The L^1-relaxed area of the graph of the vortex map

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Abstract

We compute the value of the L^1-relaxed area of the graph of the map $u : \mathrm{B}_l(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$, $u(x) := x/|x|$, $x \neq 0$, for all values of $l > 0$. Interestingly, for l in a certain range, in particular l not too large, a Plateau-type problem, having as solution a sort of catenoid constrained to contain a segment, has to be solved.

Key words: Relaxation, Cartesian currents, area functional, minimal surfaces, Plateau problem.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

Contents

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1 Introduction

Determining the domain and the expression of the relaxed area functional for graphs of nonsmooth maps in codimension greater than 1 is a challenging problem whose solution is far from being reached. Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a map $v : \Omega \to \mathbb{R}^N$ of class C^1, the area of the graph of v over Ω is given by the classical formula

$$\mathcal{A}(v, \Omega) = \int_\Omega |\mathcal{M}(\nabla v)| \, dx, \tag{1.1}$$

where $\mathcal{M}(\nabla v)$ is the vector whose entries are the determinants of the minors of the gradient ∇v of v of all orders[1] k, $0 \leq k \leq \min\{n, N\}$. Classical methods of relaxation suggest to consider the functional defined, for any $v \in L^1(\Omega, \mathbb{R}^N)$, as

$$\overline{\mathcal{A}}(v, \Omega) := \inf \Big\{ \liminf_{k \to +\infty} \mathcal{A}(v_k, \Omega) \Big\}, \tag{1.2}$$

[1] By convention, the determinant of order 0 is 1.

and called *(sequential) relaxed area functional.* The infimum is computed over all sequences of maps $v_k \in C^1(\Omega, \mathbb{R}^N)$ approaching v in $L^1(\Omega, \mathbb{R}^N)$. The results of Acerbi and Dal Maso [1] show that $\overline{\mathcal{A}}(\cdot, \Omega)$ extends $\mathcal{A}(\cdot, \Omega)$ and is L^1-lower-semicontinuous. This procedure of relaxation, besides extending the notion of graph's area to non-smooth maps, is needed also because $\mathcal{A}(\cdot, \Omega)$ is not L^1-lower-semicontinuous[2], in contrast with similar polyconvex functionals that enjoy a growth condition of the form $F(u) \geq C|\mathcal{M}(\nabla u)|^p$ for some $C > 0$, and suitable $p > 1$ (see, *e.g.*, [11,17,26]).

When $N = 1$ it is possible to characterize the domain of $\overline{\mathcal{A}}(\cdot, \Omega)$ and its expression [12]: $\overline{\mathcal{A}}(v, \Omega)$ is finite if and only if $v \in BV(\Omega)$, in which case

$$\overline{\mathcal{A}}(v, \Omega) = \int_\Omega \sqrt{1 + |\nabla v|^2} dx + |D^s v|(\Omega), \tag{1.3}$$

∇v and $D^s v$ representing the absolutely continuous and singular parts of the distributional gradient Dv of v. Formula (1.3) is a basic example of non-parametric variational integral that is a measure when considered as a function of Ω [20], and is crucial, among others, in the study of capillarity problems [16], and in the analysis of the Cartesian Plateau problem [19]. The case $N > 1$ (referred here to as the case of codimension greater than 1) is much more involved. Again, one of its main motivations is the study of the Cartesian Plateau problem in higher codimension; in addition, from the point of view of Calculus of Variations, it is of interest in those vector minimum problems involving nonconvex integrands with nonstandard growth [3], [11], [18].

Let us restrict our attention to the case $n = N = 2$. For a map $v \in C^1(\Omega, \mathbb{R}^2)$ and $\Omega \subset\subset \mathbb{R}^2$, $\mathcal{A}(v, \Omega)$ coincides with the area of the graph $G_v := \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$ of v seen as a Cartesian surface of codimension 2 in $\Omega \times \mathbb{R}^2$, and is given by

$$\mathcal{A}(v, \Omega) = \int_\Omega \sqrt{1 + |\nabla v(x_1, x_2)|^2 + |Jv(x_1, x_2)|^2}\ dx_1 dx_2.$$

Here ∇v is the gradient of v, a 2×2 matrix, $|\nabla v|^2$ is the sum of the squares of all elements of ∇v, and Jv is the Jacobian determinant of v, *i.e.*, the determinant of ∇v. It is worth to point out once more a couple of relevant difficulties arising when the codimension is greater than 1: the functional $\mathcal{A}(\cdot, \Omega)$ is no longer convex, but just polyconvex; in addition it has a sort of unilateral linear growth, in the sense that it is bounded below, but not necessarily above, by the total variation. A characterization of the domain of $\overline{\mathcal{A}}(\cdot, \Omega)$ and of its expression is, at the moment, not available. Specifically, it is only known that the domain of $\overline{\mathcal{A}}(\cdot, \Omega)$ is a proper subset of $BV(\Omega, \mathbb{R}^2)$, and that integral representation formulas such as (1.3) (on the domain of $\overline{\mathcal{A}}(\cdot, \Omega)$) are not possible. This is due to the additional difficulty that in general, for a fixed map v, the set function $A \subseteq \Omega \mapsto \overline{\mathcal{A}}(v, A)$ may be not subadditive, and in particular it cannot be a measure (as opposite to what happens in codimension 1 for a large class of non-parametric variational integrals [20]). This interesting phenomenon was conjectured by De Giorgi [13] for the triple junction map $u_T : \Omega = \mathrm{B}_l(0) \to \mathbb{R}^2$, and proved in [1], where the authors exhibited three subsets $\Omega_1, \Omega_2, \Omega_3$ of the open disk $\mathrm{B}_l(0)$ of radius l centered at 0, such that

$$\Omega_1 \subset \Omega_2 \cup \Omega_3 \qquad \text{and} \qquad \overline{\mathcal{A}}(u_T, \Omega_1) > \overline{\mathcal{A}}(u_T, \Omega_2) + \overline{\mathcal{A}}(u_T, \Omega_3). \tag{1.4}$$

The triple junction map $u_T \in BV(\Omega, \mathbb{R}^2)$ takes only three values $\alpha, \beta, \gamma \in \mathbb{R}^2$, the vertices of an equilateral triangle, in three circular 120^o-degree sectors of Ω meeting at 0. The same authors

[2]When $n = N = 2$, there are sequences $(v_k) \subset W^{1,p}(\Omega, \mathbb{R}^2)$, with $p \in [1, 2)$, weakly converging in $W^{1,p}(\Omega, \mathbb{R}^2)$ to a smooth map v for which $\mathcal{A}(v, \Omega) > \limsup_{k\to+\infty} \mathcal{A}(v_k, \Omega)$, where $\mathcal{A}(v_k, \Omega)$ is defined as for C^1-maps in (1.1), with the determinant of ∇v_k intended in the almost everywhere pointwise sense; see [4, Counterexample 7.4] and [1]. This counterexample must be slightly modified, considering $u_k(x) = kx + \lambda(x/\|x\|_\infty - x)$ for $x \in [-1/k, 1/k]$, with $\lambda > 0$ satisfying $(1 + \lambda^2)/2 > \sqrt{1 + \lambda^2}$, in order to get the strict inequality above.

show that the non-locality property (1.4) holds also for the Sobolev map $u(x) = \frac{x}{|x|}$, called here the vortex map, where Ω is a ball of radius l centered at the origin, the singular point, and $n = N \geq 3$. For these two maps u_T and u much effort has been done to understand the exact value of the area functional; the corresponding geometric problem stands in finding the optimal way, in terms of area, to "fill the holes" of the graph of u_T and u (two non-smooth 2-dimensional sets of codimension two) with limits of sequences of smooth two-dimensional graphs. In [1] it is proved that both u_T and u have finite relaxed area, but only lower and upper bounds were available for u_T, whereas the sharp estimate for u is provided only for l large enough. For the triple junction map u_T an improvement is obtained in [6], where it is exhibited a sequence (u_k) of Lipschitz maps $u_k : \mathrm{B}_l(0) \to \mathbb{R}^2$ converging to u in $L^1(\Omega, \mathbb{R}^2)$, such that

$$\lim_{k\to+\infty} \mathcal{A}(u_k, \mathrm{B}_l(0)) = |\mathcal{G}_{u_T}| + 3m_l,$$

where $|\mathcal{G}_{u_T}|$ is the area of the graph of u_T out of the jump set, and m_l is the area of an area-minimizing surface, solution of a Plateau-type problem in $\mathbb{R}^3$. Roughly speaking, three entangled area-minimizing surfaces with area m_l (each sitting in a copy of $\mathbb{R}^3 \subset \mathbb{R}^4$, the three $\mathbb{R}^3$'s being mutually nonparallel) are needed in $\mathrm{B}_l(0) \times \mathbb{R}^2$ to "fill the holes" left by the graph $\mathcal{G}_{u_T}$ of u_T, which is not boundaryless (*i.e.*, the boundary as a current is nonzero). The optimality of (u_k) was also conjectured in [6], and proven subsequently in [28], where a crucial tool is a symmetrization technique for boudaryless integral currents.

In the present paper we compute the value of the relaxed area functional for the *vortex map* u in two dimensions. That is,

$$u(x) := \frac{x}{|x|}, \qquad x \in \Omega \setminus \{0\}, \ \Omega = \mathrm{B}_l(0) \subset \mathbb{R}^2. \tag{1.5}$$

Observe that u belongs to $W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p \in [1, 2)$, and that the image of u is the one-dimensional unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, so that $Ju(x) = \det(\nabla u(x)) = 0$ for all $x \in \Omega \setminus \{0\}$. In [1, Lemma 5.2], the authors show[3] that, for l large enough,

$$\overline{\mathcal{A}}(u, \mathrm{B}_l(0)) = |\mathcal{G}_u| + \pi = \int_{\mathrm{B}_l(0)} \sqrt{1 + |\nabla u|^2} dx + \pi. \tag{1.6}$$

With the aid of an example, they also show that $\overline{\mathcal{A}}(u, \mathrm{B}_l(0))$ must be strictly smaller than the right-hand side of (1.6), since there is a sequence of C^1-maps approximating u and having, asymptotically, a lower value of $\mathcal{A}(\cdot, \Omega)$. We anticipate here that, when l is small, the above mentioned sequence is not optimal, and the construction of a recovery sequence for $\overline{\mathcal{A}}(u, \mathrm{B}_l)$ is much more involved and requires to solve a sort of Plateau-type problem in $\mathbb{R}^3$ with singular boundary, with a part of multiplicity 2. Equivalently, with a reflection argument with respect to a plane, it can be seen as a non-parametric Plateau-type problem with a partial free boundary; one of our results (Theorem 12.16, valid for any $l > 0$) consists in the analysis of solutions of this problem, in particular we show that, excluding a singular configuration[4], there is a non-parametric solution attaining a zero boundary condition on the free part.

In order to give an idea of how the value π in (1.6) pops up, it is convenient to introduce the tool of Cartesian currents. One can regard the graphs $G_v = \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$ of C^1 maps $v : \Omega \to \mathbb{R}^2$ as integer multiplicity 2-currents in $\Omega \times \mathbb{R}^2$. It is seen that a sequence (G_{u_k}) with u_k approaching u and with $\sup_k \mathcal{A}(u_k, \Omega) < +\infty$, converges[5], up to subsequences, to a Cartesian

[3] In [1] the proof of (1.6) is given also for $N = n \geq 2$, where now π in (1.6) is replaced by ω_n.

[4] This corresponds to assumption (iii) in Lemma 12.13.

[5] This is a consequence of Federer-Fleming closure theorem.

current T which splits as $T = \mathcal{G}_u + S$, with S a vertical integral current such that $\partial S = -\partial \mathcal{G}_u$. A direct computation shows that

$$\partial \mathcal{G}_u = -\delta_0 \times \partial [\![B_1]\!]$$

(see [18, Section 3.2.2]), so that the problem of determining the value of $\overline{\mathcal{A}}(u, \Omega)$ is somehow related to the computation of the mass of a mass-minimizing vertical current $S_{\min} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ satisfying

$$\partial S_{\min} = \delta_0 \times \partial [\![B_1]\!] \qquad \text{in } \mathcal{D}_1(\Omega \times \mathbb{R}^2). \tag{1.7}$$

In some cases, and in particular for l large, these two problems are related, and it turns out that $S_{\min} = \delta_0 \times [\![B_1]\!]$, whose mass is π. However $S_{\min} \neq \delta_0 \times [\![B_1]\!]$ for l small. Moreover, the two problems of determining $S_{\min}$ and the value of the relaxed area functional are, unfortunately, not related in general. This is mainly due to the following two obstructions:

- we have to guarantee that the current $\mathcal{G}_u + S_{\min}$ is obtained as a limit of smooth graphs, that is not easy to establish since not all Cartesian currents can be obtained as such limits (see [18, Section 4.2.2]);
- even if $\mathcal{G}_u + S_{\min}$ is limit of graphs $\mathcal{G}_{u_k}$ of smooth maps u_k, nothing ensures that $\mathcal{A}(u_k, \Omega) \to \overline{\mathcal{A}}(u, \Omega)$, due to possible cancellations of the currents $\mathcal{G}_{u_k}$ that, in the limit, might overlap with opposite orientation.

Actually, in many cases, as in the one considered in this paper, for an optimal sequence (u_k) realizing the value of $\overline{\mathcal{A}}(u, \Omega)$, it holds

$$\mathcal{G}_{u_k} \rightharpoonup \mathcal{G}_u + S_{\mathrm{opt}} \neq \mathcal{G}_u + S_{\min}, \tag{1.8}$$

and the limit vertical part S_{opt} satisfies $|S_{\mathrm{opt}}| > |S_{\min}|$. For instance, if l is small, it is possible to construct a sequence $(\widehat{u}_k)$ approaching u which is not a recovery sequence for $\overline{\mathcal{A}}(u, \Omega)$, but whose limit vertical part $S_{\min}$ has mass strictly smaller than the one of S_{opt} (see Section 10.2). In this case, a suitable projection of $S_{\min}$ in $\mathbb{R}^3$ is half of a classical area-minimizing catenoid between two unit circles at distance $2l$ from each other.

An additional source of difficulties in the computation of $\overline{\mathcal{A}}(u, \Omega)$ is due to an example [28] valid for the triple junction map u_T, and showing that the equality

$$\overline{\mathcal{A}}(u_T, \Omega) = |\mathcal{G}_{u_T}| + 3m_l \tag{1.9}$$

holds only under some additional requirements; for instance if the triple junction point is exactly located at the origin 0 and the domain is a disc $\Omega = \mathrm{B}_l(0)$ around it. In particular, for different domains, (1.9) is no longer valid, and S_{opt} is a vertical current whose support projection on Ω is a set connecting the triple point with $\partial\Omega$, and which does not coincide with (neither is a subset of) the jump set of u_T (see [28, Example in Section 6] and also [5] for other non-symmetric settings).

A similar behaviour of the vertical part S_{opt} holds for u: when l is small, the projection of S_{opt} on $\mathrm{B}_l(0)$ concentrates over a radius connecting 0 to $\partial \mathrm{B}_l(0)$, see Fig. 2, left. However, if the domain Ω loses its symmetry, almost nothing is known about S_{opt}.

This kind of phenomena have been observed also in other cases, as in [7, 8] where BV-maps $u : \Omega \to \mathbb{R}^2$ with a prescribed discontinuity on a curve (jump set) are considered. The creation of such "phantom bridges" between the singularities of the map u and the boundary of the domain is very specific of the choice of the L^1 topology in the computation of $\overline{\mathcal{A}}(\cdot, \Omega)$. Other choices are possible, giving rise to different relaxed functionals[6] (see [7,8]).

[6]Relaxing $\mathcal{A}(\cdot, \Omega)$ in stronger topologies τ is possible; however, this would make more difficult to prove, eventually, τ-coercivity of $\overline{\mathcal{A}}(\cdot, \Omega)$. In addition, it could destroy the interesting nonlocal phenomena related to the appearence of certain nonstandard Plateau problems, which are the focus of this paper.

The nonlocality and the uncontrollability of $S_{\rm opt}$ are more and more evident if we try to generalize (1.9) dropping the assumption that the range of u_T consists of the vertices of an equilateral triangle. If we assume that u_T takes values in $\{\alpha,\beta,\gamma\}$, three generic (not aligned) points in $\mathbb{R}^2$ then, also if the domain of u_T is symmetric, there is no sharp computation of $\overline{\mathcal{A}}(u_T,\Omega)$. In this case, the analysis is related to an entangled Plateau problem, where three area-minimizing discs have as partial free boundary three curves connecting the couples of points in $\{\alpha,\beta,\gamma\}$, respectively, and where these three curves are forced to overlap. Some partial results had been obtained in [5], where the authors find an upper bound for $\overline{\mathcal{A}}(u_T,\Omega)$. However the question of finding the value of $\overline{\mathcal{A}}(\cdot,\Omega)$ for this piecewise constant maps seems to be difficult.

Before stating our main results we need to fix some notation[7]. For $l>0$ we denote $R_{2l}:=(0,2l)\times(-1,1)$ and let $\partial_D R_{2l}:=(\{0,2l\}\times[-1,1])\cup((0,2l)\times\{-1\})$ be what we call the Dirichlet boundary of R_{2l}. Define $\varphi:\partial_D R_{2l}\to[0,1]$ as $\varphi(t,s):=\sqrt{1-s^2}$ if $(t,s)\in\{0,2l\}\times[-1,1]$ and $\varphi(t,s):=0$ if $(t,s)\in(0,2l)\times\{-1\}$. Let

$$\begin{aligned}&\widetilde{\mathcal{H}}_{2l}:=\{h:[0,2l]\to[-1,1],\ h \text{ continuous},\ h(0)=h(2l)=1\},\\ &\mathcal{X}_{D,\varphi}:=\{\psi\in W^{1,1}(R_{2l}):\psi=\varphi \text{ on } \partial_D R_{2l}\},\end{aligned}$$

and for any $h\in\widetilde{\mathcal{H}}_{2l}$ set $G_h:=\{(t,s)\in R_{2l}:s=h(t)\}$ and $SG_h:=\{(t,s)\in R_{2l}:s\le h(t)\}$ (see Fig. 18 for a view of the setting). The main result of the present paper (see Theorems 11.16 and 13.2) reads as follows:

Theorem 1.1. *Let $N=n=2$, $l>0$ and $u:\mathrm{B}_l(0)\to\mathbb{R}^2$ be the vortex map defined in* (1.5)*. Then*

$$\overline{\mathcal{A}}(u,\mathrm{B}_l(0))=\int_{\mathrm{B}_l(0)}\sqrt{1+|\nabla u|^2}dx+\inf\{\mathcal{A}(\psi,SG_h):(h,\psi)\in\widetilde{\mathcal{H}}_{2l}\times\mathcal{X}_{D,\varphi},\ \psi=0 \ on\ G_h\}. \quad (1.10)$$

We show that for l large enough the infimum is not attained in $\widetilde{\mathcal{H}}_{2l}\times\mathcal{X}_{D,\varphi}$ and equals π. We prove that a minimizer instead exists for l small, hence ψ is real analytic in the interior of SG_h; furthermore, we show that h is smooth and convex, and ψ has vanishing trace on the graph of h (Theorem 12.16).

We also show that the infimum on the right-hand side of (1.10) can be equivalently rewritten in many ways. Let us first point out that the functional $\mathcal{A}(\cdot,SG.)$ is not lower-semicontinuous, so also at this stage we need a relaxation of the infimum problem, and we introduce the functional $\mathcal{F}_{2l}$, defined on a new class $X_{2l}^{\rm conv}$ of admissible pairs of functions, which is obtained after specializing the choice of $h\in\widetilde{\mathcal{H}}_{2l}$ and then generalizing the choice of $\psi\in\mathcal{X}_{D,\varphi}$ (see Definition 12.2 in Section 12).

An equivalent formulation for this minimum problem is the following: Let us consider any Lipschitz simple curve $\gamma:[0,1]\to\overline{R}_{2l}$ with $\gamma(0)=(0,1,0)$ and $\gamma(1)=(2l,1,0)$, and then the closed curve $\Gamma\subset\mathbb{R}^3$ defined by glueing the trace of γ with the graph of φ over $\partial_D R_{2l}$. We can then consider an area-minimizing disc Σ^+ spanning Γ, solution of the classical Plateau problem. Assuming for simplicity that $\gamma([0,1])$ is the graph of a function $h\in\widetilde{\mathcal{H}}_{2l}$, then

$$\inf\{\mathcal{A}(\psi,SG_h):(h,\psi)\in\widetilde{\mathcal{H}}_{2l}\times\mathcal{X}_{D,\varphi},\ \psi=0 \text{ on } G_h\}=\inf\mathcal{H}^2(\Sigma^+), \quad (1.11)$$

where the infimum on the right-hand side is computed over the set of all such curves γ (see Corollary 12.17). For l sufficiently small, say $l\in(0,l_0)$, the infimum is attained by a disc-type surface Σ^+,

[7]The relation with the map u will be clear after Section 11; at this point we remark that R_{2l} has its first coordinate which is essentially the radial coordinate in the source $\Omega=\mathrm{B}_l(0)$, and the second coordinate is instead the first coordinate ρ in the target space $\mathbb{R}^2$. The graph of the function φ is (half of) the lateral boundary of a cylinder, which coincides with (one half of) the image of the map $\Omega\ni x=(\rho,\theta)\mapsto(\rho,u(x))$.

and $\gamma([0,1])$ coincides with the graph of a smooth convex function $h \in \widetilde{\mathcal{H}}_{2l}$. On the contrary, for $l \geq l_0$, γ is degenerate, in the sense that if $\sigma_n \subset R_{2l}$ is the free-boundary of Σ_n^+, where (Σ_n^+) is a minimizing sequence of discs for the Plateau problem, then σ_n converges to the set $\partial_D R_{2l}$ and Σ_n^+ converges to two distinct half-circles of radius 1, whose total area is π.

We do not know the explicit value of the threshold l_0. However, it is clear that $l_0 > \frac{1}{2}$ (see the discussion at the end of Section 2.6 and Remark 2.2). Furthermore, doubling the surface Σ^+ by considering its symmetric with respect to the plane containing R_{2l}, and then taking the union Σ of these two area-minimizing surfaces, it turns out that Σ solves a non-standard Plateau problem, spanning a nonsimple curve which shows self-intersections (this is the union of Γ with its symmetric with respect to R_{2l}, the obtained curve is the union of two circles connected by a segment, see Section 2.6 and Fig. 1). Again, the obtained area-minimizing surface is a sort of catenoid forced to contain a segment (see Fig. 2, left) for l small, and two distinct discs spanning the two circles for l large (Fig. 2, right). The restriction of Σ to the set $\overline{B}_1 \times [0,l]$ is a suitable projection in $\mathbb{R}^3$ of the aforementioned vertical current S_{opt}.

We will discuss on the appearence of this Plateau problem in the end of this introduction: Let us first spend some words on how we prove Theorem 1.1. The proof is divided into two parts, namely the lower bound (Sections 3-11 excluding Section 10) and the upper bound (Sections 12 and 13). The proof of the lower bound, *i.e.*, the inequality $\geq$ in (1.10), is extremely involved: we assume (u_k) to be a recovery sequence converging to u, so that $\mathcal{A}(u_k, \mathrm{B}_l(0)) \to \overline{\mathcal{A}}(u, \mathrm{B}_l(0))$, and we analyse the behaviour of the graphs G_{u_k} over two distinct subsets of $\mathrm{B}_l(0)$, respectively one on which u_k converges uniformly to u, and one where concentration phenomena are allowed (let us call this the "bad set", denoted D_k in the sequel). In the former, studied in Section 4, we see that, up to small errors, the contribution of the areas of G_{u_k} gives the first term on the right-hand side of (1.10). In the set D_k, the graphs G_{u_k} might behave very badly. In order to detect their behaviour we introduce suitable projections in $\mathbb{R}^3$ (the maps Ψ_k in Definition 5.1 and the maps π_{λ_k} in Definition 5.3) and use them to reduce the currents carried by the graphs G_{u_k} to integral 2-currents supported in the cylinder $[0,l] \times \overline{\mathrm{B}}_1(0) \subset \mathbb{R}^3$. It is necessary to use a cylindrical Steiner-type symmetrization technique for these integral currents, described in Section 3. Afterwards, an additional partition of the domain is needed, and we focus on what happens far from the origin and in a neighbourhood $\mathrm{B}_\varepsilon(0)$ of it. The first analysis is carried on in Sections 5, 6, and 7. The analysis around 0 is instead done in Section 8. Roughly speaking, we construct a cylindrically symmetric integral 2-current in $[0,l] \times \overline{\mathrm{B}}_1(0)$ whose area, up to small errors, is equal to the area of G_{u_k} over D_k. In order to relate the area of this current with the second term on the right-hand side of (1.10), we have to artificially add some rectifiable sets to this current (see Section 9)[8], in such a way to force the new integral current to be a candidate for the minimum problem on the left-hand side of (1.11). Some additional rearrangements are needed here, which are described in Section 11. The passage to the limit as $k \to +\infty$ is then performed in Theorem 11.16, where we also show that all the errors in the estimates of the previous sections are negligible.

The second part of the paper concerns the upper bound in (1.10). This consists in a careful definition of a recovery sequence (u_k) converging to the vortex map, and thus such that $\mathcal{A}(u_k, \mathrm{B}_l(0))$ approaches the value on the right-hand side of (1.10) as $k \to +\infty$. In order to explicitely construct u_k, we need first to show that the minimum problem stated in Theorem 11.16 is in fact equivalent to the non-parametric Plateau-type problem in (1.10), *i.e.*, we have to prove (1.11). This is done in Section 12, where we exploit the convexity of the domain together with some well-known regularity results for the solution of the Plateau problem in this setting. This analysis leads us to Theorem

[8] To elucidate the meaning of all the objects we introduce, we have complemented this section with some examples contained in Section 10. Note that the construction in Section 10.1, as well as the catenoid with flap in Fig. 16 analysed in Section 10.2, does not lead to a recovery sequence, for any value of l. Nevertheless, we believe the examples to be useful in order to follow the construction made to prove the lower bound.

12.6, which characterizes the solution of (1.11), and which is based on a regularity result for the minimizing pair $(h^\star, \psi^\star) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi}$. Finally, thanks to the regularity results that we have obtained (especially, boundary regularity), in Section 13 we define explicitely the maps u_k, making a crucial use of rescaled versions of the area-minimizing surface Σ in a vertical copy of $\mathbb{R}^3$ inside $\mathbb{R}^4$, and prove the upper bound in Theorem 13.2.

From this discussion the appearence of the aforementioned nonstandard Plateau problem should be more clear. The shape of the solution Σ of this problem (after a suitable projection from $\mathbb{R}^4$ to $\mathbb{R}^3$) is related to the graph $\mathcal{G}_u$ upon the "limit of the bad set" (in turn related to S_{opt} in (1.8)). More precisely, let us fix a map u_k in the recovery sequence for $\overline{\mathcal{A}}(u, \mathrm{B}_l(0))$, and let us call D_{u_k} the corresponding bad set, roughly the set where the values of u_k remain "far" from those of u. Essentially, the slice of the half catenoid-type (containing the segment) surface $\Sigma \subset \overline{\mathrm{B}}_1(0) \times [0, l]$ with respect to a plane $\mathbb{R}^2 \times \{t\}$, $t \in (0, l)$, is a closed curve touching the lateral boundary of $\overline{\mathrm{B}}_1(0) \times [0, l]$ at a point. This will be the limit of the image of the restriction $u_{k|D_{u_k} \cap \partial \mathrm{B}_t(0)}$ in $\mathbb{R}^2$ (identified with $\mathbb{R}^2 \times \{t\}$). Indeed, $u_k((\mathrm{B}_1(0) \setminus D_{u_k}) \cap \partial \mathrm{B}_t(0))$ is a closed curve that lies very close to $\mathbb{S}^1$, whereas $u_k(D_{u_k} \cap \partial \mathrm{B}_t(0))$ makes a trip in $(\overline{\mathrm{B}}_1(0) \times [0, l]) \cap (\mathbb{R}^2 \times \{t\})$ in order to approach the shape of a t-slice of Σ. Since $u_k(\partial \mathrm{B}_t(0))$ is a closed curve, after passing to the limit as $k \to +\infty$, we obtain a closed curve which overlaps $\mathbb{S}^1$ (limit of the images of the complements of the bad sets) and then is a closed curve (limit of the images of D_{u_k}) attached to $\mathbb{S}^1$ at a point whose shape is the slice of Σ.

2 Preliminaries

2.1 Notation and conventions

The symbol $\mathcal{A}(v, \Omega)$ denotes the classical area of the graph of a smooth map $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, given by (1.1). We will deal with the case $n = 2$ and mostly with the cases $(n, N) = (2, 1)$ and $(n, N) = (2, 2)$. The relaxed area functional (with respect to the L^1-convergence) is denoted by $\overline{\mathcal{A}}(v, \Omega)$ and is defined in (1.2).

We first remark that the infimum in (1.2) can be considered as taken over the class of sequences $v_k \in \mathrm{Lip}(\Omega; \mathbb{R}^2)$. This does not change the value of $\overline{\mathcal{A}}(\cdot, \Omega)$, as observed in [6].

Recall that in formula (1.1) the symbol $\mathcal{M}(\nabla v)$ denotes the vector whose entries are all determinants of the minors of ∇v. Precisely, let α and β be subsets of $\{1, 2\}$, let $\bar{\alpha}$ denote the complementary set of α, namely $\bar{\alpha} = \{1, 2\} \setminus \alpha$, let $|\cdot|$ denote the cardinality, and let $A \in \mathbb{R}^{2\times 2}$ be a matrix. Then, if $|\alpha| + |\beta| = 2$, we denote by

$$M^{\beta}_{\bar{\alpha}}(A) \tag{2.1}$$

the determinant of the submatrix of A whose lines are those with index in β, and columns with index in $\bar{\alpha}$. By convention $M^{\emptyset}_{\emptyset}(A) = 1$ and moreover

$$M^i_j = a_{ij}, \qquad i, j \in \{1, 2\}, \qquad\qquad M^{\{1,2\}}_{\{1,2\}}(A) = \det A,$$

and the vector $\mathcal{M}(A)$ will take the form

$$\mathcal{M}(A) = (M^{\beta}_{\bar{\alpha}})(A)) = (1, a_{11}, a_{12}, a_{21}, a_{22}, \det A),$$

where α and β run over all the subsets of $\{1, 2\}$ with the constraint $|\alpha| + |\beta| = 2$. We will identify α and β as multi-indeces in $\{1, 2\}$.

2.1.1 Area in cylindrical coordinates

Polar coordinates in $\mathbb{R}^2_{\rm source}$ are denoted by (r,α). Polar coordinates in the target space $\mathbb{R}^2_{\rm target}$ are denoted by (ρ,θ).

Assume that $B = \{(r,\alpha) \in \mathbb{R}^2 : r \in (r_0,r_1),\ \alpha \in (\alpha_0,\alpha_1)\}$; then the area of the graph of $v = (v_1,v_2)$ in polar coordinates is given by

$$\mathcal{A}(v,B) = \int_{r_0}^{r_1}\int_{\alpha_0}^{\alpha_1} |\mathcal{M}(\nabla v)|(r,\alpha)\ r dr d\alpha.$$

Recall that, for $i \in \{1,2\}$, we have

$$\partial_{x_1} v_i = \cos\alpha\partial_r v_i - \frac{1}{r}\sin\alpha\partial_\alpha v_i, \qquad \partial_{x_2} v_i = \sin\alpha\partial_r v_i + \frac{1}{r}\cos\alpha\partial_\alpha v_i.$$

Hence

$$\begin{aligned}|\nabla v_i|^2 &= (\partial_r v_i)^2 + \frac{1}{r^2}(\partial_\alpha v_i)^2, \qquad i \in \{1,2\},\\ \partial_{x_1} v_1\partial_{x_2} v_2 - \partial_{x_2} v_1\partial_{x_1} v_2 &= \frac{1}{r}\Big(\partial_r v_1\partial_\alpha v_2 - \partial_\alpha v_1\partial_r v_2\Big).\end{aligned} \tag{2.2}$$

Thus the area of the graph of v on B is given by

$$\begin{aligned}&\mathcal{A}(v,B)\\ &= \int_{r_0}^{r_1}\int_{\alpha_0}^{\alpha_1}\sqrt{1+(\partial_r v_1)^2+(\partial_r v_2)^2+\frac{1}{r^2}\left\{(\partial_\alpha v_1)^2+(\partial_\alpha v_2)^2+\Big(\partial_r v_1\partial_\alpha v_2-\partial_\alpha v_1\partial_r v_2\Big)^2\right\}}\ r dr d\alpha.\end{aligned} \tag{2.3}$$

We denote by $\mathrm{B}_r = \mathrm{B}_r(0) \subset \mathbb{R}^2 = \mathbb{R}^2_{\rm source}$ the open disc centered at 0 with radius $r > 0$ in the source space. Our reference domain is $\Omega = \mathrm{B}_l \subset \mathbb{R}^2_{\rm source} = \mathbb{R}^2_{(x_1,x_2)}$ where $l > 0$ is fixed once for all. The symbol u will be used to note the vortex map in (1.5), which we assume to be defined on B_l.

For any $\varrho > 0$, it is convenient to introduce the (portion of) cylinder $C_l(\varrho)$, as

$$C_l(\varrho) := (-1,l)\times B_\varrho = \{(t,\rho,\theta) \in (-1,l)\times\mathbb{R}^+\times(-\pi,\pi] : \rho < \varrho\} \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{\rm target}, \tag{2.4}$$

where (t,ρ,θ) are cylindrical coordinates in $\mathbb{R}^3$, with the cylinder axis the t-axis. For $\varrho = 1$ we simply write

$$C_l(1) = C_l. \tag{2.5}$$

For a fixed parameter $\varepsilon \in (0,l)$, we introduce the cylinders

$$C_l^\varepsilon(\varrho) := (\varepsilon,l)\times B_\varrho = \{(t,\rho,\theta) \in (0,l)\times\mathbb{R}^+\times(-\pi,\pi] : \varepsilon < \rho < \varrho\} \subset \mathbb{R}_t \times \mathbb{R}^2_{\rm target}. \tag{2.6}$$

Also in this case we use the notation

$$C_l^\varepsilon(1) = C_l^\varepsilon. \tag{2.7}$$

The closure of $C_l(\rho)$ (resp. $C_l^\varepsilon(\rho)$) is denoted by $\overline{C}_l(\rho)$ (resp. $\overline{C}_l^\varepsilon(\rho)$), and the lateral boundary of $C_l(\rho)$ (resp. $C_l^\varepsilon(\rho)$) is denoted by $\partial_{\rm lat} C_l(\rho)$ (resp. $\partial_{\rm lat} C_l^\varepsilon(\rho)$).

We will often deal with integral currents whose support is in the cylinder

$$[0,l]\times\overline{B}_1 \subset \overline{C}_l.$$

Remark 2.1. The choice of $C_l = (-1,l)\times B_1$ covering also certain negative values of the first coordinate t is useful to control and detect the behaviour of these currents on the plane $\{t = 0\}$.

2.1.2 Area formula

Let $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be Lipschitz continuous, with $k \le n$. The area of the image $f(U)$ of U by f is given by

$$\int_U Jf(x)dx$$

with the Jacobian matrix of f given by

$$Jf = \sqrt{\det\left((\nabla f)^T \nabla f\right)} = \sqrt{\sum (\det A)^2} \qquad \text{a.e.in } U,$$

where, for almost every $x \in U$, the sum is made on all submatrices $A(x)$ of $\nabla f(x)$ of dimension $k \times k$.

2.2 Currents

For the reader convenience we recall some basic notion on currents. We refer to [23] and [18] for a more exhaustive discussion (see also [15]).

Given an open set $U \subset \mathbb{R}^n$ we denote by $\mathcal{D}^k(U)$ the space of smooth k-forms compactly supported in U and by $\mathcal{D}_k(U)$ the space of k-dimensional currents, for $0 \le k \le n$. If $T \in \mathcal{D}_k(\mathbb{R}^n)$, the symbol $|T|$ denotes the mass of the current T, and if $U \subset \mathbb{R}^n$ is an open set, the symbol $|T|_U$ will denote the mass of T in U, namely

$$|T|_U := \sup T(\omega),$$

the supremum being over all $\omega \in \mathcal{D}^k(U)$ with $\|\omega\| \le 1$.

For $k \ge 1$ it is defined the boundary $\partial T \in \mathcal{D}_{k-1}(U)$ of a current $T \in \mathcal{D}_k(U)$ by the formula

$$\partial T(\omega) := T(d\omega) \text{ for all } \omega \in \mathcal{D}^{k-1}(U),$$

where $d\omega$ is the external differential of ω. For $T \in \mathcal{D}_0(U)$ one sets $\partial T := 0$.

If $F : U \to V$ a Lipschitz map between open sets, and $T \in \mathcal{D}_k(U)$, we denote by $F_\sharp T \in \mathcal{D}_k(V)$ the push-forward of T by F (see [23, Section 7.4.2]).

Given a k-dimensional rectifiable set $S \subset U$ and a tangent unit simple k-vector τ to it, we denote by $[\![S]\!]$ the current given by integration over S, namely

$$[\![S]\!](\omega) = \int_S <\tau(x), \omega(x)> \, d\mathcal{H}^k(x), \qquad \omega \in \mathcal{D}^k(U).$$

We will often omit specifying which is the vector τ if it is clear from the context. We will often deal with the case $k = 2$, and $U \subset \mathbb{R}^3$ where there are only two possible orientations. Moreover in the case $k = 3$ and $U \subset \mathbb{R}^3$ the current $[\![S]\!]$ reduces to the integration over the 3-dimensional set $S \subset \mathbb{R}^3$, and $\tau = e_1 \wedge e_2 \wedge e_3$.

We call $T \in \mathcal{D}_k(U)$ an integral current if it is rectifiable with integer multiplicity and if both $|T|_U$ and $|\partial T|_U$ are finite. The Federer-Fleming theorem for integral currents then states that a sequence of integral currents $T_i \in \mathcal{D}_k(U)$ with $\sup_i(T_i| + |\partial T_i|) < +\infty$ admits a subsequence converging weakly in the sense of currents to an integral current T.

A finite perimeter set is a subset $E \subset \mathbb{R}^n$ such that the current $[\![E]\!] \in \mathcal{D}_n(U)$ is integral. The symbol $\partial^* E$ denotes the reduced boundary of E. E is unique up to negligible sets, so that we always choose a representative of E for which the closure of the reduced boundary equals the topological boundary [24].

An integral current $T \in \mathcal{D}_k(U)$ is called indecomposable if there is no integral current $R \in \mathcal{D}_k(U)$ such that $R \neq 0 \neq T - R$ with

$$|T|_U + |\partial T|_U = |R|_U + |\partial R|_U + |T - R|_U + |\partial(T - R)|_U.$$

We will often use the following decomposition theorem for integer multiplicity currents: For every integral current $T \in \mathcal{D}_k(U)$ there is a sequence of indecomposable integral currents $T_i \in \mathcal{D}_k(U)$ with $T = \sum_i T_i$ and $|T| + |\partial T| = \sum_i |T_i| + \sum_i |\partial T_i|$ (see [15, Section 4.2.25]). In the case that $T \in \mathcal{D}_n(U)$, $U \subseteq \mathbb{R}^n$, the previous decomposition theorem can be stated as follows: There is a sequence of finite perimeter sets with $\{E_i\}_{i\in\mathbb{Z}}$ such that $T = \sum_{i\geq 0} [\![E_i \cap U]\!] - \sum_{i<0} [\![E_i \cap U]\!]$ with $\sum_i |E_i \cap U| + \sum_i \mathcal{H}^{n-1}(U \cap \partial^* E_i) = |T| + |\partial T|$ (see [23, Theorem 7.5.5] and its proof). Moreover, the decomposition theorem applied to E_i allows us to assume that the sequence $([\![E_i]\!])$ consists of indecomposable currents. In the case of 1-dimensional currents, it is possible also to characterize indecomposable currents, namely $T \in \mathcal{D}_1(\mathbb{R}^n)$ is indecomposable if $T = \gamma_\sharp [\![0, |T|]\!]$ with $\gamma : [0, |T|] \to \mathbb{R}^n$ a 1-Lipschitz simple curve, *i.e.*, injective on $[0, |T|)$. If moreover $\partial T = 0$ then $\gamma(0) = \gamma(|T|)$.

We will exploit the property that any boudaryless current $T \in \mathcal{D}_{n-1}(\mathbb{R}^n)$ is the boundary of a sum of currents given by integration over locally finite perimeter sets E_i, *i.e.*, $T = \sum_i \partial [\![E_i]\!]$. This is a consequence of the cone construction, and for integral currents can be obtained also from the isoperimetric inequality (see [23, Formula (7.26)] and [23, Theorem 7.9.1]).

We need also the concept of slice of an integral current with respect to a Lipschitz function f (see [23, Section 7.6]). Since we only employ it for slices with respect to parallel planes, the function f will be $f(x) = x_h$ where x_h is the coordinate in $\mathbb{R}^n$ whose axis is orthogonal to the considered planes. We denote by $T_t \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ the slices of $T \in \mathcal{D}_k(\mathbb{R}^n)$ on the plane $\{x_h = t\}$, which will be supported on this plane. We will also use that, if T is boundaryless, then

$$\partial(T \llcorner \{x_h < t\}) = T_t \qquad \text{for a.e. } t \in \mathbb{R}.$$

2.3 Generalized graphs in codimension 1

Let $v \in L^1(\Omega)$. We denote by $R_v \subseteq \Omega$ the set of regular points of v, *i.e.*, the set consisting of points x which are Lebesgue points for v, $v(x)$ coincides with the Lebesgue value of v at x, and v is approximately differentiable at x. We also set

$$\begin{aligned} G_v^R &:= \{(x, v(x)) \in R_v \times \mathbb{R}\}, \\ SG_v^R &:= \{(x, y) \in R_v \times \mathbb{R} : y < v(x)\}. \end{aligned}$$

We often will identify SG_v^R with the integral 3-current $[\![SG_v]\!] \in \mathcal{D}_3(\Omega \times \mathbb{R})$. If v is a function of bounded variation, $\Omega \setminus R_v$ has zero Lebesgue measure, so that the current $[\![SG_v]\!]$ coincides with the integration over the subgraph

$$SG_v := \{(x, y) \in \Omega \times \mathbb{R} : y < v(x)\}.$$

For this reason we often identify $SG_v = SG_v^R$. It is well-known that the perimeter of SG_v in $\Omega \times \mathbb{R}$ coincides with $\overline{\mathcal{A}}(v, \Omega)$.

The support of the boundary of $[\![SG_v]\!]$ includes the graph G_v^R, but in general consists also of additional parts, called vertical. We denote by

$$\mathcal{G}_v := \partial [\![SG_v]\!] \llcorner (\Omega \times \mathbb{R}),$$

the generalized graph of u, which is a 2-integral current supported on $\partial^* SG_v$, the reduced boundary of SG_v in $\Omega \times \mathbb{R}$.

Let $\widehat{\Omega} \subset \mathbb{R}^2$ be a bounded open set such that $\Omega \subseteq \widehat{\Omega}$, and suppose that $L := \widehat{\Omega} \cap \partial\Omega$ is a rectifiable curve. Given $\psi \in BV(\Omega)$ and a $W^{1,1}$ function $\varphi : \widehat{\Omega} \to \mathbb{R}$, we can consider

$$\overline{\psi} := \begin{cases} f & \text{on } \Omega, \\ \varphi & \text{on } \widehat{\Omega} \setminus \Omega. \end{cases}$$

Then (see [19], [2])

$$\overline{\mathcal{A}}(\overline{\psi}, \widehat{\Omega}) = \overline{\mathcal{A}}(\psi, \Omega) + \int_L |\psi - \varphi| d\mathcal{H}^1 + \overline{\mathcal{A}}(\varphi, \widehat{\Omega} \setminus \overline{\Omega}).$$

2.4 Polar graphs in a cylinder

Consider the (portion of) cylinder $C_l = (-1, l) \times B_1$ defined in (2.5), endowed with cylindrical coordinates $(t, \rho, \theta) \in (-1, l) \times [0, 1) \times (-\pi, \pi]$. Take the rectangle $H = \{(t, \rho, \theta) \in C_l : \theta = 0\}$, which is endowed with Cartesian coordinates $(t, \rho) \in (-1, l) \times (0, 1)$. If $\Theta : H \to [0, \pi]$ is a function defined on H, we can associate to it the map $\mathrm{id} \bowtie \Theta : H \to C_l$ defined as

$$(t, \rho) \to (t, \rho, \Theta(t, \rho)), \qquad (t, \rho) \in H.$$

The polar graph of Θ is defined as

$$G^{\mathrm{pol}}_{\Theta} := \{(t, \rho, \Theta(t, \rho)) : t \in (-1, l),\ \rho \in (0, 1)\} = \mathrm{id} \bowtie \Theta(H),$$

where again we have used cylindrical coordinates.

