HARMONIC DIPOLES AND THE RELAXATION OF THE NEO-HOOKEAN ENERGY IN 3D ELASTICITY

MARCO BARCHIESI, DUVAN HENAO, CARLOS MORA-CORRAL, AND RÉMY RODIAC

Abstract. We consider the problem of minimizing the neo-Hookean energy in $3D$. The difficulty of this problem is that the space of maps without cavitation is not compact, as shown by Conti & De Lellis with a pathological example involving a dipole. In order to rule out this behaviour we consider the relaxation of the neo-Hookean energy in the space of axisymmetric maps without cavitation. We propose a minimization space and a new explicit energy penalizing the creation of dipoles. This new energy, which is a lower bound of the relaxation of the original energy, bears strong similarities with the relaxed energy of Bethuel-Brezis-Hélein in the context of harmonic maps into the sphere.

1. Introduction

1.1. **A regularity problem for the well-posedness of the neo-Hookean model.** We consider the problem of the existence of minimizers for the neo-Hookean energy, i.e.,

$$E(\boldsymbol{u}) = \int_\Omega \left[|D\boldsymbol{u}|^2 + H(\det D\boldsymbol{u}) \right] \mathrm{d}\boldsymbol{x} \tag{1.1}$$

where $H : (0, \infty) \to [0, \infty)$ is a convex function such that

$$\lim_{t\to\infty} \frac{H(t)}{t} = \lim_{s\to 0} H(s) = \infty, \tag{1.2}$$

$\Omega \subset \mathbb{R}^3$ represents the reference configuration of an elastic body, and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is the deformation map. The neo-Hookean energy is widely used in physics, engineering and materials science and it can be derived from first principles [50, 52] assuming that the gradient of the deformation remains bounded. Since interpenetration of matter is physically unrealistic, minimizers are sought in a suitable subclass of

$$\mathcal{A} := \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3) : E(\boldsymbol{u}) < \infty,\ \boldsymbol{u} \text{ is injective a.e.}\}.$$

In his celebrated existence theory, Ball [2] was able to apply the direct method of the calculus of variations to general polyconvex energies

$$\int_\Omega W(\boldsymbol{x}, D\boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}, \tag{1.3}$$

where $W : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{\infty\}$ is the elastic stored-energy function of the material. His approach is based on the identity

$$\det D\boldsymbol{u} = \operatorname{Det} D\boldsymbol{u}, \quad \langle \operatorname{Det} D\boldsymbol{u}, \varphi\rangle := -\frac{1}{3}\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \big((\operatorname{cof} D\boldsymbol{u}) D\varphi\big)\, \mathrm{d}\boldsymbol{x}, \quad \varphi \in C_c^1(\Omega),$$

whereby the Jacobian determinant can be written as a distributional divergence, and an analogous identity for cof $D\boldsymbol{u}$, the matrix of 2×2 cofactors of the deformation gradient. The identities

Date: June 6, 2023.

are obtained from a coercivity assumption on the stored-energy function, the sharpest version of it (due to Müller, Tang & Yan [45]) being that

$$W(\boldsymbol{x}, \boldsymbol{F}) \geq c_1|\boldsymbol{F}|^2 + c_2|\operatorname{cof} \boldsymbol{F}|^{3/2} + H(\det \boldsymbol{F}), \qquad (\boldsymbol{x}, \boldsymbol{F}) \in \Omega \times \mathbb{R}^{3\times 3}.$$

However, this coercivity excludes the neo-Hookean materials. In fact, for neo-Hookean materials the hypothesis of finite energy alone is insufficient to ensure that $\det D\boldsymbol{u} = \operatorname{Det} D\boldsymbol{u}$, as shown in the models for cavitation [3, 49, 44, 48]. Because of that, the neo-Hookean energy is not H^1-quasiconvex, which is necessary for (1.3) to be H^1-weakly lower semicontinuous in $\mathcal{A}$, as proved by Ball & Murat [4].

In order to overcome the lack of H^1-weakly lower semicontinuity of the neo-Hookean energy in $\mathcal{A}$ due to cavitation, one may look for minimizers in the smaller class $\mathcal{A}^r$ of maps in $\mathcal{A}$ for which the divergence identities

$$\operatorname{Div}\big((\operatorname{adj} D\boldsymbol{u})\boldsymbol{g}\circ\boldsymbol{u}\big) = (\operatorname{div}\boldsymbol{g})\circ\boldsymbol{u}\,\det D\boldsymbol{u} \quad \forall \boldsymbol{g}\in C^1_c(\mathbb{R}^3,\mathbb{R}^3) \tag{1.4}$$

(of which $\det D\boldsymbol{u} = \operatorname{Det} D\boldsymbol{u}$ is a particular case) are satisfied, see e.g. [22], [42], [27]. Unluckily, one has then to face a problem of lack of compactness: Conti & De Lellis [11] constructed a sequence of deformations satisfying the divergence identities (1.4), weakly converging in H^1 but such that the limit does not satisfy (1.4). In this paper we try to overcome this obstruction.

Since we want to rule out the formation of anomalies at the boundary we assume that $\Omega \Subset \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a smooth bounded domain of $\mathbb{R}^3$, and require the deformations $\boldsymbol{u}$ to coincide with a bounded C^1 orientation-preserving diffeomorphism $\boldsymbol{b} : \widetilde{\Omega} \to \mathbb{R}^3$ not only on $\partial\Omega$ but on the whole of $\widetilde{\Omega}\setminus\Omega$, and to be injective a.e. on the whole of $\widetilde{\Omega}$. This setting was used before in elasticity [49, 48, 32], and can nevertheless be avoided with the techniques of [31].

Set

$$\Omega_{\boldsymbol{b}} := \boldsymbol{b}(\Omega), \qquad \widetilde{\Omega}_{\boldsymbol{b}} := \boldsymbol{b}(\widetilde{\Omega}).$$

Since the example of Conti & De Lellis is axisymmetric, we assume that Ω, $\widetilde{\Omega}$ and $\boldsymbol{b}$ are axisymmetric (see the definition (2.5) in Section 2.3). Define

$$\mathcal{A}_s := \{\boldsymbol{u}\in H^1(\widetilde{\Omega},\mathbb{R}^3) : \boldsymbol{u} \text{ is injective a.e. and axisymmetric,} \\ \det D\boldsymbol{u} > 0 \text{ a.e., } \boldsymbol{u} = \boldsymbol{b} \text{ in } \widetilde{\Omega}\setminus\Omega, \text{ and } E(\boldsymbol{u}) \leq E(\boldsymbol{b})\}. \tag{1.5}$$

Interestingly, in the axisymmetric setting we can prove that E is weakly lower semicontinuous and then that it has a minimum in $\mathcal{A}_s$ (see Proposition 3.1). However, the weak limit in the Conti-De Lellis example belongs to $\mathcal{A}_s$. That example exhibits a *dipole singularity*, i.e., a cavity opened at a point is filled by material coming from a small neighbourhood of another point. Such a flagrant interpenetration of matter can hardly be accepted as physical. Because of that, in building an existence theory for the neo-Hookean energy we would like to prove more regularity on minimizers by showing their existence in the class

$$\mathcal{A}^r_s := \{\boldsymbol{u}\in\mathcal{A}_s : \text{the divergence identities (1.4) are satisfied}\}. \tag{1.6}$$

In order to minimize E in $\mathcal{A}^r_s$ we employ a relaxation process. The reader can think of the minimization of a functional in $W^{1,1}$. Since that space is not weakly compact, one sets up the problem in the larger space BV of functions of bounded variation, and relaxes the functional by adding a term that takes into account the singular part of the distributional gradient. Here we propose something similar, but the singular part appears on the inverse of the deformation. More precisely, we set up the problem in the space

$$\mathcal{B} := \{\boldsymbol{u}\in\mathcal{A}_s : \ \widetilde{\Omega}_{\boldsymbol{b}} = \operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega}) \text{ a.e. and } \boldsymbol{u}^{-1} = (u_1^{-1}, u_2^{-1}, u_3^{-1}) \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^2)\times BV(\widetilde{\Omega}_{\boldsymbol{b}})\}, \tag{1.7}$$

where the geometric image $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \widetilde{\Omega})$ is that of Definition 2.5. The space $\mathcal{B}$ contains the weak H^1 closure of $\mathcal{A}_s^r$ (see Theorem 5.1). The fact that in $\mathcal{B}$ the geometric image coincides with $\widetilde{\Omega}_{\boldsymbol{b}}$ means that any cavitation produced by a map in this class must be filled with the image of some other part of the body (as happens in the example of Conti & De Lellis).

In $\mathcal{B}$ we provide a lower bound for the relaxation of E:

$$F(\boldsymbol{u}) := \int_\Omega \left[|D\boldsymbol{u}|^2 + H(\det D\boldsymbol{u})\right] \mathrm{d}\boldsymbol{x} + 2\left|D^s u_3^{-1}\right|(\tilde{\Omega}_{\boldsymbol{b}}),$$

where $|D^s u_3^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}})$ denotes the total variation of the singular part of the distributional gradient of u_3^{-1}. We believe that this lower bound is sharp, i.e., that F coincides with the relaxation of E. Indeed, in the companion paper [6] we improve the construction of Conti & De Lellis, by showing that the relaxation of E on that limit map (with a dipole) coincides with F. Proving the sharpness is important in order to get, eventually, a negative result: if the minimizers of the relaxed energy do not belong to $\mathcal{A}_s^r$, then E has no minimizers in $\mathcal{A}_s^r$. In any case, the energy and the space we propose can serve to the purpose of providing a positive result, i.e., existence of minimizers for E.

Theorem 1.1. *The energy F has a minimizer in $\mathcal{B}$. Moreover, if it belongs to $\mathcal{A}_s^r$, then it is also a minimizer of the original neo-Hookean energy E of* (1.1).

The new term $2|D^s u_3^{-1}|(\tilde{\Omega}_{\boldsymbol{b}})$ is the main contribution of this work. The strategy we propose to answer the question of existence of minimizers of the neo-Hookean energy in a class of regular maps is to obtain Sobolev regularity for the inverse $\boldsymbol{u}^{-1}$ of a minimizer $\boldsymbol{u}$ of F in $\mathcal{B}$. This is left for future investigations. We remark that the advantage of our results is that both $\mathcal{B}$ and F are explicit.

1.2. **The singular energy.** We explain here how the singular term $|D^s u_3^{-1}|(\tilde{\Omega}_{\boldsymbol{b}})$ appears in Theorem 1.1 and why we believe it is the adequate term to add to minimize the neo-Hookean energy, at least in the axisymmetric setting. Indeed, the only way in which the divergence identities do not pass to the limit is when the cofactors are not equiintegrable. In this, the behaviour of the sequence in the example of Conti & De Lellis is generic: a 'stack' of surfaces with smaller and smaller diameters, orthogonal to the axis of symmetry, are stretched without control. Due to the $2D$ nature of the axisymmetric setting (see Lemma 6.1), we have

$$\int_{C_\delta} |D\boldsymbol{u}_j|^2 \,\mathrm{d}\boldsymbol{x} \geq 2\int_{C_\delta} |(\operatorname{cof} D\boldsymbol{u}_j)\boldsymbol{e}_3| \,\mathrm{d}\boldsymbol{x} = 2\int_{\boldsymbol{u}_j(C_\delta)} \left|D(u_3^{-1})_j\right| \mathrm{d}\boldsymbol{y}$$

where $\boldsymbol{e}_3$ is the direction of the symmetry axis and C_δ is a small δ-cylinder around it. The sets $\boldsymbol{u}_j(C_\delta)$ collapse to a set with zero volume (thanks to the equiintegrability of the determinants); in the example by Conti & De Lellis, they collapse to a sphere, which is exactly the jump set of the vertical component of the inverse u_3^{-1} for the limit map $\boldsymbol{u}$. However,

$$\int_\Omega |D\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = \lim_{\delta \searrow 0} \int_{\Omega\setminus C_\delta} |D\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \leq \liminf_{\delta\searrow 0} \liminf_{j\to\infty} \int_{\Omega\setminus C_\delta} |D\boldsymbol{u}_j|^2 \,\mathrm{d}\boldsymbol{x},$$

so the original formula for the neo-Hookean energy completely misses out the concentration of the Dirichlet energy if applied directly to the singular map $\boldsymbol{u}$.

Note that in the reference configuration all evidence of the abnormal activity of the regular sequence is lost and hidden in the one-dimensional symmetry axis; in contrast, in the deformed configuration large (two-dimensional or fractal) structures can remain, which make the singular map remember the energy spent in their formation. All in all, the singular term $2|D^s u_3^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}})$ is *not artificial*, it emerges naturally from the Dirichlet energy. At the very least, we prove, cf. equation (6.11), that it is a lower bound of the abstract relaxed energy functional.

We want to make a last comment about recovering the regularity in order to obtain minimizers of the neo-Hookean energy in the original space $\mathcal{A}_s^r$. In this task we will be confronted not with singularities that are physically relevant but with pathological deformations that we would prefer to exclude. Indeed, in the companion paper [6], under the additional assumption $\mathcal{E}(\boldsymbol{u}) < \infty$ for the surface energy functional defined in Section 2.2, we prove that for any weak limit of regular maps there exist a countable family of dipoles $\boldsymbol{\xi}_i, \boldsymbol{\xi}_i'$ lying on the axis of symmetry and a countable family of sets of finite perimeter whose reduced boundaries Γ_i satisfy

$$|D^s u_3^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}) = \sum_{i\in\mathbb{N}} \left|\boldsymbol{\xi}_i - \boldsymbol{\xi}_i'\right| \mathcal{H}^2(\Gamma_i). \tag{1.8}$$

The discussion is therefore about how to reach a contradiction from the assumption that a minimizing sequence of regular maps ends up forming those dipoles, and a successful argument could probably use that if that were the case then the regular maps in the sequence would produce an energy concentration of (at least)

$$2|D^s u_3^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}) = 2 \cdot (\text{area of the bubble}) \cdot (\text{length of the dipole}), \tag{1.9}$$

which is presumably more than what a minimizer can afford.

1.3. **Connection with harmonic map theory.** Bethuel, Brezis and Coron [7] (see also [23]) also derived a relaxed energy to treat a problem of lack of compactness in the theory of harmonic maps from a $3D$ domain with values into $\mathbb{S}^2$. The expression (1.9) shows that the energy we obtain and the relaxed energy in the context of harmonic maps are very similar. In particular, the right-hand side of (1.8) is the analogue of the 'length of minimal connection', introduced in [8], connecting singularities of harmonic maps. Besides, the supplementary term in the harmonic map relaxed energy can be expressed in terms of this length of minimal connection in the case where the map has a finite number of singularities. This reveals a strong connection between the problem of minimizing the neo-Hookean energy and finding a smooth minimizing harmonic map from $\mathbb{B}^3$ into $\mathbb{S}^2$ with a smooth boundary data with zero degree. This problem was raised by Hardt and Lin in [25] and is still open. For the study of partial regularity and prescribed singularities problem for harmonic maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ in the axisymmetric setting we refer to [26] and [39].

1.4. **Recent related results.** During the process of revision of this paper, we became aware of the recent works [16, 17]. Both exhibit coercivity conditions on W so that condition INV (see Definition 4.5) is preserved under the weak limit in $W^{1,N-1}$: through a quick enough growth to infinity when the determinant goes to zero in [16], and through the equiintegrability of the cofactors in [17]. In fact, that the equiintegrability of the cofactors implies the stability of condition INV was shown in [29]. In addition, in [16] they construct an example of a map in the Conti–De Lellis style.

1.5. **Outline of the paper.** The paper is organized as follows. In Section 2 we introduce some notation and definitions that will be used in the sequel. More precisely, we define the geometric image and the surface energy of a map. The latter notion quantifies the failure of the divergence identities (1.4). We then make precise the axisymmetry and specify the boundary condition. We also introduce the notion of family of 'good open sets' and show how to relate the properties of a $3D$ axisymmetric map to the properties of its associated $2D$ map.

Section 3 is devoted to the proof of existence of minimizers of E in the class $\mathcal{A}_s$. These minimizers could, in principle, be irregular, forming pathologies similar to that of the example of Conti–De Lellis. This leads to the question of whether conditions exist under which such behaviour can be ruled out. With this motivation in mind, our main focus in this paper is to derive an explicit energy playing the role of a relaxed energy for the neo-Hookean problem, in which the cost of creating pathological singularities can be made visible.

In order to do that, in Section 4 we describe fine properties of maps in $\mathcal{A}_s$. We start with regularity properties of general axisymmetric maps, then we define the topological degree and topological image of maps. We will need both definitions of the classical degree for continuous functions and of the Brezis–Nirenberg degree for Sobolev maps. A particular role is played by the topological image of the segment formed by the intersection of the domain Ω and the symmetry axis. Then we focus on the invertibility property of maps in $\mathcal{A}_s$. The main result of that section states that the first two components of the inverse of a map in $\mathcal{A}_s$ are Sobolev.

In Section 5, we focus first on regularity properties of weak limits of maps in $\mathcal{A}_s^r$. It is of crucial importance for the rest of the paper that their geometric image equals (up to a null $\mathcal{L}^3$-set) the entire target domain, and that their inverses are in $BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$, with the first two components in $W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}})$. These results rest on the preliminary analysis done in Section 4.

Finally, in Section 6, we give a lower semicontinuity result for our candidate relaxed energy, hence proving a lower bound on the actual relaxed energy. We also obtain various existence results thanks to the previous analysis and give a proof of Theorem 1.1.

2. Notation and preliminaries

2.1. **Geometric image and area formula.** In this section Ω is a bounded open set of $\mathbb{R}^N$. We use the following notation for the density of a measurable set $A \subset \mathbb{R}^N$ at $\boldsymbol{x} \in \mathbb{R}^N$:

$$D(A, \boldsymbol{x}) = \lim_{r \to 0} \frac{|B(\boldsymbol{x}, r) \cap A|}{|B(\boldsymbol{x}, r)|}.$$

Here we use $|\cdot|$ for the Lebesgue measure in $\mathbb{R}^N$. An alternative notation is $\mathcal{L}^N$. The Hausdorff measure of dimension d is denoted by $\mathcal{H}^d$. The abbreviation *a.e.* for *almost everywhere* or *almost every* will be intensively used. It refers to the Lebegue measure, unless otherwise stated. Given two sets A, B of $\mathbb{R}^N$, we write $A \subset B$ a.e. if $\mathcal{L}^N(A \setminus B) = 0$, while $A = B$ a.e. or $A \overset{\text{a.e.}}{=} B$ a.e. both mean $A \subset B$ a.e. and $B \subset A$ a.e. An analogous meaning is given to the expression $\mathcal{H}^d$-a.e.

The definition of approximate differentiability can be found in many places (see, e.g., [19, Sect. 3.1.2.], [44, Def. 2.3] or [29, Sect. 2.3]).

We recall the area formula of Federer ([44, Prop. 2.6] and [19, Thm. 3.2.5 and Thm. 3.2.3]). We will use the notation $\mathcal{N}(\boldsymbol{u}, A, \boldsymbol{y})$ for the number of preimages of a point $\boldsymbol{y}$ in the set A under $\boldsymbol{u}$.

Proposition 2.1. *Let $\boldsymbol{u} \in W^{1,1}(\Omega, \mathbb{R}^N)$, and denote the set of approximate differentiability points of $\boldsymbol{u}$ by Ω_d. Then, for any measurable set $A \subset \Omega$ and any measurable function $\varphi : \mathbb{R}^N \to \mathbb{R}$,*

$$\int_A (\varphi \circ \boldsymbol{u})\, |\det D\boldsymbol{u}|\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N} \varphi(\boldsymbol{y})\, \mathcal{N}(\boldsymbol{u}, \Omega_d \cap A, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

whenever either integral exists. Moreover, if a map $\psi : A \to \mathbb{R}$ is measurable and $\bar{\psi} : \boldsymbol{u}(\Omega_d \cap A) \to \mathbb{R}$ is given by

$$\bar{\psi}(\boldsymbol{y}) := \sum_{\boldsymbol{x} \in \Omega_d \cap A,\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}} \psi(\boldsymbol{x})$$

then $\bar{\psi}$ is measurable and

$$\int_A \psi(\varphi \circ \boldsymbol{u})\, |\det D\boldsymbol{u}|\, \mathrm{d}\boldsymbol{x} = \int_{\boldsymbol{u}(\Omega_d \cap A)} \bar{\psi} \varphi\, \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{y} \in \boldsymbol{u}(\Omega_d \cap A), \tag{2.1}$$

whenever the integral on the left-hand side of (2.1) *exists.*

Definition 2.2. Let $\boldsymbol{u} \in W^{1,1}(\Omega, \mathbb{R}^N)$ be such that $\det D\boldsymbol{u} > 0$ a.e. We define Ω_0 as the set of $\boldsymbol{x} \in \Omega$ for which the following are satisfied:

i) the approximate differential of $\boldsymbol{u}$ at $\boldsymbol{x}$ exists and equals $D\boldsymbol{u}(\boldsymbol{x})$.
ii) there exist $\boldsymbol{w} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and a compact set $K \subset \Omega$ of density 1 at $\boldsymbol{x}$ such that $\boldsymbol{u}|_K = \boldsymbol{w}|_K$ and $D\boldsymbol{u}|_K = D\boldsymbol{w}|_K$,
iii) $\det D\boldsymbol{u}(\boldsymbol{x}) > 0$.

We note that the set Ω_0 is a set of full Lebesgue measure in Ω, i.e., $|\Omega \setminus \Omega_0| = 0$. This follows from Theorem 3.1.8 in [19], Rademacher's Theorem and Whitney's Theorem.

Two important properties for a map are Lusin's properties (N) and (N^{-1}).

Definition 2.3. Let $X \subset \mathbb{R}^N$ be a measurable set. We say that a measurable function $\boldsymbol{u} : X \to \mathbb{R}^N$ satisfies Lusin's condition (N) if for every $A \subset X$ such that $|A| = 0$ we have $|\boldsymbol{u}(A)| = 0$. We say that $\boldsymbol{u}$ satisfies condition (N^{-1}) if for every $A \subset \mathbb{R}^N$ such that $|A| = 0$ we have $|\boldsymbol{u}^{-1}(A)| = 0$.

We will use the following consequence of Proposition 2.1 (see, e.g., [5, Lemma 2.8]).

Lemma 2.4. *Let $\boldsymbol{u} \in W^{1,1}(\Omega, \mathbb{R}^N)$. Then $\boldsymbol{u}|_{\Omega_0}$ satisfies Lusin's condition (N). Moreover, if $\det D\boldsymbol{u}(\boldsymbol{x}) \neq 0$ for a.e. $\boldsymbol{x} \in \Omega$, then $\boldsymbol{u}$ satisfies Lusin's (N^{-1}) condition.*

Definition 2.5. For any measurable set A of Ω, the geometric image of A under $\boldsymbol{u}$ is

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, A) := \boldsymbol{u}(A \cap \Omega_0),$$

with Ω_0 as in Definition 2.2.

2.2. **The surface energy.** In this subsection $N \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ is a bounded open set. Let $\boldsymbol{u} \in W^{1,N-1}(\Omega, \mathbb{R}^N)$ be such that $\det D\boldsymbol{u} \in L^1(\Omega)$.

The adjugate matrix $\operatorname{adj} \boldsymbol{F}$ of $\boldsymbol{F} \in \mathbb{R}^{N \times N}$ satisfies $(\det \boldsymbol{F})\boldsymbol{I} = \boldsymbol{F} \operatorname{adj} \boldsymbol{F}$, where $\boldsymbol{I}$ denotes the identity matrix. The transpose of $\operatorname{adj} \boldsymbol{F}$ is the cofactor $\operatorname{cof} \boldsymbol{F}$. We start by observing that, when $N = 3$, $|\operatorname{cof} \boldsymbol{F}|$ is controlled in terms of $|\boldsymbol{F}|^2$. The proof of the following result is elementary and based on singular value decomposition.

Lemma 2.6. $|\boldsymbol{F}|^2 \geq \sqrt{3}\,|\operatorname{cof} \boldsymbol{F}|$ *for all $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$, with optimal constant.*

Definition 2.7. Let $\boldsymbol{u} \in W^{1,1}(\Omega, \mathbb{R}^N)$ be such that $\operatorname{cof} D\boldsymbol{u} \in L^1(\Omega, \mathbb{R}^{N\times N})$ and $\det D\boldsymbol{u} \in L^1(\Omega)$.

a) For every $\phi \in C^1_c(\Omega)$ and $\boldsymbol{g} \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ we define

$$\overline{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) = \int_\Omega [\boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x}) D\phi(\boldsymbol{x})) + \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \det D\boldsymbol{u}(\boldsymbol{x})] \,\mathrm{d}\boldsymbol{x}.$$

b) For all $\boldsymbol{f} \in C^1_c(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ we define

$$\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) = \int_\Omega [D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \cdot \operatorname{cof} D\boldsymbol{u}(\boldsymbol{x}) + \operatorname{div}_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \det D\boldsymbol{u}(\boldsymbol{x})] \,\mathrm{d}\boldsymbol{x}$$

and

$$\mathcal{E}(\boldsymbol{u}) = \sup\{\mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f}) : \boldsymbol{f} \in C^1_c(\Omega \times \mathbb{R}^N, \mathbb{R}^N),\ \|\boldsymbol{f}\|_{L^\infty} \leq 1\}.$$

Sometimes we will use the notation $\mathcal{E}(\boldsymbol{u}, V)$ to refer to $\mathcal{E}(\boldsymbol{u}|_V)$, where V is an open subset of Ω. Clearly, $\overline{\mathcal{E}} \leq \mathcal{E}$. The following result shows that if $\overline{\mathcal{E}}$ vanishes, so does $\mathcal{E}$. This implies that the divergence identities (1.4) are satisfied if and only if the surface energy $\mathcal{E}$ is identically zero. The proof consists in using the continuity of $\boldsymbol{f} \mapsto \mathcal{E}_{\boldsymbol{u}}(\boldsymbol{f})$ and the density of the linear span of products of functions of separated variables in $C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ (see, e.g., [37, Cor. 1.6.5]).

Lemma 2.8. *Let $\boldsymbol{u} \in W^{1,N-1}(\Omega, \mathbb{R}^N)$ be such that $\det D\boldsymbol{u} \in L^1(\Omega)$. If $\overline{\mathcal{E}}(\boldsymbol{u}) = 0$ then $\mathcal{E}(\boldsymbol{u}) = 0$.*

In the rest of this section we take $N = 3$. The following result is a particular case of the calculation of the energy $\mathcal{E}$ for the Cartesian product of two functions.

