PHASE-FIELD APPROXIMATION OF FUNCTIONALS DEFINED ON PIECEWISE-RIGID MAPS

MARCO CICALESE*, MATTEO FOCARDI, AND CATERINA IDA ZEPPIERI

Abstract. We provide a variational approximation of Ambrosio-Tortorelli type for brittle fracture energies of piecewise-rigid solids. Our result covers both the case of geometrically nonlinear elasticity and that of linearised elasticity.

Keywords: phase-field models, elliptic approximation, free-discontinuity problems, Γ-convergence, geometric rigidity, piecewise-rigid maps, linearised elasticity, brittle fracture.

2000 Mathematics Subject Classification: 49J45, 49Q20, 74B20, 74G65.

1. Introduction

According to the Griffith theory of crack-propagation in brittle materials [42], the equilibrium configuration of a fractured body is determined by balancing the reduction in bulk elastic energy $\mathcal{E}^e$ stored in the material with the increment in fracture energy $\mathcal{E}^f$ due to the formation of a new free surface. For those materials for which crack-growth can be seen as a quasi-static process, the equilibrium configurations are obtained, at each time, by solving a minimisation problem involving the total free energy of the system; *i.e.*, $\mathcal{E} := \mathcal{E}^e + \mathcal{E}^f$.

For hyperelastic brittle materials a prototypical elastic energy $\mathcal{E}^e$ is of the form

$$\mathcal{E}^e(u, K) = \mu \int_{\Omega\setminus K} W(\nabla u)\, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is open, bounded and represents the reference configuration of a body which is fractured along a sufficiently regular closed surface $K \subset \Omega$, and $u : \Omega\setminus K \to \mathbb{R}^3$ is the deformation map, which is smooth outside K. In (1.1) the constant $\mu > 0$ represents the shear modulus of the material and $W : \mathbb{M}^{3\times 3} \to [0, +\infty)$ the stored elastic energy density. In the setting of nonlinear elasticity W is assumed to be frame indifferent and to vanish only on $SO(3)$, the set of 3×3 rotation matrices; moreover, close to $SO(3)$ the function $W(\cdot)$ behaves like $\mathrm{dist}^2(\cdot, SO(3))$.

In a simplified isotropic setting, the fracture energy of a brittle material obeys the Griffith criterion and is proportional to the area of the crack-surface K; *i.e.*,

$$\mathcal{E}^f(K) = \gamma\, \mathcal{H}^2(K), \tag{1.2}$$

where the proportionality constant $\gamma > 0$ measures the fracture toughness (or fracture resistance) of the material.

Choosing $\mathcal{E}^e$ and $\mathcal{E}^f$ as in (1.1) and (1.2), respectively, the total energy $\mathcal{E}$ takes the form

$$\mathcal{E}(u, K) = \mu \int_{\Omega\setminus K} W(\nabla u)\, dx + \gamma\, \mathcal{H}^2(K).$$

The functional $\mathcal{E}$ can then be minimised by resorting to a weak formulation of the problem in De Giorgi and Ambrosio's space of *special functions of bounded variation* $SBV(\Omega)$ [31]. In this way the pair (u, K) is replaced by a single variable u which can be discontinuous on a lower-dimensional set J_u, which now plays the role of the crack-surface K. Moreover if $u \in SBV(\Omega)$ the distributional derivative Du can be decomposed into a volume contribution ∇u, which is to be interpreted as the deformation gradient outside the crack, and a surface contribution concentrated along the crack-set J_u. In the $SBV(\Omega)$-setting the energy $\mathcal{E}$ then becomes

$$\mathcal{F}(u) = \mu \int_{\Omega} W(\nabla u)\, dx + \gamma\, \mathcal{H}^2(J_u), \tag{1.3}$$

and its minimisation can be carried out by applying the direct methods to $\mathcal{F}$, or to its relaxed functional (cf. [4]).

Functionals as in (1.3) are commonly referred to as *free-discontinuity functionals* and play a central role both in fracture mechanics [37, 11, 12] and in computer vision [52] and have been extensively studied in the last decades [4, 14].

In this paper we are interested in the case when the material parameters in (1.3) satisfy the relation $\mu/\gamma \gg 1$, or, up to a renormalisation, when $\mu \gg 1$ and $\gamma = O(1)$. This parameter-regime is typical of rigid solids; *i.e.,* of solids which deform without storing any elastic energy. In fact, being μ large (and $\gamma = O(1)$), a deformation u shall satisfy $W(\nabla u) = 0$ which is equivalent to asking $\nabla u \in SO(3)$ almost everywhere in Ω. However, since $u \in SBV(\Omega)$, the differential constraint $\nabla u \in SO(3)$ does not prevent u to jump and thus fracture to occur. Hence, for $\mu \gg 1$ the energy-functional $\mathcal{F}$ models those brittle solids which exhibit a rigid behaviour in a number of subregions of Ω which are separated from one another by a discontinuity surface. Mathematically, these configurations are described by the so-called *piecewise-rigid* maps on Ω and are denoted by $PR(\Omega)$. Namely, $u \in PR(\Omega)$ if

$$u(x) = \sum_{i \in \mathbb{N}} (\mathbb{A}_i x + b_i) \chi_{E_i}(x), \tag{1.4}$$

where, for every $i \in \mathbb{N}$, $\mathbb{A}_i \in SO(3)$, $b_i \in \mathbb{R}^3$, and (E_i) is a Caccioppoli partition of Ω. Then, up to a lower-oder bulk-contribution, in the regime $\mu \gg 1$ the total energy $\mathcal{E}$ of a brittle rigid solid can be identified with its fracture energy; the latter coincides with the surface term in (1.3) where now the deformation-variable u belongs to the space $PR(\Omega)$ (see [39]).

Despite their simple analytical expression, energy functionals of type $\gamma\, \mathcal{H}^{n-1}(J_u)$ are notoriously difficult to be treated numerically, due to their explicit dependence on the discontinuity surface J_u. To develop efficient methods to compute their energy minimisers and to analyse phenomena like crack-initiation, crack-branching or arrest in nonlinearly elastic brittle materials, in the engineering community suitable "regularisations" have been recently proposed, where the surface J_u is replaced by an additional phase-field variable $v \in [0, 1]$ (see *e.g.*, [22, 60] and references therein). In these models the phase-filed variable v interpolates between the sound state (corresponding to $v = 1$) and the fractured state of the material (corresponding to $v = 0$) and it is to be interpreted as a damage variable in the spirit of [53, 54, 55].

It is well-known that the relationship between variational models for brittle fracture and (gradient) damage models can be made rigorous building upon the seminal approximation result of Ambrosio and Tortorelli [5, 6] as shown in [34, 18, 7, 8, 9, 59, 44, 19], just to mention a few examples. Furthermore, damage models á la Ambrosio-Tortorelli can be also used to approximate fracture models of cohesive-type as shown, *e.g.*, in [1, 2, 24, 30, 36, 43, 16, 26].

The purpose of the present paper is to establish a rigorous mathematical connection between damage models of Ambrosio-Tortorelli type and variational fracture models for brittle piecewise-rigid solids. In other words, in this work we provide an elliptic approximation of functionals of type

$$F(u) = \gamma\, \mathcal{H}^{n-1}(J_u), \quad u \in PR(\Omega), \tag{1.5}$$

where the constraint $u \in PR(\Omega)$ is reminiscent of the nonlinear elastic energy density W, which satisfy $W^{-1}(\{0\}) = SO(n)$.

Namely, we show that for $(u,v) \in W^{1,2}(\Omega;\mathbb{R}^n) \times W^{1,2}(\Omega)$, $0 \leq v \leq 1$ the family of functionals

$$F_\varepsilon(u,v) = \int_\Omega k_\varepsilon\, v^2\, W(\nabla u)\, dx + \frac{\gamma}{2}\int_\Omega \left(\frac{(v-1)^2}{\varepsilon} + \varepsilon|\nabla v|^2\right) dx, \tag{1.6}$$

converges, in the sense of De Giorgi's Γ-convergence [27, 13], to the functional (1.5), under the assumption that $k_\varepsilon \to +\infty$, as $\varepsilon \to 0$. As in the case of the Modica-Mortola functional [48, 49] and of the Ambrosio-Tortorelli approximation [5, 6], in (1.6) the singular-perturbation parameter $\varepsilon > 0$ determines the thickness of the diffuse interface around the limit discontinuity surface J_u, while the diverging parameter k_ε is proportional to the stiffness of the material, hence to the constant μ appearing in (1.3). More precisely, in Theorem 3.3 we prove that if the zero-set of the bulk energy density W coincides with $SO(n)$ and for every $\mathbb{A} \in \mathbb{M}^{n\times n}$ it holds

$$W(\mathbb{A}) \geq \alpha\, \mathrm{dist}^2(\mathbb{A}, SO(n)),$$

for some $\alpha > 0$, then the family (F_ε) Γ-converges to F, in the $L^1(\Omega;\mathbb{R}^n) \times L^1(\Omega)$ topology. The proof of Theorem 3.3 takes advantage of a number of analytical tools. First, to determine the set of the limit deformations we use a piecewise-rigidity result in $SBV(\Omega)$ by Chambolle, Giacomini and Ponsiglione [20] (cf. Theorem 2.3). The latter is the counterpart of the Liouville rigidity Theorem for deformations of brittle elastic materials and provides a characterisation of discontinuous deformations with zero elastic energy as a collection of an at most countable family of rigid motions defined on an underlying Caccioppoli partition of Ω. To match the assumptions of Chambolle, Giacomini, and Ponsiglione's result we use a global argument of Ambrosio [3] which is based on the co-area formula and is tailor-made to gain compactness in SBV. Namely, starting from a pair $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(\Omega,\mathbb{R}^n) \times W^{1,2}(\Omega)$ with equi-bounded energy F_ε we use the co-area formula to find a suitable sublevel set of v_ε in which u_ε can be modified to obtain a new sequence $(\tilde u_\varepsilon) \subset SBV(\Omega)$ which differs from u_ε on a set of vanishing measure and moreover satisfies

$$\sup_{\varepsilon>0}\left(k_\varepsilon \int_\Omega \mathrm{dist}^2(\nabla \tilde u_\varepsilon, SO(n))\, dx + \mathcal{H}^{n-1}(J_{\tilde u_\varepsilon})\right) < +\infty.$$

The estimate above, combined with a result of Zhang which guarantees that the zero-set of the quasiconvexification of $\mathrm{dist}(\cdot, SO(n))$ coincides with $SO(n)$, proves that any L^1 limit u of $\tilde u_\varepsilon$ satisfies $\nabla u \in SO(n)$ a.e. in Ω. Eventually, the Chambolle-Giacomini-Ponsiglione piecewise-rigidity Theorem yields that u is a piecewise-rigid map. The construction of Ambrosio additionally provides us with the sharp lower bound. Indeed, the perimeter of the sublevel sets of v_ε chosen as above prove to be asymptotically larger than the interfacial energy-contribution of the piecewise rigid limit deformation. As in the case of the Ambrosio-Tortorelli functional, the sharp interfacial energy is defined in terms of a one-dimensional optimal profile problem. The upper bound is then proven first by resorting to a density argument and then by an explicit construction. Namely, we use the density in $PR(\Omega)$ of *finite* partitions subordinated to Caccioppoli sets which are *polyhedral* [15]. Then, for these partitions, a recovery sequence matching asymptotically the sharp lower bound can be contructed by creating a layer of order ε around the jump set of the target function u, in which the transition is one-dimensional and is obtained by a suitable scaling of the optimal profile.

