EXISTENCE RESULTS IN LARGE-STRAIN MAGNETOELASTICITY WITH ASYMMETRIC EXCHANGE ENERGY

MARCO BRESCIANI, ELISA DAVOLI, AND MARTIN KRUŽÍK

Abstract. We investigate problems in large-strain magnetoelasticity when a Dzyaloshinskii-Moriya interaction term is included, too. While the magnetoelastic stored energy density is described in the Lagrangean setting, purely magnetic terms are considered in the Eulerian one. This requires careful treatment of the invertibility of admissible elastic deformations and of the regularity of the inverse maps. Besides the existence of a minimizer in the static case, we also show that the model can be extended to an evolutionary situation and enriched by a rate-independent dissipation. In this case, we prove that an energetic solution exists.

1. Introduction

Magnetic skyrmions are spin textures emerging in magnetic systems lacking inversion symmetry, and which are therefore chiral. From a mathematical point of view, they can be regarded as topological defects in the magnetic texture, carrying a suitable topological charge, known as skyrmion winding number.

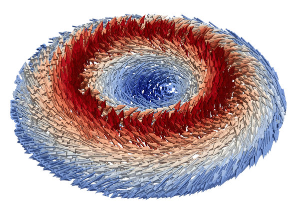

Figure 1. Numerical simulation of a magnetic skyrmion in a thin three-dimensional film (see [12]). Courtesy of G. Di Fratta, D. Praetorius, and M. Ruggeri

The notion of skyrmion has been named after the high-energy physicist T. Skyrme, who initially introduced it as a tool for describing the stability of hadrons. Ever since, skyrmions have played a central role in the description of multiple condensed-matter phenomena, ranging from Bose-Einstein condensates, to liquid crystals, and to quantum Hall systems. The presence of helical structures in magnetic crystals was originally predicted by I. Dzyaloshinskii. Magnetic skyrmions were then identified both in magnetic systems lacking inversion symmetry (such as MnSi) [21], as well as in ultrathin films and multilayers. The chirality of these structures is determined by asymmetric exchange interactions known as Dzyaloshinskii-Moriya Interactions (DMI) terms [20, 46].

Due to their small size, high stability, and to the fact that they can be written or deleted individually on magnetic stripes, these quasiparticles are reckoned as the most promising carrier of information for future storage and computing devices. As such, they are currently regarded as one of the emerging technologies in next-generation spintronics, and the question of how to manipulate them using mechanical loads is the focus of an intense research activity [31]. This naturally calls for a mathematical analysis of chiral effects in the magnetoelastic framework.

Date: March 30, 2021.

2000 *Mathematics Subject Classification.* 49J45; 74C99; 74F15.

Key words and phrases. magnetoelasticity, chirality, Eulerian-Lagrangean energies, quasistatic evolution, large-strain theories.

In this paper, we initiate a study of chirality in active materials at large strains by proving an existence result for optimal configurations of a magnetostrictive material in which DMI terms are also taken into account, both for the static problem, and for an associated quasistatic evolution.

In order to describe our results, we need to specify our mathematical setting. The variational theory of static magnetostriction [9, 15, 16, 17, 32] is based on the assumption that equilibrium configurations of the body are minimizers of an energy functional that depends on the deformation of the reference domain $\boldsymbol{y}:\Omega\to\Omega^{\boldsymbol{y}}\subset\mathbb{R}^3$ and on the magnetization $\boldsymbol{m}:\Omega^{\boldsymbol{y}}\to\mathbb{S}^2$, where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$ and $\Omega^{\boldsymbol{y}}$ is the deformed set. This energy functional is defined, for $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$, by setting

$$E(\boldsymbol{q}) \coloneqq \int_\Omega W(\nabla\boldsymbol{y},\boldsymbol{m}\circ\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}+\alpha\int_{\Omega^{\boldsymbol{y}}}|\nabla\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi}+\frac{\mu_0}{2}\int_{\mathbb{R}^3}|\nabla\zeta_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi}+\kappa\int_{\Omega^{\boldsymbol{y}}}\mathrm{curl}\,\boldsymbol{m}\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}. \tag{1.1}$$

Here, W denotes a nonlinear, frame-indifferent, magnetostrictive energy density. The second term in (1.1) is the so-called exchange energy, penalizing spatial changes of $\boldsymbol{m}$; $\alpha>0$ is the exchange constant. The third contribution in (1.1) encodes the magnetostatic energy and favors divergence-free states of the magnetization; $\mu_0>0$ is the permeability of the vacuum. In particular, the stray-field potential $\zeta_{\boldsymbol{m}}\colon\mathbb{R}^3\to\mathbb{R}$ is defined as a weak solution of the magnetostatic Maxwell equation [8, 19]:

$$\Delta\zeta_{\boldsymbol{m}}=\mathrm{div}(\chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m})\text{ in }\mathbb{R}^3.$$

Eventually, the possible lack of centrosymmetry in the crystalline structure of the material is accounted for by the last contribution in (1.1), describing bulk DMI exchange, in turn defined via the trace of the chirality tensor $\nabla\boldsymbol{m}\times\boldsymbol{m}$. Note that the sign of the constant $\kappa\in\mathbb{R}$ is not prescribed. According to its value, this energy term alone would be minimized by configurations satisfying $\mathrm{curl}\,\boldsymbol{m}=\pm\boldsymbol{m}$, or equivalently, $\pm\boldsymbol{m}=-\Delta\boldsymbol{m}$. Nevertheless, the sum of the symmetric and DMI exchange is optimized by helical fields $\boldsymbol{m}$ describing a rotation of constant frequency κ orthogonal to one of the coordinate axes, and rotating clockwise or counter-clockwise according to the sign of κ (see [18, 40]). For simplicity, in this paper we will neglect the energy contributions due to crystalline anisotropy and to Zeeman energies [8], for they behave as continuous perturbations and could easily be included in our analysis without further mathematical difficulties.

In the absence of DMI exchange, existence of minimizers for the functional in (1.1) has been proven in [51] in the case of non-simple materials, in [36] without higher-order terms but under incompressibility of the admissible deformations, as well as in [5] under weaker growth conditions on the energy density and relying on the notion of topological image. To complete our review on magnetoelasticity, we also mention a few recent works dealing with the analysis of magnetoelastic thin films. In particular, in [35] magnetoelastic plates and their corresponding quasistatic evolutions are studied within the purview of linearized elasticity. A large-strain analysis of magnetoelastic plates has been initiated in [38], under a priori constraints on the Jacobian of deformations (see also [39, 41] for some numerical results). The membrane regime for non-simple materials has been recently characterized in [13], whereas von Kármán theories starting from a nonlinear elastic setting have been identified in [7] in an incompressible framework.

The literature involving the mathematical analysis of micromagnetic models including DMI terms in the absence of elastic couplings is vast. Among the many contributions, we single out the seminal works [43, 49] (see also [37] and the references therein). We refer to [1] and the references therein for a study of effective theories and chirality transitions in the discrete-to-continuous setting. We also mention [30] for recent results on the numerics of chiral magnets, as well as [11] and [12] and the references therein, for periodic homogenization of chiral magnetic materials and for the static, dynamic, and numerical study of reduced models obtained starting from three-dimensional theories including asymmetric exchange, respectively.

In this paper, we combine the above perspectives by proving existence of equilibrium configurations for magnetoelastic energies including a chiral contribution. A simplified version of our first result reads as follows, we refer to Theorem 3.2 and Section 3 for the precise statement and assumptions.

Theorem 1.1. *Assume that the energy density W is continuous, polyconvex, blows up under extreme compressions, and is p-coercive, $p>3$. Assume also that interpenetration of matter is prevented. Then, the magnetoelastic energy functional in (1.1) admits a minimizer.*

A peculiar feature of the energy in (1.1) consists in its mixed Eulerian-Lagrangean structure. Whereas the elastic energy is evaluated on the reference configuration, and is hence Lagrangean, in fact, all micromagnetic contributions are set on the actual deformed set, thus being Eulerian. This leads to three main mathematical difficulties in the proof of Theorem 1.1. First, the minimization problem needs to be formulated in a class of admissible deformations and magnetizations for which the notion of deformed set $\Omega^{\boldsymbol{y}}$ is well-defined, and in which natural modeling assumptions such as impenetrability of matter are fulfilled. This is ensured by Lemma 2.1 and Proposition 2.5 below. Second, the class of admissible deformations must be stable with respect to the natural convergence enforced by the coercivity assumptions satisfied by W. Third, all Lagrangean terms should be lower-semicontinuous with respect to this aforementioned topology. As a result of these two latter challenges, the main ingredient in the proof of Theorem 1.1 is a compactness study for sequences of admissible states with equi-bounded energies, cf. Proposition 3.3 below.

Exploiting the global invertibility of admissible deformations, in Proposition 3.3 we prove the convergence of compositions of magnetizations with deformations. For a similar argument relying on equiintegrability of Jacobian of the inverses we refer to [25]. Note that this could not be achieved with the techniques in [4] and [5], and that here, in contrast with [36], this is not an easy task as we are not assuming incompressibility of the material. This convergence of compositions proves to be crucial in the evolutionary setting as well, in order to show the lower semicontinuity of the dissipation distance. An essential ingredient of the proof is the notion of topological degree: we refer to [14, Chapter 1] or [53, Chapter 3] for an overview of its main properties.

We mention for completeness that the existence of static minimizers could alternatively be shown with the arguments in [4] and [5], which are in turn based on local invertibility results. We have chosen not to pursue this different strategy here, as it would allow for larger classes of admissible deformations but at the cost of a significant increase in technicality, and would not guarantee convergence of compositions of magnetizations and deformations, which is instead essential for studying time-evolution. Besides, global invertibility is physically realistic.

The second part of our paper consists in a study of the quasi-static evolution of our model driven by a slight variant of the energy functional in (1.1) complemented by time-dependent applied loads representing external body forces, surface forces, and magnetic fields, respectively, as well as by dissipative effects. Our analysis is set within the theory of rate-independent processes [45] and energetic solutions.

A key difference with the study in [36] consists in our definition of dissipation. Arguing as in [50], we introduce the notion of Lagrangean magnetization. For $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$, we set

$$\mathcal{Z}(\boldsymbol{q}) \coloneqq (\mathrm{adj}\nabla\boldsymbol{y})\,\boldsymbol{m}\circ\boldsymbol{y}, \tag{1.2}$$

where adj is the transpose of the cofactor matrix.

The dissipation distance $\mathcal{D}\colon \mathcal{Q}\times\mathcal{Q}\to[0,+\infty)$ is then defined as

$$\mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}}) \coloneqq \int_\Omega |\mathcal{Z}(\boldsymbol{q})-\mathcal{Z}(\widehat{\boldsymbol{q}})|\,\mathrm{d}\boldsymbol{x}. \tag{1.3}$$

Note that a rigid body rotation does not create any dissipation. Indeed, let us take a state $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$ and a rigid motion $\boldsymbol{T}(\boldsymbol{\xi})\coloneqq\boldsymbol{R}\boldsymbol{\xi}+\boldsymbol{c}$. Given the new state $\widetilde{\boldsymbol{q}}=(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{m}})$, where $\widetilde{\boldsymbol{y}}\coloneqq\boldsymbol{T}\circ\boldsymbol{y}$ and $\widetilde{\boldsymbol{m}}\coloneqq\boldsymbol{R}\,(\boldsymbol{m}\circ\boldsymbol{T}^{-1})$, we have $\mathcal{D}(\widetilde{\boldsymbol{q}},\boldsymbol{q})=0$.

Existence of time-continuous solutions associated to the energy functional in (1.1) and the dissipation (1.3) is out-of-reach in our framework, due to a lack of compactness of the Lagrangean magnetizations in (1.2). In this latter part of the paper, we thus resort to a regularized counterpart to (1.1), in which the energy functional is augmented by the total variation of the cofactor matrix of the deformations. Namely, for every $\boldsymbol{q}\in\mathcal{Q}$, we assume the internal energy of the system to be given by

$$\widetilde{E}(\boldsymbol{q}) \coloneqq E(\boldsymbol{q}) + |D(\mathrm{cof}\,\nabla\boldsymbol{y})|(\Omega). \tag{1.4}$$

This regularization brings us to the theory of nonsimple materials initiated by Toupin [55, 56] and later extended by many authors. See [3, 45, 50], for instance. The idea is to assume that the stored energy density depends also on higher gradients of the deformations. More regularity allows us to work in a stronger topology and pass to the limit in the dissipation. In this work, we apply a fairly weak concept of nonsimple materials introduced in [6] under the name of *gradient polyconvex materials*, and assume only that $\operatorname{cof}\nabla\boldsymbol{y}\in BV(\Omega;\mathbb{R}^{3\times3})$. See also [34].

We present below a simplified statement of our second main result. We refer to Theorem 4.6 for its precise formulation.

Theorem 1.2. *Let $\boldsymbol{q}^0$ be a suitably well-prepared initial datum. Then, there exists a quasistatic evolution $t\mapsto\boldsymbol{q}(t)$ associated to the energy in (1.4) augmented by external loads and to the dissipation in (1.3), such that $\boldsymbol{q}(0)=\boldsymbol{q}^0$.*

Proofs of both main theorems rely on weak lower semicontinuity, polyconvexity, convexity and fine properties of injective deformations. Our proof of Theorem 1.2 is based on semidiscretization in time and passage to the "time-continuous" limit in the spirit of [45]. We point out that existence of time-discrete solutions can be proven without higher order terms, cf. Subsection 4.1. The regularization is only needed to pass to the time-continuous setting.

The paper is organized as follows. In Section 2 we recall some preliminary results on the invertibility of Sobolev functions. Section 3 is devoted to the proof of Theorem 1.1, whereas Section 4 describes the quasistatic problem and contains the proof of Theorem 1.2.

2. Preliminaries

In this section we collect some results regarding the invertibility of Sobolev maps with supercritical integrability. Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain. We consider maps in $W^{1,p}(\Omega;\mathbb{R}^3)$ with $p>3$. Any such map admits a representative in $C^0(\overline{\Omega};\mathbb{R}^3)$ which has the *Lusin property* (N) [42, Corollary 1], i.e. it maps sets of zero Lebesgue measure to sets of zero Lebesgue measure. Henceforth, *we will always tacitly consider this representative*. In this case, the image of measurable sets is measurable and the area formula holds [42, Corollary 2 and Theorem 2]. As a consequence, if the Jacobian determinant is different from zero almost everywhere, then the map has also the *Lusin property* (N^{-1}), i.e. the preimage of every set with zero Lebesgue measure has zero Lebesgue measure.

Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$. To make up for the fact that $\boldsymbol{y}(\Omega)$ might not be open, even if $\det\nabla\boldsymbol{y}>0$ almost everywhere, we introduce the *deformed configuration*, which is defined as $\Omega^{\boldsymbol{y}}:=\boldsymbol{y}(\Omega)\setminus\boldsymbol{y}(\partial\Omega)$. To prove that this set is actually open, we employ the *topological degree*. Recall that the degree of $\boldsymbol{y}$ on Ω is a continuous map $\deg(\boldsymbol{y},\Omega,\cdot)\colon\mathbb{R}^3\setminus\boldsymbol{y}(\partial\Omega)\to\mathbb{Z}$. For its definition and main properties, we refer to [14, Chapter 1] or [53, Chapter 3].

Lemma 2.1 (Deformed configuration). *Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\det\nabla\boldsymbol{y}>0$ almost everywhere in Ω. Then, the deformed configuration $\Omega^{\boldsymbol{y}}$ is an open set that differs from $\boldsymbol{y}(\Omega)$ by at most a set of zero Lebesgue measure. Moreover $\overline{\Omega^{\boldsymbol{y}}}=\boldsymbol{y}(\overline{\Omega})$ and $\partial\Omega^{\boldsymbol{y}}=\boldsymbol{y}(\partial\Omega)$.*

Proof. We claim that $\Omega^{\boldsymbol{y}}=\{\boldsymbol{\xi}\in\mathbb{R}^3\setminus\boldsymbol{y}(\partial\Omega):\ \deg(\boldsymbol{y},\Omega,\boldsymbol{\xi})>0\}$. Once the claim is proved, we deduce that $\Omega^{\boldsymbol{y}}$ is open. Indeed, the set on the right-hand side is open by the continuity of the degree.

Let $\boldsymbol{\xi}_0\in\mathbb{R}^3\setminus\boldsymbol{y}(\partial\Omega)$ be such that $\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}_0)>0$. Then, by the solvability property of the degree, $\boldsymbol{\xi}_0\in\boldsymbol{y}(\Omega)$ and, in turn, $\boldsymbol{\xi}_0\in\Omega^{\boldsymbol{y}}$. Conversely, let $\boldsymbol{\xi}_0\in\Omega^{\boldsymbol{y}}$. Denote by V the connected component of $\mathbb{R}^3\setminus\boldsymbol{y}(\partial\Omega)$ containing $\boldsymbol{\xi}_0$ and consider $R>0$ such that $B(\boldsymbol{\xi}_0,R)\subset\subset V$. Let $\psi\in C^\infty_c(\mathbb{R}^3)$ be such that $\psi\geq0$, $\operatorname{supp}\psi\subset\overline{B}(\boldsymbol{\xi}_0,R)\subset V$ and $\int_{\mathbb{R}^3}\psi\,\mathrm{d}\boldsymbol{\xi}=1$. Then, by the integral formula for the degree, we compute

$$\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi})=\int_\Omega\psi\circ\boldsymbol{y}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}=\int_{\boldsymbol{y}^{-1}(B(\boldsymbol{\xi}_0,R))}\psi\circ\boldsymbol{y}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

As $\psi\circ\boldsymbol{y}>0$ on $\boldsymbol{y}^{-1}(B(\boldsymbol{\xi}_0,R))$ and $\det\nabla\boldsymbol{y}>0$ almost everywhere, we obtain $\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi})>0$ and this proves the claim.