We define a sort of polar subgraph of Θ as

$$SG^{\mathrm{pol}}_{\Theta} := \{(t, \rho, \theta) : t \in (-1, l),\ \rho \in [0, 1),\ \theta \in (-\eta, \Theta(t, \rho))\}.$$

Here $\eta > 0$ is a small number introduced for convenience, and it will suffice to take $\eta < 1$. If the set $SG^{\mathrm{pol}}_{\Theta}$ has finite perimeter, its reduced boundary in $\{-\eta < \theta < \pi + \eta\} \cap C_l$ coincides with the generalized polar graph $\mathcal{G}_{\Theta}$ of Θ,

$$\mathcal{G}_{\Theta} = (\partial^* SG^{\mathrm{pol}}_{\Theta}) \cap (\{-\eta < \theta < \pi + \eta\} \cap C_l). \tag{2.8}$$

This set includes, up to $\mathcal{H}^2$-negligible sets, the polar graph $G^{\mathrm{pol}}_{\Theta}$. When $SG^{\mathrm{pol}}_{\Theta}$ has finite perimeter, we see that the current $[\![SG^{\mathrm{pol}}_{\Theta}]\!] \in \mathcal{D}_3(C_l)$ is integral and its boundary in $\{-\eta < \theta < \pi + \eta\} \cap C_l$ is the integration over the generalized polar graph of Θ, *i.e.*,

$$\partial [\![SG^{\mathrm{pol}}_{\Theta}]\!] \llcorner (\{-\eta < \theta < \pi + \eta\} \cap C_l) = [\![\mathcal{G}_{\Theta}]\!],$$

where $\mathcal{G}_{\Theta}$ is naturally oriented by the outer normal to $\partial^* SG^{\mathrm{pol}}_{\Theta}$.

Notice also that since $\Theta \in [0, \pi]$ the current $[\![\mathcal{G}_{\Theta}]\!]$ carried by the generalized polar graph $\mathcal{G}_{\Theta}$ is supported in $\{0 \leq \theta \leq \pi\} \cap C_l$.

2.5 Plateau problem in parametric form

We report here some results about the classical solution to the disc-type Plateau problem. If $\Gamma \subset \mathbb{R}^3$ is a closed rectifiable Jordan curve, the Plateau problem consists into minimize the functional

$$\mathcal{P}_{\Gamma}(X) := \int_{B_1} |\partial_{x_1} X \wedge \partial_{x_2} X| dx_1 dx_2, \tag{2.9}$$

on the class of all functions $X \in C^0(\overline{B}_1;\mathbb{R}^3)\cap H^1(B_1;\mathbb{R}^3)$ with $X\llcorner\partial B_1$ being a weakly monotonic parametrization of the curve Γ. The functional (2.9) measures the area (with multiplicity) of the surface $X(B_1)$. We can always associate to a map X the current $X_\sharp[\![B_1]\!]$, the integration over the surface $X(B_1)$. Notice in general

$$|X_\sharp[\![B_1]\!]| \leq \mathcal{P}_\Gamma(X),$$

and strict inequality can occur if for instance the map X parametrizes two times and with opposite orientation a part of the surface $X(B_1)$.

A solution X_Γ to the Plateau problem exists and satisfies the properties: it is harmonic (hence analytic)

$$\Delta X_\Gamma = 0 \qquad \text{in } B_1,$$

it is a conformal parametrization

$$|\partial_{x_1}X_\Gamma|^2 = |\partial_{x_2}X_\Gamma|^2, \qquad \partial_{x_1}X_\Gamma\cdot\partial_{x_2}X_\Gamma = 0 \qquad \text{in } B_1,$$

and $X_\Gamma\llcorner\partial B_1$ is a strictly monotonic parametrization of Γ. We will say that the surface $X_\Gamma(B_1)$ has the topology of the disc.

Thanks to the properties above it is always possible, with the aid of a conformal change of variables, to parametrize $X(B_1)$ over any simply connected bounded domain. In other words, if U is any such domain, and if $\Phi : U \to B_1$ is any conformal homeomorphism, then $X\circ\Phi$ is a solution to the Plateau problem on U.

2.6 A Plateau problem for a self-intersecting boundary space curve

The classical disc-type Plateau problem is solved for boundary value a simple Jordan space curve, in particular Γ does not have self-intersections. Here we will treat a specific Plateau problem where the curve Γ has non-trivial intersections, and it overlaps itself on a segment which is parametrized two times with opposite directions.

Specifically, we consider the cylinder $(0, 2l)\times B_1$ and two circles $\mathcal{C}_1, \mathcal{C}_2$ which are the boundaries of its two circular bases, namely $\mathcal{C}_1 := \{0\}\times\partial B_1$ and $\mathcal{C}_2 := \{2l\}\times\partial B_1$. Then we take the segment $(0, 2l)\times\{1\}\times\{0\}$. If γ_0 is a monotonic parametrization of this segment, starting from $(0, 1, 0)$ up to $(2l, 1, 0)$, γ_1 is a monotonic parametrization of $\mathcal{C}_1$ starting from the point $(0, 1, 0)$ and ending at the same point, and γ_2 a parametrization of $\mathcal{C}_2$ with initial and final point $(2l, 1, 0)$ with the same orientation of $\mathcal{C}_1$, then we consider the parametrization

$$\gamma := \gamma_1 \star \gamma_0 \star (-\gamma_2) \star (-\gamma_0), \tag{2.10}$$

(read from left to right) which is a closed curve in $\mathbb{R}^3$ which travels two times across the segment $(0, 2l)\times\{1\}\times\{0\}$ with opposite directions (the orientation of this curve is depicted in Fig. 1). We want to solve the Plateau problem with Γ to be the image of γ.

The existence of solutions to the Plateau problem spanning self-intersecting boundaries has been addressed in [21], whose results have been recently improved in [10]. Without entering deeply into the details, it is known that, depending on the geometry of γ (in this case, depending on the distance between the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$) two kind of solutions are expected:

(a) The solution consists of two discs filling $\mathcal{C}_1$ and $\mathcal{C}_2$, see Fig. 2, right. In this case, a parametrization of it $X : \overline{B}_1 \to \mathbb{R}^3$ can be chosen so that, if L_1 and L_2 are two parallel chords in B_1 dividing B_1 in three sectors, then X restricted to the sector enclosed between L_1 and L_2 parametrizes the segment γ_0 (and then its resulting area is null), $X(L_1) = P_1$ and

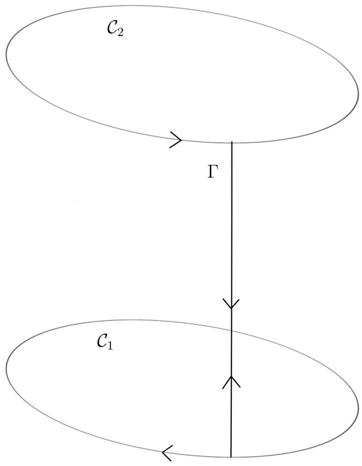

Figure 1: The self-overlapping curve Γ with its orientation.

$X(L_2) = P_2$ are the two endpoints of γ_0, and X restricted to the sectors between L_i, $i = 1, 2$, and ∂B_1 parametrizes the disc filling $\mathcal{C}_i$, $i = 1, 2$. Moreover the map X can be still taken Sobolev regular (see [10] for details).

(b) There is a classical solution, *i.e.*, there is a harmonic and conformal map $X : B_1 \to \mathbb{R}^3$, continuous up to the boundary of B_1, such that $X \llcorner \partial B_1$ is a weakly monotonic parametrization of Γ. In this case the resulting minimal surface is a sort of catenoid attached to the segment $(0, 2l) \times \{(1, 0)\}$ (see Fig. 2 left).

Remark 2.2. We expect that there is a threshold l_0 such that if $l < l_0$ an area-minimizing disc with boundary γ is of the form (b), and for values $l > l_0$ the two discs have minimal area. We do not find explicitly l_0 but it is easy to see that if $l \leq \frac{1}{2}$ an area-minimizing disc with boundary γ has always less area than the solution with two discs. Indeed, the area of the two discs is 2π, whereas we can always compare the area of the surface Σ as in (b) with the area of the lateral surface of the cylinder $(0, 2l) \times B_1$, that is $4l\pi$. Hence $\mathcal{H}^2(\Sigma) < 4l\pi \leq 2\pi$ for $l \leq \frac{1}{2}$.

3 Cylindrical Steiner symmetrization

In this section we introduce the cylindrical Steiner symmetrization of a finite perimeter[9] set $U \subseteq C_l = (-1, l) \times B_1(0)$. This rearrangement is obtained slice by slice by spherical (two dimensional) symmetrization, a technique introduced first by Pòlya. We refer to [9] and references therein for a exhaustive description of the subject. Here we collect the main properties we will use in the sequel of the paper. Furthermore we will introduce a generalization of this symmetrization in order to apply it to 2-integral currents.

Let us recall that C_l is endowed with cylindrical coordinates $(t, \rho, \theta) \in (-1, l) \times [0, 1) \times (-\pi, \pi]$. If x_1, x_2, x_3 are cartesian coordinates, we have $x_1 = t$, $x_2 = \rho \cos \theta$, $x_2 = \rho \sin \theta$. Sometimes it will be convenient to extend 2π-periodically the values of θ on the whole of $\mathbb{R}$.

[9] Recall that we choose a representative of U such that the closure of its reduced boundary $\partial^* U$ equals the topological boundary.

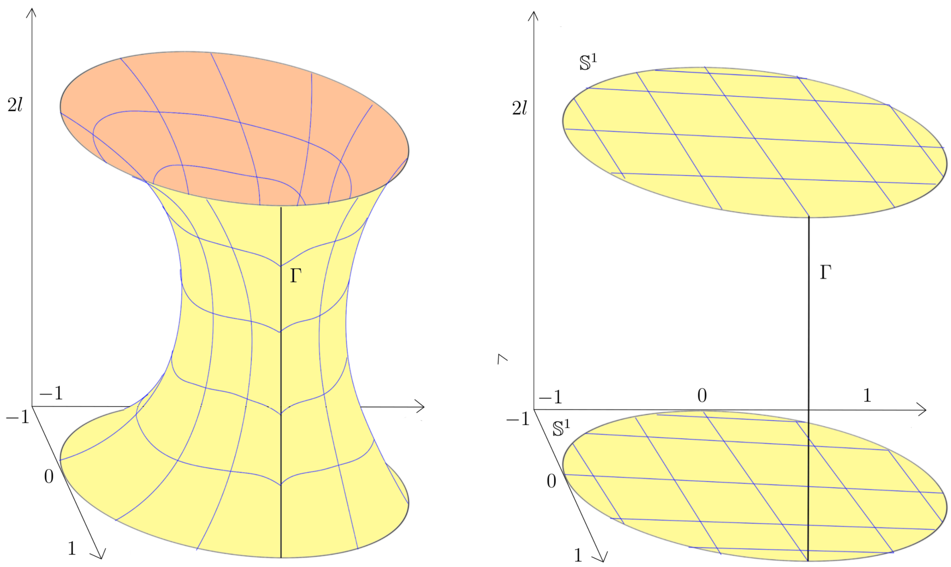

Figure 2: on the left the shape of a possible solution to the Plateau problem with boundary Γ. On the right another solution to the Plateau problem with boundary Γ. See Section 2.6

For every $t \in (-1, l)$ let $U_t := U \cap (\{t\} \times \mathbb{R}^2)$ the slice of U on the plane with first coordinate t, and for every $\rho \in (0, 1)$ let $U_t(\rho) := U_t \cap (\{t\} \times \{\rho\} \times (-\pi, \pi])$ be the slice of U_t with the circle of radius ρ.

Definition 3.1 (**Symmetrization of solid sets in** C_l)**.** *For every $t \in (-1, l)$ and $\rho \in (0, 1)$ we let*

$$\Theta(t, \rho) = \Theta_U(t, \rho) := \frac{1}{\rho}\mathcal{H}^1(U_t(\rho)), \tag{3.1}$$

and we define the cylindrically symmetrized set $\mathbb{S}(U) \subseteq C_l$ as

$$\mathbb{S}(U) := \big\{(t, \rho, \theta) : t \in (-1, l),\ \rho \in (0, 1),\ \theta \in \big(-\Theta(t, \rho)/2, \Theta(t, \rho)/2\big)\big\}\,. \tag{3.2}$$

Notice that $\Theta_U = \Theta_{\mathbb{S}(U)}$. The set $\mathbb{S}(U)$ enjoys the following properties:

(1) $\mathcal{H}^2(\mathbb{S}(U)_t) = \mathcal{H}^2(U_t)$ and $\mathcal{H}^1(\partial^*(\mathbb{S}(U)_t)) \leq \mathcal{H}^1(\partial^*(U_t))$ for every $t \in (-1, l)$;

(2) $|\mathbb{S}(U)| = |U|$ and $\mathcal{H}^2(\partial^*\mathbb{S}(U)) \leq \mathcal{H}^2(\partial^* U)$.

A proof of these properties is contained in [9, Theorem 1.4]. In particular, since U has finite perimeter, so is $\mathbb{S}(U)$ and its perimeter cannot increase after symmetrization. We will need to apply it to 3-integral currents in C_l. That is, (possibly infinite) sums of finite perimeter sets with integer coefficients. For this reason we introduce the following generalization of cylindrical symmetrization.

Let $\mathcal{E} \in \mathcal{D}_3(C_l)$ be an integral 3-current. By Federer decomposition theorem [15, Section 4.2.25, p. 420] (see also [15, Section 4.5.9] and [23, Theorem 7.5.5]) it follows that there is a sequence $(E_i)_{i\in\mathbb{N}}$ of finite perimeter sets such that

$$\mathcal{E} = \sum_i (-1)^{\sigma_i}[\![E_i]\!], \tag{3.3}$$

for suitable $\sigma_i \in \{0,1\}$. We can also assume the decomposition is done in undecomposable components, so that

$$|\mathcal{E}| = \sum_i |E_i| \qquad \text{and} \qquad |\partial\mathcal{E}| = \sum_i \mathcal{H}^2(\partial^* E_i). \tag{3.4}$$

According to Definition 3.1, we can symmetrize each set E_i into $\mathbb{S}(E_i)$.

Definition 3.2 (**Symmetrization of an integer** 3**-current**). *Let $E := supp(\mathcal{E})$ denote the support of the current $\mathcal{E} \in \mathcal{D}_3(C_l)$. We let*

$$\mathbb{S}(E) := \bigcup_i \mathbb{S}(E_i),$$

which will be referred to as the symmetrized support of $\mathcal{E}$. The symmetrized current $\mathbb{S}(\mathcal{E}) \in \mathcal{D}_3(C_l)$ is defined as

$$\mathbb{S}(\mathcal{E}) := [\![\mathbb{S}(E)]\!]. \tag{3.5}$$

Notice that the multiplicity of $[\![\mathbb{S}(E)]\!]$ is one, hence $[\![\mathbb{S}(E)]\!]$ is the integration over a finite perimeter set.

3.1 Cylindrical symmetrization of a two-current. Slicings

Let us focus on a slice $\mathcal{E}_t$ of the current $\mathcal{E}$ with respect to a plane $\{t\} \times \mathbb{R}^2_{\text{target}}$. Suppose for the moment that $\mathcal{E}$ is the integration over a finite perimeter set (that we identify with E) in C_l; $\mathcal{E}_t$ is the integration over the slice E_t of E, and suppose that the boundary of E_t is the trace σ of a rectifiable Jordan curve. Applying Definition 3.2 to the set E we see that E_t is transformed into the symmetrized set $\mathbb{S}(E_t)$ whose boundary is again[10] the trace σ_s of a Jordan curve. By the properties of the symmetrization we infer $\mathcal{H}^1(\sigma) \geq \mathcal{H}^1(\sigma_s)$.

However, if the boundary of E_t is the trace σ of a nonsimple curve, then the procedure is more involved. More generally, from Definition 3.2, we see that for a.e. $t \in (-1, l)$ the slice $\mathcal{E}_t$ of $\mathcal{E}$ is an integral 2-current, and it can be represented by integration over finite perimeter sets $(E_i)_t$ (with suitable signs) which are exactly the slices of the sets E_i in (3.3). Moreover for a.e. $t \in (-1, l)$ the boundary of $\mathcal{E}_t$ is a 1-dimensional integral current with finite mass, and it coincides with the integration (with suitable signs) over the boundaries of $(E_i)_t$, namely

$$\partial\mathcal{E}_t = \sum_i (-1)^{\sigma_i} \partial[\![(E_i)_t]\!].$$

Let us call this boundary σ (which, with a little abuse of notation, we identify with an integral 1-current, an at most countable sum of simple curves), and set $\sigma_s := \partial[\![\mathbb{S}(E)_t]\!]$. By Definition 3.2 it then follows that $\mathbb{S}(\mathcal{E})_t = [\![\mathbb{S}(E)_t]\!]$. Now, by the properties of the symmetrization, we see that $\mathcal{H}^1(\text{supp}(\sigma)) \geq \mathcal{H}^1(\text{supp}(\sigma_s))$. Also in this case it turns out that σ_s is the integration over countable many simple curves (with suitable orientation).

We have described so far how the boundary of $\mathcal{E}$ is trasformed slice by slice. In general if $\mathcal{E}$ is a 3-integral current in C_l, then the current $\mathcal{S} := \partial\mathcal{E}$ has the property that

$$|\mathcal{S}| \geq \mathcal{H}^2(\partial^* \mathbb{S}(E)).$$

There is also a viceversa. Precisely assume that $\mathcal{S}$ is any boundaryless integral 2-current in C_l. Then there is an integral 3-current $\mathcal{E}$ whose boundary is $\mathcal{S}$. So that we can define the symmetrization of $\mathcal{S}$ by symmetrizing $\mathcal{E}$.

[10] $\mathbb{S}(E_t)$ is simply connected. Indeed the support of $\rho \mapsto \Theta(t, \rho)$ is a connected subset of $(0, 1)$.

Definition 3.3 (**Cylindrical symmetrization of the boundary of a three-current**). *The symmetrization of $\mathcal{S} = \partial\mathcal{E}$ is defined as*

$$\mathbb{S}(\mathcal{S}) := \partial\mathbb{S}(\mathcal{E}).$$

The next lemma will be useful in Section 8.

Lemma 3.4. *Let $t \in (-1, l)$ be such that $\mathcal{S}\llcorner(\{t\} \times \mathbb{R}^2) = 0$. Then*

$$\mathbb{S}(\mathcal{S})\llcorner(\{t\} \times \mathbb{R}^2) = 0. \tag{3.6}$$

Proof. We know that $\mathcal{S} = \partial\mathcal{E}$. By the properties of the cylindrical symmetrization (see item (2) above) for each set E_i we have

$$\mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^* E_i\Big) \geq \mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^*\mathbb{S}(E_i)\Big).$$

From our assumption it follows[11] that for all i we have $\mathcal{H}^2((\{t\} \times \mathbb{R}^2) \cap \partial^* E_i) = 0$, and thus

$$\mathcal{H}^2((\{t\} \times \mathbb{R}^2) \cap \partial^*\mathbb{S}(E_i)) = 0, \qquad i \in \mathbb{N}.$$

To conclude the proof we have to show that

$$\mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^*\mathbb{S}(E)\Big) = \mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^*(\cup_i\mathbb{S}(E_i))\Big) = 0. \tag{3.7}$$

The conclusion easily follows if the family $\{E_i\}$ is finite, since in this case $\partial(\cup_i\mathbb{S}(E_i)) \subseteq \cup_i\partial\mathbb{S}(E_i)$. If this family is not finite we argue as follows: fix $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ so that (see (3.4))

$$\sum_{i=N_\varepsilon+1}^{+\infty} \mathcal{H}^2(\partial^* E_i) \leq \varepsilon. \tag{3.8}$$

We have

$$\mathbb{S}(E) = \cup_i\mathbb{S}(E_i) = \big(\cup_{i=1}^{N_\varepsilon} \mathbb{S}(E_i)\big) \cup \big(\cup_{i=N_\varepsilon+1}^{+\infty} \mathbb{S}(E_i)\big) =: A \cup B,$$

thus

$$(\{t\} \times \mathbb{R}^2) \cap \partial\mathbb{S}(E) \subseteq \Big((\{t\} \times \mathbb{R}^2) \cap \partial^* A\Big) \cup \Big((\{t\} \times \mathbb{R}^2) \cap \partial^* B\Big).$$

By the previous observations $\mathcal{H}^2((\{t\} \times \mathbb{R}^2) \cap \partial^* A) = 0$; we will prove that

$$\mathcal{H}^2((\{t\} \times \mathbb{R}^2) \cap \partial^* B) = \mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^*(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i))\Big) \leq \varepsilon,$$

so that (3.7) follows by arbitrariness of $\varepsilon > 0$. To do so, it suffices to write

$$\mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^*(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i))\Big) \leq \mathcal{H}^2\big(\partial^*(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i))\big) \leq \sum_{i=N_\varepsilon+1}^{+\infty} \mathcal{H}^2(\partial^*\mathbb{S}(E_i)) \leq \varepsilon.$$

The last inequality follows from (3.8) and from the fact that symmetrization does not increase the perimeter. As for the second inequality, it follows from the lower semicontinuity of the perimeter. Indeed, setting $F_k := \cup_{i=N_\varepsilon+1}^{k}\mathbb{S}(E_i)$ for $k \geq N_\varepsilon + 1$, we see that $F_k \to F_\infty := \cup_{i=N_\varepsilon+1}^{\infty}\mathbb{S}(E_i)$ in $L^1(C_l)$, and since F_k has finite perimeter we infer

$$\mathcal{H}^2(\partial^* F_\infty) \leq \liminf_{k\to+\infty} \mathcal{H}^2(\partial^* F_k) \leq \liminf_{k\to+\infty} \sum_{i=N_\varepsilon+1}^{k} \mathcal{H}^2(\partial^*\mathbb{S}(E_i)).$$

□

[11] This follows since the decomposition is done in undecomposable components: if there is some boundary of some E_i then it cannot cancel with some other boundary (opposite oriented) of some E_j.

As before, we can look at what happens to the current $\mathcal{S}$ slice by slice. If $\partial\mathcal{E} = \mathcal{S}$, then $\mathcal{S}_t = -\partial(\mathcal{E}_t)$ for a.e. $t \in (-1, l)$. Assume that $\mathcal{E}$ decomposes as in (3.3), then

$$\mathcal{E}_t = \sum_i (-1)^{\sigma_i} [\![(E_i)_t]\!] \qquad \text{for a.e. } t \in (-1, l). \tag{3.9}$$

Now the sets $(E_i)_t$ are symmetrized as before, and their union, denoted $\mathbb{S}(E_t)$ (so that $\mathbb{S}(\mathcal{E})_t = [\![\mathbb{S}(E_t)]\!]$) satisfies

$$\partial [\![\mathbb{S}(E_t)]\!] = -\mathbb{S}(\mathcal{S})_t$$

and

$$|\mathcal{S}_t| \geq \mathcal{H}^1(\partial^* \mathbb{S}(E)_t).$$

Let us go back to (3.9). In general

$$|\mathcal{E}_t| \leq \sum_i \mathcal{H}^2((E_i)_t); \tag{3.10}$$

however, since the decomposition is made of undecomposable components, (3.4) holds and hence

$$|\mathcal{E}_t| = \sum_i \mathcal{H}^2((E_i)_t) \qquad \text{for a.e. } t \in (-1, l). \tag{3.11}$$

This can be seen integrating in t formula (3.10), so that if strict inequality holds for a positive measured set of $t \in (-1, l)$ we would get strict inequality in the first equation of (3.4), which is a contradiction.

Moreover, by construction, $\mathcal{H}^2((E_i)_t) = \mathcal{H}^2(\mathbb{S}(E_i)_t)$ for all i, and since $\mathbb{S}(E)_t = \cup_i \mathbb{S}(E_i)_t$ it also follows

$$|\mathcal{E}_t| = \sum_i \mathcal{H}^2((E_i)_t) = \sum_i \mathcal{H}^2(\mathbb{S}(E_i)_t) \geq \mathcal{H}^2(\mathbb{S}(E)_t).$$

Now we fix t such that (3.11) holds and set $F_i := (E_i)_t$, $\mathcal{F} := \mathcal{E}_t$, $F := \mathrm{supp}(\mathcal{F})$, $\mathbb{S}(F) = \mathbb{S}(E)_t$. The set $F_i \in B_1$ can be sliced with respect to the radial coordinate $\rho \in (0, 1)$, so that exploiting that

$$(\mathcal{E}_t)_\rho = \sum_i (-1)^{\sigma_i} [\![((E_i)_t)_\rho]\!]$$

holds for a.e. ρ, we can repeat the same argument as before to obtain

$$|\mathcal{F}_\rho| = \sum_i \mathcal{H}^1((F_i)_\rho) \qquad \text{for a.e. } \rho \in (0, 1).$$

Again we have $\sum_i \mathcal{H}^1((F_i)_\rho) \geq \mathcal{H}^1(\mathbb{S}(F)_\rho)$. Recalling that $\mathbb{S}(F)_\rho = \mathbb{S}(E)_t \cap \partial B_\rho$, we conclude that, for a.e. $t \in (-1, l)$ and for a.e. $\rho \in (0, 1)$ the slice $(\mathcal{E}_t)_\rho$ satisfies

$$|(\mathcal{E}_t)_\rho| \geq \mathcal{H}^1(\mathbb{S}(E)_t \cap \partial B_\rho) = \rho\Theta(t, \rho), \tag{3.12}$$

where we have defined $\Theta(t, \rho) := \rho^{-1}\mathcal{H}^1(\mathbb{S}(E)_t \cap \partial B_\rho)$ the measure in radiants of the arc $\mathbb{S}(E)_t \cap \partial B_\rho$.

Remark 3.5. In the sequel we are going to apply the cylindrical symmetrization to a current supported in the portion of the cylinder $(0, l) \times B_1 \subset C_l$. The fact that we set the symmetrization in $C_l = (-1, l) \times B_1$ will be useful to avoid possible creation of boundary on the disc $\{0\} \times B_1$.

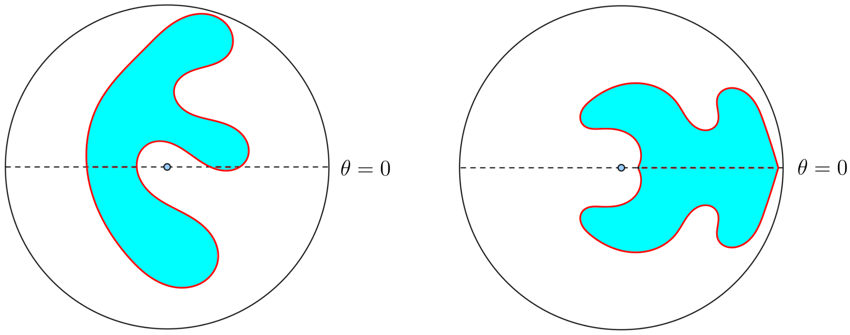

Figure 3: The symmetrization of a subset of B_1 bounded by a Jordan curve, with the respect to the radius $\{\theta = 0\}$; see formula (3.1).

4 Lower bound: first reductions on a recovery sequence

Let $u(x) = x/|x|$, $x \neq 0$, be the vortex map and $\Omega = \mathrm{B}_l$; we aim to prove that

$$\overline{\mathcal{A}}(u, \Omega) \geq \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \frac{1}{2}\mathcal{P}_\Gamma(X),$$

where Γ is the image of the self-intersecting curve parametrized in (2.10), see Fig. 2, and X is a disc-type solution of the Plateau problem for Γ.

Let $(u_k) \subset C^1(\Omega, \mathbb{R}^2)$ be a recovery sequence for the area of the graph of u, *i.e.*, $u_k \to u$ in $L^1(\Omega, \mathbb{R}^2)$ and

$$\liminf_{k \to +\infty} \mathcal{A}(u_k, \Omega) = \overline{\mathcal{A}}(u, \Omega);$$

with no loss of generality we can suppose that $u_k \to u$ almost everywhere in Ω and

$$\liminf_{k \to +\infty} \mathcal{A}(u_k, \Omega) = \lim_{k \to +\infty} \mathcal{A}(u_k, \Omega) < +\infty. \tag{4.1}$$

If $\Pi : \mathbb{R}^2_{\mathrm{target}} \to \overline{B}_1 \subset \mathbb{R}^2_{\mathrm{target}}$ is the projection map onto $\overline{B}_1$,

$$\Pi(x) := \begin{cases} \frac{x}{|x|} & \text{if } |x| > 1 \\ x & \text{if } |x| \leq 1, \end{cases} \tag{4.2}$$

then

$$\mathcal{A}(v, \Omega) \geq \mathcal{A}(\Pi \circ v, \Omega) \qquad \forall v \in C^1(\Omega, \mathbb{R}^2).$$

Notice that in general $\Pi \circ v \notin C^1(\Omega, \mathbb{R}^2)$; however $\Pi \circ v$ is of class C^1 on the set $\{x \in \Omega : |v(x)| < 1\}$ and Lipschitz continuous in Ω. Therefore, possibly replacing u_k by $\Pi \circ u_k$, we can assume that u_k takes values in $\overline{B}_1$ for all $k \in \mathbb{N}$.

We start by dividing the source disc Ω in several suitable subsets. First we observe that from (4.1) there exists a constant $C > 0$ such that

$$C \geq \int_\Omega |\nabla u_k| \, dx = \int_0^l \int_{\partial \mathrm{B}_r} |\nabla u_k(r, \alpha)| \, d\mathcal{H}^1(\alpha) dr \qquad \forall k \in \mathbb{N}. \tag{4.3}$$

By Fatou's lemma, we then infer

$$\int_0^l L(r)\ dr \leq C,$$

where

$$L(r) := \liminf_{k\to+\infty} \int_{\partial \mathrm{B}_r} |\nabla u_k(r,\alpha)|\ d\mathcal{H}^1(\alpha) \qquad \text{for a.e. } r \in (0,l).$$

In particular, $L(r)$ is finite for almost every $r \in (0,l)$. Since $u_k \to u$ almost everywhere in Ω, we have that for almost every $r \in (0,l)$

$$u_k(r,\alpha) \to u(r,\alpha) \qquad \text{for } \mathcal{H}^1-\text{a.e. } \alpha \in \partial \mathrm{B}_r.$$

Thus we can choose $\varepsilon \in (0,1)$ arbitrarily small such that the two following properties are satisfied:

$$L(\varepsilon) \leq C_\varepsilon \quad \text{for a constant } C_\varepsilon > 0 \text{ depending on } \varepsilon; \tag{4.4}$$

$$\lim_{k\to+\infty} u_k(\varepsilon,\alpha) = u(\varepsilon,\alpha) \quad \text{for } \mathcal{H}^1-\text{a.e. } \alpha \in \partial \mathrm{B}_\varepsilon. \tag{4.5}$$

4.1 The functions d_k, the subdomains A_n and D_k^δ, and selection of (λ_k)

By Egorov lemma, there exists a sequence (A_n) of measurable subsets of Ω such that, for any $n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$,

$$|A_n| < \frac{1}{n}, \tag{4.6}$$

and

$$u_k \to u \ \text{ in } L^\infty(\Omega \setminus A_n, \mathbb{R}^2) \text{ as } k \to +\infty. \tag{4.7}$$

Definition 4.1 (The function d_k and the set D_k^δ). *We indicate by $d_k : \Omega \setminus \{0\} \to [0,2]$ the function*

$$d_k := \big|u_k - u\big|, \tag{4.8}$$

and for any $\delta > 0$ we set

$$D_k^\delta := \{x \in \Omega \setminus \{0\} : d_k(x) > \delta\} =: \{d_k > \delta\}. \tag{4.9}$$

Notice that

$$\partial D_k^\delta \subseteq \{x \in \Omega \setminus \{0\} : d_k(x) = \delta\} =: \{d_k = \delta\}. \tag{4.10}$$

For ε satisfying (4.4) and (4.5), we have $d_k \in \mathrm{Lip}(\Omega \setminus \overline{\mathrm{B}}_\varepsilon; \mathbb{R}^2) \cap W^{1,1}(\Omega; \mathbb{R}^2)$. For any $n \in \mathbb{N}$, from (4.7) it follows that for any $\delta > 0$ there exists $k_{\delta,n} \in \mathbb{N}$ such that $d_k < \frac{\delta}{2}$ in $\Omega \setminus A_n$ for any $k \geq k_{\delta,n}$, and thus

$$\Omega \setminus A_n \subseteq \left\{d_k < \frac{\delta}{2}\right\} \subseteq \Omega \setminus D_k^\delta \qquad \forall k > k_{\delta,n}.$$

Passing to the complement, from (4.10) and the inclusion $\{d_k = \delta\} \subseteq \{d_k \geq \delta/2\}$, we get

$$D_k^\delta \subseteq A_n \qquad \text{and} \qquad \partial D_k^\delta \subseteq A_n \qquad \forall k > k_{\delta,n}. \tag{4.11}$$

Lemma 4.2 (Choice of λ_k and estimates on $D_k^{\lambda_k}$). *Let $\varepsilon \in (0,1)$ satisfy* (4.4) *and* (4.5)*. Let $n > 0$ and $A_n \subset \Omega$ be a measurable set satisfying* (4.6) *and* (4.7)*. Then there are a (not relabelled) subsequence of (u_k) and a decreasing infinitesimal sequence (λ_k) of positive numbers, both depending on n and ε, such that the following properties hold:*

(i) for all $k \in \mathbb{N}$ we have $\lambda_k \neq 1 - |u_k(0)|$ and the boundary of the set $D_k^{\lambda_k} = \{d_k > \lambda_k\}$ consists of an at most countable number of continuous curves which are either closed or with endpoints on $\partial\Omega$, and whose total length is finite;

(ii) $D_k^{\lambda_k} \cup \partial D_k^{\lambda_k} \subseteq A_n$ for all $k \in \mathbb{N}$;

(iii) $\displaystyle\lim_{k\to+\infty} \int_{\partial D_k^{\lambda_k}} d_k \; d\mathcal{H}^1 = \lim_{k\to+\infty} \Big(\lambda_k \mathcal{H}^1(\partial D_k^{\lambda_k})\Big) = 0;$

(iv) $\partial D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon$ consists of a finite set of points. Hence[12]*, also the relative boundary of $D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon$ in $\partial \mathrm{B}_\varepsilon$ consists of a finite set $\{x_i\}_{i\in I_k}$ of points which are the endpoints of the corresponding finite number of arcs forming $D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon$, and*

$$\lim_{k\to+\infty} \sum_{i\in I_k} d_k(x_i) = 0; \tag{4.12}$$

(v) $\mathcal{H}^1(D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon) \leq \frac{1}{n}$ for all $k \in \mathbb{N}$.

Proof. Let

$$I := (0,2) \setminus \bigcup_{k\in\mathbb{N}} \{1 - |u_k(0)|\},$$

which is of full measure in $(0,2)$.

We have, for an absolute positive constant α, recalling the definition of d_k in (4.8),

$$\int_\Omega |\nabla u_k - \nabla u| \; dx \geq \alpha \int_\Omega |\nabla d_k| \; dx = \alpha \int_0^2 \mathcal{H}^1(\{d_k = \lambda\}) \; d\lambda, \tag{4.13}$$

where the last equality follows from the coarea formula, recalling also that u_k takes values in $\overline{B}_1$. The left-hand side is uniformly bounded with respect to k, thanks to (4.3) and the fact that $\nabla u \in L^1(\Omega, \mathbb{R}^2)$. Thus, denoting

$$\varphi_k(\cdot) := \mathcal{H}^1(\{d_k = \cdot\}), \qquad \varphi := \liminf_{k\to+\infty} \varphi_k, \tag{4.14}$$

we get, from Fatou's lemma,

$$\int_0^2 \varphi(\lambda) \; d\lambda = \int_I \varphi(\lambda) \; d\lambda \leq C_1, \tag{4.15}$$

for some constant $C_1 > 0$.

Let us now focus attention on the set $\partial \mathrm{B}_\varepsilon$. We apply the tangential coarea formula to $\partial \mathrm{B}_\varepsilon$ (see for instance [24, Theorems 11.4, 18.8]) so that, if ∂_{tg} stands for the tangential derivative along $\partial \mathrm{B}_\varepsilon$, we have

$$\int_{\partial \mathrm{B}_\varepsilon} \big|\partial_{\mathrm{tg}} d_k\big| \; d\mathcal{H}^1 = \int_0^2 \mathcal{H}^0(\{d_k = \lambda\} \cap \partial \mathrm{B}_\varepsilon) \; d\lambda.$$

Arguing in a similar manner as before, denoting

$$\psi_k(\cdot) := \mathcal{H}^0(\{d_k = \cdot\} \cap \partial \mathrm{B}_\varepsilon), \qquad \psi := \liminf_{k\to+\infty} \psi_k, \tag{4.16}$$

[12] The relative boundary of $D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon$ is contained in $\partial D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon$.

it follows that, exploiting condition (4.4), there exists a constant $C'_\varepsilon > 0$ such that

$$\int_I \psi(\lambda)\ d\lambda \leq C'_\varepsilon. \tag{4.17}$$

We now claim that

$$\exists (\lambda_m) \subset I : \lim_{m\to+\infty} \lambda_m = 0, \quad \lim_{m\to+\infty} (\varphi(\lambda_m)\lambda_m) = 0 = \lim_{m\to+\infty} (\psi(\lambda_m)\lambda_m). \tag{4.18}$$

Recalling that I is of full measure, assume (4.18) is false, so that either there are $c_0 > 0$ and $\delta_0 > 0$ such that

$$\varphi(\lambda) > \frac{c_0}{\lambda} \qquad \forall \lambda \in (0, \delta_0) \cap I, \tag{4.19}$$

or there are $c'_0 > 0$ and $\delta'_0 > 0$ such that

$$\psi(\lambda) > \frac{c'_0}{\lambda} \qquad \forall \lambda \in (0, \delta'_0) \cap I. \tag{4.20}$$

Suppose for instance we are in case (4.19): since I has full measure, this contradicts (4.15); the same argument applied to (4.20) leads to contradict (4.17). Hence claim (4.18) is proven, and therefore, upon passing to a (not relabelled) subsequence we might assume that (λ_m) is decreasing, and

$$\varphi(\lambda_m)\lambda_m < \frac{1}{m}, \qquad \psi(\lambda_m)\lambda_m < \frac{1}{m} \qquad\qquad \forall m \in \mathbb{N}.$$

Thus, recalling (4.14) and (4.16), for any $m \in \mathbb{N}$ there are infinitely many $l \in \mathbb{N}$ such that

$$\varphi_l(\lambda_m)\lambda_m < \frac{2}{m}, \qquad \psi_l(\lambda_m)\lambda_m < \frac{2}{m}. \tag{4.21}$$

Moreover, for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ there exists $k(n, \lambda_m) \in \mathbb{N}$ such that

$$D_h^{\lambda_m} \cup \partial D_h^{\lambda_m} \subseteq A_n \qquad \text{and} \qquad \mathcal{H}^1(D_h^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon) \leq \frac{1}{n} \qquad \forall h \geq k(n, \lambda_m), \tag{4.22}$$

where the inclusion follows from (4.11) and the inequality being a consequence of (4.5). For any $m \in \mathbb{N}$ we can choose $h_m \in \mathbb{N}$ (depending also on n) such that $h_m < h_{m+1}$, $h_m \geq k(n, \lambda_m)$, and (4.21) is verified for $l = h_m$. Therefore

$$\lim_{m\to+\infty} (\varphi_{h_m}(\lambda_m)\lambda_m) = 0, \tag{4.23}$$

$$D_{h_m}^{\lambda_m} \cup \partial D_{h_m}^{\lambda_m} \subseteq A_n \quad \text{for all } n, m \in \mathbb{N}, \tag{4.24}$$

$$\lim_{m\to+\infty} (\psi_{h_m}(\lambda_m)\lambda_m) = 0, \tag{4.25}$$

$$\mathcal{H}^1(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon) \leq \frac{1}{n} \quad \text{for all } n, m \in \mathbb{N}. \tag{4.26}$$

Notice also that from (4.21) we have $\psi_{h_m}(\lambda_m) < +\infty$, so that $\{d_{h_m} = \lambda_m\} \cap \partial \mathrm{B}_\varepsilon$ is a finite set $\{\widetilde{x}_i\}$ of points. The relative boundary $\partial(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon)$ of $D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon$ in $\partial \mathrm{B}_\varepsilon$ must belong to $\partial D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon \subseteq \{d_{h_m} = \lambda_m\} \cap \partial \mathrm{B}_\varepsilon = \{\widetilde{x}_i\}$. Hence, let $\{x_i\} \subseteq \{\widetilde{x}_i\}$ be the set of boundary points of $D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon$ in $\partial \mathrm{B}_\varepsilon$.

Since $D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon$ is open in $\partial \mathrm{B}_\varepsilon$, we have that (whenever it is nonempty) it consists either of the union of arcs with endpoints $\{x_i\}$ or is the whole of $\partial \mathrm{B}_\varepsilon$, and statements (ii) and (v) follow. Notice also that

$$\sum_{x \in \partial(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon)} d_{h_m}(x) = \mathcal{H}^0(\partial(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon))\lambda_m \le \psi_{h_m}(\lambda_m)\lambda_m,$$

and (4.12) follows from (4.25).

To prove (iii) we see that, by definition of φ_k in (4.14) and recalling (4.23), we obtain

$$\lim_{m\to+\infty} \int_{\{d_{h_m}=\lambda_m\}} d_{h_m}\ d\mathcal{H}^1 = \lim_{m\to+\infty} \big(\mathcal{H}^1(\{d_{h_m} = \lambda_m\})\lambda_m\big) = \lim_{m\to+\infty} (\varphi_{h_m}(\lambda_m)\lambda_m) = 0.$$

A similar argument holds for ψ_k using (4.25), and also (v) follows.

It remains to prove (i). The first assertion follows since $\lambda_m \in I$ from (4.18). As for the second assertion, we see that $D_{h_m}^{\lambda_m}$ is a subset of $\Omega \setminus \{0\}$ whose perimeter is finite: indeed, by definition the reduced boundary of $D_{h_m}^{\lambda_m}$ is a subset of $\{d_{h_m} = \lambda_m\}$, which has finite $\mathcal{H}^1$ measure by (4.21). Thus $\partial D_{h_m}^{\lambda_m}$ is a closed 1-integral current in $\Omega \setminus \{0\}$ and by the decomposition theorem for 1-dimensional currents it is the sum of integration on simple curves [15, pag. 420, 421], either closed or with endopoints on the boundary of $\Omega \setminus \{0\}$, *i.e.*, $\{0\} \cup \partial\Omega$. The finiteness of the total length of these curves follows, since $D_{h_m}^{\lambda_m}$ is a set of finite perimeter. This concludes the proof of (i), and of the lemma. $\square$

Corollary 4.3. *Let ε, n and (λ_k) be as in Lemma 4.2. Then*

$$\lim_{k\to+\infty} \big(\mathcal{H}^1(\{d_k = \lambda_k\})\lambda_k\big) = 0, \qquad \lim_{k\to+\infty} \big(\mathcal{H}^0\big(\{d_k = \lambda_k\} \cap \partial \mathrm{B}_\varepsilon(0)\big)\lambda_k\big) = 0.$$

Proof. It follows from the proof of Lemma 4.2. $\square$

Once for all we fix the sequence (λ_k) as in Lemma 4.2 and, in order to shorten the notation, we give the following:

Definition 4.4 (Definite choice of D_k). *We set*

$$D_k := D_k^{\lambda_k}. \tag{4.27}$$

Let us recall that

$$\partial D_k \subseteq \{d_k = \lambda_k\}. \tag{4.28}$$

Also, observe that, upon extracting a further (not relabelled) subsequence, we might assume that the characteristic functions χ_{D_k} converge weakly* in $L^\infty(\Omega)$ to some $\zeta_n \in L^\infty(\Omega; [0, 1])$ (the sequence (D_k) depends on n, and so ζ_n depends on n). Since the limit holds also weakly in $L^1(\Omega)$ we see that

$$\|\zeta_n\|_{L^1(\Omega)} \le \liminf_{k\to+\infty} \|\chi_{D_k}\|_{L^1(\Omega)} \le \frac{1}{n}. \tag{4.29}$$

Recalling the definition of $M_\alpha^\beta(A)$ in (2.1), we prove the following statement.