As for the Ambrosio-Tortorelli approximation of the Mumford-Shah functionals (see also, e.g., [34, 7, 8, 9, 59]) also in our case the regularised bulk and surface energy in (1.6) separately converge to their sharp counterparts. Namely, in this case the bulk term in (1.6) vanishes in the limit due to the presence of the diverging parameter k_ε, that is, equivalently, limit deformations have (approximate) gradients in $SO(n)$ a.e. in Ω. Similarly, the Modica-Mortola term in (1.6) approximates the limit surface energy, which in our model carries the whole energy contribution.

It is worth mentioning that the arguments in Theorem 3.3 can be extended (resorting to by-now standard modifications) to cover the case of anisotropic surface-integrals which model the presence of preferred cleavage planes in single crystals (cf. Remark 3.4).

In Theorem 4.4 we generalise the approximation result Theorem 3.3 to the case of energy densities W vanishing on a compact set $\mathcal{K} \subset \mathbb{M}^{n\times n}$ for which a piecewise-rigidity result analogous to the one for $SO(n)$-valued discontinuous deformations holds true. In fact in [20] piecewise rigidity is proven, more in general, for those $\mathcal{K}$ for which a quantitative L^p-rigidity estimate holds (see Section 4 for more details). In this way, multiple incompatible wells can be also taken into account. From a mechanical point of view, the incompatibility describes those solids for which no fine-scale phase-mixtures are allowed in solid-solid transformations (see [51]). A list of non trivial examples of possible compact sets $\mathcal{K}$ fulfilling the assumptions of Theorem 4.4 is also included.

Finally, in Theorem 4.9 a further approximation result is provided, which covers the case of linearised elasticity.

2. Notation and preliminaries

2.1. **Notation.** In what follows $\Omega \subset \mathbb{R}^n$ denotes a bounded domain (*i.e.*, an open and connected set) with Lipschitz boundary. We use a standard notation for Lebesgue and Sobolev spaces, and for the Hausdorff measure. The Euclidean scalar product in $\mathbb{R}^n$ is denoted by $\langle\cdot,\cdot\rangle$.

We refer the reader to the book [4] for a comprehensive introduction to the theory of functions of bounded variation and of (generalised) special functions of bounded variation $(G)SBV(\Omega)$ and to [28] for the definition and main properties of generalised special functions of bounded deformation $GSBD(\Omega)$. In any of these cases we shall deal with the proper subspaces of these functional spaces in $L^1(\Omega,\mathbb{R}^n)$.

2.2. **Caccioppoli-affine functions.** We recall here the definition of Caccioppoli-affine and piecewise-rigid function. Moreover we also recall the piecewise-rigidity result [20, Theorem 1.1] in a variant which is useful for our purposes.

Definition 2.1. *A map $u\colon \Omega \to \mathbb{R}^n$ is called Caccioppoli-affine if there exist matrices $\mathbb{A}_i \in \mathbb{M}^{n\times n}$ and vectors $b_i \in \mathbb{R}^n$ such that*

$$u(x) = \sum_{i\in\mathbb{N}} (\mathbb{A}_i x + b_i)\chi_{E_i}(x), \tag{2.1}$$

with (E_i) Caccioppoli partition of Ω. In particular, if $\mathbb{A}_i \in SO(n)$ for every $i \in \mathbb{N}$, then u as in (2.1) *is called piecewise rigid. The set of piecewise-rigid functions on Ω will be denoted by $PR(\Omega)$.*

The measure theoretic properties of Caccioppoli-affine functions are collected in the result below (cf. [23, Theorem 2.2]).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and with Lipschitz boundary. Let $u\colon \Omega \to \mathbb{R}^n$ be Caccioppoli-affine, then $u \in (GSBV(\Omega))^n$. Moreover,*

(1) $\nabla u = \mathbb{A}_i$ $\mathcal{L}^n$-a.e. on E_i, for every $i \in \mathbb{N}$;
(2) $J_u = \bigcup_{i\in\mathbb{N}} \partial^* E_i \cap \Omega$, up to a set of zero $\mathcal{H}^{n-1}$-measure.

Below we recall a slight generalisation of the piecewise-rigidity result by Chambolle, Giacomini, and Ponsiglione [20, Theorem 1.1] originally stated in the SBV-setting. The proof follows from, *e.g.*, [38, Theorem 2.3].

Theorem 2.3. *Let $u \in GSBV(\Omega,\mathbb{R}^n)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$ and $\nabla u \in SO(n)$ a.e. in Ω. Then, $u \in PR(\Omega)$.*

3. Setting of the problem and main result

In this section we introduce a family of functionals of Ambrosio-Tortorelli type (cf. [5, 6]) and we prove that this family converges to a surface functional of perimeter type which is finite only on piecewise-rigid maps.

Let $W\colon \Omega\times\mathbb{M}^{n\times n}\to[0,+\infty)$ be a Borel function such that $W(x,\mathbb{A})=0$ for every $\mathbb{A}\in SO(n)$. Assume moreover that for every $x\in\Omega$ and every $\mathbb{A}\in\mathbb{M}^{n\times n}$ it holds

$$W(x,\mathbb{A})\geq\alpha\,\mathrm{dist}^2(\mathbb{A},SO(n)), \tag{3.1}$$

for some $\alpha>0$.

Let $\Phi\colon[0,1]\to[0,1]$ be an increasing and lower semicontinuous function such that $\Phi(0)=0$, $\Phi(1)=1$, $\Phi(t)>0$ for $t>0$; let moreover $V\colon[0,1]\to[0,+\infty)$ be a continuous function with $V^{-1}(\{0\})=\{1\}$.

For $\varepsilon>0$ and $\beta>0$ we introduce the auxiliary functionals $G^\beta_\varepsilon\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)\longrightarrow[0,+\infty]$ defined as

$$G^\beta_\varepsilon(u,v):=\begin{cases}\displaystyle\int_\Omega\Big(\beta\,\Phi(v)\,Q(\mathrm{dist}^2(\cdot,SO(n))(\nabla u)+\frac{V(v)}{\varepsilon}+\varepsilon|\nabla v|^2\Big)\,dx & u\in W^{1,2}(\Omega,\mathbb{R}^n)\\ & v\in W^{1,2}(\Omega),\ 0\leq v\leq 1\\ +\infty & \text{otherwise.}\end{cases} \tag{3.2}$$

where $Q(\mathrm{dist}^2(\cdot,SO(n))(\mathbb{A})$ denotes the quasi-convex envelope of $\mathrm{dist}^2(\cdot,SO(n))$ computed at $\mathbb{A}\in\mathbb{M}^{n\times n}$ (cf. [41, Section 5.3]). The following Theorem is a consequence of [34, Theorem 3.1].

Theorem 3.1. *The family of functionals (G^β_ε) defined in (3.2) $\Gamma(L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega))$-converges to the functional $G^\beta\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)\longrightarrow[0,+\infty]$ given by*

$$G^\beta(u,v):=\begin{cases}\displaystyle\int_\Omega\beta\,Q(\mathrm{dist}^2(\cdot,SO(n))(\nabla u)\,dx+2C_V\mathcal{H}^{n-1}(J_u) & u\in(GSBV^2(\Omega))^n,\\ & v=1\ \text{a.e. on }\Omega,\\ +\infty & \text{otherwise,}\end{cases} \tag{3.3}$$

where $C_V:=2\int_0^1\sqrt{V(s)}\,ds$ and $GSBV^2(\Omega):=\{u\in GSBV(\Omega)\colon \nabla u\in L^2(\Omega,\mathbb{R}^n),\ \mathcal{H}^{n-1}(S_u)<+\infty\}$.

Proof. The proof immediately follows by [34, Theorem 3.1] also noticing that $GSBV^2(\Omega,\mathbb{R}^n)$ coincides with $(GSBV^2(\Omega))^n$ (see [29, Proposition 2.3]). ☐

For $\varepsilon>0$ let $k_\varepsilon\to+\infty$, as $\varepsilon\to0$. We consider the phase-field functionals $F_\varepsilon\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)\longrightarrow[0,+\infty]$ defined as

$$F_\varepsilon(u,v):=\begin{cases}\displaystyle\int_\Omega\Big(k_\varepsilon\Phi(v)\,W(x,\nabla u)+\frac{V(v)}{\varepsilon}+\varepsilon|\nabla v|^2\Big)\,dx & (u,v)\in W^{1,2}(\Omega,\mathbb{R}^n)\times W^{1,2}(\Omega),\\ & 0\leq v\leq 1\ \text{ a.e. on }\Omega,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.4}$$

In the following proposition we show that the Γ-limit of (F_ε) (if it exists) is finite only on the set of piecewise-rigid maps.

Proposition 3.2 (Domain of the Γ-limit)**.** *Let $(u_\varepsilon,v_\varepsilon)\subset W^{1,2}(\Omega,\mathbb{R}^n)\times W^{1,2}(\Omega)$, $0\leq v_\varepsilon\leq1$ a.e. in Ω, be such that*

$$\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,v_\varepsilon)<+\infty\quad\textit{and}\quad u_\varepsilon\to u\ \textit{ in }\ L^1(\Omega,\mathbb{R}^n).$$

Then, $v_\varepsilon \to 1$ *in* $L^1(\Omega)$ *and* $u \in PR(\Omega)$.