Lemma 2.9. *Let $\omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$ be both open and bounded. Let $\boldsymbol{v} \in H^1(\omega, \mathbb{R}^2)$. Then* $\det D\boldsymbol{v} \in L^1(\omega)$ *and $\mathcal{E}(\boldsymbol{v}) = 0$. Define $\boldsymbol{w} : \omega \times I \to \mathbb{R}^3$ as $\boldsymbol{w}(x_1, x_2, x_3) = (\boldsymbol{v}(x_1, x_2), x_3)$. Then $\mathcal{E}(\boldsymbol{w}) = 0$.*

Proof. Clearly, $|\det D\boldsymbol{v}| \leq \frac{1}{2}|D\boldsymbol{v}|^2 \in L^1(\omega)$. The fact that $\mathcal{E}(\boldsymbol{v}) = 0$ is standard and can be shown by approximation by smooth maps and integration by parts (e.g., [41, Lemma 2] or [45, Th. 4.2]). For $\boldsymbol{x} \in \mathbb{R}^3$, write $\boldsymbol{x} = (\hat{\boldsymbol{x}}, x_3)$ with $\hat{\boldsymbol{x}} = (x_1, x_2)$, and analogously for $\boldsymbol{y} \in \mathbb{R}^3$. Using Lemma 2.8, it suffices to show that $\bar{\mathcal{E}}_{\boldsymbol{w}}(\phi, \boldsymbol{g}) = 0$ for $\phi \in C^1_c(\omega \times I)$ and $\boldsymbol{g} \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$ of the form

$$\phi(\boldsymbol{x}) = \phi_1(\hat{\boldsymbol{x}})\, \phi_3(x_3) \quad \text{and} \quad \boldsymbol{g}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{g}_{11}(\hat{\boldsymbol{y}})\, g_{13}(y_3) \\ g_{31}(\hat{\boldsymbol{y}})\, g_{33}(y_3) \end{pmatrix},$$

for some

$$\phi_1 \in C^1_c(\omega),\ \phi_3 \in C^1_c(I),\ \boldsymbol{g}_{11} \in C^1_c(\mathbb{R}^2, \mathbb{R}^2),\ g_{13} \in C^1_c(\mathbb{R}),\ g_{31} \in C^1_c(\mathbb{R}^2),\ g_{33} \in C^1_c(\mathbb{R}).$$

For such ϕ and $\boldsymbol{g}$ we have

$$D\phi(\boldsymbol{x}) = \begin{pmatrix} \phi_3(x_3)\, D\phi_1(\hat{\boldsymbol{x}}) \\ \phi_1(\hat{\boldsymbol{x}})\, \phi_3'(x_3) \end{pmatrix}, \qquad \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) = g_{13}(y_3) \operatorname{div} \boldsymbol{g}_{11}(\hat{\boldsymbol{y}}) + g_{31}(\hat{\boldsymbol{y}})\, g_{33}'(y_3).$$

On the other hand, for a.e. $\boldsymbol{x} \in \omega \times I$

$$D\boldsymbol{w}(\boldsymbol{x}) = \begin{pmatrix} D\boldsymbol{v}(\hat{\boldsymbol{x}}) & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix},\ \operatorname{cof} D\boldsymbol{w}(\boldsymbol{x}) = \begin{pmatrix} \operatorname{cof} D\boldsymbol{v}(\hat{\boldsymbol{x}}) & \boldsymbol{0} \\ \boldsymbol{0} & \det D\boldsymbol{v}(\hat{\boldsymbol{x}}) \end{pmatrix},\ \det D\boldsymbol{w}(\boldsymbol{x}) = \det D\boldsymbol{v}(\hat{\boldsymbol{x}}).$$

Therefore, for a.e. $\boldsymbol{x} \in \omega \times I$,

$$\begin{aligned} &\boldsymbol{g}(\boldsymbol{w}(\boldsymbol{x})) \cdot (\operatorname{cof} D\boldsymbol{w}(\boldsymbol{x}) D\phi(\boldsymbol{x})) + \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{w}(\boldsymbol{x})) \det D\boldsymbol{w}(\boldsymbol{x}) \\ &= \phi_3(x_3)\, g_{13}(x_3) \Big[\boldsymbol{g}_{11}(\boldsymbol{v}(\hat{\boldsymbol{x}})) \cdot \Big(\operatorname{cof} D\boldsymbol{v}(\hat{\boldsymbol{x}})\, D\phi_1(\hat{\boldsymbol{x}})\Big) + \phi_1(\hat{\boldsymbol{x}}) \operatorname{div} \boldsymbol{g}_{11}(\boldsymbol{v}(\hat{\boldsymbol{x}})) \det D\boldsymbol{v}(\hat{\boldsymbol{x}})\Big] \\ &\quad + g_{31}(\boldsymbol{v}(\hat{\boldsymbol{x}}))\, \phi_1(\hat{\boldsymbol{x}}) \det D\boldsymbol{v}(\hat{\boldsymbol{x}}) \left[g_{33}(x_3)\, \phi_3'(x_3) + \phi_3(x_3)\, g_{33}'(x_3)\right]. \end{aligned} \tag{2.2}$$

Now, since $\mathcal{E}(\boldsymbol{v}) = 0$ and $\mathcal{E}(\mathbf{id}_I) = 0$ we have that

$$\int_\omega \Big[\boldsymbol{g}_{11}(\boldsymbol{v}(\hat{\boldsymbol{x}})) \cdot \Big(\operatorname{cof} D\boldsymbol{v}(\hat{\boldsymbol{x}})\, D\phi_1(\hat{\boldsymbol{x}})\Big) + \phi_1(\hat{\boldsymbol{x}}) \operatorname{div} \boldsymbol{g}_{11}(\boldsymbol{v}(\hat{\boldsymbol{x}})) \det D\boldsymbol{v}(\hat{\boldsymbol{x}})\Big]\, \mathrm{d}\hat{\boldsymbol{x}} = 0$$

and

$$\int_I \left[g_{33}(x_3)\, \phi_3'(x_3) + \phi_3(x_3)\, g_{33}'(x_3)\right] \mathrm{d}x_3 = 0$$

so an integration of (2.2) in $\omega \times I$ yields $\mathcal{E}_{\boldsymbol{w}}(\phi, \boldsymbol{g}) = 0$. □

In the following, we show that the precomposition of a map of zero energy with a regular map has also zero energy; a related result was shown in the proof of [28, Th. 7].

Lemma 2.10. *Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ be such that* $\det D\boldsymbol{u} \in L^1(\Omega)$ *and $\mathcal{E}(\boldsymbol{u}) = 0$. Let $\boldsymbol{L} : \mathbb{R}^3 \to \mathbb{R}^3$ be locally Lipschitz. Then $\mathcal{E}(\boldsymbol{L} \circ \boldsymbol{u}) = 0$.*

Proof. First assume that $\boldsymbol{L}$ is of class C^2. Let $\phi \in C^1_c(\Omega)$ and $\boldsymbol{g} \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$. The key of the proof consists in showing that

$$\bar{\mathcal{E}}_{\boldsymbol{L} \circ \boldsymbol{u}}(\phi, \boldsymbol{g}) = \bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, (\operatorname{adj} D\boldsymbol{L})(\boldsymbol{g} \circ \boldsymbol{L})). \tag{2.3}$$

The integrand corresponding to $\bar{\mathcal{E}}_{\boldsymbol{L} \circ \boldsymbol{u}}(\phi, \boldsymbol{g})$ is, for a.e. $\boldsymbol{x} \in \Omega$,

$$\begin{aligned} &\boldsymbol{g}((\boldsymbol{L} \circ \boldsymbol{u})(\boldsymbol{x})) \cdot (\operatorname{cof} D(\boldsymbol{L} \circ \boldsymbol{u})(\boldsymbol{x})\, D\phi(\boldsymbol{x})) + \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}((\boldsymbol{L} \circ \boldsymbol{u})(\boldsymbol{x})) \det D(\boldsymbol{L} \circ \boldsymbol{u})(\boldsymbol{x}) \\ &= (\operatorname{adj} D\boldsymbol{L}(\boldsymbol{u}(\boldsymbol{x}))\, \boldsymbol{g}(\boldsymbol{L}(\boldsymbol{u}(\boldsymbol{x})))) \cdot (\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x}))\, D\phi(\boldsymbol{x})) \\ &\quad + \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{L}(\boldsymbol{u}(\boldsymbol{x}))) \det D\boldsymbol{L}(\boldsymbol{u}(\boldsymbol{x})) \det D\boldsymbol{u}(\boldsymbol{x}). \end{aligned} \tag{2.4}$$

Now define $\bar{\boldsymbol{g}} := (\operatorname{adj} D\boldsymbol{L})(\boldsymbol{g} \circ \boldsymbol{L})$. Then, for all $\boldsymbol{y} \in \mathbb{R}^3$, by Piola's identity,

$$\operatorname{div} \bar{\boldsymbol{g}}(\boldsymbol{y}) = \sum_{i,j=1}^{3} \frac{\partial}{\partial y_i} \left[\operatorname{adj} D\boldsymbol{L}(\boldsymbol{y})_{ij}\, g_j(\boldsymbol{L}(\boldsymbol{y}))\right] = \sum_{i,j=1}^{3} \operatorname{adj} D\boldsymbol{L}(\boldsymbol{y})_{ij} \frac{\partial}{\partial y_i} \left[g_j(\boldsymbol{L}(\boldsymbol{y}))\right],$$

so, thanks to the matrix identity $\boldsymbol{F} \operatorname{adj} \boldsymbol{F} = (\det \boldsymbol{F})\boldsymbol{I}$ valid for $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}\operatorname{div} \bar{\boldsymbol{g}}(\boldsymbol{y}) &= \sum_{i,j=1}^{3} \operatorname{adj} D\boldsymbol{L}(\boldsymbol{y})_{ij} \frac{\partial}{\partial y_i} \left[g_j(\boldsymbol{L}(\boldsymbol{y}))\right] = \sum_{i,j,k=1}^{3} \operatorname{adj} D\boldsymbol{L}(\boldsymbol{y})_{ij}\, D\boldsymbol{g}(\boldsymbol{L}(\boldsymbol{y}))_{jk}\, D\boldsymbol{L}(\boldsymbol{y})_{ki} \\ &= \operatorname{tr}\left(D\boldsymbol{L}(\boldsymbol{y}) \operatorname{adj} D\boldsymbol{L}(\boldsymbol{y})\, D\boldsymbol{g}(\boldsymbol{L}(\boldsymbol{y}))\right) = \operatorname{tr}\left(\det D\boldsymbol{L}(\boldsymbol{y})\, D\boldsymbol{g}(\boldsymbol{L}(\boldsymbol{y}))\right) \\ &= \det D\boldsymbol{L}(\boldsymbol{y}) \operatorname{div} \boldsymbol{g}(\boldsymbol{L}(\boldsymbol{y})).\end{aligned}$$

With this, we find that the integrand of $\bar{\mathcal{E}}_{\boldsymbol{u}}(\phi, \bar{\boldsymbol{g}})$ coincides with (2.4), so (2.3) is proved. As $\mathcal{E}(\boldsymbol{u}) = 0$ we obtain that $\bar{\mathcal{E}}_{\boldsymbol{L}\circ\boldsymbol{u}}(\phi, \boldsymbol{g}) = 0$.

Now assume, as in the statement, that $\boldsymbol{L}$ is only locally Lipschitz, and take a sequence $\{\boldsymbol{L}_n\}$ in $C^2(\mathbb{R}^3, \mathbb{R}^3)$ such that $\boldsymbol{L}_n \to \boldsymbol{L}$ a.e., $D\boldsymbol{L}_n \to D\boldsymbol{L}$ a.e. and

$$\sup_{n\in\mathbb{N}} \|\boldsymbol{L}_n\|_{W^{1,\infty}(B(\boldsymbol{0}, \|\boldsymbol{u}\|_{L^\infty(\Omega,\mathbb{R}^3)}))} < \infty.$$

The existence of such approximating sequence follows from a classic result (see, e.g., [18, Th. 6.6.1]). By the first part of the proof,

$$\begin{aligned}0 = \int_\Omega \Big[&(\operatorname{adj} D\boldsymbol{L}_n(\boldsymbol{u}(\boldsymbol{x}))\, \boldsymbol{g}(\boldsymbol{L}_n(\boldsymbol{u}(\boldsymbol{x})))) \cdot (\operatorname{cof} D(\boldsymbol{u}(\boldsymbol{x})\, D\phi(\boldsymbol{x})) \\ &+ \phi(\boldsymbol{x}) \operatorname{div} \boldsymbol{g}(\boldsymbol{L}_n(\boldsymbol{u}(\boldsymbol{x}))) \det D\boldsymbol{L}_n(\boldsymbol{u}(\boldsymbol{x})) \det D\boldsymbol{u}(\boldsymbol{x})\Big] \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Taking limits, we obtain that $\bar{\mathcal{E}}_{\boldsymbol{L}\circ\boldsymbol{u}}(\phi, \boldsymbol{g}) = 0$. By Lemma 2.8, $\mathcal{E}(\boldsymbol{L} \circ \boldsymbol{u}) = 0$. □

We recall now the definition of the distributional Jacobian determinant.

Definition 2.11. Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$. The distributional Jacobian $\operatorname{Det} D\boldsymbol{u}$ of $\boldsymbol{u}$ is the distribution defined by

$$\langle \operatorname{Det} D\boldsymbol{u}, \varphi \rangle := -\frac{1}{3} \langle \operatorname{adj} D\boldsymbol{u}\, \boldsymbol{u}, D\varphi \rangle = -\frac{1}{3} \int_\Omega \operatorname{adj} D\boldsymbol{u}\, \boldsymbol{u} \cdot D\varphi, \qquad \varphi \in C_c^1(\Omega).$$

2.3. **The axisymmetric setting.** In most of the paper we will work with axisymmetric (with respect to the x_3-axis) maps and domains, which are defined as follows. We say that the set $\Omega \subset \mathbb{R}^3$ is axisymmetric if

$$\bigcup_{\boldsymbol{x}\in\Omega} \left(\partial B_{\mathbb{R}^2}((0,0), |(x_1, x_2)|) \times \{x_3\}\right) \subset \Omega.$$

When we define

$$\begin{aligned}\pi : \mathbb{R}^3 &\to [0, \infty) \times \mathbb{R} & \boldsymbol{P} : [0, \infty) \times \mathbb{R} \times \mathbb{R} &\to \mathbb{R}^3 \\ \boldsymbol{x} &\mapsto (|(x_1, x_2)|, x_3) & (r, \theta, x_3) &\mapsto (r\cos\theta, r\sin\theta, x_3),\end{aligned}$$

the axisymmetry of Ω is equivalent to the equality

$$\Omega = \{\boldsymbol{P}(r, \theta, x_3) : (r, x_3) \in \pi(\Omega),\ \theta \in [0, 2\pi)\}. \tag{2.5}$$

Given an axisymmetric set Ω, we say that $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is axisymmetric if there exists $\boldsymbol{v} : \pi(\Omega) \to [0, \infty) \times \mathbb{R}$ such that

$$\begin{aligned}&(\boldsymbol{u} \circ \boldsymbol{P})(r, \theta, x_3) = \boldsymbol{P}\left(v_1(r, x_3), \theta, v_2(r, x_3)\right), \text{ i.e.,} \\ &\qquad \boldsymbol{u}(r\cos\theta, r\sin\theta, x_3) = v_1(r, x_3)(\cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2) + v_2(r, x_3)\boldsymbol{e}_3 \end{aligned} \tag{2.6}$$

for all $(r, x_3, \theta) \in \pi(\Omega) \times [0, 2\pi)$. We will say that $\boldsymbol{v}$ is the function corresponding to $\boldsymbol{u}$. This $\boldsymbol{v}$ is uniquely determined by $\boldsymbol{u}$. Note that if Ω and $\boldsymbol{u}$ are axisymmetric then so is $\boldsymbol{u}(\Omega)$.

Given $\delta > 0$, we define C_δ as the (open, infinite, solid) cylinder of radius δ:

$$C_\delta := \{\boldsymbol{P}(r, \theta, x_3) : (r, \theta, x_3) \in [0, \delta) \times [0, 2\pi) \times \mathbb{R}\}. \tag{2.7}$$

We have $\pi(C_\delta) = [0, \delta) \times \mathbb{R}$.

2.4. **Prescribing the boundary data.** As mentioned in the Introduction, we fix a smooth bounded open set $\widetilde{\Omega}$ of $\mathbb{R}^3$ such that $\Omega \Subset \widetilde{\Omega}$ and consider an orientation-preserving C^1 diffeomorphism $\boldsymbol{b}$ from the closure of $\widetilde{\Omega}$ to $\mathbb{R}^3$. We assume that Ω, $\widetilde{\Omega}$, and $\boldsymbol{b}$ are axisymmetric. The function $\boldsymbol{u}$, originally defined in Ω, is extended to $\widetilde{\Omega}$ by setting $\boldsymbol{u} = \boldsymbol{b}$ in $\Omega_D := \widetilde{\Omega} \setminus \overline{\Omega}$.

We assume that the extension to $\widetilde{\Omega}$, still called $\boldsymbol{u}$, is in $H^1(\widetilde{\Omega}, \mathbb{R}^3)$, as stated in the definition (1.5) of our function space $\mathcal{A}_s$. Regarding the class $\mathcal{A}_s^r$ of regular maps defined in (1.6), its definition is the set the maps in $\mathcal{A}_s$ having zero surface energy in the extended domain $\widetilde{\Omega}$:

$$\mathcal{A}_s^r = \{\boldsymbol{u} \in \mathcal{A}_s : \mathcal{E}(\boldsymbol{u}) = 0 \text{ in } \widetilde{\Omega}\}.$$

This way we avoid cavitation at the boundary $\partial\Omega$. Observe that the condition $\det D\boldsymbol{u} > 0$ a.e. in definition (1.5) is satisfied for any $\boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$ such that $E(\boldsymbol{u}) \leq E(\boldsymbol{b})$, thanks to the blow-up behaviour (1.2) of the neo-Hookean energy as the Jacobian vanishes.

2.5. **A family of good open sets.** Given a nonempty open set $U \Subset \widetilde{\Omega}$ with a C^2 boundary, we denote by $d : \widetilde{\Omega} \to \mathbb{R}$ the signed distance function to ∂U given by

$$d(\boldsymbol{x}) := \begin{cases} \operatorname{dist}(\boldsymbol{x}, \partial U) & \text{if } \boldsymbol{x} \in U \\ 0 & \text{if } \boldsymbol{x} \in \partial U \\ -\operatorname{dist}(\boldsymbol{x}, \partial U) & \text{if } \boldsymbol{x} \in \Omega \setminus \overline{U} \end{cases}$$

and

$$U_t := \{\boldsymbol{x} \in \widetilde{\Omega} : \ d(\boldsymbol{x}) > t\}, \tag{2.8}$$

for each $t \in \mathbb{R}$. It is a classical result (see e.g. [15]) that there exists $\delta > 0$ such that for all $t \in (-\delta, \delta)$, the set U_t is open, compactly contained in $\widetilde{\Omega}$ and has a C^2 boundary.

Definition 2.12. Let $\boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$ be such that $\det D\boldsymbol{u} > 0$ a.e. We define $\mathcal{U}_{\boldsymbol{u}}$ as the family of nonempty open sets $U \Subset \widetilde{\Omega}$ with a C^2 boundary that satisfy

a) $\boldsymbol{u}_{|\partial U} \in H^1(\partial U, \mathbb{R}^3)$, and $(\operatorname{cof} \nabla \boldsymbol{u})_{|\partial U} \in L^1(U, \mathbb{R}^{3\times 3})$,
b) $\partial U \subset \Omega_0$, $\mathcal{H}^2$-a.e., where Ω_0 is the set in Definition 2.2, and $(\nabla \boldsymbol{u}_{|\partial U})(\boldsymbol{x}) = \nabla \boldsymbol{u}(\boldsymbol{x})_{|T_{\boldsymbol{x}}\partial U}$ for $\mathcal{H}^2$-a.e. $\boldsymbol{x} \in \partial U$,
c) $\lim_{\varepsilon \to 0} \int_0^\varepsilon \left| \int_{\partial U_t} |\operatorname{cof} \nabla \boldsymbol{u}| \mathrm{d}\mathcal{H}^2 - \int_{\partial U} |\operatorname{cof} \nabla \boldsymbol{u}| \mathrm{d}\mathcal{H}^2 \right| \mathrm{d}t = 0$,
d) For every $\boldsymbol{g} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with $(\operatorname{adj} D\boldsymbol{u})(\boldsymbol{g} \circ \boldsymbol{u}) \in L^1_{\text{loc}}(\widetilde{\Omega}, \mathbb{R}^3)$,

$$\lim_{\varepsilon \to 0} \int_0^\varepsilon \left| \int_{\partial U_t} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}) \boldsymbol{\nu}_t(\boldsymbol{x})) \, \mathrm{d}\mathcal{H}^2 - \int_{\partial U} \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x})) \cdot (\operatorname{cof} \nabla \boldsymbol{u}(\boldsymbol{x}) \boldsymbol{\nu}(\boldsymbol{x})) \, \mathrm{d}\mathcal{H}^2 \right| \mathrm{d}t = 0$$

where $\boldsymbol{\nu}_t$ denotes the unit outward normal to U_t for each $t \in (0, \varepsilon)$, and $\boldsymbol{\nu}$ the unit outward normal to U.

Since we are imposing that $\boldsymbol{u}$ coincides with the C^1-diffeomorphism $\boldsymbol{b}$ in the exterior Dirichlet neighbourhood Ω_D of $\partial\Omega$, without loss of generality we may assume that $\Omega \in \mathcal{U}_{\boldsymbol{u}}$.

The following result is obtained using the coarea formula and Lebesgue's differentiation theorem for the first part, and Uryshon functions and Sard's lemma for the second. It guarantees that there are enough sets in $\mathcal{U}_{\boldsymbol{u}}$.

Lemma 2.13 (Lemma 2.11 in [30])**.** *Let $\boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$ be such that $\det D\boldsymbol{u} > 0$ a.e. Let $U \Subset \widetilde{\Omega}$ be a nonempty open set with a C^2 boundary. Then $U_t \in \mathcal{U}_{\boldsymbol{u}}$ for a.e. $t \in (-\delta, \delta)$. Moreover, for each compact $K \subset \widetilde{\Omega}$ there exists $U \in \mathcal{U}_{\boldsymbol{u}}$ such that $K \subset U$.*

2.6. **Regularity, injectivity and weak convergence of the planar function.** Let C_δ be as in (2.7). The link between the regularity of $\boldsymbol{u}$ and its associated $2D$ map $\boldsymbol{v}$ is as follows.

Lemma 2.14. *Let Ω be an axisymmetric domain, and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ an axisymmetric map with corresponding function $\boldsymbol{v}$. Let $\delta > 0$. Then, $\boldsymbol{u} \in H^1(\Omega \setminus \overline{C}_\delta, \mathbb{R}^3)$ if and only if $\boldsymbol{v} \in H^1(\pi(\Omega) \setminus ([0,\delta] \times \mathbb{R}), \mathbb{R}^2)$. Moreover, in this case,*

$$\|\boldsymbol{u}\|^2_{H^1(\Omega\setminus\overline{C}_\delta,\mathbb{R}^3)} \leq 2\pi \max\{\|\boldsymbol{x}\|_{L^\infty(\Omega,\mathbb{R}^3)}, \delta^{-1}\} \, \|\boldsymbol{v}\|^2_{H^1(\pi(\Omega)\setminus([0,\delta]\times\mathbb{R}),\mathbb{R}^2)},$$

$$\|\boldsymbol{v}\|^2_{H^1(\pi(\Omega)\setminus([0,\delta]\times\mathbb{R}),\mathbb{R}^2)} \leq (2\pi\delta)^{-1} \, \|\boldsymbol{u}\|^2_{H^1(\Omega\setminus\bar{C}_\delta,\mathbb{R}^3)}$$

and for a.e. (r, θ, x_3) with $(r, x_3) \in \pi(\Omega) \setminus ([0, \delta] \times \mathbb{R})$ and $\theta \in \mathbb{R}$,

$$\det D\boldsymbol{u}(\boldsymbol{P}(r, \theta, x_3)) = \frac{v_1(r, \theta)}{r} \det D\boldsymbol{v}(r, \theta).$$

This can be proved by using that the change of variables from Cartesian to cylindrical coordinates is a diffeomorphism when restricted to suitable domains and by using the formula for the Dirichlet energy in cylindrical coordinates given in the Appendix.

The orientation-preserving and injectivity conditions of $\boldsymbol{u}$ and $\boldsymbol{v}$ are related as follows. We recall that a function is injective a.e. if its restriction to a set of full measure is injective.

Lemma 2.15. *Let Ω be an axisymmetric domain. Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$ be axisymmetric, and let $\boldsymbol{v}$ be its corresponding function. The following hold:*

a) $\det D\boldsymbol{u} > 0$ a.e. in Ω if and only if $\det D\boldsymbol{v} > 0$ a.e. in $\pi(\Omega)$.
b) $\boldsymbol{u}$ is injective a.e. in Ω if and only if $\boldsymbol{v}$ is injective a.e in $\pi(\Omega)$.

The proof uses again the change of variables in cylindrical coordinates and is left to the reader.

The relationship between the weak convergences (hereafter denoted by $\rightharpoonup$) of a sequence of axisymmetric functions and their associated functions is contained in the two following lemmas, whose proofs are left again to the reader since they just rely on a manipulation of the axisymmetry in cylindrical coordinates.

Lemma 2.16. *Let Ω be an axisymmetric domain. For each $j \in \mathbb{N}$, let $\boldsymbol{u}_j \in H^1(\Omega, \mathbb{R}^3)$ be axisymmetric. Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$, and assume that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$ as $j \to \infty$. The following statements hold:*

i) $\boldsymbol{u}$ is axisymmetric.
ii) Let $\boldsymbol{v}_j$ and $\boldsymbol{v}$ be the corresponding functions of $\boldsymbol{u}_j$ and $\boldsymbol{u}$, respectively. Then $\boldsymbol{v}_j \rightharpoonup \boldsymbol{v}$ in $H^1(\pi(\Omega) \setminus ([0, \delta] \times \mathbb{R}), \mathbb{R}^2)$ for each $\delta > 0$.

