder continuous. □

We prove now the partial $C^{2,\vartheta}$-regularity of minimizers.

Theorem 7.3.6 ($C^{2,\vartheta}$-regularity). *Given $N \geq 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exists $\varepsilon_{\text{reg}} = \varepsilon_{\text{reg}}(N, A, \vartheta) > 0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), $Q + \beta + K + \frac{1}{K} \leq A$, $x_0 \in \partial E$, and*

$$r + \mathbf{e}_E(x_0, r) + Q^2\, D_E(x_0, r) \leq \varepsilon_{\text{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epi-graph of a $C^{2,\vartheta}$-function f.

In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a $C^{2,\vartheta}$ $(N-1)$-dimensional manifold and

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C\,(N, A, r, \vartheta)\,. \tag{7.3.38}$$

Proof. Choose ε_{reg} as in Theorem 7.1.3. Then there exists $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \{x = (x', x_N) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} \,:\, x_N < f(x')\}\,.$$

By Corollary 7.3.5 we have

$$-\operatorname{div}(\nabla M(\nabla f)\, \nabla \partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2)$$

for every $i = 1, \ldots, n$, with $\nabla M(\nabla f)$ uniformly elliptic and G - Hölder continuous. We also have that $\nabla M(\nabla f)$ is Hölder continuous. Hence the following Schauder estimates hold in this case

$$[\nabla \partial_i f]_{C^{0,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C \left\{ \|\partial_i f\|_{L^2(\mathbf{D}(x_0', r/2))} + [G]_{C^{0,\eta}(\mathbf{D}(x_0', r/2))} \right\} \quad \forall i = 1, \ldots, n,$$

for some constant C depending on r. In particular, f is $C^{2,\vartheta}$ and

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0',r/2))} \leq C\left\{\|\nabla f\|_{L^2(\mathbf{D}(x_0',r/2))} + [G]_{C^{0,\eta}(\mathbf{C}(x_0,r/2))}\right\}.$$

By the definition of G, recalling (7.3.20) and Theorem 7.1.2 (i), using Poincaré inequality and since f is Lipschitz, one can easily see that there exists $C = C(N, A, \vartheta, r) > 0$ such that

$$[G]_{C^{0,\vartheta}(\mathbf{C}(x_0,r/2))} \leq C(N, A, \vartheta, r).$$

By the Lipschitz approximation theorem it follows that

$$\frac{1}{r^{N-1}} \int_{\mathbf{D}(x_0',r/2)} |\nabla f|^2 \, dz \leq C_L \, \mathbf{e}_E(x_0, r) \leq C_L \, \varepsilon_{\text{reg}}, \tag{7.3.39}$$

which implies (7.3.38). □

Remark 7.3.7. A minimizer E_Q of the problem ($\mathcal{P}_{\beta,K,Q,R}$) satisfies the hypothesis of Theorems 7.3.6 and 7.4.3 whenever $Q > 0$ is small enough. Indeed, assume $x_0 \in \partial B_1$. Then, by the regularity of ∂B_1, there exists a radius $r = r(N) > 0$ such that

$$r + \mathbf{e}_{B_1}(x_0, 2r) \leq \frac{\varepsilon_{\text{reg}}}{2}, \tag{7.3.40}$$

where ε_{reg} is as in Theorem 7.4.3. On the other hand, by Proposition 7.2.2 we have that E_Q converges to B_1 in the Kuratowski sense when $Q \to 0$. Hence, by properties of the excess function, $\mathbf{e}_{E_Q}(x_0, 2r) \to \mathbf{e}_{B_1}(x_0, 2r)$ when $Q \to 0$. By Theorem 7.1.2 (iii) we also have $Q^2 \, D_{E_Q}(x_0, 2r) \to 0$ when $Q \to 0$. Therefore,

$$r + \mathbf{e}_{E_Q}(x_0, 2r) + Q^2 \, D_{E_Q}(x_0, 2r) \leq \varepsilon_{\text{reg}}, \tag{7.3.41}$$

when $Q > 0$ is small enough.

7.4 Smooth regularity

In this section, by a bootstrap argument, we obtain the smooth partial regularity of minimizers. Since this result is not necessary for the proof of the main theorem, the reader may skip it unless interested.

Improving the regularity from $C^{2,\eta}$ to C^∞ is easier then from $C^{1,\eta}$ to $C^{2,\eta}$, because we can straighten the boundary in a nice way once it is C^2. More precisely, we have the following lemma.

Lemma 7.4.1. *Let $k \in \mathbb{N}$, $k \geq 2$ and f is $C^{k,\vartheta}(\mathbf{D})$. There exists $\varepsilon > 0$ such that if*

$$\|f\|_{C^{2,\vartheta}(\mathbf{D})} \leq \varepsilon \quad \textit{and} \quad f(0) = 0,$$

then there exists a diffeomorphism $\Phi \in C^{k-1,\vartheta}$, $\Phi : \mathbf{C}_{1-\varepsilon} \to \mathbf{C}_{1-\varepsilon}$, such that

$$\Phi(\Gamma_f \cap \mathbf{C}_{1-\varepsilon}) = \{x = (x', x_N) \in \mathbf{D}_{1-\varepsilon} \times \mathbb{R} \, : \, x_N = 0\},$$

where Γ_f is the graph of f. Moreover,

$$\begin{aligned} &\left(\nabla\Phi(\Phi^{-1}(x))\,(\nabla\Phi(\Phi^{-1}(x)))^T\right)_{jN} = 0 \ \ \forall j \neq n; \\ &\left(\nabla\Phi(\Phi^{-1}(x))\,(\nabla\Phi(\Phi^{-1}(x)))^T\right)_{NN} \neq 0. \end{aligned} \tag{7.4.1}$$