Proof. Let $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(\Omega,\mathbb{R}^n) \times W^{1,2}(\Omega)$, $0 \le v_\varepsilon \le 1$ a.e. in Ω, be such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty \quad \text{and} \quad u_\varepsilon \to u \ \text{ in } \ L^1(\Omega,\mathbb{R}^n).$$

We first note that up to subsequences (not relabelled) we have

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty, \tag{3.5}$$

from which the convergence $v_\varepsilon \to 1$ in $L^1(\Omega)$ easily follows. In fact, for $\eta > 0$ we have

$$\mathcal{L}^n(\{1-\eta > v_\varepsilon\}) \min\{V(s)\colon s \in [0, 1-\eta)\} \le \int_\Omega V(v_\varepsilon)\, dx \le \varepsilon \sup_\varepsilon F_\varepsilon(u_\varepsilon, v_\varepsilon). \tag{3.6}$$

Since $V^{-1}(\{0\}) = 1$, the minimum in the left hand side of (3.6) is strictly positive. Therefore, gathering (3.6) and (3.5) implies that $v_\varepsilon \to 1$ in measure. The latter, together with the uniform bound satisfied by (v_ε) immediately gives $v_\varepsilon \to 1$ in $L^1(\Omega)$.

We are then left to show that $u \in PR(\Omega)$. Since $\mathrm{dist}^2(\mathbb{A}, SO(n)) \ge Q(\mathrm{dist}^2(\cdot, SO(n))(\mathbb{A})$ and $k_\varepsilon \to +\infty$, appealing to Theorem 3.1, for every $\beta > 0$ we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \ &\ge \ \liminf_{\varepsilon\to 0} \int_\Omega \left(\beta\, \Phi(v)\, \mathrm{dist}^2(\nabla u, SO(n)) + \frac{V(v)}{\varepsilon} + \varepsilon|\nabla v|^2\right) dx \\ &\ge \ \liminf_{\varepsilon\to 0} G^\beta_\varepsilon(u_\varepsilon, v_\varepsilon) \ge G^\beta(u, v).\end{aligned} \tag{3.7}$$

As a consequence, by (3.5), we obtain that $u \in (GSBV^2(\Omega))^n$ and therefore

$$\mathcal{H}^{n-1}(J_u) < +\infty. \tag{3.8}$$

Furthermore, by the arbitrariness of $\beta > 0$, (3.5), and the very definition of G^β, we also get

$$Q(\mathrm{dist}^2(\cdot, SO(n))(\nabla u) = 0 \quad \text{a.e. in} \ \ \Omega. \tag{3.9}$$

Since by [62, Theorem 1.1] (see also [61, Theorem 1.1]) it holds

$$\big\{\mathbb{A} \in \mathbb{M}^{n\times n}\colon Q(\mathrm{dist}^2(\cdot, SO(n))(\mathbb{A}) = 0\big\} = SO(n), \tag{3.10}$$

from (3.9) we obtain that $\nabla u \in SO(n)$ a.e. in Ω. Eventually, since u belongs to $(GSBV(\Omega))^n$ and satisfies (3.9) and (3.8), by (3.10) we can invoke Theorem 2.3 to get that $u \in PR(\Omega)$ and thus the claim. □

The next theorem establishes a Γ-convergence result for the functionals F_ε.

Theorem 3.3. *The family of functionals* (F_ε) *defined in* (3.4) $\Gamma(L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega))$*-converges to the functional* $F\colon L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega) \longrightarrow [0,+\infty]$ *given by*

$$F(u, v) := \begin{cases} 2C_V \mathcal{H}^{n-1}(J_u) & \text{if } u \in PR(\Omega) \ \text{ and } \ v = 1 \text{ a.e. in } \ \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.11}$$

where $C_V := 2\int_0^1 \sqrt{V(s)}\, ds$.

Proof. We divide the proof into two steps.

Step 1: Ansatz-free lower bound.

Let $(u, v) \in L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega)$ be arbitrary; we need to show that for all sequences $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega)$, $0 \le v_\varepsilon \le 1$ a.e. in Ω, we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \ge F(u, v). \tag{3.12}$$

Without loss of generality, up to the extraction of a subsequence, we may assume that the liminf in (3.12) is a limit; therefore we have

$$\sup_{\varepsilon} F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty \tag{3.13}$$

Then Proposition 3.2 readily implies that $u \in PR(\Omega)$ and $v = 1$ a.e. in Ω. For such u and v, using again the estimate (3.7) in the proof of Proposition 3.2 we obtain the lower bound inequality (3.12) by observing that $G^\beta(u,1) = 2C_V\mathcal{H}^{n-1}(J_u)$ for every $\beta > 0$.

Step 2: Existence of a recovery sequence. Let $(u,v) \in L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega)$ be arbitrary, in this step we will construct a sequence $(u_\varepsilon, v_\varepsilon) \to (u,v)$ in $L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega)$ such that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \leq F(u,v). \tag{3.14}$$

We start by noticing that the inequality in (3.14) is trivial unless we additionally assume that $u \in PR(\Omega)$ and $v = 1$ a.e. in Ω. Therefore, in particular we can write u as

$$u(x) = \sum_{i\in\mathbb{N}} (\mathbb{A}_i x + b_i)\chi_{E_i}(x), \tag{3.15}$$

where $\mathbb{A}_i \in SO(n)$, $b_i \in \mathbb{R}^n$ for every $i \in \mathbb{N}$, and (E_i) is Caccioppoli partition of Ω.

By standard density and continuity arguments (cf. [13, Remark 1.29]) we notice that it is enough to prove (3.14) in a subset X of $PR(\Omega)$, which is dense in $PR(\Omega)$ in the following sense: for every $u \in PR(\Omega)$ there exists $(u_j) \subset X$ such that

$$u_j \to u \;\text{ in }\; L^1(\Omega,\mathbb{R}^n) \;\text{ and }\; \mathcal{H}^{n-1}(J_{u_j}) \to \mathcal{H}^{n-1}(J_u), \tag{3.16}$$

for $j \to +\infty$.

We now claim that X is given by those $u \in PR(\Omega)$ of the form

$$u(x) = \sum_{i=1}^{N} (\widehat{\mathbb{A}}_i x + \hat{b}_i)\chi_{\widehat{E}_i}(x), \tag{3.17}$$

where $\widehat{\mathbb{A}}_i \in SO(n)$, $\hat{b}_i \in \mathbb{R}^n$, and, for every $i = 1,\dots,N$, $\widehat{E}_i$ is a polyhedral set in Ω. We recall that a set $\widehat{E} \subset \Omega$ is called *polyhedral* if there exist a finite number of $(n-1)$-dimensional simplexes $M_1,\dots,M_N \subset \mathbb{R}^n$, such that $\partial\widehat{E} \cap \Omega$ coincides with $\Omega \cap \cup_{j=1}^N M_j$ up to a $\mathcal{H}^{n-1}$-null set.

Indeed given u as in (3.15) the sequence (u_N) defined as

$$u_N(x) = \sum_{i=1}^{N-1} (\mathbb{A}_i x + b_i)\chi_{E_i}(x) + (\mathbb{A}_1 x + b_1)\chi_{\Omega\setminus\bigcup_i^{N-1} E_i},$$

clearly satisfies $u_N \to u$ in $L^1(\Omega,\mathbb{R}^n)$, as $N \to +\infty$. Moreover, by lower semicontinuity we have that $\mathcal{H}^{n-1}(J_u) \leq \liminf_N \mathcal{H}^{n-1}(J_{u_N})$, hence since $\mathcal{H}^{n-1}(J_{u_N}) \leq \mathcal{H}^{n-1}(J_u)$ for every $N \in \mathbb{N}$, we obtain

$$\mathcal{H}^{n-1}(J_{u_N}) \to \mathcal{H}^{n-1}(J_u),$$

as $N \to +\infty$.

Further, given the finite partition of Ω into sets of finite perimeter $E'_1,\dots,E'_N$, with $E'_i := E_i$ for $i = 1,\dots,N-1$ and $E'_N := \Omega \setminus \bigcup_i^{N-1} E_i$, we can invoke [15, Corollary 2.5] to deduce the existence of a partition of Ω into polyhedral sets $\widehat{E}^j_1,\dots,\widehat{E}^j_N$ such that, for every $i = 1,\dots,N$,

$$\mathcal{L}^n(\widehat{E}^j_i \triangle E'_i) \to 0 \quad\text{and}\quad \mathcal{H}^{n-1}(\partial^*\widehat{E}^j_i) \to \mathcal{H}^{n-1}(\partial^* E'_i),$$

as $j \to +\infty$. Eventually, the desired sequence (u_j) satisfying (3.16)-(3.17) can be obtained by a standard diagonal argument.

We now construct a recovery sequence $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(\Omega, \mathbb{R}^n) \times W^{1,2}(\Omega)$ for F_ε when u is as in (3.17). Therefore we have that, in particular, up to a set of zero $\mathcal{H}^{n-1}$-measure

$$J_u = \bigcup_{i=1}^{M} S_i \cap \Omega, \tag{3.18}$$

where $S_1, \dots, S_M \subset \mathbb{R}^n$ are a finite number of $(n-1)$-dimensional simplexes.

For every $i \in \{1, \dots, M\}$ we denote with Π_i the $(n-1)$-dimensional hyperplane containing the simplex S_i; we have that $\Pi_i \neq \Pi_\ell$, for $i \neq \ell$.

We start by constructing v_ε. To this end, we recall that

$$C_V = 2\int_0^1 \sqrt{V(s)} = \inf_{T>0} \min\left\{ \int_0^T (V(w) + |w'|^2)\, dt \colon w \in \mathcal{A}(0,T) \right\}; \tag{3.19}$$

where

$$\mathcal{A}(0,T) := \{w \in W^{1,\infty}(0,T) \colon 0 \le w \le 1,\, w(0) = 0,\, w(T) = 1\},$$

(see e.g. [13, Remark 6.1]) Hence for every fixed $\eta > 0$ there exists $T_\eta > 0$ and $w_\eta \in W^{1,\infty}(0, T_\eta)$ with $0 \le w_\eta \le 1$, $w_\eta(0) = 0$ and $w_\eta(T_\eta) = 1$ such that

$$\int_0^{T_\eta} \big(V(w_\eta) + |w_\eta'|^2\big)\, dt \le C_V + \eta. \tag{3.20}$$

Let now $0 < \xi_\varepsilon \ll \varepsilon$ and define the function $h_\varepsilon \colon [0, +\infty) \to [0,1]$ as

$$h_\varepsilon(t) := \begin{cases} 0 & \text{if } 0 \le t \le \xi_\varepsilon, \\ w_\eta\left(\frac{t-\xi_\varepsilon}{\varepsilon}\right) & \text{if } \xi_\varepsilon \le t \le \xi_\varepsilon + \varepsilon T_\eta, \\ 1 & \text{if } t \ge \xi_\varepsilon + \varepsilon T_\eta. \end{cases} \tag{3.21}$$

Let $\pi_i \colon \mathbb{R}^n \to \Pi_i$ denote the orthogonal projection onto Π_i and set $d_i(x) := \mathrm{dist}(x, \Pi_i)$, we notice that

$$\nabla d_i(x) = \frac{x - \pi_i(x)}{|x - \pi_i(x)|},$$

for every $x \in \mathbb{R}^n \setminus \Pi_i$.