Lemma 2.17. *Let Ω be an axisymmetric domain. For each $j \in \mathbb{N}$, let $\boldsymbol{u}_j : \Omega \to \mathbb{R}^3$ be an axisymmetric map with corresponding function $\boldsymbol{v}_j$. Let $\boldsymbol{v} : \pi(\Omega) \setminus (\{0\} \times \mathbb{R}) \to \mathbb{R}^3$ and assume that $\boldsymbol{v} \in H^1(\pi(\Omega) \setminus ([0, \delta] \times \mathbb{R}), \mathbb{R}^3)$ and that $\boldsymbol{v}_j \rightharpoonup \boldsymbol{v}$ in $H^1(\pi(\Omega) \setminus ([0, \delta] \times \mathbb{R}), \mathbb{R}^3)$ for every $\delta > 0$. Define $\boldsymbol{u} : \Omega \setminus \mathbb{R}\boldsymbol{e}_3 \to \mathbb{R}^3$ by (2.6). Then $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega \setminus \overline{C}_\delta)$ for every $\delta > 0$.*

2.7. **Regularity of maps in $\mathcal{A}_s$ outside the axis of symmetry.** We recall that orientation-preserving H^1 maps in $2D$ are continuous and satisfy Lusin's condition (N) and the divergence identities. These properties are inherited by the $3D$ axisymmetric maps $\boldsymbol{u}$ in $\mathcal{A}_s$ away from the symmetry axis.

Lemma 2.18. *Let Ω be an axisymmetric domain. Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$ be axisymmetric and satisfy $\det D\boldsymbol{u} > 0$ a.e., and let $\boldsymbol{v}$ be its corresponding function. Then:*

a) $\boldsymbol{v}$ has a representative that is continuous at each point of $\pi(\Omega) \setminus (\{0\} \times \mathbb{R})$, differentiable a.e., and satisfies condition (N) in $\pi(\Omega) \setminus (\{0\} \times \mathbb{R})$. Moreover, $\mathcal{E}(\boldsymbol{v}, \pi(\Omega) \setminus (\{0\} \times \mathbb{R})) = 0$.

b) $\boldsymbol{u}$ has a representative that is continuous at each point of $\Omega \setminus \mathbb{R}\boldsymbol{e}_3$, differentiable a.e., and satisfies condition (N) in $\Omega \setminus \mathbb{R}\boldsymbol{e}_3$. Moreover, $\mathcal{E}(\boldsymbol{u}, \Omega \setminus \mathbb{R}\boldsymbol{e}_3) = 0$.

c) For each $j \in \mathbb{N}$, let $\boldsymbol{u}_j \in H^1(\Omega, \mathbb{R}^3)$ be axisymmetric. Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$, and assume that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$ as $j \to \infty$. If $\det D\boldsymbol{u}_j > 0$ a.e. for all $j \in \mathbb{N}$ then $\boldsymbol{v}_j \to \boldsymbol{v}$ uniformly in compact subsets of $\pi(\Omega) \setminus (\{0\} \times \mathbb{R})$ and $\boldsymbol{u}_j \to \boldsymbol{u}$ uniformly in compact subsets of $\Omega \setminus (\{(0,0)\} \times \mathbb{R})$.

Proof. Let $\delta > 0$. By Lemma 2.14, $\boldsymbol{v}$ is in $H^1(\pi(\Omega) \setminus ([0,\delta] \times \mathbb{R}), \mathbb{R}^2)$ and, by Lemma 2.15, $\det D\boldsymbol{v} > 0$ a.e. By classical results on maps of finite distorsion, originally due to [46, 51] (see also, e.g., [24, Th. 2.5.4, Th. 5.3.5 and its Cors. 1 and 3] or [34, Th. 2.3, Cor. 2.25 and Th. 4.5]), $\boldsymbol{v}$ has a representative $\bar{\boldsymbol{v}}$ in $\pi(\Omega) \setminus ([0,\delta] \times \mathbb{R})$ that is continuous, differentiable a.e. and satisfies the (N) property. That $\mathcal{E}(\boldsymbol{v}, \pi(\Omega) \setminus ([0,\delta] \times \mathbb{R})) = 0$ is also a classical result (see, e.g., [41, 49, 43, 45]). As this is true for every $\delta > 0$, property *a)* is proved.

We define the representative $\bar{\boldsymbol{u}}$ of $\boldsymbol{u}$ through formula (2.6), but changing $\boldsymbol{u}, \boldsymbol{v}$ by $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$, respectively. As in Lemma 2.15, we readily obtain that $\bar{\boldsymbol{u}}$ is continuous in $\Omega \setminus \mathbb{R}\boldsymbol{e}_3$ and differentiable a.e. We now show the (N) property for $\bar{\boldsymbol{u}}$. Let A be a null set in $\Omega \setminus \mathbb{R}\boldsymbol{e}_3$, and for each $\theta \in \mathbb{R}$ define $A_\theta := \{(r, x_3) \colon \boldsymbol{P}(r, \theta, x_3) \in A\}$. Since $\mathcal{L}^3(\boldsymbol{P}^{-1}(A)) = 0$, we have that $\mathcal{L}^2(A_\theta) = 0$ for a.e. $\theta \in \mathbb{R}$. For any such θ we have that $\bar{\boldsymbol{v}}(A_\theta)$ is $\mathcal{L}^2$-null. By (2.6), $\boldsymbol{P}(s, \theta, y_3) \in \bar{\boldsymbol{u}}(A)$ if and only if $(s, y_3) \in \bar{\boldsymbol{v}}(A_\theta)$. Consequently,

$$\mathcal{L}^3(\bar{\boldsymbol{u}}(A)) = \int_{\mathbb{R}} \int_0^{2\pi} \int_0^\infty \chi_{\bar{\boldsymbol{u}}(A)}(\boldsymbol{P}(s, \theta, y_3))\, s \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}y_3 = \int_{\mathbb{R}} \int_0^{2\pi} \int_0^\infty \chi_{\bar{\boldsymbol{v}}(A_\theta)}(s, y_3)\, s \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}y_3.$$

Therefore, for any $R > 0$,

$$\mathcal{L}^3(\bar{\boldsymbol{u}}(A) \cap B(\boldsymbol{0}, R)) \leq R \int_{\mathbb{R}} \int_0^{2\pi} \int_0^\infty \chi_{\bar{\boldsymbol{v}}(A_\theta)}(s, y_3) \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}y_3 = R \int_0^{2\pi} \mathcal{L}^2(\bar{\boldsymbol{v}}(A_\theta)) \, \mathrm{d}\theta = 0.$$

As this is true for every $R > 0$, we obtain that $\mathcal{L}^3(\bar{\boldsymbol{u}}(A)) = 0$. Thus, $\bar{\boldsymbol{u}}$ satisfies condition (N) in $\Omega \setminus \mathbb{R}\boldsymbol{e}_3$.

Let $I \subset \mathbb{R}$ be an open interval of length less than 2π and define the function $\boldsymbol{w}$ in the set $\{(r, \theta, x_3) : (r, x_3) \in \pi(\Omega) \setminus (\{0\} \times \mathbb{R}),\ \theta \in I\}$ as $\boldsymbol{w}(r, \theta, x_3) := (v_1(r, x_3), \theta, v_2(r, x_3))$. As shown above, $\mathcal{E}(\boldsymbol{v}, \pi(\Omega) \setminus (\{0\} \times \mathbb{R})) = 0$, so by Lemma 2.9, $\mathcal{E}(\boldsymbol{w}) = 0$. By Lemma 2.10, $\mathcal{E}(\boldsymbol{P} \circ \boldsymbol{w}) = 0$, so by (2.6), $\boldsymbol{u} \circ \boldsymbol{P}$ has zero surface energy in $\{(r, \theta, x_3) : (r, x_3) \in \pi(\Omega) \setminus (\{0\} \times \mathbb{R}),\ \theta \in I\}$. Let $\delta > 0$. As $\boldsymbol{P}|_{(\delta,\infty) \times I \times \mathbb{R}}$ is a diffeomorphism that admits an extension to an open set containing the closure of $(\delta, \infty) \times I \times \mathbb{R}$, by [28, Sect. 8] or [33, Sect. 6], $\boldsymbol{u}$ has zero surface energy in $\Omega \cap \boldsymbol{P}\left((\delta, \infty) \times I \times \mathbb{R}\right)$. Considering now two open intervals I_1 and I_2 of length less than 2π, it is easy to check (see, if necessary, the proof of [40, Lemma 4.8]) that $\boldsymbol{u}$ has zero surface energy in $\Omega \cap \boldsymbol{P}\left((\delta, \infty) \times (I_1 \cup I_2) \times \mathbb{R}\right)$. Taking, additionally I_1 and I_2 such that $[0, 2\pi] \subset I_1 \cup I_2$, we obtain that $\boldsymbol{u}$ has zero surface energy in $\Omega \setminus \overline{C}_\delta$. Consequently, $\mathcal{E}(\boldsymbol{u}, \Omega \setminus \mathbb{R}\boldsymbol{e}_3) = 0$.

Now we show *c)* and assume that $\det D\boldsymbol{u}_j > 0$ a.e. for all $j \in \mathbb{N}$. By Lemma 2.15, we have that $\det D\boldsymbol{v}_j > 0$ a.e. for all $j \in \mathbb{N}$. Since the H^1 norm of $\{\boldsymbol{v}_j\}_{j \in \mathbb{N}}$ is bounded in $\pi(\Omega) \setminus ([0, \delta] \times \mathbb{R})$ for each $\delta > 0$, by a classic result on maps of finite distortion (see, e.g., [21, Lemma 2.1] for a precise reference), the family $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}}$ is equicontinuous in each compact set of $\pi(\Omega)\setminus([0,\delta]\times\mathbb{R})$, hence in each compact set of $\pi(\Omega)\setminus(\{0\}\times\mathbb{R})$. By the Ascoli–Arzelà theorem, and part *ii)* of Lemma 2.16, $\boldsymbol{v}_j\to\boldsymbol{v}$ uniformly in each compact set of $\pi(\Omega)\setminus(\{0\}\times\mathbb{R})$, in principle up to a subsequence, but in fact the convergence holds for the whole sequence because $\boldsymbol{v}$ is uniquely determined. Having in mind (2.6), we obtain that $\boldsymbol{u}_j\circ\boldsymbol{P}\to\boldsymbol{u}\circ\boldsymbol{P}$ uniformly in each compact subset of

$$\{(r,\theta,x_3):\,(r,x_3)\in\pi(\Omega)\setminus(\{0\}\times\mathbb{R}),\,\theta\in\mathbb{R}\}\,.$$

Then, as before, $\boldsymbol{u}_j\to\boldsymbol{u}$ uniformly in each compact subset of $\Omega\setminus\{(0,0)\}\times\mathbb{R}$. □

The assumptions of Lemma 2.18 will hold in most of the paper, and, in this case, without further mention, $\boldsymbol{u}$ and $\boldsymbol{v}$ are taken to be the continuous representative of themselves in the sets $\Omega\setminus\mathbb{R}\boldsymbol{e}_3$ and $\pi(\Omega)\setminus(\{0\}\times\mathbb{R})$, respectively.

3. Existence of minimizers of the neo-Hookean energy in the class $\mathcal{A}_s$

We prove in this section that, although the results of Ball & Murat show that E is not weakly lower semicontinuous in the full $3D$ setting because of the phenomenon of cavitation, when restricted to the axisymmetric setting we can prove that E is weakly lower semicontinuous. The reason for that is that maps in $\mathcal{A}_s$ are continuous outside the axis of symmetry, so cavitation can only occur on the axis of symmetry. Hence axisymmetric cavitation can be viewed as a cavitation on the boundary of the $2D$ subdomain $\pi(\Omega)$ and does not contradict $W^{1,2}$-quasiconvexity.

Proposition 3.1. *Let $\{\boldsymbol{u}_n\}_n$ be a sequence in $\mathcal{A}_s$. Then there exists $\boldsymbol{u}\in\mathcal{A}_s$ such that, up to a subsequence, $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in $H^1(\widetilde{\Omega},\mathbb{R}^3)$,*

$$\det D\boldsymbol{u}_n\rightharpoonup\det D\boldsymbol{u}\ \text{in}\ L^1(\widetilde{\Omega}),$$

$\chi_{\operatorname{im_G}(\boldsymbol{u}_n,\widetilde{\Omega})}\to\chi_{\operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega})}$ *a.e. as $n\to\infty$, and*

$$E(\boldsymbol{u})\le\liminf_{n\to\infty}E(\boldsymbol{u}_n).\tag{3.1}$$

Proof. Since $E(\boldsymbol{u}_n)\le E(\boldsymbol{b})$ for all $n\in\mathbb{N}$, we have, thanks to (1.1) and Poincaré's inequality together with the boundary condition $\boldsymbol{u}_n=\boldsymbol{b}$ in Ω_D, that $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\widetilde{\Omega},\mathbb{R}^3)$. Thus there exists $\boldsymbol{u}\in H^1(\widetilde{\Omega},\mathbb{R}^3)$ such that, up to a subsequence $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in H^1 and $\boldsymbol{u}_n\to\boldsymbol{u}$ a.e. Therefore, $\boldsymbol{u}=\boldsymbol{b}$ a.e. on $\widetilde{\Omega}\setminus\Omega$. Moreover, by Lemma 2.16, $\boldsymbol{u}$ is axisymmetric. Besides, we have $\sup_n\int_{\widetilde{\Omega}}H(\det D\boldsymbol{u}_n)<+\infty$. By using the De La Vallée Poussin criterion, we find that there exists $d\in L^1(\widetilde{\Omega})$ such that

$$\det D\boldsymbol{u}_n\rightharpoonup d\ \text{ in }\ L^1(\widetilde{\Omega}).$$

A standard argument based on (1.2) and Fatou's lemma (see, e.g., [44, Th. 5.1]) shows that $d>0$ a.e. in $\widetilde{\Omega}$. Let $\boldsymbol{v}_n$ the $2D$ map associated to $\boldsymbol{u}_n$. From 2.15 we have $\det D\boldsymbol{v}_n>0$ a.e. in $\pi(\widetilde{\Omega})$. By Lemma 2.16 we also have $\boldsymbol{v}_n\rightharpoonup\boldsymbol{v}$ in $H^1_{\mathrm{loc}}(\pi(\widetilde{\Omega}\setminus(\{0\}\times\mathbb{R})),\mathbb{R}^2)$. We can thus apply the result about higher integrability of the Jacobians due to Müller [42] to obtain that $\det D\boldsymbol{v}_n\rightharpoonup\det D\boldsymbol{v}$ in $L^1_{\mathrm{loc}}(\pi(\widetilde{\Omega}\setminus(\{0\}\times\mathbb{R})))$. From Sobolev injections we find that $v_1^n\to v_1$ in $L^2_{\mathrm{loc}}(\pi(\widetilde{\Omega}\setminus(\{0\}\times\mathbb{R})))$ and a.e. (up to a subsequence). Fix a small $\delta>0$. Since $\boldsymbol{u}=\boldsymbol{b}$ in $\widetilde{\Omega}\setminus\Omega$, a consequence of Lemma 2.18 applied to $\widetilde{\Omega}$ is that $\boldsymbol{v}$ is bounded and $(\boldsymbol{v}_n)$ is uniformly bounded in $\pi(\widetilde{\Omega})\setminus([0,\delta]\times\mathbb{R})$. Also, in the axisymmetric setting we have that

$$\det D\boldsymbol{u}_n(\boldsymbol{x})=\frac{1}{r}v_1^n(r,x_3)\det D\boldsymbol{v}_n(r,x_3)$$

(see, e.g., (6.12) in the Appendix). Hence the use of Egorov's theorem, see e.g. [48, Lemma 6.7], implies that

$$\det D\boldsymbol{u}_n \rightharpoonup \det D\boldsymbol{u} \text{ in } L^1_{\text{loc}}(\widetilde{\Omega} \setminus L).$$

Since L has zero Lebesgue measure, we find that $\det D\boldsymbol{u} = d > 0$ a.e. in $\widetilde{\Omega}$ and $\det D\boldsymbol{u}_n \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\widetilde{\Omega})$.

By Lemma 2.18, $\mathcal{E}(\boldsymbol{u}_n, \widetilde{\Omega} \setminus \mathbb{R}\boldsymbol{e}_3) = 0$ for all $n \in \mathbb{N}$. Then, by [27, Th. 2], $\boldsymbol{u}$ is injective a.e. and for a.e. $\delta > 0$ we have $\chi_{\text{im}_{\text{G}}(\boldsymbol{u}_n, \widetilde{\Omega} \setminus \overline{C}_\delta)} \to \chi_{\text{im}_{\text{G}}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_\delta)}$ a.e. as $n \to \infty$. From here, using the equiintegrability of the Jacobians, it is easy to prove that $\chi_{\text{im}_{\text{G}}(\boldsymbol{u}_n, \widetilde{\Omega})} \to \chi_{\text{im}_{\text{G}}(\boldsymbol{u}, \widetilde{\Omega})}$ in $L^1(\widetilde{\Omega})$. Passing to a subsequence we obtain the stated a.e. convergence.

Thanks to the weak continuity of the Jacobian and the convexity of H, we have that E is sequentially lower semicontinuous for the weak convergence in H^1, i.e., (3.1) holds, and, in particular, $E(\boldsymbol{u}) \leq E(\boldsymbol{b})$. □

Theorem 3.2. *The energy E has a minimizer in $\mathcal{A}_s$.*

Proof. It follows from the direct method of calculus of variations, since Proposition 3.1 shows the weak lower semicontinuity of E and the sequential weak compactness of $\mathcal{A}_s$. □

4. Regularity of inverses of maps in $\mathcal{A}_s$

The goal of this section is to give the appropriate definition of the inverse of maps in $\mathcal{A}_s$ and to prove that its first two components enjoy Sobolev regularity. This property is crucial in the proof of Proposition 6.2, which is the key for our main result Theorem 1.1.

4.1. **Topological degree.** We first recall how to define the classical Brouwer degree for continuous functions [14, 20]. Let $N \geq 2$. Let $U \subset \mathbb{R}^N$ be a bounded open set. If $\boldsymbol{u} \in C^1(\overline{U}, \mathbb{R}^N)$ then for every regular value $\boldsymbol{y}$ of $\boldsymbol{u}$ we set

$$\deg(\boldsymbol{u}, U, \boldsymbol{y}) = \sum_{\boldsymbol{x} \in \boldsymbol{u}^{-1}(y) \cap U} \det D\boldsymbol{u}(\boldsymbol{x}). \tag{4.1}$$

This sum is finite thanks to the inverse function theorem. We can show that the right-hand side of (4.1) is invariant by homotopies. This allows to extend the definition (4.1) to every $\boldsymbol{y} \notin \boldsymbol{u}(\partial U)$ and to show its depends only on the boundary values of $\boldsymbol{u}$. If $\boldsymbol{u}$ is only in $C(\partial U, \mathbb{R}^N)$, we may extend $\boldsymbol{u}$ to a continuous map in $\overline{U}$ by Tietze's theorem and set

$$\deg(\boldsymbol{u}, U, \cdot) = \deg(\boldsymbol{v}, U, \cdot),$$

where $\boldsymbol{v}$ is any map in $C^1(\overline{U}, \mathbb{R}^N)$ which is homotopic to the extension of $\boldsymbol{u}$.

If U is of class C^1 and $\boldsymbol{u} \in C^1(\partial U, \mathbb{R}^N)$, by using (4.1), Sard's theorem and the divergence identities (1.4), we can make a change of variables and integrate by parts to obtain

$$\int_{\mathbb{R}^N} \deg(\boldsymbol{u}, U, \boldsymbol{y}) \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} = \int_{\partial U} (\boldsymbol{g} \circ \boldsymbol{u}) \cdot (\operatorname{cof} D\boldsymbol{u}\, \boldsymbol{\nu}) \,\mathrm{d}\mathcal{H}^{N-1}. \tag{4.2}$$

This formula can be used as the definition of the degree for maps in $W^{1,N-1} \cap L^\infty(\partial U, \mathbb{R}^N)$ as noticed by Brezis & Nirenberg [9]. For any open set U having a positive distance away from the symmetry axis $\mathbb{R}^3$ it is possible to use the classical degree since there every map in $\mathcal{A}_s$ has a continuous representative (Lemma 2.18). However, for open sets U crossing the axis (where maps in $\mathcal{A}_s$ may have singularities) we use the Brezis–Nirenberg degree.

Definition 4.1. Let $U \subset \mathbb{R}^N$ be a bounded open set. For any $\boldsymbol{u} \in C(\partial U, \mathbb{R}^N)$ and any $\boldsymbol{y} \in \mathbb{R}^N \setminus \boldsymbol{u}(\partial U)$ we denote by $\deg(\boldsymbol{u}, U, \boldsymbol{y})$ the classical topological degree of $\boldsymbol{u}$ with respect to $\boldsymbol{y}$. Suppose now that $U \subset \mathbb{R}^N$ is a C^1 bounded open set and $\boldsymbol{u} \in W^{1,N-1}(\partial U, \mathbb{R}^N) \cap L^\infty(\partial U, \mathbb{R}^N)$.

Then the degree of $\boldsymbol{u}$, denoted by $\deg(\boldsymbol{u}, U, \cdot)$, is defined as the only L^1 function that satisfies (4.2) for all $\boldsymbol{g} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

To see that this definition makes sense we refer to [9] or [11, Rk. 3.3]. Also, using (4.2) for a sequence of smooth maps approximating $\boldsymbol{u}$ we can see that for any $\boldsymbol{u} \in C(\partial U, \mathbb{R}^N) \cap W^{1,N-1}(\partial U, \mathbb{R}^{N-1})$ such that $\mathcal{L}^N\big(\boldsymbol{u}(\partial U)\big) = 0$ the two definitions are consistent (as stated in [44, Prop. 2.1.2]). Thanks to Lemma 2.18, our maps $\boldsymbol{u}$ satisfiy the (N) property, so the condition on $\boldsymbol{u}(\partial U)$ is satisfied for all regular open sets U that are a distance apart from the axis. On the one hand, by the continuity property of the degree we can see that the topological image (Definition 4.2 below) of a bounded open set is open. This will give us adequate ambient spaces to work with in the deformed configuration, see Equation (4.10). On the other hand, it is by working with the Brezis–Nirenberg degree that the Sobolev regularity of the inverses (crucial for the lower semicontinuity result presented in this paper) is obtained in the presence of singularities (Proposition 4.12).

We will invoke many previous results about the degree and related concepts (such as the topological image or the condition INV; see below). Most of the references that we cite use the degree with slightly different assumptions on $\boldsymbol{u}$. Nevertheless, their proofs apply to our case with only small modifications.

4.2. **Topological image for the classical degree.** An important part of our analysis refers only to open sets U that either are a distance apart from the symmetry axis or enclose entirely the closed segment

$$L := \overline{\Omega} \cap \mathbb{R}\boldsymbol{e}_3 \tag{4.3}$$

where the singularities can occur. To be precise, we use the setting of Section 2.4 and frequently deal with open sets $U \subset \widetilde{\Omega}$ such that $\partial U \cap L = \varnothing$. Since $\boldsymbol{u} = \boldsymbol{b}$ in the Dirichlet region $\Omega_D = \widetilde{\Omega} \setminus \overline{\Omega}$, the map $\boldsymbol{u}$ is continuous in ∂U and, hence, the classical degree $\deg(\boldsymbol{u}, U, \cdot)$ is well defined. For those sets U the topological image is defined as follows.

Definition 4.2. Let $N \geq 2$. Let U be a bounded open set of $\mathbb{R}^N$ and let $\boldsymbol{u} \in C(\partial U, \mathbb{R}^N)$. We define $\operatorname{im_T}(\boldsymbol{u}, U)$, the topological image of U under $\boldsymbol{u}$, as the set of $\boldsymbol{y} \in \mathbb{R}^N \setminus \boldsymbol{u}(\partial U)$ such that $\deg(\boldsymbol{u}, U, \boldsymbol{y}) \neq 0$.

In the $2D$ case, the topological image through an orientation-preserving H^1 map enjoys some nice geometric properties. Let A be a bounded domain of $\mathbb{R}^2$ and let $\boldsymbol{v} \in H^1(A, \mathbb{R}^2)$ be a map such that $\det D\boldsymbol{u} > 0$ a.e. in A. As recalled in the proof of Lemma 2.18, $\boldsymbol{v}$ has a continuous representative. In what follows, we identify $\boldsymbol{v}$ with that representative. Let V be compactly included in A. We assume that $V \in \mathcal{U}_{\boldsymbol{v}}$, i.e., all the analogous properties of Definition 2.12 are satisfied for the planar map $\boldsymbol{v}$. In [5, Lemma 5.4] it was proved the following result:

$$\overline{\operatorname{im_T}(\boldsymbol{v}, V)} = \operatorname{im_T}(\boldsymbol{v}, V) \cup \boldsymbol{v}(\partial V) \quad \text{and} \quad \partial \operatorname{im_T}(\boldsymbol{v}, V) = \boldsymbol{v}(\partial V). \tag{4.4}$$

Define (cf. [5, Def. 5.6]) the topological image of a point $\boldsymbol{x}$ as

$$\operatorname{im_T}\big(\boldsymbol{v}, \boldsymbol{x}\big) := \bigcap_{\rho>0,\, B(\boldsymbol{x},\rho)\in\mathcal{U}_{\boldsymbol{v}}} \overline{\operatorname{im_T}\big(\boldsymbol{v}, B(\boldsymbol{x}, \rho)\big)}.$$

Since our map $\boldsymbol{v}$ is continuous, we have the characterization

$$\operatorname{im_T}\big(\boldsymbol{v}, \boldsymbol{x}\big) = \{\boldsymbol{v}(\boldsymbol{x})\} \text{ for every } \boldsymbol{x} \in V.$$

Indeed, by [49, Cor. 1], $\boldsymbol{v}(B(\boldsymbol{x}, \rho))$ is included a.e. in $\overline{\operatorname{im_T}(\boldsymbol{v}, B(\boldsymbol{x}, \rho))}$, for each $\rho > 0$ with $B(\boldsymbol{x}, \rho) \in \mathcal{U}_{\boldsymbol{v}}$. Therefore, since $\overline{\operatorname{im_T}(\boldsymbol{v}, B(\boldsymbol{x}, \rho))}$ is compact, the continuity implies that $\boldsymbol{v}(\boldsymbol{x})$ belongs to $\overline{\operatorname{im_T}(\boldsymbol{v}, B(\boldsymbol{x}, \rho))}$ and therefore to $\operatorname{im_T}\big(\boldsymbol{v}, \boldsymbol{x}\big)$, since ρ is arbitrary. On the other hand, again by the continuity, $\operatorname{im_T}(\boldsymbol{v}, \boldsymbol{x})$ is a singleton as proved in [49, Lemma 4].