Proof. Define

$$\Psi(x', x_N) := (x', f(x')) + x_N \frac{(-\nabla f(x'), 1)}{\sqrt{1 + |\nabla f(x')|^2}} \qquad \forall x = (x', x_N) \in \mathbf{C}_{1-\varepsilon},$$

then $\Phi := \Psi^{-1}$ is the desired diffeomorphism. □

Lemma 7.4.2. *Let k be a positive integer and let f be a $C^{k+1,\vartheta}$-Hölder continuous function defined on $\mathbf{D}(x_0, r)$ such that $\|f\|_{C^{k+1,\vartheta}} \leq \varepsilon$ for some $\varepsilon > 0$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_N) \in \mathbf{D}(x_0', r) \times \mathbb{R} \,:\, x_N < f(x')\} \cap \mathbf{C}(x_0, r).$$

Suppose v is a solution of

$$-\operatorname{div}(a_E \nabla v) = h \quad \text{in } \mathcal{D}'(B_r(x_0)), \quad a_E := \mathbf{1}_{E^c} + \beta\,\mathbf{1}_E,$$

with h^+ and h^- $C^{k,\eta}$-Hölder continuous respectively on $\overline{E} \cap \mathbf{C}(x_0, r)$ and $\overline{E^c} \cap \mathbf{C}(x_0, r)$, where $h^+ = h\,1_E$, $h^- = h\,1_{E^c}$. Then v^+, v^- are $C^{k+1,\eta}$-Hölder continuous respectively on $\overline{E} \cap \mathbf{C}(x_0, r)$ and $\overline{E^c} \cap \mathbf{C}(x_0, r)$.

Moreover,

$$\|v_1\|_{C^{k+1,\eta}(\overline{E}^c \cap \mathbf{C}(x_0,r))} \leq C \quad \text{and} \quad \|v_\beta\|_{C^{k+1,\eta}(\overline{E} \cap \mathbf{C}(x_0,r))} \leq C \qquad (7.4.2)$$

for some constant $C \geq 0$ which depends on the $C^{k,\eta}$- Hölder norms of h^+ and h^- and on the $C^{k+1,\vartheta}$ norm of f.

Proof. Assume $x_0 = 0$. Let $H := \left\{x \in \mathbb{R}^N \,:\, x_N = x \cdot e_N \leq 0\right\}$ be the half space in $\mathbb{R}^N$. By Lemma 7.4.1, we can assume that

$$\Gamma_f \cap \mathbf{C}_r = \partial H \cap \mathbf{C}_r,$$

where $\Gamma_f \cap \mathbf{C}_{r/2} := \{(x', f(x')) \,:\, x' \in \mathbf{D}_r\}$, $f(0) = 0$ and that v solves the following equation

$$-\operatorname{div}(a_H A \nabla v) = h, \qquad (7.4.3)$$

where by (7.4.1), A is a $C^{k-1,\vartheta}$-continuous elliptic matrix such that $A_{jN} = 0$ for every $j \neq N$, $A_{NN} \neq 0$.

We continue the proof by induction on k. For clarity, we do the detailed computations for the case $k = 1$ and we explain how the formulas look like for bigger k.

Case k = 1. By taking the derivatives with respect to the tangential coordinates $j \neq n$ of (7.4.3) we deduce

$$\begin{aligned} -\operatorname{div}\big(a_H A \nabla \partial_j v\big) &= \partial_j h + \operatorname{div}\big(\partial_j(a_H A) \nabla v\big) \\ &= \operatorname{div}(h\, e_j + \partial_j(a_H A) \nabla v) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \end{aligned} \qquad (7.4.4)$$

Notice that a_H is constant along tangential directions and that $(a_H\, A)^+$, $(a_H\, A)^-$ have coefficients respectively in $C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r)$ and $C^{0,\eta}(\overline{H} \cap C_r)$. Furthermore,

$$(h\, e_j + \partial_j(a_H A) \nabla v)^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r) \text{ and } (h\, e_j + \partial_j(a_H A) \nabla v)^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r).$$

Hence, exploiting Lemma A.0.4 we deduce

$$\partial_j v^+ \in C^{1,\eta}(\overline{H} \cap \mathbf{C}_r) \quad \text{and} \quad \partial_j v^- \in C^{1,\eta}(\overline{H^c} \cap \mathbf{C}_r) \qquad \forall j \neq n. \tag{7.4.5}$$

Furthermore, by (7.4.3) we have

$$-\sum_{i,j=1}^{N} \{a_H\, A_{ij} \partial_{ij} v + \partial_i (a_H\, A_{ij}) \partial_j v\} = h.$$

Thanks to the form of the matrix A we obtain

$$-\, a_H\, A_{NN} \partial_{NN} v = \sum_{i,j \neq N} \{a_H\, A_{ij} \partial_{ij} v + \partial_i (a_H\, A_{ij})\, \partial_j v\} + h. \tag{7.4.6}$$

Since the right hand side of the previous equation is Hölder continuous, we have

$$\partial_{NN} v^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{NN} v^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r).$$

Moreover, (7.4.5) implies

$$\partial_{Nj} v^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{Nj} v^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r)$$

for every $j \neq n$. Therefore,

$$v^+ \in C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad v^- \in C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_r).$$

By Lemma A.0.4 we deduce also that

$$\|\nabla v^+\|_{C^{1,\eta}(\overline{H} \cap \mathbf{C}_r)} \qquad \text{and} \qquad \|\nabla v^-\|_{C^{1,\eta}(\overline{H^c} \cap \mathbf{C}_r)}$$

are bounded by a constant which depends on the Hölder norms of ∇h^+, ∇h^-, the coefficients of $(a_H A)^+$ and $(a_H A)^-$.