By the Lusin property (N), we have $\mathscr{L}^3(\boldsymbol{y}(\Omega)\setminus\Omega^{\boldsymbol{y}})\leq\mathscr{L}^3(\boldsymbol{y}(\partial\Omega))=0$. For simplicity, set $U:=\boldsymbol{y}^{-1}(\Omega^{\boldsymbol{y}})=\Omega\setminus\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))$, so that $\boldsymbol{y}(U)=\Omega^{\boldsymbol{y}}$. Then, $\Omega\setminus U=\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))$, so that $\mathscr{L}^3(\Omega\setminus U)=0$ by the Lusin properties (N) and (N^{-1}). In particular, U is dense in Ω.

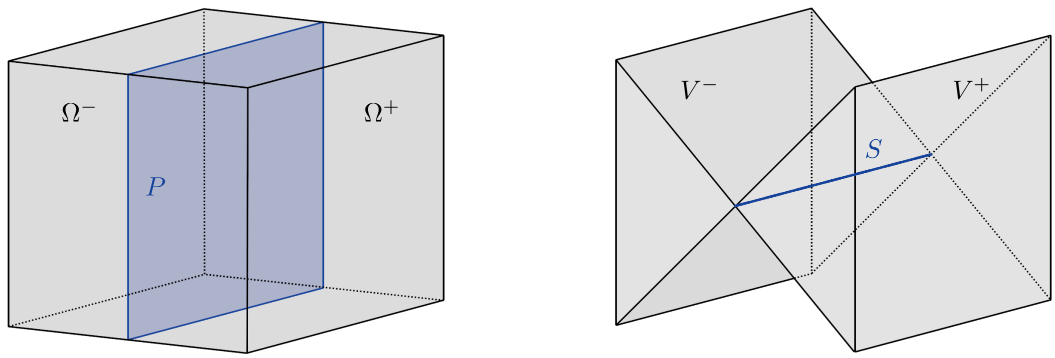

Figure 2. The deformation in Example 2.2.

We prove that $\overline{\Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\overline{\Omega})$. As $\Omega^{\boldsymbol{y}} \subset \boldsymbol{y}(\Omega)$, we immediately have $\overline{\Omega^{\boldsymbol{y}}} \subset \overline{\boldsymbol{y}(\Omega)} = \boldsymbol{y}(\overline{\Omega})$. Let $\boldsymbol{\xi} \in \boldsymbol{y}(\overline{\Omega})$ and consider $\boldsymbol{x} \in \overline{\Omega}$ such that $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{\xi}$. By density, $\overline{U} = \overline{\Omega}$. Thus, there exists $(\boldsymbol{x}_n) \subset U$ such that $\boldsymbol{x}_n \to \boldsymbol{x}$ and, in turn, $\boldsymbol{\xi}_n \coloneqq \boldsymbol{y}(\boldsymbol{x}_n) \to \boldsymbol{\xi}$. As $(\boldsymbol{\xi}_n) \subset \Omega^{\boldsymbol{y}}$, this yields $\boldsymbol{\xi} \in \overline{\Omega^{\boldsymbol{y}}}$.

Finally, we prove that $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$. This follows combining

$$\partial\Omega^{\boldsymbol{y}} = \overline{\Omega^{\boldsymbol{y}}} \setminus (\Omega^{\boldsymbol{y}})^\circ = \boldsymbol{y}(\overline{\Omega}) \setminus \Omega^{\boldsymbol{y}} = (\boldsymbol{y}(\overline{\Omega}) \setminus \boldsymbol{y}(\Omega)) \cup (\boldsymbol{y}(\overline{\Omega}) \cap \boldsymbol{y}(\partial\Omega)) \subset \boldsymbol{y}(\partial\Omega)$$

and

$$\partial\Omega^{\boldsymbol{y}} = \overline{\Omega^{\boldsymbol{y}}} \cap \overline{\mathbb{R}^3 \setminus \Omega^{\boldsymbol{y}}} = \boldsymbol{y}(\overline{\Omega}) \cap \overline{(\mathbb{R}^3 \setminus \boldsymbol{y}(\Omega)) \cup \boldsymbol{y}(\partial\Omega)} \supset \boldsymbol{y}(\overline{\Omega}) \cap \boldsymbol{y}(\partial\Omega) = \boldsymbol{y}(\partial\Omega).$$

□

The next example clarifies the difference between the sets $\boldsymbol{y}(\Omega)$ and $\Omega^{\boldsymbol{y}}$.

Example 2.2 (Ball's example). *The following is inspired by* [2, Example 1]. *Let* $\Omega = (-1,1)^3$ *and write* $\Omega = \Omega^+ \cup P \cup \Omega^-$*, where*

$$\Omega^+ \coloneqq (0,1) \times (-1,1)^2, \quad P \coloneqq \{0\} \times (-1,1)^2, \quad \Omega^- \coloneqq (-1,0) \times (-1,1)^2.$$

Define $\boldsymbol{y}\colon \Omega \to \mathbb{R}^3$ *by* $\boldsymbol{y}(\boldsymbol{x}) = (x_1, x_2, |x_1|\, x_3)$*, where* $\boldsymbol{x} = (x_1, x_2, x_3)$*. The corresponding deformed set is depicted in Figure 2. Then* $\boldsymbol{y} \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ *and for every* $\boldsymbol{x} \in \Omega \setminus P$ *we have*

$$\nabla \boldsymbol{y}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1\, x_3/|x_1| & 0 & |x_1| \end{pmatrix}.$$

In particular, $\det \nabla \boldsymbol{y} > 0$ *on* $\Omega \setminus P$*. We have* $\boldsymbol{y}(\Omega^+) = V^+$*,* $\boldsymbol{y}(P) = S$ *and* $\boldsymbol{y}(\Omega^-) = V^-$*, where, for* $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$*, we set*

$$\begin{aligned} V^+ &\coloneqq \{\boldsymbol{\xi} \in \mathbb{R}^3 : 0 < \xi_1 < 1,\ -1 < \xi_2 < 1,\ |\xi_3| < \xi_1\}, \\ S &\coloneqq \{0\} \times (-1,1) \times \{0\}, \\ V^- &\coloneqq \{\boldsymbol{\xi} \in \mathbb{R}^3 : -1 < \xi_1 < 0,\ -1 < \xi_2 < 1,\ |\xi_3| < -\xi_1\}. \end{aligned}$$

Note that $\boldsymbol{y}|_{\Omega \setminus P}$ *is injective, but* $\boldsymbol{y}$ *is not a homeomorphism. Also,* $\boldsymbol{y}(\Omega) = V^+ \cup S \cup V^-$ *is not open. Instead,* $\Omega^{\boldsymbol{y}} = V^+ \cup V^-$*, since* $S \subset \boldsymbol{y}\left(\overline{P} \cap \partial\Omega\right)$*, and this set is open. Note also that, while* $\boldsymbol{y}(\Omega)$ *is necessarily connected, the deformed configuration* $\Omega^{\boldsymbol{y}}$ *is not.*

Remark 2.3 (Topological image). Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$. The *topological image* of $\boldsymbol{y}$ is given by the set $\mathrm{im}_T(\boldsymbol{y},\Omega) \coloneqq \{\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega) : \deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) \neq 0\}$. Note that $\deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) = 0$ for every $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \boldsymbol{y}(\overline{\Omega})$, so that $\mathrm{im}_T(\boldsymbol{y},\Omega) \subset \boldsymbol{y}(\Omega)$. In relation with the problem of invertibility of deformations in elasticity, the topological image was first considered in [54] and then in several other contributions [5, 27, 28, 29, 47, 48, 52]. Note that in Lemma 2.1, we prove that, if $\det \nabla \boldsymbol{y} > 0$ almost everywhere, then

$$\Omega^{\boldsymbol{y}} = \mathrm{im}_T(\boldsymbol{y},\Omega) = \{\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \boldsymbol{y}(\partial\Omega) :\ \deg(\boldsymbol{y},\Omega,\boldsymbol{\xi}) > 0\}.$$

For more information about the topological properties of Sobolev maps with supercritical integrability, we refer to [33].

We now consider the invertibility of Sobolev maps with supercritical integrability $p > 3$. Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\det \nabla \boldsymbol{y} > 0$ almost everywhere. Assume that $\boldsymbol{y}$ is *almost everywhere injective*, i.e. there exists a set $X \subset \Omega$ with $\mathscr{L}^3(X) = 0$ such that $\boldsymbol{y}|_{\Omega\setminus X}$ is injective. In this case, we can consider the inverse $\boldsymbol{y}|^{-1}_{\Omega\setminus X} \colon \boldsymbol{y}(\Omega \setminus X) \to \Omega \setminus X$. Note that $\mathscr{L}^3(\boldsymbol{y}(X)) = 0$ by the Lusin property (N). We define the map $\boldsymbol{v} \colon \Omega^{\boldsymbol{y}} \to \mathbb{R}^3$ by setting

$$\boldsymbol{v}(\boldsymbol{\xi}) := \begin{cases} \boldsymbol{y}|^{-1}_{\Omega\setminus X}(\boldsymbol{\xi}), & \text{if } \boldsymbol{\xi} \in \Omega^{\boldsymbol{y}} \setminus \boldsymbol{y}(X), \\ \boldsymbol{a}, & \text{if } \boldsymbol{\xi} \in \Omega^{\boldsymbol{y}} \cap \boldsymbol{y}(X), \end{cases} \tag{2.1}$$

where $\boldsymbol{a} \in \mathbb{R}^3$ is arbitrarily fixed. The map $\boldsymbol{v}$ satisfies $\boldsymbol{v} \circ \boldsymbol{y} = \boldsymbol{id}$ almost everywhere in Ω and $\boldsymbol{y} \circ \boldsymbol{v} = \boldsymbol{id}$ almost everywhere in $\Omega^{\boldsymbol{y}}$. Since $\boldsymbol{y}$ maps measurable sets to measurable sets, the measurability of $\boldsymbol{v}$ follows. As $\boldsymbol{y}$ has both Lusin properties (N) and (N^{-1}), the map $\boldsymbol{v}$ has the same properties. Moreover, $\boldsymbol{v} \in L^\infty(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$ since $\boldsymbol{v}(\Omega^{\boldsymbol{y}}) \subset \Omega \cup \{\boldsymbol{a}\}$ and Ω is bounded.

We remark that the definition of $\boldsymbol{v}$ in (2.1) depends on the choice of the set X where $\boldsymbol{y}$ is not injective and of the value $\boldsymbol{a} \in \mathbb{R}^3$. However, as $\boldsymbol{y}$ has the Lusin property (N), its equivalence class is uniquely determined and coincides with the one of the classical inverse $\boldsymbol{y}^{-1}$, where the latter is defined out of a subset of $\boldsymbol{y}(\Omega)$ with zero Lebesgue measure. Hence, with a slight abuse of notation, we will denote this equivalence class of functions defined on $\Omega^{\boldsymbol{y}}$ by $\boldsymbol{y}^{-1}$ and we will refer to it as *the* inverse of $\boldsymbol{y}$.

Remark 2.4 (Ciarlet-Nečas condition). Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$ be such that $\det \nabla \boldsymbol{y} > 0$ almost everywhere. Then, $\boldsymbol{y}$ is almost everywhere injective if and only if it satisfies the *Ciarlet-Nečas condition* [10], which reads

$$\int_\Omega \det \nabla \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \leq \mathscr{L}^3(\boldsymbol{y}(\Omega)).$$

This equivalence easily follows from the area formula [10, p. 185]. Note that the Ciarlet-Nečas condition is preserved under weak convergence in $W^{1,p}(\Omega;\mathbb{R}^3)$ thanks to the weak continuity of minors and the Morrey embedding. As a consequence, given $(\boldsymbol{y}_n) \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that each $\boldsymbol{y}_n$ is almost everywhere injective with $\det \nabla \boldsymbol{y}_n > 0$ almost everywhere, if $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ for some $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\det \nabla \boldsymbol{y} > 0$ almost everywhere, then $\boldsymbol{y}$ is almost everywhere injective. Note that the condition $\det \nabla \boldsymbol{y} > 0$ almost everywhere has to be assumed a priori.

The inverse $\boldsymbol{y}^{-1}$ of $\boldsymbol{y}$ turns out to have Sobolev regularity. Note that this makes sense since, by definition, $\boldsymbol{y}^{-1}$ is defined on the deformed configuration $\Omega^{\boldsymbol{y}}$, which is open by Lemma 2.1. The Sobolev regularity of the inverse has been proved for more general classes of deformations, such as in [5, Proposition 5.3], [29, Theorem 9.3], [52, Theorem 4.6], and [54, Theorem 8]. For convenience of the reader, we recall the proof. Note that here the almost everywhere injectivity is assumed a priori.

Proposition 2.5 (Global invertibility). *Let $\boldsymbol{y} \in W^{1,p}(\Omega;\mathbb{R}^3)$ be almost everywhere injective with $\det \nabla \boldsymbol{y} > 0$ almost everywhere. Then, $\boldsymbol{y}^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$ with $\nabla \boldsymbol{y}^{-1} = (\nabla \boldsymbol{y})^{-1} \circ \boldsymbol{y}^{-1}$ almost everywhere in $\Omega^{\boldsymbol{y}}$. Moreover, $\mathrm{cof}\, \nabla \boldsymbol{y}^{-1} \in L^1(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})$ and $\det \nabla \boldsymbol{y}^{-1} \in L^1(\Omega^{\boldsymbol{y}})$.*

Proof. By the Piola identity [24, Proposition 3, p. 235], we have

$$\int_\Omega \mathrm{cof}\, \nabla \boldsymbol{y} : \nabla \boldsymbol{\zeta} \,\mathrm{d}\boldsymbol{x} = 0 \tag{2.2}$$

for every $\boldsymbol{\zeta} \in C^\infty_c(\Omega;\mathbb{R}^3)$. By density, this actually holds for $\boldsymbol{\zeta} \in W^{1,q'}_0(\Omega;\mathbb{R}^3)$, where $q := p/2$. Let $\varphi \in C^\infty(\overline{\Omega})$ and $\boldsymbol{\psi} \in C^\infty_c(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$. Choosing $\boldsymbol{\zeta} = \varphi\, \boldsymbol{\psi} \circ \boldsymbol{y}$ in (2.2), after some simple algebraic manipulations, we obtain the following identity:

$$-\int_\Omega \varphi \operatorname{div}\boldsymbol{\psi}\circ\boldsymbol{y}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}\circ\boldsymbol{y}\otimes\nabla\varphi : \operatorname{cof}\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}. \tag{2.3}$$

Let $X\subset\Omega$ with $\mathscr{L}^3(X)=0$ be such that $\boldsymbol{y}|_{\Omega\setminus X}$ is injective. For clarity, let us consider the representative $\boldsymbol{v}$ of $\boldsymbol{y}^{-1}$ in (2.1) and let us fix a representative of $\nabla\boldsymbol{y}$. Set $D\coloneqq\Omega\setminus(\boldsymbol{y}^{-1}(\boldsymbol{y}(\partial\Omega))\cup\{\det\nabla\boldsymbol{y}\leq 0\}\cup X)$, so that $\boldsymbol{v}=\boldsymbol{y}|_D^{-1}$ on $\boldsymbol{y}(D)$ and $\nabla\boldsymbol{y}$ is invertible on D. Let $\boldsymbol{\Phi}\in C^\infty_c(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times3})$ and denote its rows by $\boldsymbol{\Phi}^i=(\Phi^i_1,\Phi^i_2,\Phi^i_3)^\top$, where $i=1,2,3$. Using the change-of-variable formula, we compute

$$\begin{aligned}-\int_{\Omega^{\boldsymbol{y}}}\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{\xi} &= -\int_{\boldsymbol{y}(D)}\boldsymbol{y}|_D^{-1}\cdot\operatorname{div}\boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{\xi} = -\int_D \boldsymbol{x}\cdot\operatorname{div}\boldsymbol{\Phi}\circ\boldsymbol{y}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\\ &= -\int_\Omega \boldsymbol{x}\cdot\operatorname{div}\boldsymbol{\Phi}\circ\boldsymbol{y}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = -\sum_{i=1}^3\int_\Omega x_i\operatorname{div}\boldsymbol{\Phi}^i\circ\boldsymbol{y}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Then using (2.3) with $\varphi(\boldsymbol{x})=x_i$ for every $\boldsymbol{x}\in\Omega$ and $\boldsymbol{\psi}(\boldsymbol{\xi})=\boldsymbol{\Phi}^i(\boldsymbol{\xi})$ for every $\boldsymbol{\xi}\in\Omega^{\boldsymbol{y}}$, we obtain

$$\begin{aligned}-\int_{\Omega^{\boldsymbol{y}}}\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{\xi} &= \sum_{i,j=1}^3\int_\Omega \Phi^i_j\circ\boldsymbol{y}\,(\operatorname{cof}\nabla\boldsymbol{y})^j_i\,\mathrm{d}\boldsymbol{x} = \sum_{i,j=1}^3\int_\Omega \Phi^i_j\circ\boldsymbol{y}\,(\operatorname{adj}\nabla\boldsymbol{y})^i_j\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \boldsymbol{\Phi}\circ\boldsymbol{y}:\operatorname{adj}\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\Phi}\circ\boldsymbol{y}:(\nabla\boldsymbol{y})^{-1}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\\ &= \int_D \boldsymbol{\Phi}\circ\boldsymbol{y}:(\nabla\boldsymbol{y})^{-1}\,\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}(D)}\boldsymbol{\Phi}:(\nabla\boldsymbol{y})^{-1}\circ\boldsymbol{y}|_D^{-1}\,\mathrm{d}\boldsymbol{\xi},\end{aligned}$$

where, in the last line, we used again the change-of-variable formula. Hence, as $\mathscr{L}^3(\Omega^{\boldsymbol{y}}\setminus\boldsymbol{y}(D))=0$, we deduce that $\boldsymbol{v}$ admits a weak gradient with a representative given by

$$\nabla\boldsymbol{v}(\boldsymbol{\xi})\coloneqq\begin{cases}(\nabla\boldsymbol{y})^{-1}\circ\boldsymbol{y}|_D^{-1}(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi}\in\boldsymbol{y}(D),\\ \boldsymbol{A} & \text{if } \boldsymbol{\xi}\in\Omega^{\boldsymbol{y}}\setminus\boldsymbol{y}(D),\end{cases}$$

where $\boldsymbol{A}\in\mathbb{R}^{3\times3}$ is arbitrary. Thanks to the Lusin property (N), the equivalence class of $\nabla\boldsymbol{v}$ is uniquely determined. Moreover, it belongs to $L^1(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times3})$. Indeed, by the change-of-variable formula

$$\begin{aligned}\int_{\Omega^{\boldsymbol{y}}}|\nabla\boldsymbol{v}|\,\mathrm{d}\boldsymbol{\xi} &= \int_{\boldsymbol{y}(D)}|(\nabla\boldsymbol{y})^{-1}|\circ\boldsymbol{y}|_D^{-1}\,\mathrm{d}\boldsymbol{\xi} = \int_D|(\nabla\boldsymbol{y})^{-1}|\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\\ &= \int_D|\operatorname{adj}\nabla\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} = \int_\Omega|\operatorname{adj}\nabla\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Thus, $\boldsymbol{v}\in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$. Similarly, using the identity $\operatorname{adj}(\boldsymbol{F}^{-1})=(\det\boldsymbol{F})^{-1}\boldsymbol{F}$ for every $\boldsymbol{F}\in\mathbb{R}^{3\times3}_+$, we compute

$$\int_{\Omega^{\boldsymbol{y}}}|\operatorname{adj}\nabla\boldsymbol{v}|\,\mathrm{d}\boldsymbol{\xi} = \int_{\boldsymbol{y}(D)}(\det\nabla\boldsymbol{y})^{-1}\circ\boldsymbol{y}|_D^{-1}\,|\nabla\boldsymbol{y}|\circ\boldsymbol{y}|_D^{-1}\,\mathrm{d}\boldsymbol{\xi} = \int_D|\nabla\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} = \int_\Omega|\nabla\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x},$$

while, using the identity $\det(\boldsymbol{F}^{-1})=(\det\boldsymbol{F})^{-1}$ in $\boldsymbol{F}\in\mathbb{R}^{3\times3}_+$, we obtain

$$\int_{\Omega^{\boldsymbol{y}}}\det\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}(D)}(\det\nabla\boldsymbol{y})^{-1}\circ\boldsymbol{y}|_D^{-1}\,\mathrm{d}\boldsymbol{\xi} = \mathscr{L}^3(D)=\mathscr{L}^3(\Omega).$$