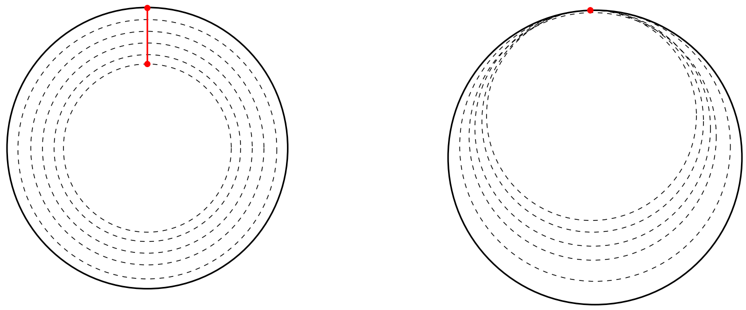

FIGURE 1. On the left the sets V and in red the set $G(\boldsymbol{y})$, on the right the set $\operatorname{im_T}(\boldsymbol{v}, V)$ and in red the point $\boldsymbol{y}$ on the boundary.

Assume now $\boldsymbol{v}$ is injective a.e. in A. Let

$$G(\boldsymbol{y}) := \{\boldsymbol{x} \in \overline{V} \colon \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{y}\} \quad \text{and} \quad T := \{\boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{v}, V) \colon G(\boldsymbol{y}) \text{ is not a singleton}\}.$$

Thus, T is the image of the points where $\boldsymbol{v}$ is not injective. By [5, Lemma 5.13 and Prop. 5.14] we have that if $\boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{v}, V)$, then $G(\boldsymbol{y}) \subset V$ and $G(\boldsymbol{y})$ is connected. Moreover, $\operatorname{im_T}(\boldsymbol{v}, V) \subset \bigcup_{\boldsymbol{x} \in V} \operatorname{im_T}(\boldsymbol{v}, \boldsymbol{x})$ and $G(\boldsymbol{y}) \cap V \neq \varnothing$ for every $\boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{v}, V)$. This means that whenever we take $\boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{v}, V)$, there is at least an $\boldsymbol{x} \in V$ such that $\boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{v}, \boldsymbol{x})$. By [49, Th. 7], we have $\mathcal{H}^1(T) = 0$. We will use this last property later in order to define the inverse. To sum up: ∂V is mapped by $\boldsymbol{v}$ onto $\partial \operatorname{im_T}(\boldsymbol{v}, V)$, while V is mapped in $\overline{\operatorname{im_T}(\boldsymbol{v}, V)}$, with the possibility that a point $\boldsymbol{x}$ is mapped to a point $\boldsymbol{y} \in \partial \operatorname{im_T}(\boldsymbol{v}, V)$. However, when this happens, there exists a set $G(\boldsymbol{y}) \subset \overline{V}$ such that $\boldsymbol{x} \in G(\boldsymbol{y})$ and $G(\boldsymbol{y}) \cap \partial V \neq \varnothing$. Roughly speaking, the deformation $\boldsymbol{v}$ may pinch an internal part of the domain to the boundary (see Figure 1).

Lemma 4.3. *Let A be a bounded domain of $\mathbb{R}^2$ and let $\boldsymbol{v} \in H^1(A, \mathbb{R}^2)$ be injective a.e. and such that $\det D\boldsymbol{v} > 0$ a.e. Moreover, let $V \in \mathcal{U}_{\boldsymbol{v}}$ and let be U an open set such that $U \Subset V$. Then $\operatorname{im_T}(\boldsymbol{v}, U) \subset \operatorname{im_T}(\boldsymbol{v}, V)$.*

Proof. By [47, Lemma 6.2], $\overline{\operatorname{im_T}(\boldsymbol{v}, U)} \subset \overline{\operatorname{im_T}(\boldsymbol{v}, V)}$. Assume by contradiction that there exists $\boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{v}, U)$ such that $\boldsymbol{y} \in \partial \operatorname{im_T}(\boldsymbol{v}, V)$. Let $\boldsymbol{x}_1 \in U$ and $\boldsymbol{x}_2 \in \partial V$ be such that $\boldsymbol{v}(\boldsymbol{x}_1) = \boldsymbol{v}(\boldsymbol{x}_2) = \boldsymbol{y}$. Since $\boldsymbol{v}$ is continuous and $\operatorname{im_T}(\boldsymbol{v}, U)$, $\operatorname{im_T}(\boldsymbol{v}, V)$ are open, there exist $\boldsymbol{x}_1' \in U$ and $\boldsymbol{x}_2' \in V \setminus \overline{U}$ such that $\boldsymbol{v}(\boldsymbol{x}_1') = \boldsymbol{v}(\boldsymbol{x}_2') = \boldsymbol{y}'$ for some $\boldsymbol{y}' \in \operatorname{im_T}(\boldsymbol{v}, U) \cap \operatorname{im_T}(\boldsymbol{v}, V)$ close to $\boldsymbol{y}$. Then, since $G(\boldsymbol{y}')$ is connected, there exists $\boldsymbol{x}_3' \in \partial U \cap G(\boldsymbol{y}')$. Therefore, $\boldsymbol{v}(\boldsymbol{x}_1') = \boldsymbol{v}(\boldsymbol{x}_3') \in \partial \operatorname{im_T}(\boldsymbol{v}, U)$, and the initial assumption must be false since $\operatorname{im_T}(\boldsymbol{v}, U)$ is open. □

Some of the properties of the topological image in the $2D$ case can be transposed to the axisymmetric setting, thanks to the following result.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^3$ be an axisymmetric domain. Let $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$ be axisymmetric and satisfy $\det D\boldsymbol{u} > 0$ a.e., and let $\boldsymbol{v}$ be its corresponding function. Let $U \subset \Omega$ be an axisymmetric open set such that $\overline{U} \cap \mathbb{R}\boldsymbol{e}_3 = \varnothing$. Then $\operatorname{im_T}(\boldsymbol{v}, \pi(U)) = \pi(\operatorname{im_T}(\boldsymbol{u}, U))$.*

Proof. Let $\boldsymbol{z} \in \mathbb{R}^2 \setminus \boldsymbol{v}(\partial \pi(U))$ and $\theta \in \mathbb{R}$. Let I be an open interval of length less than 2π containing θ. Set $U_I := \{(r, \theta', x_3) : (r, x_3) \in \pi(U),\ \theta' \in I\}$. By the product property for the degree (see, e.g., [10, Th. 8.7]),

$$\deg\left((v_1, \mathbf{id}_{\mathbb{R}}, v_2), U_I, (z_1, \theta, z_2)\right) = \deg\left(\boldsymbol{v}, \pi(U), \boldsymbol{z}\right) \deg(\mathbf{id}_{\mathbb{R}}, I, \theta) = \deg\left(\boldsymbol{v}, \pi(U), \boldsymbol{z}\right),$$

where $\mathbf{id}_{\mathbb{R}}$ is the identity map in $\mathbb{R}$. As $\boldsymbol{P} : \overline{U}_I \to \mathbb{R}^3$ can be extended to an orientation-preserving diffeomorphism in an open set containing $\overline{U}_I$ we obtain by the composition formula for the degree (see, e.g., [14, Th. 5.1]) that

$$\begin{aligned}\deg\left((v_1, \mathbf{id}_{\mathbb{R}}, v_2), U_I, (z_1, \theta, z_2)\right) &= \deg\left(\boldsymbol{P} \circ (v_1, \mathbf{id}_{\mathbb{R}}, v_2), U_I, \boldsymbol{P}(z_1, \theta, z_2)\right) \\ &= \deg\left(\boldsymbol{u} \circ \boldsymbol{P}, U_I, \boldsymbol{P}(z_1, \theta, z_2)\right),\end{aligned}$$

where in the last formula we have used (2.6). Applying again the composition formula we obtain

$$\deg\left(\boldsymbol{u} \circ \boldsymbol{P}, U_I, \boldsymbol{P}(z_1, \theta, z_2)\right) = \deg\left(\boldsymbol{u}, \boldsymbol{P}(U_I), \boldsymbol{P}(z_1, \theta, z_2)\right).$$

Altogether, we have shown that

$$\deg\left(\boldsymbol{v}, \pi(U), \boldsymbol{z}\right) = \deg\left(\boldsymbol{u}, \boldsymbol{P}(U_I), \boldsymbol{P}(z_1, \theta, z_2)\right).$$

Now we show that

$$\boldsymbol{P}(z_1, \theta, z_2) \notin \boldsymbol{u}\left(\overline{U} \setminus \boldsymbol{P}(U_I)\right). \tag{4.5}$$

Indeed, let $\boldsymbol{x} \in \overline{U} \setminus \boldsymbol{P}(U_I)$. Then $\boldsymbol{x} = \boldsymbol{P}(r\cos\theta', r\sin\theta', x_3)$ with $(r, x_3) \in \pi(\overline{U})$ and $\theta' \in \mathbb{R} \setminus (I + 2\pi\mathbb{Z})$. By (2.6), $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{P}(v_1(r, x_3), \theta', v_2(r, x_3))$, which implies (4.5). In turn, (4.5) and the excision property of the degree (see, e.g., [14, Th. 3.1] or [10, Th. 8.4]) yield

$$\deg\left(\boldsymbol{u}, \boldsymbol{P}(U_I), \boldsymbol{P}(z_1, \theta, z_2)\right) = \deg\left(\boldsymbol{u}, U, \boldsymbol{P}(z_1, \theta, z_2)\right),$$

which, together with (4.5) shows that

$$\deg\left(\boldsymbol{v}, \pi(U), \boldsymbol{z}\right) = \deg\left(\boldsymbol{u}, U, \boldsymbol{P}(z_1, \theta, z_2)\right). \tag{4.6}$$

Recapitulating, we have shown that if $\boldsymbol{z} \in \mathbb{R}^2 \setminus \boldsymbol{v}(\partial\pi(U))$ and $\theta \in \mathbb{R}$ then $\boldsymbol{P}(z_1, \theta, z_2) \notin \boldsymbol{u}(\partial U)$ and formula (4.6) holds.

Now let $\boldsymbol{z} \in \operatorname{im_T}(\boldsymbol{v}, \pi(U))$. Then $\boldsymbol{z} \notin \boldsymbol{v}(\partial\pi(U))$ and $\deg(\boldsymbol{v}, \pi(U), \boldsymbol{z}) \neq 0$. By (4.6), $\deg(\boldsymbol{u}, U, \boldsymbol{P}(z_1, \theta, z_2)) \neq 0$ for any $\theta \in \mathbb{R}$, so $\boldsymbol{P}(z_1, \theta, z_2) \in \operatorname{im_T}(\boldsymbol{u}, U)$. Consequently, $\boldsymbol{z} = (\pi \circ \boldsymbol{P})(z_1, \theta, z_2) \in \pi\left(\operatorname{im_T}(\boldsymbol{u}, U)\right)$.

To prove the converse inclusion, we start with the following simple facts:

i) If $(r, x_3) \in \pi(U)$ and $\theta \in \mathbb{R}$ then $\boldsymbol{P}(r, \theta, x_3) \in U$.
ii) If $(r, x_3) \notin \pi(U)$ and $\theta \in \mathbb{R}$ then $\boldsymbol{P}(r, \theta, x_3) \notin U$.
iii) If $(r, x_3) \in \partial\pi(U)$ and $\theta \in \mathbb{R}$ then $\boldsymbol{P}(r, \theta, x_3) \in \partial U$.

Property ii) is obvious, while i) is a consequence of the axisymmetry of U. Let us show iii). Let $(r, x_3) \in \partial\pi(U)$ and $\theta \in \mathbb{R}$. Note that $\pi(U)$ is an open set in $\mathbb{R}^2$, so

$$\partial\pi(U) = \overline{\pi(U)} \setminus \pi(U).$$

As $(r, x_3) \in \overline{\pi(U)}$, an elementary argument based on i) and the continuity of $\boldsymbol{P}$ shows that $\boldsymbol{P}(r, \theta, x_3) \in \overline{U}$. On the other hand, property ii) shows that $\boldsymbol{P}(r, \theta, x_3) \notin U$, so $\boldsymbol{P}(r, \theta, x_3) \in \partial U$ and iii) is proved.

We are in a position to show the inclusion $\pi(\operatorname{im_T}(\boldsymbol{u}, U)) \subset \operatorname{im_T}(\boldsymbol{v}, \pi(U))$. Let $\boldsymbol{z} \in \pi(\operatorname{im_T}(\boldsymbol{u}, U))$. Then there exists $\theta \in \mathbb{R}$ such that $\boldsymbol{P}(z_1, \theta, z_2) \in \operatorname{im_T}(\boldsymbol{u}, U)$. Therefore,

$$\boldsymbol{P}(z_1, \theta, z_2) \notin \boldsymbol{u}(\partial U) \text{ and } \deg\left(\boldsymbol{u}, U, \boldsymbol{P}(z_1, \theta, z_2)\right) \neq 0. \tag{4.7}$$

We shall show that $\boldsymbol{z} \notin \boldsymbol{v}(\partial\pi(U))$ by assuming that $\boldsymbol{z} = \boldsymbol{v}(r, x_3)$ for some $(r, x_3) \in \overline{\pi(U)}$. Due to (2.6),

$$\boldsymbol{P}(z_1, \theta, z_2) = \boldsymbol{P}(v_1(r, x_3), \theta, v_2(r, x_3)) = \boldsymbol{u} \circ \boldsymbol{P}(r, \theta, x_3),$$

so, by (4.7), $\boldsymbol{P}(r, \theta, x_3) \notin \partial U$, and, by iii), $\boldsymbol{z} \notin \boldsymbol{v}(\partial\pi(U))$. Thanks to (4.6) and (4.7), $\deg(\boldsymbol{v}, \pi(U), \boldsymbol{z}) \neq 0$, so $\boldsymbol{z} \in \operatorname{im_T}(\boldsymbol{v}, \pi(U))$. □

4.3. **Topological image of the singular segment.** Throughout this section assume that $\boldsymbol{b}$, Ω, $\widetilde{\Omega}$, and Ω_D are as in Section 2.4. Note that all $\boldsymbol{u}\in\mathcal{A}_s$ satisfy the properties stated in Lemma 4.6.

Away from the segment $L=\overline{\Omega}\cap\mathbb{R}\boldsymbol{e}_3$ condition INV is defined as follows.

Definition 4.5. Let U be a bounded open set in $\mathbb{R}^3$. If $\boldsymbol{u}\in C(U,\mathbb{R}^3)$, we say that $\boldsymbol{u}$ satisfies property (INV) in U provided that for every point $\boldsymbol{x}_0\in U$ and a.e. $r\in(0,\operatorname{dist}(\boldsymbol{x}_0,\partial U))$:
(a) $\boldsymbol{u}(\boldsymbol{x})\in\operatorname{im_T}(\boldsymbol{u},B(\boldsymbol{x}_0,r))$ for a.e. $\boldsymbol{x}\in B(\boldsymbol{x}_0,r)$
(b) $\boldsymbol{u}(\boldsymbol{x})\notin\operatorname{im_T}(\boldsymbol{u},B(\boldsymbol{x}_0,r))$ for a.e. $\boldsymbol{x}\in\widetilde{\Omega}\setminus B(\boldsymbol{x}_0,r)$.

The degree of any map $\boldsymbol{u}$ in $\mathcal{A}_s$ with respect to any open set U separated from the symmetry axis coincides a.e. with the number of preimages (at which $\boldsymbol{u}$ is approximately differentiable) by $\boldsymbol{u}$ in that open set. This is shown now and relies on the fine regularity properties satisfied away from the axis and on the preservation of orientation.

Lemma 4.6. *Suppose that $\boldsymbol{u}\in H^1(\widetilde{\Omega},\mathbb{R}^3)$ is axisymmetric, satisfies $\det D\boldsymbol{u}>0$ a.e. and $\boldsymbol{u}=\boldsymbol{b}$ in Ω_D. Then:*
(a) $\boldsymbol{u}$ is continuous in $\widetilde{\Omega}\setminus L$ and $\mathcal{E}(\boldsymbol{u},\widetilde{\Omega}\setminus L)=0$.
(b) For any $U\in\mathcal{U}_{\boldsymbol{u}}$ (see Definition 2.12) such that $\overline{U}\cap L=\varnothing$

$$\deg(\boldsymbol{u},U,\cdot)=\mathcal{N}(\boldsymbol{u},\Omega_d\cap U,\cdot)\;\; a.e. \quad and \quad \operatorname{im_G}(\boldsymbol{u},U)=\operatorname{im_T}(\boldsymbol{u},U)\;\; a.e., \tag{4.8}$$

where Ω_d is the set of approximate differentiability.
(c) $\boldsymbol{u}$ satisfies condition (INV) in $\widetilde{\Omega}\setminus L$ if and only if $\boldsymbol{u}$ is injective a.e. In particular, all maps in $\mathcal{A}_s$ satisfy (INV) in $\widetilde{\Omega}\setminus L$.

Proof. Part (a) follows from Lemma 2.18 and [40, Lemma 4.8]. Parts (b) and (c) can be obtained with the same proof of [5, Th. 4.1 and Lemma 5.1.(a)]. ☐

Recall that $\Omega_{\boldsymbol{b}}:=\boldsymbol{b}(\Omega)$ and $\widetilde{\Omega}_{\boldsymbol{b}}:=\boldsymbol{b}(\widetilde{\Omega})$.

Definition 4.7. Let $\boldsymbol{u}\in\mathcal{A}_s$ and let $\mathcal{U}^s_{\boldsymbol{u}}:=\{U\in\mathcal{U}_{\boldsymbol{u}}$ is axisymmetric and $U\Subset\widetilde{\Omega}\setminus\mathbb{R}\boldsymbol{e}_3\}$.
a) We define the topological image of $\widetilde{\Omega}\setminus L$ by $\boldsymbol{u}$ as

$$\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus L):=\boldsymbol{b}(\Omega_D)\cup\bigcup_{U\in\mathcal{U}^s_{\boldsymbol{u}}}\operatorname{im_T}(\boldsymbol{u},U).$$

b) We define the topological image of L by $\boldsymbol{u}$ as

$$\operatorname{im_T}(\boldsymbol{u},L):=\widetilde{\Omega}_{\boldsymbol{b}}\setminus\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus L).$$

Lemma 4.8. *If $\boldsymbol{u}\in\mathcal{A}_s$, then:*
(a) $\boldsymbol{u}(\boldsymbol{x})\in\overline{\Omega_{\boldsymbol{b}}}$ for every $\boldsymbol{x}\in\Omega\setminus L$,
(b) $\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus L)\subset\widetilde{\Omega}_{\boldsymbol{b}}$, and
(c) $\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus L)=\operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega}\setminus L)=\operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega})$ a.e.

Proof. Part (a): the map $\boldsymbol{u}$ cannot send a point $\boldsymbol{x}\in\Omega\setminus L$ in $\boldsymbol{b}(\Omega_D)$. Indeed, by continuity, a ball centered at $\boldsymbol{x}$ should be mapped into $\boldsymbol{b}(\Omega_D)$, against the fact that $\boldsymbol{u}$ is injective a.e.

Let us show that $\boldsymbol{u}$ cannot send a point $\boldsymbol{x}\in\Omega\setminus L$ outside $\widetilde{\Omega}_{\boldsymbol{b}}$. Observe that Ω_D wraps Ω: for $\eta>0$ small enough, $\{\boldsymbol{x}\in\mathbb{R}^3\setminus\overline{\Omega}\colon \operatorname{dist}(\boldsymbol{x},\overline{\Omega})<\eta\}\subset\Omega_D$. Since $\boldsymbol{b}$ is a homeomorphism, $\boldsymbol{b}(\Omega_D)$ wraps $\Omega_{\boldsymbol{b}}$: for $\eta>0$ small enough, $\{\boldsymbol{y}\in\mathbb{R}^3\setminus\overline{\Omega_{\boldsymbol{b}}}\colon \operatorname{dist}(\boldsymbol{y},\overline{\Omega_{\boldsymbol{b}}})<\eta\}\subset\boldsymbol{b}(\Omega_D)$. Therefore, in order to 'exit' from $\Omega_{\boldsymbol{b}}$ one has to pass through $\boldsymbol{b}(\Omega_D)$.

Given $\boldsymbol{x}\in\Omega\setminus L$, let γ be a continuous curve with endpoints $\boldsymbol{x}$ and $\boldsymbol{x}'\in\partial\Omega$ such that $\gamma\setminus\{\boldsymbol{x}'\}\subset\Omega\setminus L$. We have that $\boldsymbol{u}(\gamma)$ is connected since $\boldsymbol{u}$ is continuous in $\widetilde{\Omega}\setminus L$. Moreover,

$\boldsymbol{u}(\boldsymbol{x}') \in \boldsymbol{u}(\partial\Omega) \Subset \widetilde{\Omega}_{\boldsymbol{b}}$. If $\boldsymbol{u}(\boldsymbol{x}) \notin \widetilde{\Omega}_{\boldsymbol{b}}$, then $\boldsymbol{u}(\gamma)$ has to cross $\boldsymbol{b}(\Omega_D)$. This implies that $\boldsymbol{u}$ maps at least one point $\boldsymbol{x}'' \in \Omega$ in $\boldsymbol{b}(\Omega_D)$. This contradicts what was proved at the beginning, finishing the proof of (a).

Part (b): let $\boldsymbol{v}$ be the planar function corresponding to $\boldsymbol{u}$. Given $U \in \mathcal{U}_{\boldsymbol{u}}^s$, let $V \in \mathcal{U}_{\boldsymbol{v}}$ be such that and $\pi(U) \subset V \Subset \pi(\widetilde{\Omega} \setminus \mathbb{R}\boldsymbol{e}_3)$. Thanks to what was observed in Subsection 4.2, $\operatorname{im_T}(\boldsymbol{v}, V) \subset \boldsymbol{v}(V)$. Moreover, by Lemmas 4.3 and 4.4 we have

$$\pi(\operatorname{im_T}(\boldsymbol{u}, U)) = \operatorname{im_T}(\boldsymbol{v}, \pi(U)) \subset \operatorname{im_T}(\boldsymbol{v}, V).$$

Therefore, by (a), $\operatorname{im_T}(\boldsymbol{u}, U) \subset \widetilde{\Omega}_{\boldsymbol{b}}$.

Part (c): for any $k \in \mathbb{N}$, let $U_k \in \mathcal{U}_{\boldsymbol{u}}^s$ be an axisymmetric open set containing

$$\{\boldsymbol{x} \in \widetilde{\Omega} \colon \operatorname{dist}(\boldsymbol{x}, \mathbb{R}\boldsymbol{e}_3) > 1/k\}$$

and such that $V_k := \pi(U_k) \in \mathcal{U}_{\boldsymbol{v}}$. Any set $U \in \mathcal{U}_{\boldsymbol{u}}^s$ is included in some U_k. Moreover, since by Lemma 4.3 $\operatorname{im_T}(\boldsymbol{v}, \pi(U)) \subset \operatorname{im_T}(\boldsymbol{v}, V_k)$, by Lemma 4.4 we have $\operatorname{im_T}(\boldsymbol{u}, U) \subset \operatorname{im_T}(\boldsymbol{u}, U_k)$. This proves that

$$\bigcup_{U \in \mathcal{U}_{\boldsymbol{u}}^s} \operatorname{im_T}(\boldsymbol{u}, U) = \bigcup_{k \in \mathbb{N}} \operatorname{im_T}(\boldsymbol{u}, U_k).$$

By Lemma 4.6, $\operatorname{im_T}(\boldsymbol{u}, U_k) = \operatorname{im_G}(\boldsymbol{u}, U_k)$ a.e. for all $k \in \mathbb{N}$. Hence, since $|L| = 0$,

$$\begin{aligned} \operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega}) &\overset{\text{a.e.}}{=} \operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega} \setminus L) = \boldsymbol{b}(\Omega_D) \cup \bigcup_{k \in \mathbb{N}} \operatorname{im_G}(\boldsymbol{u}, U_k) \\ &\overset{\text{a.e.}}{=} \boldsymbol{b}(\Omega_D) \cup \bigcup_{k \in \mathbb{N}} \operatorname{im_T}(\boldsymbol{u}, U_k) = \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L), \end{aligned}$$

where Lemma 2.4 was used in the first equality. □

Note that $\operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L)$ is open as a union of open sets and hence $\operatorname{im_T}(\boldsymbol{u}, L)$ is closed. Also, by Lemma 4.8.(b),

$$\operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L) = \widetilde{\Omega}_{\boldsymbol{b}} \setminus \operatorname{im_T}(\boldsymbol{u}, L). \tag{4.9}$$

For example, in the construction of Conti & De Lellis [11] $\operatorname{im_T}(\boldsymbol{u}, L)$ consists (apart from the corresponding segment in the symmetry axis) of the sphere $\partial B\big((0, 0, \frac{1}{2}), \frac{1}{2}\big)$, which may be regarded as new surface inside the elastic body created by the singular map $\boldsymbol{u}$.