Moreover, for every $\delta > 0$ we define

$$S_i^\delta := \{y \in \Pi_i \colon \mathrm{dist}(y, S_i) \le \delta\}.$$

Now let γ_ε^i be a cut-off function between S_i^ε and $S_i^{2\varepsilon}$; *i.e.*, $\gamma_\varepsilon^i \in C_0^\infty(S_i^{2\varepsilon})$, $0 \le \gamma_\varepsilon^i \le 1$, $\gamma_\varepsilon^i \equiv 1$ in S_i^ε, and $|\nabla \gamma_\varepsilon^i| \le c/\varepsilon$ in Π_i, for some $c > 0$. For every $i \in \{1, \dots, M\}$ set

$$v_\varepsilon^i(x) := \gamma_\varepsilon^i(\pi_i(x)) h_\varepsilon(d_i(x)) + 1 - \gamma_\varepsilon^i(\pi_i(x)). \tag{3.22}$$

From the very definition of v_ε^i we have that $0 \le v_\varepsilon^i \le 1$ and $v_\varepsilon^i \in W^{1,\infty}(\mathbb{R}^n)$. Moreover, by using the following facts: $|\nabla \gamma_\varepsilon^i| \le c/\varepsilon$, π_i is Lipschitz with constant 1, and $|\nabla d_i| = 1$, we also get

$$\|\nabla v_\varepsilon^i\|_{L^\infty(\mathbb{R}^n)} \le \frac{c}{\varepsilon}. \tag{3.23}$$

Additionally, by definition, $v_\varepsilon^i \to 1$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and

$$v_\varepsilon^i \equiv 0 \ \text{ in } \ A_i^\varepsilon \ \text{ and } \ v_\varepsilon^i \equiv 1 \ \text{ in } \ \mathbb{R}^n \setminus B_i^\varepsilon, \tag{3.24}$$

where

$$A_i^\varepsilon := \{x \in \mathbb{R}^n \colon \pi_i(x) \in S_i^\varepsilon \text{ and } d_i(x) \le \xi_\varepsilon\}$$

and

$$B_i^\varepsilon := \{x \in \mathbb{R}^n \colon \pi_i(x) \in S_i^{2\varepsilon} \text{ and } d_i(x) \le \xi_\varepsilon + \varepsilon T_\eta\}.$$

Therefore, in view of (3.24) we have

$$\int_{B_i^\varepsilon}\left(\frac{V(v_\varepsilon^i)}{\varepsilon}+\varepsilon|\nabla v_\varepsilon^i|^2\right)dx=\frac{V(0)}{\varepsilon}\,\mathcal{L}^n(A_i^\varepsilon)+\int_{B_i^\varepsilon\setminus A_i^\varepsilon}\left(\frac{V(v_\varepsilon^i)}{\varepsilon}+\varepsilon|\nabla v_\varepsilon^i|^2\right)dx \tag{3.25}$$

moreover, we notice that

$$\lim_{\varepsilon\to0}\frac{V(0)}{\varepsilon}\,\mathcal{L}^n(A_i^\varepsilon)=2V(0)\lim_{\varepsilon\to0}\frac{\xi_\varepsilon}{\varepsilon}\,\mathcal{H}^{n-1}(S_i^\varepsilon)=0, \tag{3.26}$$

where to establish the last equality we have used that $\xi_\varepsilon \ll \varepsilon$ and $\mathcal{H}^{n-1}(S_i^\varepsilon)\to\mathcal{H}^{n-1}(S_i)$, as $\varepsilon\to0$. We now estimate the second term in the right-hand side of (3.25). To do so it is convenient to write

$$B_i^\varepsilon\setminus A_i^\varepsilon=H_i^\varepsilon\cup I_i^\varepsilon$$

where

$$H_i^\varepsilon:=\{x\in\mathbb{R}^n\colon \pi_i(x)\in S_i^\varepsilon \text{ and } \xi_\varepsilon\le d_i(x)\le\xi_\varepsilon+\varepsilon T_\eta\} \tag{3.27}$$

and

$$I_i^\varepsilon:=\{x\in\mathbb{R}^n\colon \pi_i(x)\in S_i^{2\varepsilon}\setminus S_i^\varepsilon \text{ and } d_i(x)\le\xi_\varepsilon+\varepsilon T_\eta\}. \tag{3.28}$$

By definition of v_ε^i, in the set H_i^ε it holds

$$v_\varepsilon^i(x)=w_\eta\Big(\frac{d_i(x)-\xi_\varepsilon}{\varepsilon}\Big)$$

and thus

$$\nabla v_\varepsilon^i(x)=\frac{1}{\varepsilon}w_\eta'\Big(\frac{d_i(x)-\xi_\varepsilon}{\varepsilon}\Big)\nabla d_i(x).$$

Therefore, since $|\nabla d_i|=1$ a.e., we have

$$\begin{aligned}\int_{H_i^\varepsilon}\left(\frac{V(v_\varepsilon^i)}{\varepsilon}+\varepsilon|\nabla v_\varepsilon^i|^2\right)dx &= \int_{H_i^\varepsilon}\left(\frac{1}{\varepsilon}V\Big(w_\eta\Big(\frac{d_i(x)-\xi_\varepsilon}{\varepsilon}\Big)\Big)+\varepsilon\Big|\frac{1}{\varepsilon}w_\eta'\Big(\frac{d_i(x)-\xi_\varepsilon}{\varepsilon}\Big)\nabla d_i(x)\Big|^2\right)dx\\ &= 2\int_{S_i^\varepsilon}d\mathcal{H}^{n-1}\int_{\xi_\varepsilon}^{\xi_\varepsilon+\varepsilon T_\eta}\left(\frac{1}{\varepsilon}V\Big(w_\eta\Big(\frac{t-\xi_\varepsilon}{\varepsilon}\Big)\Big)+\frac{1}{\varepsilon}\Big|w_\eta'\Big(\frac{t-\xi_\varepsilon}{\varepsilon}\Big)\Big|^2\right)dt\\ &= 2\int_{S_i^\varepsilon}d\mathcal{H}^{n-1}\int_0^{T_\eta}\big(V(w_\eta(t))+|w_\eta'(t)|^2\big)\,dt\\ &\le 2(C_V+\eta)\,\mathcal{H}^{n-1}(S_i)+o(1),\end{aligned} \tag{3.29}$$

as $\varepsilon\to0$, where to establish the last inequality we have used (3.20).

Furthermore, from (3.23) it is immediate to show that

$$\lim_{\varepsilon\to0}\int_{I_i^\varepsilon}\left(\frac{V(v_\varepsilon^i)}{\varepsilon}+\varepsilon|\nabla v_\varepsilon^i|^2\right)dx\le\lim_{\varepsilon\to0}\frac{c}{\varepsilon}\,\varepsilon\,\mathcal{H}^{n-1}(S_i^{2\varepsilon}\setminus S_i^\varepsilon)=0. \tag{3.30}$$

Eventually, gathering (3.25)-(3.30) yields

$$\int_{B_i^\varepsilon}\left(\frac{V(v_\varepsilon^i)}{\varepsilon}+\varepsilon|\nabla v_\varepsilon^i|^2\right)dx\le2(C_V+\eta)\,\mathcal{H}^{n-1}(S_i)+o(1),$$

as $\varepsilon\to0$.

Now the idea is to combine together the sequences (v_ε^i) in order to define a new sequence (v_ε) which belongs to $W^{1,2}(\Omega)$ and in every B_i^ε coincides with (v_ε^i), up to a set where the surface energy is negligible. Moreover the sequence (v_ε) shall satisfy: $v_\varepsilon\to1$ in $L^1(\Omega)$ and

$$\limsup_{\varepsilon\to0}\int_{\Omega}\left(\frac{V(v_\varepsilon^i)}{\varepsilon}+\varepsilon|\nabla v_\varepsilon^i|^2\right)dx\le2(C_V+\eta)\,\mathcal{H}^{n-1}(J_u), \tag{3.31}$$

where u is as in (3.17).

To this end, we define

$$v_\varepsilon := \min\{v_\varepsilon^1, \dots, v_\varepsilon^M\}; \tag{3.32}$$

clearly, $0 \le v_\varepsilon \le 1$, $(v_\varepsilon) \subset W^{1,2}(\Omega)$, and $v_\varepsilon \to 1$ in $L^1(\Omega)$ hence, in particular, $v_\varepsilon \to 1$ in $L^1(\Omega)$. Further, setting

$$A^\varepsilon := \bigcup_{i=1}^M A_i^\varepsilon \ \text{ and } \ B^\varepsilon := \bigcup_{i=1}^M B_i^\varepsilon,$$

by (3.24) and (3.32) we readily deduce that

$$v_\varepsilon \equiv 0 \ \text{ in } \ A^\varepsilon \ \text{ and } \ v_\varepsilon \equiv 1 \ \text{ in } \ \mathbb{R}^n \setminus B^\varepsilon. \tag{3.33}$$

Then, writing $\Omega = (\Omega \setminus B^\varepsilon) \cup (\Omega \cap (B^\varepsilon \setminus A^\varepsilon)) \cup (\Omega \cap A^\varepsilon)$, in view of (3.33) we get

$$\begin{aligned} \int_\Omega \left(\frac{V(v_\varepsilon^i)}{\varepsilon} + \varepsilon |\nabla v_\varepsilon^i|^2 \right) dx &\le \int_{\Omega \cap (B^\varepsilon \setminus A^\varepsilon)} \left(\frac{V(v_\varepsilon^i)}{\varepsilon} + \varepsilon |\nabla v_\varepsilon^i|^2 \right) dx \\ &+ 2V(0) \frac{\xi_\varepsilon}{\varepsilon} \sum_{i=1}^M \mathcal{H}^{n-1}(S_i^\varepsilon). \end{aligned} \tag{3.34}$$