Lemma 4.5 (The currents T_k and the limit current $\mathcal{T}_n$). *Let $n \in \mathbb{N}$ be fixed and let A_n satisfy (4.6) and (4.7). For any $k \in \mathbb{N}$ define the current $T_k \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ as*

$$T_k(\omega) := \begin{cases} \int_{\Omega \setminus D_k} \varphi(x, u_k(x)) M_{\bar{\alpha}}^{\beta}(\nabla u_k(x)) \, dx & \text{if } |\beta| \leq 1, \\ 0 & \text{if } |\beta| = 2, \end{cases}$$

where $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ is a 2-form that writes as

$$\omega(x, y) = \varphi(x, y) dx^{\alpha} \wedge dy^{\beta}, \quad \varphi \in C_c^{\infty}(\Omega \times \mathbb{R}^2), \quad |\alpha| + |\beta| = 2. \tag{4.30}$$

Then

$$\lim_{k \to +\infty} T_k = \mathcal{T}_n \in \mathcal{D}_2(\Omega \times \mathbb{R}^2) \qquad \text{weakly in the sense of currents,}$$

where

$$\mathcal{T}_n(\omega) := \int_{\Omega} \varphi(x, u(x)) M_{\bar{\alpha}}^{\beta}(\nabla u(x))(1 - \zeta_n(x)) \, dx \qquad \forall \omega \text{ as in (4.30).}$$

Proof. Since the Jacobian of u vanishes almost everywhere it follows that $\mathcal{T}_n(\varphi dy^1 \wedge dy^2) = 0$ for all φ as in (4.30). Then for 2-forms $\omega = \varphi dy^1 \wedge dy^2$ the convergence $T_k(\omega) \to \mathcal{T}_n(\omega)$ is achieved. We are then left to prove that for all 2-forms ω with $\omega(x, y) = \varphi(x, y) dx^{\alpha} \wedge dy^{\beta}$, $\varphi \in C_c^{\infty}(\Omega \times \mathbb{R}^2)$, $|\alpha| + |\beta| = 2$, and $|\beta| \leq 1$, it holds

$$\lim_{k \to +\infty} \int_{\Omega \setminus D_k} \varphi(x, u_k(x)) M_{\bar{\alpha}}^{\beta}(\nabla u_k(x)) \, dx = \int_{\Omega} \varphi(x, u(x)) M_{\bar{\alpha}}^{\beta}(\nabla u)(1 - \zeta_n(x)) \, dx. \tag{4.31}$$

To simplify the argument we treat separately the cases $\omega = \varphi(x, y) dx^1 \wedge dx^2$ and $\omega = \varphi(x, y) dx^i \wedge dy^j$ for some $i, j \in \{1, 2\}$. In the former case we simply have

$$\int_{\Omega \setminus D_k} \varphi(x, u_k(x)) \, dx = \int_{\Omega} \varphi(x, u_k(x)) \chi_{\Omega \setminus D_k}(x) \, dx.$$

Then, using that $u_k \to u$ uniformly in $\Omega \setminus D_k$ (see (4.7), Lemma 4.2(ii) and (4.27)) and $\chi_{\Omega \setminus D_k} \to \chi_{\Omega} - \zeta_n$ weakly* in $L^{\infty}(\Omega)$, it follows

$$\lim_{k \to +\infty} \int_{\Omega \setminus D_k} \varphi(x, u_k(x)) \, dx = \int_{\Omega} \varphi(x, u(x))(1 - \zeta_n(x)) \, dx = \mathcal{T}_n(\omega).$$

Assume now $\omega = \varphi(x, y) dx^i \wedge dy^j$, $i, j \in \{1, 2\}$, $i \neq j$. In this case (4.31) reads as

$$\lim_{k \to +\infty} \Big(\int_{\Omega \setminus D_k} \varphi(x, u_k(x)) D_{\bar{i}}[(u_k(x))_j] \, dx - \int_{\Omega} \varphi(x, u(x)) D_{\bar{i}} u_j(x)(1 - \zeta_n(x)) \, dx \Big) = 0,$$

with $\bar{i} = \{1, 2\} \setminus \{i\}$. Since $\chi_{D_k} \to \zeta_n$ weakly* in $L^{\infty}(\Omega)$, this is equivalent to proving

$$\lim_{k \to +\infty} \Big(\int_{\Omega \setminus D_k} \varphi(x, u_k(x)) D_{\bar{i}}[(u_k(x))_j] \, dx - \int_{\Omega \setminus D_k} \varphi(x, u(x)) D_{\bar{i}} u_j(x) \, dx \Big) = 0.$$

The quantity between parentheses on the left-hand side can be written as

$$\int_{\Omega \setminus D_k} \Big(\varphi(x, u_k(x)) - \varphi(x, u(x)) \Big) D_{\bar{i}}[(u_k(x))_j] \, dx + \int_{\Omega \setminus D_k} \varphi(x, u(x)) \Big(D_{\bar{i}}[(u_k(x))_j] - D_{\bar{i}} u_j(x) \Big) \, dx,$$

and we see that the first integral tends to zero as $k \to +\infty$, since $u_k \to u$ uniformly in $\Omega \setminus D_k$, φ is Lipschitz continuous, and the $L^1(\Omega)$-norm of $D_{\bar{i}}[(u_k)_j]$ is uniformly bounded with respect to k. The second integral can be instead integrated by parts[13], obtaining

$$\begin{aligned}
&\int_{\Omega\setminus D_k} \varphi(x,u(x))(D_{\bar{i}}[(u_k(x))_j] - D_{\bar{i}}u_j(x))\ dx \\
&= \int_{\partial D_k} \varphi(x,u(x))((u_k(x))_j - u_j(x))\nu_{\bar{i}}(x)\ d\mathcal{H}^1(x) - \int_{\Omega\setminus D_k} D_{\bar{i}}(\varphi(x,u(x)))((u_k(x))_j - u_j(x))\ dx \\
&=: \mathrm{I}_k + \mathrm{II}_k.
\end{aligned}$$

Thanks to the fact that φ is bounded and that $|(u_k)_j(x) - u_j(x)| \leq d_k(x) = \lambda_k$ on ∂D_k, we conclude by Corollary 4.3 that $\lim_{k\to+\infty} \mathrm{I}_k = 0$. Moreover

$$\begin{aligned}
\mathrm{II}_k = &-\int_{\Omega\setminus D_k} \partial_{x_{\bar{i}}}\varphi(x,u(x))((u_k(x))_j - u_j(x))dx \\
&-\sum_{l=1}^{2}\int_{\Omega\setminus D_k} \partial_{y_l}\varphi(x,u(x))D_{\bar{i}}u_l(x)((u_k)_j(x) - u_j(x))dx =: \mathrm{II}_{k,1} + \mathrm{II}_{k,2}.
\end{aligned}$$

Then $\lim_{k\to+\infty} \mathrm{II}_{k,1} = \lim_{k\to+\infty} \mathrm{II}_{k,2} = 0$, since the partial derivatives of φ are bounded, $D_{\bar{i}}u \in L^1(\Omega \setminus D_k, \mathbb{R}^2)$, $|(u_k)_j - u_j| \leq d_k \leq \lambda_k$ on $\Omega \setminus D_k$, and $\lim_{k\to+\infty} \lambda_k = 0$. $\square$

Remark 4.6. The mass of the current T_k is given by

$$|T_k| = \int_{\Omega\setminus D_k} \sqrt{1 + |\nabla u_k|^2}dx. \tag{4.32}$$

To see (4.32) we choose a 2-form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ as

$$\omega := \sum_{|\alpha|+|\beta|=2} \varphi_{\bar{\alpha}\beta} dx^\alpha \wedge dy^\beta, \qquad \|\omega\| \leq 1,$$

set[14] $\widehat{\omega}(x,y) =: (\varphi_{\bar{\alpha}\beta}(x,y)) \in \mathbb{R}^6$, and

$$\widetilde{\mathcal{M}}(\nabla u_k(x)) := (1, D_1[(u_k(x))_1], D_2[(u_k(x))_1], D_1[(u_k(x))_2], D_2[(u_k(x))_2], 0) \in \mathbb{R}^6 = \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R},$$

so that

$$\begin{aligned}
T_k(\omega) &= \int_{\Omega\setminus D_k} \langle \widehat{\omega}(x,u_k(x)), \widetilde{\mathcal{M}}(\nabla u_k(x))\rangle dx \\
&\leq \|\widehat{\omega}\| \int_{\Omega\setminus D_k} |\widetilde{\mathcal{M}}(\nabla u_k(x))| dx \leq \int_{\Omega\setminus D_k} \sqrt{1 + |\nabla u_k|^2}dx.
\end{aligned} \tag{4.33}$$

To prove the converse inequality, choosing $\widehat{\omega}(x,y) = \frac{\widetilde{\mathcal{M}}(\nabla u_k(x))}{|\widetilde{\mathcal{M}}(\nabla u_k(x))|}$ would give the equality in (4.33). However, $\frac{\widetilde{\mathcal{M}}(\nabla u_k)}{|\widetilde{\mathcal{M}}(\nabla u_k)|}$ is not necessarily of class C_c^∞, so we need to use the density of $C_c^\infty(\Omega \times \mathbb{R}^2)$ in $L^1(\Omega \times \mathbb{R}^2)$ (here we use that $\widetilde{\mathcal{M}}(\nabla u_k) \in L^\infty(\Omega, \mathbb{R}^6)$ since u_k is Lipschitz continuous).

With a similar argument, setting

$$\widetilde{\mathcal{M}}(\nabla u(x)) := (1 - \zeta_n(x))\mathcal{M}(\nabla u(x)) \in \mathbb{R}^6, \quad x \in \Omega \setminus \overline{\mathrm{B}}_\varepsilon$$

we can show that the total mass of $\mathcal{T}_n$ in $(\Omega \setminus \overline{\mathrm{B}}_\varepsilon) \times \mathbb{R}^2$ is given by

$$|\mathcal{T}_n|_{(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)\times\mathbb{R}^2} = \int_{\Omega\setminus\overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)||1 - \zeta_n|\ dx. \tag{4.34}$$

[13] From Lemma 4.2(i), D_k has rectifiable boundary; moreover, $\varphi(\cdot, u_k(\cdot))$ is Lipschitz. We can then apply a version of the Gauss-Green theorem, see for instance [24, pag. 124, exercise 12.12].

[14] Here α and β run over all the multi-indeces in $\{1,2\}$ with the constraint $|\alpha| + |\beta| = 2$.

4.2 Estimate of the mass of $[\![G_{u_k}]\!]$ over $\Omega \setminus D_k$

We denote by $\Phi_k = \Phi_{u_k} = \mathrm{Id} \bowtie u_k : \Omega \to \Omega \times \mathbb{R}^2$ the map

$$\Phi_k(x) = (x, u_k(x)), \tag{4.35}$$

in such a way that $\Phi_k(\Omega) = G_{u_k}$, with $G_{u_k} = \{(x,y) \in \Omega \times \mathbb{R}^2 : y = u_k(x)\}$ the graph of u_k.

We denote as usual by

$$[\![G_{u_k}]\!] \in \mathcal{D}_2(\Omega \times \mathbb{R}^2) \tag{4.36}$$

the integral current supported by the graph of u_k.

We now want to estimate the area of the graph of u_k over the set $(\Omega \setminus \overline{\mathrm{B}}_\varepsilon) \setminus D_k$.

Proposition 4.7. *Let $\varepsilon \in (0, l)$ satisfy* (4.4) *and* (4.5)*, $n \in \mathbb{N}$, (λ_k) be as in Lemma 4.2, and let D_k be as in* (4.27)*. Then*

$$\liminf_{k \to +\infty} \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)|\, dx \geq \int_{\Omega \setminus \overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)|\, dx - \frac{1}{n} - \frac{2}{\varepsilon n}. \tag{4.37}$$

Proof. Set $\Omega_\varepsilon := \Omega \setminus \overline{\mathrm{B}}_\varepsilon$. Since, by definition, T_k vanishes on smooth 2-forms supported in $(D_k \cap \Omega_\varepsilon) \times \mathbb{R}^2$, we employ (4.32) to obtain

$$\begin{aligned} &\liminf_{k \to +\infty} \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)|\, dx \geq \quad \liminf_{k \to +\infty} \int_{\Omega \setminus D_k} \sqrt{1 + |\nabla u_k|^2}\, dx \geq \liminf_{k \to +\infty} |T_k|_{(\Omega_\varepsilon \setminus D_k) \times \mathbb{R}^2} \\ =&\liminf_{k \to +\infty} |T_k|_{\Omega_\varepsilon \times \mathbb{R}^2} \geq |\mathcal{T}_n|_{\Omega_\varepsilon \times \mathbb{R}^2}, \end{aligned} \tag{4.38}$$

where we use that (T_k) weakly converges to $\mathcal{T}_n$ (Lemma 4.5), and the weak lower semicontinuity of the mass. In turn, from (4.34) and (4.29),

$$\begin{aligned} |\mathcal{T}_n|_{\Omega_\varepsilon \times \mathbb{R}^2} &= \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)||1 - \zeta_n|\, dx \geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\, dx - \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)||\zeta_n|\, dx \\ &\geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\, dx - \|\mathcal{M}(\nabla u)\|_{L^\infty(\Omega_\varepsilon)} \|\zeta_n\|_{L^1(\Omega_\varepsilon)} \\ &\geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\, dx - \frac{1}{n} \|\mathcal{M}(\nabla u)\|_{L^\infty(\Omega_\varepsilon)}. \end{aligned} \tag{4.39}$$

Next, using $\sqrt{1+z^2} \leq 1 + |z|$ and[15] $|\nabla u(x)| \leq \frac{2}{|x|}$ which, on Ω_ε, is bounded by $2/\varepsilon$, we also get

$$\|\mathcal{M}(\nabla u)\|_{L^\infty(\Omega_\varepsilon)} = \|\sqrt{1 + |\nabla u|^2}\|_{L^\infty(\Omega_\varepsilon)} \leq 1 + \frac{2}{\varepsilon}.$$

We deduce

$$|\mathcal{T}_n|_{\Omega_\varepsilon \times \mathbb{R}^2} \geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\, dx - \frac{1}{n} - \frac{2}{\varepsilon n}.$$

From (4.39) and (4.38) inequality (4.37) follows. $\square$

[15] $D_i u_j(x) = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}$, hence $\sum_{ij} (D_i u_j(x))^2 = \frac{2}{|x|^2} + 2\frac{x_1^2 x_2^2}{|x|^6} \leq \frac{4}{|x|^2}$.

5 The maps Ψ_k, π_{λ_k}, and the currents $\mathfrak{D}_k$, $\widehat{\mathfrak{D}}_k$, $\mathcal{E}_k$

Recalling that D_k is defined in (4.27) and (4.9), in Section 4.2 we have estimated the area of the graph of u_k over $\Omega \setminus D_k$. The next step, which is considerably more difficult, is to estimate this area over D_k, and this will be splitted in several parts (Sections 6-9). After introducing some preliminaries in Section 5.1, the first step is to reduce the graph of u_k (a surface of codimension 2 in $\mathbb{R}^4$) to a suitable rectifiable set ($\Psi_k(D_k)$ and their projections) of codimension 1 sitting in $\overline{C}_l \subset \mathbb{R}^3$. In this section we introduce all various objects needed to prove the lower bound.

Definition 5.1 (The map Ψ_k). *For all $k \in \mathbb{N}$, we define the map $\Psi_k = \Psi_{u_k} : \Omega \to \mathbb{R}^3 = \mathbb{R}_{|x|} \times \mathbb{R}^2_{\rm target}$ as*

$$\Psi_k(x) :=(|x|, u_k(x)) \qquad \forall x \in \Omega. \tag{5.1}$$

Notice that Ψ_k takes values in $\overline{C}_l$, and is Lipschitz continuous. Moreover $\Psi_k = R \circ \Phi_k$, where $\Phi_k = \mathrm{Id} \bowtie u_k : \Omega \to \mathbb{R}^4$ is defined in (4.35), and $R : \mathbb{R}^4 \ni (x, y) \mapsto (|x|, y) \in \mathbb{R}^3$. By the area formula and since $\mathrm{Lip}(R) = 1$ we have

$$\int_B (\nabla\Psi_k^T \nabla\Psi_k)^{\frac12}\, dx \le \int_B (\nabla\Phi_k^T \nabla\Phi_k)^{\frac12}\, dx = \int_B |\mathcal{M}(\nabla u_k)|\, dx = |\mathcal{G}_{u_k}|_{B\times\mathbb{R}^2},$$

for any Borel set $B \subseteq \Omega$.

5.1 The sets $\Psi_k(D_k)$ and the currents $(\Psi_k)_\sharp[\![D_k]\!]$

We start noticing that

$$\Psi_k(\Omega \setminus D_k) \subset \overline{C}_l \setminus C_l(1-\lambda_k), \qquad k \in \mathbb{N}, \tag{5.2}$$

where we recall that $C_l(1-\lambda_k)$ is defined in (2.4). Indeed, since $\Omega \setminus D_k \subseteq \{d_k \le \lambda_k\}$ for any $k \in \mathbb{N}$ we have

$$\lambda_k \ge |u_k(x) - \frac{x}{|x|}| \ge \mathrm{dist}(u_k(x), \mathbb{S}^1) = 1 - |u_k(x)|, \qquad x \in \Omega \setminus D_k, \tag{5.3}$$

so that $|u_k(x)| \ge 1 - \lambda_k$. In particular

$$\Psi_k(\partial D_k) \subset \overline{C}_l \setminus C_l(1-\lambda_k), \qquad k \in \mathbb{N}. \tag{5.4}$$

As a consequence, since the map Ψ_k is Lipschitz continuous, we have:

Corollary 5.2. *For all $k \in \mathbb{N}$ the integral 2-current $(\Psi_k)_\sharp[\![D_k]\!]$ is boundaryless in $C_l(1-\lambda_k)$.*

Observe that $\Psi_k(D_k)$ is rectifiable and contains[16] the support of $(\Psi_k)_\sharp[\![D_k]\!]$; also $\Psi_k(D_k)$ is contained in $[0, l) \times \overline{B}_1$. Specifically, the fact that C_l has axial coordinate in $(-1, l)$ and not in $(0, l)$ will be convenient in order to control the behaviour of $(\Psi_k)_\sharp[\![D_k]\!]$ on $\{0\} \times \mathbb{R}^2$.

Definition 5.3 (The projection π_{λ_k}). *We let*

$$\pi_{\lambda_k} = \pi_{\lambda_k} : \mathbb{R}^3 \to \overline{C}_l(1-\lambda_k) \tag{5.5}$$

be the orthogonal projection onto the compact convex set $\overline{C}_l(1-\lambda_k)$.

[16]It could be different because of possible cancellations.

In Section 5.2 we project $\Psi_k(D_k)$ on $\overline{C}_l(1-\lambda_k)$ in order to get a rectifiable set (and its associated current) whose area (counted with multiplicity) is less than or equal to the area of the original set; the area of the projected set, in turn, gives a lower bound for the mass of $[\![G_{u_k}]\!]$ over D_k (see formulas (5.7) and (5.11)). Then, as a second step, we symmetrize $\pi_{\lambda_k} \circ \Psi_k(D_k)$ using the cylindrical rearrangement introduced in Section 3 to get a still smaller (in area) object. The estimate of the area of the symmetrized object is divided in two parts: the first one (Section 7) deals with $\pi_{\lambda_k} \circ \Psi_k(D_k \cap (\Omega \setminus \mathrm{B}_\varepsilon))$ whose symmetrized set can be seen as the generalized graph of a suitable polar function. In Section 8 we deal with the second part, where we estimate the area of the symmetrization obtained from $\pi_{\lambda_k} \circ \Psi_k(D_k \cap \mathrm{B}_\varepsilon)$. In Sections 9 and 11, we collect our estimates and we utilize the symmetrized object as a competitor for a suitable non-parametric Plateau problem. To do this we need to glue to the obtained rectifiable set some artificial surfaces, whose areas are controlled and are infinitesimal in the limit as $k \to +\infty$. This limit is taken only at the end of Section 11, allowing us to analyse a non-parametric Plateau problem whose boundary condition does not depend on k, so that also its solution does not depend on k. The area of such a solution will be the lower bound for the area of the rectifiable set $\pi_{\lambda_k} \circ \Psi_k(D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \cup \pi_{\lambda_k} \circ \Psi_k(D_k \cap \mathrm{B}_\varepsilon)$, and then finally for the area of the graph of u_k on D_k.

5.2 Construction of the current $\widehat{\mathfrak{D}}_k$ via the currents $\mathcal{D}_k$ and $\mathcal{W}_k$

We are interested in the part of the set $\Psi_k(D_k)$ included in $\overline{C}_l(1-\lambda_k)$; we need an explicit description of the boundary of $\Psi_k(D_k)$, and to this aim we compose Ψ_k with the projection π_{λ_k} in (5.5).

Definition 5.4 (Projection of $\Psi_k(D_k)$: the current $\mathfrak{D}_k$). *We define the current $\mathfrak{D}_k \in \mathcal{D}_2(C_l)$ as*

$$\mathfrak{D}_k := (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k]\!]. \tag{5.6}$$

Remark 5.5. In general $\Psi_k(D_k) \subseteq \overline{C_l}(1-\lambda_k) \cup (\overline{C_l} \setminus C_l(1-\lambda_k))$, while $\mathrm{spt}(\mathfrak{D}_k) \subseteq \overline{C_l}(1-\lambda_k)$.

Since $\mathrm{Lip}(\pi_{\lambda_k}) = 1$, the map π_{λ_k} does not increase the area, and therefore

$$\int_{D_k} |J(\pi_{\lambda_k} \circ \Psi_k)| \, dx \le \int_{D_k} |J(\Psi_k)| \, dx \le |[\![G_{u_k}]\!]|_{D_k \times \mathbb{R}^2}, \tag{5.7}$$

$$\int_{D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k} \circ \Psi_k)| \, dx \le \int_{D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)} |J(\Psi_k)| \, dx \le |[\![G_{u_k}]\!]|_{(D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \times \mathbb{R}^2}, \tag{5.8}$$

The same holds for the mass of the current $\mathfrak{D}_k$, *i.e.*,

$$|\mathfrak{D}_k| \le |(\Psi_k)_\sharp [\![D_k]\!]|,$$

and recalling also (2.7),

$$|\mathfrak{D}_k|_{\overline{C}_l^\varepsilon} \le |(\Psi_k)_\sharp [\![D_k]\!]|_{\overline{C}_l^\varepsilon} \le |[\![G_{u_k}]\!]|_{(D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \times \mathbb{R}^2}. \tag{5.9}$$

Remark 5.6. The area, counted with multiplicity, of the 2-rectifiable set $\pi_{\lambda_k} \circ \Psi_k(D_k)$ is greater than or equal to the mass of the current $\mathfrak{D}_k$, more specifically

$$\int_{D_k} |J(\pi_{\lambda_k} \circ \Psi_k)| \, dx \ge |\mathfrak{D}_k|_{\overline{C}_l(1-\lambda_k)} \qquad \text{and} \qquad \int_{D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k} \circ \Psi_k)| \, dx \ge |\mathfrak{D}_k|_{\overline{C}_l^\varepsilon(1-\lambda_k)}. \tag{5.10}$$

This is due to the fact that $\pi_{\lambda_k} \circ \Psi_k(D_k)$ might overlap with opposite orientations so that the multiplicity of $\mathfrak{D}_k$ vanishes, and the overlappings do not contribute to its mass. In particular, $\mathrm{spt}(\mathfrak{D}_k) \subseteq \pi_{\lambda_k} \circ \Psi_k(D_k)$.

From (5.7) and (5.10) it follows

$$|[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2} \geq |\mathfrak{D}_k|_{\overline{C}_l(1-\lambda_k)}, \qquad |[\![G_{u_k}]\!]|_{(D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon))\times\mathbb{R}^2} \geq |\mathfrak{D}_k|_{\overline{C}_l^\varepsilon(1-\lambda_k)}. \tag{5.11}$$

We now analyse the boundary of $\mathfrak{D}_k$. Up to small modifications, we will prove that it is boundaryless in $C_l(1-\lambda'_k)$ (see (5.20) and (5.23), where λ'_k are suitable small numbers in $(0,\lambda_k)$ chosen below in Definition 5.12) and so $\mathfrak{D}_k$ can be symmetrized according to Definition 3.3. Before proceeding to the symmetrization we need some preliminaries. We build suitable currents $\mathcal{W}_k$, with their support sets denoted by W_k (see (5.17) and (5.16)), with $\partial\mathcal{W}_k$ coinciding with $\partial\mathfrak{D}_k$ (see (5.21), (5.22), and (5.23)).

Remark 5.7. By (5.4), $\pi_{\lambda_k}\circ\Psi_k(\partial D_k)$ is contained in $\partial_{\mathrm{lat}}C_l(1-\lambda_k)$. By Lemma 4.2(i), $\pi_{\lambda_k}\circ\Psi_k(\partial D_k)$ is the union of the image of at most countably many curves, and this union, counted with multiplicities, has finite $\mathcal{H}^1$ measure: specifically, if we define

$$M(\pi_{\lambda_k}\circ\Psi_k(\partial D_k)) := \int_{\partial D_k} \left|\partial_{\mathrm{tg}}\ (\pi_{\lambda_k}\circ\Psi_k)\right|\, d\mathcal{H}^1,$$

where ∂_{tg} stands for the tangential derivative along ∂D_k, then $M(\pi_{\lambda_k}\circ\Psi_k(\partial D_k)) < +\infty$ since $\mathcal{H}^1(\partial D_k) < +\infty$ (still by Lemma 4.2(i)) and u_k is Lipschitz continuous.

Moreover

$$\partial\mathfrak{D}_k = (\pi_{\lambda_k}\circ\Psi_k)_\sharp\partial[\![D_k]\!] \in \mathcal{D}_1(C_l) \qquad \text{in } C_l. \tag{5.12}$$

It is convenient to introduce a suitable map τ parametrizing the region $\overline{C}_l\setminus C_l(1-\lambda_k)$ in between the two concentric cylinders; this map can then be pulled back by $\pi_{\lambda_k}\circ\Psi_k$, but only in $\Omega\setminus D_k$, to get the map $\widetilde{\tau}$.

Definition 5.8 (The maps τ, $\widetilde{\tau}$). *We set*

$$\begin{aligned}
&\tau = \tau_{\lambda_k} : [1-\lambda_k,1]\times\partial C_l(1-\lambda_k) \to \overline{C}_l\setminus C_l(1-\lambda_k) \subset \mathbb{R}^3,\\
&\tau(\rho,t,y) := \Big(t, \frac{y}{|y|}\rho\Big), \qquad\quad \rho\in[1-\lambda_k,1], \quad (t,y)\in\partial C_l(1-\lambda_k) = [-1,l]\times\partial B_{1-\lambda_k}.
\end{aligned} \tag{5.13}$$

By (5.2) *it follows* $\pi_{\lambda_k}\circ\Psi_k(\Omega\setminus D_k)\subset\partial C_l(1-\lambda_k)$, *hence we can also set*

$$\widetilde{\tau}(\rho,x) = \widetilde{\tau}_{u_k,\lambda_k}(\rho,x) := \tau(\rho,\pi_{\lambda_k}\circ\Psi_k(x)), \qquad \rho\in[1-\lambda_k,1],\ x\in\Omega\setminus D_k. \tag{5.14}$$

Notice that $\tau(\rho,\cdot,\cdot)$ takes values in $\partial C_l(\rho)$ for any $\rho\in[1-\lambda_k,1]$, that $\tau(\cdot,t,y)$ moves along the normal to the lateral boundary of $\partial C_l(1-\lambda_k)$ at the point (t,y), and $\tau(1-\lambda_k,\cdot,\cdot)$ is the identity. We also observe that, due to the fact that $\pi_{\lambda_k}\circ\Psi_k$ takes values in $[0,l)\times\overline{B}_1$, the same holds for $\widetilde{\tau}$.

Remark 5.9. If $\lambda_k>0$ is small enough (which is true for k large enough), the Jacobian of τ is close to 1 so that the $\mathcal{H}^1$-measure, counted with multiplicities, of the set $\tau(\rho,\pi_{\lambda_k}\circ\Psi_k(\partial D_k))$ is, for fixed ρ, bounded by two times the $\mathcal{H}^1$-measure of $\pi_{\lambda_k}\circ\Psi_k(\partial D_k)$, still counted with multiplicities. More precisely,

$$2\int_{\partial D_k}\left|\partial_{\mathrm{tg}}\ (\pi_{\lambda_k}\circ\Psi_k)\right|\, d\mathcal{H}^1 \geq \int_{\partial D_k}\left|\partial_{\mathrm{tg}}\ \tau(\rho,\pi_{\lambda_k}\circ\Psi_k)\right|\, d\mathcal{H}^1, \qquad \rho\in[1-\lambda_k,1], \tag{5.15}$$

$$2\int_{(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)\cap\partial D_k}\left|\partial_{\mathrm{tg}}\ (\pi_{\lambda_k}\circ\Psi_k)\right|\ d\mathcal{H}^1 \geq \int_{(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)\cap\partial D_k}\left|\partial_{\mathrm{tg}}\ \tau(\rho,\pi_{\lambda_k}\circ\Psi_k)\right|\, d\mathcal{H}^1,$$

for all $\rho\in[1-\lambda_k,1]$ and $k\in\mathbb{N}$ large enough, where we recall that, from Lemma 4.2(i), ∂D_k is rectifiable.

Now we take a sequence[17] of numbers $\lambda_k' \in (0, \lambda_k)$, which will be fixed in the sequel (see Definition 5.13).

Definition 5.10 (The set W_k and the current $\mathcal{W}_k$). *We define the 2-rectifiable set*[18]

$$W_k := \tau\big([1-\lambda_k, 1-\lambda_k'] \times \pi_{\lambda_k} \circ \Psi_k(\partial D_k)\big) = \widetilde{\tau}\big([1-\lambda_k, 1-\lambda_k'] \times \partial D_k\big), \tag{5.16}$$

and the 2-current

$$\mathcal{W}_k := \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda_k'] \times \partial D_k]\!] \in \mathcal{D}_2(C_l). \tag{5.17}$$

Clearly $\mathrm{spt}(\mathcal{W}_k) \subseteq W_k$; Again, although $\mathcal{W}_k$ is defined as a current in C_l, it is supported in $[0, l] \times B_1$.

Remark 5.11 (Use of $[\![\cdot]\!]$ for not top-dimensional currents). ∂D_k is endowed with a natural orientation, inherited from the fact that it is the boundary of the set D_k; consistently, we sometimes use the identification $[\![\partial D_k]\!] = \partial [\![D_k]\!]$. With a little abuse of notation we have noted the current integration over $[1-\lambda_k, 1-\lambda_k'] \times \partial D_k$, meaning that ∂D_k is endowed with this natural orientation. Finally, recalling that $\tilde{\tau}(\rho, \cdot)$ takes values in $\partial C_l(\rho)$, we can do the following identification:

$$\widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda_k'] \times \partial D_k]\!] = \widetilde{\tau}_\sharp \partial [\![[1-\lambda_k, 1-\lambda_k'] \times D_k]\!] \llcorner \Big(C_l(1-\lambda_k') \setminus \overline{C}_l(1-\lambda_k) \Big).$$

We denote

$$M(W_k) := \int_{[1-\lambda_k, 1-\lambda_k'] \times \partial D_k} |J(\widetilde{\tau}(\rho, x))| \; d\rho \; d\mathcal{H}^1(x) \tag{5.18}$$

the area of W_k counted with multiplicities. By the area formula and using (5.15) we infer

$$|\mathcal{W}_k| \leq M(W_k) \leq 2(\lambda_k - \lambda_k') \int_{\partial D_k} \Big| \partial_{\mathrm{tg}} \; (\pi_{\lambda_k} \circ \Psi_k) \Big| \; d\mathcal{H}^1 = 2(\lambda_k - \lambda_k') M(\pi_{\lambda_k} \circ \Psi_k(\partial D_k)). \tag{5.19}$$

Then we are led to the following

Definition 5.12 (The sequence (λ_k')). *We select $\lambda_k' \in (0, \lambda_k)$ so that*

$$2(\lambda_k - \lambda_k') M(\pi_{\lambda_k} \circ \Psi_k(\partial D_k)) \leq \frac{1}{n} \qquad \forall k \in \mathbb{N}. \tag{5.20}$$

Finally we observe that

$$\partial \mathcal{W}_k = \tau(1-\lambda_k', \cdot, \cdot)_\sharp \Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial [\![D_k]\!] \Big) - (\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial [\![D_k]\!]. \tag{5.21}$$

Definition 5.13 (The current $\widehat{\mathfrak{D}}_k$). *We define*

$$\widehat{\mathfrak{D}}_k := \mathfrak{D}_k + \mathcal{W}_k \in \mathcal{D}_2(C_l). \tag{5.22}$$

The next result will be useful to select a primitive of $\widehat{\mathfrak{D}}_k$.

Corollary 5.14. *The current $\widehat{\mathfrak{D}}_k$ is supported in $[0, l] \times \overline{B}_{1-\lambda_k'}$ and*

$$\widehat{\mathfrak{D}}_k \quad \text{is boundaryless in the open cylinder } C_l(1-\lambda_k'). \tag{5.23}$$

In particular $\partial \widehat{\mathfrak{D}}_k = 0$ in $\mathcal{D}_1((-\infty, l) \times B_{1-\lambda_k'})$.

Proof. The statement follows by construction, and noticing that, since $\tau(1-\lambda_k', \cdot, \cdot)_\sharp \Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial [\![D_k]\!] \Big)$ has support in $\partial_{\mathrm{lat}} C_l(1-\lambda_k')$, one can use (5.12) to deduce (5.23). □

[17] The sequence (λ_k') depends on ε and n.

[18] The set W_k consists of "vertical" walls, normal to $\partial C_l(1-\lambda_k)$, build on $\pi_{\lambda_k} \circ \Psi_k(\partial D_k)$, with height $\lambda_k' - \lambda_k$: see Fig. 11.

5.3 The 3-current $\mathcal{E}_k$ and the symmetrization of $\widehat{\mathfrak{D}}_k$

Since we want to symmetrize $\widehat{\mathfrak{D}}_k$ according to Definition 3.3, we need to identify a unique primitive 3-current $\mathcal{E}_k$ such that $\partial\mathcal{E}_k = \widehat{\mathfrak{D}}_k$.

The restriction of the map $\pi_{\lambda_k} \circ \Psi_k$ to $\Omega \setminus D_k$ takes $\Omega \setminus D_k$ into $\partial C_l(1-\lambda_k)$ (see (5.2)), and can also be written as

$$\pi_{\lambda_k} \circ \Psi_k(x) = \Big(|x|, \frac{u_k(x)}{|u_k(x)|}(1-\lambda_k)\Big), \qquad x \in \Omega \setminus D_k. \tag{5.24}$$

The current $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]$ has boundary

$$\partial(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!] = -(\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial [\![D_k]\!]. \tag{5.25}$$

Definition 5.15 (The currents $\mathcal{Y}_k$ and $\mathcal{X}_k$). *Recalling the definition of τ (see (5.14), (5.13)) we set*

$$\mathcal{Y}_k := \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times (\Omega \setminus D_k)]\!] \in \mathcal{D}_3(C_l), \tag{5.26}$$

$$\mathcal{X}_k := [\![C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)]\!] - \mathcal{Y}_k \in \mathcal{D}_3(C_l). \tag{5.27}$$

Notice that $\mathcal{X}_k$ cannot be directly defined as a push-forward via the map $\widetilde{\tau}$, for part of $\Psi_k(D_k)$ could be contained in $C_l(1-\lambda_k)$, and for this reason we are led to define it as a difference.

The current $\mathcal{Y}_k$ could have multiplicity different from 0 and 1, and in particular could not be the integration over a finite perimeter set. This depends on the fact that the map Ψ_k could generate overlappings and self-intersections of the set $\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus D_k)$. If the multiplicity of $\mathcal{Y}_k$ is only 1 or 0 then the same holds for $\mathcal{X}_k$. Also, $\mathcal{Y}_k$ might be null, and in this case $\mathcal{X}_k$ coincides with the integration over the region $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$. A finer description of these two currents will be necessary later, and this will be done by a slicing argument in Lemma 6.4 below.

Recalling (5.17),

$$\partial\mathcal{Y}_k = -\mathcal{W}_k = -\partial\mathcal{X}_k \qquad \text{in } C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k),$$

as it can be seen by considering the push-forward by τ of (5.25). We proceed to the symmetrization in $C_l(1-\lambda'_k)$ of the current $\widehat{\mathfrak{D}}_k$ in (5.22). By (5.23) it follows the existence of an integer multiplicity 3-current $\mathcal{E}_k \in \mathcal{D}_3(C_l(1-\lambda'_k))$ such that

$$\partial\mathcal{E}_k = \widehat{\mathfrak{D}}_k \qquad \text{in } C_l(1-\lambda'_k). \tag{5.28}$$

The current $\mathcal{E}_k$ is unique up to a constant, that we might assume to be integer, since $\mathcal{E}_k$ has integer multiplicity. Hence we choose such a constant[19] so that

$$\mathcal{E}_k \llcorner \Big(C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)\Big) = \mathcal{X}_k. \tag{5.29}$$

Let E_k denote the support of $\mathcal{E}_k$; by decomposition,

$$\mathcal{E}_k = \sum_i (-1)^{\sigma_i} [\![E_{k,i}]\!] \qquad \text{in } C_l(1-\lambda'_k), \tag{5.30}$$

[19] The fact that this choice is possible is a consequence of the constancy theorem (see for instance [23, Proposition 7.3.1]). Indeed, let $\widehat{\mathcal{E}}_k$ have the same boundary (*i.e.*, $\mathcal{W}_k$) of $\mathcal{X}_k$ in $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$. Thus $\widehat{\mathcal{E}}_k - \mathcal{X}_k$ is boundaryless, and must be an integer multiple of the integration over $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$, *i.e.*, $\widehat{\mathcal{E}}_k - \mathcal{X}_k = h[\![C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)]\!]$. We then set $\mathcal{E}_k := \widehat{\mathcal{E}}_k - h[\![C_l(1-\lambda'_k)]\!]$ so that $\mathcal{E}_k = \mathcal{X}_k$ in $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$.

with $E_{k,i} \subset C_l(1-\lambda'_k)$ finite perimeter sets, the decomposition done with undecomposable components, see (3.3), (3.4). We denote

$$\mathbb{S}(E_k) := \cup_i \mathbb{S}(E_{k,i})$$

the union of the cylindrical symmetrizations of the sets $E_{k,i}$, see (3.2). Recalling (5.28), Definition 3.3 and (3.5), the symmetrization of the current $\widehat{\mathfrak{D}}_k$ is

$$\mathbb{S}(\widehat{\mathfrak{D}}_k) = \partial \mathbb{S}(\mathcal{E}_k) = \partial [\![\mathbb{S}(E_k)]\!] \llcorner C_l(1-\lambda'_k). \tag{5.31}$$

Formula (5.31) contains the needed information about the symmetrization of $\Psi_k(D_k)$, since by construction $\mathfrak{D}_k = \widehat{\mathfrak{D}}_k \llcorner \overline{C_l}(1-\lambda_k)$ (recall (5.6)).

We have

$$\widehat{\mathfrak{D}}_k = \sum_i (-1)^{\sigma_i} [\![\partial^* E_{k,i}]\!] \qquad \text{in } C_l(1-\lambda'_k), \tag{5.32}$$

and since the decomposition in (5.30) is done by undecomposable components, by (3.4) it follows, in $C_l(1-\lambda'_k)$,

$$|\widehat{\mathfrak{D}}_k| = \sum_i \mathcal{H}^2(\partial^* E_{k,i}) \qquad \text{and} \qquad \partial^* E_{k,i} \subseteq \operatorname{spt}(\widehat{\mathfrak{D}}_k).$$

Remark 5.16 (Nonuniqueness of the decomposition). Once the decomposition (5.30) is fixed, the symmetrization is uniquely determined. However, the decomposition might not be unique, and the resulting symmetrized current in general depends on the choice of the decomposition. This will not be an issue, since our procedure will lead to a minimization problem which will not depend on this step.

Since

$$\mathcal{H}^2(\partial^* \mathbb{S}(E_{k,i})) \leq \mathcal{H}^2(\partial^* E_{k,i}) \qquad \text{for all } i \in \mathbb{N},$$

and $\mathbb{S}(E_k) = \cup_i \mathbb{S}(E_{k,i})$, we also have

$$\mathcal{H}^2(\partial^* \mathbb{S}(E_k)) \leq \sum_i \mathcal{H}^2(\partial^* E_{k,i}) = |\widehat{\mathfrak{D}}_k|. \tag{5.33}$$

The same inequalities hold if we restrict the mass to the set C_l^ε, namely

$$|\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C_l^\varepsilon} \leq |\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon}. \tag{5.34}$$

Now we want to understand whether $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ has some boundary on $\{0\} \times \mathbb{R}^2$. We have already observed (Corollary 5.14) that $\widehat{\mathfrak{D}}_k$ has no boundary in $C_l(1-\lambda'_k)$. The same holds for the symmetrized current:

Corollary 5.17 (Closedness of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ in $(-\infty, l) \times B_{1-\lambda'_k}$). *The current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is supported in $[0,l] \times \overline{B}_{1-\lambda'_k}$ and $\partial \mathbb{S}(\widehat{\mathfrak{D}}_k) = 0$ in $\mathcal{D}_1((-\infty, l) \times B_{1-\lambda'_k})$.*

Proof. By definition, $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is the boundary of the current carried by the integration over the finite perimeter set $\mathbb{S}(E_k)$ in $C_l(1-\lambda'_k)$. Hence $\partial \mathbb{S}(\widehat{\mathfrak{D}}_k) = 0$ in $\mathcal{D}_1(C_l(1-\lambda'_k))$. The conclusion then follows from the fact that $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is supported in $[0,l) \times B_{1-\lambda'_k} \subset C_l(1-\lambda'_k)$. $\square$

6 Towards an estimate of $|\mathbb{S}(\widehat{\mathfrak{D}}_k)|$: two useful lemmas

Now that the symmetrization $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ of the current $\widehat{\mathfrak{D}}_k$ in $C_l(1-\lambda'_k)$ is obtained (see (5.31)), we need to estimate its mass. This will be done separately in $\overline{C}_l^\varepsilon(1-\lambda_k) = [\varepsilon, l] \times \overline{B}_{1-\lambda_k}$ and in $\overline{C}_\varepsilon(1-\lambda_k) = [-1, \varepsilon] \times \overline{B}_{1-\lambda_k}$. In formula (7.4) of Section 7 we express the restriction of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ to C_l^ε as generalized graph of suitable functions $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ and estimate the area of these graphs (see Proposition 7.9, below). In addition, we need a fine description of the trace of the symmetrized set boundary $\partial\mathbb{S}(E_k)$ on the lateral part of $\partial C_l(1-\lambda'_k)$; this will be done in Section 8.

We start by collecting in Lemma 6.3 and Lemma 6.4 two important preliminary estimates; we need to introduce the functions $|u_k|^-$, $|u_k|^+$.

For any $r \in (\varepsilon, l)$ we consider the closed curve[20] $\alpha \in (0, 2\pi] \mapsto \Psi_k(r, \alpha) \in \{r\} \times \overline{B}_1$; the image of $\Psi_k(r, \cdot)$ is the slice of $\Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)$ with the plane $\{t = r\}$.

Definition 6.1 (**The functions** $|u_k|^\pm$)**.** *For all $r \in (\varepsilon, l)$ we define*

$$|u_k|^-(r) := \min_{\alpha \in (0,2\pi]} |u_k(r, \alpha)|, \qquad |u_k|^+(r) := \max_{\alpha \in (0,2\pi]} |u_k(r, \alpha)|, \tag{6.1}$$

Thus the map $\Psi_k(r, \cdot)$ defined in (5.1) takes values in

$$\{r\} \times (\overline{B}_{|u_k|^+(r)} \setminus B_{|u_k|^-(r)}).$$

Let us remark that $|u_k|^-(r)$ might be equal to 0, that $|u_k|^+(r) \le 1$, and that it might happen that $|u_k|^+(r) = |u_k|^-(r)$, see Fig. 4. Moreover, from (5.4),

$$|u_k| \ge 1 - \lambda_k \qquad \text{in } \Omega \setminus D_k, \tag{6.2}$$

so that

$$|u_k|^+(r) \ge 1 - \lambda_k \qquad \text{if } \ r \text{ is such that } \ (\Omega \setminus D_k) \cap \partial\mathrm{B}_r \neq \emptyset,$$

whereas it might happen that

$$|u_k|^+(r) < 1 - \lambda_k \qquad \text{if } \ r \text{ is such that } \ (\Omega \setminus D_k) \cap \partial\mathrm{B}_r = \emptyset. \tag{6.3}$$

In such a case, since $D_k \subseteq A_n$ (Lemma 4.2 (ii)), this can happen only if $\partial\mathrm{B}_r \subseteq A_n$.