General k. As in the case $k = 1$, we start by taking the derivatives of (7.4.3) with respect to the tangential coordinates $j \neq n$. We get an equation similar to (7.4.4):

$$-\mathrm{div}\big(a_H A\, \nabla \partial_{i_1, i_2, \dots, i_k} v\big) = \mathrm{div}\left(\partial_{i_2, \dots, i_k} h\, e_{i_1} + \sum \partial_{j_1, j_2, \dots, j_l} (a_H A) \nabla \left(\partial_{\{i_1, i_2, \dots, i_k\} \setminus \{j_1, j_2, \dots, j_l\}} v\right)\right)$$

in $\mathcal{D}'(\mathbb{R}^N)$. This gives us

$$\partial_{i_1, i_2, \dots, i_k} v^+ \in C^{1,\eta}(\overline{H} \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{i_1, i_2, \dots, i_k} v^- \in C^{1,\eta}(\overline{H^c} \cap \mathbf{C}_r) \tag{7.4.7}$$

for all $i_1 \neq n$, $i_2 \neq n$, $\dots$, $i_k \neq n$.

By (7.4.7)

$$\partial_{i_1, i_2, \dots, i_k, n} v^+ \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{i_1, i_2, \dots, i_k, n} v^- \in C^{0,\eta}(\overline{H^c} \cap \mathbf{C}_r)$$

for all $i_1 \neq n$, $i_2 \neq n$, $\dots$, $i_k \neq n$, and thus, taking derivatives of (7.4.6) in tangential directions, we get

$$\partial_{i_1, i_2, \dots, i_{k-1}, n, n} v^+ \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{i_1, i_2, \dots, i_{k-1}, n, n} v^- \in C^{0,\eta}(\overline{H^c} \cap \mathbf{C}_r).$$

Induction on the number of normal directions yields

$$v^+ \in C^{k+1,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad v^- \in C^{k+1,\eta}(\overline{H}^c \cap \mathbf{C}_r).$$

□

Theorem 7.4.3 (**C^∞-regularity**). *Given $N \geq 3$ and $A > 0$, there exists $\varepsilon_{\rm reg} = \varepsilon_{\rm reg}(N, A) > 0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) with $Q + \beta + K + \frac{1}{K} \leq A$, $x_0 \in \partial E$, and*

$$r + \mathbf{e}_E(x_0, r) + Q^2\, D_E(x_0, r) \leq \varepsilon_{\rm reg},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epi-graph of a C^∞-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a C^∞ $(N-1)$-dimensional manifold. Moreover, for every $\vartheta \in (0, \frac{1}{2})$ there exists a constant $C(N, A, k, r, \vartheta) > 0$ such that

$$[f]_{C^{k,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C(N, A, k, r, \vartheta) \tag{7.4.8}$$

for every $k \in \mathbb{N}$.

Proof. If we choose $\varepsilon_{\rm reg}$ as in Theorem 7.3.6, then there exists $f \in C^{2,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \{x = (x', x_N) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} \,:\, x_N < f(x')\}\,.$$

By Corollary 7.3.5 we have

$$-\operatorname{div}\left(\nabla M(\nabla f)\, \nabla \partial_i f\right) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2) \tag{7.4.9}$$

for every $i = 1, \ldots, n$, with $\nabla M(\nabla f)$ uniformly elliptic and Hölder continuous and G - Hölder continuous.

Now we argue by induction on k. The induction step is divided into two parts:

Claim 1:

$$f \text{ is } C^k\text{-Hölder continuous} \implies u^+, u^- \text{ are } C^k\text{-Hölder continuous respectively on } \overline{E} \cap \mathbf{C}(x_0, r/2) \text{ and } \overline{E^c} \cap \mathbf{C}(x_0, r/2).$$

Moreover, there exists a universal constant $C = C(N, A) > 0$ and $\vartheta \in (0, \frac{1}{2})$ such that

$$\|Q\, u^+\|_{C^{k,\vartheta}(\overline{E} \cap \mathbf{C}(x_0, r/2))} \leq C \quad \text{and} \quad \|Q\, u^-\|_{C^{k,\vartheta}(\overline{E^c} \cap \mathbf{C}(x_0, r/2))} \leq C. \tag{7.4.10}$$

Claim 2:

$$f \text{ is } C^k\text{-Hölder continuous} \implies f \text{ is } C^{k+1}\text{-Hölder continuous}.$$

To proof Claim 1, we apply Lemma (7.4.2) to $v = Qu$ and $h = Q\rho$. By (7.3.20) the norms

$$\|Q\, \nabla u^+\|_{C^{0,\vartheta}(\overline{H} \cap \mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\, \nabla u^-\|_{C^{0,\vartheta}(\overline{H^c} \cap \mathbf{C}_{r/2})}$$

and bounded by a universal constant. That gives us (7.4.10).

As for Claim 2, notice that by the definition of M, since f is C^k-Hölder continuous, we have that $\nabla M(\nabla f)$ in (7.4.9) is C^{k-1}-Hölder continuous. By Claim 1 we deduce that G is C^{k-1}-Hölder continuous with its norm uniformly bounded. Then, using Schauder estimates for (7.4.9), we get that f is C^{k+1}-Hölder continuous.