Since $\xi_\varepsilon \ll \varepsilon$ and

$$\sum_{i=1}^M \mathcal{H}^{n-1}(S_i^\varepsilon) \to \sum_{i=1}^M \mathcal{H}^{n-1}(S_i) = \mathcal{H}^{n-1}(J_u),$$

as $\varepsilon \to 0$, the second term in the right hand side of (3.34) is negligible. Hence, to get (3.31) we are left to estimate the surface energy in $\Omega \cap (B^\varepsilon \setminus A^\varepsilon)$. We claim that

$$\limsup_{\varepsilon \to 0} \mathcal{S}_\varepsilon := \limsup_{\varepsilon \to 0} \int_{\Omega \cap (B^\varepsilon \setminus A^\varepsilon)} \left(\frac{V(v_\varepsilon)}{\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2 \right) dx \le 2(C_V + \eta)\, \mathcal{H}^{n-1}(J_u). \tag{3.35}$$

We notice that

$$B^\varepsilon \setminus A^\varepsilon = \bigcup_{i=1}^M (H_i^\varepsilon \cup I_i^\varepsilon) \setminus A^\varepsilon,$$

where the sets H_i^ε and I_i^ε are defined as in (3.27) and (3.28), respectively. Since moreover

$$(H_i^\varepsilon \cup I_i^\varepsilon) \setminus A^\varepsilon \subset \bigcup_{j \ne i} \Big((H_i^\varepsilon \cup I_i^\varepsilon) \cap (H_j^\varepsilon \cup I_j^\varepsilon) \Big) \cup \bigcap_{j \ne i} \Big((H_i^\varepsilon \cup I_i^\varepsilon) \setminus B_j^\varepsilon \Big),$$

we have

$$\begin{aligned} \mathcal{S}_\varepsilon &\le \sum_{i=1}^M \int_{\Omega \cap ((H_i^\varepsilon \cup I_i^\varepsilon) \setminus A^\varepsilon)} \left(\frac{V(v_\varepsilon)}{\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2 \right) dx \\ &\le \sum_{i=1}^M \int_{\Omega \cap \bigcap_{j \ne i} \left((H_i^\varepsilon \cup I_i^\varepsilon) \setminus B_j^\varepsilon \right)} \left(\frac{V(v_\varepsilon)}{\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2 \right) dx \\ &+ \sum_{i=1}^M \int_{\Omega \cap \bigcup_{j \ne i} \left((H_i^\varepsilon \cup I_i^\varepsilon) \cap (H_j^\varepsilon \cup I_j^\varepsilon) \right)} \left(\frac{V(v_\varepsilon)}{\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2 \right) dx \\ &=: \mathcal{S}_\varepsilon^1 + \mathcal{S}_\varepsilon^2. \end{aligned} \tag{3.36}$$

We now estimate the terms $\mathcal{S}_\varepsilon^1$ and $\mathcal{S}_\varepsilon^2$ separately. To this end, we start observing that

$$\bigcap_{j \ne i} \Big((H_i^\varepsilon \cup I_i^\varepsilon) \setminus B_j^\varepsilon \Big) \subset \bigcap_{j \ne i} \{x \in \mathbb{R}^n \colon v_\varepsilon^i(x) \le v_\varepsilon^j(x)\},$$

hence, invoking (3.29) and (3.30), we readily get

$$\begin{aligned}\mathcal{S}^1_\varepsilon &= \sum_{i=1}^{M} \int_{\Omega\cap\bigcap_{j\neq i}\left((H_i^\varepsilon\cup I_i^\varepsilon)\setminus B_j^\varepsilon)\right)} \left(\frac{V(v^i_\varepsilon)}{\varepsilon} + \varepsilon|\nabla v^i_\varepsilon|^2\right) dx \\ &\leq (2C_V+\eta)\sum_{i=1}^{M}\mathcal{H}^{n-1}(S_i) + o(1) \\ &= (2C_V+\eta)\,\mathcal{H}^{n-1}(J_u) + o(1), \end{aligned} \tag{3.37}$$

as $\varepsilon \to 0$. Moreover, appealing to (3.23) easily gives

$$\mathcal{S}^2_\varepsilon \leq \sum_{i=1}^{M}\sum_{j\neq i}\frac{c}{\varepsilon}\,\mathcal{L}^n\big((H_i^\varepsilon\cup I_i^\varepsilon)\cap(H_j^\varepsilon\cup I_j^\varepsilon)\big). \tag{3.38}$$

We now claim that

$$\lim_{\varepsilon\to 0}\frac{c}{\varepsilon}\,\mathcal{L}^n\big((H_i^\varepsilon\cup I_i^\varepsilon)\cap(H_j^\varepsilon\cup I_j^\varepsilon)\big) = 0, \tag{3.39}$$

for every $i, j \in \{1,\dots,M\}$. Indeed, since $\Pi_i \neq \Pi_j$ then the set $S_i \cap S_j$ is contained in an $(n-2)$-dimensional affine subspace of $\mathbb{R}^n$, so that by (3.27) and (3.28) we can deduce that

$$\mathcal{L}^n\big((H_i^\varepsilon\cup I_i^\varepsilon)\cap(H_j^\varepsilon\cup I_j^\varepsilon)\big) \leq c(\xi_\varepsilon+\varepsilon T_\eta)^2 = O(\varepsilon^2), \tag{3.40}$$

as $\varepsilon \to 0$, where the constant $c > 0$ depends only on the angle between Π_i and Π_j and on $\mathcal{H}^{n-2}(S_i\cap S_j)$. Hence, (3.40) immediately yields (3.39).

Finally, gathering (3.36) and (3.37) entails (3.35), as desired.

Therefore, to conclude the proof of the upper bound we now have to exhibit a sequence $(u_\varepsilon) \subset W^{1,2}(\Omega,\mathbb{R}^n)$ such that $u_\varepsilon \to u$ in $L^1(\Omega,\mathbb{R}^n)$ and

$$\limsup_{\varepsilon\to 0}\int_\Omega k_\varepsilon\Phi(v_\varepsilon)\,W(x,\nabla u_\varepsilon)\,dx = 0. \tag{3.41}$$

To this end, set

$$(A^\varepsilon)' := \bigcup_{i=1}^{M}\Big\{x\in\mathbb{R}^n\colon \pi_i(x)\in S_i^{\varepsilon/2} \text{ and } d_i(x)\leq\frac{\xi_\varepsilon}{2}\Big\},$$

let $\varphi_\varepsilon \in C_0^\infty(A^\varepsilon)$ be a cut-off function between $(A^\varepsilon)'$ and A^ε, and define

$$u_\varepsilon := (1-\varphi_\varepsilon)u.$$

Then, clearly $(u_\varepsilon) \subset W^{1,\infty}(\Omega,\mathbb{R}^n)$, moreover $u_\varepsilon \to u$ in $L^1(\Omega,\mathbb{R}^n)$. Moreover, since $v_\varepsilon \equiv 0$ in A^ε, and Φ vanishes at zero, it holds

$$\int_\Omega k_\varepsilon\Phi(v_\varepsilon)\,W(x,\nabla u_\varepsilon)\,dx = \int_{\Omega\setminus A^\varepsilon} k_\varepsilon\Phi(v_\varepsilon)\,W(x,\nabla u)\,dx$$

hence using that $u \in PR(\Omega)$ together with the fact that for every $x \in \Omega$ the function $W(x,\cdot)$ vanishes in $SO(n)$ we immediately get

$$\Phi(v_\varepsilon)\,W(x,\nabla u) = 0 \;\text{ a.e. in } \Omega\setminus A^\varepsilon$$

and hence the claim. □

Remark 3.4 (Approximation of inhomogeneous anisotropic perimeter functionals)**.** Arguing as in the proof of [59] (see also [34, Theorem 3.1]), in view of Proposition 3.2 one can establish a Γ-convergence result for functionals of the form

$$F_\varepsilon^\phi(u,v) := \begin{cases} \displaystyle\int_\Omega \left(k_\varepsilon \Phi(v)\, W(x,\nabla u) + \frac{V(v)}{\varepsilon} + \varepsilon\phi^2(x,\nabla v)\right) dx & (u,v) \in W^{1,2}(\Omega,\mathbb{R}^n) \times W^{1,2}(\Omega), \\ & 0 \le v \le 1 \text{ a.e. on } \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.42}$$

where the euclidean norm in F_ε is now replaced by a Finsler norm ϕ. That is, $\phi\colon \Omega \times \mathbb{R}^n \to [0,+\infty)$ is a continuous function which is convex in its second variable and satisfies the two following properties:

i. for every $(x,z) \in \Omega \times \mathbb{R}^n$ and for every $t \in \mathbb{R}$

$$\phi(x,tz) = |t|\phi(x,z);$$

ii. for every $(x,z) \in \Omega \times \mathbb{R}^n$ there exist $0 < m \le M < +\infty$ such that

$$m|z| \le \phi(x,z) \le M|z|.$$

In this case it can be proven that the family of functionals (F_ε^ϕ) $\Gamma(L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega))$-converges to the following inhomogeneous and anisotropic functional $F^\phi\colon L^1(\Omega,\mathbb{R}^n) \times L^1(\Omega) \longrightarrow [0,+\infty]$ defined on piecewise rigid maps as:

$$F^\phi(u,v) := \begin{cases} 2C_V \displaystyle\int_{J_u} \phi(x,\nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in PR(\Omega) \text{ and } v = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise,} \end{cases}$$

where ν_u denotes the exterior unit normal to J_u.

4. Incompatible wells and linearised elasticity

In this section we are going to address two possible extensions of Theorem 3.3. We first discuss a generalisation of Theorem 3.3 to the case where the zeros of the potential W lie in a suitable nonempty compact set $\mathcal{K}$. Then, we show that our proof-strategy also applies to the case of linearised elasticity. Similarly as in Section 3, also in these cases the key tools for the analysis are two suitable variants of the piecewise-rigidity property stated in Theorem 2.3 (cf. Theorem 4.2 and Theorem 4.6).

4.1. The case of $\mathcal{K}$ piecewise-rigid maps.

Let $U \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, and let $\mathcal{K} \subset \mathbb{M}^{n\times n}$ be a nonempty compact set satisfying the following L^p-quantitative rigidity estimate for some $p \in (1, n/(n-1))$: there exists a constant $C > 0$ (depending only on p and n) such that for every $u \in W^{1,p}(U,\mathbb{R}^n)$

$$\min_{\mathbb{A}\in\mathcal{K}} \|\nabla u - \mathbb{A}\|_{L^p(U,\mathbb{M}^{n\times n})} \le C\|\text{dist}(\nabla u,\mathcal{K})\|_{L^p(U)}. \tag{4.1}$$

We notice that (4.1) implies the rigidity of the differential inclusion

$$v \in W^{1,\infty}(U,\mathbb{R}^n) \text{ and } \nabla v(x) \in \mathcal{K} \quad \text{a.e. } U, \tag{4.2}$$

in the sense explained in Lemma 4.1 below. In the statement of Lemma 4.1 we use the same terminology adopted in [56, Chapter 8] (see also [51, Section 1.4] and [46]).