4.4. **Sobolev regularity of inverses.** As $\boldsymbol{b} : \tilde{\Omega} \to \mathbb{R}^3$ is a diffeomorphism, $\Omega_{\boldsymbol{b}} = \operatorname{im_T}(\boldsymbol{b}, \Omega)$. Moreover, for $\boldsymbol{u} \in \mathcal{A}_s$, as $\boldsymbol{u} = \boldsymbol{b}$ in Ω_D, the traces of $\boldsymbol{u}$ and $\boldsymbol{b}$ on $\partial\Omega$ coincide. As $\boldsymbol{b}|_{\partial\Omega}$ is continuous, it is a representative of $\boldsymbol{u}|_{\partial\Omega}$, hence the degree $\deg(\boldsymbol{u}, \Omega, \cdot)$ is defined and equals $\deg(\boldsymbol{b}, \Omega, \cdot)$. In particular,

$$\Omega_{\boldsymbol{b}} = \boldsymbol{b}(\Omega) = \operatorname{im_T}(\boldsymbol{b}, \Omega) = \operatorname{im_T}(\boldsymbol{u}, \Omega), \qquad \widetilde{\Omega}_{\boldsymbol{b}} = \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega}). \tag{4.10}$$

Let Ω_0 be the set of Definition 2.2. Note that since maps in $\mathcal{A}_s$ are defined in $\widetilde{\Omega}$, then Ω_0 also contains points outside Ω; in fact, it is of full measure in $\widetilde{\Omega}$. It was proved in [28, Lemma 3] that if $\boldsymbol{u}$ is one-to-one a.e. then $\boldsymbol{u}|_{\Omega_0}$ is injective.

Definition 4.9. Let $\boldsymbol{u} \in \mathcal{A}_s$. We define its inverse as the map $\boldsymbol{u}^{-1} : \operatorname{im_G}(\boldsymbol{u}, \Omega) \to \mathbb{R}^3$ that sends every $\boldsymbol{y} \in \operatorname{im_G}(\boldsymbol{u}, \Omega)$ to the only $\boldsymbol{x} \in \Omega_0$ such that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}$.

Let A be a bounded domain of $\mathbb{R}^2$ and let $\boldsymbol{v} \in H^1(A, \mathbb{R}^2)$ be injective a.e. and such that $\det D\boldsymbol{v} > 0$ a.e. From the comments in Subsection 4.2, given $V \in \mathcal{U}_{\boldsymbol{v}}$, we can define on $\operatorname{im_T}(\boldsymbol{v}, V)$

$$\boldsymbol{v}^{-1}(\boldsymbol{y}) = \text{any element of } G(\boldsymbol{y}).$$

The definition is well posed; moreover, for every $\boldsymbol{y} \in \operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, V) \setminus T$ we have $\boldsymbol{y} = \boldsymbol{v}(\boldsymbol{v}^{-1}(\boldsymbol{y}))$. Finally, by [49, Lemma 6] we have $\boldsymbol{v}^{-1}$ is continuous at any point of $\operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, V) \setminus T$, while by [49, Th. 8] we have $\boldsymbol{v}^{-1} \in W^{1,1}(\operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, V), \mathbb{R}^2)$ and

$$D\boldsymbol{v}^{-1}(\boldsymbol{v}(\boldsymbol{x})) = D\boldsymbol{v}(\boldsymbol{x})^{-1} = \frac{\operatorname{adj} D\boldsymbol{v}(\boldsymbol{x})}{\det D\boldsymbol{v}(\boldsymbol{x})} \quad \text{for a.e. } \boldsymbol{x} \in V.$$

All the properties above can be transposed to the axisymmetric case. Let Ω be an axisymmetric domain, $\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3)$ be axisymmetric, and let $\boldsymbol{v}$ be its corresponding planar function. Assume that $\boldsymbol{u}$ is injective a.e. and that $\det D\boldsymbol{u} > 0$ a.e. in Ω. By Lemma 2.15 the same properties hold for $\boldsymbol{v}$ in $\pi(\Omega)$. Moreover, by Lemma 2.14, if $U \subset \Omega$ is an axisymmetric open set such that $\overline{U} \cap \mathbb{R}\boldsymbol{e}_3 = \varnothing$, then $\boldsymbol{v} \in H^1(\pi(U), \mathbb{R}^2)$. Thanks to Lemma 2.18, we will consider a representative of $\boldsymbol{u}$ that is continuous at each point of $\Omega \setminus \mathbb{R}\boldsymbol{e}_3$, and a representative of $\boldsymbol{v}$ that is continuous at each point of $\pi(\Omega) \setminus (\{0\} \times \mathbb{R})$. We have $\pi(U) \subset V$ for some $V \in \mathcal{U}_{\boldsymbol{v}}$ (using the analogue of Lemma 2.13 in $2D$). Let $\boldsymbol{v}^{-1}$ be the map defined in $\operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, V)$ as above. Recalling that by Lemma 4.4, $\operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, \pi(U)) = \pi(\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U))$, while by Lemma 4.3, $\operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, \pi(U)) \subset \operatorname{im}_{\mathrm{T}}(\boldsymbol{v}, V)$, we can define on $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U)$ a map $\boldsymbol{u}^{-1}$ through formula (2.6), changing $\boldsymbol{u}, \boldsymbol{v}$ by $\boldsymbol{u}^{-1}, \boldsymbol{v}^{-1}$, respectively. Let R be the axisymmetric set such that $\pi(R) = T$; we have $\mathcal{H}^2(R) = 0$ and for every $\boldsymbol{y} \in \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U) \setminus R$ we have $\boldsymbol{y} = \boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))$. Since by (4.8), $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) = \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U)$ a.e., if $\boldsymbol{u} \in \mathcal{A}_s$, then $\boldsymbol{u}^{-1}$ in $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U)$ is a specific representative of the inverse defined in Definition 4.9. As in Lemma 2.18, we obtain that $\boldsymbol{u}^{-1}$ is continuous at any point of $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U) \setminus R$. Moreover, as in Lemma 2.14, we obtain that $\boldsymbol{u}^{-1} \in W^{1,1}_{\mathrm{loc}}(\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U), \mathbb{R}^3)$ and

$$D\boldsymbol{u}^{-1}(\boldsymbol{u}(\boldsymbol{x})) = D\boldsymbol{u}(\boldsymbol{x})^{-1} = \frac{\operatorname{adj} D\boldsymbol{u}(\boldsymbol{x})}{\det D\boldsymbol{u}(\boldsymbol{x})} \quad \text{for a.e. } \boldsymbol{x} \in U. \tag{4.11}$$

Lemma 4.10. *Let $\boldsymbol{u} \in \mathcal{A}_s$. Then $\boldsymbol{u}^{-1} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}} \setminus \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, L), \mathbb{R}^3)$ and formula* (4.11) *holds for a.e. $\boldsymbol{x} \in \widetilde{\Omega}$.*

Proof. Let $\{U_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{U}^s_{\boldsymbol{u}}$ as in the proof of Lemma 4.8. From what we wrote before, $\boldsymbol{u}^{-1} \in W^{1,1}(\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U_k), \mathbb{R}^3)$, and this implies that $\boldsymbol{u}^{-1} \in W^{1,1}_{\mathrm{loc}}(\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, \widetilde{\Omega} \setminus L), \mathbb{R}^3)$. Using (4.9) we find that $\boldsymbol{u}^{-1} \in W^{1,1}_{\mathrm{loc}}(\widetilde{\Omega}_{\boldsymbol{b}} \setminus \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, L), \mathbb{R}^3)$. Moreover, formula (4.11) holds for a.e. $\boldsymbol{x} \in U_k$, from which we get immediately that it also holds for a.e. $\boldsymbol{x} \in \widetilde{\Omega}$. Using that formula, and a change of variables, we find that

$$\int_{\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, \widetilde{\Omega} \setminus L)} |D\boldsymbol{u}^{-1}(\boldsymbol{y})| \, \mathrm{d}\boldsymbol{y} = \int_{\widetilde{\Omega}} |\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} < \infty$$

since $\boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$. Therefore, $\boldsymbol{u}^{-1} \in W^{1,1}(\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, \widetilde{\Omega} \setminus L), \mathbb{R}^3)$. □

We now prove that when $\boldsymbol{u}$ has zero surface energy in $\widetilde{\Omega}$ then also the geometric image of $\widetilde{\Omega}$ (not only its topological image) coincides with $\widetilde{\Omega}_{\boldsymbol{b}}$ (up to a Lebesgue-null set). The first step for the proof is to establish (4.8) also for open sets U enclosing the singular segment L.

Proposition 4.11. *Suppose that $\boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$ is axisymmetric and satisfies $\det D\boldsymbol{u} > 0$ a.e., $\boldsymbol{u} = \boldsymbol{b}$ in Ω_D, and $\mathcal{E}(\boldsymbol{u}) = 0$ in $\widetilde{\Omega}$. Then*

(a) $\boldsymbol{u} \in L^\infty(\Omega, \mathbb{R}^3)$, $\operatorname{Det} D\boldsymbol{u} = \det D\boldsymbol{u}$, and $\boldsymbol{u}$ is injective a.e.

(b) $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \widetilde{\Omega}) = \widetilde{\Omega}_{\boldsymbol{b}}$ a.e.

(c) For any $U \in \mathcal{U}_{\boldsymbol{u}}$,

$$\deg(\boldsymbol{u}, U, \cdot) = \chi_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)} \quad \text{a.e.} \tag{4.12}$$

In particular, when $\partial U \cap L = \varnothing$ (so that the classical degree and the topological image is well defined),

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) = \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U) \quad \text{a.e.} \tag{4.13}$$

Proof. The results were proved in [5, Th. 4.1] for maps $\boldsymbol{u} \in W^{1,p}$ with $p > 2$. Here we only explain the very minor modifications needed for the generalization to our H^1 setting. Now that we have $\mathcal{E}(\boldsymbol{u}) = 0$ in $\widetilde{\Omega}$ (as opposed to Lemma 4.6 where essentially any axisymmetric map was considered, so that $\mathcal{E}(\boldsymbol{u}) = 0$ only in $\widetilde{\Omega} \setminus L$), arguing exactly as in [5, Th. 4.1] we obtain both (4.13) and

$$\deg(\boldsymbol{u}, U, \cdot) = \mathcal{N}(\boldsymbol{u}, \Omega_d \cap U, \cdot) \quad \text{a.e.}, \tag{4.14}$$

where Ω_d is the set of approximate differentiability, for any $U \in \mathcal{U}_{\boldsymbol{u}}$ such that $\partial U \cap L = \varnothing$ (as opposed to Lemma 4.6 where the stronger restriction $\overline{U} \cap L = \varnothing$ was imposed). In particular, applying (4.14) to any $U \in \mathcal{U}_{\boldsymbol{u}}$ such that $\Omega \Subset U \Subset \widetilde{\Omega}$, we find that for a.e. $\boldsymbol{x} \in \Omega_0$

$$\boldsymbol{u}(\boldsymbol{x}) \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) \overset{\text{a.e.}}{=} \operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, U) = \operatorname{im}_{\mathrm{T}}(\boldsymbol{b}, U) \subset \widetilde{\Omega}_{\boldsymbol{b}},$$

thus proving the L^∞ bound. In addition,

$$\mathcal{N}(\boldsymbol{u}, \Omega_d \cap U, \cdot) \overset{\text{a.e.}}{=} \deg(\boldsymbol{u}, U, \cdot) = \deg(\boldsymbol{b}, U, \cdot).$$

As $\boldsymbol{b}$ is an orientation-preserving diffeomorphism,

$$\deg(\boldsymbol{b}, U, \cdot) = \begin{cases} 1 & \text{in } \boldsymbol{b}(U), \\ 0 & \text{in } \mathbb{R}^3 \setminus \boldsymbol{b}(\overline{U}), \\ \text{undefined} & \text{in } \boldsymbol{b}(\partial U) \end{cases}$$

and $\boldsymbol{b}(\partial U)$ has measure zero. We conclude that $\mathcal{N}(\boldsymbol{u}, \Omega_d \cap U, \cdot) = \deg(\boldsymbol{b}, U, \cdot) = \chi_{\boldsymbol{b}(U)}$ a.e., which implies that $\boldsymbol{u}$ is injective a.e. in U. As this is true for all $U \in \mathcal{U}_{\boldsymbol{u}}$ with $\Omega \Subset U$, and $\widetilde{\Omega}$ can be written as the union of countably many such U, we conclude that $\boldsymbol{u}$ is injective a.e. and $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \widetilde{\Omega}) = \boldsymbol{b}(\widetilde{\Omega})$ a.e. Since the L^∞ bound has already been established, the identity $\operatorname{Det} D\boldsymbol{u} = \det D\boldsymbol{u}$ can be proved exactly as in [5, Th. 4.1].

Take now an arbitrary $U \in \mathcal{U}_{\boldsymbol{u}}$ (on whose boundary $\boldsymbol{u}$ is not necessarily continuous). Proceeding as in [5, Thm. 4.1] one still obtains that there exists $c \in \mathbb{Z}$ such that

$$\mathcal{N}(\boldsymbol{u}, \Omega_d \cap U, \cdot) - \deg(\boldsymbol{u}, U, \cdot) = c \quad \text{a.e.},$$

where $\deg(\boldsymbol{u}, U, \cdot)$ is now the Brezis–Nirenberg degree (see Definition 4.1 and the remark after it). Since $\boldsymbol{u}$ is injective a.e., the first term coincides a.e. (see Lemma 2.4) with $\chi_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)}$. As $\boldsymbol{u} \in L^\infty(\Omega, \mathbb{R}^3)$, there exists a set $N \subset \Omega$ of measure zero such that $\boldsymbol{u}(\Omega \setminus N) \subset B(\boldsymbol{0}, \|\boldsymbol{u}\|_{L^\infty})$. Therefore, $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \Omega) \subset B(\boldsymbol{0}, \|\boldsymbol{u}\|_{L^\infty}) \cup \boldsymbol{u}(\Omega_0 \cap N)$. The set $\boldsymbol{u}(\Omega_0 \cap N)$ has measure zero thanks to Lemma 2.4. Thus, $\chi_{\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U)} = 0$ a.e. outside $\mathbb{R}^3 \setminus B(\boldsymbol{0}, \|\boldsymbol{u}\|_{L^\infty})$. By Definition 4.1, $\deg(\boldsymbol{u}, \partial U, \cdot) = 0$ a.e. outside $B(\boldsymbol{0}, \|\boldsymbol{u}\|_{L^\infty})$. Consequently, $c = 0$ and (4.12) holds. □

Next, we show that the inverse of a map $\boldsymbol{u}$ in $\mathcal{A}_s^r$ has $W^{1,1}$ regularity. This follows essentially from [30, Th. 3.4]; nevertheless, some clarifying statements are in order due to the potential lack of continuity of $\boldsymbol{u}$ on L.

Proposition 4.12. *Let $\boldsymbol{u} \in \mathcal{A}_s^r$. Then $\boldsymbol{u}^{-1} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$.*

Proof. Following [11], in the H^1 setting the sets

$$A_{\boldsymbol{u},U} := \{\boldsymbol{y} \in \mathbb{R}^3 : \deg(\boldsymbol{u}, U, \boldsymbol{y}) \neq 0\}$$

are given an auxiliary role, the actual topological images being now defined as

$$\operatorname{im}_{\mathrm{T}}^{\mathrm{BN}}(\boldsymbol{u}, U) := \{\boldsymbol{y} \in \mathbb{R}^3 : D\big(A_{\boldsymbol{u},U}, \boldsymbol{y}\big) = 1\}.$$

The superscript 'BN' has been added to indicate that use is made of the Brezis–Nirenberg degree. That notation and definition will only appear in this proof.

Part I: if $\boldsymbol{u} \in \mathcal{A}_s^r$ then for any $U \in \mathcal{U}_{\boldsymbol{u}}$,

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) = \operatorname{im}_{\mathrm{T}}^{\mathrm{BN}}(\boldsymbol{u}, U) \text{ a.e.} \tag{4.15}$$

The proof consists in recalling (4.12), which implies that

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) = A_{\boldsymbol{u},U} \quad \text{a.e.} \tag{4.16}$$

Now, by Lebesgue's differentiation theorem,

$$\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U) = \{\boldsymbol{y} \in \mathbb{R}^3 : D(\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, U), \boldsymbol{y}) = 1\} \quad \text{a.e.}$$

Using (4.16) as well, we conclude (4.15).

Part II: maps in $\mathcal{A}_s^r$ satsify INV in the whole $\widetilde{\Omega}$. Let $U \in \mathcal{U}_{\boldsymbol{u}}$ and assume that $\boldsymbol{u}|_{U \setminus N}$ is injective for some set $N \subset U$ of measure zero. Take $\boldsymbol{a} \in U$ and define $r_{\boldsymbol{a}} := \operatorname{dist}(\boldsymbol{a}, \partial U)$. Then $B(\boldsymbol{a}, r) \in \mathcal{U}_{\boldsymbol{u}|_U}$ for a.e. $r \in (0, r_{\boldsymbol{a}})$. Fix any such r. For all $\boldsymbol{x} \in B(\boldsymbol{a}, r) \cap \Omega_0$ we have that $\boldsymbol{u}(\boldsymbol{x}) \in \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, B(\boldsymbol{a}, r))$. By (4.15) and Lemma 2.4, we infer that $\boldsymbol{u}(\boldsymbol{x}) \in \operatorname{im}_{\mathrm{T}}^{\mathrm{BN}}(\boldsymbol{u}, B(\boldsymbol{a}, r))$ for a.e. $\boldsymbol{x} \in B(\boldsymbol{a}, r)$. Now, for all $\boldsymbol{x} \in U \setminus (B(\boldsymbol{a}, r) \cup N)$, by the injectivity, $\boldsymbol{u}(\boldsymbol{x}) \notin \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, B(\boldsymbol{a}, r))$. As before, $\boldsymbol{u}(\boldsymbol{x}) \notin \operatorname{im}_{\mathrm{T}}^{\mathrm{BN}}(\boldsymbol{u}, B(\boldsymbol{a}, r))$ for a.e. $\boldsymbol{x} \in U \setminus B(\boldsymbol{a}, r)$. We have then shown that $\boldsymbol{u}|_U$ satisfies the condition INV for H^1 maps (with the topological images defined using the Brezis–Nirenberg degree). As $\widetilde{\Omega}$ can be written as the union of countably many $U \in \mathcal{U}_{\boldsymbol{u}}$, we conclude that $\boldsymbol{u}$ satisfies condition INV.

Part III: regularity of the inverse when $\boldsymbol{u} \in \mathcal{A}_s^r$. Once condition INV for H^1 maps has been established, we obtain by [30, Th. 3.4] that the extension of $\boldsymbol{u}^{-1}$ by zero to all $\mathbb{R}^3$ is in SBV and the restriction of $D\boldsymbol{u}^{-1}$ to $\operatorname{im}_{\mathrm{T}}^{\mathrm{BN}}(\boldsymbol{u}, U)$ is absolutely continuous with respect to the Lebesgue measure for any $U \in \mathcal{U}_{\boldsymbol{u}}$. Apply this to any $U \in \mathcal{U}_{\boldsymbol{u}}$ such that $\Omega \Subset U \Subset \widetilde{\Omega}$. Since $\boldsymbol{b}$ is a C^1 orientation-preserving diffeomorphism, applying the measure-theoretic inverse function theorem in [44, Lemma 2.5] we find that $\operatorname{im}_{\mathrm{T}}^{\mathrm{BN}}(\boldsymbol{b}, U) = \operatorname{im}_{\mathrm{T}}(\boldsymbol{b}, U)$. As $\operatorname{im}_{\mathrm{T}}(\boldsymbol{b}, U)$ is open, we obtain that $\boldsymbol{u}^{-1} \in W^{1,1}(\operatorname{im}_{\mathrm{T}}(\boldsymbol{b}, U), \mathbb{R}^3)$. Since $\boldsymbol{u}^{-1} = \boldsymbol{b}^{-1}$ in $\widetilde{\Omega}_{\boldsymbol{b}} \setminus \operatorname{im}_{\mathrm{T}}(\boldsymbol{b}, U)$, it follows that $\boldsymbol{u}^{-1} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$, as desired. □

4.5. **Pointwise convergence of inverses.** The following result shows that the inverse is stable under the weak limit in H^1.

Lemma 4.13. *For each $j \in \mathbb{N}$, let $\boldsymbol{u}_j, \boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3)$ be axisymmetric. Assume that $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $H^1(\widetilde{\Omega}, \mathbb{R}^3)$ as $j \to \infty$. Suppose that $\det D\boldsymbol{u}_j > 0$ a.e. for all $j \in \mathbb{N}$ and $\det D\boldsymbol{u} > 0$ a.e., and that $\boldsymbol{u}_j$ and $\boldsymbol{u}$ are invertible a.e. Then $\boldsymbol{u}_j^{-1} \to \boldsymbol{u}^{-1}$ a.e.*

Proof. By Lemma 2.16, $\boldsymbol{u}$ is axisymmetric. Let $\boldsymbol{v}_j$ and $\boldsymbol{v}$ be the corresponding $2D$ functions to $\boldsymbol{u}_j$ and $\boldsymbol{u}$, respectively. By Lemmas 2.14, 2.15 and 2.18, $\det D\boldsymbol{v}_j > 0$ a.e., $\boldsymbol{v}_j$ is injective a.e., $\boldsymbol{v}_j \in H^1(\pi(\widetilde{\Omega}) \setminus ([0, \delta] \times \mathbb{R}))$ for any $\delta > 0$, and analogously for $\boldsymbol{v}$. Moreover, $\boldsymbol{v}_j \rightharpoonup \boldsymbol{v}$ in $H^1(\pi(\widetilde{\Omega}) \setminus ([0, \delta] \times \mathbb{R}))$ for each $\delta > 0$, and $\mathcal{E}(\boldsymbol{v}_j, \pi(\widetilde{\Omega}) \setminus (\{0\} \times \mathbb{R})) = 0$, and analogously for $\boldsymbol{v}$. By [5, Th. 6.3], $\boldsymbol{v}_j^{-1} \to \boldsymbol{v}^{-1}$ a.e. Arguing as in Lemma 2.16 (on the inverses) shows that $\boldsymbol{u}_j^{-1} \to \boldsymbol{u}^{-1}$ a.e. □

4.6. **The horizontal components of the inverse have no singular parts on $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, L)$.** For general maps in $\mathcal{A}_s$ the equality $\widetilde{\Omega}_{\boldsymbol{b}} = \operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \widetilde{\Omega})$ does not hold in general, as can be seen by the classical example of a radial cavitation. Furthermore, even when $\operatorname{im}_{\mathrm{G}}(\boldsymbol{u}, \widetilde{\Omega})$ does coincide a.e. with $\widetilde{\Omega}_{\boldsymbol{b}}$, the inverse is not necessarily in $W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ as shown by the example of Conti & De Lellis [11] where $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u}, L)$ consists (apart from the symmetry axis) of the sphere $\partial B\big((0, 0, \frac{1}{2}), \frac{1}{2}\big)$ in the deformed configuration and $\boldsymbol{u}^{-1}$ has a jump across this sphere. Nevertheless, in the following lemma we show that any such singularities in $\boldsymbol{u}^{-1}$ are due to the vertical component u_3^{-1} of the inverse, whereas its horizontal components u_1^{-1} and u_2^{-1} enjoy a Sobolev regularity.

From now on, for $\alpha \in \{1,2,3\}$, we denote by u_α^{-1} the α-th component of $\boldsymbol{u}^{-1}$. Recall that equalities (4.10) hold and that

$$\widetilde{\Omega}_{\boldsymbol{b}} = \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L) \cup \operatorname{im_T}(\boldsymbol{u}, L) \tag{4.17}$$

by Definition 4.7 and Lemma 4.8.(b). We shall need the following gluing theorem for BV functions [1, Th. 3.84].

Proposition 4.14. *Let $N, m \geq 1$. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $\boldsymbol{u}, \boldsymbol{v} \in BV(\Omega, \mathbb{R}^m)$ and let E be a set of finite perimeter in Ω, with $\partial^* E \cap \Omega$ oriented by $\boldsymbol{\nu}_E$. Let $\boldsymbol{u}^+_{\partial^* E}$, $\boldsymbol{v}^-_{\partial^* E}$ be the traces of $\boldsymbol{u}$ and $\boldsymbol{v}$ on $\partial^* E$, which are defined for $\mathcal{H}^2$-a.e. point of $\partial^* E$. Set $\boldsymbol{w} = \boldsymbol{u}\chi_E + \boldsymbol{v}\chi_{\Omega \setminus E}$. Then*

$$\boldsymbol{w} \in BV(\Omega, \mathbb{R}^m) \quad \Leftrightarrow \quad \int_{\partial^* E \cap \Omega} |\boldsymbol{u}^+_{\partial^* E} - \boldsymbol{v}^-_{\partial^* E}| \, \mathrm{d}\mathcal{H}^{N-1} < \infty,$$

and in this case,

$$D\boldsymbol{w} = D\boldsymbol{u} \llcorner E^1 + (\boldsymbol{u}^+_{\partial^* E} - \boldsymbol{v}^-_{\partial^* E}) \otimes \boldsymbol{\nu}_E \mathcal{H}^{N-1} \llcorner (\partial^* E \cap \Omega) + D\boldsymbol{v} \llcorner E^0,$$

where E^0 and E^1, respectively, denote the set of points at which E has density 0 and 1.

Proposition 4.15. *Let $\boldsymbol{u} \in \mathcal{A}_s$ and $\alpha = 1, 2$. Denote by $\widehat{u_\alpha^{-1}} : \widetilde{\Omega}_{\boldsymbol{b}} \to \mathbb{R}$ the map*

$$\widehat{u_\alpha^{-1}} = \begin{cases} u_\alpha^{-1} & \text{in } \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L) = \widetilde{\Omega}_b \setminus \operatorname{im_T}(\boldsymbol{u}, L), \\ 0 & \text{in } \operatorname{im_T}(\boldsymbol{u}, L). \end{cases}$$

Then, $\widehat{u_\alpha^{-1}} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}})$ and $\widehat{u_\alpha^{-1}}$ has a precise representative whose restriction to the complement of a certain $\mathcal{H}^2$-null set is continuous.

Proof. Let $\boldsymbol{v}$ be the planar function corresponding to $\boldsymbol{u}$. As usual we identify $\boldsymbol{v}$ with its continuous representative in $\pi(\widetilde{\Omega}) \setminus (\{0\} \times \mathbb{R})$.

Part I: covering the $2D$ domain $\pi(\Omega) \setminus (\{0\} \times \mathbb{R})$ with an increasing sequence of good open sets, ever closer to the singular segment.