Definition 6.2 (**The set** $Q_{k,\varepsilon}$)**.** *We define*

$$Q_{k,\varepsilon} := \{r \in (\varepsilon, l) : |u_k|^+(r) < 1 - \lambda_k\}. \tag{6.4}$$

Then

$$Q_{k,\varepsilon} \subseteq \{r \in (\varepsilon, l) : \partial\mathrm{B}_r \subseteq A_n\}. \tag{6.5}$$

The next lemma, that will be used in Section 9, shows that the measure of $Q_{k,\varepsilon}$ is small (see Fig. 4).

Lemma 6.3 (**Estimate of** $Q_{k,\varepsilon}$)**.** *We have*

$$\mathcal{H}^1(Q_{k,\varepsilon}) < \frac{1}{2\pi\varepsilon n}.$$

Proof. If $t \in Q_{k,\varepsilon}$ then $\partial\mathrm{B}_t \subseteq A_n$. Then

$$\mathcal{H}^1(Q_{k,\varepsilon}) = \int_{Q_{k,\varepsilon}} 1 dt \le \frac{1}{2\pi\varepsilon} \int_{Q_{k,\varepsilon}} 2\pi t \, dt = \frac{1}{2\pi\varepsilon} \int_{Q_{k,\varepsilon}} \mathcal{H}^1(\partial\mathrm{B}_t) dt \le \frac{1}{2\pi\varepsilon} |A_n|,$$

where the last inequality is a consequence of the coarea formula and (6.5). The thesis then follows recalling that $|A_n| < \frac{1}{n}$, see (4.6). $\square$

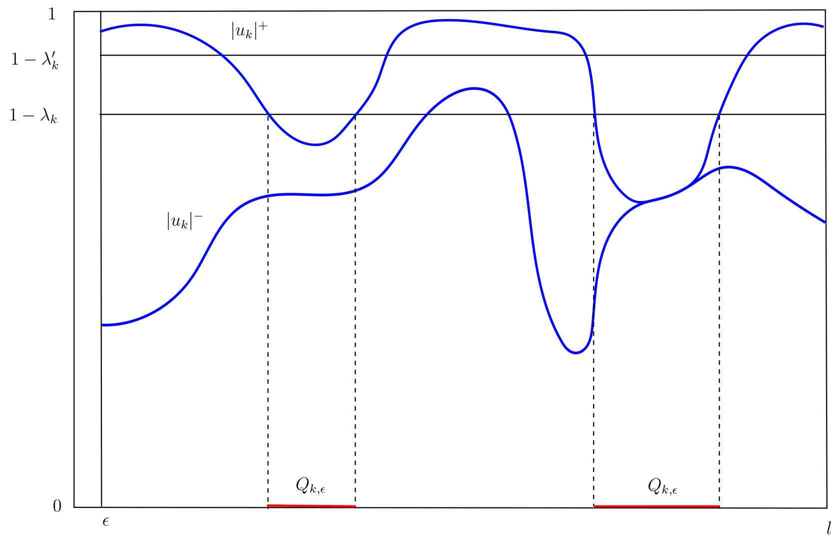

Figure 4: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ defined in (6.1), and the set $Q_{k,\varepsilon}$ in (6.4).

By slicing and from (5.29), (5.30), we have for almost every $t \in (0,l)$ and almost every $\rho \in (1-\lambda_k, 1-\lambda_k')$,

$$(\mathcal{X}_k)_{t,\rho} = \sum_i (-1)^{\sigma_i} [\![E_{k,i} \cap (\{t\} \times \partial B_\rho)]\!],$$

and

$$\mathcal{H}^1(\mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)) \le \sum_i \mathcal{H}^1(E_{k,i} \cap (\{t\} \times \partial B_\rho)) = |(\mathcal{X}_k)_{t,\rho}|, \tag{6.6}$$

since the decomposition is done in undecomposable components (see (3.12)).

Recalling the definition of Θ in (3.1) we have, for fixed $t \in (0,l)$ and for any $\rho \in (0, 1-\lambda_k']$,

$$\Theta_k(t,\rho) := \Theta_{\mathbb{S}(E_k)}(t,\rho) = \frac{1}{\rho}\mathcal{H}^1(\mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)) \tag{6.7}$$

denotes the measure (in radiants) of the slice $\mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)$. By construction,

$$\Theta_k(t,\rho) = \Theta_k(t,\varrho) \qquad \text{for any } \rho, \varrho \in (1-\lambda_k, 1-\lambda_k'),$$

since the slices of $\mathcal{X}_k$, and hence of the sets $E_{k,i}$, are radially symmetric[21] in $C_l(1-\lambda_k') \setminus \overline{C_l}(1-\lambda_k)$. Also, the right-hand side of (6.7) vanishes for $\rho \in (1-\lambda_k, 1-\lambda_k')$.

We now look for an estimate of $\Theta_k(t,\rho)$, for $t \in (\varepsilon, l)$ and $\rho \in (1-\lambda_k, 1-\lambda_k')$: the next lemma will be used in Section 9.

Lemma 6.4 (**L^1-estimate of the angular slices**). *We have*

$$\int_\varepsilon^l \Theta_k(t,\rho)\, dt \le \frac{1}{\varepsilon n} + o_k(1) \qquad \forall \rho \in (1-\lambda_k, 1-\lambda_k'), \tag{6.8}$$

where $o_k(1)$ is a nonnegative function, depending on ε and n, and infinitesimal as $k \to +\infty$.

[20]We use here polar coordinates (r,α).

[21]Each radial section is (suitably rescaled) the same since, by definition, function τ in (5.13) is radial.

Proof. It is convenient to set

$$H_{k,t} := D_k \cap \partial \mathrm{B}_t, \qquad H^c_{k,t} := (\Omega \setminus D_k) \cap \partial \mathrm{B}_t \qquad \forall t \in (\varepsilon, l). \tag{6.9}$$

Observe that the relative boundary of $H_{k,t}$, *i.e.*, the boundary of $H_{k,t}$ when considered as a subset of $\partial \mathrm{B}_t$, is contained in $\partial D_k \cap \partial \mathrm{B}_t$.

We fix $t \in (\varepsilon, l)$ such that the relative boundary of $H_{k,t}$ is a finite set of points (this happens for $\mathcal{H}^1$-a.e. t, since $\mathcal{H}^1(\partial D_k) < +\infty$ from Lemma 4.2(i)) and fix any $\rho \in (1-\lambda_k, 1-\lambda'_k)$. By inequality (6.6) we have

$$\Theta_k(t,\rho) \leq \frac{1}{\rho}|(\mathcal{X}_k)_{t,\rho}|, \tag{6.10}$$

so it is sufficient to estimate the mass of a (slice of a) slice of the 3-current $\mathcal{X}_k$ defined in (5.27). We recall that by (5.27) we have[22]

$$(\mathcal{X}_k)_{t,\rho} = [\![\{t\} \times \partial B_\rho]\!] - (\mathcal{Y}_k)_{t,\rho}, \tag{6.11}$$

where $[\![\{t\} \times \partial B_\rho]\!]$ has a natural orientation[23] inherited by the fact that it is the boundary of $[\![\{t\} \times B_\rho]\!]$ in $\{t\} \times \mathbb{R}^2$, which in turn is a slice of $[\![C_l(\rho)]\!]$. By (5.26)

$$(\mathcal{Y}_k)_{t,\rho} = \widetilde{\tau}_\sharp [\![\{\rho\} \times ((\Omega \setminus D_k) \cap \partial \mathrm{B}_t)]\!] = \tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![H^c_{k,t}]\!], \tag{6.12}$$

see (5.13), (5.14), (5.24), and Remark 5.11 for the orientation of $[\![\{\rho\} \times ((\Omega \setminus D_k) \cap \partial \mathrm{B}_t)]\!]$. As for $[\![H^c_{k,t}]\!]$ we endow the set $H^c_{k,t} \subset \partial \mathrm{B}_t$ with the orientation inherited by $\partial \mathrm{B}_t$, *i.e.*, by a counterclockwise tangent unit vector. Now, since the restriction of $\tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\cdot))$ to $\partial \mathrm{B}_t$ takes values in $\{t\} \times \partial B_\rho$, the current $(\mathcal{Y}_k)_{t,\rho}$ is the integration over arcs[24] in $\{t\} \times \partial B_\rho$. To identify these arcs we distinguish the following three cases (A), (B), (C):

(A) $H^c_{k,t} = \emptyset$. From (6.12) it follows $(\mathcal{Y}_k)_{t,\rho} = 0$ and $(\mathcal{X}_k)_{t,\rho} = [\![\{t\} \times \partial B_\rho]\!]$ from (6.11). Thus

$$\Theta_k(t,\rho) = 2\pi \leq 2\pi \frac{t}{\varepsilon} = \frac{1}{\varepsilon}\mathcal{H}^1(H_{k,t}). \tag{6.13}$$

(B) $H^c_{k,t} = \partial \mathrm{B}_t \subset \Omega \setminus D_k$, hence $(\mathcal{Y}_k)_{t,\rho} = \tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![\partial \mathrm{B}_t]\!]$ from (6.12). Then

$$(\mathcal{Y}_k)_{t,\rho} = [\![\{t\} \times \partial B_\rho]\!]. \tag{6.14}$$

Indeed, fix three points $x_1, x_2, x_3 \in \partial \mathrm{B}_t$ in counterclockwise order such that $|\frac{x_i}{|x_i|} - \frac{x_j}{|x_j|}| > 4\lambda_k$ for $i \neq j$. Since $d_k(x) = |\frac{x}{|x|} - u_k(x)| < \lambda_k$ for $x \in \Omega \setminus D_k$, $x \neq 0$, the points $z_i := \pi_{\lambda_k} \circ \Psi_k(x_i)$ are still in counterclockwise order in $\{t\} \times \partial B_{1-\lambda_k}$ (the image of the arc $\overline{x_1 x_2}$ covers the arc $\overline{z_1 z_2}$ that does not contain z_3). Therefore $(\pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![\overline{x_i x_{i+1}}]\!] = [\![\overline{z_i z_{i+1}}]\!]$ for $i = 1,2,3$[25] (with the convention $x_4 = x_1$, $z_4 = z_1$), and hence

$$(\pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![\partial \mathrm{B}_t]\!] = \sum_{i=1}^{3} (\pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![\overline{x_i x_{i+1}}]\!] = \sum_{i=1}^{3} [\![\overline{z_i z_{i+1}}]\!] = [\![\{t\} \times \partial B_{1-\lambda_k}]\!].$$

[22] The orientation of ∂B_ρ is taken counterclockwise.

[23] The orientation of the 3-current $[\![C_l(\rho)]\!]$ induces an orientation of its slice $[\![\{t\} \times B_\rho]\!]$. This orientation induces an orientation of $[\![\{t\} \times \partial B_\rho]\!]$, which coincides with the orientation of $[\![\{t\} \times \mathbb{R}^2]\!]_\rho$ induced by the slicing by ρ.

[24] Such arcs could overlap, since in general the multiplicity of $\mathcal{Y}_k$ might be different from 1.

[25] The boundary of $(\pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![\overline{x_i x_{i+1}}]\!]$ is $\delta_{z_{i+1}} - \delta_{z_i}$, hence $(\pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![\overline{x_i x_{i+1}}]\!]$ is an arc connecting z_i and z_{i+1}. Since this arc cannot contain the third point, it must be $[\![\overline{z_i z_{i+1}}]\!]$, counterclockwise oriented.

Taking the push-forward by τ we get (6.14). From this and (6.11) we deduce $(\mathcal{X}_k)_{t,\rho} = 0$, and $\Theta_k(t,\rho) = 0$.

Before passing to case (C), we anticipate an observation which will be useful to deal with it. Let $\overline{x_1x_2} \subset (\Omega \setminus D_k) \cap \partial \mathrm{B}_t$ be an arc oriented counterclockwise. We want to identify the current $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\overline{x_1x_2}]\!]$; to do that we consider three different cases for $\overline{x_1x_2}$. Case 1: $|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}| > 2\lambda_k$. Hence $z_1 := \pi_{\lambda_k} \circ \Psi_k(x_1)$ and $z_2 := \pi_{\lambda_k} \circ \Psi_k(x_2)$ must have the same order on $\partial B_{1-\lambda_k}$ of x_1 and x_2, moreover $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\overline{x_1x_2}]\!] = [\![\overline{z_1z_2}]\!]$, where $\overline{z_1z_2}$ is the arc connecting z_1, z_2, starting from z_1 and oriented counterclockwise. Case 2: $|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}| \le 2\lambda_k$ (that implies $|z_1 - z_2| \le 4\lambda_k$, and z_1, z_2 could have reversed order of x_1 and x_2). Let z_1, z_2 have the same order of x_1, x_2, then $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\overline{x_1x_2}]\!] = [\![\overline{z_1z_2}]\!]$ where $\overline{z_1z_2}$ is the arc connecting z_1, z_2, starting from z_1 and oriented counterclockwise. Now let z_1, z_2 have the reversed order of x_1, x_2. If $\overline{x_1x_2}$ is the short path arc connecting x_1, x_2, then $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\overline{x_1x_2}]\!] = [\![\overline{z_1z_2}]\!]$, where $\overline{z_1z_2}$ is the (short path) arc connecting z_1, z_2, starting from z_1, and oriented clockwise. If $\overline{x_1x_2}$ is instead the long path arc joining x_1, x_2, then $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\overline{x_1x_2}]\!] = [\![\{t\} \times \partial B_{1-\lambda_k}]\!] + [\![\overline{z_1z_2}]\!]$, where $[\![\{t\} \times \partial B_{1-\lambda_k}]\!]$ is oriented counterclockwise, and $\overline{z_1z_2}$ is the (short path) arc starting from z_1 and oriented counterclockwise. Notice also that in case 2 we always have $\mathcal{H}^1(\overline{z_1z_2}) < 8\lambda_k$.

Now, we analyse the third case.

(C) $H^c_{k,t}$ is union of finitely many arcs. Let us denote by $\left\{\overline{x_1^ix_2^i}\right\}_i$ these distinct arcs with endpoints[26] $x_j^i = (t, \alpha_j^i) \in \partial \mathrm{B}_t$, with the index $i \in \{1, \dots, h = h_{k,t}\}$ varying in a finite set, so that

$$[\![H^c_{k,t}]\!] = \sum_{i=1}^h [\![\overline{x_1^ix_2^i}]\!] \qquad \text{and} \qquad [\![H_{k,t}]\!] = \sum_{i=1}^h [\![\overline{x_2^ix_1^{i+1}}]\!],$$

where again the orientation of $\overline{x_1^ix_2^i}$ is the one inherited by the counterclockwise orientation of $\partial \mathrm{B}_t$ and, by convention, $h+1 = 1$. Being $H^c_{k,t}$ relatively closed set in $\partial \mathrm{B}_t$, it might happen that $x_1^i = x_2^i$ for some i. Notice that x_j^i belongs to the relative boundary of $H_{k,t}$ which, in turn, is a subset of $\partial D_k \cap \partial \mathrm{B}_t$.

We denote

$$z_j^i := \pi_{\lambda_k} \circ \Psi_k(x_j^i) \in \{t\} \times \partial B_{1-\lambda_k}.$$

After applying $\pi_{\lambda_k} \circ \Psi_k(\cdot)$, the points x_j^i might also reverse their order, *i.e.*, the orientation of the arc $\pi_{\lambda_k} \circ \Psi_k \left(\overline{x_1^ix_2^i}\right)$ could be the opposite of the orientation of $\overline{x_1^ix_2^i}$.

In order to describe the current $(\mathcal{X}_k)_{t,\rho}$ we need first to extend $\tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\cdot))$ to $H_{k,t}$: note carefully that $\pi_{\lambda_k} \circ \Psi_k(\cdot)$ is well-defined in $H^c_{k,t}$, but not necessarily in $H_{k,t}$, since $\pi_{\lambda_k} \circ \Psi_k(H_{k,t}) \cap C_l(1 - \lambda_k)$ may not be empty, and in such a case it is not in the domain of $\tau(\rho, \cdot)$. The extension we get (see (6.16)) will allow to write a specific double slice of $\mathcal{X}_k$ as push-forward, see (6.25). We stress that this extension is done for a fixed slice $\{t\} \times \mathbb{R}^2$ and in general it cannot be done globally[27] for all $t \in (\varepsilon, l)$.

For t fixed such that case (C) holds, we extend the function $\pi_{\lambda_k} \circ \Psi_k(\cdot)$ to $H_{k,t}$ as follows. Let $\overline{x_2^ix_1^{i+1}}$ be an arc of $H_{k,t}$; we want to map this arc on an arc in $\{t\} \times \partial B_{1-\lambda_k}$ joining the two image points z_2^i, z_1^{i+1}, with the orientation from z_2^i to z_1^{i+1}. However there are infinitely many[28] choices of an arc connecting z_2^i to z_1^{i+1}. To specify which arc we choose we distinguish two possibilities: $|z_2^i - z_1^{i+1}| \le 2\lambda_k$, and $|z_2^i - z_1^{i+1}| > 2\lambda_k$. Notice that $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ is the only case in which the

[26] In polar coordinates.

[27] We do not need a global extension since we aim to obtain an estimate which holds for a fixed t.

[28] We can for instance join z_2^i to z_1^{i+1} travelling along an oriented arc connecting them, and then travelling along the whole circle an arbitrary number of times (thus considering a self-overlapping arc).

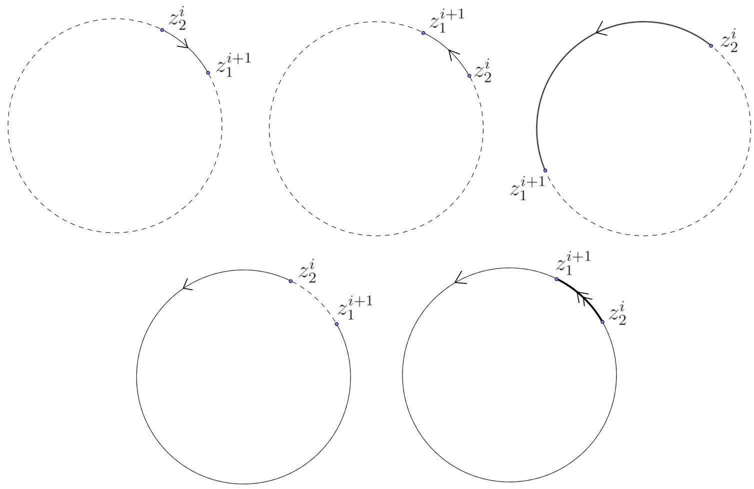

Figure 5: The choice of the arc between z_2^i and z_1^{i+1}. The correct arc is the one in bold on the dashed circle $\{t\} \times \partial B_{1-\lambda_k}$. On top left the case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ and the arc is clockwise oriented; on top center again case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ and the arc counterclockwise oriented; on top right the case $|z_2^i - z_1^{i+1}| > 2\lambda_k$; on bottom left again the case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ when the oriented arc $\overline{z_2^i z_1^{i+1}}$ is the long one. Finally on bottom right it is depicted again the case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ but the counterclockwise arc between z_2^i and z_1^{i+1} has reversed order with respect to α_2^i and α_1^{i+i}, so that β_i is the long arc; in this case the correct arc such that $|\widehat{\beta}_i - \beta_i| \le 2\widehat{\lambda}_k$ is the short one connecting z_2^i and z_1^{i+1} (double bold) together with a complete turn around the circle.

points x_2^i and x_1^{i+1} could have images z_2^i and z_1^{i+1} with a reversed order on $\{t\} \times \partial B_{1-\lambda_k}$. Indeed, since $x_2^i, x_1^{i+1} \in \partial \mathrm{B}_t \cap \partial D_k$, we have $d_k(x_j^i) = |\frac{x_j^i}{|x_j^i|} - u_k(x_j^i)| = \lambda_k$. In particular, if the distance between z_2^i and z_1^{i+1} is larger than $2\lambda_k$, it means that the distance between $u_k(x_2^i)$ and $u_k(x_1^{i+1})$ were larger than $2\lambda_k$ (π_{λ_k} does not increase the distance), so that z_2^i and z_1^{i+1} must have the same order of $\frac{x_2^i}{|x_2^i|}$ and $\frac{x_1^{i+1}}{|x_1^{i+1}|}$ on ∂B_1, which is the same order of x_2^i and x_1^{i+1} on $\partial \mathrm{B}_t$.

We are now in a position to specify the arc: when $|z_2^i - z_1^{i+1}| > 2\lambda_k$ we define $\overline{z_2^i z_1^{i+1}}$ to be the counterclockwise oriented arc[29] from z_2^i to z_1^{i+1}. When $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ we argue as follows: Let β_i be the angular amplitude of the arc $\overline{x_2^i x_1^{i+1}}$. We define $\overline{z_2^i z_1^{i+1}}$ as the unique oriented arc from z_2^i to z_1^{i+1} satisfying the following property: If $\widehat{\beta}_i$ is its oriented angular amplitude (positive if counterclockwise oriented, negative otherwise), then

$$|\widehat{\beta}_i - \beta_i| \le 2\widehat{\lambda}_k, \tag{6.15}$$

where $\widehat{\lambda}_k$ is the angular amplitude of a chord on $\partial B_{1-\lambda_k}$ of length λ_k (see Fig. 5). It is easy to check that there is a unique arc $\overline{z_2^i z_1^{i+1}}$ satisfying this property. Moreover the same property holds for β_i and $\widehat{\beta}_i$ in the case that $|z_2^i - z_1^{i+1}| > 2\lambda_k$, since $\pi_{\lambda_k} \circ \Psi_k(\cdot)$ does not change the angular coordinate of a point x_j^i of a quantity larger than $\widehat{\lambda}_k$.

Once we have specified the image arc, we can define $\widehat{P}_{k,i} : \overline{x_2^i x_1^{i+1}} \to \overline{z_2^i z_1^{i+1}}$ to be the affine (with respect to the angular coordinate) function mapping x_2^i to z_2^i and x_1^{i+1} to z_1^{i+1}. We then introduce

[29] Likewise the orientation from x_2^i to x_1^{i+1}.

$P_k = P_{k,t} : \partial \mathrm{B}_t \to \{t\} \times \partial B_{1-\lambda_k}$ as follows:

$$P_k(x) := \begin{cases} \pi_{\lambda_k} \circ \Psi_k(x) & \text{if } x \in H^c_{k,t}, \\ \\ \widehat{P}_{k,i}(x) & \text{if } x \in \overline{x_2^i x_1^{i+1}} \text{ for some } i. \end{cases} \tag{6.16}$$

We claim that

$$\tau(\rho, P_k(\cdot))_\sharp [\![\partial \mathrm{B}_t]\!] = [\![\{t\} \times \partial B_\rho]\!]. \tag{6.17}$$

Since the map $\tau(\rho, \cdot)$ is an orientation preserving homeomorphism between $\partial B_{1-\lambda_k}$ and ∂B_ρ, it is sufficient to show that

$$P_k(\cdot)_\sharp [\![\partial \mathrm{B}_t]\!] = [\![\{t\} \times \partial B_{1-\lambda_k}]\!]. \tag{6.18}$$

Equivalently, we will prove that

$$\sum_{i=1}^h ([\![\overline{z_1^i z_2^i}]\!] + [\![\overline{z_2^i z_1^{i+1}}]\!]) = [\![\{t\} \times \partial B_{1-\lambda_k}]\!]. \tag{6.19}$$

Let ω_i (resp. β_i) be the angular amplitude, in counterclockwise order, of the arc $\overline{x_1^i x_2^i}$ (resp. $\overline{x_2^i x_1^{i+1}}$). Trivially we have $\sum_{i=1}^h (\omega_i + \beta_i) = 2\pi$. If $\widehat{\omega}_i$ (resp. $\widehat{\beta}_i$) is the angular amplitude of $\overline{z_1^i z_2^i}$ (resp. $\overline{z_2^i z_1^{i+1}}$), taken with sign ± 1 according to their orientation, we see that to prove (6.19) it suffices to show

$$\sum_{i=1}^h (\widehat{\omega}_i + \widehat{\beta}_i) = 2\pi. \tag{6.20}$$

To do this we use (6.15); notice first that the counterpart of (6.15) holds for the arc between x_1^i and x_2^i: Namely the map $\pi_{\lambda_k} \circ \Psi_k$ transforms the arc $\overline{x_1^i x_2^i}$ of angular amplitude ω_i, in the arc $\overline{z_1^i z_2^i}$ of amplitude $\widehat{\omega}_i$ in such a way that

$$|\widehat{\omega}_i - \omega_i| \leq 2\widehat{\lambda}_k. \tag{6.21}$$

Now, if θ_j^i is the angular coordinate of z_j^i, and α_j^i is the angular coordinate of x_j^i, we know that

$$\theta_j^i = \alpha_j^i + r_j^i, \qquad \text{with } |r_j^i| \leq \widehat{\lambda}_k. \tag{6.22}$$

Here again $\widehat{\lambda}_k$ is the angle of a chord of length λ_k on $\partial B_{1-\lambda_k}$. To prove (6.20) we reduce ourselves to show that

$$\widehat{\omega}_i = \omega_i + r_2^i - r_1^i, \tag{6.23}$$

$$\widehat{\beta}_i = \beta_i + r_1^{i+1} - r_2^i, \tag{6.24}$$

for all i. Fix i; we can assume $\alpha_2^i = \alpha_1^i + \omega_i$, and by (6.22) we get

$$\widehat{\omega}_i = \omega_i + r_2^i - r_1^i + 2k_i\pi,$$

with $k_i \in \mathbb{Z}$ accordingly to the number of oriented complete turns around the circle $\partial B_{1-\lambda_k}$. From (6.21) we have $k_i = 0$ for all i, and (6.23) follows. A similar argument, using (6.15), leads to (6.24), hence (6.20) is proved, and (6.17) follows at once. Define

$$y_j^i := \tau(\rho, z_j^i) \in \{t\} \times \partial B_\rho.$$

From (6.17), (6.11), and (6.12) it follows that

$$(\mathcal{X}_k)_{t,\rho} = \tau(\rho, P_k(\cdot))_\sharp [\![\partial \mathrm{B}_t]\!] - \tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\cdot))_\sharp [\![H^c_{k,t}]\!] = \tau(\rho, P_k(\cdot))_\sharp [\![H_{k,t}]\!], \tag{6.25}$$

so that, since the maps $\tau(\rho,\cdot)$ send the arcs $\overline{z_2^i z_1^{i+1}}$ onto $\overline{y_2^i y_1^{i+1}}$, we have

$$(\mathcal{X}_k)_{t,\rho} = \sum_{i=1}^h [\![\overline{y_2^i y_1^{i+1}}]\!], \tag{6.26}$$

hence

$$|(\mathcal{X}_k)_{t,\rho}| \leq \sum_{i=1}^h \mathcal{H}^1(\overline{y_2^i y_1^{i+1}}). \tag{6.27}$$

We now estimate the length of the arcs $\overline{y_2^i y_1^{i+1}}$. For simplicity we fix i and set $Y_1 := y_2^i$, $Y_2 := y_1^{i+1}$, $X_1 := x_2^i$ and $X_2 := x_1^{i+1}$. Let $\mathrm{d}(\cdot,\cdot)$ denote the distance between points of $\{t\} \times \partial B_\rho$ (*i.e.*, the length of the minimal arc connecting the two points), let π_ρ be the orthogonal projection of $\mathbb{R}^2_{\mathrm{target}}$ onto the convex set $\overline{B}_\rho$, and write $Y_i = (t, \widetilde{Y}_i)$ with $\widetilde{Y}_i \in \overline{B}_\rho$, for $i = 1, 2$. Then, setting $\widehat{X}_i := \frac{X_i}{|X_i|}$ and denoting $\overline{\widehat{X}_1 \widehat{X}_2}$ the arc between $\widehat{X}_1$ and $\widehat{X}_2$ on $\{t\} \times \partial B_1$, we have

$$\begin{aligned}
\mathcal{H}^1(\overline{Y_1 Y_2}) &\leq \mathcal{H}^1\Big(\overline{\pi_\rho(X_1)\pi_\rho(X_2)}\Big) + \mathrm{d}(\pi_\rho(X_1), \widetilde{Y}_1) + \mathrm{d}(\pi_\rho(X_2), \widetilde{Y}_2) \\
&= \rho \mathcal{H}^1\Big(\overline{\widehat{X}_1 \widehat{X}_2}\Big) + \mathrm{d}(\pi_\rho(X_1), \widetilde{Y}_1) + \mathrm{d}(\pi_\rho(X_2), \widetilde{Y}_2) \\
&\leq \rho \mathcal{H}^1\Big(\overline{\widehat{X}_1 \widehat{X}_2}\Big) + \frac{\pi}{2}|\pi_\rho(X_1) - \widetilde{Y}_1| + \frac{\pi}{2}|\pi_\rho(X_2) - \widetilde{Y}_2| \\
&\leq \rho \mathcal{H}^1\Big(\overline{\widehat{X}_1 \widehat{X}_2}\Big) + \frac{\pi}{2}|\pi_\rho(X_1) - \pi_\rho \circ u_k(X_1)| + \frac{\pi}{2}|\pi_\rho \circ u_k(X_1) - \widetilde{Y}_1| \\
&\quad + \frac{\pi}{2}|\pi_\rho(X_2) - \pi_\rho \circ u_k(X_2)| + \frac{\pi}{2}|\pi_\rho \circ u_k(X_2) - \widetilde{Y}_2| \\
&\leq \frac{\rho}{\varepsilon}\mathcal{H}^1\Big(\overline{X_1 X_2}\Big) + \frac{\pi}{2}\big(d_k(X_1) + d_k(X_2)\big) + \pi(\lambda_k - \lambda_k'),
\end{aligned} \tag{6.28}$$

where we use that, for $x \neq 0$,

$$d_k(x) = |\frac{x}{|x|} - u_k(x)| = |u(x) - u_k(x)| \geq |\pi_\rho \circ u(x) - \pi_\rho \circ u_k(x)|,$$

because $\mathrm{Lip}(\pi_\rho) = 1$, $|\pi_\rho \circ u_k(X_i) - \widetilde{Y}_i| \leq \lambda_k - \lambda_k'$ for $i = 1, 2$, and $X_i \in \partial \mathrm{B}_t,\ \ t > \varepsilon$. By (6.10) (6.27) and (6.28), we infer

$$\Theta_k(t, \rho) \leq \frac{1}{\varepsilon}\mathcal{H}^1(H_{k,t}) + \frac{\pi}{2\rho}\sum_{x \in \partial H_{k,t}} (d_k(x) + \lambda_k). \tag{6.29}$$

Estimate (6.29) holds for $\mathcal{H}^1$-almost every $t \in (\varepsilon, l)$ such that neither case (A) nor (B) happens. Moreover, by (6.13) it holds also in case (A). Case (B) does not contribute to the L^1 norm of $\Theta_k(\cdot, \rho)$, and therefore (6.29) holds for $\mathcal{H}^1$-almost every $t \in (\varepsilon, l)$.

Denoting by $m(x) = |x|$, so that $|\nabla m| = 1$ out of the origin, the coarea formula allows us to write

$$\int_{\partial D_k} d_k(\sigma) d\mathcal{H}^1(\sigma) \geq \int_{\partial D_k} |\frac{\partial m}{\partial \sigma}| d_k(\sigma) d\mathcal{H}^1(\sigma) = \int_\varepsilon^l \sum_{x \in m^{-1}(t) \cap \partial D_k} d_k(x) dt = \int_\varepsilon^l \sum_{x \in \partial H_{k,t}} d_k(x) dt.$$

Similarly

$$\int_{\partial D_k} \lambda_k \, d\mathcal{H}^1(\sigma) \geq \int_\varepsilon^l \lambda_k \mathcal{H}^0(\{x \in \partial H_{k,t}\}) \, dt.$$

Recalling (4.6), from (6.29) we finally get

$$\int_\varepsilon^l \Theta_k(t,\rho)dt \leq \frac{1}{\varepsilon}|D_k| + \frac{\pi}{2(1-\lambda_k)} \int_{\partial D_k} (d_k(\sigma) + 2\lambda_k) \; d\mathcal{H}^1(\sigma) \leq \frac{1}{\varepsilon n} + o_k(1),$$

where $o_k(1)$ depends on ε and n (since λ_k does) and vanishes as $k \to +\infty$, thanks to Lemma 4.2 (iii). □

7 Estimate from below of the mass of $[\![G_{u_k}]\!]$ over $D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)$

Now we want to identify the current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ in (5.31) as sum of polar graphs (Section 2.4), and to do this we need some preliminaries.

Definition 7.1 (The function $\vartheta_{k,\varepsilon}$). *Recalling the definition of $\Theta_k = \Theta_{\mathbb{S}(E_k)}$ in (6.7), we set*

$$\vartheta_{k,\varepsilon} : (\varepsilon, l) \times (0, 1-\lambda'_k] \times \{0\} \to [0,\pi], \qquad \vartheta_{k,\varepsilon}(t,\rho,0) := \frac{\Theta_k(t,\rho)}{2}. \tag{7.1}$$

Note that $\mathrm{dom}(\vartheta_{k,\varepsilon}) \subsetneq \mathrm{dom}(\Theta_k)$. The polar graph of $\vartheta_{k,\varepsilon}$ is the set $G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} = \{(t,\rho,\vartheta_{k,\varepsilon}(t,\rho,0)) : (t,\rho,0) \in (\varepsilon,l) \times (0,1-\lambda'_k] \times \{0\}\}$. By construction $\mathbb{S}(E_k)$ is the polar subgraph of $\vartheta_{k,\varepsilon}$ restricted to the half-cylinder $\{(t,\rho,\theta) : t \in (\varepsilon,l), \theta \in (0,\pi)\}$. More precisely, let η be any number[30] with $0 < \eta < \frac{\pi}{4}$; then the polar subgraph

$$SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} := \{(t,\rho,\theta) \in (\varepsilon,l) \times (0,1-\lambda'_k] \times (-\pi/4,\pi) : \theta \in (-\eta, \vartheta_{k,\varepsilon}(t,\rho,0))\}$$

satisfies

$$SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} \cap \{\theta \in (0,\pi)\} = \mathbb{S}(E_k) \cap \{\theta \in (0,\pi)\}, \tag{7.2}$$

and similarly (for the polar epigraph), setting

$$UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}} := \{(t,\rho,\theta) \in (\varepsilon,l) \times (0,1-\lambda'_k] \times (-\pi,\pi/4) : \theta \in (-\vartheta_{k,\varepsilon}(t,\rho,0), \eta)\},$$

we have

$$UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}} \cap \{\theta \in (-\pi,0)\} = \mathbb{S}(E_k) \cap \{\theta \in (-\pi,0)\}. \tag{7.3}$$

Remark 7.2 (The sets $\vartheta_{k,\varepsilon} = 0$, $\vartheta_{k,\varepsilon} = \pi$). Careful attention must be paid to the sets $\{\vartheta_{k,\varepsilon} = 0\}$ and $\{\vartheta_{k,\varepsilon} = \pi\}$. Indeed on such sets the two graphs of $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ overlap and then, when considered as integral currents, they cancel each other. Moreover the set $\partial^*\mathbb{S}(E_k)$ includes the two graphs of $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ with the exception of these two sets. In other words, from (7.2) and (7.3) we have

$$\mathbb{S}(E_k) \cap C_l^\varepsilon = \big(SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} \cap \{\theta \in (0,\pi)\}\big) \cup \big(UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}} \cap \{\theta \in (-\pi,0)\}\big) \tag{7.4}$$

up to $\mathcal{H}^3$-negligible sets. From this formula it is evident that the graphs of $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ over $\{\Theta_k = 0\} \cup \{\Theta_k = 2\pi\}$ cancel each other, and thus they do not belong to the reduced boundary of $\mathbb{S}(E_k)$. Moreover, the polar subgraph and the polar epigraph are sets of finite perimeter, as is their union in (7.4).

[30] $\eta = 0$ is not allowed, since in this case the boundary of the subgraph (as a current) does not include the set where $\theta = 0$.

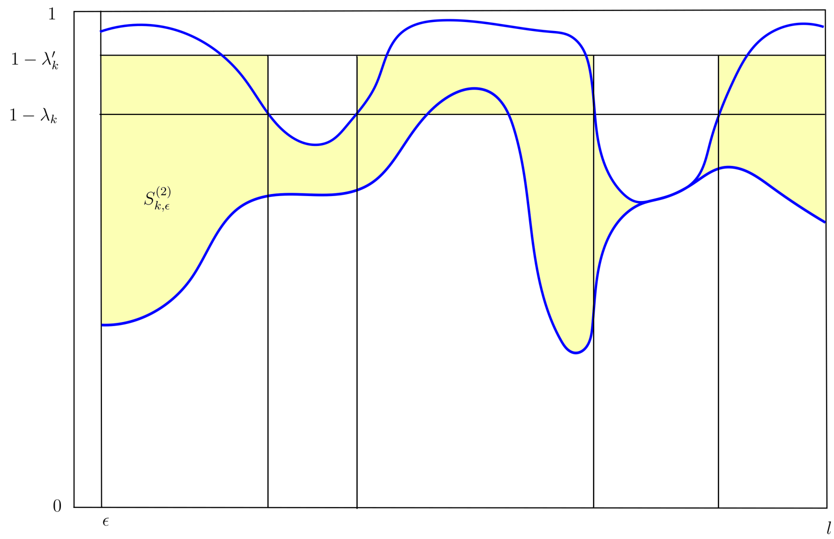

Figure 6: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ and the set $S^{(2)}_{k,\varepsilon}$ in Definition 7.4. See also Fig. 4.

Definition 7.3 (**The polar projection map** $\pi_0^{\rm pol}$)**.** *We let* $\pi_0^{\rm pol} = \pi^{\rm pol}_{0,\lambda'_k,\varepsilon} : \overline{C}^\varepsilon_l(1-\lambda'_k) \to \overline{C}^\varepsilon_l(1-\lambda'_k) \cap \{\theta = 0\}$ *be the polar projection defined by*

$$\pi_0^{\rm pol}(t,\rho,\theta) := (t,\rho,0). \tag{7.5}$$

We now introduce various subsets of $(0,l) \times (0,1) \times \{0\}$ in cylindrical coordinates, namely $\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \subseteq S^{(2)}_{k,\varepsilon} \subseteq S^{(2)}_{k,\varepsilon} \cup J_{Q_{k,\varepsilon}} \subseteq S^{(4)}_{k,\varepsilon}$. We start with $S^{(2)}_{k,\varepsilon}$ (see also formulas (7.16) and (9.4) below), and note preliminarly that

$$\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \tag{7.6}$$

coincides with

$$\left\{(t,\rho,0) \in C_l : t \in (\varepsilon,l),\ \rho \in [|u_k|^-(t) \wedge (1-\lambda_k), |u_k|^+(t) \wedge (1-\lambda_k)]\right\},$$

$|u_k|^-, |u_k|^+$ being the functions introduced in (6.1).

Definition 7.4 (**The set** $S^{(2)}_{k,\varepsilon}$)**.** *Recalling the expression of* $Q_{k,\varepsilon}$ *in* (6.4)*, we define*

$$S^{(2)}_{k,\varepsilon} := \pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \cup \Big(((\varepsilon,l) \setminus Q_{k,\varepsilon}) \times [1-\lambda_k, 1-\lambda'_k] \times \{0\}\Big), \tag{7.7}$$

see Fig. 6.

We have $S^{(2)}_{k,\varepsilon} = \pi_0^{\rm pol}\Big(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon) \cup \tau\big([1-\lambda_k, 1-\lambda'_k] \times A\big)\Big)$, where τ is defined in (5.13), and $A := \{(t,y) \in \pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon) : t \in (\varepsilon,l), y \in \partial B_{1-\lambda_k}\}$, since $(t,y) \in \pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)$ and $y \in \partial B_{1-\lambda_k}$ implies that $t \in (\varepsilon,l) \setminus Q_{k,\varepsilon}$, see Fig. 4 and (5.5).

Remark 7.5. (i) It might happen that $\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\mathrm{B}}_\varepsilon) = \emptyset$. By construction we have

$$\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \cap \Big(((\varepsilon,l) \setminus Q_{k,\varepsilon}) \times [1-\lambda_k, 1-\lambda'_k] \times \{0\}\Big) = ((\varepsilon,l) \setminus Q_{k,\varepsilon}) \times \{1-\lambda_k\} \times \{0\}.$$

The two functions $|u_k|^-$ and $|u_k|^+$ could coincide in large portions of (ε, l) (and even everywhere), so that $\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon))$ could collapse to a curve (for instance if $\Psi_k(\Omega) \subset \partial C_l$); see also the example in Section 10.1. On the other hand, $\mathcal{H}^2(S^{(2)}_{k,\varepsilon}) > 0$ (see Lemma 6.3).

(ii) Notice that $A \supseteq \pi_{\lambda_k} \circ \Psi_k(\partial D_k \setminus \overline{\rm B}_\varepsilon) \cup \pi_{\lambda_k} \circ \Psi_k((\Omega \setminus \overline{\rm B}_\varepsilon) \setminus D_k)$. Moreover $A \cap \pi_{\lambda_k} \circ \Psi_k(D_k)$ may not be empty.

(iii) Inside the cylinder $C^\varepsilon_l(1-\lambda_k)$, $S^{(2)}_{k,\varepsilon}$ is exactly the $\pi_0^{\rm pol}$-projection of $\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\rm B}_\varepsilon)$; remember also that $(\pi_{\lambda_k} \circ \Psi_k)_\sharp([\![D_k \setminus \overline{\rm B}_\varepsilon]\!]) = (\partial \mathcal{E}_k) \llcorner C^\varepsilon_l(1-\lambda_k)$, by (5.6), (5.22) and (5.28).

(iv) Recalling the definition of W_k in (5.16),

$$\pi_0^{\rm pol}\Big(\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\rm B}_\varepsilon) \cup W_k\Big) \subseteq S^{(2)}_{k,\varepsilon}, \tag{7.8}$$

and the above inclusion might be strict.

(v) If $\vartheta_{k,\varepsilon}(t,\rho,0) \in (0,\pi)$ then $(t,\rho,0) \in S^{(2)}_{k,\varepsilon}$, by (7.8). Indeed in this case the circle $(\pi_0^{\rm pol})^{-1}(t,\rho,0)$ intersects both some sets in $\{E_{k,i}\}$ (see (5.30)) and their complement, so in particular $(\pi_0^{\rm pol})^{-1}(t,\rho,0)$ must intersect the reduced boundary of some of the sets in $\{E_{k,i}\}$, namely $\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\rm B}_\varepsilon) \cup W_k$, for $\mathcal{H}^2$-a.e. $(t,\rho,0) \in S^{(2)}_{k,\varepsilon}$. Furthermore

$$\{(t,\rho,0) : \vartheta_{k,\varepsilon}(t,\rho,0) \in (0,\pi)\} = \pi_0^{\rm pol}\big({\rm spt}(\mathbb{S}(\widehat{\mathfrak{D}}_k))\big),$$

up to $\mathcal{H}^2-$ negligible sets[31]

Remark 7.6. (i) $\Theta_k = 2\pi$ on $\{(t,\rho,0) : t \in Q_{k,\varepsilon}, \rho \in (|u_k|^+(t), 1-\lambda'_k)\}$. Notice that the part of the cylinder $\{(t,\rho,\theta) : t \in Q_{k,\varepsilon}, \rho \in (|u_k|^+(t), 1-\lambda'_k), \theta \in (-\pi,\pi]\}$ does not intersect $\pi_{\lambda_k} \circ \Psi_k(D_k)$, and neither W_k, by construction. As a consequence it does not intersect ${\rm spt}(\mathbb{S}(\widehat{\mathfrak{D}}_k))$.

(ii) We write $\{(t,\rho,0) \in S^{(2)}_{k,\varepsilon} :$ either $\Theta_k(t,\rho) = 0$ or $\Theta_k(t,\rho) = 2\pi\} = S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$. Then $S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$ corresponds to the values of t and ρ for which $(t,\rho,0) \in S^{(2)}_{k,\varepsilon}$ and the slice $\mathbb{S}(E_k)_{(t,\rho)} = \mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)$ is either empty or the whole circle $\{t\} \times \partial B_\rho$ (up to $\mathcal{H}^1$-negligible sets). Notice also that the intersection $\pi_0^{\rm pol}\Big(\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\rm B}_\varepsilon) \cup W_k\Big) \cap S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$ may not be empty on a set of positive $\mathcal{H}^2-$measure. Indeed in the proof of Proposition 7.9, we show that the $\pi_0^{\rm pol}$-projection of[32] $\Big(\pi_{\lambda_k} \circ \Psi_k(D_k)\Big) \setminus {\rm spt}(\widehat{\mathfrak{D}}_k)$ is contained in $S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$.