Lemma 4.1. *Let $U \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\mathcal{K} \subset \mathbb{M}^{n\times n}$ be a nonempty compact set satisfying* (4.1)*. Then, the following statements hold true:*

(1) the differential inclusion (4.2) *is rigid for exact solutions; i.e., the only solutions to* (4.2) *are affine functions;*

(2) the differential inclusion (4.2) *is rigid for approximate solutions; i.e., if* $\mathrm{dist}(\nabla u_j, \mathcal{K}) \to 0$ *in measure in* U, (u_j) *converges to* u *weakly* in* $W^{1,\infty}(U, \mathbb{R}^n)$, $u_j = \mathbb{A}x$ *on* ∂U *for some* $\mathbb{A} \in \mathbb{M}^{n\times n}$, *then* (∇u_j) *converges in measure to* ∇u *in* U *and* u *is affine;*

(3) the differential inclusion (4.2) *is strongly rigid; i.e., if* $\mathrm{dist}(\nabla u_j, \mathcal{K}) \to 0$ *in measure in* U *and* (u_j) *converges to* u *weakly* in* $W^{1,\infty}(U, \mathbb{R}^n)$, *then* (∇u_j) *converges in measure to* ∇u *in* U *and* u *is affine;*

(4) we have

$$\mathcal{K} = \mathcal{K}^{\mathrm{qc}}\,, \tag{4.3}$$

where $\mathcal{K}^{\mathrm{qc}}$ *denotes the quasiconvex envelope of* $\mathcal{K}$*; i.e.,*

$$\mathcal{K}^{\mathrm{qc}} := \{\mathbb{A} \in \mathbb{M}^{n\times n} :\ f(\mathbb{A}) \le \sup_{\mathcal{K}} f,\ \forall f : \mathbb{M}^{n\times n} \to \mathbb{R}\ \textit{quasiconvex}\}\,.$$

For the readers' convenience the proof of Lemma 4.1 is included in the Appendix A.

Below we give a list of nonempty compact sets $\mathcal{K} \subset \mathbb{M}^{n\times n}$ for which (4.1) holds true. The most prominent examples are due to Ball and James [10, Proposition 2] and to Friesecke, James, and Müller [40, Theorem 3.1] and correspond, respectively, to the case of two non rank-1 connected matrices and to that of $SO(n)$.

We notice that in the examples (1) and (3) below, property (4.1) directly follows from an incompatibility condition for the approximate solutions of (4.2), as shown in [21] (see also [32, Theorem 1.2]). This condition reduces rigidity for multiple-wells to a single-well rigidity statement. We recall here that two disjoint compact sets $K_1, K_2 \in \mathbb{M}^{n\times n}$ are incompatible for the differential inclusion (4.2), with $\mathcal{K} = K_1 \cup K_2$, if for any sequence $(u_j) \subset W^{1,\infty}(U, \mathbb{R}^n)$ such that $\mathrm{dist}(\nabla u_j, K_1 \cup K_2) \to 0$ in measure, then either $\mathrm{dist}(\nabla u_j, K_1) \to 0$ or $\mathrm{dist}(\nabla u_j, K_2) \to 0$ in measure. In this case K_1 and K_2 are also called incompatible energy-wells.

In the examples (4) and (5) listed below, property (4.1) is instead a consequence of the Friesecke, James, and Müller rigidity estimate [40, Theorem 3.1] for (2), and of the above mentioned incompatibility for approximate solutions of (4.2). Although equality (4.3) is a consequence of (4.1) as established by Lemma 4.1, for each example in the list below we also give a precise reference to a direct proof of (4.3). We refer the reader to [51], [46] and [56, Chapter 8] for more details on these topics.

(1) $\mathcal{K} = \{\mathbb{A}_1, \mathbb{A}_2\}$, where $\mathbb{A}_1, \mathbb{A}_2 \in \mathbb{M}^{n\times n}$ are not rank-1 connected, see [10, Proposition 2], (see also [61, Example 4.3]);

(2) $\mathcal{K} = SO(n)$ [40, Theorem 3.1] (for (4.3) see [45] and also [61, Example 4.4]);

(3) $\mathcal{K} = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3\}$, where $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \in \mathbb{R}^{n\times n}$ are such that $\mathcal{K}$ has no rank-1 connections, see [57, Section 4];

(4) $\mathcal{K} = \bigcup_{i=1}^{N} \mathbb{A}_i SO(2)$, where $\mathbb{A}_i \in \mathbb{R}^{2\times 2}$ are such that $\det\mathbb{A}_i > 0$ for all $i \in \{1, 2, 3\}$ and $\mathcal{K}$ has no rank-1 connections, see [58, Theorem 2 and Remark 1];

(5) $\mathcal{K} = SO(3) \cup SO(3)\mathbb{H}$, where $\mathbb{H} = \mathrm{diag}(h_1, h_2, h_3)$, $h_1 \ge h_2 \ge h_3 > 0$ and $h_2 \ne 1$ (the latter condition is equivalent to $\mathcal{K}$ having no rank-1 connections). Additionally, one of the following two conditions must hold true:
   (i) there exists i such that $(h_i - 1)(h_{i-1}h_{i+1} - 1) \ge 0$ (here the indices are counted modulo 3);
   (ii) $h_1 \ge h_2 > 1 > h_3 > \frac{1}{3}$ or $3 > h_1 > 1 > h_2 \ge h_3 > 0$;

   see [33, Theorem 1.2].

We now recall an extension of the piecewise-rigidity result contained in Theorem 2.3 to the case of a compact set $\mathcal{K}$ for which (4.1) holds true. We state this result for $GSBV$-functions and we refer the reader to [20, Theorem 2.1] for the original statement in the SBV-setting.

Theorem 4.2. *Let $\mathcal{K} \subset \mathbb{M}^{n\times n}$ be a nonempty compact set for which (4.1) holds true and let $u \in GSBV(\Omega, \mathbb{R}^n)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$ and $\nabla u \in \mathcal{K}$ a.e. in Ω. Then, $u \in PR_{\mathcal{K}}(\Omega)$; i.e.,*

$$u(x) = \sum_{i\in\mathbb{N}} (\mathbb{A}_i x + b_i)\chi_{E_i}(x), \tag{4.4}$$

with $\mathbb{A}_i \in \mathcal{K}$ for every $i \in \mathbb{N}$, $b_i \in \mathbb{R}^n$, and (E_i) Caccioppoli partition of Ω.

For $\varepsilon > 0$ we consider the functionals $F_\varepsilon^{\mathcal{K}} \colon L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega) \longrightarrow [0, +\infty]$ defined as

$$F_\varepsilon^{\mathcal{K}}(u, v) := \begin{cases} \displaystyle\int_\Omega \left(k_\varepsilon \Phi(v)\, W(x, \nabla u) + \frac{V(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx & (u, v) \in W^{1,2}(\Omega, \mathbb{R}^n) \times W^{1,2}(\Omega), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise}, \end{cases} \tag{4.5}$$

where $k_\varepsilon \to +\infty$, as $\varepsilon \to 0$, $W \colon \Omega \times \mathbb{M}^{n\times n} \to [0, +\infty)$ is a Borel function such that $W(x, \mathbb{A}) = 0$ for every $\mathbb{A} \in \mathcal{K}$. We assume moreover that for every $x \in \Omega$ and every $\mathbb{A} \in \mathbb{M}^{n\times n}$ it holds $W(x, \mathbb{A}) \ge \alpha \operatorname{dist}^2(\mathbb{A}, \mathcal{K})$, for some $\alpha > 0$.

By combining Proposition 4.3 and Theorem 4.4 below we can identify the Γ-limit of $F_\varepsilon^{\mathcal{K}}$. Now using Theorem 4.2 in place of Theorem 2.3, these results can be proven by following exactly the same arguments employed in the proofs of Proposition 3.2 and Theorem 3.3, respectively.

The following proposition shows that the Γ-limit of $(F_\varepsilon^{\mathcal{K}})$ (if it exists) is finite only on $PR_{\mathcal{K}}(\Omega)$.

Proposition 4.3. *Let $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(\Omega, \mathbb{R}^n) \times W^{1,2}(\Omega)$, $0 \le v_\varepsilon \le 1$ a.e. in Ω, be such that*

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{\mathcal{K}}(u_\varepsilon, v_\varepsilon) < +\infty \quad \text{and} \quad u_\varepsilon \to u \ \text{ in } \ L^1(\Omega, \mathbb{R}^n).$$

Then, $v_\varepsilon \to 1$ in $L^1(\Omega)$ and $u \in PR_{\mathcal{K}}(\Omega)$.

In addition, the following Γ-convergence result holds true.

Theorem 4.4. *The family $(F_\varepsilon^{\mathcal{K}})$ defined in (4.5) $\Gamma(L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega))$-converges to the functional $F^{\mathcal{K}} \colon L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega) \longrightarrow [0, +\infty]$ given by*

$$F^{\mathcal{K}}(u, v) := \begin{cases} 2C_V \mathcal{H}^{n-1}(J_u) & \text{if } u \in PR_{\mathcal{K}}(\Omega),\ v = 1 \ \text{ a.e. in } \Omega \\ +\infty & \text{otherwise}, \end{cases} \tag{4.6}$$

where $C_V := 2\int_0^1 \sqrt{V(s)}\, ds$.

4.2. **The case of linearised elasticity.** An approximation result similar to that proven in Theorems 3.3 and 4.4 can be established for interfacial energies appearing in the context of linearised elasticity. In this case the energy functionals are defined on piecewise-rigid displacements; *i.e.*, on displacements of a linearly elastic body which does not store elastic energy.

Definition 4.5. *A map $u \colon \Omega \to \mathbb{R}^n$ is called a piecewise-rigid displacement if there exist skew-symmetric matrices $\mathbb{A}_i \in \mathbb{M}^{n\times n}_{\text{skew}}$ and vectors $b_i \in \mathbb{R}^n$ such that*

$$u(x) = \sum_{i\in\mathbb{N}} (\mathbb{A}_i x + b_i)\chi_{E_i}(x), \tag{4.7}$$

with (E_i) Caccioppoli partition of Ω. The set of piecewise-rigid displacements on Ω will be denoted by $PRD(\Omega)$.

The next piecewise-rigidity result corresponds to Theorems 2.3 and 4.2 in the setting of linearised elasticity. We state it here for $GSBD$ maps, the original version [20, Theorem A.1] being stated in the SBD-setting, though the method of proof works also in the more general $GSBD$-setting (see also [38, Theorem 2.1, Remark 2.2 (i)] for the two-dimensional case).