Therefore, $\operatorname{cof}\nabla\boldsymbol{v}\in L^1(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times3})$ and $\det\nabla\boldsymbol{v}\in L^1(\Omega^{\boldsymbol{y}})$. □

Remark 2.6 (Area formula for the inverse). Let $\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3)$ be almost everywhere injective with $\det\nabla\boldsymbol{y}>0$ almost everywhere. Let $X\subset\Omega$ with $\mathscr{L}^3(X)=0$ be such that $\boldsymbol{y}|_{\Omega\setminus X}$ is injective and let $\boldsymbol{v}$ be the representative of $\boldsymbol{y}^{-1}$ in (2.1). By Proposition 2.5, $\boldsymbol{v}\in W^{1,1}(\Omega^{\boldsymbol{y}};\mathbb{R}^3)$. Since $\boldsymbol{y}$ has the Lusin property (N^{-1}), the map $\boldsymbol{v}$ has the Lusin property (N). Moreover, $\boldsymbol{v}$ is almost everywhere injective. Thus, we can use the area formula to estimate the measure of preimages of sets via $\boldsymbol{y}$. Let $F\subset\mathbb{R}^3$ be measurable. Then $\boldsymbol{y}^{-1}(F)\coloneqq\{\boldsymbol{x}\in\Omega:\ \boldsymbol{y}(\boldsymbol{x})\in F\}$. We assume that $F\subset\boldsymbol{y}(\Omega)$ and we write

$$F=(F\cap\boldsymbol{y}(\partial\Omega))\cup(F\cap\boldsymbol{y}(X))\cup(F\setminus(\boldsymbol{y}(\partial\Omega)\cup\boldsymbol{y}(X))),$$

so that

$$\begin{aligned}\boldsymbol{y}^{-1}(F) &= \boldsymbol{y}^{-1}(F\cap \boldsymbol{y}(\partial\Omega))\cup \boldsymbol{y}^{-1}((F\cap \boldsymbol{y}(X)))\cup \boldsymbol{y}^{-1}(F\setminus (\boldsymbol{y}(\partial\Omega)\cup \boldsymbol{y}(X)))\\ &= \boldsymbol{y}^{-1}(F\cap \boldsymbol{y}(\partial\Omega))\cup \boldsymbol{y}^{-1}((F\cap \boldsymbol{y}(X)))\cup \boldsymbol{v}(F\setminus (\boldsymbol{y}(\partial\Omega)\cup \boldsymbol{y}(X))),\end{aligned}$$

where, in the last line, we used (2.1). Exploiting both Lusin properties (N) and (N^{-1}) of $\boldsymbol{y}$ and the Lusin property (N) of $\boldsymbol{v}$, we have $\mathscr{L}^3(\boldsymbol{y}^{-1}(F))=\mathscr{L}^3(\boldsymbol{v}(F\setminus(\boldsymbol{y}(\partial\Omega)\cup\boldsymbol{y}(X))))=\mathscr{L}^3(\boldsymbol{v}(F))$. Finally, applying the area formula [26, Theorem 2] with $\boldsymbol{v}$, we compute

$$\mathscr{L}^3(\boldsymbol{y}^{-1}(F))=\mathscr{L}^3(\boldsymbol{v}(F))=\int_F \det\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{\xi}. \tag{2.4}$$

3. Static Setting

Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain. For $p>3$ fixed, the class of admissible deformations is given by

$$\mathcal{Y}\coloneqq\left\{\boldsymbol{y}\in W^{1,p}(\Omega;\mathbb{R}^3):\ \det\nabla\boldsymbol{y}>0\text{ a.e., }\boldsymbol{y}\text{ a.e. injective, }\boldsymbol{y}=\overline{\boldsymbol{y}}\ \text{ on }\Gamma\right\}, \tag{3.1}$$

where $\Gamma\subset\partial\Omega$ relatively open with $\mathscr{H}^2(\Gamma)>0$ and $\overline{\boldsymbol{y}}\in C^0(\Gamma;\mathbb{R}^3)$ are given.

Example 3.1. *Let Ω and $\boldsymbol{y}$ be as in Example 2.2. Given $\Gamma\coloneqq\{-1,1\}\times(-1,1)^2$ and $\overline{\boldsymbol{y}}\coloneqq\boldsymbol{id}$, we have $\boldsymbol{y}\in\mathcal{Y}$. In particular, this is a case in which $\mathcal{Y}\neq\emptyset$.*

Henceforth, we identify each $\boldsymbol{y}\in\mathcal{Y}$ with its continuous representative and we set $\Omega^{\boldsymbol{y}}\coloneqq\boldsymbol{y}(\Omega)\setminus\boldsymbol{y}(\partial\Omega)$. Then, admissible magnetizations are given by maps $\boldsymbol{m}\in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)$. Note that this makes sense as $\Omega^{\boldsymbol{y}}$ is open by Lemma 2.1. Thus, the class of admissible states is defined as

$$\mathcal{Q}\coloneqq\left\{(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}:\ \boldsymbol{y}\in\mathcal{Y},\ \boldsymbol{m}\in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)\right\}. \tag{3.2}$$

We endow the set $\mathcal{Q}$ with the topology that makes the map $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\mapsto(\boldsymbol{y},\chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m},\chi_{\Omega^{\boldsymbol{y}}}\nabla\boldsymbol{m})$ from $\mathcal{Q}$ to $W^{1,p}(\Omega;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})$ a homeomorphism onto its image, where the latter space is equipped with the weak product topology. Hence $\boldsymbol{q}_n\to\boldsymbol{q}$ in $\mathcal{Q}$ if and only if the following convergences hold:

$$\boldsymbol{y}_n\rightharpoonup\boldsymbol{y}\text{ in }W^{1,p}(\Omega;\mathbb{R}^3), \tag{3.3}$$

$$\chi_{\Omega^{\boldsymbol{y}_n}}\boldsymbol{m}_n\rightharpoonup\chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m}\text{ in }L^2(\mathbb{R}^3;\mathbb{R}^3), \tag{3.4}$$

$$\chi_{\Omega^{\boldsymbol{y}_n}}\nabla\boldsymbol{m}_n\rightharpoonup\chi_{\Omega^{\boldsymbol{y}}}\nabla\boldsymbol{m}\text{ in }L^2(\mathbb{R}^3;\mathbb{R}^{3\times3}). \tag{3.5}$$

In this case, up to subsequences, we actually have $\chi_{\Omega^{\boldsymbol{y}_n}}\boldsymbol{m}_n\to\chi_{\Omega^{\boldsymbol{y}}}\boldsymbol{m}$ in $L^a(\mathbb{R}^3;\mathbb{R}^3)$ for every $1\leq a<\infty$.

The energy functional $E\colon\mathcal{Q}\to\mathbb{R}$ is defined, for $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$, by setting

$$E(\boldsymbol{q})\coloneqq\int_\Omega W(\nabla\boldsymbol{y},\boldsymbol{m}\circ\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}+\alpha\int_{\Omega^{\boldsymbol{y}}}|\nabla\boldsymbol{m}|^2\,\mathrm{d}\boldsymbol{\xi}+\frac{\mu_0}{2}\int_{\mathbb{R}^3}|\nabla\zeta_{\boldsymbol{m}}|^2\,\mathrm{d}\boldsymbol{\xi}+\kappa\int_{\Omega^{\boldsymbol{y}}}\operatorname{curl}\boldsymbol{m}\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}. \tag{3.6}$$

The first term represents the *magnetoelastic energy* of the system. Note that, as $\boldsymbol{y}$ satisfies the Lusin property (N^{-1}), the composition $\boldsymbol{m}\circ\boldsymbol{y}$ is measurable and its equivalence class does not depend on the choice of the representative of $\boldsymbol{m}$. The nonlinear magnetoelastic energy density $W\colon\mathbb{R}^{3\times3}_+\times\mathbb{S}^2\to[0,+\infty)$ is continuous and satisfies the following two assumptions:

(coercivity) there exist a constant $K>0$ and a Borel function $\gamma\colon[0,+\infty)\to[0,+\infty)$ satisfying $\lim_{h\to0^+}\gamma(h)=+\infty$ such that

$$W(\boldsymbol{F},\boldsymbol{\lambda})\geq K|\boldsymbol{F}|^p+\gamma(\det\boldsymbol{F}) \tag{3.7}$$

for every $\boldsymbol{F}\in\mathbb{R}^{3\times3}_+$ and $\boldsymbol{\lambda}\in\mathbb{S}^2$;

(polyconvexity) there exists a function $\widehat{W}\colon\mathbb{R}^{3\times3}_+\times\mathbb{R}^{3\times3}_+\times\mathbb{R}_+\times\mathbb{S}^2\to[0,+\infty)$ such that $\widehat{W}(\cdot,\cdot,\cdot,\boldsymbol{\lambda})$ is convex for every $\boldsymbol{\lambda}\in\mathbb{S}^2$ and there holds

$$W(\boldsymbol{F},\boldsymbol{\lambda})=\widehat{W}(\boldsymbol{F},\operatorname{cof}\boldsymbol{F},\det\boldsymbol{F},\boldsymbol{\lambda}) \tag{3.8}$$

for every $\boldsymbol{F}\in\mathbb{R}^{3\times3}_+$ and $\boldsymbol{\lambda}\in\mathbb{S}^2$.

The second term is the *exchange energy* and comprises the parameter $\alpha > 0$. The third term is called *magnetostatic energy* and involves the function $\zeta_{\boldsymbol{m}} \colon \mathbb{R}^3 \to \mathbb{R}$ which is a weak solution of the magnetostatic Maxwell equation:

$$\Delta \zeta_{\boldsymbol{m}} = \mathrm{div}(\chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \text{ in } \mathbb{R}^3. \tag{3.9}$$

This means that $\zeta_{\boldsymbol{m}}$ belongs to the homogeneous Sobolev space

$$V^{1,2}(\mathbb{R}^3) := \{\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) : \nabla\varphi \in L^2(\mathbb{R}^3; \mathbb{R}^3)\}$$

and satisfies the following:

$$\forall\, \varphi \in V^{1,2}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla \zeta_{\boldsymbol{m}} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{\xi}.$$

Note that such weak solutions exist and are unique up to additive constants [5, Proposition 8.8], so that their gradient is uniquely defined. The fourth term describes the *Dzyaloshinskii-Moriya interaction energy* and it is characterized by the parameter $\kappa \in \mathbb{R}$. In particular, the energy E can assume negative values.

The main result of this section is the existence of minimizers of the energy E in (3.6). Recall the definition of the class of admissible states in (3.1) and (3.2).

Theorem 3.2 (Existence of minimizers). *Assume $p > 3$ and $\mathcal{Y} \neq \emptyset$. Suppose that W is continuous and satisfies (3.7) and (3.8). If there is $\boldsymbol{q} \in \mathcal{Q}$ such that $E(\boldsymbol{q}) < +\infty$, then the functional E admits a minimizer in $\mathcal{Q}$.*

We begin by proving a compactness result. Recall the definition of the function γ in (3.7).

Proposition 3.3 (Compactness). *Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ satisfy*

$$||\nabla \boldsymbol{y}_n||_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq C, \qquad ||\nabla \boldsymbol{m}_n||_{L^2(\Omega^{\boldsymbol{y}_n};\mathbb{R}^{3\times 3})} \leq C, \qquad ||\gamma(\det \nabla \boldsymbol{y}_n)||_{L^1(\Omega)} \leq C \tag{3.10}$$

for every $n \in \mathbb{N}$. Then, there exists $\boldsymbol{q} \in \mathcal{Q}$ with $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$ such that, up to subsequences, we have $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$ and $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ in $L^a(\Omega; \mathbb{R}^3)$ for every $1 \leq a < \infty$.

Proof. For convenience of the reader, the proof is subdivided into three steps. $C > 0$ will be a generic constant, whose value may change from line to line.

Step 1 (Compactness). By (3.10), using the Poincaré inequality with boundary terms, we deduce that $(\boldsymbol{y}_n)$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^3)$. Thus, up to subsequences, (3.3) holds for some $\boldsymbol{y} \in W^{1,p}(\Omega; \mathbb{R}^3)$.

We claim that $\boldsymbol{y} \in \mathcal{Y}$. Given Remark 2.4 and the compactness of the trace operator, we only have to prove that $\det \nabla \boldsymbol{y} > 0$ almost everywhere in Ω. By the weak continuity of minors, $\det \nabla \boldsymbol{y}_n \rightharpoonup \det \nabla \boldsymbol{y}$ in $L^{p/3}(\Omega)$. Then, for every $F \subset \Omega$ measurable, we have

$$\int_F \det \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} = \lim_n \int_F \det \nabla \boldsymbol{y}_n \, \mathrm{d}\boldsymbol{x} \geq 0,$$

and, given the arbitrariness of F, we deduce that $\det \nabla \boldsymbol{y} \geq 0$ almost everywhere in Ω. By contradiction, suppose that $\det \nabla \boldsymbol{y} = 0$ on a measurable set $G \subset \Omega$ with $\mathscr{L}^3(G) > 0$. In this case, up to subsequences, $\det \nabla \boldsymbol{y}_n \to 0$ almost everywhere in G, and, taking into account (3.7), we obtain $\gamma(\det \nabla \boldsymbol{y}_n) \to +\infty$ almost everywhere in G. Then, by the Fatou lemma, we obtain $\liminf_n \int_G \gamma(\det \nabla \boldsymbol{y}_n) \, \mathrm{d}\boldsymbol{x} = +\infty$, which contradicts (3.10). Therefore, $\det \nabla \boldsymbol{y} > 0$ almost everywhere in Ω.

The compactness of the sequence $(\boldsymbol{q}_n)$ is proved as in [36, Proposition 2.1]. By the Morrey embedding, we have $\boldsymbol{y}_n \to \boldsymbol{y}$ uniformly in Ω. From this, we obtain the following:

$$\forall\, A \subset\subset \Omega^{\boldsymbol{y}} \text{ open}, \quad A \subset \Omega^{\boldsymbol{y}_n} \quad \text{for } n \gg 1 \text{ depending on } A, \tag{3.11}$$

$$\forall\, O \supset\supset \Omega^{\boldsymbol{y}} \text{ open}, \quad O \supset \Omega^{\boldsymbol{y}_n} \quad \text{for } n \gg 1 \text{ depending on } O. \tag{3.12}$$

To see (3.11), let $A \subset\subset \Omega^{\boldsymbol{y}}$ be open so that $\mathrm{dist}(\partial A; \partial \Omega^{\boldsymbol{y}}) > 0$. Recall that $\partial \Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial \Omega)$ by Lemma 2.1. Then, for $n \gg 1$ depending on A, we have

$$||\boldsymbol{y}_n - \boldsymbol{y}||_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq \mathrm{dist}(\partial A; \boldsymbol{y}(\partial\Omega)).$$

Let $\boldsymbol{\xi} \in A$. We obtain

$$||\boldsymbol{y}_n - \boldsymbol{y}||_{C^0(\overline{\Omega};\mathbb{R}^3)} \leq \operatorname{dist}(\boldsymbol{\xi}; \boldsymbol{y}(\partial\Omega)),$$

and, by the stability property of the degree, we deduce $\boldsymbol{\xi} \notin \boldsymbol{y}_n(\partial\Omega)$ and $\deg(\boldsymbol{y}_n, \Omega, \boldsymbol{\xi}) = \deg(\boldsymbol{y}, \Omega, \boldsymbol{\xi})$ for $n \gg 1$. As $\deg(\boldsymbol{y}, \Omega, \boldsymbol{\xi}) > 0$ by Remark 2.3, the solvability property of the degree gives $\boldsymbol{\xi} \in \Omega^{\boldsymbol{y}_n}$ for $n \gg 1$. This proves (3.11), while (3.12) is immediate.