□

7.5 Proof of Theorem 1.1.9

Finally, we are ready to prove that the only minimizers of $\mathcal{F}$ are balls if the charge Q is small enough.

Proof of Theorem 1.1.9. Argue by contradiction. Suppose there exists a sequence of minimizers E_h corresponding to $Q_h \to 0$ such that E_h are not balls. Translate the sets E_h if needed so that their barycenters are at the origin. Arguing in a similar way to the proof of Theorem 5.2.2 (using Proposition 7.2.2 instead of Lemma 5.3.4 and Theorem 7.3.6 instead of Theorem 5.4.18) we have that starting from a certain h the sets are nearly-spherical parametrized by φ_h with $\|\varphi_h\|_{C^{2,\vartheta}(\partial B_1)} < \delta$, where δ is the one of Theorem 4.3.14.

Now we apply Theorem 4.3.14 to see that $\mathcal{F}(E_h) > F(B_1)$ for h big enough, contradicting the minimality of E_h. □

We can now prove Corollary 1.1.10, which follows from Theorem 1.1.9 and properties of minimizers established in [DPHV19].

Proof of Corollary 1.1.10. Let Q_0 be the one of Theorem 1.1.9. Let E be an open set such that $|E| = |B_1|$. Let us show that $\mathcal{F}(E) \geq \mathcal{F}(B_1)$. If E is bounded, then $\mathcal{F}(E_h) \geq F(B_1)$ by Theorem 1.1.9. Assume now that E is unbounded.

We can assume that E is of finite perimeter, since otherwise $\mathcal{F}(E) = \infty$. Then, by [Mag12, Remark 13.12], there exists a sequence $R_h \to \infty$ such that $E \cap B_{R_h} \to E$ in L^1, $P(E \cap B_{R_h}) \to P(E)$. Rescale the sets so that their volumes are the same as the one of the ball, i.e.

$$\Omega_h = \alpha_h \left(E \cap B_{R_h}\right) \text{ with } \alpha_h = \left(\frac{|B_1|}{|E \cap B_{R_h}|}\right)^{1/N}.$$

Note that since $|E| = |B_1|$, $\alpha_h \to 1$, so also for Ω_h we have $|\Omega_h \Delta E| \to 0$, $P(\Omega_h) \to P(E)$. Now, by the continuity of the functional $\mathcal{G}$ in L^1 (see [DPHV19, Proposition 2.6]), we get

$$F(\Omega_h) = P(\Omega_h) + \mathcal{G}(\Omega_h) \to P(E) + \mathcal{G}(E) = \mathcal{F}(E). \tag{7.5.1}$$

On the other hand, $\Omega_h \subset \alpha_h B_{R_h}$, so it is bounded and hence, by Theorem 1.1.9,

$$\mathcal{F}(\Omega_h) \geq \mathcal{F}(B_1) \text{ for every } h.$$

Combining the last inequality with (7.5.1), we get $\mathcal{F}(E) \geq \mathcal{F}(B_1)$. Thus, the infimum in the problem ($\mathcal{P}_{\beta,K,Q}$) is achieved on balls.

Let us show that the only minimizers are the balls. Let E be a minimizer for ($\mathcal{P}_{\beta,K,Q}$). If E is bounded, then by Theorem 1.1.9 it should be a ball of radius 1. We now explain why E cannot be unbounded. Indeed, suppose the contrary holds. Then there we can find a sequence of points x_k such that $x_k \in E$, $|x_k - x_j| \geq 1$ for $k \neq j$ (for example, we can define $x_k := E \backslash B_{\max\{|x_1|,|x_2|,\ldots,|x_{k-1}|\}+1}$). Now, by density estimates for minimizers (Theorem 7.1.2 (v)), we have

$$\frac{|B_r(x) \cap E|}{|B_r|} \geq \frac{1}{C} \text{ for } x \in E,\ r \in (0, \overline{r}). \tag{7.5.2}$$

Note that even though Theorem 7.1.2 (v) deals with minimizers of ($\mathcal{P}_{\beta,K,Q,R}$), the constants C and $\overline{r}$ do not depend on R, so it applies in our case. It remains to use (7.5.2) for $x = x_k$ and $r = \min(1/2\overline{r}, 1/2)$ to see that

$$|E| \geq \sum_{k=1}^{\infty} |B_r(x_k) \cap E| \geq \sum_{k=1}^{\infty} \frac{|B_r|}{C} = \infty,$$

which contradicts the fact that $|E| = |B_1|$. Thus, E is bounded and it is a ball of radius 1. □

Appendices

Appendix A

Some regularity results

Here we collect some of the results we are using in Chapters 5, 6, and 7.