In what follows $e(u)$ denotes the symmetrized approximate gradient of $u \in GSBD(\Omega)$ (cf. [28]).

Theorem 4.6. *Let $u \in GSBD(\Omega)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$ and $e(u) = 0$ a.e. in Ω. Then, $u \in PRD(\Omega)$.*

To approximate interfacial energies defined on $PRD(\Omega)$, for $\varepsilon > 0$, we consider the functionals $E_\varepsilon \colon L^1(\Omega, \mathbb{R}^n) \times L^1(\Omega) \longrightarrow [0, +\infty]$ defined as

$$E_\varepsilon(u, v) := \begin{cases} \displaystyle\int_\Omega \Big(k_\varepsilon \Phi(v)\, W(x, \nabla u)\, dx + \frac{V(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \Big)\, dx & (u, v) \in W^{1,2}(\Omega, \mathbb{R}^n) \times W^{1,2}(\Omega), \\ & 0 \le v \le 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{4.8}$$

where $k_\varepsilon \to +\infty$, as $\varepsilon \to 0$, $W : \Omega \times \mathbb{M}^{n \times n} \to [0, +\infty)$ is a Borel function such that $W(\cdot, \mathbb{A}) = 0$ if $\mathbb{A}$ is skew-symmetric. Moreover for every $x \in \Omega$ and every $\mathbb{A} \in \mathbb{M}^{n \times n}$ it holds

$$\alpha |\mathbb{A}^{\rm sym}|^2 \le W(x, \mathbb{A}),$$

for some $\alpha > 0$. Here we denote by $\mathbb{A}^{\rm sym}$ the symmetric part of $\mathbb{A}$, namely $\mathbb{A}^{\rm sym} := \frac{\mathbb{A} + \mathbb{A}^T}{2}$. We use standard notation for the strain $e(u) = (\nabla u)^{\rm sym}$ of $u \in W^{1,2}(\Omega, \mathbb{R}^n)$.

Remark 4.7. We refer the reader to [25, Remark 4.14] for an explicit example of a nonconvex, polyconvex function which depends non-trivially on the skew-symmetric part of $\mathbb{A}$ and satisfies the bounds

$$\alpha |\mathbb{A}^{\rm sym}|^2 \le W(x, \mathbb{A}) \le \beta (|\mathbb{A}^{\rm sym}|^2 + 1)$$

for some α, $\beta > 0$, for every $x \in \Omega$ and for every $\mathbb{A} \in \mathbb{M}^{n \times n}$.

Another example can be obtained by taking $W(\mathbb{A}) = h^2(\mathbb{A})$, $\mathbb{A} \in \mathbb{M}^{n \times n}$, where h is a one-homogeneous quasiconvex function such that for every $\mathbb{A} \in \mathbb{M}^{n \times n}$ and for some α, $\beta > 0$

$$\alpha |\mathbb{A}^{\rm sym}| \le h(\mathbb{A}) \le \beta |\mathbb{A}^{\rm sym}|\,,$$

with h depending non-trivially on $\mathbb{A}^{\rm skew} := \frac{\mathbb{A} - \mathbb{A}^T}{2}$. We notice that, in particular, h (and therefore W) is not convex. A function as above can be obtained by slightly modifying Müller's celebrated example [50] similarly as in [17, Section 7].

In the following proposition we show that the Γ-limit of (E_ε) (if it exists) is finite only on piecewise-rigid displacements.

Proposition 4.8. *Let $(u_\varepsilon, v_\varepsilon) \subset W^{1,2}(\Omega, \mathbb{R}^n) \times W^{1,2}(\Omega)$ be such that*

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty \quad \text{and} \quad u_\varepsilon \to u \ \text{ in } \ L^1(\Omega, \mathbb{R}^n).$$

Then, $v_\varepsilon \to 1$ in $L^1(\Omega)$ and $u \in PRD(\Omega)$.

Proof. We argue by comparison as in the proof of Proposition 3.2 to infer that v_ε converges to 1 strongly in $L^1(\Omega)$, and that the limit function u satisfies

$$e(u) = 0 \ \text{ a.e. in } \ \Omega, \quad \text{and} \quad \mathcal{H}^{n-1}(J_u) < +\infty$$

now invoking [19, Theorem 5.1] (see also [44, Theorem 7]) in place of Theorem 3.1. Eventually, the conclusion follows by Theorem 4.6. □

Arguing as in the proof of Theorem 3.3, on account of Proposition 4.8 we can now prove the following Γ-convergence result for the family (E_ε).

Theorem 4.9. *The family of functionals* (E_ε) *defined in* (4.8) $\Gamma(L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega))$*-converges to the functional* $E\colon L^1(\Omega,\mathbb{R}^n)\times L^1(\Omega)\longrightarrow[0,+\infty]$ *given by*

$$E(u,v):=\begin{cases}2C_V\mathcal{H}^{n-1}(J_u) & \text{if } u\in PRD(\Omega) \text{ and } v=1 \text{ a.e. in } \Omega,\\ +\infty & \text{otherwise},\end{cases} \tag{4.9}$$

where $C_V:=2\int_0^1\sqrt{V(s)}\,ds$.

Proof. The lower bound inequality follows thanks to [19, Theorem 5.1] (see also [44, Theorem 7]).

The proof of the upper bound inequality in Step 2 of Theorem 3.3 remains unchanged up to replacing rotation matrices with skew-symmetric ones. □

Appendix A. Proof of the rigidity lemma

Proof of Lemma 4.1. We start by showing statement (3), from which (1) and (2) immediately follow (we notice that actually the validity of (1) and (2) is equivalent to (3), as shown in [56, Corollary 8.9]). To this end let (u_j) and u be as in (3); then $(\operatorname{dist}(\nabla u_j,\mathcal{K}))$ converges to 0 in $L^p(U)$, indeed it converges to 0 in measure and ∇u_j is bounded in $L^\infty(U,\mathbb{M}^{n\times n})$. Thanks to (4.1) we can find $\mathbb{A}_j\in\mathcal{K}$ such that

$$\|\nabla u_j-\mathbb{A}_j\|_{L^p(U,\mathbb{M}^{n\times n})}\le C\|\operatorname{dist}(\nabla u_j,\mathcal{K})\|_{L^p(U)}\,.$$

For an arbitrary subsequence (j_k), we extract a further subsequence (j_{k_h}) such that $\mathbb{A}_{j_{k_h}}$ converges to some $\mathbb{A}\in\mathcal{K}$. Therefore, $(\nabla u_{j_{k_h}})$ converges to $\mathbb{A}$ in $L^p(U,\mathbb{M}^{n\times n})$. This convergence combined with the weak* convergence of (u_j) to u in $W^{1,\infty}(U,\mathbb{R}^n)$ immediately gives $\nabla u=\mathbb{A}$ a.e. on U. Being the limit independent of the subsequence, the Urysohn property implies that the whole sequence (∇u_j) converges to $\mathbb{A}$ in $L^p(U,\mathbb{M}^{n\times n})$ and hence the claim.

We finally prove (4) directly from (4.1), despite its validity is well-known in literature as a consequence of (2) (cf. for instance [51, Theorem 4.10]). To conclude we only need to prove that $\mathcal{K}^{\rm qc}\subseteq\mathcal{K}$. To do so we use that

$$\mathcal{K}^{\rm qc}=\{\mathbb{A}\in\mathbb{M}^{n\times n}\colon Q(\operatorname{dist}^q(\cdot,\mathcal{K}))(\mathbb{A})=0\}$$

for every $q\in[1,+\infty)$ (cf. [61, Proposition 2.14], and also [51, Theorem 4.10]). Let $\mathbb{A}\in\mathcal{K}^{\rm qc}$; then the definition of quasi-convex envelope of the distance function yields the existence of $\varphi_j\in W^{1,\infty}_0(U,\mathbb{R}^n)$ such that

$$\lim_{j\to+\infty}\int_U\operatorname{dist}(\mathbb{A}+\nabla\varphi_j(x),\mathcal{K})dx=0.$$

Moreover, the Zhang Lemma (cf. [61], and also [51, Lemma 4.21]) provides us with a sequence $(\phi_j)\subset W^{1,\infty}_0(U,\mathbb{R}^n)$ such that

$$\sup_{j\in\mathbb{N}}\|\nabla\phi_j\|_{L^\infty(U,\mathbb{M}^{n\times n})}<+\infty,\qquad\lim_{j\to+\infty}\mathcal{L}^n(\{\phi_j\neq\varphi_j\})=0\,.$$

Thus, being $\mathcal{K}$ compact and $(\nabla\phi_j)$ bounded in $L^\infty(U,\mathbb{M}^{n\times n})$, we obtain

$$\begin{aligned}\int_U\operatorname{dist}^p(\mathbb{A}+\nabla\phi_j(x),\mathcal{K})dx&\le\int_U\operatorname{dist}^p(\mathbb{A}+\nabla\varphi_j(x),\mathcal{K})dx+C\mathcal{L}^n(\{\phi_j\neq\varphi_j\})\\&\le C\int_U\operatorname{dist}(\mathbb{A}+\nabla\varphi_j(x),\mathcal{K})dx+C\mathcal{L}^n(\{\phi_j\neq\varphi_j\})\,,\end{aligned}$$

thus, eventually,

$$\lim_{j\to+\infty}\int_U \operatorname{dist}^p(\mathbb{A}+\nabla\phi_j(x),\mathcal{K})dx = 0\,.$$

For $x\in U$ let $u_j(x)=\mathbb{A}x+\phi_j(x)$. By (4.1), there exists $\mathbb{A}_j\in\mathcal{K}$ such that

$$\|\nabla u_j-\mathbb{A}_j\|_{L^p(U,\mathbb{M}^{n\times n})}\le C\|\operatorname{dist}(\nabla u_j,\mathcal{K})\|_{L^p(U)}\,,$$

then by the Jensen Inequality we get

$$\limsup_{j\to+\infty}\mathcal{L}^n(U)|\mathbb{A}-\mathbb{A}_j|^p\le\lim_{j\to+\infty}\int_U|\mathbb{A}-\mathbb{A}_j+\nabla\phi_j|^p dx = 0\,,$$

and therefore $\mathbb{A}\in\mathcal{K}$. □

Acknowledgments

The work of M. Cicalese was supported by the DFG Collaborative Research Center TRR 109, "Discretization in Geometry and Dynamics". The work of C. I. Zeppieri was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under the Germany Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure. M. Focardi has been partially supported by GNAMPA.