Let $A \subset\subset \Omega^{\boldsymbol{y}}$ be open and $n \gg 1$ as in (3.11). From (3.10), we have

$$\int_A |\nabla \boldsymbol{m}_n|^2 \, \mathrm{d}\boldsymbol{\xi} \leq \int_{\Omega^{\boldsymbol{y}_n}} |\nabla \boldsymbol{m}_n|^2 \, \mathrm{d}\boldsymbol{\xi} \leq C, \tag{3.13}$$

for every $n \gg 1$. Recalling that magnetizations are sphere-valued, we deduce that $(\boldsymbol{m}_n)$ is bounded in $W^{1,2}(A;\mathbb{R}^3)$, so that, up to subsequences, $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ in $W^{1,2}(A;\mathbb{R}^3)$ for some $\boldsymbol{m} \in W^{1,2}(A;\mathbb{R}^3)$. By the Rellich embedding, $\boldsymbol{m}_n \to \boldsymbol{m}$ in $L^2(A;\mathbb{R}^3)$ and, in turn, $|\boldsymbol{m}| = 1$ almost everywhere in A. The map $\boldsymbol{m} \in W^{1,2}_{\mathrm{loc}}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)$ does not depend on A. In particular, as the right-hand side of (3.13) does not depend on A, we actually have $\boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}};\mathbb{S}^2)$. Therefore, $\boldsymbol{q} := (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$. Moreover, arguing with a sequence (A_j) of open sets such that $A_j \subset\subset A_{j+1} \subset\subset \Omega^{\boldsymbol{y}}$ for every $j \in \mathbb{N}$ and $\Omega^{\boldsymbol{y}} = \bigcup_{j=1}^{\infty} A_j$, we select a (not relabeled) subsequence of $(\boldsymbol{m}_n)$ such that

$$\forall A \subset\subset \Omega^{\boldsymbol{y}} \text{ open}, \quad \boldsymbol{m}_n \rightharpoonup \boldsymbol{m} \text{ in } W^{1,2}(A), \quad \boldsymbol{m}_n \to \boldsymbol{m} \text{ almost everywhere in } A. \tag{3.14}$$

We remark that, for every $A \subset\subset \Omega^{\boldsymbol{y}}$ open, the sequence $(\boldsymbol{m}_n) \subset W^{1,2}(A;\mathbb{S}^2)$ is defined only for $n \gg 1$ depending on A.

Step 2 (Convergence in $\mathcal{Q}$). In order to prove that $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$, we are left to show (3.4) and (3.5). To prove the first claim, we consider $\boldsymbol{\varphi} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. We need to show that

$$\lim_n \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} = 0. \tag{3.15}$$

Let $A, O \subset \mathbb{R}^3$ be open such that $A \subset\subset \Omega^{\boldsymbol{y}} \subset\subset O$. We write

$$\begin{aligned}\int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} &= \int_A (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{O \setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\mathbb{R}^3 \setminus O} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.16}$$

For the first integral on the right-hand side of (3.16), by (3.11) for $n \gg 1$ we have

$$\int_A (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} = \int_A (\boldsymbol{m}_n - \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}, \tag{3.17}$$

where, as $n \to \infty$, the right-hand side goes to zero since $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ in $W^{1,2}(A;\mathbb{R}^3)$ by (3.14). Using the Hölder inequality, the second integral on the right-hand side of (3.16) is estimated as follows

$$\left| \int_{O \setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \right| \leq 2 \sqrt{\mathscr{L}^3(O \setminus A)} \, ||\boldsymbol{\varphi}||_{L^2(\mathbb{R}^3;\mathbb{R}^3)}. \tag{3.18}$$

By (3.12), the third integral on the right-hand side of (3.16) equals zero for $n \gg 1$. Therefore, we obtain

$$\limsup_n \left| \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \right| \leq 2 \sqrt{\mathscr{L}^3(O \setminus A)} \, ||\boldsymbol{\varphi}||_{L^2(\mathbb{R}^3;\mathbb{R}^3)},$$

from which, letting $O \searrow \overline{\Omega^{\boldsymbol{y}}}$ and $A \nearrow \Omega^{\boldsymbol{y}}$ so that $\mathscr{L}^3(O \setminus A) \to \mathscr{L}^3(\partial\Omega^{\boldsymbol{y}}) = 0$, we deduce (3.15). Here, we used that $\partial\Omega^{\boldsymbol{y}} = \boldsymbol{y}(\partial\Omega)$ by Lemma 2.1 and that $\mathscr{L}^3(\boldsymbol{y}(\partial\Omega)) = 0$ thanks to the Lusin property (N). Thus (3.4) is proved.

For the second claim, we proceed in a similar way. Given $\boldsymbol{\Phi} \in L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$, we need to show

$$\lim_n \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} = 0. \tag{3.19}$$

As before, we consider $A, O \subset \mathbb{R}^3$ open with $A \subset\subset \Omega^{\boldsymbol{y}} \subset\subset O$ and we write

$$\begin{aligned}\int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} &= \int_A (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \\ &+ \int_{O \setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \\ &+ \int_{\mathbb{R}^3 \setminus O} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.20}$$

For the first integral on the right-hand side of (3.20), by (3.11), for $n \gg 1$ we have

$$\int_A (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} = \int_A (\nabla \boldsymbol{m}_n - \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x},$$

and, as $n \to \infty$, the right-hand side goes to zero since $\boldsymbol{m}_n \rightharpoonup \boldsymbol{m}$ in $W^{1,2}(A;\mathbb{R}^3)$ by (3.14). Note that the sequence $(\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n) \subset L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ is bounded by (3.10). Using the Hölder inequality, the second integral on the right-hand side of (3.20) is estimated as follows:

$$\begin{aligned}&\left| \int_{O\setminus A} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \right| \\ &\qquad \leq \left(||\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n||_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} + ||\chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}||_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \right) ||\boldsymbol{\Phi}||_{L^2(O\setminus A;\mathbb{R}^{3\times 3})} \\ &\qquad \leq \left(C + ||\chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}||_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \right) ||\boldsymbol{\Phi}||_{L^2(O\setminus A;\mathbb{R}^{3\times 3})}.\end{aligned}$$

By (3.12), the third integral on the right-hand side of (3.20) equals zero for $n \gg 1$. Therefore, we obtain

$$\limsup_n \left| \int_{\mathbb{R}^3} (\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n - \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}) : \boldsymbol{\Phi} \, \mathrm{d}\boldsymbol{x} \right| \leq \left(C + ||\chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}||_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \right) ||\boldsymbol{\Phi}||_{L^2(O\setminus A;\mathbb{R}^{3\times 3})}.$$

From this, letting $O \searrow \overline{\Omega^{\boldsymbol{y}}}$ and $A \nearrow \Omega^{\boldsymbol{y}}$ so that $\mathscr{L}^3(O \setminus A) \to \mathscr{L}^3(\partial \Omega^{\boldsymbol{y}}) = 0$ and, in turn, $||\boldsymbol{\Phi}||_{L^2(O\setminus A;\mathbb{R}^{3\times 3})} \to 0$, we deduce (3.19). Thus also (3.5) is proved.

Step 3 (Convergence of the compositions). By Proposition 2.5, $\boldsymbol{y}_n^{-1} \in W^{1,1}(\Omega^{\boldsymbol{y}_n};\mathbb{R}^3)$ with $\det \nabla \boldsymbol{y}_n^{-1} \in L^1(\Omega^{\boldsymbol{y}_n})$ for every $n \in \mathbb{N}$. Let $A \subset\subset \Omega^{\boldsymbol{y}}$ be open and recall (3.11). We claim that the sequence $(\det \nabla \boldsymbol{y}_n^{-1}) \subset L^1(A)$ is equi-integrable. To show this, we argue as in [5, Proposition 7.8]. Define $\widehat{\gamma}\colon (0,+\infty) \to [0,+\infty)$ by setting $\widehat{\gamma}(k) \coloneqq k\,\gamma(1/k)$. In this case

$$\lim_{k\to+\infty} \frac{\widehat{\gamma}(k)}{k} = \lim_{k\to+\infty} \gamma(1/k) = \lim_{h\to 0^+} \gamma(h) = +\infty,$$

where we used (3.7). Using the change-of-variable formula, we compute

$$\begin{aligned}\int_{\Omega^{\boldsymbol{y}_n}} \widehat{\gamma}(\det \nabla \boldsymbol{y}_n^{-1}) \, \mathrm{d}\boldsymbol{\xi} &= \int_{\Omega^{\boldsymbol{y}_n}} \gamma(1/\det \nabla \boldsymbol{y}_n^{-1}) \det \nabla \boldsymbol{y}_n^{-1} \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\Omega^{\boldsymbol{y}_n}} \gamma(\det \nabla \boldsymbol{y}_n) \circ \boldsymbol{y}_n^{-1} \, (\det \nabla \boldsymbol{y}_n)^{-1} \circ \boldsymbol{y}_n^{-1} \, \mathrm{d}\boldsymbol{\xi} \\ &= \int_{\Omega} \gamma(\det \nabla \boldsymbol{y}_n) \, \mathrm{d}\boldsymbol{x},\end{aligned}$$

where the right-hand side is uniformly bounded by (3.10). Thus, $(\det \nabla \boldsymbol{y}_n^{-1}) \subset L^1(A)$ is equi-integrable by the de la Vallée-Poussin Criterion [22, Theorem 2.29] for $n \gg 1$. In particular, using the area formula as in Remark 2.6, we deduce the following:

$$\begin{gathered}\text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that for every } F \subset A \text{ measurable} \\ \text{with } \mathscr{L}^3(F) < \delta \text{ and for every } n \in \mathbb{N} \text{ we have } \mathscr{L}^3(\boldsymbol{y}_n^{-1}(F)) < \varepsilon.\end{gathered} \tag{3.21}$$

We now prove that $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ in $L^1(\Omega;\mathbb{R}^3)$. Let $\eta > 0$. Take $A \subset\subset \Omega^{\boldsymbol{y}}$ open such that $\mathscr{L}^3(\Omega \setminus \boldsymbol{y}^{-1}(A)) < \eta$. We compute

$$\int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} = \int_{\Omega \setminus \boldsymbol{y}^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{y}^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x}. \tag{3.22}$$

As magnetizations are sphere-valued, the first integral on the right-hand side of (3.22) is bounded by $2\mathscr{L}^3(\Omega \setminus \boldsymbol{y}^{-1}(A)) < 2\eta$. For the second integral on the right-hand side of (3.22), we split it as

$$\int_{\boldsymbol{y}^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{y}^{-1}(A)\setminus \boldsymbol{y}_n^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{y}^{-1}(A)\cap \boldsymbol{y}_n^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}. \tag{3.23}$$

We claim that $\mathscr{L}^3(\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A)) \to 0$, as $n \to \infty$. To see this, let $V \subset \mathbb{R}^3$ be open and such that $A \subset\subset V \subset\subset \Omega^{\boldsymbol{y}}$. In this case, $\boldsymbol{y}(\boldsymbol{y}^{-1}(A)) = A \subset\subset V$ so that, by uniform convergence, $\boldsymbol{y}_n(\boldsymbol{y}^{-1}(A)) \subset V$ for $n \gg 1$ which, in turn, gives $\boldsymbol{y}^{-1}(A) \subset \boldsymbol{y}_n^{-1}(V)$ for $n \gg 1$. Then, we have

$$\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A) \subset \boldsymbol{y}_n^{-1}(V) \setminus \boldsymbol{y}_n^{-1}(A) = \boldsymbol{y}_n^{-1}(V \setminus A), \tag{3.24}$$

for $n \gg 1$. In particular, for $\varepsilon > 0$ arbitrary, $\mathscr{L}^3(V \setminus A) < \delta$ with $\delta > 0$ given by (3.21). Hence, for $n \gg 1$ depending only on ε, from (3.21) and (3.24), we obtain $\mathscr{L}^3(\boldsymbol{y}^{-1}(A) \setminus \boldsymbol{y}_n^{-1}(A)) < \varepsilon$ and the claim is proved. Thus, as magnetizations are sphere-valued, the first integral on the right-hand side of (3.23) goes to zero, as $n \to \infty$.

To estimate the second integral on the right-hand side of (3.23) we proceed as follows. Take $\varepsilon = \eta$ and let $\delta > 0$ be given by (3.21). We assume that δ is sufficiently small in order to have $\mathscr{L}^3(\boldsymbol{y}^{-1}(F)) < \eta$ for every $F \subset A$ measurable with $\mathscr{L}^3(F) < \delta$. By the Lusin Theorem, there exists $B_1 \subset A$ closed with $\mathscr{L}^3(B_1) < \delta/2$ such that $\boldsymbol{m}|_{A\setminus B_1}$ is continuous while, by the Egorov Theorem, there exists $B_2 \subset A$ closed with $\mathscr{L}^3(B_2) < \delta/2$ such that $\boldsymbol{m}_n \to \boldsymbol{m}$ uniformly on $A \setminus B_2$. Set $B \coloneqq B_1 \cup B_2$, so that $B \subset A$ is closed with $\mathscr{L}^3(B) < \delta$. We write

$$\boldsymbol{y}^{-1}(A) \cap \boldsymbol{y}_n^{-1}(A) = \left(\left(\boldsymbol{y}^{-1}(A) \cap \boldsymbol{y}_n^{-1}(A)\right) \cap \left(\boldsymbol{y}^{-1}(B) \cup \boldsymbol{y}_n^{-1}(B)\right)\right) \cup \left(\boldsymbol{y}^{-1}(A \setminus B) \cap \boldsymbol{y}_n^{-1}(A \setminus B)\right)$$

and we accordingly split the second integral on the right-hand side of (3.23) as

$$\int_{\boldsymbol{y}^{-1}(A)\cap \boldsymbol{y}_n^{-1}(A)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} = \int_{\left(\boldsymbol{y}^{-1}(A)\cap \boldsymbol{y}_n^{-1}(A)\right)\cap\left(\boldsymbol{y}^{-1}(B)\cup \boldsymbol{y}_n^{-1}(B)\right)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{y}^{-1}(A\setminus B)\cap \boldsymbol{y}_n^{-1}(A\setminus B)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}. \tag{3.25}$$

The first integral on the right-hand side of (3.25) is simply estimated by

$$\begin{aligned}\int_{\left(\boldsymbol{y}^{-1}(A)\cap \boldsymbol{y}_n^{-1}(A)\right)\cap\left(\boldsymbol{y}^{-1}(B)\cup \boldsymbol{y}_n^{-1}(B)\right)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} &\leq 2\mathscr{L}^3(\boldsymbol{y}^{-1}(B) \cup \mathscr{L}^3(\boldsymbol{y}_n^{-1}(B))) \\ &\leq 2\left(\mathscr{L}^3(\boldsymbol{y}^{-1}(B)) + \mathscr{L}^3(\boldsymbol{y}_n^{-1}(B)\right) \\ &< 4\eta,\end{aligned} \tag{3.26}$$

where, in the last line, we used (3.21). For the second integral on the right-hand side of (3.25), we have

$$\int_{\boldsymbol{y}^{-1}(A\setminus B)\cap \boldsymbol{y}_n^{-1}(A\setminus B)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} \leq \int_{\boldsymbol{y}^{-1}(A\setminus B)\cap \boldsymbol{y}_n^{-1}(A\setminus B)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}_n|\,\mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{y}^{-1}(A\setminus B)\cap \boldsymbol{y}_n^{-1}(A\setminus B)} |\boldsymbol{m} \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}. \tag{3.27}$$

Given our choice of B, for $n \gg 1$ depending only on η, we have $\sup_{A\setminus B} |\boldsymbol{m}_n - \boldsymbol{m}| < \eta$. In particular, for every $\boldsymbol{x} \in \boldsymbol{y}_n^{-1}(A \setminus B)$, we have $|\boldsymbol{m}_n(\boldsymbol{y}_n(\boldsymbol{x})) - \boldsymbol{m}(\boldsymbol{y}_n(\boldsymbol{x}))| < \eta$, so that

$$\int_{\boldsymbol{y}^{-1}(A\setminus B)\cap \boldsymbol{y}_n^{-1}(A\setminus B)} |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}_n|\,\mathrm{d}\boldsymbol{x} < \eta\mathscr{L}^3(\Omega).$$

On the other hand

$$\int_{\boldsymbol{y}^{-1}(A\setminus B)\cap \boldsymbol{y}_n^{-1}(A\setminus B)} |\boldsymbol{m} \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x} = \int_\Omega \chi_{\boldsymbol{y}^{-1}(A\setminus B)}\, \chi_{\boldsymbol{y}_n^{-1}(A\setminus B)}\, |\boldsymbol{m} \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}.$$

Let $\boldsymbol{x} \in \boldsymbol{y}^{-1}(A \setminus B)$, so that $\boldsymbol{y}(\boldsymbol{x}) \in A \setminus B$. As A is open and B is closed, by uniform convergence we have $\boldsymbol{y}_n(\boldsymbol{x}) \in A \setminus B$ or, equivalently, $\boldsymbol{x} \in \boldsymbol{y}_n^{-1}(A \setminus B)$ for $n \gg 1$. By our choice of B, we also have $\boldsymbol{m}(\boldsymbol{y}_n(x)) \to \boldsymbol{m}(\boldsymbol{y}(\boldsymbol{x}))$, as $n \to \infty$. Thus, by the Dominated Convergence Theorem, the second integral on the right-hand side of (3.26) tends to zero, as $n \to \infty$. Combining (3.22)–(3.23) and (3.25)–(3.27), we obtain

$$\limsup_n \int_\Omega |\boldsymbol{m}_n \circ \boldsymbol{y}_n - \boldsymbol{m} \circ \boldsymbol{y}| \, \mathrm{d}\boldsymbol{x} \leq (6 + \mathscr{L}^3(\Omega))\, \eta,$$

which concludes the proof. The convergence of $(\boldsymbol{m}_n \circ \boldsymbol{y}_n)$ in $L^a(\Omega; \mathbb{R}^3)$ for $1 < a < +\infty$ follows by an analogous argument. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. First, we prove that the energy E is bounded from below. Let $\boldsymbol{q} \in \mathcal{Q}$ with $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$ and, for simplicity, set $r := p/3$. By the Hölder inequality and the Young inequality, we have

$$\begin{aligned} \mathscr{L}^3(\Omega^{\boldsymbol{y}}) &\leq \int_\Omega \det \nabla \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \leq C \int_\Omega |\nabla \boldsymbol{y}|^3 \, \mathrm{d}\boldsymbol{x} \\ &\leq C ||\nabla \boldsymbol{y}||^3_{L^p(\Omega;\mathbb{R}^{3\times 3})} \mathscr{L}^3(\Omega)^{1/r'} \\ &\leq \frac{C\varepsilon^r}{r} ||\nabla \boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + \frac{C}{r'\varepsilon^{r'}} \mathscr{L}^3(\Omega), \end{aligned} \tag{3.28}$$

where $\varepsilon > 0$ is arbitrary.