7.1 The current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ as sum of a polar subgraph and a polar epigraph

Let $G^{\rm pol}_{\pm\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)}$ be the polar graph of $\pm\vartheta_{k,\varepsilon} \llcorner \Big(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\Big)$; these two sets, by symmetry, overlap, and

$$[\![G^{\rm pol}_{-\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)}]\!] + [\![G^{\rm pol}_{\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)}]\!] = 0,$$

[31]There could be $(t,\rho,0) \in \pi_0^{\rm pol}\big({\rm spt}(\mathbb{S}(\widehat{\mathfrak{D}}_k))\big)$ such that $\vartheta_{k,\varepsilon}(t,\rho,0) \notin (0,\pi)$. Indeed take $t \in (\varepsilon, l)$ such that $\partial {\rm B}_t \subset D_k$ and assume that $\Psi_k(\partial {\rm B}_t) = \{t\} \times \partial B_\rho$, $\rho < 1-\lambda_k$. Then $\vartheta_{k,\varepsilon}(t,\rho) = \pi$ and $\{t\} \times \partial B_\rho \subset \partial^* \mathbb{S}(E_k)$; however this can only happen for (t,ρ) in a negligible $\mathcal{H}^2$-set.

[32]This is the set where $\pi_{\lambda_k} \circ \Psi_k(D_k)$ overlaps itself with opposite orientation; this set might have positive area, see Fig. 10.

due to the fact that $[\![G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ and $[\![G^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ are oriented in opposite way. Indeed we endow $[\![G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ with the orientation inherited by looking at it as the boundary of the polar subgraph of $\vartheta_{k,\varepsilon}$, and we endow $[\![G^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ with the opposite orientation, since we look at it as boundary of an epigraph.

Definition 7.7 (The currents $\mathcal{G}^{\pm}_{k,\varepsilon}$). *We set*

$$\begin{aligned}
\mathcal{G}^{+}_{k,\varepsilon} &:= (\partial[\![SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}}]\!])\llcorner\left(\{\theta\in(0,\pi)\}\cap C^{\varepsilon}_l(1-\lambda'_k)\right)+[\![G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!],\\
\mathcal{G}^{-}_{k,\varepsilon} &:= (\partial[\![UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}}]\!])\llcorner\left(\{\theta\in(-\pi,0)\}\cap C^{\varepsilon}_l(1-\lambda'_k)\right)+[\![G^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!].
\end{aligned}\tag{7.9}$$

The non standard orientation of $\mathcal{G}^{-}_{k,\varepsilon}$ is chosen in such a way that condition (7.10) in Proposition 7.9 below takes place. In this proposition we will also see that, being $\mathbb{S}(E_k)$ a finite perimeter set in C^{ε}_l, its reduced boundary, seen as a current, has finite mass. In turn, the integration on its boundary is exactly $\mathcal{G}^{+}_{k,\varepsilon}+\mathcal{G}^{-}_{k,\varepsilon}$ (see also (7.4)).

Remark 7.8. The generalized polar graph of $\vartheta_{k,\varepsilon}$ might have large parts on which $\vartheta_{k,\varepsilon}\in\{0,\pi\}$; for this reason we neglected this part in the currents introduced in (7.9) by restricting the boundary of the subgraph in $\{\theta\in(0,\pi)\}\cap C^{\varepsilon}_l(1-\lambda'_k)$ (and similarly for the epigraph). However we want to consider the graph above the set $\vartheta_{k,\varepsilon}\in\{0,\pi\}$ on the strip $S^{(2)}_{k,\varepsilon}$, in particular the projection of the set where $\pi_{\lambda_k}\circ\Psi_k(D_k)$ overlaps itself (which may have positive area)[33], for this reason we have to add the term $[\![G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ in formulas (7.9). The reason why we have to get rid of the graph of $\vartheta_{k,\varepsilon}$ on $\{\vartheta_{k,\varepsilon}\in\{0,\pi\}\}$ outside $S^{(2)}_{k,\varepsilon}$ is that this term is not controlled by the area of $(\pi_{\lambda_k}\circ\Psi_k(\Omega\setminus\overline{\mathrm{B}}_\varepsilon))$ (see also Remark 7.6 (iv)).

Proposition 7.9 (Estimate of the mass of $\mathcal{G}^{\pm}_{k,\varepsilon}$). *Let ε be fixed as in* (4.4) *and* (4.5)*, and recall the definition* (5.31) *of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$. Then the following properties hold:*

$$\mathcal{G}^{+}_{k,\varepsilon}+\mathcal{G}^{-}_{k,\varepsilon}=\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner C^{\varepsilon}_l(1-\lambda'_k),\tag{7.10}$$

$$|\mathcal{G}^{+}_{k,\varepsilon}|+|\mathcal{G}^{-}_{k,\varepsilon}|=|\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C^{\varepsilon}_l(1-\lambda'_k)}+2\mathcal{H}^2\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right),\tag{7.11}$$

$$|\mathcal{G}^{+}_{k,\varepsilon}|+|\mathcal{G}^{-}_{k,\varepsilon}|\le\int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)}|J(\pi_{\lambda_k}\circ\Psi_k)|\,dx+\frac{1}{n}+o_k(1),\tag{7.12}$$

where $o_k(1)$ is a nonnegative infinitesimal sequence as $k\to+\infty$, depending on n and ε.

Proof. Identity (7.10) follows by definition and from (7.4). Concerning (7.11), setting for simplicity

$$J^{0,2\pi}_{k,\varepsilon}:=S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\},\tag{7.13}$$

it is sufficient to observe that $\mathrm{spt}([\![\mathcal{G}^{+}_{k,\varepsilon}]\!])$ and $\mathrm{spt}([\![\mathcal{G}^{-}_{k,\varepsilon}]\!])$ coincide on the set $\vartheta_{k,\varepsilon}(J^{0,2\pi}_{k,\varepsilon})$ (whose measure is equal[34] to the measure of $J^{0,2\pi}_{k,\varepsilon}$). Thus, the currents $\mathcal{G}^{+}_{k,\varepsilon}$ and $\mathcal{G}^{-}_{k,\varepsilon}$ cancel each other on this set, since they are endowed with opposite orientation. Hence

$$|\mathcal{G}^{+}_{k,\varepsilon}|+|\mathcal{G}^{-}_{k,\varepsilon}|=|\mathcal{G}^{+}_{k,\varepsilon}+\mathcal{G}^{-}_{k,\varepsilon}|+2\mathcal{H}^2(J^{0,2\pi}_{k,\varepsilon}),\tag{7.14}$$

[33] See the example in Section 10.2.

[34] Indeed $\vartheta_{k,\varepsilon}$ restricted to $J^{0,2\pi}_{k,\varepsilon}\cap\{\Theta_k=0\}$ is the identity map and $\vartheta_{k,\varepsilon}$ restricted to $J^{0,2\pi}_{k,\varepsilon}\cap\{\Theta_k=2\pi\}$ is a π-rotation.

and (7.11) follows from (7.10).

Let us prove (7.12). We recall that the rectifiable set $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ includes the support of the current $\widehat{\mathfrak{D}}_k$. There might be parts of $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ where the multiplicity of $\widehat{\mathfrak{D}}_k$ is zero, and this happens for instance where two pieces of $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ overlap with opposite orientations. We decompose $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ as follows:

$$\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k = Z_k^0 \cup \mathrm{spt}(\widehat{\mathfrak{D}}_k) = Z_k^0 \cup \mathrm{spt}(\mathfrak{D}_k) \cup \mathrm{spt}(\mathcal{W}_k), \tag{7.15}$$

where

$$Z_k^0 := \Big(\pi_{\lambda_k} \circ \Psi_k(D_k) \setminus \mathrm{spt}(\mathfrak{D}_k)\Big) \cup \Big(W_k \setminus \mathrm{spt}(\mathcal{W}_k)\Big)$$

is the set where $\widehat{\mathfrak{D}}_k$ has vanishing multiplicity. It is convenient to introduce the following notation for the set in (7.6):

$$S^{(1)}_{k,\varepsilon} := \pi_0^{\mathrm{pol}}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)). \tag{7.16}$$

We claim that

$$S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon} \subseteq \pi_0^{\mathrm{pol}}\big(Z_k^0\big), \tag{7.17}$$

where π_0^{pol} is the projection introduced in (7.5) (again, here the inclusion is intended up to $\mathcal{H}^2$-negligible sets). To prove this we argue by slicing: for $t \in (\varepsilon, l)$ set

$$(S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon})_t := (S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon}) \cap (\{t\} \times \mathbb{R}^2).$$

It is sufficient to show that

$$(S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon})_t \subseteq \pi_0^{\mathrm{pol}}\big(Z_k^0\big) \qquad \text{for } \mathcal{H}^1-\text{a.e. } t \in (\varepsilon, l). \tag{7.18}$$

In turn, denoting $(Z_k^0)_t := Z_k^0 \cap (\{t\} \times \mathbb{R}^2)$ we will prove[35]

$$(S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon})_t \subseteq \pi_0^{\mathrm{pol}}\big((Z_k^0)_t\big) \qquad \text{for } \mathcal{H}^1-\text{a.e. } t \in (\varepsilon, l). \tag{7.19}$$

Now, $(Z_k^0)_t$ is, for $\mathcal{H}^1$-a.e. $t \in (\varepsilon, l)$, the set where the coefficient of the integral current $(\widehat{\mathfrak{D}}_k)_t$ is zero. Recalling (7.13), we have that[36] $\Theta_k(t,\rho) \in \{0, 2\pi\}$ for $\rho \in [|u_k|^-(t) \wedge (1-\lambda_k), |u_k|^+(t) \wedge (1-\lambda_k)]$ such that $(t,\rho,0) \in J^{0,2\pi}_{k,\varepsilon}$. This means that either

- for all i the intersection between $E_{k,i}$ (see (5.30)) and $\{t\} \times \partial B_\rho$ is empty (up to $\mathcal{H}^1$-negligible sets), or
- for at least one i, it happens $E_{k,i} \cap (\{t\} \times \partial B_\rho) = \{t\} \times \partial B_\rho$ (up to $\mathcal{H}^1$-negligible sets).

In both cases, for $\mathcal{H}^1$-a.e. $\rho \in (S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon})_t$, the current $(\mathbb{S}(\widehat{\mathfrak{D}}_k))_t$ is null on the set

$$\{(t,\rho,\theta) : \theta \in (-\pi,\pi), \rho \in (S^{(1)}_{k,\varepsilon} \cap J^{0,2\pi}_{k,\varepsilon})_t\}.$$

Indeed, recalling that $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \partial[\![\mathbb{S}(E_k)]\!]$, in the first case this is obvious, in the second one it is sufficient to remember that $E_k = \cup_i E_{k,i}$. In other words, the set $(\pi_{\lambda_k} \circ \Psi_k(D_k))_t$ must overlap itself with opposite directions in this set, because the multiplicity of $(\widehat{\mathfrak{D}}_k)_t$ is null there. Hence we have proved (7.19), and claim (7.17) follows.

[35] The only fact we will use is that the π_0^{pol}-projection of the set $\pi_{\lambda_k} \circ \Psi_k(D_k)$ is surjective on $S^{(2)}_{k,\varepsilon}$ (essentially by definition) and then the inverse image of a point where $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is null is covered at least two times.

[36] See Fig. 10, the two bold segments: on one $\Theta_k = 0$ and on the other one $\Theta_k = 2\pi$

As a consequence of (7.17) and of its proof, we have

$$
\begin{aligned}
2\mathcal{H}^2(J^{0,2\pi}_{k,\varepsilon}) &\leq 2\mathcal{H}^2\big(\pi_0^{\rm pol}(Z^0_k \cap C^\varepsilon_l(1-\lambda_k))\big) + 2(\lambda_k - \lambda'_k)l \\
&\leq \int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + M(W_k\cap C^\varepsilon_l(1-\lambda'_k)) - |\widehat{\mathfrak{D}}_k|_{C^\varepsilon_l(1-\lambda'_k)} + o_k(1), \qquad (7.20)
\end{aligned}
$$

see (5.18). Indeed, the first inequality is easy to see, recalling that $J^{0,2\pi}_{k,\varepsilon}$ is the union of $S^{(1)}_{k,\varepsilon}\cap J^{0,2\pi}_{k,\varepsilon}$ and $J^{0,2\pi}_{k,\varepsilon}\setminus S^{(1)}_{k,\varepsilon}$, and the latter has measure less than $(\lambda_k-\lambda'_k)l$ that is infinitesimal as $k\to+\infty$ (we denote it by $o_k(1)$). To see the second inequality we use decomposition (7.15). Since Z^0_k is covered at least two times (with opposite directions), the area $M((\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k)\cap C^\varepsilon_l(1-\lambda'_k))$ of $\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k$ in $C^\varepsilon_l(1-\lambda'_k)$ counted with multiplicity, *i.e.*,

$$
M((\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k)\cap C^\varepsilon_l(1-\lambda'_k)) := \int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + M(W_k\cap C^\varepsilon_l(1-\lambda'_k)),
$$

satisfies

$$
\begin{aligned}
M((\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k)\cap C^\varepsilon_l(1-\lambda'_k)) \geq& 2\mathcal{H}^2(Z^0_k\cap C^\varepsilon_l(1-\lambda'_k)) + |\widehat{\mathfrak{D}}_k|_{C^\varepsilon_l(1-\lambda'_k)} \\
\geq& 2\mathcal{H}^2\big(\pi_0^{\rm pol}(Z^0_k\cap C^\varepsilon_l(1-\lambda'_k))\big) + |\widehat{\mathfrak{D}}_k|_{C^\varepsilon_l(1-\lambda'_k)}, \\
\geq& 2\mathcal{H}^2\big(\pi_0^{\rm pol}(Z^0_k\cap C^\varepsilon_l(1-\lambda_k))\big) + |\widehat{\mathfrak{D}}_k|_{C^\varepsilon_l(1-\lambda'_k)},
\end{aligned}
$$

and (7.20) follows.

In order to prove (7.12) it is now sufficient to observe that

$$
\begin{aligned}
&|\mathcal{G}^+_{k,\varepsilon}| + |\mathcal{G}^-_{k,\varepsilon}| = |\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C^\varepsilon_l(1-\lambda'_k)} + 2\mathcal{H}^2(J^{0,2\pi}_{k,\varepsilon}) \\
\leq& |\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C^\varepsilon_l(1-\lambda'_k)} + \int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + M(W_k\cap C^\varepsilon_l(1-\lambda'_k)) - |\widehat{\mathfrak{D}}_k|_{C^\varepsilon_l(1-\lambda'_k)} + o_k(1) \\
\leq& \int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + \frac{1}{n} + o_k(1),
\end{aligned}
$$

where we have used (5.20) and (5.34) localized in the cylinder $C^\varepsilon_l(1-\lambda'_k)$.

□

Corollary 7.10. *We have*

$$
|[\![G_{u_k}]\!]|_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon))\times\mathbb{R}^2} \geq |\mathcal{G}^+_{k,\varepsilon}| + |\mathcal{G}^-_{k,\varepsilon}| - \frac{1}{n} - o_k(1).
$$

Proof. It follows from (7.12) and (5.8). □

Now we restrict our attention to the rectifiable sets ${\rm spt}(\mathcal{G}^\pm_{k,\varepsilon})$, the supports of the currents in (7.9). We recall that the function $\vartheta_{k,\varepsilon}$ might take values in $(0,\pi)$ only in the "strip" $S^{(2)}_{k,\varepsilon}$, see Remark 7.5 (v), and

$$
S^{(2)}_{k,\varepsilon} \subset (\varepsilon, l)\times[0,1]\times\{0\}\subset C_l.
$$

Now we add to $\mathcal{G}^+_{k,\varepsilon}$ a graph on some additional set outside $S^{(2)}_{k,\varepsilon}$, see Fig.8.

Definition 7.11. *We let*

$$
J_{Q_{k,\varepsilon}} := \{(t,\rho,0)\in C_l : t\in Q_{k,\varepsilon},\ \rho\in[|u_k|^+(t), 1-\lambda'_k]\}. \qquad (7.21)
$$

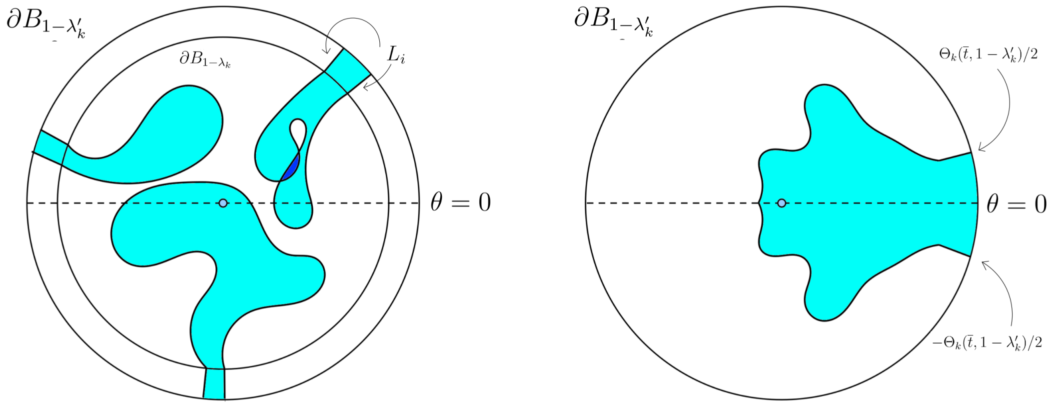

Figure 7: Intersection of the cylinder $C_l(1-\lambda_k)$ with $\{t=\bar{t}\} \times \mathbb{R}^2$. The symmetrization of a closed current in $B_{1-\lambda'_k}$, which on the left is emphasized in grey, and with dark grey the area in which the multiplicity of the current is 2. The set is bounded by a generic curve with endpoints on $\partial B_{1-\lambda_k}$, in turn these endpoints have been joined with $\partial B_{1-\lambda'_k}$ by radial segments L_i. The area emphasized has been symmetrized with the respect to the radius $\{\theta = 0\}$ in the right picture. In the picture on the right, we have indicated the angles $\pm\Theta_k(\bar{t}, 1-\lambda'_k)/2$.

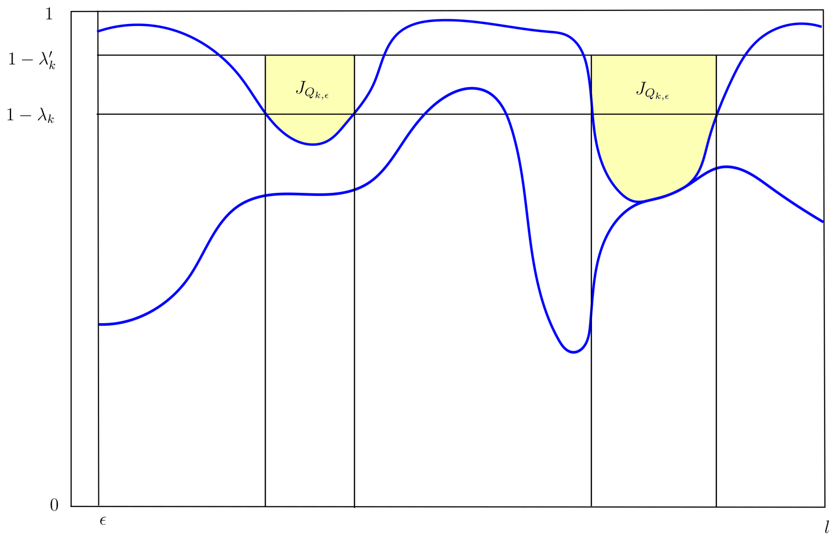

Figure 8: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ and the set $J_{Q_{k,\varepsilon}}$ in (7.21). See also Fig. 4.

By definition of $Q_{k,\varepsilon}$ in (6.4), we have that for $\mathcal{H}^2$-a.e. $(t,\rho,0) \in J_{Q_{k,\varepsilon}}$ it holds $(\pi_0^{\rm pol})^{-1}((t,\rho,0)) \cap \mathrm{spt}(\mathbb{S}(\widehat{\mathfrak{D}}_k)) = \emptyset$, so that $\vartheta_{k,\varepsilon} \in \{0,\pi\}$ on $J_{Q_{k,\varepsilon}}$. Recalling (5.29) and (5.27), it is not difficult to see that $(\pi_0^{\rm pol})^{-1}(J_{Q_{k,\varepsilon}}) \subseteq \mathbb{S}(E_k)$. Hence

$$\vartheta_{k,\varepsilon} = \pi \qquad \text{in } J_{Q_{k,\varepsilon}}$$

(see also Remark 7.6).

Now, we want to add to the currents $\mathcal{G}^{\pm}_{k,\varepsilon}$ in (7.7) a new part above a region that becomes, in Section 11, the subgraph of the function h.

Definition 7.12 (The currents $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}$ and $\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$). *We define*

$$\begin{aligned}
\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} :=& \mathcal{G}^{+}_{k,\varepsilon} + [\![G^{\rm pol}_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!] \in \mathcal{D}_2(C_l^\varepsilon(1-\lambda_k')),\\
\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} :=& \mathcal{G}^{-}_{k,\varepsilon} + [\![G^{\rm pol}_{-\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!] \in \mathcal{D}_2(C_l^\varepsilon(1-\lambda_k')).
\end{aligned}$$

Lemma 7.13. *The following assertions hold:*

(i)

$$|\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}| = |\mathcal{G}^{+}_{k,\varepsilon}| + \mathcal{H}^2(J_{Q_{k,\varepsilon}}); \tag{7.22}$$

(ii)

$$\mathcal{H}^2(J_{Q_{k,\varepsilon}}) \le |Q_{k,\varepsilon}| \le \frac{1}{2\pi\varepsilon n}; \tag{7.23}$$

(iii)

$$\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} = \partial[\![\mathbb{S}(E_k)]\!] \llcorner C_l^\varepsilon(1-\lambda_k'). \tag{7.24}$$

Proof. (i) follows from the fact that $\mathcal{G}^{+}_{k,\varepsilon}$ and $[\![G_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]$ have disjoint supports, and $|[\![G_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]| = \mathcal{H}^2(J_{Q_{k,\varepsilon}})$. (ii) follows from

$$\mathcal{H}^2(J_{Q_{k,\varepsilon}}) = \int_{Q_{k,\varepsilon}} (1-\lambda_k' - |u_k|^+(t))\, dt \le |Q_{k,\varepsilon}| \le \frac{1}{2\pi\varepsilon n},$$

where the last inequality is a consequence of Lemma 6.3. (iii) follows as in Proposition 7.9 using the fact that $[\![G_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]$ and $[\![G_{-\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]$ have opposite orientation. □

Current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ is closed in $C_l^\varepsilon(1-\lambda_k')$. We can look at its boundary as a current in $\mathcal{D}_2((\varepsilon,l)\times\mathbb{R}^2)$, which stands on the lateral boundary of the cylinder $C_l^\varepsilon(1-\lambda_k')$. To this aim we study the trace of $\vartheta_{k,\varepsilon}$ (that is $\Theta_k(t,\rho)/2$) on the segment

$$(\varepsilon,l)\times\{1-\lambda_k'\}\times\{0\}. \tag{7.25}$$

Observe that by definition

$$\vartheta_{k,\varepsilon} = \pi \quad \text{on } Q_{k,\varepsilon}\times\{1-\lambda_k'\}\times\{0\} \subseteq (\varepsilon,l)\times\{1-\lambda_k'\}\times\{0\},$$

whereas on $((\varepsilon,l)\setminus Q_{k,\varepsilon})\times\{1-\lambda_k'\}\times\{0\}$ we have

$$\vartheta_{k,\varepsilon}(t,1-\lambda_k',0) = \Theta_k(t,1-\lambda_k')/2 = \Theta_k(t,\rho)/2, \qquad t\in(\varepsilon,l)\setminus Q_{k,\varepsilon},$$

for all $\rho \in (1-\lambda_k, 1-\lambda_k')$.

Definition 7.14 (The 2-rectifiable set $\Sigma_{k,\varepsilon}$). *We let*

$$\Sigma_{k,\varepsilon} := \Big\{(t,\rho,\theta) : t \in (\varepsilon, l),\ \rho = 1-\lambda'_k,\ \theta \in (-\Theta_k(t,1-\lambda'_k)/2, \Theta_k(t,1-\lambda'_k)/2)\Big\}. \tag{7.26}$$

Referring to the right picture in Figure 7, the section of $\Sigma^{\bar{t}}_k$ is the short arc connecting the points $(\bar{t}, 1-\lambda'_k, -\Theta_k(\bar{t},1-\lambda'_k)/2)$ and $(\bar{t},1-\lambda'_k,\Theta_k(\bar{t},1-\lambda'_k)/2)$; see also Fig. 11.

If we denote by $[\![\Sigma_{k,\varepsilon}]\!]$ the current given by integration over $\Sigma_{k,\varepsilon}$ (suitably oriented), its boundary coincides with the boundary of $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ on $\partial_{\rm lat} C^{\varepsilon}_l(1-\lambda'_k)$.

Lemma 7.15 (Properties of $\Sigma_{k,\varepsilon}$). *$\Sigma_{k,\varepsilon}$, oriented by the outward unit normal to the lateral boundary of $C_l(1-\lambda'_k)$, is such that*

$$\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] \in \mathcal{D}_2((\varepsilon,l)\times\mathbb{R}^2) \qquad \text{is boundaryless.}$$

Moreover

$$\mathcal{H}^2(\Sigma_{k,\varepsilon}) \le \frac{1}{\varepsilon n} + o_k(1), \tag{7.27}$$

where the sequence $o_k(1)\ge 0$ depends on n and ε, and is infinitesimal as $k\to+\infty$. Finally

$$(\partial[\![\Sigma_{k,\varepsilon}]\!]) \llcorner (\{\varepsilon\}\times\mathbb{R}^2) = [\![\{\varepsilon\}\times\{1-\lambda'_k\}\times\Big[\frac{-\Theta_k(\varepsilon,1-\lambda'_k)}{2}, \frac{\Theta_k(\varepsilon,1-\lambda'_k)}{2}\Big]]\!], \tag{7.28}$$

oriented counterclockwise[37].

Proof. The fact that the current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!]$ is boundaryless in $\mathcal{D}_2((\varepsilon,l)\times\mathbb{R}^2)$ is a consequence of the fact that $\Sigma_{k,\varepsilon}$ is a subset of the polar subgraph of the trace of $\vartheta_{k,\varepsilon}$ on $(\varepsilon,l)\times\{1-\lambda'_k\}\times\{0\}$. Concerning (7.27) we have

$$\mathcal{H}^2(\Sigma_{k,\varepsilon}) = \int_\varepsilon^l \int_{-\Theta_k(t,1-\lambda'_k)/2}^{\Theta_k(t,1-\lambda'_k)/2} (1-\lambda'_k)\,d\theta dt \le \frac{1}{\varepsilon n} + o_k(1), \tag{7.29}$$

where the last inequality follows from Lemma 6.4. As for the last assertion, we have to understand which is the orientation of $[\![\Sigma_{k,\varepsilon}]\!]$, which has been chosen in such a way that $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] = (\partial[\![\mathbb{S}(E_k)]\!])\llcorner((\varepsilon,l)\times[0,1-\lambda'_k]\times\{\theta\in(-\pi,\pi]\})$. Hence, since $\mathbb{S}(E_k)$ is contained in $C_l(1-\lambda'_k)$, the orientation of $[\![\Sigma_{k,\varepsilon}]\!]$ is the one inherited by the external normal to $\partial[\![\mathbb{S}(E_k)]\!]$, namely the outward unit normal to the lateral boundary of $C_l(1-\lambda'_k)$. $\square$

8 Estimate from below of the mass of $[\![G_{u_k}]\!]$ over $D_k\cap \mathrm{B}_\varepsilon$

We now analyse the image of $D_k\cap\mathrm{B}_\varepsilon$ through Ψ_k. We want to reduce this set to a current $\mathcal{V}_k\in\mathcal{D}_2(\{\varepsilon\}\times\mathbb{R}^2)$ (defined in (8.12)), in order that it contains the necessary information on the area of $\Psi_k(D_k\cap\mathrm{B}_\varepsilon)$. To this aim we need first to describe the boundary of $\mathcal{V}_k$ and then show that its mass gives a lower bound for the area of the graph of u_k (see formula (8.13)).

Borrowing the notation from the proof of Lemma 6.4, the set $\partial\mathrm{B}_\varepsilon$ is splitted as:

$$\partial\mathrm{B}_\varepsilon = (D_k\cap\partial\mathrm{B}_\varepsilon)\cup((\Omega\setminus D_k)\cap\partial\mathrm{B}_\varepsilon) =: H_{k,\varepsilon}\cup H^c_{k,\varepsilon}. \tag{8.1}$$

[37] Looking at the plane $\{\varepsilon\}\times\mathbb{R}^2$ from the side $t>\varepsilon$.

We denote by

$$\{x_i\}_{i=1}^{I_k} \subseteq \{\widehat{x}_i\}_{i=1}^{J_k} := \partial \mathrm{B}_\varepsilon \cap \partial D_k, \tag{8.2}$$

the finite family of points (see Lemma 4.2 (v)) which represents the relative boundary of $H_{k,\varepsilon}$ in $\partial\mathrm{B}_\varepsilon$. Recall that $\{\widehat{x}_i\}_{i=1}^{J_k}$ is finite as well by Lemma 4.2 (iv). For notational simplicity, we skip the dependence on ϵ.

Recalling the definition of W_k in (5.16), the following crucial lemma states that $(\pi_{\lambda_k} \circ \Psi_k(D_k)) \cup W_k$ does not intersect the plane $\{\varepsilon\} \times \mathbb{R}^2$ in a set of positive $\mathcal{H}^2$-measure.

Lemma 8.1. *The rectifiable set $(\pi_{\lambda_k} \circ \Psi_k(D_k)) \cup W_k$ satisfies*

$$\mathcal{H}^2\Big((\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k) \cap \{t = \varepsilon\}\Big) = 0.$$

Proof. It is sufficient to show that $\mathcal{H}^2((\pi_{\lambda_k} \circ \Psi_k(D_k)) \cap \{t = \varepsilon\}) = 0$ and $\mathcal{H}^2(W_k \cap \{t = \varepsilon\}) = 0$. To show the first equality, suppose $\mathcal{H}^2(\pi_{\lambda_k} \circ \Psi_k(D_k) \cap \{t = \varepsilon\}) > 0$. Since $\mathrm{Lip}(\pi_{\lambda_k}) = 1$ and π_{λ_k} takes on the plane $\{t = \varepsilon\}$ into itself, we have $\mathcal{H}^2(\Psi_k(D_k) \cap \{t = \varepsilon\}) > 0$. Again, being Ψ_k Lipschitz continuous, we deduce that $\Psi_k^{-1}(\Psi_k(D_k) \cap \{t = \varepsilon\})$ has positive measure. But $\Psi_k^{-1}(\Psi_k(D_k) \cap \{t = \varepsilon\}) \subset \Psi_k^{-1}(\{t = \varepsilon\}) = \partial\mathrm{B}_\varepsilon$ which has obviously $\mathcal{H}^2$ null measure.

Let us prove that $\mathcal{H}^2(W_k \cap \{t = \varepsilon\}) = 0$. Recalling (see (5.16)) that $W_k = \tau([1 - \lambda_k, 1 - \lambda'_k] \times \gamma_k)$ with $\gamma_k := \pi_{\lambda_k} \circ \Psi_k(\partial D_k)$, and since $\tau(\cdot, z)$ in (5.13) does not change the axial coordinate of z, we see[38] that $\tau([1 - \lambda_k, 1 - \lambda'_k] \times \gamma_k) \cap \{t = \varepsilon\}$ has positive $\mathcal{H}^2$ measure only if $\gamma_k \cap \{t = \varepsilon\}$ has positive $\mathcal{H}^1$ measure. Again, since also π_{λ_k} does not change the axial coordinate, as before this happens only if $\Psi_k^{-1}(\widehat{\gamma}_k \cap \{t = \varepsilon\})$ has positive $\mathcal{H}^1$-measure, where $\widehat{\gamma}_k := \Psi_k(\partial D_k)$; by Lemma 4.2, this is not possible, since we know that $\widehat{\gamma}_k \cap \{t = \varepsilon\} = \{\Psi_k(\widehat{x}_i)\}$ (see (8.2)), and then $\Psi_k^{-1}(\widehat{\gamma}_k \cap \{t = \varepsilon\}) = \{\widehat{x}_i\}$ which is a finite set. ☐

We recall from (5.6) and (5.22) that

$$\widehat{\mathfrak{D}}_k = (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k]\!] + \mathcal{W}_k. \tag{8.3}$$

An immediate consequence of Lemma 8.1, formula (8.3), and the fact that $\widehat{\mathfrak{D}}_k$ is boundaryless in $C_l(1 - \lambda'_k)$, is the following:

Corollary 8.2. *We have $\widehat{\mathfrak{D}}_k \llcorner \{t = \varepsilon\} = 0$. In particular*

$$\partial(\widehat{\mathfrak{D}}_k \llcorner \{\varepsilon < t < l\}) \llcorner (\{t = \varepsilon\}) = -\partial(\widehat{\mathfrak{D}}_k \llcorner \{-1 < t < \varepsilon\}) \llcorner (\{t = \varepsilon\}) \qquad \text{in } C_l(1 - \lambda'_k).$$

If $\{E_{k,i}\}_{i \in \mathbb{N}}$, $E_{k,i} \in C_l$ are the sets which we have symmetrized (see (5.30)), $\mathbb{S}(E_k)$ is the symmetrized set, and $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is the symmetrized current, we have to understand the behaviour of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ on $\{\varepsilon\} \times \mathbb{R}^2$. We have observed that $\widehat{\mathfrak{D}}_k \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = 0$ because $\mathcal{H}^2\left((\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k) \cap \{\varepsilon\} \times \mathbb{R}^2\right) = 0$. The same holds for the symmetrized current, as a particular consequence of Lemma 3.4:

$$\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = 0.$$

8.1 Description of the boundary of the current $\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner ((-1, \varepsilon) \times B_{1-\lambda'_k})$

Our first aim is to describe the boundary of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ on $\{\varepsilon\} \times \mathbb{R}^2$ (Corollary 8.5). To do so, let us recall that π_{λ_k} is given in Definition 5.3 and that the points x_i are defined in (8.2).

[38] For instance, using the coarea formula.

Definition 8.3 (The current $\mathcal{H}_{k,\varepsilon}$). *Recalling* (8.1), *we set*

$$\mathcal{H}_{k,\varepsilon} := (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![H_{k,\varepsilon}]\!] \in \mathcal{D}_1(\{\varepsilon\} \times B_1), \tag{8.4}$$

where $H_{k,\varepsilon}$ is oriented counterclockwise.

Let us denote by $\{\widetilde{x}_i\} \subseteq \{x_i\}$ the points which represent the support of the current $\partial[\![H_{k,\varepsilon}]\!]$. We can consider the orthogonal projection[39] onto the lateral boundary of $C_l(1-\lambda'_k)$, and we denote by $L_{k,i}$ the segment connecting $\pi_{\lambda_k}(\Psi_k(\widetilde{x}_i))$ (which belongs to the lateral boundary of $C_l(1-\lambda_k)$) to the image point of $\Psi_k(\widetilde{x}_i)$ through this projection.

We consider the 1-integral current in $\{\varepsilon\} \times B_{1-\lambda'_k}$ given by

$$\mathcal{H}_{k,\varepsilon} + \sum_i [\![L_{k,i}]\!] \in \mathcal{D}_1(\{\varepsilon\} \times B_{1-\lambda'_k}), \tag{8.5}$$

where $[\![L_{k,i}]\!]$ are the integrations over the segments $L_{k,i}$ taken with suitable orientation in order that

$$\partial\Big(\mathcal{H}_{k,\varepsilon} + \sum_i [\![L_{k,i}]\!]\Big) = 0 \qquad \text{in } \{\varepsilon\} \times B_{1-\lambda'_k}. \tag{8.6}$$

Before stating the following crucial lemma, we recall that the current $\widehat{\mathfrak{D}}_k$ is defined in C_l but is supported in $[0,l] \times \overline{B}_{1-\lambda'_k}$.

Lemma 8.4 (Boundary of $\widehat{\mathfrak{D}}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)$ in $\{\varepsilon\} \times B_{1-\lambda'_k}$). *We have*

$$\partial\big(\widehat{\mathfrak{D}}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\big) = \mathcal{H}_{k,\varepsilon} + \sum_i [\![L_{k,i}]\!] \qquad \text{in } \mathcal{D}_1(\{\varepsilon\} \times B_{1-\lambda'_k}). \tag{8.7}$$

Proof. We recall that

$$\widehat{\mathfrak{D}}_k = \mathfrak{D}_k + \mathcal{W}_k,$$

where $\mathfrak{D}_k$ is defined in (5.6) and, by (5.17), $\mathcal{W}_k = \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times \partial D_k]\!]$. Observe that

$$\partial\left(\mathcal{W}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\right) = \sum_i [\![L_{k,i}]\!] \qquad \text{in the annulus } \{\varepsilon\} \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}).$$

Indeed, this follows from the definition of $L_{k,i}$, the equality[40]

$$\begin{aligned}\mathcal{W}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2) =& \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times (\mathrm{B}_\varepsilon \cap \partial D_k)]\!] \\ =& \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times \partial(D_k \cap \mathrm{B}_\varepsilon)]\!] - \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times H_{k,\varepsilon}]\!],\end{aligned}$$

and (8.6). Moreover, from (5.6),

$$\begin{aligned}\partial\Big(\mathfrak{D}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\Big) \llcorner (\{\varepsilon\} \times B_1) &= \partial\Big(((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k]\!]) \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\Big) \llcorner (\{\varepsilon\} \times B_1) \\ &= \partial\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k \cap \mathrm{B}_\varepsilon]\!]\Big) \llcorner (\{\varepsilon\} \times B_1) \\ &= \Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial [\![D_k \cap \mathrm{B}_\varepsilon]\!]\Big) \llcorner (\{\varepsilon\} \times B_1) \\ &= \mathcal{H}_{k,\varepsilon} \qquad \text{on } \{\varepsilon\} \times B_1,\end{aligned}$$

where in the last equality we use[41] $[\![\partial(D_k \cap \mathrm{B}_\varepsilon)]\!] = [\![D_k \cap \partial \mathrm{B}_\varepsilon]\!] = [\![H_{k,\varepsilon}]\!]$ on $\partial \mathrm{B}_\varepsilon$. $\square$

[39] Defined at least in the region $C_l(1-\lambda'_k) \setminus C_l(1-\lambda_k)$.

[40] Notice that $\partial(D_k \cap \mathrm{B}_\varepsilon) = (\partial D_k \cap \mathrm{B}_\varepsilon) \cup (\partial D_k \cap \partial \mathrm{B}_\varepsilon) \cup (D_k \cap \partial \mathrm{B}_\varepsilon)$; recall also that, by Lemma 4.2 (iv), $\partial D_k \cap \partial \mathrm{B}_\varepsilon$ consists of a finite set of points.

[41] Here we take the boundary of D_k in $\partial \mathrm{B}_\varepsilon$ in the sense of currents, so that isolated points are neglected.

Thanks to Corollary 5.17, both $\widehat{\mathfrak{D}}_k$ and $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ have no boundary in $(-\infty,l)\times B_{1-\lambda'_k}$. Now, we need to describe the boundary of the symmetrized current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ restricted to $(-1,\varepsilon)\times B_{1-\lambda'_k}$, see (8.11). We recall the definitions of $\mathcal{X}_k$ and $\mathcal{Y}_k$ in (5.27) and (5.26), and for $t\in(-1,\varepsilon]$ and $\rho\in(1-\lambda_k,1-\lambda'_k)$ the function $\Theta_k(t,\rho)$ defined in (6.10). Also in this case

$$\Theta_k(t,\rho)=\Theta_k(t,\varrho)\qquad\text{for all }\rho,\varrho\in(1-\lambda_k,1-\lambda'_k).$$

In cylindrical coordinates, if

$$X_1:=(\varepsilon,1-\lambda_k,\Theta_k(\varepsilon,1-\lambda_k)/2),\qquad X_2:=(\varepsilon,1-\lambda_k,-\Theta_k(\varepsilon,1-\lambda_k)/2),$$

we denote the two 1-currents

$$\mathbb{S}(L)_1:=\tau(\cdot,X_1)_\sharp[\![(1-\lambda_k,1-\lambda'_k)]\!]\qquad\text{and}\qquad\mathbb{S}(L)_2:=\tau(\cdot,X_2)_\sharp[\![(1-\lambda_k,1-\lambda'_k)]\!],\tag{8.8}$$

see Fig. 9. Set

$$Y_1=\tau(1-\lambda'_k,X_1),\qquad Y_2=\tau(1-\lambda'_k,X_2).\tag{8.9}$$

We know, by construction and definition of Θ_k, that

$$\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times(B_{1-\lambda'_k}\setminus\overline{B}_{1-\lambda_k}))\big)\llcorner(\{\varepsilon\}\times\mathbb{R}^2)=\mathbb{S}(L)_1-\mathbb{S}(L)_2$$

in $\mathcal{D}_1(\{\varepsilon\}\times(B_{1-\lambda'_k}\setminus\overline{B}_{1-\lambda_k}))$. We define

$$\mathbb{S}(\mathcal{H}_{k,\varepsilon}):=\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times B_{1-\lambda'_k})\big)\llcorner(\{\varepsilon\}\times\mathbb{R}^2)-\mathbb{S}(L)_1+\mathbb{S}(L)_2\tag{8.10}$$

in $\mathcal{D}_1(\{\varepsilon\}\times B_{1-\lambda_k})$, see again Fig. 9. With these definitions at our disposal we can now write

$$\begin{aligned}&\partial\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times B_{1-\lambda'_k})\Big)\\=&\mathbb{S}(\mathcal{H}_{k,\varepsilon})+\mathbb{S}(L)_1-\mathbb{S}(L)_2+\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t\in(-1,\varepsilon)\}\big)\llcorner(\{-1\}\times B_{1-\lambda'_k})\\=&\mathbb{S}(\mathcal{H}_{k,\varepsilon})+\mathbb{S}(L)_1-\mathbb{S}(L)_2.\end{aligned}\tag{8.11}$$

Here we have used once again that $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is supported in $[0,l]\times B_1$, and then its boundary on $\{t=-1\}$ is always null.

We can clarify the meaning of the last term in formula (8.11).

Corollary 8.5. *We have*

$$\mathbb{S}(\mathcal{H}_{k,\varepsilon})+\mathbb{S}(L)_1-\mathbb{S}(L)_2=-\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((\varepsilon,l)\times B_{1-\lambda'_k})\big)\llcorner(\{\varepsilon\}\times\mathbb{R}^2).$$

Proof. It follows from (8.11), Corollary 8.2, and Lemma 3.4. □

8.2 Construction of the current $\mathcal{V}_{k,\varepsilon}$

Let $\Pi_\varepsilon:\mathbb{R}^3\to\{\varepsilon\}\times\mathbb{R}^2$ be the orthogonal projection on $\{\varepsilon\}\times\mathbb{R}^2$.