Lemma A.0.1 (Campanato's lemma, [AFP00, Theorem 7.51]). *Let $p \geq 1$ and $g \in L^p(B_{2R}(x_0))$. Assume that there exist $\sigma \in (0,1)$ and $A > 0$ such that for every $x \in B_R(x_0)$*

$$\frac{1}{|B_r|}\int_{B_r(x)} |g(y) - [g]_{x,r}|^p \, dy \leq A^p \left(\frac{r}{R}\right)^{p\sigma}, \quad \forall B_r(x) \subset B_R(x_0). \tag{A.0.1}$$

Then there exists a constant $C = C(N, p, \sigma)$ such that g is σ-Hölder continuous in $B_R(x_0)$ with a constant $C\frac{A}{R^\sigma}$ and

$$\max_{x \in B_R(x_0)} |g(x)| \leq CA + |[g]_{x_0,R}| \,.$$

Lemma A.0.2 ([AFP00, Lemma 7.54]). *Let $0 < q < p$, $s > 0$. Suppose that $h : (0, a) \to [0, +\infty)$ is an increasing function such that*

$$h(r) \leq c_1 \left(\frac{r}{R}\right)^p (h(R) + R^s) + c_2 \, R^q \quad \textit{for every } 0 < r < R,$$

where c_1 and c_2 are positive constants. Then there exists $c = c(p, q, s, c_1, c_2) > 0$ such that

$$h(r) \leq c \left\{ \left(\frac{r}{R}\right)^q h(R) + r^q \right\} \quad \textit{for every } 0 < r < R.$$

Lemma A.0.3 ([AFP00, Theorem 7.53]). *Let v be a solution of*

$$-\mathrm{div}(a_H \nabla v) = \rho_H \quad \textit{in } \mathcal{D}'\left(B_1(x_0)\right),$$

where $\rho_H \in L^\infty\left(B_1(x_0)\right)$ and

$$H := \{y \in \mathbb{R}^N \, : \, (y - x_0) \cdot e_N \leq 0\}, \quad a_H = \beta \mathbf{1}_H + \mathbf{1}_{H^c}.$$

Then there exist $\gamma \in (0,1)$ and a constant $C_0 = C_0(N, \beta, \|\rho_H\|_\infty) > 0$ such that

$$\int_{B_{\lambda r}(x_0)} |T_H v - [T_H v]_{x_0,\lambda r}|^2 \, dx \leq C_0 \lambda^{N+2\gamma} \int_{B_r(x_0)} |T_H v - [T_H v]_{x_0,r}|^2 \, dx + C_0 \, r^{N+1},$$

for all $\lambda \in (0,1)$ *small enough. Note that*

$$T_H v := \big(\partial_1 v, \dots, \partial_{N-1} v, (1 + (\beta - 1)\mathbf{1}_H)\partial_N v\big)$$

.

By arguing similarly to the proof of Theorem 7.53 in [AFP00], we can show the following lemma.

Lemma A.0.4. *Let* $H \subset \mathbb{R}^N$ *be the half space. Let* $v \in W^{1,2}(B_1)$ *be a solution of*

$$-\operatorname{div}(A\nabla v) = \operatorname{div} G \quad \text{in } \mathcal{D}'(B_1), \tag{A.0.2}$$

where

$$G^+ := G\,\mathbf{1}_H \in C^{0,\alpha}(H), \quad G^- := G\,\mathbf{1}_{H^c} \in C^{0,\alpha}(H^c),$$

A *is an elliptic matrix and* $A^+ = A\,\mathbf{1}_H$*,* $A^- = A\,\mathbf{1}_{H^c}$ *have coefficients respectively in* $C^{0,\alpha}(B_r \cap \overline{H})$ *and* $C^{0,\alpha}(B_1 \cap \overline{H^c})$*. Then*

$$v^+ := v\,\mathbf{1}_H \in C^{1,\alpha}(B_{1/2} \cap \overline{H}), \quad v^- := v\,\mathbf{1}_{H^c} \in C^{1,\alpha}(B_{1/2} \cap \overline{H^c}).$$

Moreover, there exists a constant $C = C\left(\|G^\pm\|_{C^{0,\alpha}}, \|A^\pm\|_{C^{0,\alpha}}\right) > 0$ *such that*

$$[\nabla v^+]_{C^{0,\alpha}(\overline{H} \cap B_{1/2})} \le C \quad \text{and} \quad [\nabla v^-]_{C^{0,\alpha}(\overline{H^c} \cap B_{1/2})} \le C. \tag{A.0.3}$$

Proof. Fix $x_0 \in B_{1/2}$, and let r be such that $B_r(x_0) \subset B_1$. We denote by a^+ and a^- the averages of A in $B_r(x_0) \cap H$ and $B_r(x_0) \cap H^c$ respectively. In an analogous way we define g^+ and g^- as the averages of G in $B_r(x_0) \cap H$ and $B_r(x_0) \cap H^c$. For $x \in B_r(x_0)$ we set

$$\overline{A} := \begin{cases} a^+ \text{ if } x_N > 0 \\ a^- \text{ if } x_N < 0 \end{cases} \quad \text{and} \quad \overline{G} := \begin{cases} g^+ \text{ if } x_N > 0 \\ g^- \text{ if } x_N < 0 \end{cases}.$$

By the assumptions of the lemma,

$$|A(x) - \overline{A}(x)| \le cr^\alpha \quad \text{and} \quad |G(x) - \overline{G}(x)| \le cr^\alpha. \tag{A.0.4}$$

Let w be the solution of

$$\begin{cases} -\operatorname{div}(\overline{A}\nabla w) = \operatorname{div} \overline{G} \text{ in } B_r, \\ w = v \text{ on } \partial B_r(x_0). \end{cases}$$