Using again the Hölder inequality and the Young inequality and (3.28), we estimate

$$\begin{aligned} |E^{\mathrm{DMI}}(\boldsymbol{q})| &\leq |\kappa| \int_{\Omega^{\boldsymbol{y}}} |\mathrm{curl}\, \boldsymbol{m}| \, \mathrm{d}\boldsymbol{\xi} \leq 2|\kappa| \int_{\Omega^{\boldsymbol{y}}} |\nabla \boldsymbol{m}| \, \mathrm{d}\boldsymbol{\xi} \\ &\leq 2|\kappa| \, ||\nabla \boldsymbol{m}||_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} \mathscr{L}^3(\Omega^{\boldsymbol{y}})^{1/2} \\ &\leq \delta ||\nabla \boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} + \frac{\kappa^2}{\delta} \mathscr{L}^3(\Omega^{\boldsymbol{y}}) \\ &\leq \delta ||\nabla \boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} + \frac{C\kappa^2\varepsilon^r}{r\delta} ||\nabla \boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + \frac{C\kappa^2}{r'\varepsilon^{r'}\delta} \mathscr{L}^3(\Omega), \end{aligned} \tag{3.29}$$

where $\delta > 0$ is arbitrary. Hence, from (3.7) and (3.29), we deduce

$$\begin{aligned} E(\boldsymbol{q}) &\geq E^{\mathrm{el}}(\boldsymbol{q}) + E^{\mathrm{exc}}(\boldsymbol{q}) + E^{\mathrm{DMI}}(\boldsymbol{q}) \\ &\geq \left(K - \frac{C\kappa^2\varepsilon^r}{r\delta} \right) ||\nabla \boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + (\alpha - \delta) \, ||\nabla \boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} \\ &\quad + ||\gamma(\det \nabla \boldsymbol{y})||_{L^1(\Omega)} - \frac{C\kappa^2}{r'\varepsilon^{r'}\delta} \mathscr{L}^3(\Omega), \end{aligned}$$

so that, for $\delta < \alpha$ and $\varepsilon < (C^{-1} r K \delta \kappa^{-2})^{1/r}$, we obtain

$$E(\boldsymbol{q}) \geq C_1 \, ||\nabla \boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + C_2 \, ||\nabla \boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})} - C_3 + ||\gamma(\det \nabla \boldsymbol{y})||_{L^1(\Omega)}, \tag{3.30}$$

where $C_1(p, K, \kappa) > 0$, $C_2 > 0$ and $C_3(\Omega, p, \kappa) > 0$. This yields $I := \inf_{\mathcal{Q}} E \geq -C_3$.

Let $(\boldsymbol{q}_n) \subset \mathcal{Q}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n, \boldsymbol{m}_n)$ be a minimizing sequence for E, namely such that $E(\boldsymbol{q}_n) \to I$, as $n \to \infty$. In particular, $E(\boldsymbol{q}_n) \leq C$ for every $n \in \mathbb{N}$. Thanks to (3.30), we deduce (3.10) and we can apply Proposition 3.3. Then, there exist a subsequence of $(\boldsymbol{q}_n)$ (not relabeled) and an admissible state $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{Q}$ such that $\boldsymbol{q}_n \to \boldsymbol{q}$ in $\mathcal{Q}$ and $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ in $L^a(\Omega; \mathbb{R}^3)$ for every $1 \leq a < \infty$.

We claim that

$$E(\boldsymbol{q}) \leq \liminf_n E(\boldsymbol{q}_n), \tag{3.31}$$

so that $\boldsymbol{q}$ is a minimizer of E, as $E(\boldsymbol{q}_n) \to I$. We focus on the elastic energy first. We have $\nabla \boldsymbol{y}_n \rightharpoonup \nabla \boldsymbol{y}$ in $L^p(\Omega; \mathbb{R}^{3\times 3})$ and, by the weak continuity of minors, also $\mathrm{cof}\, \nabla \boldsymbol{y}_n \rightharpoonup \mathrm{cof}\, \nabla \boldsymbol{y}$ in $L^{p/2}(\Omega; \mathbb{R}^{3\times 3})$, and

$\det \nabla \boldsymbol{y}_n \rightharpoonup \det \nabla \boldsymbol{y}$ in $L^r(\Omega)$. Moreover, the subsequence can be chosen in order to have $\boldsymbol{m}_n \circ \boldsymbol{y}_n \to \boldsymbol{m} \circ \boldsymbol{y}$ almost everywhere in Ω. Thus, given (3.8), applying [3, Theorem 5.4] we prove that

$$E^{\mathrm{el}}(\boldsymbol{q}) \leq \liminf_n E^{\mathrm{el}}(\boldsymbol{q}_n). \tag{3.32}$$

We have $\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n \to \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ and $\chi_{\Omega^{\boldsymbol{y}_n}} \nabla \boldsymbol{m}_n \rightharpoonup \chi_{\Omega^{\boldsymbol{y}}} \nabla \boldsymbol{m}$ in $L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$. The lower semicontinuity of the norm gives

$$E^{\mathrm{exc}}(\boldsymbol{q}) \leq \liminf_n E^{\mathrm{exc}}(\boldsymbol{q}_n), \tag{3.33}$$

while, as

$$\operatorname{curl} \boldsymbol{m}_n \cdot \boldsymbol{m}_n = (\partial_2 m_n^3 - \partial_3 m_n^2) m_n^1 + (\partial_3 m_n^1 - \partial_1 m_n^3) m_n^2 + (\partial_1 m_n^2 - \partial_2 m_n^1) m_n^3,$$

we have

$$E^{\mathrm{DMI}}(\boldsymbol{q}) = \lim_n E^{\mathrm{DMI}}(\boldsymbol{q}_n). \tag{3.34}$$

We focus on the magnetostatic energy. Denote by ζ_n a weak solutions of the Maxwell equation corresponding to $\boldsymbol{q}_n$. Thus, for every $n \in \mathbb{N}$ and for every $\varphi \in V^{1,2}(\mathbb{R}^3)$, there holds

$$\int_{\mathbb{R}^3} \nabla \zeta_n \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{\xi}. \tag{3.35}$$

Denote by $V^{1,2}(\mathbb{R}^3)/\mathbb{R}$ the quotient of $V^{1,2}(\mathbb{R}^3)$ with respect to constant functions and recall that this is an Hilbert space with inner product given by

$$([\varphi],[\psi]) \mapsto \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi \,\mathrm{d}\boldsymbol{\xi}.$$

Testing (3.35) with $\varphi = \zeta_n$ and using that $||\chi_{\Omega^{\boldsymbol{y}_n}} \boldsymbol{m}_n||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq C$ for every $n \in \mathbb{N}$ by (3.28), we obtain that $||[\zeta_n]||_{V^{1,2}(\mathbb{R}^3)/\mathbb{R}} = ||\nabla \zeta_n||_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq C$ for every $n \in \mathbb{N}$. Therefore, there exists $\zeta \in V^{1,2}(\mathbb{R}^3)$ such that, up to subsequences, we have $[\zeta_n] \rightharpoonup [\zeta]$ in $V^{1,2}(\mathbb{R}^3)/\mathbb{R}$, or equivalently, $\nabla \zeta_n \rightharpoonup \nabla \zeta$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. Passing to the limit, as $n \to \infty$, in (3.35), we obtain that

$$\int_{\mathbb{R}^3} \nabla \zeta \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^3} \chi_{\Omega^{\boldsymbol{y}}} \boldsymbol{m} \cdot \nabla \varphi \,\mathrm{d}\boldsymbol{\xi},$$

for every $\varphi \in V^{1,2}(\mathbb{R})$. Thus ζ is a weak solution of the Maxwell equation corresponding to $\boldsymbol{q}$, so that $E^{\mathrm{mag}}(\boldsymbol{q}) = \frac{\mu_0}{2} ||\nabla \zeta||^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$. By the lower semicontinuity of the norm, we conclude

$$E^{\mathrm{mag}}(\boldsymbol{q}) \leq \liminf_n E^{\mathrm{mag}}(\boldsymbol{q}_n). \tag{3.36}$$

Finally, combining (3.32)-(3.34) and (3.36), we obtain (3.31). □

Remark 3.4 (Existence with applied loads). The existence result given in Theorem 3.2 can be extended to include applied loads. Consider $\boldsymbol{f} \in L^{p'}(\Omega;\mathbb{R}^3)$, $\boldsymbol{g} \in L^{p'}(\Sigma;\mathbb{R}^3)$ with $\Sigma \subset \partial\Omega$ relatively open such that $\mathscr{H}^2(\partial\Omega \setminus (\Gamma \cup \Sigma)) = 0$, and $\boldsymbol{h} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, representing the applied body force, surface force and magnetic field, respectively. Then, the functional

$$\boldsymbol{q} \mapsto E(\boldsymbol{q}) - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{y} \,\mathrm{d}\boldsymbol{x} - \int_\Sigma \boldsymbol{g} \cdot \boldsymbol{y} \,\mathrm{d}\mathscr{H}^2 - \int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h} \cdot \boldsymbol{m} \,\mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, admits a minimizer in $\mathcal{Q}$. Indeed, the functional determined by the applied loads is continuous with respect to the topology of $\mathcal{Q}$.

4. Quasistatic Setting

In this section we prove the existence of quasi-static evolutions of the system driven by the energy E under time-dependent applied loads and dissipative effects. The framework is the theory of *rate-independent processes* [45] with the notion of *energetic solutions*. We start describing the general setting.

The applied loads are determined by the functions

$$\boldsymbol{f} \in C^1([0,T];L^{p'}(\Omega;\mathbb{R}^3)), \quad \boldsymbol{g} \in C^1([0,T];L^{p'}(\Sigma;\mathbb{R}^3)), \quad \boldsymbol{h} \in C^1([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3)), \tag{4.1}$$

representing the external body force, surface force and magnetic field, respectively. Define the functional $\mathcal{L}\colon [0,T]\times\mathcal{Q}\to\mathbb{R}$ by setting

$$\mathcal{L}(t,\boldsymbol{q}) \coloneqq \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} + \int_\Sigma \boldsymbol{g}(t)\cdot\boldsymbol{y}\,\mathrm{d}\mathscr{H}^2 + \int_{\Omega^{\boldsymbol{y}}} \boldsymbol{h}(t)\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}, \tag{4.2}$$

where $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$. The total energy of the system is given by the functional $\mathcal{E}\colon[0,T]\times\mathcal{Q}\to\mathbb{R}$ defined by

$$\mathcal{E}(t,\boldsymbol{q}) \coloneqq E(\boldsymbol{q}) - \mathcal{L}(t,\boldsymbol{q}). \tag{4.3}$$

Starting from (3.30), by a repeated application of the Hölder inequality and the Young inequality and using (3.28), we prove

$$\mathcal{E}(t,\boldsymbol{q}) \geq C_1 ||\nabla\boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times3})} + C_2||\nabla\boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times3})} + \int_\Omega \gamma(\det\nabla\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} - C_3 \tag{4.4}$$

for every $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$. Here, $C_1(\Omega,p,K,\kappa)>0$, where $K>0$ was introduced in (3.7), $C_2>0$ and $C_3(\Omega,p,\kappa,\overline{M},M_{\boldsymbol{f}},M_{\boldsymbol{g}},M_{\boldsymbol{h}})>0$, where $\overline{M}\coloneqq||\overline{\boldsymbol{y}}||_{L^{p'}(\Sigma;\mathbb{R}^3)}$ takes into account the boundary datum in (3.1) and we set

$$M_{\boldsymbol{f}} \coloneqq ||\boldsymbol{f}||_{C^0([0,T];L^{p'}(\Omega;\mathbb{R}^3))}, \quad M_{\boldsymbol{g}} \coloneqq ||\boldsymbol{g}||_{C^0([0,T];L^{p'}(\Sigma;\mathbb{R}^3))}, \quad M_{\boldsymbol{h}} \coloneqq ||\boldsymbol{h}||_{C^0([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}.$$

In particular, from (4.4), we deduce $\inf_{[0,T]\times\mathcal{Q}}\mathcal{E}\geq -C_3$.

Given the regularity of the applied loads, for every $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$, the map $t\mapsto\mathcal{L}(t,\boldsymbol{q})$ belongs to $C^1([0,T])$. In particular, for every $t\in[0,T]$, we compute

$$\partial_t\mathcal{E}(t,\boldsymbol{q}) = -\partial_t\mathcal{L}(t,\boldsymbol{q}) = -\int_\Omega \dot{\boldsymbol{f}}(t)\cdot\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} - \int_\Sigma \dot{\boldsymbol{g}}(t)\cdot\boldsymbol{y}\,\mathrm{d}\mathscr{H}^2 - \int_{\Omega^{\boldsymbol{y}}} \dot{\boldsymbol{h}}(t)\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{\xi}. \tag{4.5}$$

Employing again the Hölder inequality and the Young inequality and exploiting (4.4), we prove the estimate

$$|\partial_t\mathcal{E}(t,\boldsymbol{q})| \leq L\,(\mathcal{E}(t,\boldsymbol{q})+M), \tag{4.6}$$

where $L\coloneqq C(\Omega,p,K,\kappa,\overline{M},L_{\boldsymbol{f}},L_{\boldsymbol{g}},L_{\boldsymbol{h}})>0$ and $M\coloneqq C(\Omega,p,K,\kappa,\overline{M},M_{\boldsymbol{f}},M_{\boldsymbol{g}},M_{\boldsymbol{h}})>0$, where we set

$$L_{\boldsymbol{f}} \coloneqq ||\dot{\boldsymbol{f}}||_{C^0([0,T];L^{p'}(\Omega;\mathbb{R}^3))}, \quad L_{\boldsymbol{g}} \coloneqq ||\dot{\boldsymbol{g}}||_{C^0([0,T];L^{p'}(\Sigma;\mathbb{R}^3))}, \quad L_{\boldsymbol{h}} \coloneqq ||\dot{\boldsymbol{h}}||_{C^0([0,T];L^2(\mathbb{R}^3;\mathbb{R}^3))}.$$

From this, using the Gronwall inequality, we obtain

$$\mathcal{E}(t,\boldsymbol{q}) + M \leq (\mathcal{E}(s,\boldsymbol{q})+M)e^{L(t-s)}, \tag{4.7}$$

for every $\boldsymbol{q}\in\mathcal{Q}$ and $s,t\in[0,T]$ with $s<t$.

As in [50], we introduce the *Lagrangean magnetization* given, for $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})\in\mathcal{Q}$, by

$$\mathcal{Z}(\boldsymbol{q}) \coloneqq (\mathrm{adj}\nabla\boldsymbol{y})\,\boldsymbol{m}\circ\boldsymbol{y}. \tag{4.8}$$

The dissipation distance $\mathcal{D}\colon\mathcal{Q}\times\mathcal{Q}\to[0,+\infty)$ is defined as

$$\mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}}) \coloneqq \int_\Omega |\mathcal{Z}(\boldsymbol{q})-\mathcal{Z}(\widehat{\boldsymbol{q}})|\,\mathrm{d}\boldsymbol{x}. \tag{4.9}$$

Moreover, the variation of any map $\boldsymbol{q}\colon[0,T]\to\mathcal{Q}$ with respect to $\mathcal{D}$ on the interval $[s,t]\subset[0,T]$ is defined by

$$\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[s,t]) \coloneqq \sup\left\{\sum_{i=1}^N \mathcal{D}(\boldsymbol{q}(t_i),\boldsymbol{q}(t_{i-1}))\,:\,\Pi=(t_0,\dots,t_N)\text{ partition of }[s,t]\right\}. \tag{4.10}$$

Here, by a partition of the interval $[s,t]$ we mean any finite ordered set $\Pi=(t_0,\dots,t_N)\subset[0,T]^N$ with $s=t_0<t_1<\cdots<t_N=t$. Note that in (4.10) each partition can have different cardinality.

The existence of energetic solutions is usually proved in two steps: first, one constructs time-discrete solutions corresponding to a given partition of the time interval, then one obtains the desired solution from the piecewise constant interpolants by compactness arguments considering a sequence of partitions of vanishing size. These two steps will be addressed in the next two subsections. We point out that in Subsection 4.1 we argue without higher-order terms and that the regularization is only added in the passage to the time-continuous setting.

4.1. **Time-discrete setting.** Let $\Pi=(t_0,\dots,t_N)$ be a partition of $[0,T]$. We consider the *incremental minimization problem* determined by Π with initial data $\boldsymbol{q}^0\in\mathcal{Q}$, which reads as follows:

$$\begin{gathered}\text{find } (\boldsymbol{q}^1,\dots,\boldsymbol{q}^N)\in\mathcal{Q}^N \text{ such that each } \boldsymbol{q}^i \text{ is a minimizer}\\ \text{of } \boldsymbol{q}\mapsto\mathcal{E}(t_i,\boldsymbol{q})+\mathcal{D}(\boldsymbol{q}^{i-1},\boldsymbol{q}) \text{ for } i=1,\dots,N.\end{gathered}\tag{4.11}$$

The next result states the existence of solutions of (4.11) and collects their main properties. Recall the definition of the total energy $\mathcal{E}$ and of the dissipation distance $\mathcal{D}$ in (4.3) and (4.9), respectively. Recall also (4.6).