Definition 8.6. *We set*

$$\mathcal{V}_{k,\varepsilon}:=(\Pi_\varepsilon)_\sharp\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times B_{1-\lambda'_k})\Big)\in\mathcal{D}_2(C_l).\tag{8.12}$$

Lemma 8.7. *We have*

$$|[\![G_{u_k}]\!]|_{(D_k\cap\mathrm{B}_\varepsilon)\times\mathbb{R}^2}\geq|\mathcal{V}_{k,\varepsilon}|-2\pi(\lambda_k-\lambda'_k).$$

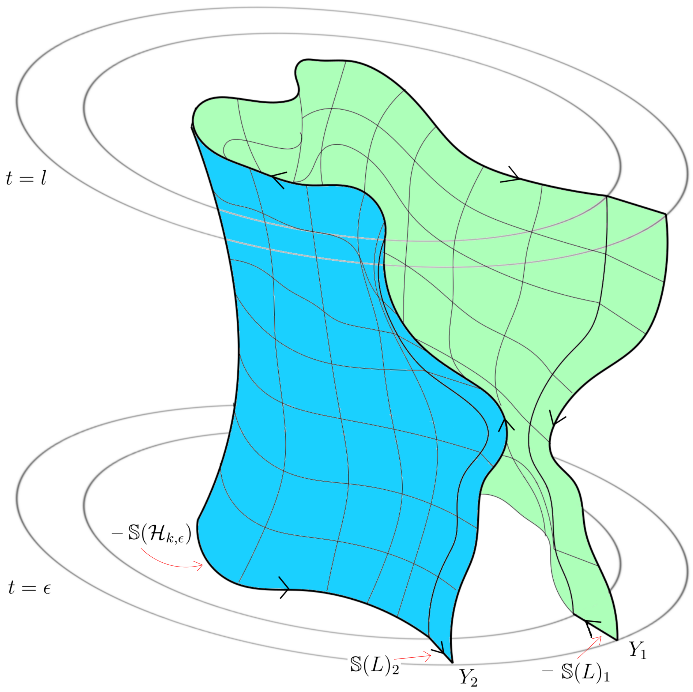

Figure 9: The current $\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((\varepsilon,l)\times B_{1-\lambda'_k})$ is depicted. At $t=\varepsilon$ we emphasized the various objects composing its boundary, taken with their orientation.

Proof. By (8.12), since $\mathrm{Lip}(\Pi_\varepsilon)=1$, we have, using (8.3), (5.17),

$$\begin{aligned}|\mathcal{V}_{k,\varepsilon}| =&|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})}+|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times B_{1-\lambda_k}}\\ \leq&|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})}+|\widehat{\mathfrak{D}}_k|_{(-1,\varepsilon)\times B_{1-\lambda_k}}\\ =&|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})}+|\mathfrak{D}_k|_{(-1,\varepsilon)\times B_{1-\lambda_k}}\\ \leq&2\pi(\lambda_k-\lambda'_k)+|[\![G_{u_k}]\!]|_{(D_k\cap \mathrm{B}_\varepsilon)\times\mathbb{R}^2},\end{aligned} \tag{8.13}$$

where we have also used a localized version of (5.34) in $(-1,\varepsilon)\times B_{1-\lambda_k}$. □

By Corollary 8.5 it holds[42]

$$\partial\mathcal{V}_{k,\varepsilon}=\mathbb{S}(\mathcal{H}_{k,\varepsilon})+\mathbb{S}(L)_1-\mathbb{S}(L)_2\qquad\text{in }\{\varepsilon\}\times B_{1-\lambda'_k}. \tag{8.14}$$

Clearly the above current is boundaryless in $\{\varepsilon\}\times B_{1-\lambda'_k}$; more precisely it is an oriented curve connecting Y_2 to Y_1 (defined in (8.9)) as soon as $Y_2\neq Y_1$, with $\mathbb{S}(\mathcal{H}_{k,\varepsilon})$ clockwise oriented[43]. If we extend $\mathcal{V}_{k,\varepsilon}$ to 0 on the whole plane $\{\varepsilon\}\times\mathbb{R}^2$ (keeping the same notation) we have

$$\partial\mathcal{V}_{k,\varepsilon}=\mathcal{L}_k+\mathbb{S}(\mathcal{H}_{k,\varepsilon})+\mathbb{S}(L)_1-\mathbb{S}(L)_2\qquad\text{on }\{\varepsilon\}\times\mathbb{R}^2, \tag{8.15}$$

for some current $\mathcal{L}_k$ supported on $\{\varepsilon\}\times\partial B_{1-\lambda'_k}$ and whose boundary is two deltas, with suitable signs, on Y_1 and Y_2. In particular $\mathcal{L}_k$ is the integration between Y_1 to Y_2 on the circle $\{\varepsilon\}\times\partial B_{1-\lambda'_k}$.

[42]Recall that $\partial(\Pi_\varepsilon)_\sharp\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t<\varepsilon\}\Big)=(\Pi_\varepsilon)_\sharp\partial\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t<\varepsilon\}\Big)$ and that the map Π_ε does not move the plane where $\partial(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t<\varepsilon\})$ is supported.

[43]When looking at the plane $\{\varepsilon\}\times\mathbb{R}^2$ from $t>\varepsilon$.

However there are two arcs which connect these two points, namely (in cylindrical coordinates)

$$\{\varepsilon\} \times \{1-\lambda_k'\} \times \left[\frac{-\Theta_k(\varepsilon, 1-\lambda_k')}{2}, \frac{\Theta_k(\varepsilon, 1-\lambda_k')}{2}\right] \tag{8.16}$$

oriented clockwise and

$$\left(\{\varepsilon\} \times \partial B_{1-\lambda_k'}\right) \setminus \left\{\{\varepsilon\} \times \{1-\lambda_k'\} \times \left[\frac{-\Theta_k(\varepsilon, 1-\lambda_k')}{2}, \frac{\Theta_k(\varepsilon, 1-\lambda_k')}{2}\right]\right\} \tag{8.17}$$

oriented counterclockwise. We have to identify $\mathcal{L}_k$ with the integration over one of these two arcs.

Proposition 8.8. *$\mathcal{L}_k$ is the counterclockwise integration over the arc connecting Y_1 and Y_2 given by* (8.17).

Before proving this proposition we anticipate a useful observation.

Remark 8.9. We set

$$\mathbb{S}(E_k)_\varepsilon := \mathbb{S}(E_k) \cap \{t = \varepsilon\}.$$

Since $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is the boundary of the integration over $\mathbb{S}(E_k)$, the current $\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner ((-1,\varepsilon) \times B_{1-\lambda_k'}) + [\![\mathbb{S}(E_k)_\varepsilon]\!]$ is boundaryless in $\mathcal{D}_2(C_l(1-\lambda_k'))$ (with $\mathbb{S}(E_k)_\varepsilon$ suitably oriented). It follows, invoking Corollary 8.5, that

$$\partial [\![\mathbb{S}(E_k)_\varepsilon]\!] = -\mathbb{S}(\mathcal{H}_{k,\varepsilon}) - \mathbb{S}(L)_1 + \mathbb{S}(L)_2 \qquad \text{in } \{\varepsilon\} \times B_{1-\lambda_k'}.$$

The fact that

$$\partial \mathcal{V}_{k,\varepsilon} = \mathcal{L}_k + \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 \qquad \text{in } \{\varepsilon\} \times \mathbb{R}^2, \tag{8.18}$$

(where $\mathcal{L}_k$ is as in Proposition 8.8) means that $\mathcal{V}_{k,\varepsilon}$ is the integration over the set

$$B_{1-\lambda_k'} \setminus \mathbb{S}(E_k)_\varepsilon.$$

In particular $\mathcal{V}_{k,\varepsilon}$ has coefficient 1 in $B_{1-\lambda_k'} \setminus \mathbb{S}(E_k)_\varepsilon$ and zero in $\mathbb{S}(E_k)_\varepsilon$. On the other hand, if $\mathcal{L}_k$ were the integration over (8.16) oriented clockwise, then we would have that $\mathcal{V}_{k,\varepsilon}$ had coefficient -1 in $\mathbb{S}(E_k)_\varepsilon$ and 0 in $B_{1-\lambda_k'} \setminus \mathbb{S}(E_k)_\varepsilon$.

We can now prove Proposition 8.8.

Proof. Appealing to Remark 8.9, it is sufficient to show that the coefficient of $\mathcal{V}_{k,\varepsilon}$ is 1 in $B_{1-\lambda_k'} \setminus \mathbb{S}(E_k)_\varepsilon$. Equivalently we can show that this coefficient is zero in $B_{1-\lambda_k'} \cap \mathbb{S}(E_k)_\varepsilon$.

Let us recall, by definitions (5.26) and (5.27),

$$(\mathcal{Y}_k)_t = \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda_k'] \times ((\Omega \setminus D_k) \cap \partial \mathrm{B}_t)]\!] \qquad \text{for a.e. } t \in (0,\varepsilon], \tag{8.19}$$

$$(\mathcal{X}_k)_t = [\![\{t\} \times (B_{1-\lambda_k'} \setminus \overline{B}_{1-\lambda_k})]\!] - (\mathcal{Y}_k)_t \qquad \text{for a.e. } t \in (0,\varepsilon]. \tag{8.20}$$

Recalling Lemma 4.2(i), we now divide our analysis in two cases:

(1) $|u_k(0)| < 1-\lambda_k$.

(2) $|u_k(0)| > 1-\lambda_k$.

We notice that, in both cases, by continuity of u_k, for all $\delta \in (0,1)$ there is $t_k^\delta > 0$ such that

$$u_k(\mathrm{B}_t) \subset B_\delta(u_k(0)) \qquad \forall t \in (0, t_k^\delta]. \tag{8.21}$$

Case (1): If δ is sufficiently small, we can also assume that

$$B_\delta(u_k(0)) \subset B_{1-\lambda_k}(0), \tag{8.22}$$

and therefore

$$u_k(\mathrm{B}_t) \subset B_{1-\lambda_k}(0) \qquad \forall t \in (0, t_k^\delta]. \tag{8.23}$$

In this case it turns out that if $t \le t_k^\delta$ then the current $(\mathcal{Y}_k)_t$ in (8.19) is null, because $|u_k|^+(t) < 1 - \lambda_k$, hence $(\Omega \setminus D_k) \cap \partial \mathrm{B}_t = \emptyset$ by (6.2). In particular, by (8.20),

$$(\mathcal{X}_k)_t = [\![\{t\} \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})]\!] \qquad \text{for a.e. } t \le t_k^\delta.$$

Eventually, since $\overline{\mathrm{B}}_t \subset D_k$ for any $t \in [0, t_k^\delta]$, from (8.21), we also deduce $u_k(\overline{\mathrm{B}}_{t_k^\delta} \cap D_k) = u_k(\overline{\mathrm{B}}_{t_k^\delta}) \subset B_\delta(u_k(0))$, so that

$$\Psi_k(D_k) \cap ([0, t_k^\delta] \times B_1) = \pi_{\lambda_k} \circ \Psi_k(D_k) \cap ([0, t_k^\delta] \times B_1) \subset [0, t_k^\delta] \times B_\delta(u_k(0)). \tag{8.24}$$

Now, consider the decomposition (5.30) of $\mathcal{E}_k$. By the crucial identification (5.29) and (8.22) we infer that there must be a set $E_{k,h} \in \{E_{k,i}\}_{i \in \mathbb{N}}$ with[44]

$$\begin{aligned} \mathcal{X}_k \llcorner ((-1, t_k^\delta) \times B_{1-\lambda'_k}) &= [\![(-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})]\!] \\ &= [\![E_{k,h} \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big)]\!]. \end{aligned}$$

Therefore

$$E_{k,h} \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big) = (-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}).$$

This has the following consequence: denoting as usual $\mathbb{S}(E_{k,h})$ the cylindrical symmetrization of $E_{k,h}$ we infer

$$\mathbb{S}(E_{k,h}) \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big) = (-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}),$$

and since $\mathbb{S}(E_{k,h}) \subset \mathbb{S}(E_k)$ we also have

$$\mathbb{S}(E_k) \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big) = (-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}). \tag{8.25}$$

We now consider two subcases.

(1A) $\mathcal{H}^2\Big(\big(\{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\big) \setminus \mathbb{S}(E_k)_\varepsilon\Big) > 0$. To conclude the proof it is sufficient to show that

$$\text{the multiplicity of } \mathcal{V}_{k,\varepsilon} \text{ on } \big(\{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\big) \setminus \mathbb{S}(E_k)_\varepsilon \text{ is } 1, \tag{8.26}$$

because $(\{\varepsilon\} \times B_{1-\lambda'_k}) \setminus \mathbb{S}(E_k)_\varepsilon$ is, by definition, outside the finite perimeter set $\mathbb{S}(E_k)$.

We argue by slicing, and consider the lines $l_{\rho,\theta}$ in $\mathbb{R}^3$ given by $l_{\rho,\theta} = \mathbb{R} \times \{\rho\} \times \{\theta\}$, with ρ and θ fixed. Consider any point p_0 of coordinates $\rho \in (1 - \lambda_k, 1 - \lambda'_k)$ and $\theta \in (-\pi, \pi]$ such that

$$p_0 \in (\{\varepsilon\} \times B_{1-\lambda'_k}) \setminus \mathbb{S}(E_k)_\varepsilon. \tag{8.27}$$

[44] Since the decomposition in (5.30) is done in undecomposable components, such a set is unique.

For a.e. such $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and $\theta \in (-\pi,\pi]$ the slice of $\widehat{\mathfrak{D}}_k \llcorner ((-1,\varepsilon)\times B_{1-\lambda'_k})$ with respect to this line is the sum of some Dirac deltas with suitable signs, according to the orientation of $\widehat{\mathfrak{D}}_k$. Indeed $\widehat{\mathfrak{D}}_k$ is the integration over the boundary of the finite perimeter set $\mathbb{S}(E_k)$, so it turns out that, for a.e. $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and $\theta \in (-\pi,\pi]$ the slice of $[\![\mathbb{S}(E_k)]\!]$ with respect to the line $l_{\rho,\theta}$ is exactly

$$[\![\mathbb{S}(E_k)_{\rho,\theta}]\!] = [\![\mathbb{S}(E_k)\cap l_{\rho,\theta}]\!], \tag{8.28}$$

that is the integration over some disjoint intervals. If $p_1, p_2, \dots p_m$ are the intervals endpoints (written in order[45] on $l_{\rho,\theta}$) and if we assume that the last interval between the points p_1 and $p_0 = (\varepsilon,\rho,\theta)$ is outside $\mathbb{S}(E_k)$, then it results

$$\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!] = -\sum_{\substack{i>0\\ i \text{ even}}} \delta_{p_i} + \sum_{\substack{i>0\\ i \text{ odd}}} \delta_{p_i}. \tag{8.29}$$

If instead the last interval $[p_1,p_0]$ is inside $\mathbb{S}(E_k)$ we have

$$\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!] = \sum_{\substack{i>0\\ i \text{ even}}} \delta_{p_i} - \sum_{\substack{i>0\\ i \text{ odd}}} \delta_{p_i}. \tag{8.30}$$

Let us now prove claim (8.26). We have obtained that, for a.e. $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and any $\theta \in (-\pi,\pi]$ such that (8.27) holds, the slice $\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!]$ is the sum in (8.29), and thanks to (8.25) we deduce that the total number of points involved in (8.29) must be odd. As a consequence, the push-forward by Π_ε of $\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!]$ is a Dirac delta with coefficient -1. Since this holds for a.e. $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and any $\theta \in (-\pi,\pi]$, the conclusion follows.

(1B) Suppose $\mathcal{H}^2\Big(\big(\{\varepsilon\}\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})\big)\setminus\mathbb{S}(E_k)_\varepsilon\Big) = 0$. In this case we pass to the complementary set; namely, if $\{\varepsilon\}\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k}) = \mathbb{S}(E_k)_\varepsilon \cap \big(\{\varepsilon\}\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})\big)$, up to $\mathcal{H}^2$-negligible sets, we show that the multiplicity of $\mathcal{V}_{k,\varepsilon}$ on this set is null. To do so it is sufficient to repeat the slicing argument above for a.e. (ρ,θ) such that $p_0 = (\varepsilon,\rho,\theta) \in \{\varepsilon\}\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})$. For these points (8.30) takes place, since by (8.25) the number of points involved in the sum is even. The conclusion follows.

Case (2): Choosing $\delta \in (0,1)$ small enough,

$$\Psi_k(\mathrm{B}_{t_k^\delta}) \subset [0,t_k^\delta]\times B_\delta(u_k(0)), \tag{8.31}$$

and, using $|u_k(0)| > 1-\lambda_k$,

$$\pi_{\lambda_k}\circ\Psi_k(D_k\cap \mathrm{B}_{t_k^\delta}) \subset [0,t_k^\delta]\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k}).$$

Recalling the definition of $\widehat{\mathfrak{D}}_k$, it is not difficult to see that the current $\widehat{\mathfrak{D}}_k\llcorner((-1,t_k^\delta)\times B_{1-\lambda'_k})$ is supported in $[0,t_k^\delta]\times(\overline{B}_{1-\lambda'_k}\setminus B_{1-\lambda_k})$. By the properties of cylindrical symmetrization, we have also that $\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,t_k^\delta)\times B_{1-\lambda'_k})$ is supported in $[0,t_k^\delta]\times(\overline{B}_{1-\lambda'_k}\setminus B_{1-\lambda_k})$.

Obviously, being $\mathcal{Y}_k$ null on $(-1,0)\times B_{1-\lambda'_k}$, we have

$$\mathcal{X}_k\llcorner\Big((-1,0)\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})\Big) = [\![(-1,0)\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})]\!],$$

[45] p_1 is the point closer to $\{\varepsilon\}\times\mathbb{R}^2$

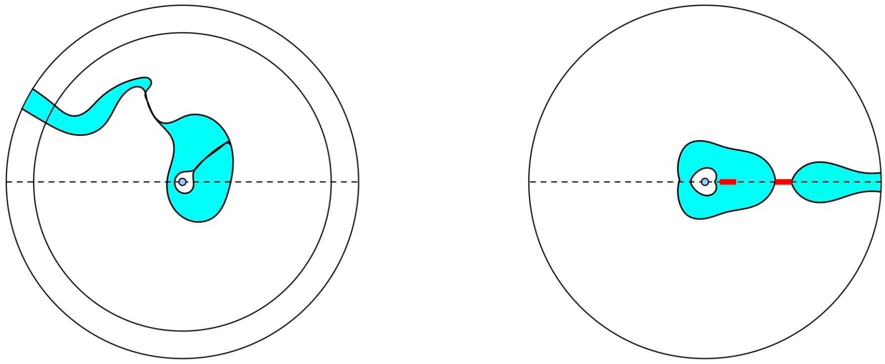

Figure 10: We represent the symmetrization of a general closed current in B_1. On the left it is visible that on two parts the curve overlaps itself in such a way that the multiplicity of the associated current is zero. In the symmetrized set, on the right picture, we have emphasized in bold the corresponding set $J^{0,2\pi}_{k,\varepsilon}$ in (7.13).

and we find a set $E_{k,h}$, such that

$$\mathcal{X}_k \llcorner \Big((-1,0)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k})\Big) = [\![E_{k,h}\cap((-1,0)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k}))]\!].$$

If we pass to the symmetrized set, arguing as in case (1A), we infer

$$\mathbb{S}(E_k)\cap\big((-1,0)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k})\big) = (-1,0)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k}).$$

In other words, $(-1,0)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k})$ is contained in $\mathbb{S}(E_k)$, and since the support of $\partial^*\mathbb{S}(E_k)$ does not intersect the set $(-1,0)\times B_{1-\lambda_k'}$, we infer that also

$$(-1,0)\times B_{1-\lambda_k'}\subset\mathbb{S}(E_k). \tag{8.32}$$

We now decompose $\{\varepsilon\}\times B_{1-\lambda_k}$ as

$$\{\varepsilon\}\times B_{1-\lambda_k} = \big((\{\varepsilon\}\times B_{1-\lambda_k})\cap\mathbb{S}(E_k)_\varepsilon\big)\cup\big((\{\varepsilon\}\times B_{1-\lambda_k})\setminus\mathbb{S}(E_k)_\varepsilon\big),$$

and one of these two sets on the right-hand side must have positive $\mathcal{H}^2$-measure. Assume that $\mathcal{H}^2((\{\varepsilon\}\times B_{1-\lambda_k})\setminus\mathbb{S}(E_k)_\varepsilon)>0$. Then we will prove that the multiplicity of $\mathcal{V}_{k,\varepsilon}$ on this set is 1 (if instead $(\{\varepsilon\}\times B_{1-\lambda_k})\setminus\mathbb{S}(E_k)_\varepsilon$ has zero measure then it is sufficient to prove that $\mathcal{V}_{k,\varepsilon}$ has zero multiplicity on $(\{\varepsilon\}\times B_{1-\lambda_k})\cap\mathbb{S}(E_k)_\varepsilon$; we drop this case being completely similar to the former).

Therefore we now proceed as in case (1), slicing with respect to lines $l_{\rho,\theta}$ with $(\varepsilon,\rho,\theta)\in\{\varepsilon\}\times(\{\varepsilon\}\times B_{1-\lambda_k})\setminus\mathbb{S}(E_k)_\varepsilon$. Since the last point $p_0=(\varepsilon,\rho,\theta)$ does not belong to $\mathbb{S}(E_k)_\varepsilon$, we are concerned with the sum in (8.29), and by (8.32) we infer that the number of $\{p_i\}$ involved in the sum is odd. The conclusion follows as in case (1). □

9 Gluing rectifiable sets

In this section we show that, up to adding to $\partial\mathbb{S}(E_k)$ a rectifiable set with small $\mathcal{H}^2$-measure, $\partial\mathbb{S}(E_k)$ can be described as a polar graph of a suitable modification of the function $\vartheta_{k,\varepsilon}$ over a subset[46] of

[46] called $S^{(4)}_{k,\varepsilon}$, see (9.4).

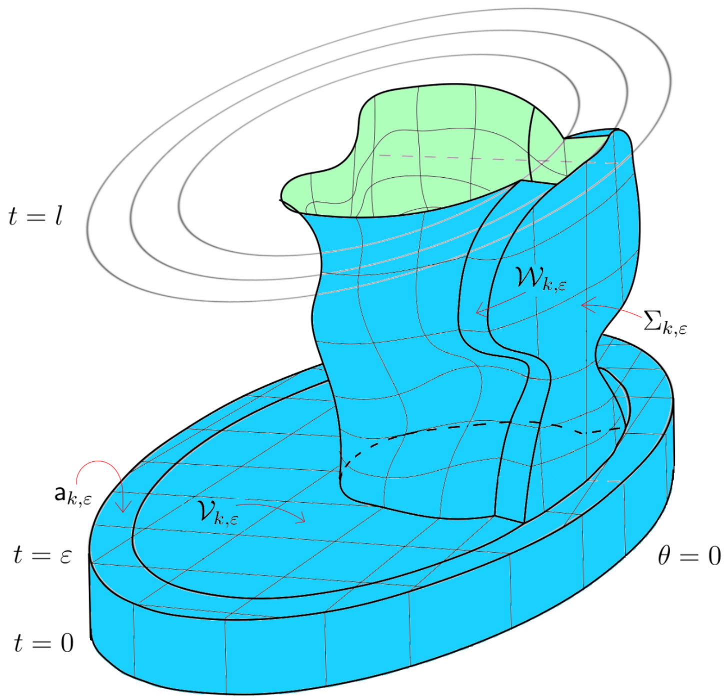

Figure 11: The largest (resp. smaller) basis circle has radius 1 (resp. $1-\lambda'_k$). The smallest top circle has radius $1-\lambda_k$. The symbol $\mathcal{W}_{k,\varepsilon}$ denotes the restriction of $\mathcal{W}_k$ to $\overline{C}^l_\varepsilon(1-\lambda'_k)$, after symmetrization. Note that $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ does not include $\Sigma_{k,\varepsilon}$ and $\mathcal{V}_{k,\varepsilon}$; see (7.24).

the rectangle[47] $(0,l) \times [0,1] \times \{0\} \subset \mathbb{R}^3$, and with Dirichlet boundary conditions independent of k. In Section 11 we will reduce the estimate of the area of the graph of u_k to an estimate for a non-parametric Plateau problem which in turn will be independent of k.

First we remark that $\mathbb{S}(E_k) \subseteq C_l(1-\lambda'_k)$ and $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \partial\mathbb{S}(\mathcal{E}_k)$ in $C_l(1-\lambda'_k)$, see (5.31). If we look at $\mathbb{S}(E_k)$ as a subset of C_l, we cannot conclude $\partial^*\mathbb{S}(E_k) = \mathbb{S}(\widehat{\mathfrak{D}}_k)$ in C_l, and $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is not a closed current in C_l. For this reason we have to identify the boundary of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ in C_l.

Recalling Corollary 8.5 and Definition 7.14,

$$\begin{aligned}
&\partial\Big(\big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!]\big) \llcorner ((\varepsilon,l)\times\mathbb{R}^2)\Big) \llcorner (\{\varepsilon\}\times\mathbb{R}^2) \\
=&\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner ((\varepsilon,l)\times\mathbb{R}^2)\big) \llcorner (\{\varepsilon\}\times\mathbb{R}^2) = -\mathbb{S}(\mathcal{H}_{k,\varepsilon}) - \mathbb{S}(L)_1 + \mathbb{S}(L)_2 - [\![\overline{Y_1Y_2}]\!],
\end{aligned}$$

in $\mathcal{D}_2((-\infty,l)\times\mathbb{R}^2)$, where $[\![\overline{Y_1Y_2}]\!]$ is the integration on $\overline{Y_1Y_2}$ (see (8.16)) oriented from Y_1 to Y_2. As a consequence, from (8.18), we obtain

$$\partial\big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon}\big) \llcorner (\{\varepsilon\}\times\mathbb{R}^2) = [\![\{\varepsilon\}\times\partial B_{1-\lambda'_k}]\!] \qquad \text{in } \mathcal{D}_2((-\infty,l)\times\mathbb{R}^2),$$

where $\partial B_{1-\lambda'_k}$ is counterclockwisely oriented.

[47] In cartesian coordinates.

9.1 Enforcing boundary conditions at $\{0\} \times \mathbb{R}^2$; a modification of $\vartheta_{k,\varepsilon}$

Let $\mathsf{a}_{k,\varepsilon}$ denote the integration over the annulus $\{\varepsilon\} \times (B_1 \setminus B_{1-\lambda'_k})$, in such a way that

$$\partial \mathsf{a}_{k,\varepsilon} = [\![\{\varepsilon\} \times \partial B_1]\!] - [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!],$$

see Fig. 11. Then

$$|\mathsf{a}_{k,\varepsilon}| = \pi(1 - (1 - \lambda'_k)^2) \leq 2\pi\lambda'_k, \tag{9.1}$$

and

$$\partial\big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon}\big) = [\![\{\varepsilon\} \times \partial B_1]\!] \qquad \text{in } \mathcal{D}_2((-\infty, l) \times \mathbb{R}^2).$$

Finally, we add to the current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon}$ the integration over the lateral boundary of the cylinder $(0, \varepsilon) \times B_1$, so that the resulting current

$$\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0, \varepsilon) \times \partial B_1]\!] \in \mathcal{D}_2((0, l) \times \mathbb{R}^2), \tag{9.2}$$

satisfies

$$\partial\Big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0, \varepsilon) \times \partial B_1]\!]\Big) = [\![\{0\} \times \partial B_1]\!] \text{ in } \mathcal{D}_2((-\infty, l) \times \mathbb{R}^2);$$

in particular it is boundaryless in $\mathcal{D}_2((0, l) \times \mathbb{R}^2)$.

Now, we want to identify the solid region that we can call the "inside" of the current in (9.2).

Definition 9.1 (**The sets** $O_{k,\varepsilon}$). *We let*

$$O_{k,\varepsilon} := \big(\mathbb{S}(E_k) \cap ((\varepsilon, l) \times \mathbb{R}^2)\big) \cup ((0, \varepsilon] \times B_1) \subset [0, l] \times \mathbb{R}^2. \tag{9.3}$$

A direct check shows that the current built in (9.2) is the integration over the boundary of $[\![O_{k,\varepsilon}]\!]$. Indeed, by Lemma 7.13(iii) and Definition 7.14 we see that the integration over $\mathbb{S}(E_k) \cap ((\varepsilon, l) \times \mathbb{R}^2)$ has as boundary $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!]$ in $(\varepsilon, l) \times \mathbb{R}^2$, whereas $(0, \varepsilon] \times B_1$ trivially has boundary $(0, \varepsilon) \times \partial B_1$ in $(0, \varepsilon) \times \mathbb{R}^2$. The current $\mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon}$ represents the boundary of $[\![O_{k,\varepsilon}]\!]$ concentrated on the plane $\{\varepsilon\} \times \mathbb{R}^2$. In turn we will see (formulas (9.6), (9.7)) that $O_{k,\varepsilon}$ is the polar subgraph of a suitable modification of $\vartheta_{k,\varepsilon}$. Thus we are going to introduce the new extra "strip" (recalling the definition of $S^{(2)}_{k,\varepsilon}$ in (7.7) and of $J_{Q_{k,\varepsilon}}$ in (7.21)):

$$\begin{aligned} S^{(4)}_{k,\varepsilon} :&= S^{(2)}_{k,\varepsilon} \cup J_{Q_{k,\varepsilon}} \cup ((\varepsilon, l) \times [1 - \lambda'_k, 1] \times \{0\}) \\ &= \{(t, \rho, \theta) : t \in (\varepsilon, l), \rho \in [|u_k|^-(t) \wedge (1 - \lambda_k), 1], \theta = 0\}, \end{aligned} \tag{9.4}$$

see Fig. 12 (and also Figs. 4, 8).

Definition 9.2 (**The function** $\widehat{\vartheta}_{k,\varepsilon}$). *We define* $\widehat{\vartheta}_{k,\varepsilon} : (0, l) \times [0, 1] \times \{0\} \to \mathbb{R}$ *as*

$$\widehat{\vartheta}_{k,\varepsilon} := \begin{cases} \vartheta_{k,\varepsilon} & \text{in } (\varepsilon, l) \times [0, 1 - \lambda'_k] \times \{0\} \\ 0 & \text{in } (\varepsilon, l) \times [1 - \lambda'_k, 1] \times \{0\} \\ \pi & \text{in } (0, \varepsilon] \times [0, 1] \times \{0\}. \end{cases} \tag{9.5}$$

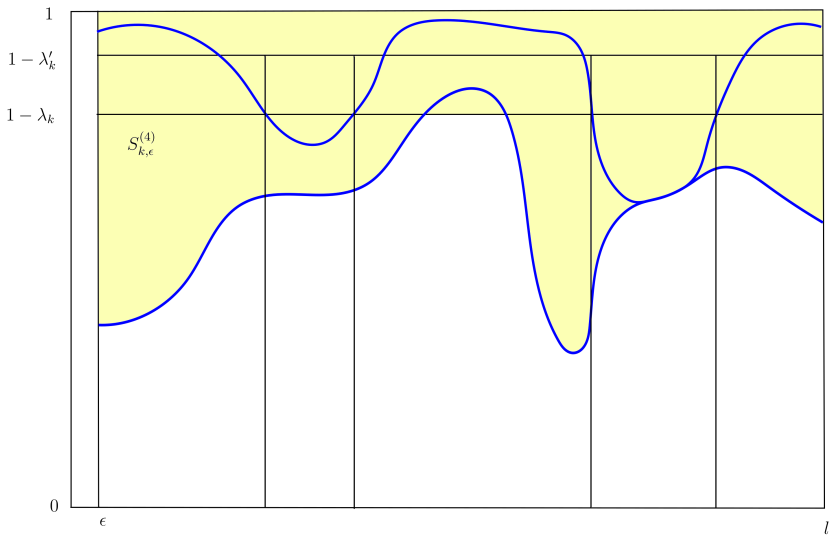

Figure 12: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ and the set $S^{(4)}_{k,\varepsilon}$ in (9.4). See also Fig. 4.

Accordingly, we extend the currents $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}$ and $\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ as follows: As in (7.2) we fix $\eta \in (0, \frac{\pi}{4})$, and set

$$
\begin{aligned}
SG^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon}} &:= \{(t,\rho,\theta) \in (0,l) \times [0,1] \times \{0\} : \theta \in (-\eta, \widehat{\vartheta}_{k,\varepsilon}(t,\rho,0))\},\\
UG^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon}} &:= \{(t,\rho,\theta) \in (0,l) \times [0,1] \times \{0\} : \theta \in (-\widehat{\vartheta}_{k,\varepsilon}(t,\rho,0), \eta)\}.
\end{aligned}
$$

Remark 9.3. By construction,

$$SG^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon}} \cap \{\theta \in (0,\pi)\} = O_{k,\varepsilon} \cap \{\theta \in (0,\pi)\}, \tag{9.6}$$

$$UG^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon}} \cap \{\theta \in (-\pi,0)\} = O_{k,\varepsilon} \cap \{\theta \in (-\pi,0)\}, \tag{9.7}$$

where the set $O_{k,\varepsilon}$ is defined in (9.3).

The next currents are constructed to reach the segment $(0,l) \times \{1\} \times \{0\}$.

Definition 9.4 (**The currents $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$**)**.** *We define the currents*

$$
\begin{aligned}
\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} &:= (\partial [\![SG^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon}}]\!]) \llcorner \{\theta \in (0,\pi)\} + [\![G^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon} \llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon} \in \{0,\pi\}\right\} \cap S^{(4)}_{k,\varepsilon}\right)}]\!],\\
\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} &:= (\partial [\![UG^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon}}]\!]) \llcorner \{\theta \in (-\pi,0)\} + [\![G^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon} \llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon} \in \{0,\pi\}\right\} \cap S^{(4)}_{k,\varepsilon}\right)}]\!].
\end{aligned} \tag{9.8}
$$

In other words, the support of $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ coincides with the generalized polar graph of $\widehat{\vartheta}_{k,\varepsilon}$ restricted to $S^{(4)}_{k,\varepsilon} \times [0,\pi]$. Notice that also in this case $[\![G^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon} \llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon} \in \{0,\pi\}\right\} \cap S^{(4)}_{k,\varepsilon}\right)}]\!] + [\![G^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon} \llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon} \in \{0,\pi\}\right\} \cap S^{(4)}_{k,\varepsilon}\right)}]\!] = 0$, and

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = [\![\partial^* O_{k,\varepsilon}]\!] \text{ in } (0,l) \times \mathbb{R}^2. \tag{9.9}$$

Moreover, by (9.8) and (7.9),

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| = |\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| + 2\mathcal{H}^2\left(\{\widehat{\vartheta}_{k,\varepsilon} \in \{0,\pi\}\} \cap S^{(4)}_{k,\varepsilon}\right). \tag{9.10}$$

Finally

$$\begin{aligned}\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} =& \mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + [\![(\varepsilon,l) \times [1-\lambda_k',1] \times \{0\}]\!] + [\![\Sigma_{k,\varepsilon} \cap \{0 \le \theta \le \pi\}]\!] + \mathcal{V}_{k,\varepsilon} \llcorner \{0 \le \theta \le \pi\} \\ &+ \mathsf{a}_{k,\varepsilon} \llcorner \{0 \le \theta \le \pi\} + [\![((0,\varepsilon) \times \partial B_1) \cap \{0 \le \theta \le \pi\}]\!],\end{aligned} \tag{9.11}$$

and

$$\begin{aligned}\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} =& \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} - [\![(\varepsilon,l) \times [1-\lambda_k',1] \times \{0\}]\!] + [\![\Sigma_{k,\varepsilon} \cap \{-\pi \le \theta \le 0\}]\!] + \mathcal{V}_{k,\varepsilon} \llcorner \{-\pi \le \theta \le 0\} \\ &+ \mathsf{a}_{k,\varepsilon} \llcorner \{-\pi \le \theta \le 0\} + [\![((0,\varepsilon) \times \partial B_1) \cap \{-\pi \le \theta \le 0\}]\!],\end{aligned}$$

so that

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = \mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0,\varepsilon) \times \partial B_1]\!].$$

Remark 9.5. The function $\widehat{\vartheta}_{k,\varepsilon}$ is defined on the whole domain $(0,l) \times [0,1] \times \{0\}$, but it might take values in $(0,\pi)$ only in $S^{(2)}_{k,\varepsilon}$, see Remark 7.5(v). Moreover, referring also to Remark 7.8, we see that the currents $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ and $\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}$ neglect the generalized polar graph of $\widehat{\vartheta}_{k,\varepsilon}$ (defined in (2.8)) on $((0,l) \times [0,1] \times \{0\}) \setminus S^{(4)}_{k,\varepsilon}$, with the only exception of the "vertical" part $[\![(0,\varepsilon) \times \partial B_1]\!]$.

An important step in the proof of the estimate from below in Theorem 11.16 is given by the next inequality.

Proposition 9.6 (Estimate from below in terms of the mass of $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ and $\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}$). *Let ε be fixed as in* (4.4) *and* (4.5). *The following inequality holds:*

$$|[\![G_{u_k}]\!]|_{D_k \times \mathbb{R}^2} \ge |\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1) \tag{9.12}$$

for an absolute constant $C > 0$, where the sequence $o_k(1) \ge 0$ depends on ε and n, and is infinitesimal as $k \to +\infty$.

Proof. By (9.11) we get

$$\begin{aligned}|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| \le& |\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}| + \lambda_k' l + |[\![\Sigma_{k,\varepsilon} \cap \{0 < \theta < \pi\}]\!]| + |\mathcal{V}_{k,\varepsilon} \llcorner \{0 < \theta < \pi\}| \\ &+ |\mathsf{a}_{k,\varepsilon} \llcorner \{0 \le \theta \le \pi\}| + |[\![((0,\varepsilon) \times \partial B_1) \cap \{0 \le \theta \le \pi\}]\!]|.\end{aligned}$$

A similar estimate holds for $|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|$ so that

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \le |\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}| + |\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}| + |[\![\Sigma_{k,\varepsilon}]\!]| + |\mathcal{V}_{k,\varepsilon}| + |\mathsf{a}_{k,\varepsilon}| + |[\![(0,\varepsilon) \times \partial B_1]\!]| + 2\lambda_k' l.$$

Coupling the above inequality with (7.22) and (7.23) gives

$$
\begin{aligned}
&|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \\
\leq &|\mathcal{G}^{+}_{k,\varepsilon}| + |\mathcal{G}^{-}_{k,\varepsilon}| + |[\![\Sigma_{k,\varepsilon}]\!]| + |\mathcal{V}_{k,\varepsilon}| + |\mathsf{a}_{k,\varepsilon}| + |[\![(0,\varepsilon)\times\partial B_1]\!]| + 2\lambda_k' l + \frac{1}{\pi\varepsilon n} \\
\leq &\int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + |[\![\Sigma_{k,\varepsilon}]\!]| + |\mathcal{V}_{k,\varepsilon}| + |\mathsf{a}_{k,\varepsilon}| + |[\![(0,\varepsilon)\times\partial B_1]\!]| \\
&\quad + 2\lambda_k' l + \frac{1}{\pi\varepsilon n} + \frac{1}{n} \\
\leq &\int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + |[\![G_{u_k}]\!]|_{(D_k\cap\mathrm{B}_\varepsilon)\times\mathbb{R}^2} + \pi\varepsilon + \frac{C}{\varepsilon n} + o_k(1),
\end{aligned}
$$

where the second inequality follows from (7.12), the last inequality follows from (7.27), (9.1) and (8.13), and $C > 0$ is an absolute constant. Here $o_k(1)$ is a nonnegative quantity infinitesimal as $k \to +\infty$, depending on ε and n. In conclusion

$$
\begin{aligned}
&|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1) \\
\leq &|[\![G_{u_k}]\!]|_{(D_k\cap(\Omega\setminus\mathrm{B}_\varepsilon))\times\mathbb{R}^2} + |[\![G_{u_k}]\!]|_{(D_k\cap\mathrm{B}_\varepsilon)\times\mathbb{R}^2} = |[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2}.
\end{aligned}
$$

□

10 Three examples

Before concluding the proof of the lower bound, we aim to explain the various geometric objects introduced in the previous sections (for a recovery sequence) through three interesting examples of sequences converging to the vortex map u. We warn the reader that the sequences of Sections 10.1 and 10.2, as well as the one of Section 10.3 for l small, are not recovery sequences; nevertheless, we believe it is useful to describe the various quantities introduced in Sections 5-9 in correspondence to these sequences, since this shades some light on the proof of the lower bound.

10.1 An approximating sequence of maps with degree zero: cylinder

In [1] the authors describe an approximating sequence (u_k) of smooth maps taking values in $\mathbb{S}^1$ which, in our context, are defined in polar coordinates as follows:

$$
u_k(r,\theta) := \begin{cases}
(\cos\theta, \sin\theta) & \text{in } \Omega_1 := \Omega \setminus (\mathrm{B}_{r_k} \cup \{\theta \in (-\theta_k, \theta_k)\}), \\
(\cos(\frac{r}{r_k}(\theta-\pi)+\pi), \sin(\frac{r}{r_k}(\theta-\pi)+\pi)) & \text{in } \mathrm{B}_{r_k} \setminus \{\theta \in (-\theta_k, \theta_k)\}, \\
(\cos(\frac{\theta_k-\pi}{\theta_k}\theta+\pi), \sin(\frac{\theta_k-\pi}{\theta_k}\theta+\pi)) & \text{in } \{\theta \in [0,\theta_k)\} \setminus \mathrm{B}_{r_k}, \\
(\cos(\frac{-\theta_k+\pi}{-\theta_k}\theta+\pi), \sin(\frac{-\theta_k+\pi}{-\theta_k}\theta+\pi)) & \text{in } \{\theta \in (-\theta_k, 0)\} \setminus \mathrm{B}_{r_k}, \\
(\cos(\frac{r}{r_k}(\frac{\theta_k-\pi}{\theta_k}\theta)+\pi), \sin(\frac{r}{r_k}(\frac{\theta_k-\pi}{\theta_k}\theta)+\pi)) & \text{in } \Omega_4 := \mathrm{B}_{r_k} \cap \{\theta \in [0,\theta_k)\}, \\
(\cos(\frac{r}{r_k}(\frac{-\theta_k+\pi}{-\theta_k}\theta)+\pi), \sin(\frac{r}{r_k}(\frac{-\theta_k+\pi}{-\theta_k}\theta)+\pi)) & \text{in } \Omega_4 := \mathrm{B}_{r_k} \cap \{\theta \in (-\theta_k, \theta_k)\},
\end{cases} \tag{10.1}
$$

where (r_k) and (θ_k) are two infinitesimal sequences of positive numbers; see Fig. 13. Recall that in the previous sections we introduced the number ε; we may assume here that $r_k << \varepsilon$ for all $k \in \mathbb{N}$. Notice that $u_k(0,0) = (-1,0) = u_k(r,0)$ for $r \in (0,l)$. Moreover for $t \in (0,l)$ we have $u_k(\partial\mathrm{B}_t) = \partial B_1 \setminus \{\theta \in (-\theta_k, \theta_k)\}$, and the degree of u_k is zero.

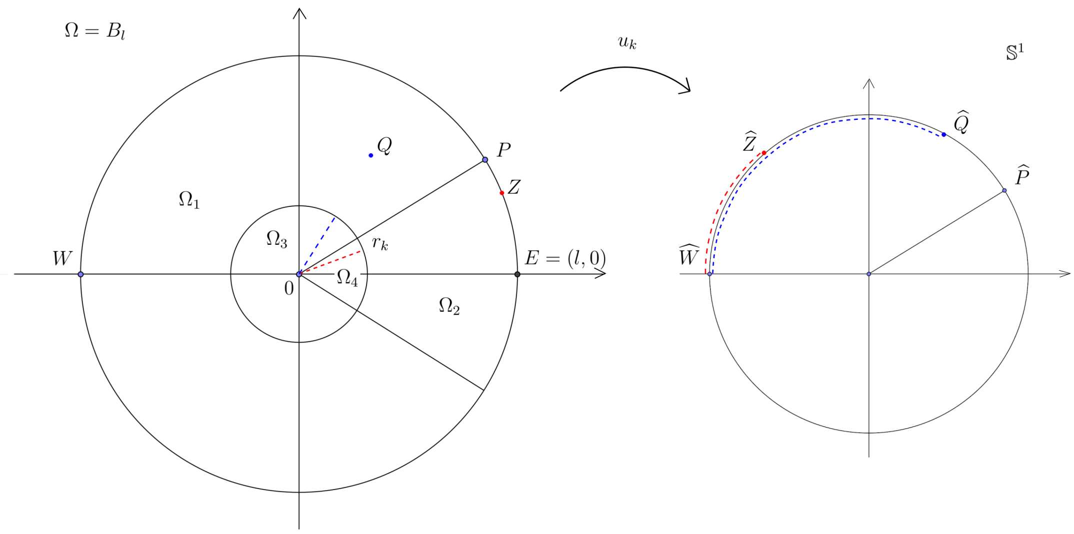

Figure 13: The map u_k in (10.1). We set $\hat{P} := P/|P| = \theta_k$, $\hat{Q} := Q/|Q|$. All points in $\Omega_1 \cup \Omega_2$ are retracted on $\mathbb{S}^1$. The image of Ω_3 through u_k is as follows: u_k sends the generic dotted segment onto the (long) dotted arc on $\mathbb{S}^1$. Finally, the image of Ω_4 through u_k is as follows: u_k sends the generic dotted segment onto the (short) dotted arc on $\mathbb{S}^1$.