Note that the last equation can be rewritten as

$$\begin{cases} -\operatorname{div}(a^+\nabla w^+) = 0 \text{ in } H \cap B_r(x_0), \\ -\operatorname{div}(a^-\nabla w^-) = 0 \text{ in } H^c \cap B_r(x_0), \\ w^+ = w^- \text{ on } \partial H \cap B_r(x_0), \\ a^+\nabla w^+ \cdot e_N - a^-\nabla w^- \cdot e_N = g^+ \cdot e_N - g^- \cdot e_N \text{ on } \partial H \cap B_r(x_0), \\ w = v \text{ on } \partial B_r(x_0), \end{cases} \tag{A.0.5}$$

where $w^+ := w\,\mathbf{1}_{H\cap B_r(x_0)}$, $w^- := w\,\mathbf{1}_{H^c\cap B_r(x_0)}$. For a function u set

$$\overline{D}_c u(x) = \sum_{i=1}^{N} \overline{A}_{i,N} \nabla_i u(x) + \overline{G} \cdot e_N; \tag{A.0.6}$$

$$D_c u(x) = \sum_{i=1}^{N} A_{i,N} \nabla_i u(x) + G \cdot e_N. \tag{A.0.7}$$

The reason for such a definition is that $D_c v$ and $\overline{D}_c w$ have no jumps on the boundary thanks to the transmission condition in (A.0.5). We are going to estimate the decay of $D_\tau w$ and $\overline{D}_c w$, which will lead to Hölder continuity of $D_\tau v$ and $D_c v$, yielding the desired estimate on ∇v.

Step 1: tangential derivatives of w. Since both $\overline{A}$ and $\overline{G}$ are constant along the tangential directions, the classical difference quotient method (see, for example, [GM12, Section 4.3]) gives us that $D_\tau w \in W^{1,2}_{loc}(B_r(x_0))$ and $\operatorname{div}(\overline{A}\nabla(D_\tau w)) = 0$ in $B_r(x_0)$. Hence, Caccioppoli's inequality holds:

$$\int_{B_\rho(x)} |\nabla(D_\tau w)|^2 dy \le C\rho^{-2} \int_{B_{2\rho}(x)} |D_\tau w - (D_\tau w)_{x,2\rho}|^2 dy \tag{A.0.8}$$

for all balls $B_{2\rho}(x) \subset B_r(x_0)$ and by De Giorgi's regularity theorem, $D_\tau w$ is Hölder-continuous and, thus, if $B_{\rho'}(x) \subset B_r(x_0)$,

$$\int_{B_\rho(x)} |D_\tau w - (D_\tau w)_{x,\rho}|^2 dy \le c \left(\frac{\rho}{\rho'}\right)^{N+2\gamma} \int_{B_{\rho'}(x)} |D_\tau w - (D_\tau w)_{x,\rho'}|^2 dy \tag{A.0.9}$$

for any $\rho \in (0, \rho'/2)$ and

$$\max_{B_{\rho'/2}(x)} |D_\tau w|^2 \le \frac{C}{(\rho')^N} \int_{B_{\rho'}(x)} |D_\tau w|^2 dy. \tag{A.0.10}$$

Step 2: regularity of $\overline{D}_c w$. First let us show that the distributional gradient of $\overline{D}_c w$ is given by the gradient of $\overline{D}_c$ on the upper half ball plus the one on the lower, i.e. that there is no contribution on the hyperplane. For that, we need to check that

$$\int_{B_r(x_0)} \overline{D}_c w \operatorname{div} \varphi \, dx = \int_{B_r(x_0)^+} \nabla \overline{D}_c w \cdot \varphi \, dx + \int_{B_r(x_0)^-} \nabla \overline{D}_c w \cdot \varphi \, dx$$

for any $\varphi \in C_c^\infty(B_r(x_0); \mathbb{R}^N)$. Indeed, if we perform integration by parts on the left hand side, we get

$$\int_{B_r(x_0)} \overline{D}_c w \operatorname{div} \varphi \, dx = \int_{B_r(x_0)^+} \nabla \overline{D}_c w \cdot \varphi \, dx + \int_{B_r(x_0)^-} \nabla \overline{D}_c w \cdot \varphi \, dx$$
$$+ \int_{\partial H \cap B_r(x_0)} \left(\sum_{i=1}^{N} a^+_{i,n} \nabla_i w(x) + g^+ \cdot e_N - \sum_{i=1}^{N} a^-_{i,n} \nabla_i w(x) - g^- \cdot e_N \right) (\varphi \cdot e_N) \, d\mathcal{H}^{N-1}$$

for any $\varphi \in C_c^\infty(B_r(x_0);\mathbb{R}^N)$ and the last term vanishes thanks to the transmission condition in (A.0.5). Thus, the distributional gradient of $\overline{D}_c w$ coincides with the point-wise one.

Since $D_\tau(\overline{D}_c w) = \overline{D}_c(D_\tau w) - \overline{G}\cdot e_N$, the tangential derivatives of $\overline{D}_c w$ are in L^2_{loc}. As for the normal derivative, by the definition (A.0.6)

$$\left|\frac{\partial \overline{D}_c w}{\partial \nu}(x)\right| \le C|\nabla D_\tau w| + 2\|\overline{G}\|_{L^\infty}.$$