Proposition 4.1 (Solutions of the incremental minimization problem). *Assume $p>3$ and $\mathcal{Y}\neq\emptyset$. Suppose that W is continuous and satisfies (3.7) and (3.8) and that the applied loads satisfy (4.1). Let $\Pi=(t_0,\dots,t_N)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0\in\mathcal{Q}$. Then, the incremental minimization problem (4.11) admits a solution $(\boldsymbol{q}^1,\dots,\boldsymbol{q}^N)\in\mathcal{Q}^N$. Moreover, if $\boldsymbol{q}^0$ is such that*

$$\mathcal{E}(0,\boldsymbol{q}^0)\leq\mathcal{E}(0,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}^0,\widehat{\boldsymbol{q}})\tag{4.12}$$

for every $\widehat{\boldsymbol{q}}\in\mathcal{Q}$, then the following holds:

$$\forall\, i=1,\dots,N,\ \forall\,\widehat{\boldsymbol{q}}\in\mathcal{Q},\ \ \mathcal{E}(t_i,\boldsymbol{q}^i)\leq\mathcal{E}(t_i,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}^i,\widehat{\boldsymbol{q}}),\tag{4.13}$$

$$\forall\, i=1,\dots,N,\ \ \mathcal{E}(t_i,\boldsymbol{q}^i)-\mathcal{E}(t_{i-1},\boldsymbol{q}^{i-1})+\mathcal{D}(\boldsymbol{q}^{i-1},\boldsymbol{q}^i)\leq\int_{t_{i-1}}^{t_i}\partial_t\mathcal{E}(\tau,\boldsymbol{q}^{i-1})\,\mathrm{d}\tau,\tag{4.14}$$

$$\forall\, i=1,\dots,N,\ \ \mathcal{E}(t_i,\boldsymbol{q}^i)+M+\sum_{j=1}^{i}\mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j)\leq\left(\mathcal{E}(0,\boldsymbol{q}^0)+M\right)e^{Lt_i}.\tag{4.15}$$

Proof. The main point is to prove the existence of solutions of (4.11). Given a solution of (4.11) where $\boldsymbol{q}^0$ satisfies (4.12), then (4.13)-(4.15) are obtained by standard computations as in [44, Theorem 3.2].

It is sufficient to show that, for $\tilde{t}\in[0,T]$ and $\widetilde{\boldsymbol{q}}\in\mathcal{Q}$ fixed, the auxiliary functional $\mathcal{F}\colon\mathcal{Q}\to\mathbb{R}$ given by $\mathcal{F}(\boldsymbol{q}):=\mathcal{E}(\tilde{t},\boldsymbol{q})+\mathcal{D}(\widetilde{\boldsymbol{q}},\boldsymbol{q})$, admits a minimizer in $\mathcal{Q}$.

First note that, as $\mathcal{D}$ is positive, from (4.4) we obtain

$$\mathcal{F}(\boldsymbol{q})\geq\mathcal{E}(\tilde{t},\boldsymbol{q})\geq C_1||\nabla\boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times3})}+C_2||\nabla\boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times3})}-C_3+||\gamma(\det\nabla\boldsymbol{y})||_{L^1(\Omega)}\tag{4.16}$$

for every $\boldsymbol{q}\in\mathcal{Q}$ with $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$. In particular, we deduce $J:=\inf_{\mathcal{Q}}\mathcal{F}\geq-C_3$.

Let $(\boldsymbol{q}_n)\subset\mathcal{Q}$ with $\boldsymbol{q}_n=(\boldsymbol{y}_n,\boldsymbol{m}_n)$ be a minimizing sequence for $\mathcal{F}$, namely such that $\mathcal{F}(\boldsymbol{q}_n)\to J$. Thus, $\mathcal{F}(\boldsymbol{q}_n)\leq C$ for every $n\in\mathbb{N}$, and (4.16) yields (3.10). By Proposition 3.3, there exist $\boldsymbol{q}\in\mathcal{Q}$ such that, up to subsequences, we have $\boldsymbol{q}_n\to\boldsymbol{q}$ in $\mathcal{Q}$ and $\boldsymbol{m}_n\circ\boldsymbol{y}_n\to\boldsymbol{m}\circ\boldsymbol{y}$ in $L^a(\Omega;\mathbb{R}^3)$ for every $1\leq a<\infty$. Arguing as in the proof of Theorem 3.2, we prove (3.31) while, using the Rellich embedding and the compactness of traces, we show

$$\mathcal{L}(\tilde{t},\boldsymbol{q})=\lim_n\mathcal{L}(\tilde{t},\boldsymbol{q}_n).\tag{4.17}$$

By the weak continuity of minors, $\operatorname{adj}\nabla\boldsymbol{y}_n\rightharpoonup\operatorname{adj}\nabla\boldsymbol{y}$ in $L^{p/2}(\Omega;\mathbb{R}^{3\times3})$. Hence, we have $\mathcal{Z}(\boldsymbol{q}_n)\rightharpoonup\mathcal{Z}(\boldsymbol{q})$ in $L^1(\Omega;\mathbb{R}^3)$, and, by the lower semicontinuity of the norm, we deduce

$$\mathcal{D}(\widetilde{\boldsymbol{q}},\boldsymbol{q})\leq\liminf_n\mathcal{D}(\widetilde{\boldsymbol{q}},\boldsymbol{q}_n).\tag{4.18}$$

Finally, combining (3.31), (4.17) and (4.18), we obtain

$$\mathcal{F}(\boldsymbol{q})\leq\liminf_n\mathcal{F}(\boldsymbol{q}_n)=J,$$

so that $\boldsymbol{q}$ is a minimizer of $\mathcal{F}$. □

Remark 4.2 (Time-discrete energetic solutions). Note that (4.13) and (4.14) can be seen as the discrete counterparts of the stability condition (4.38) and the energy balance (4.39) in Theorem 4.6, respectively.

4.2. **Time-continuous setting.** As it is common for finite strain theories, it is not possible to pass from the time-discrete formulation to the time-continuous one in our setting without further higher-order terms. Henceforth, we regularize the problem as follows. Recalling (3.1), we restrict ourselves to the class of deformations

$$\widetilde{\mathcal{Y}} := \left\{ \boldsymbol{y} \in \mathcal{Y} :\ D(\operatorname{cof} \nabla \boldsymbol{y}) \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3\times 3}) \right\}, \tag{4.19}$$

so that the corresponding class of admissible states is given by

$$\widetilde{\mathcal{Q}} := \left\{ (\boldsymbol{y}, \boldsymbol{m}) :\ \boldsymbol{y} \in \widetilde{\mathcal{Y}},\ \boldsymbol{m} \in W^{1,2}(\Omega^{\boldsymbol{y}}; \mathbb{S}^2) \right\}. \tag{4.20}$$

In (4.19), $D(\operatorname{cof} \nabla \boldsymbol{y})$ denotes the distributional gradient of $\operatorname{cof} \nabla \boldsymbol{y}$ which is assumed to be given by a bounded tensor-valued Radon measure.

Example 4.3. *Let Ω, P and $\boldsymbol{y}$ be as in Example 2.2 and recall Example 3.1. Then, $\boldsymbol{y} \in \widetilde{\mathcal{Y}}$. To see this, for every $\boldsymbol{x} \in \Omega \setminus P$ with $\boldsymbol{x} = (x_1, x_2, x_3)$, we compute*

$$\operatorname{cof} \nabla \boldsymbol{y}(\boldsymbol{x}) := \begin{pmatrix} |x_1| & 0 & -x_1\, x_3/|x_1| \\ 0 & |x_1| & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set $u(\boldsymbol{x}) := |x_1|$ and $v(\boldsymbol{x}) := -x_1\, x_3/|x_1|$. Then $u \in W^{1,\infty}(\Omega)$, while $v \in BV(\Omega)$ since

$$D_1 v = w\, \mathscr{H}^2 \llcorner \{0\} \times (-1,1)^2,$$

where D_1 denotes the distributional derivative with respect to the first variable and we set $w(\boldsymbol{x}) := 2x_3$. Therefore $\boldsymbol{y} \in \widetilde{\mathcal{Y}}$.

Recalling (3.6), the regularized energy $\widetilde{E} \colon \widetilde{\mathcal{Q}} \to \mathbb{R}$ is given by

$$\widetilde{E}(\boldsymbol{q}) := E(\boldsymbol{q}) + E^{\mathrm{reg}}(\boldsymbol{q}), \tag{4.21}$$

where, for $\boldsymbol{q} = (\boldsymbol{y}, \boldsymbol{m})$, we set

$$E^{\mathrm{reg}}(\boldsymbol{q}) := |D(\operatorname{cof} \nabla \boldsymbol{y})|(\Omega). \tag{4.22}$$

Here, $|D(\operatorname{cof} \nabla \boldsymbol{y})|(\Omega)$ denotes the total variation of the measure $D(\operatorname{cof} \nabla \boldsymbol{y}) \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3\times 3})$ over Ω.

Then, the corresponding total energy $\widetilde{\mathcal{E}} \colon [0,T] \times \widetilde{\mathcal{Q}} \to \mathbb{R}$ is defined as

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}) := \widetilde{E}(\boldsymbol{q}) - \mathcal{L}(t, \boldsymbol{q}), \tag{4.23}$$

where $\mathcal{L}$ is given by (4.2). Therefore, from (4.6) and (4.7), we deduce

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}) + M \leq (\widetilde{\mathcal{E}}(s, \boldsymbol{q}) + M) e^{L(t-s)} \tag{4.24}$$

for every $\boldsymbol{q} \in \widetilde{\mathcal{Q}}$ and $s, t \in [0,T]$ with $s < t$.

Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$. Under regularization, the *incremental minimization problem* determined by Π with initial data $\boldsymbol{q}^0 \in \mathcal{Q}$ reads as follows:

$$\begin{gathered} \text{find } (\boldsymbol{q}^1, \dots, \boldsymbol{q}^N) \in \widetilde{\mathcal{Q}}^N \text{ such that each } \boldsymbol{q}^i \text{ is a minimizer} \\ \text{of } \boldsymbol{q} \mapsto \widetilde{\mathcal{E}}(t_i, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}^{i-1}, \boldsymbol{q}) \text{ for } i = 1, \dots, N. \end{gathered} \tag{4.25}$$

Similarly to Proposition 4.1, we have the following result.

Proposition 4.4 (Solutions of the incremental minimization problem under regularization). *Assume $p > 3$ and $\widetilde{\mathcal{Y}} \neq \emptyset$. Suppose that W is continuous and satisfies (3.7) and (3.8) and that the applied loads satisfy (4.1). Let $\Pi = (t_0, \dots, t_N)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0 \in \widetilde{\mathcal{Q}}$. Then, the incremental minimization problem (4.25) admits a solution $(\boldsymbol{q}^1, \dots, \boldsymbol{q}^N) \in \widetilde{\mathcal{Q}}^N$. Moreover, if $\boldsymbol{q}^0$ satisfies*

$$\forall \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) \leq \widetilde{\mathcal{E}}(0, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^0, \widehat{\boldsymbol{q}}), \tag{4.26}$$

then the following holds:

$$\forall\, i = 1, \dots, N,\ \ \forall \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t_i, \boldsymbol{q}^i) \leq \widetilde{\mathcal{E}}(t_i, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}^i, \widehat{\boldsymbol{q}}), \tag{4.27}$$

$$\forall\, i=1,\dots,N, \quad \widetilde{\mathcal{E}}(t_i,\boldsymbol{q}^i)-\widetilde{\mathcal{E}}(t_{i-1},\boldsymbol{q}^{i-1})+\mathcal{D}(\boldsymbol{q}^{i-1},\boldsymbol{q}^i)\leq \int_{t_{i-1}}^{t_i}\partial_t\widetilde{\mathcal{E}}(\tau,\boldsymbol{q}^{i-1})\,\mathrm{d}\tau, \tag{4.28}$$

$$\forall\, i=1,\dots,N, \quad \widetilde{\mathcal{E}}(t_i,\boldsymbol{q}^i)+M+\sum_{j=1}^{i}\mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j)\leq (\widetilde{\mathcal{E}}(0,\boldsymbol{q}^0)+M)\,e^{Lt_i}. \tag{4.29}$$

Proof. Again, the main point is to prove the existence of solutions to (4.25). Hence, we show that the auxiliary functional $\widetilde{\mathcal{F}}\colon \widetilde{\mathcal{Q}}\to\mathbb{R}$ defined by $\widetilde{\mathcal{F}}(\boldsymbol{q}) \coloneqq \widetilde{\mathcal{E}}(\tilde{t},\boldsymbol{q})+\mathcal{D}(\widetilde{\boldsymbol{q}},\boldsymbol{q})$, where $\tilde{t}\in[0,T]$ and $\widetilde{\boldsymbol{q}}\in\widetilde{\mathcal{Q}}$ are fixed, admits a minimizer in $\widetilde{\mathcal{Q}}$.

First note that, by (4.4), we have

$$\begin{aligned}\widetilde{\mathcal{F}}(\boldsymbol{q})\geq \widetilde{\mathcal{E}}(\tilde{t},\boldsymbol{q}) \geq C_1||\nabla\boldsymbol{y}||^p_{L^p(\Omega;\mathbb{R}^{3\times 3})}+C_2||\nabla\boldsymbol{m}||^2_{L^2(\Omega^{\boldsymbol{y}};\mathbb{R}^{3\times 3})}-C_3\\ +||\gamma(\det\nabla\boldsymbol{y})||_{L^1(\Omega)}+||D(\mathrm{cof}\,\nabla\boldsymbol{y})||_{\mathcal{M}_b(\Omega;\mathbb{R}^{3\times3\times3})}\end{aligned} \tag{4.30}$$

for every $\boldsymbol{q}\in\widetilde{\mathcal{Q}}$ with $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$. In particular, we deduce $\widetilde{J}\coloneqq \inf_{\widetilde{\mathcal{Q}}}\widetilde{\mathcal{F}}\geq -C_3$.

Let $(\boldsymbol{q}_n)\subset\widetilde{\mathcal{Q}}$ with $\boldsymbol{q}_n=(\boldsymbol{y}_n,\boldsymbol{m}_n)$ be a minimizing sequence for $\widetilde{\mathcal{F}}$, namely such that $\widetilde{\mathcal{F}}(\boldsymbol{q}_n)\to\widetilde{J}$. As $\widetilde{\mathcal{F}}(\boldsymbol{q}_n)\leq C$ for every $n\in\mathbb{N}$, from (4.30) we obtain

$$\begin{aligned}||\nabla\boldsymbol{y}_n||_{L^p(\Omega;\mathbb{R}^{3\times3})}\leq C, \qquad & ||\nabla\boldsymbol{m}_n||_{L^2(\Omega^{\boldsymbol{y}_n};\mathbb{R}^{3\times3})}\leq C,\\ ||\gamma(\det\nabla\boldsymbol{y}_n)||_{L^1(\Omega)}\leq C, \qquad & ||D(\mathrm{cof}\,\nabla\boldsymbol{y}_n)||_{\mathcal{M}_b(\Omega;\mathbb{R}^{3\times3\times3})}\leq C.\end{aligned}$$

In this case, we have $||\mathrm{cof}\,\nabla\boldsymbol{y}_n||_{BV(\Omega;\mathbb{R}^{3\times3})}\leq C$ for every $n\in\mathbb{N}$. Thus, up to subsequence, we have

$$\mathrm{cof}\,\nabla\boldsymbol{y}_n\to\boldsymbol{G} \text{ in } L^1(\Omega;\mathbb{R}^{3\times3}), \quad D(\mathrm{cof}\,\nabla\boldsymbol{y}_n)\overset{*}{\rightharpoonup} D\boldsymbol{G} \text{ in } \mathcal{M}_b(\Omega;\mathbb{R}^{3\times3}), \tag{4.31}$$

for some $\boldsymbol{G}\in BV(\Omega;\mathbb{R}^{3\times3})$. By Proposition 3.3, there exists $\boldsymbol{q}\in\mathcal{Q}$ with $\boldsymbol{q}=(\boldsymbol{y},\boldsymbol{m})$ such that, up to subsequences, $\boldsymbol{q}_n\to\boldsymbol{q}$ in $\mathcal{Q}$ and $\boldsymbol{m}_n\circ\boldsymbol{y}_n\to\boldsymbol{m}\circ\boldsymbol{y}$ in $L^a(\Omega;\mathbb{R}^3)$ for every $1\leq a<\infty$. As $\mathrm{cof}\,\nabla\boldsymbol{y}_n\rightharpoonup\mathrm{cof}\,\nabla\boldsymbol{y}$ in $L^{p/2}(\Omega;\mathbb{R}^{3\times3})$ by the weak continuity of minors, from (4.31) we deduce $\boldsymbol{G}=\mathrm{cof}\,\nabla\boldsymbol{y}$. Hence, $\boldsymbol{y}\in\widetilde{\mathcal{Y}}$ and, in turn, $\boldsymbol{q}\in\widetilde{\mathcal{Q}}$.

Arguing as in the proof of Theorem 3.2, we prove (3.31), while, as in the proof of Proposition 4.1, we show (4.17) and (4.18). By (4.31) and the lower semicontinuity of the total variation, we have

$$E^{\mathrm{reg}}(\boldsymbol{q})\leq\liminf_n E^{\mathrm{reg}}(\boldsymbol{q}_n). \tag{4.32}$$

Combining (3.31), (4.17), (4.18) and (4.32), we deduce

$$\widetilde{\mathcal{F}}(\boldsymbol{q})\leq\liminf_n\widetilde{\mathcal{F}}(\boldsymbol{q}_n)=\widetilde{J},$$

so that $\boldsymbol{q}$ is a minimizer of $\widetilde{\mathcal{F}}$. □

Recall the definition of variation with respect to $\mathcal{D}$ in (4.10).