Thanks to Lemma 2.14, $\boldsymbol{v} \in H^1(\pi(\widetilde{\Omega}) \setminus ([0,\delta] \times \mathbb{R}), \mathbb{R}^2)$ for each $\delta > 0$. Then, by Lemma 2.13, for a.e. small $\delta > 0$ it is possible to find[1] an open set $E^{(\delta)} \Subset \pi(\widetilde{\Omega}) \setminus (\{0\} \times \mathbb{R})$, with a C^2 boundary, such that

$$\partial E^{(\delta)} \cap \pi(\Omega) = \{(r, x_3) \in \pi(\Omega) : r = \delta\}$$

and $E^{(\delta)} \in \mathcal{U}_{\boldsymbol{v}}$ (that is, all the analogous properties of Definition 2.12 are satisfied for the planar map $\boldsymbol{v}$). Indeed, it can be seen that for every small $c > 0$ it is possible to construct a C^2 open set $E \Subset \pi(\widetilde{\Omega} \setminus (\{0\} \times \mathbb{R})$ such that

$$\partial E \cap \pi(\Omega) = \{(r, x_3) \in \pi(\Omega) : r = c\}$$

(begin with the set $(\{c\} \times \mathbb{R}) \cap \pi(\Omega)$, which consists of a finite number of segments and is nonempty because c is small; stretch this set vertically so as to obtain a new finite union of segments containing the former ones but having their endpoints on $\pi(\Omega_D) = \pi(\widetilde{\Omega} \setminus \overline{\Omega})$; then

[1]In principle, we would like to use $E_\delta := \pi(\widetilde{\Omega}) \setminus ([0,\delta] \times \mathbb{R})$ itself. However, our analysis is based on previous works where having a C^2 boundary (and not just piecewise C^2), as well as being compactly contained in the working domain, were added as requirements for membership to the class of good open sets (for the sake of achieving concise statements such as the assertion '$U_t \in \mathcal{U}_{\boldsymbol{u}}$ for a.e. t' in Lemma 2.13).

close the loop—or loops—with a C^2 curve entirely contained in $\pi(\Omega_D)$). Applying Lemma 2.13 to E we find that $E_t \in \mathcal{U}_{\boldsymbol{v}}$ for a.e. $t > 0$, with E_t defined in (2.8). It can be seen that

$$\partial E_t \cap \pi(\Omega) = \{(r, x_3) \in \pi(\Omega) : r = c - t\}.$$

Recall that $E_t \in \mathcal{U}_{\boldsymbol{v}}$ for every small $c > 0$ and a.e. small $t > 0$. Since a.e. small $\delta > 0$ can be written as $\delta = c - t$ for an appropriate choice of c and t, the claim follows.

Part II: the inverse and the definition of the $\mathcal{H}^1$-null exceptional set.

From the comments in Subsection 4.4, $\boldsymbol{v}^{-1} \in W^{1,1}(\mathrm{im_T}(\boldsymbol{v}, E^{(\delta)}), \mathbb{R}^2)$ and it is continuous at every point of $\mathrm{im_T}(\boldsymbol{v}, E^{(\delta)}) \setminus T_\delta$, where T_δ has zero $\mathcal{H}^1$-measure and it is defined by

$$T_\delta := \Big\{(s, y_3) \in \mathrm{im_T}(\boldsymbol{v}, E^{(\delta)}) : \text{ there exist at least two different points } (r, x_3) \in \overline{E^{(\delta)}} \text{ such that } (s, y_3) \in \mathrm{im_T}\big(\boldsymbol{v}, (r, x_3)\big)\Big\}.$$

Observe that for every $(r, x_3) \in \pi(\widetilde{\Omega}) \setminus (\{0\} \times \mathbb{R})$,

$$\boldsymbol{v}(r, x_3) \in \mathrm{im_T}(\boldsymbol{v}, E^{(\delta)}) \setminus T_\delta \quad \Rightarrow \quad \boldsymbol{v}^{-1}\big(\boldsymbol{v}(r, x_3)\big) = (r, x_3). \tag{4.18}$$

Fix a sequence $(E_k)_{k\in\mathbb{N}}$ of the sets $E^{(\delta)}$ of Part I with $E_k = E^{(\delta_k)}$ and $(\delta_k)_{k\in\mathbb{N}}$ a sequence decreasing to zero. Without loss of generality, assume that $E_k \Subset E_{k+1}$ for all k, that each E_k is the projection by π of a good $3D$ open set $V_k \in \mathcal{U}_{\boldsymbol{u}}$, and that $\pi(\widetilde{\Omega}) \setminus (\{0\} \times \mathbb{R}) = \bigcup_{k\in\mathbb{N}} E_k$. Set

$$T := \bigcup_{k\in\mathbb{N}} T_{\delta_k} \cup \bigcup_{k\in\mathbb{N}} \boldsymbol{v}(\partial E_k \setminus \pi(\Omega_0)),$$

where Ω_0 is the set of Definition 2.2. By property (b) in Definition 2.12 and the fact that $H^1(\Gamma, \mathbb{R}^2)$ functions on a smooth curve Γ map $\mathcal{H}^1$-null sets onto $\mathcal{H}^1$-null sets (see [38], [44, Prop. 2.7]), it follows that $\mathcal{H}^1(T) = 0$.

Denote by $w : \pi(\widetilde{\Omega}_{\boldsymbol{b}}) \to \mathbb{R}$ the function

$$w = \begin{cases} v_1^{-1} & \text{in } \pi(\mathrm{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L)), \\ 0 & \text{in } \pi(\mathrm{im_T}(\boldsymbol{u}, L)), \end{cases}$$

where v_1^{-1} is the first component of the inverse of $\boldsymbol{v}$. In particular $\boldsymbol{u}^{-1} = v_1^{-1}(\cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2) + v_2^{-1}\boldsymbol{e}_3$.

From (4.4) we have $\partial\, \mathrm{im_T}(\boldsymbol{v}, E_k) \subset \boldsymbol{v}(\partial E_k)$ for all k. But $\partial E_k = \big(\partial E_k \cap \pi(\widetilde{\Omega}\setminus\Omega)\big) \cup \big((\{\delta_k\}\times\mathbb{R}) \cap \pi(\Omega)\big)$. Thus we find $\partial\, \mathrm{im_T}(\boldsymbol{v}, E_k) \subset \boldsymbol{v}\big(\pi(\widetilde{\Omega}\setminus\Omega)\big) \cup \boldsymbol{v}\left((\{\delta_k\} \times \mathbb{R}) \cap \pi(\Omega)\right)$. However, thanks to the boundary condition $\boldsymbol{b}$, we know that $\boldsymbol{v}\big(\pi(\widetilde{\Omega}\setminus\Omega)\big) = \pi\big(\boldsymbol{b}(\widetilde{\Omega}\setminus\Omega)\big)$. Hence $\pi(\Omega_{\boldsymbol{b}}) \cap \boldsymbol{v}\big(\pi(\widetilde{\Omega}\setminus\Omega)\big) = \varnothing$. But we also have $\boldsymbol{v}(\pi(\Omega)) = \pi(\Omega_{\boldsymbol{b}})$. Thus we infer that $\boldsymbol{v}\big((\{\delta_k\} \times \mathbb{R}) \cap \pi(\Omega)\big) \cap \pi(\Omega_{\boldsymbol{b}}) = \boldsymbol{v}\big((\{\delta_k\} \times \mathbb{R}) \cap \pi(\Omega)\big)$ and we deduce

$$\partial\, \mathrm{im_T}(\boldsymbol{v}, E_k) \cap \pi(\Omega_{\boldsymbol{b}}) = \boldsymbol{v}\big((\{\delta_k\} \times \mathbb{R}) \cap \pi(\Omega)\big). \tag{4.19}$$

Let $(\delta_k, x_3) \in \pi(\Omega)$ be such that $(s, y_3) := \boldsymbol{v}(\delta_k, x_3) \in \pi(\Omega_{\boldsymbol{b}}) \setminus T$. We claim that

$$(s, y_3) \in \pi\big(\,\mathrm{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L)\big), \quad w \text{ is continuous at } (s, y_3), \quad \text{and} \quad w(s, y_3) = \delta_k. \tag{4.20}$$

First, we show that $(s, y_3) \in \mathrm{im_T}(\boldsymbol{v}, E_{k+1})$. Suppose, for a contradiction, that this were false. Since, by continuity, it is easy to see that $(s, y_3) \in \overline{\mathrm{im_T}(\boldsymbol{v}, E_{k+1})}$, by (4.4) it follows that $(s, y_3) = \boldsymbol{v}(\delta_{k+1}, x_3')$ for some $(\delta_{k+1}, x_3') \in \pi(\Omega)$. But $(s, y_3) \notin T$, so both (δ_k, x_3) and (δ_{k+1}, x_3') belong to Ω_0. Thus, on the one hand, $\boldsymbol{v}(\delta_k, x_3) = \boldsymbol{v}(\delta_{k+1}, x_3')$. On the other hand, $\boldsymbol{u}$ is injective in Ω_0 [28, Lemma 3], yielding a contradiction.

By Lemma 4.4 and Definition 4.7, we conclude that (s, y_3) indeed belongs to $\operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L)$. In addition, since $(s, y_3) \in \operatorname{im_T}(\boldsymbol{v}, E_{k+1}) \setminus T_{k+1}$, $\boldsymbol{v}^{-1}$ is continuous at (s, y_3). By (4.18), $\boldsymbol{v}^{-1}(s, y_3) = (\delta_k, x_3)$ and $w(s, y_3) = \delta_k$.

Combining (4.19) with (4.20) we find that if $(s, y_3) \in \partial \operatorname{im_T}(\boldsymbol{v}, E_k) \cap \pi(\Omega_{\boldsymbol{b}}) \setminus T$ and $(s^{(j)}, y_3^{(j)})$ is any sequence converging to (s, y_3) then $w\big((s^{(j)}, y_3^{(j)})\big) \overset{j\to\infty}{\longrightarrow} \delta_k$. This is the motivation for the definition of the exceptional set T.

Part III: $\operatorname{im_T}(\boldsymbol{v}, E_k) \cup \pi\Big(\boldsymbol{b}\big(\widetilde{\Omega} \setminus (\Omega \cup \overline{C}_{\delta_k})\big)\Big) = \operatorname{im_T}\big(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus ([0, \delta_k] \times \mathbb{R})\big)$ for every $k \in \mathbb{N}$. In preparation for using the excision property of the degree, let us show first that

$$\text{if } \boldsymbol{y}_0' \in \operatorname{im_T}(\boldsymbol{v}, E_k) \quad \text{then} \quad \boldsymbol{y}_0' \notin \boldsymbol{v}\Big(\partial E_k \cup \partial\big(\pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\big)\Big). \tag{4.21}$$

By definition of topological image, $\boldsymbol{y}_0' \notin \boldsymbol{v}(\partial E_k)$. Suppose, for a contradiction, that $\boldsymbol{y}_0' \in \boldsymbol{v}\Big(\partial\big(\pi(\widetilde{\Omega}) \setminus ([0, \delta_k] \times \mathbb{R})\big) \setminus \partial E_k\Big)$. Then $\boldsymbol{y}_0' = \boldsymbol{v}(\boldsymbol{x}_0')$ for some $\boldsymbol{x}_0' = (r_0, x_3) \in \partial \pi\big(\widetilde{\Omega}\big)$ with $r_0 \geq \delta_k$, or some $\boldsymbol{x}_0' = (r_0, x_3) \in \partial\Big(\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big)\Big)$ with $r_0 = \delta_k$. In both cases, $\boldsymbol{x}_0'$ is in the region where $\boldsymbol{v}$ is defined through the boundary data $\boldsymbol{b}$. By [44, Lemma 2.5]

$$D\left(\boldsymbol{v}\Big(\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big)\Big), \boldsymbol{y}_0'\right) \geq \frac{1}{2}.$$

At the same time, since $\boldsymbol{y}_0'$ belongs to the open set $\operatorname{im_T}(\boldsymbol{v}, E_k) \overset{\text{a.e.}}{=} \operatorname{im_G}(\boldsymbol{v}, E_k)$, using (4.8) and Lemma 4.4 we find that

$$\mathcal{L}^2\left(\operatorname{im_G}(\boldsymbol{v}, E_k) \cap \boldsymbol{v}\Big(\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big)\Big)\right) > 0.$$

Since $\boldsymbol{v}$ is injective a.e. we arrive at a contradiction.

A similar proof yields that

$$\text{if } \boldsymbol{y}_0' \in \operatorname{im_T}\Big(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\Big) \setminus \pi\Big(\boldsymbol{b}\big(\widetilde{\Omega} \setminus (\Omega \cup \overline{C}_{\delta_k})\big)\Big)$$
$$\text{then} \quad \boldsymbol{y}_0' \notin \boldsymbol{v}\Big(\partial E_k \cup \partial\big(\pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\big)\Big).$$

Having both relations, let us now prove that $\operatorname{im_T}(\boldsymbol{v}, E_k) \subset \operatorname{im_T}\big(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\big)$. Let $\boldsymbol{y}_0' \in \operatorname{im_T}(\boldsymbol{v}, E_k)$. By (4.21), the excision property gives that

$$\deg(\boldsymbol{v}, E_k, \boldsymbol{y}_0') + \deg(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big), \boldsymbol{y}_0') = \deg(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big), \boldsymbol{y}_0'). \tag{4.22}$$

Since the restriction of $\boldsymbol{v}$ to $\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big)$ is an orientation-preserving diffeomorphism (given by the planar function corresponding to the axisymmetric boundary data $\boldsymbol{b}$), then a proof similar to that of (4.21) shows that $\boldsymbol{y}_0' \notin \boldsymbol{v}\Big(\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big)\Big)$. Hence, the second degree in (4.22) is zero. By definition of $\operatorname{im_T}(\boldsymbol{v}, E_k)$, the first degree in (4.22) is 1, hence $\boldsymbol{y}_0' \in \operatorname{im_T}\big(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\big)$, as desired.

The proof that $\pi\Big(\boldsymbol{b}\big(\widetilde{\Omega} \setminus (\Omega \cup \overline{C}_{\delta_k})\big)\Big) \subset \operatorname{im_T}\big(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\big)$ is easier, because in that region $\boldsymbol{v}$ is dictated by the diffeomorphism $\boldsymbol{b}$. Finally, to prove that

$$\operatorname{im_T}\Big(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big([0, \delta_k] \times \mathbb{R}\big)\Big) \setminus \pi\Big(\boldsymbol{b}\big(\widetilde{\Omega} \setminus (\Omega \cup \overline{C}_{\delta_k})\big)\Big) \subset \operatorname{im_T}(\boldsymbol{v}, E_k),$$

suppose that $\boldsymbol{y}_0'$ belongs to the set on the left. Then, by (4.22),

$$\deg(\boldsymbol{v}, E_k, \boldsymbol{y}_0') + \deg(\boldsymbol{v}, \pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0, \delta_k] \times \mathbb{R})\big), \boldsymbol{y}_0') \neq 0.$$

The second term is zero since $\boldsymbol{v}$ restricted to $\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0,\delta_k]\times\mathbb{R})\big)$ is a diffeomorphism and $\boldsymbol{y}_0' \notin \pi\Big(\boldsymbol{b}\big(\widetilde{\Omega}\setminus(\Omega\cup\overline{C}_{\delta_k})\big)\Big)$, a set which contains $\boldsymbol{v}\Big(\pi(\widetilde{\Omega}) \setminus \big(\overline{E_k} \cup ([0,\delta_k]\times\mathbb{R})\big)\Big)$. Consequently, the first degree is nonzero and $\boldsymbol{y}_0' \in \operatorname{im_T}(\boldsymbol{v}, E_k)$, finishing the proof of this Part III.

Part IV: $\mathcal{H}^2$-continuity of $\widehat{u_\alpha^{-1}}$. For each $k\in\mathbb{N}$ define $w_k : \pi(\widetilde{\Omega}_{\boldsymbol{b}}) \to \mathbb{R}$ as

$$w_k = \begin{cases} v_1^{-1} & \text{in } \operatorname{im_T}\big(\boldsymbol{v}, \pi(\widetilde{\Omega})\setminus\big([0,\delta_k]\times\mathbb{R}\big)\big), \\ \delta_k & \text{otherwise,}\end{cases}$$

where v_1^{-1} is defined in terms of $\boldsymbol{b}^{-1}$ in $\pi\Big(\boldsymbol{b}\big(\widetilde{\Omega}\setminus(\Omega\cup\overline{C}_{\delta_k})\big)\Big)$. By Parts II and III, the function w_k is continuous at every point in $\pi(\widetilde{\Omega}_{\boldsymbol{b}})\setminus T$. Since

$$\sup_{\pi(\widetilde{\Omega}_{\boldsymbol{b}})\setminus T} |w - w_k| = \delta_k,$$

we have that $w_k \to w$ uniformly in $\pi(\widetilde{\Omega}_{\boldsymbol{b}})\setminus T$ as $k\to\infty$.

For the bound $|w-w_k|\le\delta_k$ in the image of $\pi(\widetilde{\Omega})\cap\big((0,\delta_k]\times\mathbb{R}\big)$ we use that if $(r,x_3)\in\pi(\Omega)$ with $r>0$ and $\boldsymbol{v}(r,x_3)\in\pi(\Omega_{\boldsymbol{b}})\setminus T$ then $\boldsymbol{v}(r,x_3)\in\pi\big(\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus L)\big)$, w is continuous at $\boldsymbol{v}(r,x_3)$, and $w\big(\boldsymbol{v}(r,x_3)\big)=r$. That can be proved similarly as Part II, finding k such that $\delta_k<r$ and assuming that $(s,y_3)=\boldsymbol{v}(r,x_3)$ is both on $\boldsymbol{v}(\partial E_k)$ and on $\boldsymbol{v}(\partial E_{k+1})$.

Therefore, $w|_{\pi(\widetilde{\Omega}_{\boldsymbol{b}})\setminus T}$ is continuous. Since $\widehat{u_1^{-1}}\boldsymbol{e}_1+\widehat{u_2^{-1}}\boldsymbol{e}_2 = w(\cos\theta\boldsymbol{e}_1+\sin\theta\boldsymbol{e}_2)$ we have then that $\widehat{u_\alpha^{-1}}$ has a precise representative whose restriction to the complement of a certain set of zero $\mathcal{H}^2$-measure (the preimage by π of T) is continuous.

Part V: Sobolev regularity of $\widehat{u_\alpha^{-1}}$. Let $V_k\Subset\widetilde{\Omega}$ be the good (3D) open set such that $\pi(V_k)=E_k$. By [29, Prop. 2.17.(vi)], $\operatorname{im_T}(\boldsymbol{u},V_k)$ has finite perimeter and $\partial^*\operatorname{im_T}(\boldsymbol{u},V_k)=\operatorname{im_G}(\boldsymbol{u},\partial V_k)$ $\mathcal{H}^2$-a.e. The set $\boldsymbol{b}\big(\widetilde{\Omega}\setminus(\Omega\cup\overline{C}_{\delta_k})\big)$ also has finite perimeter since $\boldsymbol{b}$ is diffeomorphism up to the boundary of $\widetilde{\Omega}$. By Part III and Lemma 4.4,

$$\operatorname{im_T}(\boldsymbol{u},V_k)\cup\boldsymbol{b}\big(\widetilde{\Omega}\setminus(\Omega\cup\overline{C}_{\delta_k})\big)=\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus\overline{C}_{\delta_k}) \tag{4.23}$$

for every $k\in\mathbb{N}$. Hence, $\operatorname{im_T}(\boldsymbol{u},\widetilde{\Omega}\setminus\overline{C}_{\delta_k})$ is a set of finite perimeter. By Lemma 4.10 the map $\boldsymbol{u}_{V_k}^{-1}:\mathbb{R}^3\to\mathbb{R}^3$ given by

$$\boldsymbol{u}_{V_k}^{-1}(\boldsymbol{y})=\begin{cases}\boldsymbol{u}^{-1}(\boldsymbol{y}), & \boldsymbol{y}\in\operatorname{im_T}(\boldsymbol{u},V_k),\\ \boldsymbol{0}, & \boldsymbol{y}\in\mathbb{R}^3\setminus\operatorname{im_T}(\boldsymbol{u},V_k)\end{cases}$$

is in $SBV(\mathbb{R}^3,\mathbb{R}^3)$ and $D\boldsymbol{u}_{V_k}^{-1}\llcorner\operatorname{im_T}(\boldsymbol{u},V_k)$ is absolutely continuous. Applying Proposition 4.14 to $\boldsymbol{u}_{V_k}^{-1}$ and to $\boldsymbol{b}^{-1}$, with $\operatorname{im_T}(\boldsymbol{u},V_k)$ as the set of finite perimeter in the hypotheses of that gluing theorem, we obtain that the map

$$\boldsymbol{y}\in\widetilde{\Omega}_{\boldsymbol{b}}\mapsto\begin{cases}\boldsymbol{u}^{-1}(\boldsymbol{y}), & \boldsymbol{y}\in\operatorname{im_T}(\boldsymbol{u},V_k)\\ \boldsymbol{b}^{-1}(\boldsymbol{y}), & \boldsymbol{y}\in\widetilde{\Omega}_{\boldsymbol{b}}\setminus\operatorname{im_T}(\boldsymbol{u},V_k)\end{cases}$$

is in $SBV(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)$, with derivative given by

$$D\boldsymbol{u}_{V_k}^{-1}\llcorner\operatorname{im_T}(\boldsymbol{u},V_k)+D\boldsymbol{b}^{-1}\llcorner\big(\widetilde{\Omega}_{\boldsymbol{b}}\setminus\overline{\operatorname{im_T}(\boldsymbol{u},V_k)}\big) + \big((\boldsymbol{u}^{-1})^+-(\boldsymbol{b}^{-1})^-\big)\boldsymbol{\nu}_{\operatorname{im_T}(\boldsymbol{u},V_k)}\mathcal{H}^2\llcorner\operatorname{im_G}(\boldsymbol{u},\partial V_k).$$

(The set $\operatorname{im_T}(\boldsymbol{u}, V_k)$ has neither density zero nor one at $\mathcal{H}^2$-a.e. point in $\partial \operatorname{im_T}(\boldsymbol{u}, V_k)$ thanks to [44, Lemma 2.5]) Since $\boldsymbol{u} = \boldsymbol{b}$ in Ω_D, taking (4.23) into account, the map can be rewritten as

$$\boldsymbol{y} \in \widetilde{\Omega}_{\boldsymbol{b}} \mapsto \begin{cases} \boldsymbol{u}^{-1}(\boldsymbol{y}), & \boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k}) \\ \boldsymbol{b}^{-1}(\boldsymbol{y}), & \text{otherwise,} \end{cases}$$

with a corresponding rewriting for the derivative. At this point, taking into account (4.23), we apply Proposition 4.14 again, now to the first two components of the above map and to the function

$$\boldsymbol{y} = (s\cos\theta, s\sin\theta, y_3) \in \widetilde{\Omega}_{\boldsymbol{b}} \mapsto \delta_k(\cos\theta, \sin\theta)$$

(which belongs to $W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^2)$), with $\operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k})$ as the set of finite perimeter in the hypothesis of that gluing theorem, to find that the map $\boldsymbol{W}_k : \widetilde{\Omega}_{\boldsymbol{b}} \to \mathbb{R}^2$ given by

$$\boldsymbol{W}_k : \boldsymbol{y} = (s\cos\theta, s\sin\theta, y_3) \in \widetilde{\Omega}_{\boldsymbol{b}} \mapsto \begin{cases} \big(u_1^{-1}(\boldsymbol{y}), u_2^{-1}(\boldsymbol{y})\big) & \boldsymbol{y} \in \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k}) \\ \delta_k(\cos\theta, \sin\theta) & \text{otherwise,} \end{cases} \tag{4.24}$$

is in $SBV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^2)$, with derivative given by

$$\begin{aligned} D(u_1^{-1}, u_2^{-1}) \llcorner \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k}) + \delta_k(-\sin\theta, \cos\theta) \otimes D\theta \llcorner \big(\widetilde{\Omega}_{\boldsymbol{b}} \setminus \overline{\operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k})}\big) \\ + \big((u_1^{-1}, u_2^{-1})^+ - \delta_k(\cos\theta, \sin\theta)\big) \boldsymbol{\nu}_{\operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k})} \mathcal{H}^2 \llcorner \big(\widetilde{\Omega}_{\boldsymbol{b}} \cap \partial^* \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k})\big). \end{aligned}$$

However, by Lemma 4.4, the radial component of what would be the planar map corresponding to $\boldsymbol{W}_k$ in (4.24) is precisely w_k, so by Part IV we know that the jump $(u_1^{-1}, u_2^{-1})^+ - \delta_k(\cos\theta, \sin\theta)$ is zero for $\mathcal{H}^2$-a.e. point on $\widetilde{\Omega}_{\boldsymbol{b}} \cap \partial^* \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus \overline{C}_{\delta_k})$. Therefore, the maps $\boldsymbol{W}_k$ under consideration belong to $W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}})$.

The uniform convergence $w_k \to w$ of Part IV translates, in particular, into the a.e. convergence of the maps $\boldsymbol{W}_k$ in (4.24) to the map $(\widehat{u_1^{-1}}, \widehat{u_2^{-1}})$ in the statement of the proposition. On the other hand, Lemma 2.14 shows that the gradients of the maps $\boldsymbol{W}_k$ are equiintegrable because

$$\int_A |D\boldsymbol{W}_k| \, \mathrm{d}\boldsymbol{y} \le \int_{A \cap \operatorname{im_T}(\boldsymbol{u}, \widetilde{\Omega} \setminus L)} |D\boldsymbol{u}^{-1}| \, \mathrm{d}\boldsymbol{y} + \int_{A \cap \widetilde{\Omega}_{\boldsymbol{b}}} \delta_k |D\theta| \, \mathrm{d}\boldsymbol{y}$$

for any measurable subset $A \subset \pi(\widetilde{\Omega}_{\boldsymbol{b}})$. Therefore, the limit $(\widehat{u_1^{-1}}, \widehat{u_2^{-1}})$ also belongs to $W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^2)$, finishing the proof. □

5. Weak limits of regular maps

We investigate here the properties of maps in $\overline{\mathcal{A}_s^r}$: the weak H^1 closure of the class of regular maps. We start by proving that $\overline{\mathcal{A}_s^r}$ is contained in the space $\mathcal{B}$ defined in (1.7).