It implies

$$\left|\nabla \overline{D}_c w(x)\right| \le C\left(|\nabla D_\tau w| + \|\overline{G}\|_{L^\infty}\right).$$

and thus $\overline{D}_c w$ is in $W^{1,2}_{loc}$. Now, using Poincaré's inequality and (A.0.8), we have

$$\begin{aligned}
&\int_{B_\rho(x)} \left|\overline{D}_c w - (\overline{D}_c w)_{x,\rho}\right|^2 dy \le C\rho^2 \int_{B_\rho(x)} \left|\nabla(\overline{D}_c w)\right|^2 dy \\
&\le C\rho^2 \int_{B_\rho(x)} |\nabla(D_\tau w)|^2 dy + C\rho^{N+2} \le C \int_{B_{2\rho}(x)} \left|D_\tau w - (D_\tau w)_{x,2\rho}\right|^2 dy + C\rho^{N+2}
\end{aligned}$$

for any $B_{2\rho}(x) \subset B_r(x_0)$. Remembering (A.0.9), we obtain

$$\begin{aligned}
\int_{B_\rho(x)} \left|\overline{D}_c w - (\overline{D}_c w)_{x,\rho}\right|^2 dy &\le C\left(\frac{\rho}{r}\right)^{N+2\gamma} \int_{B_{r/2}(x)} \left|D_\tau w - (D_\tau w)_{x,r/2}\right|^2 dy + C\rho^{N+2} \\
&\le C\left(\frac{\rho}{r}\right)^{N+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2 dy + C\rho^{N+2}
\end{aligned} \tag{A.0.11}$$

for any $x \in B_{r/4}(x_0)$, $\rho \le r/4$. Hence, by Lemma A.0.1, $\overline{D}_c w$ is Hölder-continuous and

$$\max_{B_{r/4}(x_0)} \left|\overline{D}_c w\right|^2 \le \frac{C}{r^N} \int_{B_r(x_0)} \left|\overline{D}_c w\right|^2 dy + C. \tag{A.0.12}$$

Step 3: compairing v and w. Subtracting the equation for w from the equation for v we get

$$\begin{aligned}
\int_{B_r(x_0)} \overline{A}_{i,j}(y)\left(\frac{\partial v}{\partial y_i} - \frac{\partial w}{\partial y_i}\right)\frac{\partial \varphi}{\partial y_j} dy &= \int_{B_r(x_0)} \left(\overline{A}_{i,j}(y) - A_{i,j}(y)\right)\frac{\partial v}{\partial y_i}\frac{\partial \varphi}{\partial y_j} dy \\
&+ \int_{B_r(x_0)} \left(\overline{G}_i - G_i\right)\frac{\partial \varphi}{\partial y_i} dy
\end{aligned} \tag{A.0.13}$$

for any $\varphi \in W^{1,2}_0(B_r(x_0))$. We test (A.0.13) with $\varphi = v - w$ to get

$$\int_{B_r(x_0)} |\nabla v - \nabla w|^2 dy \le Cr^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{N+2\alpha},$$

which in turn gives us

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla v|^2 dy &\le 2\int_{B_\rho(x_0)} |\nabla w|^2 dy + 2\int_{B_\rho(x_0)} |\nabla v - \nabla w|^2 dy \\
&\le 2\omega_N \rho^N \sup_{B_{r/4}(x_0)} |\nabla w|^2 + Cr^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{N+2\alpha}
\end{aligned}$$

for $\rho \le r/4$. Recalling (A.0.10) and (A.0.12), we obtain

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \le C\left(\frac{\rho}{r}\right)^N \int_{B_r(x_0)} |\nabla w|^2 dy + C\rho^N + Cr^{2\alpha}\int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{N+2\alpha}$$
$$\le C\left(\frac{\rho}{r}\right)^N \int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{2\alpha}\int_{B_r(x_0)} |\nabla v|^2 dy + Cr^N.$$

Now we can apply Lemma A.0.2 and get there exists $r_0 > 0$ such that for $\rho < r/4 < r_0$

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \le C\left(\frac{\rho}{r}\right)^{N-\alpha} \int_{B_r(x_0)} |\nabla v|^2 dy + C\rho^{N-\alpha}.$$

In particular, for $\rho < r_0$ we have

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \le C\rho^{N-\alpha}, \tag{A.0.14}$$

where $C = C\left(\|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}, \|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}}\right)$. Note that the L^2 norm of ∇v in B_1 is bounded by some constant depending only on L^∞ norms of A and G, as can be seen by testing the equation (A.0.2) with v.

Step 4: Hölder-continuity of ∇v. We show local Hölder continuity of $D_c v$ and $D_\tau v$, Hölder-continuity of ∇v in $B_{1/2} \cap \overline{H}$ and in $B_{1/2} \cap \overline{H^c}$ follows immediately.

Take $\rho < r_0$, where r_0 is from the previous step. Let d be any real number. Using the definitions (A.0.6) and (A.0.7), inequalities (A.0.4), and inequality (A.0.14), we get

$$\int_{B_\rho(x_0)} |D_c v - d|^2 dy \le 2\int_{B_\rho(x_0)} |\overline{D}_c v - d|^2 dy + Cr^{2\alpha}\int_{B_\rho(x_0)} |\nabla v|^2 dy$$
$$\le 4\int_{B_\rho(x_0)} |\overline{D}_c w - d|^2 dy + Cr^{N+\alpha}$$

and hence, using (A.0.11) we have for $\rho < r/4$, $r < r_0$

$$\int_{B_\rho(x_0)} |D_c v - (D_c v)_{x_0,\rho}|^2 dy \le \int_{B_\rho(x_0)} |D_c v - (\overline{D}_c w)_{x_0,\rho}|^2 dy$$
$$\le 4\int_{B_\rho(x_0)} |\overline{D}_c w - (\overline{D}_c w)_{x_0,\rho}|^2 dy + Cr^{N+\alpha} \le C\left(\frac{\rho}{r}\right)^{N+2\gamma}\int_{B_r(x_0)} |D_\tau w|^2 dy + Cr^{N+\alpha}. \tag{A.0.15}$$

Similarly, using (A.0.9) instead of (A.0.11), we get

$$\int_{B_\rho(x_0)} |D_\tau v - (D_\tau v)_{x_0,\rho}|^2 dy \le C\left(\frac{\rho}{r}\right)^{N+2\gamma}\int_{B_r(x_0)} |D_\tau w|^2 dy + Cr^{N+\alpha}. \tag{A.0.16}$$

Applying Lemma A.0.2 to (A.0.15) and (A.0.16), we deduce that $D_c v$ and $D_\tau v$ are Hölder by Lemma A.0.1.