References

[1] R. Alicandro, A. Braides, J. Shah, Free-discontinuity problems via functionals involving the L^1-norm of the gradient and their approximations. *Interfaces Free Bound.* **1** (1999), 17–37.

[2] R. Alicandro and M. Focardi, Variational Approximation of Free-Discontinuity Energies with Linear Growth. *Comm. Cont. Math.* **4**, no. 4 (2002), 685–723.

[3] L. Ambrosio, *The space* SBV(Ω) *and free-discontinuity problems*, in Variational and Free Boundary problems, 29–45, IMA Vol. Math. Appl. **53**, Springer, New York, 1993.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, the Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Ambrosio and V. M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.* **43**, no. 8 (1990), 999-1036.

[6] L. Ambrosio and V. M. Tortorelli, On the approximation of free-discontinuity problems. *Boll. Un. Mat. Ital.* (7) **6-B** (1992), 105-123.

[7] A. Bach, A. Braides, and C. I. Zeppieri, Quantitative analysis of finite-difference approximations of free-discontinuity functionals. *Interfaces Free Bound.* **22** no. 3 (2020), 317–381.

[8] A. Bach, M. Cicalese, and M. Ruf, Random finite-difference discretizations of the Ambrosio-Tortorelli functional with optimal mesh size, *SIAM J. Math. Anal.* **53** no. 2 (2021), 2275–2318. .

[9] A. Bach, R. Marziani, and C. I. Zeppieri, Γ-convergence and stochastic homogenisation of singularly-perturbed elliptic functionals. *ArXiv preprint: 2102.09872.*

[10] J. Ball and R. James, Fine phase mixtures as minimizers of energy. *Arch. Rat. Mech. Anal.*, **100**, (1987), 13–52.

[11] B. Bourdin, G. A. Francfort, and J.-J. Marigo, Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids.* **48**, no.4 (2000), 797–826.

[12] B. Bourdin, G. A. Francfort, and J.-J. Marigo, The variational approach to fracture. *J. Elasticity* **9** (2008), 5–148.

[13] A. Braides, Γ-*convergence for Beginners.* Oxford University Press, Oxford 2002.

[14] A. Braides, *Approximation of Free discontinuity Problems.* Springer Lect. Notes in Math., Vol. **1694**, Springer 1998.

[15] A. Braides, S. Conti, and A. Garroni, Density of polyhedral partitions. *Calc. Var. Partial Differential Equations* **56**, no. 2 (2017), 1–10.

[16] M. Caroccia and N. Van Goethem, Damage-driven fracture with low-order potentials: asymptotic behavior, existence and applications. *ESAIM Math. Model. Numer. Anal.* **53** (2019), no. 4, 1305–1350.

[17] M. Caroccia, M. Focardi, and N. Van Goethem, On the integral representation of variational functionals on BD. *SIAM J. Math. Anal.* **52** (2020), no. 4, 40224067.

[18] A. Chambolle, An approximation result for special functions with bounded deformation. *J. Math. Pures Appl.* **83** (2004), 929–954.

[19] A. Chambolle and V. Crismale, A density result in $GSBD^p$ with applications to the approximation of brittle fracture energies. *Arch. Ration. Mech. Anal.* **232** (2019), no. 3, 1329–1378.

[20] A. Chambolle, A. Giacomini, and M. Ponsiglione, Piecewise rigidity. *J. Funct. Anal.*, **244** (2007), 134–153.

[21] N. Chaudhuri, S. Müller, Rigidity estimate for two incompatible wells. *Calc. Var. Partial Differ. Equ.* **19** (2004), 379–390.

[22] J. D. Clayton and J. Knap, A geometrically nonlinear phase field theory of brittle fracture. *International Journal of Fracture*, **189**, no. 2 (2014), 139–148.

[23] S. Conti, M. Focardi, and F. Iurlano, Which special functions of bounded deformation have bounded variation? *Proc. Roy. Soc. Edinburgh Sect. A* **148A** (2017), 33–50.

[24] S. Conti, M. Focardi, and F. Iurlano, Phase field approximation of cohesive fracture models. *Ann. I. H. Poincaré*, **33**, 1033–1067, (2016).

[25] S. Conti, M. Focardi, and F. Iurlano, Integral representation for functionals defined on SBD^p in dimension two. *Arch. Ration. Mech. Anal.* **223** (2017), no. 3, 1337–1374.

[26] V. Crismale, On the approximation of SBD functions and some applications. *SIAM J. Math. Anal.* **51** (2019), no. 6, 5011–5048.

[27] G. Dal Maso, *An introduction to* Γ*-convergence.* Springer Science & Business Media, **8** 2012.

[28] G. Dal Maso, Generalised functions of bounded deformation. *J. Eur. Math. Soc.* **15**, no. 5 (2013), 1943–1997.

[29] G. Dal Maso, G. Francfort, R. Toader, Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.*, **176** no. 2 (2005),165–225.

[30] G. Dal Maso and F. Iurlano, Fracture models as Γ-limits of damage models. *Commun. Pure Appl. Anal.* **12** (2013), 1657–1686.

[31] E. De Giorgi, L. Ambrosio, New functionals in the calculus of variations. (Italian) *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.

[32] C. De Lellis and L. Székelyhidi, Simple proof of two-well rigidity. *C. R. Math. Acad. Sci. Paris* **343**, no. 5 (2006), 367–370.

[33] G. Dolzmann, B. Kirchheim, S. Müller, and V. Šverák, The two-well problem in three dimensions. *Calc. Var. Partial Differential Equations* **10**, no. 1 (2000), 21–40.

[34] M. Focardi, On the variational approximation of free-discontinuity problems in the vectorial case. *Math. Models Methods Appl. Sci.* **11**, no. 4 (2001), 663–684.

[35] M. Focardi, "Variational Approximation of Vectorial Free Discontinuity Problems: the Discrete and Continuous Case". *PhD thesis*, Scuola Normale Superiore, Pisa, 2002.

[36] M. Focardi and F. Iurlano, Asymptotic analysis of Ambrosio-Tortorelli energies in linearized elasticity. *SIAM J. Math. Anal.* **46** (2014), 2936–2955.

[37] G. Francfort and J.-J. Marigo, Revisiting Brittle Fracture as an Energy Minimization Problem. *Journal of the Mechanics and Physics of Solids* **46**, (1998), 1319–1342.

[38] M. Friedrich, A piecewise Korn inequality in SBD and applications to embedding and density results. *SIAM J. Math. Anal.* **50** (2018), 3842–3918.

[39] M. Friedrich and F. Solombrino, Functionals defined on piecewise rigid functions: Integral representation and Γ -convergence. *Arch. Rational Mech. Anal.* **236** (2020), 1325–1387.

[40] G. Friesecke, R. James, and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.* **55**, no. 11 (2002), 1461–1506.

[41] E. Giusti, *Direct methods in the calculus of variations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[42] A. A. Griffith, The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of Londres*, **221** (1920),163–198.

[43] F. Iurlano, Fracture and plastic models as Γ-limits of damage models under different regimes. *Adv. Calc. Var.* **6** (2013), 165–189.

[44] F. Iurlano, A density result for $GSBD$ and its application to the approximation of brittle fracture energies. *Calc. Var. Partial Differential Equations* **51** (2014), no. 1-2, 315–342.

[45] D. Kinderlehrer, Remarks about equilibrium configurations of crystals, in *Material instabilities in continuum mechanics* (Edinburgh, 1985-1986), 217-241, Oxford Sci. Publ., Oxford Univ. Press, New York, 1988.

[46] B. Kirchheim, Rigidity and Geometry of Microstructures. Lecture notes 16, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig (2003)

[47] J. Kristensen, Lower semicontinuity in spaces of weakly differentiable functions. *Math. Ann.* **313**, no. 4 (1999), 653–710.

[48] L. Modica, The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.* **98** (1987), 123–142.

[49] L. Modica and S. Mortola, Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital.* **14-B** (1977), 285–299.

[50] S. Müller, On quasiconvex functions which are homogeneous of degree 1. *Indiana Univ. Math. J.*, **41** (1992), no. 1, 295–301.

[51] S. Müller, Variational models for microstructure and phase transitions. *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, 85–210. Lecture Notes in Mathematics, 1713. Springer, Berlin, 1999.

[52] D. Mumford, J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.* **42** (1989), no. 5, 577–685.

[53] K. Pham and J.-J. Marigo, Approche variationnelle de l'endommagement: I. Les concepts fondamentaux. *Comptes Rendus Mécanique*, **338** (2010), 191–198.
[54] K. Pham and J.-J. Marigo, Approche variationnelle de l'endommagement: II. Les modèles à gradient. *Comptes Rendus Mécanique*, **338** (2010), 199–206.
[55] K. Pham, H. Amor, J.-J. Marigo, C. Maurini, Gradient Damage Models and Their Use to Approximate Brittle Fracture. *International Journal of Damage Mechanics*, **20**(4) (2011), 618–652.
[56] F. Rindler, *Calculus of variations.* Universitext. Springer, Cham, 2018. xii+444 pp.
[57] V. Šverák, New examples of quasiconvex functions. *Arch. Rational Mech. Anal.* **119** (1992), no. 4, 293–300.
[58] V. Šverák, On the problem of two wells. *Microstructure and phase transition*, 183–189, IMA Vol. Math. Appl., 54, Springer, New York, 1993.
[59] D. Vicente, An anisotropic Mumford Shah model. *J. Math. Anal. Appl.* **447** (2017), 181–205.
[60] J.-Y. Wu, V. P. Nguyen, C. T. Nguyen, D. Sutula, S. Bordas and S. Sinaie, Phase field modeling of fracture. *Advances in applied mechancis: multi-scale theory and computation* **52** (2018).
[61] K. Zhang, A construction of quasiconvex functions with linear growth at infinity. *Ann. Scuola Norm. Sup. Pisa* Cl. Sci. (4) **19**, no. 3 (1992), 313–326.
[62] K. Zhang, Quasiconvex functions, $SO(n)$ and two elastic wells. *Ann. Inst. H. Poincarè Anal. Non Linèaire* **14**, no. 6 (1997), 759–785.

(M. Cicalese) Zentrum Mathematik - M7, Technische Universität München, Boltzmannstrasse 3, 85747 Garching, Germany
E-mail address: cicalese@ma.tum.de

(M. Focardi) DiMaI "U. Dini", Università di Firenze, V.le G.B. Morgagni 67/A, 50134 Firenze, Italia
E-mail address: matteo.focardi@unifi.it

(C.I. Zeppieri) Angewandte Mathematik, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
E-mail address: caterina.zeppieri@uni-muenster.de