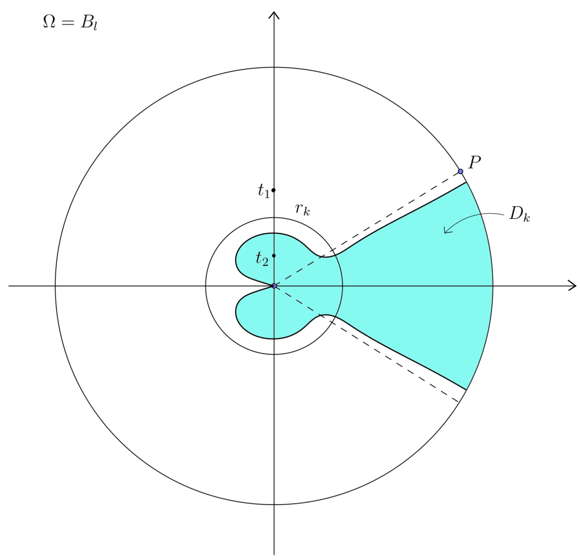

Figure 14: The set emphasized is D_k for the sequence in (10.1).

Now we fix an infinitesimal sequence (λ_k) of positive numbers. Inspecting (10.1), it turns out that the set D_k defined in (4.27) and (4.9) satisfies $D_k \subsetneq \mathrm{B}_{r_k} \cup \{\theta \in (-\theta_k, \theta_k)\} \cap \Omega$, see Fig. 14.

Notice that $\mathrm{spt}(u_{k\sharp}[\![(\Omega \setminus D_k) \cap \partial \mathrm{B}_t]\!]) \subseteq u_k((\Omega \setminus D_k) \cap \partial \mathrm{B}_t)$ with strict inclusion whenever t is such that $(\Omega \setminus \overline{D_k}) \cap \{\theta = \pm\theta_k\} \cap \partial \mathrm{B}_t \neq \emptyset$. For instance, for $t \in (r_k, l)$, we have $u_k((\Omega \setminus D_k) \cap \partial \mathrm{B}_t) = \partial B_1 \setminus \{\theta \in (-\theta_k, \theta_k)\}$ and $u_k(D_k \cap \partial \mathrm{B}_t) = \partial B_1 \setminus \{\theta \in (-\theta_k - a_k, \theta_k + a_k)\}$, where a_k is a positive small angle; on the other hand[48] $\mathrm{spt}(u_{k\sharp}[\![(\Omega \setminus D_k) \cap \partial \mathrm{B}_t]\!]) = \partial B_1 \setminus \{\theta \in (-\theta_k - a_k, \theta_k + a_k)\} = \mathrm{spt}(u_{k\sharp}[\![D_k \cap \partial \mathrm{B}_t]\!])$. Similarly we have $\mathrm{spt}(u_{k\sharp}[\![D_k \cap \partial \mathrm{B}_t]\!]) \subseteq u_k(D_k \cap \partial \mathrm{B}_t)$ with strict inclusion[49] whenever $D_k \cap \{\theta = \pm\theta_k\} \cap \partial \mathrm{B}_t \neq \emptyset$ (thus, $t \in (0, r_k)$).

The support of the current $\mathfrak{D}_k = (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k]\!]$ in (5.6) is a connected set contained in $\partial_{\mathrm{lat}}(C_l(1-\lambda_k))$. But in this example we also have $\mathrm{spt}(\mathfrak{D}_k) = \mathrm{spt}\left((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]\right)$; moreover

$$\mathrm{spt}(\mathfrak{D}_k) \cap C_l^\varepsilon = (\varepsilon, l) \times \{1 - \lambda_k\} \times ((-\pi, \pi) \setminus (-\theta_k - a_k, \theta_k + a_k)).$$

The support of the current $\mathcal{W}_k$ in (5.17), an orthogonal "wall" over $C_l(1-\lambda_k)$ of height $\lambda_k - \lambda'_k$, built on $\partial \mathfrak{D}_k$, divides $C_l(1-\lambda'_k) \setminus C_l(1-\lambda_k)$ into two connected sets; one of them, $\mathrm{spt}(\mathcal{Y}_k)$ (see (5.26)), has $\mathrm{spt}(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]$ as part of its boundary, and the other one, $\mathrm{spt}(\mathcal{X}_k)$ (see (5.27)), has $\partial C_l(1-\lambda_k) \setminus \mathrm{spt}((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!])$ as part of its boundary. Hence the support of $\widehat{\mathfrak{D}}_k = \mathfrak{D}_k + \mathcal{W}_k$ (Definition 5.13) divides $C_l(1-\lambda'_k)$ into two connected sets, one of them, E_k, is $C_l(1-\lambda_k) \cup \mathrm{spt}(\mathcal{X}_k)$, and the second one is its complement in $C_l(1-\lambda'_k)$ and equals $\mathrm{spt}(\mathcal{Y}_k)$. In this particular example the cylindrical Steiner symmetrization introduced in Section 3 is unnecessary, since $\mathbb{S}(E_k) = E_k$ and hence $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \widehat{\mathfrak{D}}_k$. Thus the function $\vartheta_{k,\varepsilon}$ in (7.1) reads as

$$\vartheta_{k,\varepsilon}(t,\rho) = \frac{\Theta_k(t,\rho)}{2} = \frac{1}{2\rho}\mathcal{H}^1((E_k)_{t,\rho}) = \begin{cases} \theta_k + a_k & \text{for } (t,\rho) \in (\varepsilon, l) \times (1-\lambda_k, 1-\lambda'_k], \\ \pi & \text{for } (t,\rho) \in (\varepsilon, l) \times (0, 1-\lambda_k]. \end{cases}$$

Note that

$$\begin{aligned} \mathrm{spt}(\widehat{\mathfrak{D}}_k) \cap C_l^\varepsilon &= (\mathrm{spt}(\mathfrak{D}_k) \cup \mathrm{spt}(\mathcal{W}_k)) \cap C_l^\varepsilon \\ &= \Big(\mathrm{spt}(\mathfrak{D}_k) \cup \Big((\varepsilon, l) \times [1-\lambda_k, 1-\lambda'_k] \times \{-\theta_k - a_k, \theta_k + a_k\}\Big)\Big) \cap C_l^\varepsilon, \end{aligned}$$

and that $\mathrm{spt}(\widehat{\mathfrak{D}}_k) \cap (\{0\} \times \mathbb{R}^2)$ is the segment $\{0\} \times [1-\lambda_k, 1-\lambda'_k] \times \{\pi\}$.

Concerning the functions in (6.1), we have $|u_k|^+ = |u_k|^- = 1$, thus $\pi_0^{\mathrm{pol}}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) = (\varepsilon, l) \times \{1-\lambda_k\} \times \{0\}$, and the sets $Q_{k,\varepsilon}$ in (6.4) and $J_{Q_{k,\varepsilon}}$ in (7.21) are empty. Hence, for $S^{(2)}_{k,\varepsilon}$ in (7.7), we have $S^{(2)}_{k,\varepsilon} = (\varepsilon, l) \times [1-\lambda_k, 1-\lambda'_k] \times \{0\}$. Moreover, for Θ_k in (6.7), we have $\Theta_k(t,\rho) \in (0,\pi)$ for $(t, \rho, 0) \in S^{(2)}_{k,\varepsilon}$; thus, recalling Definitions 7.7 and 7.12, and since $[\![G^{\mathrm{pol}}_{\pm\vartheta_{k,\varepsilon} \llcorner S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0, 2\pi\}\}}]\!] = 0$,

$$\mathcal{G}^{(3)}_{\pm\vartheta_{k,\varepsilon}} = \mathcal{G}^{\pm}_{k,\varepsilon} = \widehat{\mathfrak{D}}_k \llcorner \{(t, \rho, \theta) : t \in (\varepsilon, l), \ \ \pm\theta \in (0, \pi)\},$$

where the last equality follows by construction (see (7.4) and (5.31)).

The set $\Sigma_{k,\varepsilon}$ in Definition 7.14 equals

$$\Sigma_{k,\varepsilon} = (\varepsilon, l) \times \{1 - \lambda'_k\} \times (-\theta_k - a_k, \theta_k + a_k),$$

hence the current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] = \widehat{\mathfrak{D}}_k + [\![\Sigma_{k,\varepsilon}]\!]$ is boundaryless in C_l^ε (with $\Sigma_{k,\varepsilon}$ suitably oriented).

[48] This is due to the fact that $u_k|_{\Omega \setminus D_k}$ covers the arcs $(\partial B_1) \cap \{\theta \in (\theta_k, \theta_k + a_k) \cup (-\theta - a_k, -\theta)\}$ twice, with opposite orientations.

[49] This could only happen for $t \in (0, r_k)$.

Recalling (8.12), we have

$$\mathcal{V}_{k,\varepsilon} = [\![\{\varepsilon\} \times [1-\lambda_k, 1-\lambda_k'] \times ((-\pi,\pi] \setminus (-\theta_k - a_k, \theta_k + a_k))]\!],$$

hence

$$\partial \mathcal{V}_{k,\varepsilon} = \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 + \mathcal{L}_k,$$

see (8.4), (8.10), (8.8), (8.15), and Proposition 8.8,

$$\begin{aligned}
\mathbb{S}(\mathcal{H}_{k,\varepsilon}) &= \mathcal{H}_{k,\varepsilon} = (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\partial \mathrm{B}_\varepsilon \cap D_k]\!] = (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\{\varepsilon\} \times (-\theta + a_k, \theta - a_k)]\!] \\
&= [\![\{\varepsilon\} \times \{1-\lambda_k\} \times ((-\pi,\pi] \setminus (-\theta_k - a_k, \theta_k + a_k))]\!] \text{ (clockwise oriented when looking at} \\
&\quad \text{the plane } \{\varepsilon\} \times \mathbb{R}^2 \text{ from } t > \varepsilon), \\
\mathbb{S}(L)_1 &= [\![\{\varepsilon\} \times [1-\lambda_k, 1-\lambda_k'] \times \{\theta_k + a_k\}]\!], \\
\mathbb{S}(L)_2 &= [\![\{\varepsilon\} \times [1-\lambda_k, 1-\lambda_k'] \times \{-\theta_k - a_k\}]\!], \\
\mathcal{L}_k &= [\![\{\varepsilon\} \times \{1-\lambda_k'\} \times ((-\pi,\pi] \setminus (-\theta_k - a_k, \theta_k + a_k))]\!] \\
&\quad \text{(counterclockwise oriented when looking at the plane } \{\varepsilon\} \times \mathbb{R}^2 \text{ from } t > \varepsilon).
\end{aligned}$$

Notice that

$$\partial \left(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] \right) \llcorner \left(\{\varepsilon\} \times \mathbb{R}^2 \right) = -\Big(\mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 \Big) + [\![\{\varepsilon\} \times \partial B_{1-\lambda_k'}]\!] - \mathcal{L}_k,$$

Hence

$$\partial \left(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} \right) \llcorner \left(\{\varepsilon\} \times \mathbb{R}^2 \right) = [\![\{\varepsilon\} \times \partial B_{1-\lambda_k'}]\!]$$

Thus, for $\widehat{\vartheta}_{k,\varepsilon}$ in (9.5),

$$\widehat{\vartheta}_{k,\varepsilon}(t,\rho) = \begin{cases} \theta_k + a_k & \text{in } (\varepsilon, l) \times (1-\lambda_k, 1-\lambda_k'] \times \{0\}, \\ 0 & \text{in } (\varepsilon, l) \times (1-\lambda_k', 1] \times \{0\}, \\ \pi & \text{in } ((0,\varepsilon] \times (0,1]) \cup ((\varepsilon, l) \times (0, 1-\lambda_k]) \times \{0\}, \end{cases}$$

and $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$ in (9.8) is given by

$$\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}} = \mathcal{G}^{(3)}_{\pm\vartheta_{k,\varepsilon}} + \Big([\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0,\varepsilon) \times \partial B_1]\!] \Big) \llcorner \{0 \le \pm\theta \le \pi\} \pm [\![(\varepsilon, l) \times [1-\lambda_k', 1] \times \{0\}]\!],$$

where $\mathsf{a}_{k,\varepsilon} = [\![\{\varepsilon\} \times (B_1 \setminus B_{1-\lambda_k'})]\!]$. Observe that

$$(\{0\} \times \{1\} \times \{\theta \in [0,\pi]\}) \cup \{[0,l] \times \{1\} \times \{0\}\} \subset \operatorname{supp}(\partial \mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}).$$

Hence

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = \mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0,\varepsilon) \times \partial B_1]\!],$$

and

$$\partial(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}) \llcorner \{t < l\} = [\![\{0\} \times \partial B_1]\!]$$

Remark 10.1. (u_k) is not a recovery sequence, due to Theorem 13.2. We have

$$\lim_{k \to +\infty} \mathcal{A}(u_k, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| \, dx + 2\pi l,$$

and $2\pi l$ has the meaning of the lateral area of the cylinder of height l and basis the unit disc. This surface is not a minimizer of the problem on the right-hand side of (11.25) (where it corresponds to $h \equiv 1$).

10.2 An approximating sequence of maps with degree zero: catenoid union a flap

In this section we discuss another example of a possible approximating sequence (u_k). We replace the cylinder lateral surface[50] $[0,l] \times \{1\} \times (-\pi,\pi]$, which contains the image of $(r_k,l) \times (-\theta_k,\theta_k)$ through the map Ψ_k in the example of Section 10.1, with half[51] of a catenoid union a flap (see Fig. 16): calling this union $CF \llcorner (0,l) \times \mathbb{R}^2$, we have

$$CF =: \{(t,\overline{\rho}(t),\theta) : t \in [0,2l],\ \theta \in (-\pi,\pi]\} \cup \{(t,r,0) : t \in (0,2l),\ r \in [\overline{\rho}(t),1]\},$$

where $\overline{\rho}(t) := a\cosh(\frac{t-l}{a})$, and $a > 0$ is such that $\overline{\rho}(0) = 1$ (and $\overline{\rho}(2l) = 0$).

Notice that CF "spans" $\Big(\{0,2l\} \times \{1\} \times (-\pi,\pi]\Big) \cup \Big([0,2l] \times \{1\} \times \{0\}\Big)$, which is the union of two unit circles connected by a segment.

Let $r_k > 0, \theta_k > 0$, $\overline{\theta}_k > \theta_k$ be such that $r_k, \theta_k, (\overline{\theta}_k - \theta_k) \to 0^+$ as $k \to +\infty$. Set

$$\rho(t) := \overline{\rho}\left(\frac{t-r_k}{l-r_k} l\right), \qquad t \in (r_k,l).$$

We define $u_k := u$ in $\Omega \setminus \Big(\mathrm{B}_{r_k} \cup \{\theta \in (-\overline{\theta}_k,\overline{\theta}_k)\}\Big)$, in particular

$$u_k(\partial \mathrm{B}_t \setminus \{\theta \in (-\overline{\theta}_k,\overline{\theta}_k)\}) = \partial B_1 \setminus \{\theta \in (-\overline{\theta}_k,\overline{\theta}_k)\}, \qquad t \in (r_k,l).$$

On $\{\theta \in (-\overline{\theta}_k,\overline{\theta}_k)\} \setminus \mathrm{B}_{r_k}$ we define u_k in such a way that for each $t \in (r_k,l)$ we have

$$\begin{aligned} &u_k\left(\partial \mathrm{B}_t \cap \{\pm\theta \in (\theta_k,\overline{\theta_k})\}\right) = \partial B_1 \cap \{\pm\theta \in (0,\overline{\theta}_k)\}, \\ &u_k\left(\partial \mathrm{B}_t \cap \{\pm\theta \in (0,\theta_k)\}\right) = \{(r,0) \in B_1 : r \in [\rho(t),1]\} \cup \Big(\partial B_{\rho(t)} \cap \{\pm\theta \in (0,\pi)\}\Big). \end{aligned}$$

See Fig. 15 for a representation of the map u_k. To define u_k on B_{r_k} we adopt a construction similar to the one in (10.1). First of all, $u_k(0,0) := (-1,0)$. Then, in $\mathrm{B}_{r_k} \cap \{\theta \in (-\pi,\pi) \setminus (-\overline{\theta}_k,\overline{\theta}_k)\}$ we impose u_k as in (10.1) with $\overline{\theta}_k$ replacing θ_k. In $\mathrm{B}_{r_k} \cap \{\theta \in (-\overline{\theta}_k,\overline{\theta}_k)\}$ we require

$$u_k([0,r_k],\alpha) := \partial B_1 \cap \{\pm\theta \in (u_k(r_k,\alpha),\pi)\}, \qquad \pm\alpha \in (0,\pi].$$

Hence

$$u_k(\partial \mathrm{B}_t) \begin{cases} \subsetneq \partial B_1 & \text{if } t \in (0,r_k], \\ = (\partial B_1) \cup \{(r,0) \in \mathrm{B}_1 : r \in [\rho(t),1]\} \cup (\partial B_{\rho(t)}) & \text{if } t \in (r_k,l). \end{cases}$$

Remark 10.2. For an explicit construction, see Section 13 with $h^\star = \overline{\rho}$ and

$$\psi^\star(s,t) = \begin{cases} 0 & \text{for } t \in (0,l),\ s \in [-1,-\overline{\rho}(t)], \\ \sqrt{(\overline{\rho}(t))^2 - s^2} & \text{for } t \in (0,l),\ s \in [-\overline{\rho}(t),\overline{\rho}(t)]. \end{cases}$$

Now we fix an infinitesimal sequence (λ_k) of positive numbers; we may also assume that $|(\cos\overline{\theta}_k, \sin\overline{\theta}_k) - (1,0)| << \lambda_k$. Hence $D_k \subsetneq (\mathrm{B}_{r_k} \cup \{\theta \in (-\theta_k,\theta_k)\})$.

[50] In polar coordinates.

[51] For convenience, we consider the doubled segment $[0,2l]$, in order to define the catenoid; then we restrict the construction to $(0,l)$.

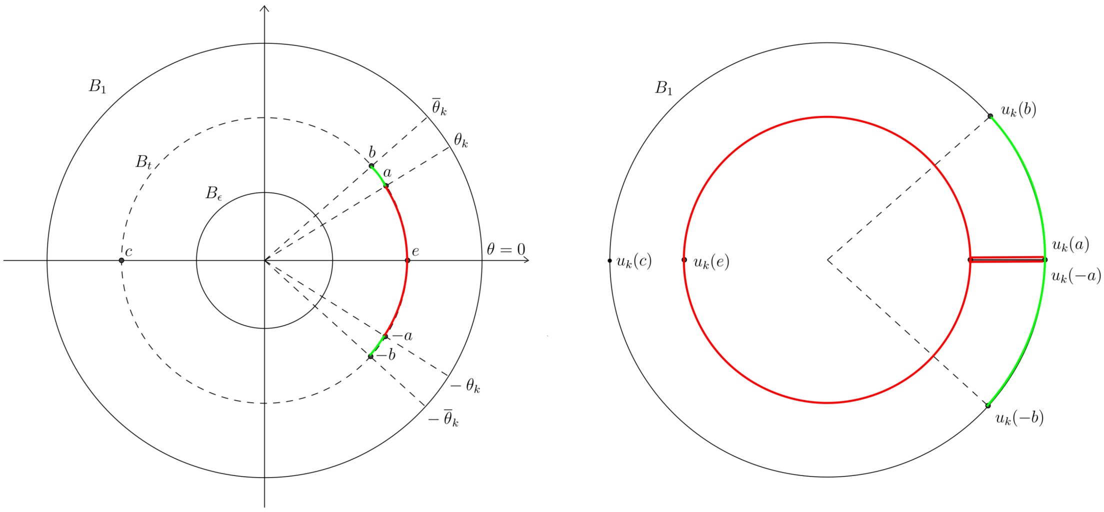

Figure 15: Source and target of the map u_k in the example of Section 10.2. The small interior circle in the right figure is a t-slice of a catenoid, whereas the horizontal segment is the t-section of the flap.

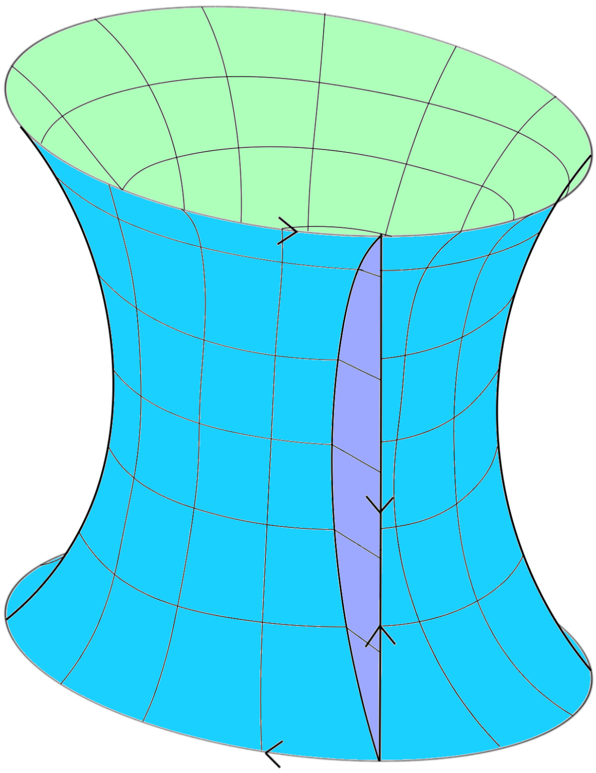

Figure 16: Catenoid union a flap (Section 10.2). This is the set CF, namely the limit (as $k \to +\infty$) of the image by $\pi_{\lambda_k} \circ \Psi_k$ of D_k.

By construction there exists $t_k \in (r_k, \varepsilon)$ such that $\rho(t_k) = 1 - \lambda_k$. Hence for $t \in [t_k, l)$ we have

$$\operatorname{spt}\Big((u_k)_\sharp(\partial \mathrm{B}_t \cap (\Omega \setminus D_k))\Big) = \partial B_1, \qquad \partial B_1 \subsetneq u_k(\partial \mathrm{B}_t \cap (\Omega \setminus D_k)),$$
$$\operatorname{spt}\Big((u_k)_\sharp(\partial \mathrm{B}_t \cap D_k)\Big) = \partial B_{\rho(t)}, \qquad \partial B_{\rho(t)} \subsetneq u_k(\partial \mathrm{B}_t \cap D_k).$$

Notice that the above strict inclusions are due to the fact that the segments $\{(r,0) \in B_1 : r \in [\rho(t), 1]\} \cap u_k(D_k)$ and $\{(r,0) \in B_1 : r \in [\rho(t), 1]\} \cap u_k(\Omega \setminus D_k)$ are covered twice with opposite orientation.

On the other hand, for $t \in [r_k, t_k)$ we have

$$\partial B_1 \subset \operatorname{spt}\Big((u_k)_\sharp(\partial \mathrm{B}_t \cap (\Omega \setminus D_k))\Big), \qquad \operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp(\partial \mathrm{B}_t \cap (\Omega \setminus D_k))\Big) \subset \partial B_{1-\lambda_k}$$
$$\partial B_{\rho(t)} \supset \operatorname{spt}\Big((u_k)_\sharp(\partial \mathrm{B}_t \cap D_k)\Big), \qquad \operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp(\partial \mathrm{B}_t \cap D_k)\Big) \subset \partial B_{1-\lambda_k},$$

where the second inclusion is due to the fact that $(\pi_{\lambda_k} \circ \Psi_k)(\partial \mathrm{B}_t \cap (\Omega \setminus D_k))$ covers two arcs of $\partial B_{1-\lambda_k}$ twice with opposite orientation. Notice also that the only cancellation that could happen on $(\pi_{\lambda_k} \circ \Psi_k)(\partial \mathrm{B}_t \cap D_k)$ due to covering more than one with opposite orientation is along the segment $\{(r,0) \in B_1 : r \in [\rho(t), 1]\}$, in particular we have

$$\operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp(\partial \mathrm{B}_t \cap (\Omega \setminus D_k))\Big) = \operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp(\partial \mathrm{B}_t \cap D_k)\Big).$$

By definition the above equality holds also for $t \in (0, r_k)$; moreover we have

$$\operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp(\partial \mathrm{B}_{t_1} \cap D_k)\Big) \subsetneq \operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp(\partial \mathrm{B}_{t_2} \cap D_k)\Big), \qquad t_1 < t_2,\ t_1, t_2 \in (0, r_k),$$

where $\pi_{\lambda_k} \circ \Psi_k(0) = \{0\} \times \{1 - \lambda_k\} \times \{\pi\}$.

From the above discussion we can see that the support of the current $\mathfrak{D}_k$ is a connected set contained in $\overline{C}_l(1 - \lambda_k)$ and the support of $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]$ is a connected set contained in $\partial_{\mathrm{lat}} C_l(1 - \lambda_k)$, in particular we have

$$\begin{aligned}
&\operatorname{spt}\Big(\mathfrak{D}_k\Big) \llcorner C_{t_k} = \operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]\Big) \llcorner C_{t_k} \ \subsetneq \partial C_{t_k}(1 - \lambda_k),\\
&\operatorname{spt}\Big(\mathfrak{D}_k\Big) \llcorner C_l^{t_k} = \{(t, r, \theta) : t \in (t_k, l),\ r = \rho(t),\ \theta \in (-\pi, \pi]\} \subsetneq C_l^{t_k}(1 - \lambda_k),\\
&\operatorname{spt}\Big(\partial \mathfrak{D}_k\Big) \subset \partial C_{t_k}(1 - \lambda_k),\\
&\operatorname{spt}\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]\Big) \llcorner C_l^{t_k} = \partial C_l^{t_k}(1 - \lambda_k).
\end{aligned}$$

The current $\mathcal{W}_k$, a normal wall over $\partial \mathfrak{D}_k$ over $C_l(1 - \lambda_k)$ of height $\lambda_k - \lambda_k'$, built on $\partial \mathfrak{D}_k$, divides $C_l(1-\lambda_k') \setminus C_l(1-\lambda_k)$ into two connected sets; one of them, $\operatorname{spt}(\mathcal{Y}_k)$, contains $C_l^{t_k}(1-\lambda_k') \setminus C_l^{t_k}(1-\lambda_k)$ and has $\operatorname{spt}\left((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]\right)$ as part of its boundary. The other one, $\operatorname{spt}(\mathcal{X}_k)$ (see (5.27)), contains $C_0(1-\lambda_k') \setminus C_0(1-\lambda_k)$ and has $\partial C_l(1-\lambda_k) \setminus \operatorname{spt}(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]$ as part of its boundary. Hence $\widehat{\mathfrak{D}}_k = \mathfrak{D}_k + \mathcal{W}_k$ divides $C_l(1 - \lambda_k')$ into two connected sets, one of them is

$$E_k = C_{t_k}(1 - \lambda_k) \cup \operatorname{spt}(\mathcal{X}_k) \cup \{(t, r, \theta) \in C_l^{t_k}(1 - \lambda_k) : r \in (0, \rho(t)]\},$$

and the second one is its complement in $C_l(1 - \lambda_k')$ and contains $\operatorname{spt}(\mathcal{Y}_k)$. Note that $\operatorname{spt}(\mathcal{W}_k) \subset [0, t_k] \times [1-\lambda_k, 1-\lambda_k'] \times (-\pi, \pi]$ and that $\operatorname{spt}(\widehat{\mathfrak{D}}_k) \cap (\{0\} \times \mathbb{R}^2)$ is the segment $\{0\} \times [1-\lambda_k, 1-\lambda_k'] \times \{\pi\}$.

In this particular example when we apply the cylindrical Steiner symmetrization introduced in Section 3 nothing changes, *i.e.*, $\mathbb{S}(E_k) = E_k$ and hence $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \widehat{\mathfrak{D}}_k$. Observe that

$$|u_k|^+ = 1, \qquad |u_k|^- = \begin{cases} 1, & r \le r_k, \\ \rho(t), & r > r_k. \end{cases}$$

Thus

$$\pi_0^{\mathrm{pol}}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) = \{(t, r, 0) : t \in (\varepsilon, l)\ r \in [\rho(t), 1 - \lambda_k]\},$$

and the sets $Q_{k,\varepsilon}$ in (6.4) and $J_{Q_{k,\varepsilon}}$ in (7.21) are empty. Hence

$$S^{(2)}_{k,\varepsilon} = \{(t, r, 0) : t \in (\varepsilon, l)\ r \in [\rho(t), 1 - \lambda'_k]\}.$$

Moreover we have $\Theta_k(t, \rho) = 0$ for $(t, \rho, 0) \in S^{(2)}_{k,\varepsilon}$ thus, recalling Definitions 7.7 and 7.12,

$$\begin{aligned} \mathcal{G}^{(3)}_{\pm\vartheta_{k,\varepsilon}} = \mathcal{G}^{\pm}_{k,\varepsilon} &= \widehat{\mathfrak{D}}_k \llcorner \{(t, \rho, \theta) : t \in (\varepsilon, l),\ \ \pm\theta \in (0, \pi)\} + [\![G^{\mathrm{pol}}_{\pm\vartheta_{k,\varepsilon} \llcorner S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0, 2\pi\}\}}]\!] \\ &= \widehat{\mathfrak{D}}_k \llcorner \{(t, \rho, \theta) : t \in (\varepsilon, l),\ \ \pm\theta \in (0, \pi)\} \pm [\![\{(t, r, 0) : t \in (\varepsilon, l),\ r \in [\rho(t), 1 - \lambda'_k]\}]\!]. \end{aligned}$$

The set $\Sigma_{k,\varepsilon}$ in Definition 7.14 is empty, hence $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] = \widehat{\mathfrak{D}}_k$ which is boundaryless in C^ε_l.

We have, recalling (8.12),

$$\mathcal{V}_{k,\varepsilon} = [\![\{\varepsilon\} \times [\rho(\varepsilon), 1 - \lambda'_k] \times (-\pi, \pi]]\!],$$

i.e.,

$$\partial \mathcal{V}_{k,\varepsilon} = \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 + \mathcal{L}_k = -[\![\{\varepsilon\} \times \partial B_{\rho(\varepsilon)}]\!] + [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!],$$

where (see (8.10) (8.4), (8.8), (8.15), and Proposition 8.8)

$$\begin{aligned} \mathbb{S}(\mathcal{H}_{k,\varepsilon}) = \mathcal{H}_{k,\varepsilon} &= (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k \cap \partial \mathrm{B}_\varepsilon]\!] \text{ (oriented counterclockwise)} = -[\![\{\varepsilon\} \times \partial B_{\rho(\varepsilon)}]\!], \\ \mathbb{S}(L)_1 = \mathbb{S}(L)_2 &= [\![\{\varepsilon\} \times [1 - \lambda_k, 1 - \lambda'_k] \times \{0\}]\!], \\ \mathcal{L}_k &= [\![\overline{Y_2 Y_1}]\!] = [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!]. \end{aligned}$$

Notice that

$$\begin{aligned} \partial(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}) \llcorner \{\varepsilon\} \times \mathbb{R}^2 &= [\![\{\varepsilon\} \times \partial B_{1-\lambda_k}]\!] \\ &= -\Big(\mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 \Big) + [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!] - \mathcal{L}_k. \end{aligned}$$

Thus

$$\partial(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + \mathcal{V}_{k,\varepsilon}) \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!],$$

and

$$\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}} = \mathcal{G}^{(3)}_{\pm\vartheta_{k,\varepsilon}} + \Big(\mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0, \varepsilon) \times \partial B_1]\!] \Big) \llcorner \{0 \le \pm\theta \le \pi\} \pm [\![(\varepsilon, l) \times [1 - \lambda'_k, 1] \times \{0\}]\!],$$

where $\mathsf{a}_{k,\varepsilon} = [\![\{\varepsilon\} \times (B_1 \setminus B_{1-\lambda'_k})]\!]$; observe that

$$(\partial B_1 \cap \{0 \le \pm\theta \le \pi\}) \cup \{[0, l] \times \{1\} \times \{0\}\} \subset \operatorname{supp}(\partial \mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}).$$

Hence

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = \mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0, \varepsilon) \times \partial B_1]\!],$$

and

$$\partial(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}) \llcorner \{t < l\} = [\![\{0\} \times \partial B_1]\!].$$

Remark 10.3. (u_k) is not a recovery sequence, due to Theorem 13.2. We have

$$\lim_{k\to+\infty} \mathcal{A}(u_k,\Omega) = \int_\Omega |\mathcal{M}(\nabla u)|\, dx + \mathcal{H}^2(\text{catenoid}) + 2\,\mathcal{H}^2(\text{flap}).$$

This surface is not a minimizer of problem on the right-hand side of (11.25).

10.3 Smoothing by convolution: the case of the two discs

In [1], the authors describe a sequence (u_k) of maps converging to the vortex map u, simply defined as follows:

$$u_k(r,\theta) := \phi_k(r)u(r,\theta), \tag{10.2}$$

where $\phi_k : [0,l] \to [0,1]$ is a smooth function such that $\phi_k = 0$ in $[0,\frac{1}{k^2}]$, $\phi_k = 1$ in $[\frac{1}{k},l]$, and $0 \le \phi_k' \le 2k$. Hence $|u_k - u| = 1 - \phi_k$. We shall assume that for all $k > 0$, we have $\frac{1}{k} << \varepsilon$.

Now we fix an infinitesimal sequence (λ_k) of positive numbers, hence for any $k \in \mathbb{N}$ there exists $r_k \in (0,1/k)$ such that $\phi_k(r_k) = 1 - \lambda_k$, and we have

$$D_k = \mathrm{B}_{r_k}. \tag{10.3}$$

Notice that $u_k(\partial \mathrm{B}_r) = \partial B_{\phi_k(r)}$,

$$\begin{aligned}
&\mathrm{spt}(\mathfrak{D}_k) = \{(t,r,\theta) : t \in [1/k^2, r_k],\ r = \phi_k(t),\ \theta \in (-\pi,\pi]\} \subsetneq \overline{C}_{r_k}(1-\lambda_k),\\
&\mathrm{spt}((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]) = \pi_{\lambda_k} \circ \Psi_k(\Omega \setminus D_k) = \partial C_l^{r_k}(1-\lambda_k).
\end{aligned}$$

Also, using (10.3), $\pi_{\lambda_k} \circ \Psi_k(D_k) \setminus \mathrm{spt}(\mathfrak{D}_k) = [0,1/k^2) \times \{0\} \times \{0\} = \pi_{\lambda_k} \circ \Psi_k(\mathrm{B}_{1/k^2})$ and, unlike the examples in Sections 10.1, 10.2, there is no cancellation due to covering the same 2-dimensional set with two opposite orientations; the fact that $\pi_{\lambda_k} \circ \Psi_k(D_k) \setminus \mathrm{spt}(\mathfrak{D}_k)$ is nonempty is due to the fact that $\pi_{\lambda_k} \circ \Psi_k(\mathrm{B}_{1/k^2})$ is one-dimensional. Moreover we have

$$\mathrm{spt}((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\partial D_k]\!]) = \mathrm{spt}(\mathfrak{D}_k) \cap \mathrm{spt}((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]) = \{r_k\} \times \{1-\lambda_k\} \times (-\pi,\pi].$$

Hence

$$\mathrm{spt}(\mathcal{W}_k) = \{r_k\} \times [1-\lambda_k, 1-\lambda_k'] \times (-\pi,\pi],$$

i.e., the current $\mathcal{W}_k$ divides $C_l(1-\lambda_k') \setminus C_l(1-\lambda_k)$ into two connected sets; one of them, $\mathrm{spt}(\mathcal{Y}_k) = C_l^{r_k}(1-\lambda_k') \setminus C_l^{r_k}(1-\lambda_k)$, and the other one, $\mathrm{spt}(\mathcal{X}_k) = C_{r_k}(1-\lambda_k') \setminus C_{r_k}(1-\lambda_k)$. Therefore $\widehat{\mathfrak{D}}_k = \mathfrak{D}_k + \mathcal{W}_k$ divides $C_l(1-\lambda_k')$ into two connected sets, one of them is

$$E_k = C_{r_k}(1-\lambda_k') \setminus \{(t,r,\theta) : t \in (1/k^2, r_k),\ r \in (0,\phi_k(t)),\ \theta \in (-\pi,\pi]\},$$

and the second one is its complement in $C_l(1-\lambda_k')$ and contains $\mathrm{spt}(\mathcal{Y}_k)$.

Also in this example, when we apply the cylindrical Steiner symmetrization introduced in Section 3, nothing changes, *i.e.,* $\mathbb{S}(E_k) = E_k$ and hence $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \widehat{\mathfrak{D}}_k$.

Note also that $|u_k|^+ = |u_k|^- = 1$ in (r_k,l), thus $\pi_0^{\mathrm{pol}}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) = (\varepsilon,l) \times \{1-\lambda_k\} \times \{0\}$, and the sets $Q_{k,\varepsilon}$ in (6.4) and $J_{Q_{k,\varepsilon}}$ in (7.21) are empty. Hence $S_{k,\varepsilon}^{(2)} = (\varepsilon,l) \times [1-\lambda_k, 1-\lambda_k'] \times \{0\}$. We have $\Theta_k(t,\rho) = 0$ for $(t,\rho,0) \in S_{k,\varepsilon}^{(2)}$ thus, recalling Definitions 7.7 and 7.12,

$$\mathcal{G}^{(3)}_{\pm\vartheta_{k,\varepsilon}} = \mathcal{G}^{\pm}_{k,\varepsilon} = \pm [\![S^{(2)}_{k,\varepsilon}]\!] = \pm [\![(\varepsilon,l) \times [1-\lambda_k, 1-\lambda_k'] \times \{0\}]\!].$$

The set $\Sigma_{k,\varepsilon}$ in Definition 7.14 is empty, hence $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] = 0$. We have, recalling (8.12),

$$\mathcal{V}_{k,\varepsilon} = [\![\{\varepsilon\} \times \overline{B}_{1-\lambda_k'}]\!],$$

and

$$\partial\mathcal{V}_{k,\varepsilon} = \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 + \mathcal{L}_k = [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!],$$

where (see (8.10) (8.4), (8.8), (8.15), and Proposition 8.8)

$$\begin{aligned}
&\mathbb{S}(\mathcal{H}_{k,\varepsilon}) = \mathcal{H}_{k,\varepsilon} = 0,\\
&\mathbb{S}(L)_1 = \mathbb{S}(L)_2 = [\![\{\varepsilon\} \times [1-\lambda_k, 1-\lambda'_k] \times \{0\}]\!],\\
&\mathcal{L}_k = [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!].
\end{aligned}$$

Notice that

$$\partial(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}) \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = -\Big(\mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2\Big) + [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!] - \mathcal{L}_k = 0,$$

and

$$\partial(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + \mathcal{V}_{k,\varepsilon}) \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!].$$

Finally we have

$$\begin{aligned}
\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}} =& \mathcal{G}^{(3)}_{\pm\vartheta_{k,\varepsilon}} + \Big(\mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0,\varepsilon) \times \partial B_1]\!]\Big) \llcorner \{0 \le \pm\theta \le \pi\} \pm [\![(\varepsilon, l) \times [1-\lambda'_k, 1] \times \{0\}]\!]\\
=& [\![\{\varepsilon\} \times \overline{B}_1]\!] + [\![(0,\varepsilon) \times \partial B_1]\!] \pm [\![(\varepsilon, l) \times [1-\lambda_k, 1] \times \{0\}]\!],
\end{aligned}$$

where $\mathsf{a}_{k,\varepsilon} = [\![\{\varepsilon\} \times (B_1 \setminus B_{1-\lambda'_k})]\!]$. Notice that

$$(\partial B_1 \cap \{0 \le \pm\theta \le \pi\}) \cup \{[0,l] \times \{1\} \times \{0\}\} \subset \mathrm{supp}(\{\partial\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}\}).$$

Hence

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = [\![\{\varepsilon\} \times \overline{B}_1]\!] + [\![(0,\varepsilon) \times \partial B_1]\!],$$

and

$$\partial(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}) \llcorner \{t < l\} = [\![\{0\} \times \partial B_1]\!].$$

Remark 10.4. (u_k) is a recovery sequence for l sufficiently large, due to [1, Lemma 4.2]. We have

$$\lim_{k\to+\infty} \mathcal{A}(u_k, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \pi,$$

and π has the meaning of the area of the unit disc. This surface, for l sufficiently large, is a minimizer of problem on the right-hand side of (11.25) (where it corresponds to $h \equiv -1$).

11 Lower bound

In this section we reduce the analysis of $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$ in Definition 9.4 to a non-parametric Plateau-type problem with a sort of free boundary. Precisely, after suitable projections, we will arrive to a Plateau-type problem on the closed rectangle $\overline{R}_l$, where

$$R_l := (0, l) \times (-1, 1) \times \{0\}$$

in Cartesian coordinates, equivalently $R_l = \{t \in (0,l),\ \rho \in [0,1),\ \theta = 0\} \cup \{t \in (0,l),\ \rho \in [0,1),\ \theta = \pi\}$ in cylindrical coordinates. The rectangle R_l will be often identified with $(0,l) \times (-1,1)$, thus neglecting the third coordinate. We will impose a Dirichlet boundary condition φ on a part

$$\partial_D R_l := (\{0\} \times [-1,1]) \cup ([0,l] \times \{-1\}) \tag{11.1}$$

of ∂R_l, while no conditions will be imposed on $\{l\} \times (-1,1)$; more involved conditions will be assigned on $(0,l) \times \{1\}$, see the mutual relations between ψ and h in (11.23) (see also the problem on the right-hand side of (11.25) and Section 12).

Then the strategy to estimate from below the relaxed area of the graph of the vortex map u will be the following: We split

$$\mathcal{A}(u_k, \Omega) = \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \, dx + \int_{D_k} |\mathcal{M}(\nabla u_k)| \, dx.$$

In order to estimate the $\liminf_{k \to +\infty}$ of the first term on the right-hand side we will employ (4.37), whereas, in order to pass to the limit as $k \to +\infty$ in the second term, we will use (9.12), so that we first want to render the right-hand side of this latter inequality independent of k. This will be done with the aid of the non-parametric Plateau-type problem studied in this section and in Section 12.

Definition 11.1 (**The projection** p)**.** *We let $p : C_l \cap \{t \geq 0\} \to \overline{R}_l$ be the othogonal projection.*

Recall that $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$ are defined in (9.8), that $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ is the generalized polar graph of $\widehat{\vartheta}_{k,\varepsilon}$ on its domain of definition (see (9.5)), and that $\widehat{\vartheta}_{k,\varepsilon}$ takes values in $[0,\pi]$. We first prove the following preliminary result:

Lemma 11.2. *Let $\varepsilon \in (0,1)$ be as in (4.4), (4.5), and $k \in \mathbb{N}$. Then there is a negligible set $C_{k,\varepsilon} \subset (0,l)$ such that for all $t \in (0,l) \setminus C_{k,\varepsilon}$*

$$p\Big(\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})\Big) \cap (\{t\} \times \mathbb{R}^2) \tag{11.2}$$

is a subinterval of the segment $\overline{R}_l \cap (\{t\} \times \mathbb{R}^2) = \{t\} \times [-1,1] \times \{0\}$ with one endpoint $(t,1,0)$. Moreover $p(\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})) = p(\mathrm{spt}(\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}))$.

Proof. The latter assertion follows by symmetry. To prove the former, we argue by slicing. For a.e. $t \in (0,l)$ the set $\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}) \cap (\{t\} \times \mathbb{R}^2)$ coincides with support of the current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$, see [23, Def. 7.6.2]. First notice that for all $t \in (0,\varepsilon)$ the conclusion follows by construction[52].