□

Appendix B

Inequalities for powers

Lemma B.0.1 ([FZ16, Lemma 2.3]). *Let $p > 1$. There exists $c(p) \geq 0$ such that if $\kappa \geq 0$ and $\xi, \eta \in \mathbb{R}^N$ then*

$$\left(\left(\kappa^2 + |\xi|^2\right)^{\frac{p-2}{2}} \xi - \left(\kappa^2 + |\eta|^2\right)^{\frac{p-2}{2}} \eta\right) \cdot (\xi - \eta) \geq c \left(\kappa^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2.$$

Moreover, there exists another constant $C(p) \geq 0$ such that if $\Omega \subset \mathbb{R}^N$ is an open set and for u, $v \in W^{1,p}(\Omega)$ and $0 \leq s \leq 1$, we set $u^s(x) = su(x) + (1-s)v(x)$, then the following two inequalities hold:

- *for $p \geq 2$*

$$\int_\Omega |\nabla u - \nabla v|^p \leq C \int_0^1 \frac{1}{s}\, ds \int_\Omega \left(|\nabla u^s|^{p-2} \nabla u^s - |\nabla v|^{p-2} \nabla v\right) \cdot \nabla(u^s - v); \tag{B.0.1}$$

- *for $1 < p < 2$*

$$\int_\Omega |\nabla u - \nabla v|^p \leq C \left(\int_0^1 \frac{1}{s}\, ds \int_\Omega \left(|\nabla u^s|^{p-2} \nabla u^s - |\nabla v|^{p-2} \nabla v\right) \cdot \nabla(u^s - v)\right)^{\frac{p}{2}} \cdot \\ \cdot \left(\int_\Omega (|\nabla u| + |\nabla v|)^p\right)^{1-\frac{p}{2}}. \tag{B.0.2}$$

Lemma B.0.2. *Let $x, y \in \mathbb{R}^N$, $p \in (1, \infty)$. Then the following inequalities hold:*

- *if $p \geq 2$, then*

$$|y|^p \geq |x|^p + p|x|^{p-2} x \cdot (y - x) + c|y - x|^p$$

  for some $c = c(p) > 0$;

- *if $1 < p < 2$, then*

$$|y|^p \geq |x|^p + p|x|^{p-2} x \cdot (y - x) + c|y - x|^2 \left(|x|^2 + |y - x|^2\right)^{\frac{p-2}{2}}$$

  for some $c = c(p) > 0$.

Proof. Consider a function $f : \mathbb{R}^N \to \mathbb{R}$ defined as $f(x) = |x|^p$. Writing Taylor expansion for f we get

$$|y|^p = |x|^p + p|x|^{p-2}x \cdot (y-x) + \int_0^1 (1-t)D^2 f(x+t(y-x))(y-x)\cdot(y-x)\,dt.$$

If $p = 2$, we thus have

$$|y|^2 = |x|^2 + 2x\cdot(y-x) + \frac{1}{2}|y-x|^2,$$

which gives us a desired inequality. We shall consider $p \neq 2$ from now on.

For $p \neq 2$ the Hessian $D^2 f(x)$ looks as follows:

$$D^2 f(x) = p|x|^{p-2} Id + p(p-2)|x|^{p-4} A,$$

where $A_{i,j} = x_i x_j$. We notice that

$$0 \le A\xi\cdot\xi \le |x|^2|\xi|^2 \text{ for any vector } \xi \in \mathbb{R}^N,$$

yielding

$$D^2 f(x)\xi\cdot\xi \ge c|x|^{p-2}|\xi|^2 \text{ for any vector } \xi \in \mathbb{R}^N,$$

where $c = c(p) > 0$ ($c = p$ for $p > 2$, $c = p(p-1)$ for $1 < p < 2$).

So, we have

$$|y|^p \ge |x|^p + p|x|^{p-2}x\cdot(y-x) + |y-x|^2\int_0^1 (1-t)|x+t(y-x)|^{p-2}\,dt.$$

Let us consider the cases of different p separately. First, we deal with $1 < p < 2$. In this case $p - 2 < 0$ and so

$$\int_0^1 (1-t)|x+t(y-x)|^{p-2}\,dt \ge \frac{1}{4}\int_{1/4}^{3/4} (|x| + t|y-x|)^{p-2}\,dt \ge c\left(|x|^2 + |y-x|^2\right)^{\frac{p-2}{2}},$$

finishing the proof of lemma in this case.

To tackle the case $p > 2$, we further consider two cases. If $|y-x| < 2|x|$, then

$$\int_0^1 (1-t)|x+t(y-x)|^{p-2}\,dt \ge c\int_0^{1/4} |x|^{p-2}\,dt \ge c|y-x|^{p-2}.$$

If instead $|y-x| \ge 2|x|$, then

$$\int_0^1 (1-t)|x+t(y-x)|^{p-2}\,dt \ge c\int_{4/7}^{6/7} |y-x|^{p-2}\,dt \ge c|y-x|^{p-2}.$$

□

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