Proposition 4.5 (Piecewise constant interpolants). *Assume $p>3$ and $\widetilde{\mathcal{Y}}\neq\emptyset$. Suppose that W is continuous and satisfies (3.7) and (3.8) and that the applied loads satisfy (4.1). Let $\Pi=(t_0,\dots,t_N)$ be a partition of $[0,T]$ and let $\boldsymbol{q}^0\in\widetilde{\mathcal{Q}}$ satisfy (4.26). Let $(\boldsymbol{q}^1,\dots,\boldsymbol{q}^N)\in\widetilde{\mathcal{Q}}^N$ be a solution of the incremental minimization problem (4.25) and define the (right-continuous) piecewise constant interpolant $\boldsymbol{q}_\Pi\colon[0,T]\to\widetilde{\mathcal{Q}}$ as*

$$\boldsymbol{q}_\Pi(t)\coloneqq\begin{cases}\boldsymbol{q}^{i-1} & \text{if } t\in[t_{i-1},t_i) \text{ for some } i=1,\dots,N,\\ \boldsymbol{q}^N & \text{if } t=T.\end{cases} \tag{4.33}$$

Then, the following holds:

$$\forall t\in\Pi,\ \forall\widehat{\boldsymbol{q}}\in\widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_\Pi(t))\leq\widetilde{\mathcal{E}}(t,\widehat{\boldsymbol{q}})+\mathcal{D}(\boldsymbol{q}_\Pi(t),\widehat{\boldsymbol{q}}), \tag{4.34}$$

$$\forall s,t\in\Pi:\ s<t, \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_\Pi(t))-\widetilde{\mathcal{E}}(s,\boldsymbol{q}_\Pi(s))+\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi;[s,t])\leq\int_s^t\partial_t\widetilde{\mathcal{E}}(\tau,\boldsymbol{q}_\Pi(\tau))\,\mathrm{d}\tau, \tag{4.35}$$

$$\forall t\in[0,T], \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_\Pi(t))+M+\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi;[0,t])\leq(\widetilde{\mathcal{E}}(0,\boldsymbol{q}^0)+M)e^{Lt}. \tag{4.36}$$

Proof. The claims (4.34) and (4.35) follow immediately from (4.27) and (4.28), respectively. We prove (4.36). Let $t \in [0,T]$ and let $i \in \{1,\dots,N\}$ be such that $t_{i-1} \le t < t_i$. In this case, we have

$$\boldsymbol{q}_\Pi(t) = \boldsymbol{q}^{i-1}, \qquad \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi;[0,t]) = \sum_{j=1}^{i-1} \mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j).$$

Thus, using (4.24) and (4.29), we compute

$$\begin{aligned}\widetilde{\mathcal{E}}(t,\boldsymbol{q}_\Pi(t)) + M + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_\Pi;[0,t]) &\le \left(\widetilde{\mathcal{E}}(t_{i-1},\boldsymbol{q}^{i-1}) + M\right) e^{L(t-t_{i-1})} + \sum_{j=1}^{i-1}\mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j) \\ &\le \left(\widetilde{\mathcal{E}}(t_{i-1},\boldsymbol{q}_{i-1}) + M + \sum_{j=1}^{i-1}\mathcal{D}(\boldsymbol{q}^{j-1},\boldsymbol{q}^j)\right) e^{L(t-t_{i-1})} \\ &\le \left(\widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + M\right) e^{Lt}.\end{aligned} \tag{4.37}$$

□

The main result of this section is the following.

Theorem 4.6 (Existence of energetic solutions under regularization). *Assume $p > 3$ and $\widetilde{\mathcal{Y}} \neq \emptyset$. Suppose that W is continuous and satisfies (3.7) and (3.8) and that the applied loads satisfy (4.1). Then, for every $\boldsymbol{q}^0 \in \widetilde{\mathcal{Q}}$ satisfying (4.26), there exists an energetic solution $\boldsymbol{q}\colon [0,T] \to \widetilde{\mathcal{Q}}$ of the regularized problem which fulfills the initial condition $\boldsymbol{q}(0) = \boldsymbol{q}^0$. Namely, the following* stability condition *and* energy balance *hold:*

$$\forall t \in [0,T],\ \forall \widehat{\boldsymbol{q}} \in \widehat{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)) \le \widetilde{\mathcal{E}}(t,\widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t),\widehat{\boldsymbol{q}}), \tag{4.38}$$

$$\forall t \in [0,T], \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q};[0,t]) = \widetilde{\mathcal{E}}(0,\boldsymbol{q}^0) + \int_0^t \partial_t\widetilde{\mathcal{E}}(\tau,\boldsymbol{q}(\tau))\,\mathrm{d}\tau. \tag{4.39}$$

In the proof, we will use the following version of the Helly Selection Principle given by [44, Theorem 5.1].

Lemma 4.7 (Helly Selection Principle). *Let Z be a Banach space and let $\mathcal{K} \subset Z$ be compact. Let $(\boldsymbol{z}_n) \subset BV([0,T];Z)$ be such that for every $n \in \mathbb{N}$ there holds*

$$\forall t \in [0,T],\ \ \boldsymbol{z}_n(t) \in \mathcal{K} \tag{4.40}$$

and

$$\mathrm{Var}(\boldsymbol{z}_n;[0,T]) \le C. \tag{4.41}$$

Then, there exist a subsequence $(\boldsymbol{z}_{n_k})$ and a map $\boldsymbol{z} \in BV([0,T];Z)$ such that there holds:

$$\forall t \in [0,T], \quad \boldsymbol{z}_{n_k}(t) \to \boldsymbol{z}(t) \ \text{in } Z. \tag{4.42}$$

The proof of Theorem 4.6 follows rigorously the well-established scheme introduced in [23]. Therefore, we simply show how to lead the argument back to the original scheme. For additional details we refer to [44, Theorem 5.2].

Proof of Theorem 4.6. Following [44, Theorem 5.2], we subdivide the proof into five steps.

Step 1 (A priori estimates). Let (Π_n) be a sequence of partitions of $[0,T]$ with $\Pi_n = (t_0^n,\dots,t_{N_n}^n)$ such that $|\Pi_n| \coloneqq \max\{t_i^n - t_{i-1}^n : i = 1,\dots,N_n\} \to 0$, as $n \to \infty$. For every $n \in \mathbb{N}$, by Proposition 4.4, the incremental minimization problem (4.25) determined by Π_n admits a solution and, by Proposition 4.5, the corresponding piecewise constant interpolant $\boldsymbol{q}_n \coloneqq \boldsymbol{q}_{\Pi_n}$ with $\boldsymbol{q}_n = (\boldsymbol{y}_n,\boldsymbol{m}_n)$ defined according to (4.33) satisfies the following:

$$\forall t \in \Pi_n,\ \forall \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) \le \widetilde{\mathcal{E}}(t,\widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_n(t),\widehat{\boldsymbol{q}}), \tag{4.43}$$

$$\forall s,t \in \Pi_n :\ s < t, \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) - \widetilde{\mathcal{E}}(s,\boldsymbol{q}_n(s)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[s,t]) \le \int_s^t \partial_t\widetilde{\mathcal{E}}(\tau,\boldsymbol{q}_n(\tau))\,\mathrm{d}\tau, \tag{4.44}$$

$$\forall t \in [0,T], \quad \widetilde{\mathcal{E}}(t, \boldsymbol{q}_n(t)) + M + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,t]) \leq (\widetilde{\mathcal{E}}(0,\boldsymbol{q}^0)+M)e^{Lt}. \tag{4.45}$$

In particular, from (4.45), we deduce that, for every $n \in \mathbb{N}$, there hold

$$\forall t \in [0,T], \quad \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t)) \leq C \tag{4.46}$$

and

$$\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n;[0,T]) \leq C \tag{4.47}$$

for some $C(\boldsymbol{q}^0,T,L,M)>0$.

Step 2 (Selection of subsequences). From (4.4) and (4.46), for every $n \in \mathbb{N}$ and $t \in [0,T]$ we have

$$||\nabla \boldsymbol{y}_n(t)||_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq C, \qquad ||\nabla \boldsymbol{m}_n(t)||_{L^2(\Omega^{\boldsymbol{y}_n(t)};\mathbb{R}^{3\times 3})} \leq C,$$
$$||\gamma(\det \nabla \boldsymbol{y}_n(t))||_{L^1(\Omega)} \leq C, \qquad ||D(\mathrm{cof}\, \nabla \boldsymbol{y}_n(t))||_{\mathcal{M}_b(\Omega;\mathbb{R}^{3\times 3\times 3})} \leq C.$$

This shows that all the maps of the sequence $(\boldsymbol{q}_n)$ take values in the set $\widetilde{\mathcal{K}} \subset \widetilde{\mathcal{Q}}$ defined as

$$\widetilde{\mathcal{K}} \coloneqq \Big\{ \widehat{\boldsymbol{q}} = (\widehat{\boldsymbol{y}},\widehat{\boldsymbol{m}}) \in \widetilde{\mathcal{Q}}: \quad ||\nabla \widehat{\boldsymbol{y}}||_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq C, \qquad ||\nabla \widehat{\boldsymbol{m}}||_{L^2(\Omega^{\widehat{\boldsymbol{y}}};\mathbb{R}^{3\times 3})} \leq C,$$
$$||\gamma(\det \nabla \widehat{\boldsymbol{y}})||_{L^1(\Omega)} \leq C, \quad ||D(\mathrm{cof}\,\nabla \widehat{\boldsymbol{y}})||_{\mathcal{M}_b(\Omega;\mathbb{R}^{3\times 3\times 3})} \leq C \Big\}.$$

Applying Proposition 3.3 and arguing as in the proof of Proposition 4.4, we prove the following:

$$\begin{gathered}\text{for every } (\widehat{\boldsymbol{q}}_n) \subset \widetilde{\mathcal{K}} \text{ with } \widehat{\boldsymbol{q}}_n = (\widehat{\boldsymbol{y}}_n, \widehat{\boldsymbol{m}}_n) \text{ there exist } (\widehat{\boldsymbol{q}}_{n_k}) \text{ and } \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}} \text{ with } \widehat{\boldsymbol{q}} = (\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) \\ \text{such that } \widehat{\boldsymbol{q}}_{n_k} \to \widehat{\boldsymbol{q}} \text{ in } \widetilde{\mathcal{Q}}, \quad \widehat{\boldsymbol{m}}_{n_k} \circ \widehat{\boldsymbol{y}}_{n_k} \to \widehat{\boldsymbol{m}} \circ \widehat{\boldsymbol{y}} \text{ in } L^a(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq a < \infty, \\ \mathrm{cof}\,\nabla \widehat{\boldsymbol{y}}_{n_k} \to \mathrm{cof}\,\nabla \widehat{\boldsymbol{y}} \text{ in } L^1(\Omega;\mathbb{R}^{3\times 3}) \text{ and } D(\mathrm{cof}\,\nabla \widehat{\boldsymbol{y}}_{n_k}) \overset{*}{\rightharpoonup} D(\mathrm{cof}\,\nabla \widehat{\boldsymbol{y}}) \text{ in } \mathcal{M}_b(\Omega;\mathbb{R}^{3\times 3\times 3}).\end{gathered} \tag{4.48}$$

From this, we deduce that the set

$$\mathcal{K} \coloneqq \Big\{ \mathcal{Z}(\widehat{\boldsymbol{q}}): \quad \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{K}} \Big\}$$

is compact with respect to the strong topology of $L^1(\Omega;\mathbb{R}^3)$. Indeed, let $(\widehat{\boldsymbol{z}}_n) \subset \mathcal{K}$ be defined by $\widehat{\boldsymbol{z}}_n = \mathcal{Z}(\widehat{\boldsymbol{q}}_n)$ with $\widehat{\boldsymbol{q}}_n \in \widetilde{\mathcal{K}}$ for every $n \in \mathbb{N}$. Then, by (4.48), we have $\mathrm{cof}\,\nabla \widehat{\boldsymbol{y}}_{n_k} \to \mathrm{cof}\,\nabla \widehat{\boldsymbol{y}}$ in $L^1(\Omega;\mathbb{R}^{3\times 3})$ and $\widehat{\boldsymbol{m}}_{n_k} \circ \widehat{\boldsymbol{y}}_{n_k} \to \widehat{\boldsymbol{m}} \circ \widehat{\boldsymbol{y}}$ in $L^1(\Omega;\mathbb{R}^3)$. Therefore, we infer that $\widehat{\boldsymbol{z}}_{n_k} \to \widehat{\boldsymbol{z}}$ in $L^1(\Omega;\mathbb{R}^3)$, where $\widehat{\boldsymbol{z}} = \mathcal{Z}(\widehat{\boldsymbol{q}})$.

Now, consider the sequence $(\boldsymbol{z}_n) \subset BV([0,T];L^1(\Omega;\mathbb{R}^3))$ with $\boldsymbol{z}_n(t) \coloneqq \mathcal{Z}(\boldsymbol{q}_n(t))$ for every $t \in [0,T]$. Setting $Z = L^1(\Omega;\mathbb{R}^3)$, the sequence $(\boldsymbol{z}_n)$ satisfies (4.40) by construction, as the the maps of the sequence $(\boldsymbol{q}_n)$ take values in $\widetilde{\mathcal{K}}$, while (4.41) holds in view of (4.47). Therefore, by Lemma 4.7, there exist a subsequence $(\boldsymbol{z}_{n_k})$ and a map $\boldsymbol{z} \in BV([0,T];L^1(\Omega;\mathbb{R}^3))$ such that (4.42) holds.

For every $n \in \mathbb{N}$, define $\vartheta_n \colon [0,T] \to \mathbb{R}$ by setting $\vartheta_n(t) \coloneqq \partial_t \widetilde{\mathcal{E}}(t,\boldsymbol{q}_n(t))$. Note that, by (4.6) and (4.46), the sequence (ϑ_n) is bounded in $L^\infty(0,T)$, hence, up to subsequences, $\vartheta_n \overset{*}{\rightharpoonup} \vartheta$ in $L^\infty(0,T)$ for some $\vartheta \in L^\infty(0,T)$. If we define $\bar{\vartheta} \colon [0,T] \to \mathbb{R}$ as $\bar{\vartheta}(t) \coloneqq \limsup_n \vartheta_n(t)$, then $\bar{\vartheta} \in L^\infty(0,T)$ and, by the Fatou Lemma, $\vartheta \leq \bar{\vartheta}$.

Finally, for every fixed $t \in [0,T]$, exploiting (4.48), we select a subsequence $(\boldsymbol{q}_{n_{k^t_\ell}}(t))$ (depending on t) such that

$$\boldsymbol{q}_{n_{k^t_\ell}}(t) \to \boldsymbol{q}(t) \text{ in } \widetilde{\mathcal{Q}}, \quad \boldsymbol{m}_{n_{k^t_\ell}} \circ \boldsymbol{y}_{n_{k^t_\ell}}(t) \to \boldsymbol{m} \circ \boldsymbol{y}(t) \text{ in } L^a(\Omega;\mathbb{R}^3) \text{ for every } 1 \leq a < \infty, \tag{4.49}$$

$$\mathrm{cof} \nabla \boldsymbol{y}_{n_{k^t_\ell}}(t) \to \mathrm{cof}\,\nabla \boldsymbol{y}(t) \text{ in } L^1(\Omega;\mathbb{R}^{3\times 3}), \quad D(\mathrm{cof} \nabla \boldsymbol{y}_{n_{k^t_\ell}}(t)) \overset{*}{\rightharpoonup} D(\mathrm{cof}\,\nabla \boldsymbol{y}(t)) \text{ in } \mathcal{M}_b(\Omega;\mathbb{R}^{3\times 3\times 3}) \tag{4.50}$$

for some $\boldsymbol{q}(t) \in \widetilde{\mathcal{Q}}$ with $\boldsymbol{q}(t) = (\boldsymbol{y}(t),\boldsymbol{m}(t))$ and $\vartheta_{n_{k^t_\ell}}(t) \to \bar{\vartheta}(t)$. Note that, from (4.49) and (4.50), we obtain $\boldsymbol{z}_{n_{k^t_\ell}}(t) = \mathcal{Z}(\boldsymbol{q}_{n_{k^t_\ell}}(t)) \to \mathcal{Z}(\boldsymbol{q}(t))$ in $L^1(\Omega;\mathbb{R}^3)$ which, combined with (4.42), yields $\boldsymbol{z}(t) = \mathcal{Z}(\boldsymbol{q}(t))$.

The candidate solution $\boldsymbol{q} \colon [0,T] \to \widetilde{\mathcal{Q}}$ is pointwise defined by this procedure.