Theorem 5.1. *Let $\boldsymbol{u} \in \overline{\mathcal{A}_s^r}$. Then*

i) $\boldsymbol{u}$ belongs to $\mathcal{A}_s$.
ii) $\operatorname{im_G}(\boldsymbol{u}, \Omega) = \Omega_{\boldsymbol{b}}$ a.e. and $\mathcal{L}^3(\operatorname{im_T}(\boldsymbol{u}, L)) = 0$.
iii) $\boldsymbol{u}^{-1} \in BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ and $\operatorname{supp} D^s\boldsymbol{u}^{-1} \subset \operatorname{im_T}(\boldsymbol{u}, L)$. Moreover, $\|\boldsymbol{u}^{-1}\|_{BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)} \le M$ for some $M > 0$ not depending on $\boldsymbol{u}$.
iv) $u_\alpha^{-1} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}})$ for $\alpha = 1, 2$.

Proof. Let $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_s^r$ satisfy $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $H^1(\widetilde{\Omega}, \mathbb{R}^3)$. By Proposition 3.1, $\boldsymbol{u} \in \mathcal{A}_s$, $\det D\boldsymbol{u}_n \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\widetilde{\Omega})$, and $\chi_{\operatorname{im_G}(\boldsymbol{u}_n, \widetilde{\Omega})} \to \chi_{\operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega})}$ a.e. Now, by Proposition 4.11.(b), $\operatorname{im_G}(\boldsymbol{u}_n, \widetilde{\Omega}) = \widetilde{\Omega}_{\boldsymbol{b}}$ a.e. for every $n \in \mathbb{N}$, hence $\boldsymbol{u}$ inherits this property. By (4.17) and Lemma 4.8 c), it then follows that $\mathcal{L}^3(\operatorname{im_T}(\boldsymbol{u}, L)) = 0$, completing the proof of *ii)*.

From Proposition 4.12 we have that $\boldsymbol{u}_n^{-1} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}\|D\boldsymbol{u}_n^{-1}\|_{L^1(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^{3\times 3})} &= \|\operatorname{cof} D\boldsymbol{u}_n\|_{L^1(\widetilde{\Omega},\mathbb{R}^{3\times 3})} \\ &\leq \|D\boldsymbol{u}_n\|^2_{L^2(\Omega,\mathbb{R}^{3\times 3})} + \|D\boldsymbol{b}\|^2_{L^2(\Omega_D;\mathbb{R}^{3\times 3})} \leq E(\boldsymbol{u}_n) + C \leq E(\boldsymbol{b}) + C,\end{aligned}$$

where we have used Lemma 2.6.

On the other hand the image of each $\boldsymbol{u}_n^{-1}$ is contained in Ω, so $\|\boldsymbol{u}_n^{-1}\|_{L^\infty(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)}$ and, hence $\|\boldsymbol{u}_n^{-1}\|_{L^1(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)}$ are bounded by a constant only depending on Ω and $\widetilde{\Omega}_{\boldsymbol{b}}$. Thus, by the theorem of compactness in BV we find that, up to a subsequence, there exists $\boldsymbol{w} \in BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ such that $\boldsymbol{u}_n^{-1} \to \boldsymbol{w}$ in $L^1(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ and a.e. in $\widetilde{\Omega}_{\boldsymbol{b}}$. By Lemma 4.13, $\boldsymbol{w} = \boldsymbol{u}^{-1}$ a.e. Finally, by Lemma 4.10 we have $\operatorname{supp} D^s\boldsymbol{u}^{-1} \subset \operatorname{im_T}(\boldsymbol{u}, L)$. This shows *iii)*.

Since $\mathcal{L}^3(\operatorname{im_T}(\boldsymbol{u}, L)) = 0$, the functions u_α^{-1} and $\widehat{u_\alpha^{-1}}$ (see Proposition 4.15) coincide a.e. Thus, by Proposition 4.15, $u_\alpha^{-1} \in W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}})$, which shows *iv)*. □

For $\boldsymbol{u} \in \overline{\mathcal{A}_s^r}$ we have, by Theorem 5.1, that $\boldsymbol{u}^{-1} \in BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ and we introduce the following standard decomposition of the distributional derivative of $\boldsymbol{u}^{-1}$:

$$D\boldsymbol{u}^{-1} = \nabla\boldsymbol{u}^{-1} + D^s\boldsymbol{u}^{-1} = \nabla\boldsymbol{u}^{-1} + D^j\boldsymbol{u}^{-1} + D^c\boldsymbol{u}^{-1};$$

see, e.g., [1, Sect. 3.9]. In this decomposition $\nabla\boldsymbol{u}^{-1}$ denotes the absolutely continuous part of $D\boldsymbol{u}^{-1}$ with respect to the Lebesgue measure, $D^s\boldsymbol{u}^{-1}$ is the singular part which can be furthermore decomposed in a jump part $D^j\boldsymbol{u}^{-1}$ and a Cantor part $D^c\boldsymbol{u}^{-1}$. Moreover, we denote by $J_{\boldsymbol{u}^{-1}}$ the set of jump points of $\boldsymbol{u}^{-1}$. We fix a Borel orientation $\boldsymbol{\nu}$ of $J_{\boldsymbol{u}^{-1}}$, and, with respect to this orientation, the lateral traces of $\boldsymbol{u}^{-1}$ are denoted by $(\boldsymbol{u}^{-1})^+$ and $(\boldsymbol{u}^{-1})^-$. Analogously, the jump is defined as $[\boldsymbol{u}^{-1}] := (\boldsymbol{u}^{-1})^+ - (\boldsymbol{u}^{-1})^-$.

The following lemma, which uses many ideas of [28, Th. 2], relates the surface energy $\mathcal{E}_{\boldsymbol{u}}$ with the singular part of $D\boldsymbol{u}^{-1}$. With a small abuse of notation, given $\phi \in C_c^1(\Omega)$, we define $[\phi \circ \boldsymbol{u}^{-1}]$ in $J_{\boldsymbol{u}^{-1}}$ as $\phi \circ (\boldsymbol{u}^{-1})^+ - \phi \circ (\boldsymbol{u}^{-1})^-$. Recall that $\boldsymbol{u}^{-1}$ initially is defined only on $\operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega})$ but if $\operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega}) = \widetilde{\Omega}_{\boldsymbol{b}}$ a.e. then $\boldsymbol{u}^{-1}$ is defined a.e. in the open set $\widetilde{\Omega}_{\boldsymbol{b}}$. Assume that $\boldsymbol{u}^{-1}$ is the precise representative of itself.

Lemma 5.2. *Let* $\boldsymbol{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ *be such that* $\det D\boldsymbol{u} \in L^1(\widetilde{\Omega})$ *and* $\det D\boldsymbol{u} > 0$ *a.e. Let* $\phi \in C_c^1(\widetilde{\Omega})$ *and* $\boldsymbol{g} \in C_c^1(\mathbb{R}^3, \mathbb{R}^3)$. *Suppose that* $\operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega}) = \widetilde{\Omega}_{\boldsymbol{b}}$ *a.e.,* $\boldsymbol{u}$ *is injective a.e. and* $\boldsymbol{u}^{-1} \in BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$. *Then*

$$\begin{aligned}\overline{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) &= -\langle D^s(\phi \circ \boldsymbol{u}^{-1}), \boldsymbol{g}\rangle \\ &= -\int_{\widetilde{\Omega}_{\boldsymbol{b}}} \nabla\phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) \otimes \boldsymbol{g}(\boldsymbol{y}) \cdot \mathrm{d}D^c\boldsymbol{u}^{-1}(\boldsymbol{y}) - \int_{J_{\boldsymbol{u}^{-1}}} [\phi \circ \boldsymbol{u}^{-1}]\, \boldsymbol{g} \cdot \boldsymbol{\nu} \,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

Proof. By the change of variables formula and using that $\operatorname{im_G}(\boldsymbol{u}, \widetilde{\Omega}) = \widetilde{\Omega}_{\boldsymbol{b}}$ a.e., we find

$$\overline{\mathcal{E}}_{\boldsymbol{u}}(\phi, \boldsymbol{g}) = \int_{\widetilde{\Omega}_{\boldsymbol{b}}} \left[\boldsymbol{g}(\boldsymbol{y}) \cdot D\boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))^{-T} D\phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) + \phi(\boldsymbol{u}^{-1}(\boldsymbol{y})) \operatorname{div} \boldsymbol{g}(\boldsymbol{y})\right] \mathrm{d}\boldsymbol{y}. \tag{5.1}$$

By the chain rule for BV functions (see, e.g., [1, Th. 3.96]), $\phi \circ \boldsymbol{u}^{-1} \in BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ and

$$\begin{aligned}&\nabla(\phi \circ \boldsymbol{u}^{-1}) = \nabla\phi(\boldsymbol{u}^{-1})\, \nabla\boldsymbol{u}^{-1}, \\ &D^s(\phi \circ \boldsymbol{u}^{-1}) = \nabla\phi(\boldsymbol{u}^{-1}) D^c\boldsymbol{u}^{-1} + [\phi \circ \boldsymbol{u}^{-1}] \otimes \boldsymbol{\nu}_{\boldsymbol{u}} \mathcal{H}^{n-1} \llcorner J_{\boldsymbol{u}^{-1}}.\end{aligned} \tag{5.2}$$

By Lemma 4.10, $\nabla\boldsymbol{u}^{-1}(\boldsymbol{u}(\boldsymbol{x})) = \nabla\boldsymbol{u}(\boldsymbol{x})^{-1}$ for a.e. $\boldsymbol{x} \in \widetilde{\Omega}$. This and (5.2) imply that, for a.e. $\boldsymbol{y} \in \tilde{\Omega}_{\boldsymbol{b}}$,

$$\nabla(\phi \circ \boldsymbol{u}^{-1})(\boldsymbol{y}) = \nabla\phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\, \nabla\boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))^{-1}.$$

Therefore,

$$\int_{\widetilde{\Omega}_{\boldsymbol{b}}} \boldsymbol{g}(\boldsymbol{y}) \cdot D\boldsymbol{u}(\boldsymbol{u}^{-1}(\boldsymbol{y}))^{-T} D\phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} = \int_{\widetilde{\Omega}_{\boldsymbol{b}}} \nabla(\phi\circ\boldsymbol{u}^{-1})(\boldsymbol{y})\cdot\boldsymbol{g}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \langle\nabla(\phi\circ\boldsymbol{u}^{-1}),\boldsymbol{g}\rangle. \quad (5.3)$$

On the other hand, by definition of distributional derivative,

$$\int_{\widetilde{\Omega}_{\boldsymbol{b}}} \phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\,\mathrm{div}\,\boldsymbol{g}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = -\langle D(\phi\circ\boldsymbol{u}^{-1}),\boldsymbol{g}\rangle. \quad (5.4)$$

Putting together (5.1), (5.3) and (5.4) we obtain

$$\overline{\mathcal{E}}_{\boldsymbol{u}}(\phi,\boldsymbol{g}) = \langle\nabla(\phi\circ\boldsymbol{u}^{-1}) - D(\phi\circ\boldsymbol{u}^{-1}),\boldsymbol{g}\rangle = -\langle D^s(\phi\circ\boldsymbol{u}^{-1}),\boldsymbol{g}\rangle.$$

This, together with (5.2), concludes the proof. □

Recall the definition of the singular segment $L = \overline{\Omega}\cap\mathbb{R}\boldsymbol{e}_3$ in (4.3).

Proposition 5.3. *Let $\boldsymbol{u} \in \overline{\mathcal{A}_s^r}$. Then $\boldsymbol{u}^{-1}(\boldsymbol{y}) \in L$ for $|D^c\boldsymbol{u}^{-1}|$-a.e. $\boldsymbol{y} \in \widetilde{\Omega}_{\boldsymbol{b}}$ and $(\boldsymbol{u}^{-1})^{\pm}(\boldsymbol{y}) \in L$ for $\mathcal{H}^2$-a.e. $\boldsymbol{y} \in J_{\boldsymbol{u}^{-1}}$.*

Proof. Without loss of generality, $\boldsymbol{u}^{-1}$ is the precise representative of itself.

Let $\boldsymbol{g} \in C_c^1(\mathbb{R}^3,\mathbb{R}^3)$ and $\phi \in C_c^1(\widetilde{\Omega}\setminus\mathbb{R}\boldsymbol{e}_3)$. Then, there exists $\delta > 0$ such that $\phi \in C_c^1(\widetilde{\Omega}\setminus\overline{C}_\delta)$. By Lemma 2.18, we have that $\overline{\mathcal{E}}_{\boldsymbol{u}}(\phi,\boldsymbol{g}) = 0$, so due to Lemma 5.2 we obtain that

$$\int_{\widetilde{\Omega}_{\boldsymbol{b}}} \nabla\phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\otimes\boldsymbol{g}(\boldsymbol{y})\cdot\mathrm{d}D^c\boldsymbol{u}^{-1}(\boldsymbol{y}) + \int_{J_{\boldsymbol{u}^{-1}}} [\phi\circ\boldsymbol{u}^{-1}]\,\boldsymbol{g}\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^2 = 0. \quad (5.5)$$

By approximation, the previous equality, which does not involve derivatives of $\boldsymbol{g}$, is also valid for every bounded Borel $\boldsymbol{g}:\mathbb{R}^3\to\mathbb{R}^3$.

Let $D^c\boldsymbol{u}^{-1} = \boldsymbol{A}\,|D^c\boldsymbol{u}^{-1}|$ be the polar decomposition of $D^c\boldsymbol{u}^{-1}$, so $\boldsymbol{A}:\widetilde{\Omega}_{\boldsymbol{b}}\to\mathbb{R}^{3\times3}$ is Borel, $|D^c\boldsymbol{u}^{-1}|$-integrable and $|\boldsymbol{A}| = 1$ in $|D^c\boldsymbol{u}^{-1}|$-a.e. $\widetilde{\Omega}_{\boldsymbol{b}}$. Let $\boldsymbol{y}_0 \in \widetilde{\Omega}_{\boldsymbol{b}}$ be a $|D^c\boldsymbol{u}^{-1}|$-Lebesgue point of $\boldsymbol{u}^{-1}$, i.e.,

$$\lim_{r\to0^+} \frac{\int_{B(\boldsymbol{y}_0,r)} |\boldsymbol{u}^{-1}(\boldsymbol{y}) - \boldsymbol{u}^{-1}(\boldsymbol{y}_0)|\,\mathrm{d}|D^c\boldsymbol{u}^{-1}|(\boldsymbol{y})}{|D^c\boldsymbol{u}^{-1}|(B(\boldsymbol{y}_0,r))} = 0, \quad (5.6)$$

and note that $|D^c\boldsymbol{u}^{-1}|$-a.e. point of $\widetilde{\Omega}_{\boldsymbol{b}}$ satisfies that. Let us suppose that $\boldsymbol{u}^{-1}(\boldsymbol{y}_0) \notin L$ and take a closed cube $Q \subset \widetilde{\Omega}$ centered at $\boldsymbol{u}^{-1}(\boldsymbol{y}_0)$ with $Q\cap L = \varnothing$. Consider the Borel set

$$U := \{\boldsymbol{y}\in\widetilde{\Omega}_{\boldsymbol{b}} \colon \boldsymbol{u}^{-1}(\boldsymbol{y}) \in Q \text{ and } \boldsymbol{u}^{-1} \text{ is approximately continuous at } \boldsymbol{y}\}.$$

Given any $\psi \in C^1(\mathbb{R})$, take $\phi \in C_c^1(\widetilde{\Omega}\setminus\mathbb{R}\boldsymbol{e}_3)$ such that $\phi(\boldsymbol{x}) = \psi(x_\alpha)$ for all $\boldsymbol{x}\in Q$. For $1\le\alpha,i\le3$ and $r>0$ fixed, we apply (5.5) to $\boldsymbol{g} = \operatorname{sgn}\psi'\operatorname{sgn}A_{\alpha i}\chi_{B(\boldsymbol{y}_0,r)\cap U}\boldsymbol{e}_i$ and deduce that

$$\int_{\{\boldsymbol{y}\in B(\boldsymbol{y}_0,r):\boldsymbol{u}^{-1}(\boldsymbol{y})\in Q\}} \operatorname{sgn}\psi'\operatorname{sgn}A_{\alpha i}\left(\nabla\phi(\boldsymbol{u}^{-1}(\boldsymbol{y}))\otimes\boldsymbol{e}_i\right)\cdot\boldsymbol{A}\,\mathrm{d}|D^c\boldsymbol{u}^{-1}| = 0.$$

This can also be written as

$$\int_{\{\boldsymbol{y}\in B(\boldsymbol{y}_0,r):\boldsymbol{u}^{-1}(\boldsymbol{y})\in Q\}} \left|\psi'(u_\alpha^{-1}(\boldsymbol{y}))\right|\,|A_{\alpha i}|\,\mathrm{d}|D^c\boldsymbol{u}^{-1}| = 0.$$

We use the previous equality first with $\psi(t) = \cos t$ and then with $\psi(t) = \sin t$. We sum the two equalities and use that $|\cos t| + |\sin t| \ge 1$ to get

$$\int_{\{\boldsymbol{y}\in B(\boldsymbol{y}_0,r):\boldsymbol{u}^{-1}(\boldsymbol{y})\in Q\}} |A_{\alpha i}|\,\mathrm{d}|D^c\boldsymbol{u}^{-1}| = 0.$$

We then sum this equality for $1\le\alpha,i\le3$ to obtain

$$|D^c\boldsymbol{u}^{-1}|\left(\{\boldsymbol{y}\in B(\boldsymbol{y}_0,r):\boldsymbol{u}^{-1}(\boldsymbol{y})\in Q\}\right) = 0.$$

This equality implies that $|\boldsymbol{u}^{-1}(\boldsymbol{y}) - \boldsymbol{u}^{-1}(\boldsymbol{y}_0)| > \operatorname{diam} Q/2$ for $|D^c\boldsymbol{u}^{-1}|$-a.e. $\boldsymbol{y} \in B(\boldsymbol{y}_0, r)$. Being true for all $r > 0$, this is a contradiction with (5.6). Therefore, $|D^c\boldsymbol{u}^{-1}|$-a.e. $\boldsymbol{y} \in \widetilde{\Omega}_{\boldsymbol{b}}$ satisfies $\boldsymbol{u}^{-1}(\boldsymbol{y}) \in L$.

Now we show that $(\boldsymbol{u}^{-1})^{\pm}(\boldsymbol{y}) \in L$ for $\mathcal{H}^2$-a.e. $\boldsymbol{y} \in J_{\boldsymbol{u}^{-1}}$. Let $\boldsymbol{y}_0 \in J_{\boldsymbol{u}^{-1}}$ be a $\mathcal{H}^2 \llcorner J_{\boldsymbol{u}^{-1}}$-Lebesgue point for both $(\boldsymbol{u}^{-1})^+$ and $(\boldsymbol{u}^{-1})^-$, i.e.,

$$\lim_{r\to 0^+} \frac{\int_{J_{\boldsymbol{u}^{-1}}\cap B(\boldsymbol{y}_0,r)} |(\boldsymbol{u}^{-1})^{\pm}(\boldsymbol{y}) - (\boldsymbol{u}^{-1})^{\pm}(\boldsymbol{y}_0)| \, \mathrm{d}\mathcal{H}^2(\boldsymbol{y})}{\mathcal{H}^2(J_{\boldsymbol{u}^{-1}} \cap B(\boldsymbol{y}_0, r))} = 0, \tag{5.7}$$

and note that $\mathcal{H}^2$-a.e. point in $J_{\boldsymbol{u}^{-1}}$ satisfies that. For each $r > 0$, we apply (5.5) to $\boldsymbol{g} = \chi_{J_{\boldsymbol{u}^{-1}}\cap B(\boldsymbol{y}_0,r)}\boldsymbol{\nu}$ and deduce that

$$\int_{J_{\boldsymbol{u}^{-1}}\cap B(\boldsymbol{y}_0,r)} [\phi \circ \boldsymbol{u}^{-1}] \, \mathrm{d}\mathcal{H}^2 = 0.$$

This and (5.7) imply that $[\phi \circ \boldsymbol{u}^{-1}](\boldsymbol{y}_0) = 0$. If $(\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) \notin L$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) \in L$, we choose $\phi \in C^1_c(\widetilde{\Omega} \setminus \mathbb{R}\boldsymbol{e}_3)$ such that $\phi(\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) \neq 0$ and reach a contradiction with $[\phi \circ \boldsymbol{u}^{-1}](\boldsymbol{y}_0) = 0$. Analogously if $(\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) \in L$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) \notin L$. If $(\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) \notin L$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) \notin L$, we choose $\phi \in C^1_c(\widetilde{\Omega} \setminus \mathbb{R}\boldsymbol{e}_3)$ such that $\phi((\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0)) \neq \phi((\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0))$, which contradicts $[\phi \circ \boldsymbol{u}^{-1}](\boldsymbol{y}_0) = 0$. Hence, the only possibility is that $(\boldsymbol{u}^{-1})^+(\boldsymbol{y}_0) \in L$ and $(\boldsymbol{u}^{-1})^-(\boldsymbol{y}_0) \in L$. □

6. Lower bound for the relaxed energy and an explicit alternative variational problem

In this section we study the energetic cost for a weak limit of functions in $\mathcal{A}^r_s$ to leave $\mathcal{A}^r_s$. Note that, by Conti-De Lellis' counterexample, condition INV is not satisfied, in general, by functions in $\overline{\mathcal{A}^r_s}$. This is due to the lack of equiintegrability of the cofactors, so the theory of [29] cannot be applied.

A standard diagonal argument shows that $\overline{\mathcal{A}^r_s}$ is closed under the weak convergence of $H^1(\Omega, \mathbb{R}^3)$. From Theorem 5.1 we see that the energy

$$F(\boldsymbol{u}) := E(\boldsymbol{u}) + 2|D^s u_3^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}). \tag{6.1}$$

is well defined on $\overline{\mathcal{A}^r_s}$. We start with the following lemma, which plays a role of an energy-area inequality and should be compared with Lemma 2.6.

Lemma 6.1. *Let $\boldsymbol{u} \in \mathcal{A}_s$. Then $|\operatorname{adj} D\boldsymbol{u}\, \boldsymbol{e}_3| \leq \frac{1}{2}|D\boldsymbol{u}|^2$. This inequality is optimal and cannot be attained by a map in $\mathcal{A}_s$.*

Proof. With the expressions of $D\boldsymbol{u}$ and $\operatorname{cof} D\boldsymbol{u}$ in terms of the associated $2D$ map $\boldsymbol{v}$, cf. (6.12), we find

$$|\operatorname{adj} D\boldsymbol{u}\, \boldsymbol{e}_3| = \frac{|v_1|}{r}\left(|\partial_r v_1|^2 + |\partial_{x_3} v_1|^2\right)^{1/2} \leq \frac{1}{2}\left(\frac{|v_1|^2}{r^2} + |\partial_r v_1|^2 + |\partial_{x_3} v_1|^2\right) \leq \frac{1}{2}|D\boldsymbol{u}|^2.$$

The equality implies $\frac{v_1}{r} = (|\partial_r v_1|^2 + |\partial_{x_3} v_1|^2)^{1/2}$ and $\nabla v_2 = 0$. This cannot be attained by a map in $\mathcal{A}_s$, since $\nabla v_2 = 0$ implies $\det D\boldsymbol{v} = 0$, so $\det D\boldsymbol{u} = 0$. Nonetheless, the constant is optimal in $\mathcal{A}_s$, as can be checked by considering $v_1(r, x_3) = r$ and $v_2(r, x_3) = \varepsilon x_3$ for $\varepsilon \searrow 0$, which corresponds to $\boldsymbol{u}(\boldsymbol{x}) = (x_1, x_2, \varepsilon x_3)$. □

The following lower semicontinuity result is the cornerstone of the strategy in this paper for the study of the regularity of the minimizers of the neo-Hookean energy.

Proposition 6.2. *The energy F defined in* (6.1) *is sequentially lower semicontinuous in $\overline{\mathcal{A}^r_s}$ for the weak convergence in $H^1(\widetilde{\Omega}, \mathbb{R}^3)$.*

Proof. Recall from Theorem 5.1 that $\overline{\mathcal{A}_s^r} \subset \mathcal{A}_s$. Let $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ be a sequence in $\overline{\mathcal{A}_s^r}$ tending weakly in $H^1(\widetilde{\Omega},\mathbb{R}^3)$ to $\boldsymbol{u} \in \overline{\mathcal{A}_s^r}$. Thanks to Theorem 5.1 iii), the BV norm of $\boldsymbol{u}_k^{-1}$ is bounded, so, due to Lemma 4.13, we have that, up to a subsequence, $\boldsymbol{u}_k^{-1} \overset{*}{\rightharpoonup} \boldsymbol{u}^{-1}$ in $BV(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)$ and a.e. By Proposition 3.1, we have that $\det D\boldsymbol{u}_k \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\widetilde{\Omega})$ and, because of the convexity of H,

$$\int_{\widetilde{\Omega}} H(\det D\boldsymbol{u}) \leq \liminf_{k\to\infty} \int_{\widetilde{\Omega}} H(\det D\boldsymbol{u}_k). \tag{6.2}$$

We first prove that the sequence $\{\det D\boldsymbol{u}_k^{-1}\}_{k\in\mathbb{N}}$ is equiintegrable. This can be proved as in [5, Prop. 7.8]. Indeed, define $H_1 : (0,\infty) \to \mathbb{R}$ as $H_1(t) := tH(1/t)$. Then H_1 grows superlinearly at infinity and

$$\int_{\widetilde{\Omega}_{\boldsymbol{b}}} H_1(\det D\boldsymbol{u}_k^{-1})\,\mathrm{d}\boldsymbol{y} = \int_{\widetilde{\Omega}} H(\det D\boldsymbol{u}_k)\,\mathrm{d}\boldsymbol{x} \leq E(\boldsymbol{b}).$$

Thus the equiintegrability follows from the De La Vallée Poussin criterion.

Now, let $\varepsilon > 0$. Recall that C_δ is given by (2.7). Choose $\delta_0 > 0$ such that

$$\int_{C_{\delta_0}\cap\widetilde{\Omega}} |D\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} < \varepsilon. \tag{6.3}$$

Because of the axial symmetry, it can be seen that the sequence $\{\chi_{\widetilde{\Omega}\setminus C_{\delta_0}} \operatorname{cof} D\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ is equiintegrable, cf. [33, Th. 1.3]. This is due to the fact that the corresponding 2D maps $\boldsymbol{v}_k$ are bounded in $H^1\big(\pi(\widetilde{\Omega} \setminus \overline{C}_{\delta_0}),\mathbb{R}^2\big)$ and then by a result of Müller [42], since we also have $\det D\boldsymbol{v}_k > 0$ a.e., $\det D\boldsymbol{v}_k$ are equiintegrable. Now we obtain the equiintegrability result for $\{\chi_{\widetilde{\Omega}\setminus C_{\delta_0}} \operatorname{cof} D\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ by expressing the cofactor matrix in terms of the 2D map $\boldsymbol{v}_k$ and observing that one entry is $\det D\boldsymbol{v}_k$ and the others are products of a sequence converging strongly in L^2 by a sequence converging weakly in L^2; cf. (6.12) in the Appendix.