Step 3 (Stability of the limiting function). We claim that $\boldsymbol{q}$ satisfies (4.38). Fix $t \in [0,T]$. Henceforth, for simplicity, we will replace the subscripts n_k and $n_{k^t_\ell}$ by k and k^t_ℓ, respectively. For every

$k \in \mathbb{N}$, set $\tau_k(t) := \max\{s \in \Pi_k : s \le t\}$ and note that $\tau_k(t) \to t$, since $|\Pi_k| \to 0$. Then, $\boldsymbol{q}_k(t) = \boldsymbol{q}_k(\tau_k(t))$ so that, by (4.27), we have

$$\forall \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(\tau_k(t), \boldsymbol{q}_k(t)) \le \widetilde{\mathcal{E}}(\tau_k(t), \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_k(t), \widehat{\boldsymbol{q}}). \tag{4.51}$$

Recall (4.49) and (4.50). Arguing as in the proof of Proposition 4.4 and exploiting the continuity of the applied loads in (4.1), we obtain

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) \le \liminf_{\ell} \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \boldsymbol{q}_{k^t_\ell}(t)). \tag{4.52}$$

Moreover, by the continuity of the applied loads in (4.1), there holds

$$\forall \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \widehat{\boldsymbol{q}}) \to \widetilde{\mathcal{E}}(t, \widehat{\boldsymbol{q}}), \tag{4.53}$$

while, as $\boldsymbol{z}_{k^t_\ell}(t) \to \boldsymbol{z}(t)$ in $L^1(\Omega;\mathbb{R}^3)$ and $\boldsymbol{z}(t) = \mathcal{Z}(\boldsymbol{q}(t))$, we have

$$\forall \widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}, \quad \mathcal{D}(\boldsymbol{q}_{k^t_\ell}(t), \widehat{\boldsymbol{q}}) = ||\mathcal{Z}(\boldsymbol{q}_{k^t_\ell}(t)) - \mathcal{Z}(\widehat{\boldsymbol{q}})||_{L^1(\Omega;\mathbb{R}^3)} \to ||\mathcal{Z}(\boldsymbol{q}(t)) - \mathcal{Z}(\widehat{\boldsymbol{q}})||_{L^1(\Omega;\mathbb{R}^3)} = \mathcal{D}(\boldsymbol{q}(t), \widehat{\boldsymbol{q}}). \tag{4.54}$$

Hence, combining (4.51)–(4.54), we deduce

$$\begin{aligned} \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) &\le \liminf_{\ell} \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \boldsymbol{q}_{k^t_\ell}(t)) \\ &\le \liminf_{\ell} \left\{ \widetilde{\mathcal{E}}(\tau_{k^t_\ell}(t), \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_{k^t_\ell}(t), \widehat{\boldsymbol{q}}) \right\} \\ &= \widetilde{\mathcal{E}}(t, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \widehat{\boldsymbol{q}}), \end{aligned}$$

for every $\widehat{\boldsymbol{q}} \in \widetilde{\mathcal{Q}}$, which gives (4.38) for t fixed. □

Step 4 (Upper energy estimate). We claim that $\boldsymbol{q}$ satisfies the upper energy estimate

$$\forall t \in [0,T], \quad \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) \le \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}(\tau))\, \mathrm{d}\tau. \tag{4.55}$$

Recall (4.46). For every $n \in \mathbb{N}$, using (4.24), we obtain

$$\forall s,t \in [0,T], \quad |\widetilde{\mathcal{E}}(t, \boldsymbol{q}_n(t)) - \widetilde{\mathcal{E}}(s, \boldsymbol{q}_n(s))| \le (C+M) \left| e^{L|t-s|} - 1 \right| =: \rho(t-s), \tag{4.56}$$

where $\rho(r) \to 0$, as $r \to 0$.

Fix $t \in [0,T]$, so that $\boldsymbol{q}_n(t) = \boldsymbol{q}_n(\tau_n(t))$ and $\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]) = \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)])$ for every $n \in \mathbb{N}$. Recall the definition of θ_n in Step 2. By (4.56) and (4.44), we have

$$\begin{aligned} \widetilde{\mathcal{E}}(t, \boldsymbol{q}_n(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]) &\le \widetilde{\mathcal{E}}(\tau_n(t), \boldsymbol{q}_n(\tau_n(t))) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0, \tau_n(t)]) + \rho(|\Pi_n|) \\ &\le \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^{\tau_n(t)} \vartheta_n(\tau)\, \mathrm{d}\tau + \rho(|\Pi_n|), \end{aligned} \tag{4.57}$$

for every $n \in \mathbb{N}$. Also, by the lower semicontinuity of the total variation, we have

$$\mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) = \mathrm{Var}(\boldsymbol{z}; [0,t]) \le \liminf_n \mathrm{Var}(\boldsymbol{z}_n; [0,t]) = \liminf_n \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_n; [0,t]), \tag{4.58}$$

as (4.42) holds and $\boldsymbol{z}(s) = \mathcal{Z}(\boldsymbol{q}(s))$ for every $s \in [0,T]$. Then, from (4.52), (4.57) and (4.58), we deduce

$$\begin{aligned} \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) &\le \liminf_{\ell} \left\{ \widetilde{\mathcal{E}}(t, \boldsymbol{q}_{k^t_\ell}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}_{k^t_\ell}; [0,t]) \right\} \\ &\le \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \liminf_{\ell} \left\{ \int_0^{\tau_{k^t_\ell}(t)} \vartheta_{k^t_\ell}(\tau)\, \mathrm{d}\tau + \rho(|\Pi_{k^t_\ell}|) \right\} \\ &= \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^t \vartheta(\tau)\, \mathrm{d}\tau \le \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^t \bar{\vartheta}(\tau)\, \mathrm{d}\tau, \end{aligned} \tag{4.59}$$

where, in the last line, we used that $\vartheta_k \overset{*}{\rightharpoonup} \vartheta$ in $L^\infty(0,T)$, $\vartheta \le \bar{\vartheta}$ almost everywhere in $(0,T)$ and $\rho(|\Pi_k|) \to 0$.

We claim that $\bar{\vartheta}(s) = \partial_t \widetilde{\mathcal{E}}(s, \boldsymbol{q}(s))$ for almost every $s \in (0,T)$. Fix $t \in (0,T)$. Testing (4.38) with $\boldsymbol{q}_{k^t_\ell}(t)$, we have

$$-\widetilde{\mathcal{E}}(t, \boldsymbol{q}_{k^t_\ell}(t)) \leq -\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathcal{D}(\boldsymbol{q}(t), \boldsymbol{q}_{k^t_\ell}(t))$$

so that, using (4.54), we compute

$$\begin{aligned}\limsup_\ell \widetilde{\mathcal{E}}(t, \boldsymbol{q}_{k^t_\ell}(t)) &= -\liminf_\ell \left(-\widetilde{\mathcal{E}}(t, \boldsymbol{q}_{k^t_\ell}(t))\right) \\ &\leq -\liminf_\ell \left(-\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathcal{D}(\boldsymbol{q}(t), \boldsymbol{q}_{k^t_\ell}(t))\right) \leq \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)).\end{aligned}$$

Given (4.52), we conclude that $\widetilde{\mathcal{E}}(t, \boldsymbol{q}_{k^t_\ell}(t)) \to \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t))$. Recalling (4.49), by [44, Proposition 5.6], we have $\vartheta_{k^t_\ell}(t) = \partial_t \widetilde{\mathcal{E}}(t, \boldsymbol{q}_{k^t_\ell}(t)) \to \partial_t \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t))$ and, as $\vartheta_{k^t_\ell}(t) \to \bar{\vartheta}(t)$, we deduce $\bar{\vartheta}(t) = \partial_t \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t))$. Therefore, (4.59) gives (4.55) for fixed t.

Step 5 (Lower energy estimate). Finally, we show that $\boldsymbol{q}$ satisfies

$$\forall t \in [0,T], \quad \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [0,t]) \geq \widetilde{\mathcal{E}}(0, \boldsymbol{q}^0) + \int_0^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}(\tau))\, \mathrm{d}\tau, \tag{4.60}$$

which, combined with (4.55), proves (4.39).

Note that, by (4.52), we have $\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) \leq C$ for every $t \in [0,T]$. Moreover, the function is $t \mapsto \partial_t \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t))$ belongs to $L^\infty(0,T)$, as it coincides almost everywhere with $\bar{\vartheta}$. Hence, by [44, Proposition 5.7], for every $s, t \in [0,T]$ with $s < t$ we have

$$\widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Var}_{\mathcal{D}}(\boldsymbol{q}; [s,t]) \geq \widetilde{\mathcal{E}}(s, \boldsymbol{q}(s)) + \int_s^t \partial_t \widetilde{\mathcal{E}}(\tau, \boldsymbol{q}(\tau))\, \mathrm{d}\tau,$$

which in turn yields (4.60).

Remark 4.8 (Regularity of the applied loads). The regularity assumptions on the applied loads in (4.1) can be relaxed. Indeed, following the scheme in [45, Theorem 2.1.6], the existence of energetic solutions as in Theorem 4.6 can still be proved if we just assume

$$\boldsymbol{f} \in W^{1,1}(0,T; L^{p'}(\Omega; \mathbb{R}^3)), \quad \boldsymbol{g} \in W^{1,1}(0,T; L^{p'}(\Sigma; \mathbb{R}^3)), \quad \boldsymbol{h} \in W^{1,1}(0,T; L^2(\mathbb{R}^3; \mathbb{R}^3)).$$

Acknowledgements

The authors acknowledge support from the Austrian Science Fund (FWF) projects F65, V 662, Y1292, from the FWF-GAČR project I 4052/19-29646L, and from the OeAD-WTZ project CZ04/2019 (MŠMTČR 8J19AT013).

References

[1] A. Bach, M. Cicalese, L. Kreutz, G. Orlando. The antiferromagnetic XY model on the triangular lattice: chirality transitions at the surface scaling. *Preprint ArXiv 2004.01416*

[2] J. M. Ball. Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), 315–328.

[3] J. M. Ball, J. C. Currie, P. J. Olver. Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Funct. Anal.* **41** (1981), 135–174.

[4] M. Barchiesi, A. De Simone. Frank energy for nematic elastomers: a nonlinear model. *ESAIM Control Optim. Calc. Var.* **2** (2015), 372–377.

[5] M. Barchiesi, D. Henao, C. Mora-Corral. Local Invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity. *Arch. Ration. Mech. Anal.* **224** (2017), 743–816.

[6] B. Benešová, M. Kružík, A. Schlömerkemper. A note on locking materials and gradient polyconvexity. *Math. Mod. Meth. Appl. Sci.* **28** (2018), 2367–2401.

[7] M. Bresciani. Linearized von Kármán theory for incompressible magnetoelastic plates. *Preprint ArXiv 2007.14122*

[8] W. F. Brown. *Magnetostatic principles in ferromagnetism.* North-Holland Publishing Company, New York, 1962.

[9] W. F. Brown. *Magnetoelastic Interactions.* Springer, Berlin, 1966.

[10] P. G. Ciarlet, J. Nečas. Injectivity and self-contact in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **97** (1987), 173–188.

[11] E. Davoli, G. Di Fratta. Homogenization of chiral magnetic materials. - A mathematical evidence of Dzyaloshinskii's predictions on helical structures. *J. Nonlinear Sci.* **30** (2020), 1229–1262.

[12] E. Davoli, G. Di Fratta, D. Praetorius, M. Ruggeri. Micromagnetics of thin films in the presence of Dzyaloshinskii-Moriya interaction. *Preprint ArXiv 2010.15541*

[13] E. Davoli, M. Kružík, P. Piovano, U. Stefanelli. Magnetoelastic thin films at large strains. *Cont. Mech. Thermodyn.* **33** (2021), 327–341.

[14] K. Deimling. *Nonlinear Functional Analysis.* Springer-Verlag, Berlin, 1985.

[15] A. DeSimone. Energy minimizers for large ferromagnetic bodies. *Arch. Ration. Mech. Anal.* **125** (1993), 99–143.

[16] A. DeSimone, G. Dolzmann. Existence of minimizers for a variational problem in two-dimensional nonlinear magnetoelasticity. *Arch. Ration. Mech. Anal.* **144** (1998), 107–120.

[17] A. DeSimone, R.D. James. A constrained theory of magnetoelasticity. *J. Mech. Phys. Solids*, **50** (2002), 283–320.

[18] G. Di Fratta, M. Innerberger, D. Praetorius. Weak–strong uniqueness for the Landau–Lifshitz–Gilbert equation in micromagnetics. *Nonlinear Analysis: Real World Applications*, **55** (2020), 103122.

[19] G. Di Fratta, C. B. Muratov, F. N. Rybakov, V. V. Slastikov. Variational principles of micromagnetics revisited. *SIAM J. Math. Anal.* **52** (2020), 3580—3599.

[20] I. Dzyaloshinsky. A thermodynamic theory of "weak" ferromagnetism of antiferromagnetics. *J. Phys. Chem. Solids*, **4** (1958), 241–255.

[21] A. Fert, N. Reyren, V. Cros. Magnetic skyrmions: Advances in physics and potential applications. *Nature Reviews Materials*, **2** (2017).

[22] I. Fonseca, G. Leoni. *Modern Methods in the Calculus of Variations: L^p spaces.* Springer, New York, 2007.

[23] G. Francfort, A. Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. Reine Angew. Math.* **595** (2004), 55–91.

[24] M. Giaquinta, G. Modica, J. Souček. *Cartesian Currents in the Calculus of Variations. I. Cartesian Currents.* Springer-Verlag, Berlin, 1998.

[25] D. Grandi, M. Kružík, E. Mainini, U. Stefanelli. A phase-field approach to eulerian interfacial energies. *Arch. Ration. Mech. Anal.* **234** (2019), 351–373.

[26] P. Hajłasz. Change of variables formula under minimal assumptions. *Colloq. Math.* **64** no. 1 (1993).

[27] D. Henao, C. Mora-Corral. Lusin's condition and the distributional determinant for deformations with finite energy. *Adv. Calc. Var.* **5** (2012), 355–409.

[28] D. Henao, C. Mora-Corral. Regularity of inverses of Sobolev deformations with finite surface energy. *J. Funct. Anal.* **268** (2015), 2356–2378.

[29] D. Henao, C. Mora-Corral, M. Oliva. Global invertibility of Sobolev maps. *Adv. Calc. Var.*, to appear.

[30] G. Hrkac, C.-M. Pfeiler, D. Praetorius, M. Ruggeri, A. Segatti, B. Stiftner. Convergent tangent plane integrators for the simulation of chiral magnetic skyrmion dynamics. *Adv. Comp. Math.* **45** (2019), 1329–1368.

[31] Y. Hu, B. Wang. Unified theory of magnetoelastic effects in B20 chiral magnets. *New J. Phys.* **19** (2017), 123002.

[32] R.D. James, D. Kinderlehrer. Frustration in ferromagnetic materials. *Continuum Mech. Thermodyn.* **2** (1990), 215–239.

[33] S. Krömer. Global invertibility for orientation-preserving Sobolev maps via invertibility on or near the boundary. *Arch. Ration. Mech. Anal.* **238** (2020), 1113–1155.

[34] M. Kružík, T. Roubíček. *Mathematical Methods in Continuum Mechanics of Solids.* Springer, Cham, 2019.

[35] M. Kružík, U. Stefanelli, C. Zanini. Quasistatic evolution of magnetoelastic plates via dimension reduction. *Discrete Contin. Dyn. Syst.* **35** (2015), 2615–2623.

[36] M. Kružík, U. Stefanelli, J. Zeman. Existence results for incompressible magnetoelasticity. *Discrete Contin. Dyn. Syst.* **35** (2015), 5999–6013.

[37] X. Li, C. Melcher. Stability of axisymmetric chiral skyrmions. *J. Funct. Anal.* **275** (2018), 2817–2844.

[38] J. Liakhova. A theory of magnetostrictive thin films with applications. Ph.D. Thesis, University of Minnesota, 1999.

[39] J. Liakova, M. Luskin, T. Zhang. Computational modeling of ferromagnetic shape memory thin films. *Ferroelectrics.* **342** (2006), 7382.

[40] R. G. Lund, C. B. Muratov. One-dimensional domain walls in thin ferromagnetic films with fourfold anisotropy. *Nonlinearity*, **29** (2016), 1716–1734.

[41] M. Luskin, T. Zhang. Numerical analysis of a model for ferromagnetic shape memory thin films. *Comput. Methods Appl. Mech. Engrg.* **196** (2007), 37–40.

[42] M. Marcus, V. J. Mizel. Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. *Bull. Am. Math. Soc.* **79** (1973), 790–795.

[43] C. Melcher. Chiral skyrmions in the plane. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **470** (2014), p. 20140394.

[44] A. Mielke. Evolution of rate-independent systems. In C. M. Dafermos, E. Feireisl. *Handbook of Differential Equations. Volume II. Evolutionary Equations.* Elsevier, Amsterdam, 2005.

[45] A. Mielke, T. Roubíček. *Rate-independent Systems. Theory and Application.* Springer, New York, 2015.

[46] T. Moriya. Anisotropic superexchange interaction and weak ferromagnetism. *Physical Review* **120** (1960), 91.

[47] S. Müller, T. Qi, B. S. Yan. On a new class of elastic deformations not allowing for cavitation. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **11** (1994) 217–243.

[48] S. Müller, S. J. Spector. An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Ration. Mech. Anal.*, **131** (1995), 1–66.

[49] C. B. Muratov, V. V. Slastikov. Domain structure of ultrathin ferromagnetic elements in the presence of Dzyaloshinskii-Moriya interaction. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **473** (2017), p. 20160666.
[50] T. Roubíček, G. Tomassetti. A thermodynamically consistent model of magneto-elastic materials under diffusion at large strains and its analysis. *Z. Angew. Math. Phys.* **69** (2018), Article number: 55.
[51] P. Rybka, M. Luskin. Existence of energy minimizers for magnetostrictive materials. *SIAM J. Math. Anal.* **36** (2005), 2004–2019.
[52] T. Qi. Almost-everywhere injectivity in nonlinear elasticity. *Proc. Roy. Soc. Edinburgh Sect. A*, **109** (1988), 79–95.
[53] J. T. Schwartz. *Nonlinear Functional Analysis.* Gordon and Breach, New York, 1969.
[54] V. Šverák. Regularity properties of deformation with finite energy. *Arch. Ration. Mech. Anal.* **100** (1988), 105–127.
[55] R.A. Toupin. Elastic materials with couple stresses. *Arch. Ration. Mech. Anal.* **11** (1962), 385–414.
[56] R.A. Toupin. Theories of elasticity with couple stress. *Arch. Ration. Mech. Anal.* **17** (1964), 85–112.

(M. Bresciani) Institute of Analysis and Scientific Computing, TU Wien, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria. Email: marco.bresciani@tuwien.ac.at.

(E. Davoli) Institute of Analysis and Scientific Computing, TU Wien, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria. Email: elisa.davoli@tuwien.ac.at.

(M. Kružík) Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou Věží 4, 182 00 Prague, Czechia. Email: kruzik@utia.cas.cz.