Hence, there exists $\eta > 0$, independent of k, such that if $A \subset \widetilde{\Omega}$ is measurable,

$$|A| < \eta \;\Rightarrow\; \int_{A\setminus C_{\delta_0}} |\operatorname{cof} D\boldsymbol{u}_k|\,\mathrm{d}\boldsymbol{x} < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{6.4}$$

Given any open subset V of $\widetilde{\Omega}_{\boldsymbol{b}}$ (which we shall later choose to be a thin neighbourhood of $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u},L)$), and any good $\delta_1 < \delta_0$,

$$\int_V |\nabla(\boldsymbol{u}_k^{-1})_3|\,\mathrm{d}\boldsymbol{y} = \int_{\boldsymbol{u}_k^{-1}(V)\cap C_{\delta_1}} |\operatorname{adj}\nabla\boldsymbol{u}_k\,\boldsymbol{e}_3|\,\mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{u}_k^{-1}(V)\setminus C_{\delta_1}} |\operatorname{adj}\nabla\boldsymbol{u}_k\,\boldsymbol{e}_3|\,\mathrm{d}\boldsymbol{x}. \tag{6.5}$$

By Lemma 6.1 the first integral in the right-hand side of (6.5) is bounded by the integral of $\frac{1}{2}|D\boldsymbol{u}_k|^2$ in $C_{\delta_1}\cap\widetilde{\Omega}$. As for the second integral, note that

$$|\boldsymbol{u}_k^{-1}(V) \setminus C_{\delta_1}| \leq \int_V \det D\boldsymbol{u}_k^{-1}\,\mathrm{d}\boldsymbol{y}. \tag{6.6}$$

Since $\mathcal{L}^3\big(\operatorname{im}_{\mathrm{T}}(\boldsymbol{u},L)\big) = 0$ combining (6.6), (6.4), and the equiintegrability of $\{\det D\boldsymbol{u}_k^{-1}\}_{k\in\mathbb{N}}$ it is possible to find $\delta_1 > 0$, with $\delta_1 < \delta_0$ and an open set $V \subset \widetilde{\Omega}_{\boldsymbol{b}}$ such that

$$\operatorname{im}_{\mathrm{T}}(\boldsymbol{u},L) \subset V \quad \text{and} \quad \int_{\boldsymbol{u}_k^{-1}(V)\setminus C_{\delta_1}} |\operatorname{cof} D\boldsymbol{u}_k|\,\mathrm{d}\boldsymbol{x} < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{6.7}$$

By (6.7), for this V we get

$$\int_V |\nabla(\boldsymbol{u}_k^{-1})_3|\,\mathrm{d}\boldsymbol{y} \leq \frac{1}{2}\int_{C_{\delta_1}\cap\widetilde{\Omega}} |D\boldsymbol{u}_k|^2\,\mathrm{d}\boldsymbol{x} + \varepsilon,$$

and therefore

$$
\begin{aligned}
|D(\boldsymbol{u}_k^{-1})_3|(V) &= \int_V |\nabla(\boldsymbol{u}_k^{-1})_3|\,\mathrm{d}\boldsymbol{y} + |D^s(\boldsymbol{u}_k^{-1})_3|(V) \\
&\leq \frac{1}{2}\int_{C_{\delta_1}\cap\widetilde{\Omega}} |D\boldsymbol{u}_k|^2\,\mathrm{d}\boldsymbol{x} + \varepsilon + |D^s(\boldsymbol{u}_k^{-1})_3|(\widetilde{\Omega}_{\boldsymbol{b}}).
\end{aligned}
$$

Observe that by Theorem 5.1 iii) the inclusions $\operatorname{supp} D^s\boldsymbol{u}^{-1} \subset \operatorname{im_T}(\boldsymbol{u}, L) \subset V$ hold. Then, by the inequality above, as $\boldsymbol{u}_k^{-1} \overset{*}{\rightharpoonup} \boldsymbol{u}^{-1}$ in $BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$ we have that

$$
|D^s(\boldsymbol{u}^{-1})_3|(\widetilde{\Omega}_{\boldsymbol{b}}) \leq \varepsilon + \liminf_{k\to\infty}\left[\frac{1}{2}\int_{C_{\delta_1}\cap\widetilde{\Omega}} |D\boldsymbol{u}_k|^2\,\mathrm{d}\boldsymbol{x} + |D^s(\boldsymbol{u}_k^{-1})_3|(\widetilde{\Omega}_{\boldsymbol{b}})\right]. \tag{6.8}
$$

On the other hand, by (6.3), as $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $H^1(\widetilde{\Omega}, \mathbb{R}^{3\times 3})$ we have also that

$$
\frac{1}{2}\int_{C_{\delta_1}\cap\widetilde{\Omega}} |D\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\widetilde{\Omega}\setminus C_{\delta_1}} |D\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} \leq \varepsilon + \liminf_{k\to\infty}\frac{1}{2}\int_{\widetilde{\Omega}\setminus C_{\delta_1}} |D\boldsymbol{u}_k|^2\,\mathrm{d}\boldsymbol{x}. \tag{6.9}
$$

Gathering (6.8) and (6.9), since $\varepsilon > 0$ is arbitrary we obtain that

$$
\frac{1}{2}\int_{\widetilde{\Omega}} |D\boldsymbol{u}|^2 + |D^s(\boldsymbol{u}^{-1})_3|(\widetilde{\Omega}_{\boldsymbol{b}}) \leq \liminf_{k\to\infty}\left[\frac{1}{2}\int_{\widetilde{\Omega}} |D\boldsymbol{u}_k|^2 + |D^s(\boldsymbol{u}_k^{-1})_3|(\widetilde{\Omega}_{\boldsymbol{b}})\right]. \tag{6.10}
$$

The proof of the proposition is concluded by gathering (6.2) and (6.10). □

Remark 6.3. Without the Sobolev regularity for the horizontal components of the inverse, the estimate of the first term of the right-hand side of (6.5) would have been made for the whole cofactor matrix, yielding in (6.1) the suboptimal prefactor $\sqrt{3}$ of Lemma 2.6 instead of the prefactor 2 coming from Lemma 6.1.

Remark 6.4. From Proposition 6.2 we get in particular that if $\boldsymbol{u}_k$ is a sequence in $\mathcal{A}_s^r$ with H^1-weak limit $\boldsymbol{u}$, then

$$
F(\boldsymbol{u}) \leq \liminf_{k\to\infty} E(\boldsymbol{u}_k).
$$

Since we are in the presence of a problem of lack of compactness it is natural to try and describe the space $\overline{\mathcal{A}_s^r}$ and the relaxed energy defined on this space by

$$
E_{\mathrm{rel}}(\boldsymbol{u}) := \inf\{\liminf_{k\to\infty} E(\boldsymbol{u}_k) : \{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \in \mathcal{A}_s^r \text{ and } \boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \text{ in } H^1(\Omega, \mathbb{R}^3)\}.
$$

It is well known that E_{rel} is the largest lower semicontinuous functional in $\overline{\mathcal{A}_s^r}$ (for the H^1-weak topology) that is below E in $\mathcal{A}_s^r$. Since F is lower semicontinuous in $\overline{\mathcal{A}_s^r}$ and

$$
E_{\mathrm{rel}} = E \quad \text{in } \mathcal{A}_s^r
$$

we conclude that

$$
E_{\mathrm{rel}} \geq F \quad \text{in } \overline{\mathcal{A}_s^r}. \tag{6.11}
$$

It is tempting to conjecture that the equality $E_{\mathrm{rel}} = F$ holds at least for some special choices of the function H. In view of Proposition 6.2 (and its consequence (6.11)), it remains to characterize $\overline{\mathcal{A}_s^r}$ and to show that for any $\boldsymbol{u} \in \overline{\mathcal{A}_s^r}$ there exists a sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_s^r$ converging weakly to $\boldsymbol{u}$ in $H^1(\Omega, \mathbb{R}^3)$ such that

$$
\lim_{n\to\infty} E(\boldsymbol{u}_n) = F(\boldsymbol{u}).
$$

There are serious difficulties in constructing this sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ (if it exists at all). One of them relies on the restrictions of being orientation-preserving and injective a.e., even though there are some partial results in this direction (see [36, 35, 12, 40, 13] and the references therein).

At any rate, the interest of defining the relaxed energy in an abstract way is to be able to prove that it attains its infimum in $\overline{\mathcal{A}_s^r}$, and that the initial energy attains its minimum in $\mathcal{A}_s^r$ if and only if there exists a minimizer of $E_{\rm rel}$ in $\overline{\mathcal{A}_s^r}$ which is in $\mathcal{A}_s^r$. These two facts are classical in the theory of relaxation and follow from abstract arguments. The energy F satisfies analogous properties and, hence, can be a substitute of $E_{\rm rel}$.

Theorem 6.5. *The energy F has a minimizer in $\overline{\mathcal{A}_s^r}$. If it belongs to $\mathcal{A}_s^r$, then it is also a minimizer of E.*

Proof. Recall that $\overline{\mathcal{A}_s^r}$ is closed for the weak convergence in $H^1(\widetilde{\Omega},\mathbb{R}^3)$. It is also bounded in $H^1(\widetilde{\Omega},\mathbb{R}^3)$. From Proposition 6.2, F is lower semicontinuous in $\overline{\mathcal{A}_s^r}$. Clearly, F is coercive in $\overline{\mathcal{A}_s^r}$. This readily implies the existence of minimizers.

As for the second part of the statement, we assume that there exists a minimizer $\boldsymbol{u}_0$ of F in $\overline{\mathcal{A}_s^r}$ such that $\boldsymbol{u}_0 \in \mathcal{A}_s^r$. We then have $F(\boldsymbol{u}_0) \leq F(\boldsymbol{w})$ for any $\boldsymbol{w} \in \overline{\mathcal{A}_s^r}$. But since $F = E$ in $\mathcal{A}_s^r$, we find that $E(\boldsymbol{u}_0) \leq E(\boldsymbol{w})$ for all $\boldsymbol{w} \in \mathcal{A}_s^r$. That is, $\boldsymbol{u}_0$ is a minimizer of E in $\mathcal{A}_s^r$. □

In the same vein, we have the following result.

Proposition 6.6. *The energy E has a minimizer in $\overline{\mathcal{A}_s^r}$.*

Proof. From Lemma 2.16 we have that $\overline{\mathcal{A}_s^r} \subset \mathcal{A}_s$ and from Proposition 3.1 that E is lower semicontinuous on $\mathcal{A}_s$. Moreover, $\overline{\mathcal{A}_s^r}$ is closed for the H^1-weak convergence. As noted before, E is coercive in $\mathcal{A}_s$. These are the three main ingredients to obtain the conclusion. □

It would be nice to have an explicit description of $\overline{\mathcal{A}_s^r}$. Although this characterization is missing, we are able to prove the existence of minimizers of the energy F in the explicit space $\mathcal{B}$ defined in (1.7) which is a priori larger than $\overline{\mathcal{A}_s^r}$. Indeed, from Theorem 5.1 we have that $\overline{\mathcal{A}_s^r} \subset \mathcal{B} \subset \mathcal{A}_s$. Besides, the energy F is well defined on $\mathcal{B}$, it controls the BV norm of the inverses, and a slight adaptation of Proposition 6.2 yields the lower semicontinuity of F in $\mathcal{B}$.

Proposition 6.7. *The energy F is sequentially lower semicontinuous in $\mathcal{B}$ for the H^1-weak convergence.*

Proof. Let $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}$ tending weakly in $H^1(\widetilde{\Omega},\mathbb{R}^3)$ to $\boldsymbol{u} \in \mathcal{B}$. We can assume that $\liminf_{k\to\infty} F(\boldsymbol{u}_k) < \infty$. In particular, $\sup_{k\in\mathbb{N}} |D\boldsymbol{u}_k^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}) < \infty$. As $\|\boldsymbol{u}_k\|_{L^\infty(\widetilde{\Omega},\mathbb{R}^3)}$ and, hence, $\|\boldsymbol{u}_k\|_{L^1(\widetilde{\Omega},\mathbb{R}^3)}$ are bounded, the BV norm of $\boldsymbol{u}_k^{-1}$ is bounded, so, due to Lemma 4.13, we have that, up to a subsequence, $\boldsymbol{u}_k^{-1} \overset{*}{\rightharpoonup} \boldsymbol{u}^{-1}$ in $BV(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)$ and a.e. From here, the proof is the same as in Proposition 6.2. □

Proof of Theorem 1.1. Let $\{\boldsymbol{u}_k\}_k$ be a minimizing sequence for F in $\mathcal{B}$. Clearly $\sup_k F(\boldsymbol{u}_k) < \infty$, and we can assume that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $H^1(\widetilde{\Omega},\mathbb{R}^3)$. Since, by Proposition 6.7, F is lower semicontinuous for the weak convergence in H^1 of maps in $\mathcal{B}$, it suffices to show that the weak limit $\boldsymbol{u}$ is in $\mathcal{B}$. We know from Proposition 3.1 that $\boldsymbol{u} \in \mathcal{A}_s$, $\det D\boldsymbol{u}_k \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\widetilde{\Omega})$ and $\operatorname{im_G}(\boldsymbol{u}_k,\widetilde{\Omega}) \to \operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega})$ a.e. In particular, $\widetilde{\Omega}_{\boldsymbol{b}} = \operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega})$ a.e. and from (4.17), $\operatorname{im_T}(\boldsymbol{u},L)$ is a null Lebesgue set. Now we use Lemma 2.6 to show that

$$
\begin{aligned}
F(\boldsymbol{u}_k) &\geq \int_{\widetilde{\Omega}} |\operatorname{cof} D\boldsymbol{u}_k| \,\mathrm{d}\boldsymbol{x} + 2|D^s\boldsymbol{u}_k^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}) \\
&\geq \int_{\operatorname{im_G}(\boldsymbol{u},\widetilde{\Omega})} |\nabla \boldsymbol{u}_k^{-1}| \,\mathrm{d}\boldsymbol{y} + |D^s\boldsymbol{u}_k^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}) = |D\boldsymbol{u}_k^{-1}|(\widetilde{\Omega}_{\boldsymbol{b}}).
\end{aligned}
$$

As $\{\boldsymbol{u}_k^{-1}\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)$, we find that $\boldsymbol{u}_k^{-1}$ is bounded in $BV(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)$. Up to a subsequence, thanks to Lemma 4.13, we have that $\boldsymbol{u}_k^{-1} \to \boldsymbol{u}^{-1}$ in $L^1(\widetilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^3)$ and a.e., with

$\boldsymbol{u}^{-1} \in BV(\widetilde{\Omega}_{\boldsymbol{b}}, \mathbb{R}^3)$. From Proposition 4.15 we also infer that u_1^{-1}, u_2^{-1} are in $W^{1,1}(\widetilde{\Omega}_{\boldsymbol{b}})$. This proves that $\boldsymbol{u}$ minimizes F in $\mathcal{B}$.

The other statement of Theorem 1.1 can be shown as in the proof of Theorem 6.5. □

The following is a summary of existence results we obtained in this article.

Spaces	$\mathcal{A}_s^r$	$\overline{\mathcal{A}_s^r}$	$\overline{\mathcal{A}_s^r}$	$\mathcal{B}$	$\mathcal{A}_s$
Energies	E	E	F	F	E
Minimizers	?	yes Prop. 6.6	yes Th. 6.5	yes Th. 1.1	yes Th. 3.2

Appendix: Working with axially symmetric maps

We recall from the Appendix in [33] that if $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ is axisymmetric and is given in cylindrical coordinates by $\boldsymbol{u}(r\cos\theta, r\sin\theta, x_3) = v_1(r, x_3)\boldsymbol{e}_r + v_2(r, x_3)\boldsymbol{e}_3$ then

$$D\boldsymbol{u} = \begin{pmatrix} \partial_r v_1 & 0 & \partial_{x_3} v_1 \\ 0 & \frac{v_1}{r} & 0 \\ \partial_r v_2 & 0 & \partial_{x_3} v_2 \end{pmatrix}, \qquad \operatorname{cof} D\boldsymbol{u} = \begin{pmatrix} \frac{v_1}{r}\partial_{x_3} v_2 & 0 & -\frac{v_1}{r}\partial_r v_2 \\ 0 & \det Dv & 0 \\ -\frac{v_1}{r}\partial_{x_3} v_1 & 0 & \frac{v_1}{r}\partial_r v_1 \end{pmatrix}, \tag{6.12}$$

$$\det D\boldsymbol{u} = \frac{1}{r} v_1 \det D\boldsymbol{v},$$

and the Dirichlet energy is given by

$$\int_\Omega |D\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = 2\pi \int_{\pi(\Omega)} \left(|\partial_r \boldsymbol{v}|^2 + |\partial_{x_3} \boldsymbol{v}|^2\right) r \,\mathrm{d}r\,\mathrm{d}x_3 + 2\pi \int_{\pi(\Omega)} \frac{v_1^2}{r}\,\mathrm{d}r\,\mathrm{d}x_3.$$

Acknowledgements

We gratefully acknowledge J. Ball for his observation (see Section 3) that in the axisymmetric setting cavitation does not truly show that the neo-Hookean energy fails to be H^1-quasiconvex.

Marco Barchiesi has been supported by project VATEXMATE. Duvan Henao has been funded by the FONDECYT project 1190018. Carlos Mora-Corral has been supported by the Agencia Estatal de Investigación of the Spanish Ministry of Research and Innovation, through project PID2021-124195NB-C32 and the Severo Ochoa Programme for Centres of Excellence in R&D CEX2019-000904-S, by the Madrid Government (Comunidad de Madrid, Spain) under the multiannual Agreement with UAM in the line for the Excellence of the University Research Staff in the context of the V PRICIT (Regional Programme of Research and Technological Innovation), and by the ERC Advanced Grant 834728. Rémy Rodiac has been partially supported by the ANR project BLADE Jr. ANR-18-CE40-0023.

There is no conflict of interest and there is no data attached to this manuscript.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, New York, 2000.

[2] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal., 63 (1977), pp. 337–403.

[3] ———, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Philos. Trans. Roy. Soc. London Ser. A, 306 (1982), pp. 557–611.

[4] J. M. Ball and F. Murat, *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal., 58 (1984), pp. 225–253.

[5] M. Barchiesi, D. Henao, and C. Mora-Corral, *Local invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity*, Arch. Rational Mech. Anal., 224 (2017), pp. 743–816.

[6] M. Barchiesi, D. Henao, C. Mora-Corral, and R. Rodiac, *On the lack of compactness problem in the axisymmetric neo-Hookean model.* https://arxiv.org/abs/2111.07112.

[7] F. Bethuel, H. Brezis, and J.-M. Coron, *Relaxed energies for harmonic maps*, in Variational methods (Paris, 1988), vol. 4 of Progr. Nonlinear Differential Equations Appl., Birkhäuser Boston, Boston, MA, 1990, pp. 37–52.

[8] H. Brezis, J.-M. Coron, and E. H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys., 107 (1986), pp. 649–705.

[9] H. Brezis and L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.), 1 (1995), pp. 197–263.

[10] R. F. Brown, *A topological introduction to nonlinear analysis*, Birkhäuser Boston, Inc., Boston, MA, 1993.

[11] S. Conti and C. De Lellis, *Some remarks on the theory of elasticity for compressible Neohookean materials*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2 (2003), pp. 521–549.

[12] S. Conti and G. Dolzmann, *On the theory of relaxation in nonlinear elasticity with constraints on the determinant*, Arch. Rational Mech. Anal., 217 (2015), pp. 413–437.

[13] G. De Philippis and A. Pratelli, *The closure of planar diffeomorphisms in Sobolev spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 37 (2020), pp. 181–224.

[14] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1985.

[15] J. Dieudonné, *Treatise on analysis. Vol. III*, Academic Press, New York, 1972.

[16] A. Doležalová, S. Hencl, and J. Malý, *Weak limit of homeomorphisms in $W^{1,n-1}$ and (INV) condition.* ArXiv preprint 2112.08041, 2021.

[17] A. Doležalová, S. Hencl, and A. Molchanova, *Weak limit of homeomorphisms in $W^{1,n-1}$: invertibility and lower semicontinuity of energy.* ArXiv preprint 2212.06452, 2022.

[18] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.

[19] H. Federer, *Geometric measure theory*, Springer, New York, 1969.

[20] I. Fonseca and W. Gangbo, *Degree theory in analysis and applications*, Oxford University Press, New York, 1995.

[21] ———, *Local invertibility of Sobolev functions*, SIAM J. Math. Anal., 26 (1995), pp. 280–304.

[22] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal., 106 (1989), pp. 97–159.

[23] ———, *Cartesian currents in the calculus of variations. II*, Springer-Verlag, Berlin, 1998.

[24] V. M. Gol'dshteĭn and Y. G. Reshetnyak, *Quasiconformal mappings and Sobolev spaces*, vol. 54 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1990.

[25] R. Hardt and F.-H. Lin, *A remark on H^1 mappings*, Manuscripta Math., 56 (1986), pp. 1–10.

[26] R. Hardt, F.-H. Lin, and C.-C. Poon, *Axially symmetric harmonic maps minimizing a relaxed energy*, Comm. Pure Appl. Math., 45 (1992), pp. 417–459.

[27] D. Henao and C. Mora-Corral, *Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity*, Arch. Rational Mech. Anal, 197 (2010), pp. 619–655.

[28] ———, *Fracture surfaces and the regularity of inverses for BV deformations*, Arch. Rational Mech. Anal., 201 (2011), pp. 575–629.

[29] ———, *Lusin's condition and the distributional determinant for deformations with finite energy*, Adv. Calc. Var., 5 (2012), pp. 355–409.

[30] ———, *Regularity of inverses of Sobolev deformations with finite surface energy*, J. Funct. Anal., 268 (2015), pp. 2356–2378.

[31] D. Henao, C. Mora-Corral, and M. Oliva, *Global invertibility of Sobolev maps*, Adv. Calc. Var., 14 (2021), pp. 207–230.

[32] D. Henao, C. Mora-Corral, and X. Xu, *Γ-convergence approximation of fracture and cavitation in nonlinear elasticity*, Arch. Rational Mech. Anal., 216 (2015), pp. 813–879.

[33] D. Henao and R. Rodiac, *On the existence of minimizers for the neo-Hookean energy in the axisymmetric setting*, Discrete Con. Syn. Sist. Series A, 38 (2018), pp. 4509–4536.

[34] S. Hencl and P. Koskela, *Lectures on mappings of finite distortion*, vol. 2096 of Lecture Notes in Mathematics, Springer, Cham, 2014.

[35] S. Hencl and C. Mora-Corral, *Diffeomorphic approximation of continuous almost everywhere injective Sobolev deformations in the plane*, Q. J. Math., 66 (2015), pp. 1055–1062.

[36] T. Iwaniec, L. V. Kovalev, and J. Onninen, *Diffeomorphic approximation of Sobolev homeomorphisms*, Arch. Ration. Mech. Anal., 201 (2011), pp. 1047–1067.

[37] J. G. Llavona, *Approximation of continuously differentiable functions*, vol. 130 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1986.

[38] M. Marcus and V. J. Mizel, *Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems*, Bull. Amer. Math. Soc., 79 (1973), pp. 790–795.

[39] L. Martinazzi, *A note on n-axially symmetric harmonic maps from B^3 to S^2 minimizing the relaxed energy*, J. Funct. Anal., 261 (2011), pp. 3099–3117.
[40] C. Mora-Corral and M. Oliva, *Relaxation of nonlinear elastic energies involving the deformed configuration and applications to nematic elastomers*, ESAIM: COCV, 25 (2019), p. 19.
[41] S. Müller, *Weak continuity of determinants and nonlinear elasticity*, C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), pp. 501–506.
[42] ———, *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angew. Math., 412 (1990), pp. 20–34.
[43] ———, Det = det. *A remark on the distributional determinant*, C. R. Acad. Sci. Paris Sér. I Math., 311 (1990), pp. 13–17.
[44] S. Müller and S. J. Spector, *An existence theory for nonlinear elasticity that allows for cavitation*, Arch. Rational Mech. Anal., 131 (1995), pp. 1–66.
[45] S. Müller, Q. Tang, and B. S. Yan, *On a new class of elastic deformations not allowing for cavitation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 11 (1994), pp. 217–243.
[46] Y. G. Reshetnyak, *Spatial mappings with bounded distortion*, Sibirsk. Mat. Ž., 8 (1967), pp. 629–658.
[47] ———, *Space mappings with bounded distortion*, vol. 73 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1989.
[48] J. Sivaloganathan and S. J. Spector, *On the existence of minimizers with prescribed singular points in nonlinear elasticity*, J. Elast., 59 (2000), pp. 83–113.
[49] V. Šverák, *Regularity properties of deformations with finite energy*, Arch. Rational Mech. Anal., 100 (1988), pp. 105–127.
[50] L. R. G. Treloar, *The physics of rubber elasticity*, Clarendon Press, Oxford, 1975.
[51] S. K. Vodop'yanov and V. M. Gol'dšteĭn, *Quasiconformal mappings, and spaces of functions with first generalized derivatives*, Sibirsk. Mat. Ž., 17 (1976), pp. 515–531, 715.
[52] J. Weiner, *Statistical mechanics of elasticity*, Dover Publications, 2nd ed., 2017.

(Marco Barchiesi) Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, Via Weiss 2 - 34128 Trieste, Italy.
Email address: barchies@gmail.com

(Duvan Henao) Faculty of Mathematics & Institute for Mathematical and Computational Engineering, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile. (Current address: Instituto de Ciencias de la Ingeniería, Universidad de O'Higgins, Rancagua, Chile.)
Email address: dahenao@uc.cl

(Carlos Mora-Corral) Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain and Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain.
Email address: carlos.mora@uam.es

(Rémy Rodiac) Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405, Orsay, France,
Email address: remy.rodiac@math.universite-paris-saclay.fr