It remains to consider the case $t \in (\varepsilon, l)$. Recall that the set $S^{(4)}_{k,\varepsilon}$ in (9.4) has the form

$$\{t \in (\varepsilon, l), \ \rho \in [|u_k|^-(t) \wedge (1-\lambda_k), 1], \ \theta = 0\}.$$

Therefore, for a.e. $t \in (\varepsilon, l)$ the slice $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ is the integration over $\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})$ restricted to the plane $\{t\} \times \mathbb{R}^2$, which in turn is the integration over the generalized polar graph (see (2.8)) of $\widehat{\vartheta}_{k,\varepsilon}$ restricted to the closed set

$$\{t\} \times [|u_k|^-(t) \wedge (1-\lambda_k), 1] \times [0,\pi].$$

Namely

$$(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t = [\![\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}) \cap \big(\{t\} \times [|u_k|^-(t) \wedge (1-\lambda_k), 1] \times [0,\pi]\big)]\!],$$

[52]In this set we have $\vartheta_{k,\varepsilon} = \pi$ and the current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ is the integration over the half-circle $\{t\} \times ((\partial B_1) \cap \{\theta \in (0,\pi)\})$, whose projection through p is the whole interval with endpoints $(t,1,0)$ and $(t,1,0)$ (in cylindrical coordinates).

so that the support σ_t of $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ can also be obtained as

$$\sigma_t = \bigcap_{h=1}^{+\infty} \sigma_t^h, \tag{11.3}$$

where

$$\sigma_t^h := \mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}) \cap \Big(\{t\} \times [\big(|u_k|^-(t) \wedge (1-\lambda_k)\big) - \frac{1}{h}, 1] \times [0,\pi]\Big).$$

For $h \in \mathbb{N}$ large enough, let

$$U_h := \{t\} \times \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, 1\big) \times (-\frac{1}{h}, \pi + \frac{1}{h}),$$

which is a relatively open set in $\{t\} \times B_1$, and let $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ be the slice of $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ on $\{t\} \times B_1$. We have

$$\mathrm{spt}((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h) \subset \sigma_t^h \subset \{t\} \times [\big(|u_k|^-(t) \wedge (1-\lambda_k)\big) - \frac{1}{h}, 1] \times [0,\pi]. \tag{11.4}$$

On the other hand since $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} \llcorner (\{t\} \times B_1)$ is the boundary of the subgraph of $\widehat{\vartheta}_{k,\varepsilon}$ in $\{t\} \times B_1$, it is a closed 1-integral current in U_h and in $(\{t\} \times (B_1 \setminus \overline{B}_{(|u_k|^-(t)\wedge(1-\lambda_k))-\frac{1}{h}})$, so that the boundary $\partial((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h)$ in $\mathcal{D}_1(\{t\} \times B_1)$ is supported on $(\partial U_h) \cap ((\partial B_1) \cup \partial B_{(|u_k|^-(t)\wedge(1-\lambda_k))-\frac{1}{h}})$. From (11.4), the fact that $\widehat{\vartheta}_{k,\varepsilon} = 0$ at $(t,1)$ and that $\widehat{\vartheta}_{k,\varepsilon}$ is constant on the segment $\big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, |u_k|^-(t) \wedge (1-\lambda_k)\big)$ with value either 0 or π, we deduce that

$$\begin{aligned}
&\mathrm{spt}(\partial((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h)) \\
&\subset \big(\{t\} \times \{(|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}\} \times \{0,\pi\}\big) \bigcup \big(\{t\} \times \{1\} \times \{0\}\big).
\end{aligned} \tag{11.5}$$

Moreover, if we set

$$P_1 := (t,1,0) \text{ and } P_2^h := \big(t, (|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, \widehat{\vartheta}_{k,\varepsilon}((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h})\big),$$

from (11.5) it follows that

$$\partial((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h) = \delta_{P_2^h} - \delta_{P_1}.$$

By decomposition of the integral 1-current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h$ (see [15, Section 4.2.25]), there are at most countable Lipschitz curves $\{\alpha_i^h\}$ such that α_0^h connects P_2^h to P_1, and α_i^h is closed for $i > 0$. We claim that there cannot be closed curves α_i^h, namely $\{\alpha_i^h\}_{i\in\mathbb{N}} = \{\alpha_0^h\}$. Indeed, since α_0^h connects P_1 and P_2^h, we see that $(\{t\} \times \partial B_\rho) \cap \alpha_0^h$ consists of at least one point for $\mathcal{H}^1$-a.e. $\rho \in \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, 1\big)$. On the other hand, $(\{t\} \times \partial B_\rho) \cap \sigma_t^h$ consists of only one point[53] for $\mathcal{H}^1$-a.e. $\rho \in \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, 1\big)$. So there cannot be other curves α_i^h otherwise the last condition will be violated.

From the claim we deduce that the current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h$ is the integration over a simple curve α_0^h connecting P_2^h and P_1, and its support coincides with σ_t^h. Now, from (11.3) and the fact that σ_t is a segment on $\{t\} \times \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, |u_k|^-(t) \wedge (1-\lambda_k)\big)$, we conclude that also σ_t must be a unique curve, say α_0, connecting P_1 to $P_2 := \lim_{h\to\infty} P_2^h$. By continuity of the projection by p, α_0 is an interval with one endpoint in $p(P_1) = (t,1,0)$, for a.e. $t \in (\varepsilon, l)$. $\square$

[53]Because σ_t^h is the support of a polar graph; the points where this intersection is not a singleton coincide with the values of ρ where $\widehat{\vartheta}_{k,\varepsilon}$ has a jump.

The new coordinates (w_1, w_2, w_3)**.** In what follows, it is convenient to revert the rectangle with respect to its second coordinate: if $(t, \rho, \theta) \in [0, l] \times [0, 1] \times (-\pi, \pi]$ are the cylindrical coordinates in the cylinder C_l exploited so far, we introduce Cartesian coordinates $(w_1, w_2, w_3) \in [0, l] \times [-1, 1] \times [-1, 1]$ defined as

$$w_1 := t, \quad w_2 := -\rho \cos\theta, \quad w_3 := -\rho \sin\theta, \tag{11.6}$$

in such a way that the segment $\{0 \le t \le l, \rho = 1, \theta = 0\}$ coincides with the bottom edge $[0, l] \times \{-1\} \times \{0\}$ of the rectangle $\overline{R}_l$.

Thanks to Lemma 11.2 we are allowed to give the following

Definition 11.3 (The function $h_{k,\varepsilon}$**).** *Let $\varepsilon \in (0, 1)$ be as in (4.4), (4.5), and $k \in \mathbb{N}$. We define $h_{k,\varepsilon} : [0, l] \to [-1, 1]$ as*

$$h_{k,\varepsilon}(w_1) := \mathcal{H}^1\Big(p\big(\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})\big) \cap (\{w_1\} \times \mathbb{R}^2)\Big) - 1.$$

For all $w_1 \in (0, l)$ for which Lemma 11.2 is valid, we have that $1 + h_{k,\varepsilon}(w_1)$ equals the length of the interval in (11.2). Now the content of Lemma 11.2 is that the p-projection of $\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})$ on $\overline{R}_l$ is of the form

$$p\Big(\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})\Big) = SG_{h_{k,\varepsilon}} := \{(w_1, w_2) \in R_l : w_1 \in (0, l), w_2 \in (-1, h_{k,\varepsilon}(w_1))\}, \tag{11.7}$$

up to a set of zero $\mathcal{H}^2$-measure. The function $h_{k,\varepsilon}$ is built in such a way that $(w_1, -1)$ and $(w_1, h_{k,\varepsilon}(w_1))$ are the endopoints of the interval $p(\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})) \cap (\{w_1\} \times \mathbb{R}^2)$ for almost every $w_1 \in (0, l)$. Observe that

$$h_{k,\varepsilon} \ge -1 + \lambda'_k \qquad \text{in } (\varepsilon, l),$$

and

$$h_{k,\varepsilon} = 1 \qquad \text{in } (0, \varepsilon).$$

Indeed, from Definition 9.2, equation (9.4) and Definition 9.4, we see that the set $\Big((0, l) \times [1 - \lambda'_k, 1] \times \{0\}\Big) \cup \Big((0, \varepsilon) \times [-1, 1] \times \{0\}\Big)$ is contained in $p(\mathrm{spt}(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}))$.

We have built $O_{k,\varepsilon}$ in (9.3) as the set enclosed between $\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}$ and $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$, see formula (9.9). We now perform a (classical) Steiner symmetrization[54] of the set $O_{k,\varepsilon}$ with respect to the plane $\{w_3 = 0\}$. We denote by $\mathbb{S}_{\mathrm{cl}}(O_{k,\varepsilon})$ the symmetrized set.

Remark 11.4. We emphasize that the set $O_{k,\varepsilon}$ in $(0, \varepsilon) \times \mathbb{R}^2$ is exactly $(0, \varepsilon) \times B_1$, and is already symmetric with respect to the plane containing R_l. For this reason $O_{k,\varepsilon}$ does not change (in that region) after Steiner symmetrization,

$$O_{k,\varepsilon} \cap \{w_1 \in (0, \varepsilon)\} = \mathbb{S}_{\mathrm{cl}}(O_{k,\varepsilon}) \cap \{w_1 \in (0, \varepsilon)\}. \tag{11.8}$$

Since the perimeter does not increase when symmetrizing, from (9.9) we conclude

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \ge \mathcal{H}^2(\partial^* \mathbb{S}_{\mathrm{cl}}(O_{k,\varepsilon}) \cap ((0, l) \times \mathbb{R}^2)). \tag{11.9}$$

[54]Despite $O_{k,\varepsilon}$ is obtained by cylindrical symmetrization, it still can have "holes" (see Fig. 10 for a slice), that disappear when further performing the Steiner symmetrization.

Definition 11.5 (The function $\psi_{k,\varepsilon}$). *We introduce the function $\psi_{k,\varepsilon} : R_l \to [0, +\infty)$ as*

$$\psi_{k,\varepsilon}(w_1, w_2) := \frac{1}{2}\mathcal{H}^1(\{w_3 : (w_1, w_2, w_3) \in O_{k,\varepsilon}\}), \qquad (w_1, w_2) \in R_l. \tag{11.10}$$

We stress that the set where $\psi_{k,\varepsilon} > 0$ is contained, up to $\mathcal{H}^2$-negligible sets, in the region $SG_{h_{k,\varepsilon}}$ defined in (11.7). Notice also that $\psi_{k,\varepsilon}$ may take the value 0 in $SG_{h_{k,\varepsilon}}$ on a set of positive $\mathcal{H}^2$-measure.

Remark 11.6. (i) By definition of classical Steiner symmetrization,

$$\begin{aligned}\mathbb{S}_{\rm cl}(O_{k,\varepsilon}) &= \{w = (w_1, w_2, w_3) \in R_l \times \mathbb{R} : w_3 \in (-\psi_{k,\varepsilon}(w_1, w_2), \psi_{k,\varepsilon}(w_1, w_2))\} \\ &= \{w = (w_1, w_2, w_3) \in SG_{h_{k,\varepsilon}} \times \mathbb{R} : w_3 \in (-\psi_{k,\varepsilon}(w_1, w_2), \psi_{k,\varepsilon}(w_1, w_2))\},\end{aligned}$$

up to Lebesgue-negligible sets, the second equality following from the fact that $\psi_{k,\varepsilon} = 0$ almost everywhere in $R_l \setminus SG_{h_{k,\varepsilon}}$;

(ii) since $O_{k,\varepsilon}$ has finite perimeter, it follows that $\psi_{k,\varepsilon} \in BV(R_l)$;

(iii) since $O_{k,\varepsilon} \llcorner ([0, \varepsilon) \times \mathbb{R}^2) = C_l \llcorner ([0, \varepsilon) \times \mathbb{R}^2)$ and $O_{k,\varepsilon} \llcorner ([\varepsilon, l) \times \mathbb{R}^2)$ is contained in $C_l(1 - \lambda'_k) \llcorner ([\varepsilon, l) \times \mathbb{R}^2)$ (as a consequence of (9.5)), it follows that $\psi_{k,\varepsilon}$ has null trace on the segments $(0, l) \times \{-1\}$ and $(0, l) \times \{1\}$.

We can split $\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon})$ as

$$\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}) = ((\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon})) \cap \{w_3 > 0\}) \bigcup ((\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon})) \cap \{w_3 < 0\}) =: (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+ \cup (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^- \tag{11.11}$$

up to a set of $\mathcal{H}^2$-measure zero, in such a way that

$$(\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+ = (\partial^* SG_{\psi_{k,\varepsilon}}) \cap \big(R_l \times (0, +\infty)\big), \qquad (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^- = (\partial^* UG_{-\psi_{k,\varepsilon}}) \cap \big(R_l \times (-\infty, 0)\big), \tag{11.12}$$

where $SG_{\psi_{k,\varepsilon}}$ and $UG_{-\psi_{k,\varepsilon}}$ are, respectively, the (standard) generalized subgraph and epigraph of $\pm\psi_{k,\varepsilon}$ in $R_l \times \mathbb{R}$. Notice that, since $\psi_{k,\varepsilon} \geq 0$,

$$\begin{aligned}&(\partial^* SG_{\psi_{k,\varepsilon}}) \cap (R_l \times [0, +\infty)) \\ &\qquad = (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+ \cup \{(w_1, w_2, 0) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon} = 0\} \cup (R_l \setminus SG_{h_{k,\varepsilon}}), \\ &(\partial^* UG_{-\psi_{k,\varepsilon}}) \cap (R_l \times (-\infty, 0]) \\ &\qquad = (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^- \cup \{(w_1, w_2, 0) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon} = 0\} \cup (R_l \setminus SG_{h_{k,\varepsilon}}),\end{aligned} \tag{11.13}$$

up to $\mathcal{H}^2$-negligible sets.

We are ready to prove the following:

Lemma 11.7. *We have*

$$\begin{aligned}|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \geq &\mathcal{H}^2\left((\partial^* SG_{\psi_{k,\varepsilon}}) \cap (R_l \times [0, +\infty))\right) + \mathcal{H}^2\left((\partial^* UG_{-\psi_{k,\varepsilon}}) \cap (R_l \times (-\infty, 0])\right) \\ &- 2\mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}}).\end{aligned} \tag{11.14}$$

Moreover, $\mathcal{H}^2\left((\partial^* SG_{\psi_{k,\varepsilon}}) \cap (R_l \times [0, +\infty))\right) = \mathcal{H}^2\left((\partial^* UG_{-\psi_{k,\varepsilon}}) \cap (R_l \times (-\infty, 0])\right)$.

Proof. The last assertion follows by symmetry. Let us prove the former: By (11.13) we have

$$\begin{aligned}\mathcal{H}^2(\partial^* SG_{\psi_{k,\varepsilon}} \cap (R_l \times [0,+\infty))) =& \mathcal{H}^2\big((\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+\big) + \mathcal{H}^2(\{(w_1,w_2) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon} = 0\}) \\ &+ \mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}}), \\ \mathcal{H}^2(\partial^* UG_{-\psi_{k,\varepsilon}} \cap (R_l \times (-\infty,0])) =& \mathcal{H}^2\big((\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^-\big) + \mathcal{H}^2(\{(w_1,w_2) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon} = 0\}) \\ &+ \mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}}).\end{aligned}$$

Taking the sum of these two expressions and using (11.9), (11.11), we obtain

$$\begin{aligned}&\mathcal{H}^2(\partial^* SG_{\psi_{k,\varepsilon}} \cap (R_l \times [0,+\infty))) + \mathcal{H}^2(\partial^* UG_{-\psi_{k,\varepsilon}} \cap (R_l \times (-\infty,0])) \\ &\le |\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| + 2\mathcal{H}^2(\{(w_1,w_2) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon} = 0\}) + 2\mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}}).\end{aligned}$$

Recalling (9.4), we now claim that, up to $\mathcal{H}^2$-negligible sets,

$$\{(w_1,w_2) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon}(w_1,w_2) = 0\} \subset \big\{\widehat{\vartheta}_{k,\varepsilon} = 0\big\} \cap S^{(4)}_{k,\varepsilon}, \tag{11.15}$$

see Fig. 10. From the claim it follows that

$$\mathcal{H}^2(\{(w_1,w_2) \in SG_{h_{k,\varepsilon}} : \psi_{k,\varepsilon} = 0\}) \le \mathcal{H}^2\big(\big\{\widehat{\vartheta}_{k,\varepsilon} \in \{0,\pi\}\big\} \cap S^{(4)}_{k,\varepsilon}\big),$$

and hence by (9.10) we conclude

$$\begin{aligned}&\mathcal{H}^2(\partial^* SG_{\psi_{k,\varepsilon}} \cap (R_l \times [0,+\infty))) + \mathcal{H}^2(\partial^* UG_{-\psi_{k,\varepsilon}} \cap (R_l \times (-\infty,0])) \\ &\le |\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| + 2\mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}}),\end{aligned}$$

that is (11.14). It remains to show (11.15). As usual, we argue by slicing; hence for almost all $w_1 \in (0,l)$ we will show that (11.15) holds (up to $\mathcal{H}^1$-negligible sets). Notice that both the left and right-hand sides of (11.15) are empty for $w_1 < \varepsilon$, so we assume $w_1 > \varepsilon$. Therefore, fix $(\widetilde{w}_1,\widetilde{w}_2) \in SG_{h_{k,\varepsilon}}$ (with $\widetilde{w}_1 > \varepsilon$) such that $\psi_{k,\varepsilon}(\widetilde{w}_1,\widetilde{w}_2) = 0$ and assume, by contradiction, that $\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1,\widetilde{w}_2) > 0$. In a first step we will suppose $\widetilde{w}_2 < 0$. We might further assume that $\widetilde{w}_2$ is a Lebesgue point for the function $\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1,\cdot)$. Hence in any left-neighbourhood of this point $\widehat{\vartheta}_{k,\varepsilon}$ is strictly positive on a set of positive meausure, *i.e.*, we can find positive numbers δ_1,δ_2 such that for all $\delta \in (0,\delta_1)$, there exists a set $B \subset (\widetilde{w}_2 - \delta, \widetilde{w}_2)$, of positive measure such that

$$\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1,w) > \delta_2 > 0 \qquad \forall w \in B. \tag{11.16}$$

If $\pi_0^{\rm pol}$ is the projection in Definition (7.5), since $\widetilde{w}_2 < 0$ for $\delta_3 > 0$ small enough the segment $I := \{(\widetilde{w}_1,\widetilde{w}_2,w_3) : w_3 \in (0,\delta_3)\}$ satisfies

$$I_0 := \pi_0^{\rm pol}(I) \subset \{(\widetilde{w}_1,w_2,0) : w_2 \in (\widetilde{w}_2 - \delta_2, \widetilde{w}_2)\}.$$

We have that $\pi_0^{\rm pol} : I \to I_0$ is a homeomorphism. Now, if $\psi_{k,\varepsilon}(\widetilde{w}_1,\widetilde{w}_2) = 0$ the segment I cannot intersect the subgraph of $\widehat{\vartheta}_{k,\varepsilon}$ (on a set of positive $\mathcal{H}^1$-measure), and thus

$$\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1,w_2) \le \theta\big((\pi_0^{\rm pol}{}_{|I_0})^{-1}(\widetilde{w}_1,w_2,0)\big) \qquad \text{for } \mathcal{H}^1\text{-a.e. } (\widetilde{w}_1,w_2,0) \in I_0, \tag{11.17}$$

where θ represents the usual angular coordinate. Since $\theta\big((\pi_0^{\rm pol}{}_{|I_0})^{-1}(\widetilde{w}_1,w_2,0)\big)$ is infinitesimal as $w_2 \to \widetilde{w}_2^-$, condition (11.17) contradicts (11.16).

Let us now treat the case $\widetilde{w}_2 > 0$. This is much simpler to deal with, up to noticing that $\widehat{\vartheta}_{k,\varepsilon}$ is defined on $S^{(4)}_{k,\varepsilon} \subset \{(w_1, w_2, w_3) : w_2 \in [-1, 0]\}$. The fact that $\psi_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) = 0$ means that the line $(\widetilde{w}_1, \widetilde{w}_2) \times \mathbb{R}$ does not intersect $O_{k,\varepsilon}$ on a set of positive $\mathcal{H}^1$-measure but this contradicts the fact that $(\widetilde{w}_1, \widetilde{w}_2) \in SG_{h_{k,\varepsilon}}$. Indeed since $(\widetilde{w}_1, \widetilde{w}_2) \in SG_{h_{k,\varepsilon}}$ hence there exists $w_2 > \widetilde{w}_2$ such that $\psi_{k,\varepsilon}(\widetilde{w}_1, w_2) > 0$. Let $A := O_{k,\varepsilon} \cap (\widetilde{w}_1, w_2) \times \mathbb{R})$ then a suitable rotation of A around the axis of the cylinder shall meet $(\widetilde{w}_1, \widetilde{w}_2) \times \mathbb{R}$ on a set $\widetilde{A}$ of positive $\mathcal{H}^1$-measure (note that $\widetilde{A} \subset O_{k,\varepsilon}$), which contradicts $\psi_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) = 0$. □

Remark 11.8. By (11.8), (11.10) and (9.3), we deduce

$$\text{the trace of } \psi_{k,\varepsilon} \text{ on } \overline{R}_l \cap \{w_1 = 0\} \text{ is } \sqrt{1 - w_2^2}, \quad \text{for } w_2 \in [-1, 1]. \tag{11.18}$$

Moreover, by construction and by Remark 11.6 (iii),

$$\psi_{k,\varepsilon}(w_1, -1) = 0 \quad \text{and} \quad \psi_{k,\varepsilon}(w_1, 1) = 0, \qquad w_1 \in (0, l). \tag{11.19}$$

Remark 11.9. We can write [19]

$$\mathcal{H}^2\Big((\partial^* SG_{\psi_{k,\varepsilon}}) \cap (R_l \times [0, \infty))\Big) = \mathbb{A}(\psi_{k,\varepsilon}, R_l), \tag{11.20}$$

where

$$\mathbb{A}(\psi_{k,\varepsilon}, R_l) = \int_{R_l} \sqrt{1 + |\nabla \psi_{k,\varepsilon}|^2}\, dx + |D^s \psi_{k,\varepsilon}|(R_l)$$

is the classical area of the graph of the BV-function $\psi_{k,\varepsilon}$ in R_l. Moreover, by (11.19), it follows $|D^s\psi_{k,\varepsilon}|(R_l) = |D^s\psi_{k,\varepsilon}|\big(\overline{R}_l \setminus (\{w_1 = 0\} \cup \{w_1 = l\})\big)$ and hence

$$\mathbb{A}(\psi_{k,\varepsilon}, R_l) = \mathbb{A}\big(\psi_{k,\varepsilon}, \overline{R}_l \setminus (\{w_1 = 0\} \cup \{w_1 = l\})\big).$$

Recalling the expression (11.1) of $\partial_D R_l$, define $\varphi : \partial_D R_l \to [0, 1]$ as

$$\varphi(w_1, w_2) := \begin{cases} \sqrt{1 - w_2^2} & \text{if } (w_1, w_2) \in \{0\} \times [-1, 1], \\ 0 & \text{if } (w_1, w_2) \in (0, l) \times \{-1\}. \end{cases} \tag{11.21}$$

Definition 11.10 (The functional $\mathcal{F}_l$). *Given $h \in L^\infty([0, l], [-1, 1])$ and $\psi \in \mathrm{BV}(R_l; [0, 1])$ we define*

$$\mathcal{F}_l(h, \psi) := \mathbb{A}(\psi, R_l) - \mathcal{H}^2(R_l \setminus SG_h) + \int_{\partial_D R_l} |\psi - \varphi|\, d\mathcal{H}^1 + \int_{(0,l)\times\{1\}} |\psi|\, d\mathcal{H}^1. \tag{11.22}$$

We further define

$$X_l := \{(h, \psi) : h \in L^\infty([0, l], [-1, 1]), \psi \in \mathrm{BV}(R_l, [0, 1]), \psi = 0 \text{ in } R_l \setminus SG_h\}. \tag{11.23}$$

Remark 11.11. (i) The Borel function $h_{k,\varepsilon} : [0, l] \to [-1, 1]$ satisfies $h_{k,\varepsilon} = 1$ in $[0, \varepsilon)$, and $\psi_{k,\varepsilon} \in BV([0, l] \times [-1, 1])$ is such that $\psi_{k,\varepsilon} = 0$ almost everywhere in $R_l \setminus SG_{h_{k,\varepsilon}}$. Moreover $\psi_{k,\varepsilon}(w_1, w_2) = \sqrt{1 - w_2^2}$ for $(w_1, w_2) \in (0, \varepsilon) \times [-1, 1]$, and $\psi_{k,\varepsilon}(\cdot, -1) = 0$ in $[0, l]$. In particular

$$(h_{k,\varepsilon}, \psi_{k,\varepsilon}) \in X_l.$$

(ii) if $(h, \psi) \in X_l$, and if h is smaller than 1 almost everywhere on $(0, l)$ then the last addendum on the right-hand side of (11.22) vanishes.

(iii) Thanks to (11.18) and (11.19), it follows from Remark 11.9 that

$$\begin{aligned}&\mathcal{H}^2\big((\partial^* SG_{\psi_{k,\varepsilon}})\cap(R_l\times[0,+\infty))\big)-\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}})\\ =&\mathbb{A}(\psi_{k,\varepsilon},\overline{R}_l\setminus(\{w_1=0\}\cup\{w_1=l\}))-\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}})=\mathcal{F}_l(h_{k,\varepsilon},\psi_{k,\varepsilon}).\end{aligned}$$

As a consequence, from Lemma 11.7 we have

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}|+|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \ \geq\ 2\mathcal{F}_l(h_{k,\varepsilon},\psi_{k,\varepsilon}). \tag{11.24}$$

Notice that in minimizing $\mathcal{F}_l$ we have a free boundary condition on the edge $\{l\}\times[-1,1]$. By Remark 11.11 (i) and (11.24) we have

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}|+|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|\geq 2\inf_{(h,\psi)\in X_l}\mathcal{F}_l(h,\psi), \tag{11.25}$$

which leads to the investigation of the minimum problem on the right-hand side.

Remark 11.12. Let $(h,\psi)\in X_l$. If $t_0\in(0,l)$ is a Lebesgue point for h, and if $h(t_0)<1$, then the trace of ψ over the segment $\{w_1=t_0,\ h(t_0)\leq w_2\leq 1\}$ vanishes. Indeed for any $\eta>0$ we can find $\delta_\eta>0$ such that

$$\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}|h(w_1)-h(t_0)|\,dw_1<\eta \qquad \forall\delta\in(0,\delta_\eta). \tag{11.26}$$

Let now $s_0\in(-1,1)$ be such that $h(t_0)<s_0\leq 1$ (*i.e.*, $(t_0,s_0)\in\{w_1=t_0,\ w_2>h(t_0)\}$), and set $2\Delta:=s_0-h(t_0)$. By Chebyschev inequality and (11.26) it follows that the set $B_\Delta:=\{w_1\in(t_0-\delta,t_0+\delta):|h(w_1)-h(t_0)|>\Delta\}$ satisfies

$$\mathcal{H}^1(B_\Delta)\leq\frac{2\delta\eta}{\Delta}. \tag{11.27}$$

Then, for any $\xi\in(0,\Delta)$ we infer[55]

$$\begin{aligned}&\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}\int_{s_0-\xi}^{s_0+\xi}\psi(w_1,w_2)\,dw_2dw_1\leq\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}\int_{s_0-\xi}^{s_0+\xi}\chi_{\{\psi>0\}}(w_1,w_2)\,dw_2dw_1\\ \leq&\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}\int_{s_0-\xi}^{s_0+\xi}\chi_{SG_h}(w_1,w_2)\,dw_2dw_1\leq\frac{\xi}{\delta}\int_{t_0-\delta}^{t_0+\delta}\chi_{B_\Delta}(w_1)dw_1\leq\frac{2\xi\eta}{\Delta},\end{aligned} \tag{11.28}$$

where the penultimate inequality follows from the inclusions

$$\begin{aligned}SG_h\cap\big([t_0-\delta,t_0+\delta]\times[s_0-\xi,s_0+\xi]\big)&\subseteq SG_h\cap\big([t_0-\delta,t_0+\delta]\times[s_0-\Delta,s_0+\Delta]\big)\\ &\subseteq B_\Delta\times[s_0-\Delta,s_0+\Delta],\end{aligned}$$

and the last inequality follows from (11.27). Now (11.28) entails the claim by the arbitrariness of $\eta>0$ and since $\psi\geq 0$.

We now refine the choice of the class of pairs (h,ψ) where the infimum in (11.25) is computed.

Definition 11.13 (The classes $\mathcal{H}_l$ and $X_l^{\rm conv}$). *We set*

$$\begin{aligned}&\mathcal{H}_l:=\{h\in L^\infty([0,l],[-1,1]):\ h \textit{ convex and nonincreasing in } [0,l], h(0)=1\},\\ &X_l^{\rm conv}:=\{(h,\psi)\in X_l:\ h\in\mathcal{H}_l\}.\end{aligned}$$

[55] In the first inequality we have used that $0\leq\psi\leq 1$; in the second inequality that SG_h is the subgraph of h in $(0,l)\times(-1,1)$; in the third inequality we have used that $s_0-h(t_0)=2\Delta$ and that $\xi<\Delta$.

Proposition 11.14 (Convexifying h). *We have*

$$\inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi) = \inf_{(h,\psi)\in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi). \tag{11.29}$$

Proof. It is enough to show the inequality "$\geq$". By extending ψ outside R_l as $\psi := 0$ in $((0,l)\times\mathbb{R})\setminus R_l$, we see that

$$\mathcal{F}_l(h,\psi) = \mathbb{A}\left(\psi, \overline{R}_l \setminus \left(\{w_1=0\}\cup\{w_1=l\}\right)\right) - \mathcal{H}^2(R_l\setminus SG_h) + \int_{\{0\}\times[-1,1]} |\psi^- - \varphi|\, d\mathcal{H}^1, \tag{11.30}$$

where, with a little abuse of notation,

$$\mathbb{A}\left(\zeta, \overline{R}_l \setminus \left(\{w_1=0\}\cup\{w_1=l\}\right)\right) = \mathbb{A}(\zeta, R_l) + \int_{(0,l)\times\{1,-1\}} |\zeta^-|\, d\mathcal{H}^1,$$

ζ^- being the trace of $\zeta\in BV(R_l)$ on $(0,l)\times\{1,-1\}$.

The thesis of the proposition will follow from the next three observations:

(1) If $h\in\mathcal{H}_l$ is such that $h(t_0)=-1$ for some Lebesgue point $t_0\in(0,l)$, then the subgraph SG_h of h splits in two mutually disjoint components: $SG_h^- = SG_h\cap\{w_1<t_0\}$ and $SG_h^+ = SG_h\cap\{w_1>t_0\}$. Let $\psi\in BV(R_l,[0,1])$ be such that

$$\psi = 0 \qquad \text{a.e. in } \ R_l\setminus SG_h.$$

The trace of ψ over the segment $\{w_1=t_0,\ h(t_0)\leq w_2\leq 1\}$ is 0, as a consequence of Remark 11.12. Then the function $\psi^\star : R_l\to[0,1]$ defined as

$$\psi^\star(w_1,w_2) := \begin{cases} \psi(w_1,w_2) & \text{if } w_1<t_0,\\ 0 & \text{otherwise,}\end{cases}$$

still satisfies $(h,\psi^\star)\in X_l$, and

$$\mathcal{F}_l(h,\psi^\star) \leq \mathcal{F}_l(h,\psi).$$

Being $\psi^\star$ identically zero in $\{w_1>t_0\}$, in particular in $SG_h\cap\{w_1>t_0\}$, we can introduce

$$h^\star(w_1) := \begin{cases} h(w_1) & \text{if } w_1<t_0,\\ -1 & \text{otherwise,}\end{cases}$$

so that $(h^\star,\psi^\star)\in X_l$ and we easily see that $\mathcal{F}_l(h^\star,\psi^\star)\leq\mathcal{F}_l(h,\psi^\star)$; hence

$$\mathcal{F}_l(h^\star,\psi^\star)\leq\mathcal{F}_l(h,\psi).$$

(2) More generally, let $(h,\psi)\in X_l$ and let $t_0\in(0,l)$ be any Lebesgue point of h; we can also suppose that $h(t_0)<1$. Consider

$$h^\star(w_1) := \begin{cases} h(w_1) & \text{if } w_1<t_0,\\ h(w_1)\wedge h(t_0) & \text{otherwise,}\end{cases} \tag{11.31}$$

$$\psi^\star(w_1,w_2) := \begin{cases} \psi(w_1,w_2) & \text{if } w_1<t_0,\\ \psi(w_1,w_2) & \text{if } w_1\geq t_0,\ w_2\leq h(t_0),\\ 0 & \text{otherwise.}\end{cases}$$

We claim that $\mathcal{F}_l(h^\star, \psi^\star) \le \mathcal{F}_l(h, \psi)$. Define

$$U := \{(w_1, w_2) \in (0, l) \times (-1, 1) : w_1 > t_0,\ h(t_0) < w_2 < h(w_1)\},$$

that is the set where we have replaced ψ by 0. To prove the claim, using (11.30) and the equalities

$$\int_{\{0\}\times[-1,1]} |\psi^- - \varphi|\ d\mathcal{H}^1 = \int_{\{0\}\times[-1,1]} |\psi^{\star -} - \varphi|\ d\mathcal{H}^1,$$
$$\mathcal{H}^2(R_l \setminus SG_{h^\star}) = \mathcal{H}^2(U \cup (R_l \setminus SG_h)) = \mathcal{H}^2(U) + \mathcal{H}^2(R_l \setminus SG_h),$$

we have to show that

$$\mathbb{A}\left(\psi^\star, \overline{R}_l \setminus \left(\{w_1 = 0\} \cup \{w_1 = l\}\right)\right) \le \mathbb{A}\left(\psi, \overline{R}_l \setminus \left(\{w_1 = 0\} \cup \{w_1 = l\}\right)\right) + \mathcal{H}^2(U). \quad (11.32)$$

Assume that U is non-empty and that $\mathcal{H}^2(U) > 0$. It is convenient to introduce

$$V := \{(w_1, w_2) \in R_l : t_0 < w_1 < l,\ h(w_1) \vee h(t_0) \le w_2 < 1\},$$

so that $U \cup V = \{(w_1, w_2) : w_1 > t_0,\ h(t_0) < w_2 < 1\}$ is an open rectangle. Since we have modified ψ only in U, inequality (11.32) is equivalent to

$$\begin{aligned}
&\mathbb{A}(\psi^\star, U \cup V) + \int_{(t_0,l)\times\{h(t_0)\}} |\psi^{\star +} - \psi^{\star -}| d\mathcal{H}^1 + \int_{(t_0,l)\times\{1\}} |\psi^{\star -}| d\mathcal{H}^1 \\
\le &\mathbb{A}(\psi, U \cup V) + \int_{(t_0,l)\times\{h(t_0)\}} |\psi^+ - \psi^-| d\mathcal{H}^1 + \int_{(t_0,l)\times\{1\}} |\psi^-| d\mathcal{H}^1 + \mathcal{H}^2(U),
\end{aligned} \quad (11.33)$$

with $\psi^\pm$ (resp. $\psi^{\star\pm}$) the external and internal traces of ψ (resp. $\psi^\star$) on $\partial(U \cup V)$; here we have used from Remark 11.12 that the trace of ψ on $\{t_0\} \times (h(t_0), 1)$ is zero (hence $\int_{\{t_0\}\times(h(t_0),1)} |\psi^+ - \psi^-| d\mathcal{H}^1 = \int_{\{t_0\}\times(h(t_0),1)} |\psi^{\star +} - \psi^{\star -}| d\mathcal{H}^1 = 0$) and that the external traces ψ^+, $\psi^{\star +}$ on $(t_0, l) \times \{1\}$ vanish as well. Hence, exploiting that $\psi^\star = 0$ on $U \cup V$, so that $\mathbb{A}(\psi^\star, U \cup V) = \mathcal{H}^2(U) + \mathcal{H}^2(V)$, and that $\psi^\star = \psi$ on $R_l \setminus (U \cup V)$, inequality (11.33) is equivalent to

$$\begin{aligned}
&\mathcal{H}^2(V) + \int_{(t_0,l)\times\{h(t_0)\}} |\psi^+| d\mathcal{H}^1 \\
\le &\mathbb{A}(\psi, U \cup V) + \int_{(t_0,l)\times\{h(t_0)\}} |\psi^+ - \psi^-| d\mathcal{H}^1 + \int_{(t_0,l)\times\{1\}} |\psi^-| d\mathcal{H}^1.
\end{aligned} \quad (11.34)$$

We split

$$(t_0, l) = H_1 \cup H_2 \cup H_3,$$

with $H_1 := \{w_1 \in (t_0, l) : h(w_1) = 1\}$, $H_2 := \{w_1 \in (t_0, l) : h(t_0) \le h(w_1) < 1\}$, and $H_3 := \{w_1 \in (t_0, l) : h(w_1) < h(t_0)\}$. Since $\mathbb{A}(\psi; U \cup V) = \mathcal{H}^2(\mathcal{G}_\psi \cap ((U \cup V) \times \mathbb{R}))$, by slicing and looking at $\mathcal{G}_\psi$ as an integral current, we have[56]

$$\begin{aligned}
\mathbb{A}(\psi, U \cup V) &\geq \int_{(t_0,l)} \mathcal{H}^1\Big((\mathcal{G}_\psi)_t \cap ((t_0,l) \times (h(t_0),1) \times \mathbb{R})\Big)\, dt \\
&\geq \int_{(t_0,l)} \int_{(h(t_0),1)} |D_{w_2}\psi(t,s)| \ \ dt + \mathcal{H}^2(V) \\
&= \int_{H_1 \cup H_2} \int_{(h(t_0),1)} |D_{w_2}\psi(t,s)| \ \ dt + \mathcal{H}^2(V) \\
&\geq \int_{H_2} |\psi^-(t,h(t_0))|\, dt + \int_{H_1} |\psi^-(t,h(t_0)) - \psi^-(t,1)|\, dt + \mathcal{H}^2(V) \\
&\geq \int_{H_1 \cup H_2} |\psi^-(t,h(t_0))|\, dt - \int_{H_1} |\psi^-(t,1)|\, dt + \mathcal{H}^2(V) \\
&= \int_{(t_0,l)} |\psi^-(t,h(t_0))|\, dt - \int_{H_1} |\psi^-(t,1)|\, dt + \mathcal{H}^2(V) \\
&= \int_{(t_0,l)} |\psi^-(t,h(t_0))|\, dt - \int_{(t_0,l)} |\psi^-(t,1)|\, dt + \mathcal{H}^2(V),
\end{aligned}$$

where $(\mathcal{G}_\psi)_t$ is the slice of $\mathcal{G}_\psi$ on the plane $\{w_1 = t\}$, that is the generalized graph of the function $\psi \llcorner \{w_2 = t\}$. From the above expression, the triangular inequality implies (11.34).

(3) Let $(h,\psi) \in X_l$. Let $t_1, t_2 \in (\varepsilon, l)$ be Lebesgue points for h with $t_1 < t_2$, and let $r_{12}(t) := h(t_1) + \frac{h(t_2)-h(t_1)}{t_2-t_1}(t - t_1)$. We consider the following modifications of h and ψ:

$$h^\#(w_1) := \begin{cases} h(w_1) & \text{if } 0 < w_1 < t_1 \text{ or } l > w_1 > t_2, \\ h(w_1) \wedge r_{12}(w_1) & \text{otherwise,} \end{cases}$$

and

$$\psi^\#(w_1,w_2) := \begin{cases} \psi(w_1,w_2) & \text{if } 0 < w_1 < t_1 \text{ or } l > w_1 > t_2, \\ \psi(w_1,w_2) & \text{if } w_1 \in [t_1,t_2] \text{ and } w_2 \leq r_{12}(w_1), \\ 0 & \text{otherwise.} \end{cases}$$

In other words we set ψ equal to 0 above the segment L_{12} connecting $(t_1, h(t_1))$ to $(t_2, h(t_2))$. Also in this case we have

$$\mathcal{F}_l(h^\#, \psi^\#) \leq \mathcal{F}_l(h,\psi). \tag{11.35}$$

Indeed, if $h(t_1) = h(t_2)$ the proof is identical to the case (2). Otherwise, it can be obtained by slicing as well, parametrizing L_{12} by an arc length parameter, then slicing the region $\{(w_1,w_2) : w_1 \in (t_1,t_2),\ w_2 \in (\ell_{12}(w_1), 1)\}$[57] by lines perpendicular to L_{12}, and exploiting the fact that ψ equals zero on the segments $\{t_i\} \times (h(t_i), 1)$.

Let $(h,\psi) \in X_l$ be given; from (3) we can always replace h by its convex envelope and modifying accordingly ψ, we get two functions $h^\#$ and $\psi^\#$ such that (11.35) holds. Moreover, by (2), if $t_0 \in (0,l)$ is a Lebesgue point for $h^\#$, we can always replace $h^\#$ by $h^\star$ in (11.31), so that $h^\star$ turns out to be nonincreasing. The assertion of the proposition follows. □

[56] Here we use that $D_{w_2}\psi = 0$ in V.

[57] ℓ_{12} represents the affine function whose graph is L_{12}.

Let us rewrite the functional $\mathcal{F}_l$ in a convenient way. Let $(h,\psi) \in X_l^{\mathrm{conv}}$, and let $G_h = \{(w_1, h(w_1)) : w_1 \in (0,l)\} \subset \overline{R}_l$ be the graph of h. We have, using (11.22),

$$\mathcal{F}_l(h,\psi) = \mathbb{A}(\psi, SG_h) + \int_{G_h \setminus \{h=-1\}} |\psi| \; d\mathcal{H}^1 + \int_{\partial_D R_l} |\psi - \varphi| \; d\mathcal{H}^1, \tag{11.36}$$

where, in the integral over G_h, we consider the trace of $\psi \llcorner SG_h$ on G_h.

Corollary 11.15. *Let $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. Then for any $k \in \mathbb{N}$ we have*

$$|[\![G_{u_k}]\!]|_{D_k \times \mathbb{R}^2} \geq 2 \inf_{(h,\psi) \in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi) - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1), \tag{11.37}$$

for an absolute constant $C > 0$, and where the sequence $o_k(1)$ depends on ε and n and is infinitesimal as $k \to +\infty$.

Proof. From (11.25) and Proposition 11.14, we get

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \geq 2 \inf_{(h,\psi) \in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi). \tag{11.38}$$

Combining (11.38) with (9.12), inequality (11.37) follows. □

11.1 Lower bound: reduction to a Plateau-type problem on the rectangle R_l

We now state and prove our first main result.

Theorem 11.16 (Lower bound for the area of the vortex map). *The relaxed area of the graph of the vortex map u satisfies*

$$\overline{\mathcal{A}}(u,\Omega) \geq \int_\Omega |\mathcal{M}(\nabla u)| \; dx + 2 \inf_{(h,\psi) \in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi). \tag{11.39}$$

Proof. We write

$$\mathcal{A}(u_k,\Omega) = \mathcal{A}(u_k, \Omega \setminus D_k) + \mathcal{A}(u_k, D_k) = \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \; dx + \int_{D_k} |\mathcal{M}(\nabla u_k)| \; dx.$$

Therefore

$$\overline{\mathcal{A}}(u,\Omega) \geq \liminf_{k \to +\infty} \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \; dx + \liminf_{k \to +\infty} \int_{D_k} |\mathcal{M}(\nabla u_k)| \; dx. \tag{11.40}$$

Given $\varepsilon \in (0,l)$ and $n \in \mathbb{N}$, from (4.37) it follows

$$\liminf_{k \to +\infty} \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \; dx \geq \int_{\Omega \setminus \overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)| \; dx - \frac{1}{n} - \frac{2}{\varepsilon n}. \tag{11.41}$$

From (11.37) we have

$$\int_{D_k} |\mathcal{M}(\nabla u_k)| \; dx = |[\![G_{u_k}]\!]|_{D_k \times \mathbb{R}^2} \geq 2 \inf_{(h,\psi) \in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi) - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1). \tag{11.42}$$

Exploiting the fact that the right-hand side of (11.38) does not depend on k, we can pass to the liminf as $k \to +\infty$ in the above expression, to obtain

$$\liminf_{k \to +\infty} \int_{D_k} |\mathcal{M}(\nabla u_k)| \; dx \geq 2 \inf_{(h,\psi) \in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi) - \pi\varepsilon - \frac{C}{\varepsilon n}. \tag{11.43}$$

From (11.40), (11.41) and (11.43) we obtain

$$\overline{\mathcal{A}}(u,\Omega) \geq \int_{\Omega\setminus\overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)|\, dx + 2 \inf_{(\psi,h)\in X_l^{\mathrm{conv}}} \mathcal{F}_l(\psi,h) - \pi\varepsilon - \frac{C+2}{\varepsilon n} - \frac{1}{n}, \tag{11.44}$$

for all $n \in \mathbb{N}$ and $\varepsilon \in (0,l)$. Letting $n \to +\infty$ and then $\varepsilon \to 0^+$, by the dominated convergence theorem (since $\Omega \setminus \overline{\mathrm{B}}_\varepsilon \to \Omega$ as $\varepsilon \to 0^+$) we get

$$\begin{aligned}\overline{\mathcal{A}}(u,\Omega) &\geq \liminf_{\varepsilon\to 0^+} \Big(\int_{\Omega\setminus\overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)|\, dx + 2 \inf_{(h,\psi)\in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi) - \pi\varepsilon\Big) \\ &= \int_\Omega |\mathcal{M}(\nabla u)|\, dx + 2 \inf_{(h,\psi)\in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi).\end{aligned}$$

□

12 Structure of minimizers of $\mathcal{F}_{2l}$

In this section we analyse the minimum problem on the right-hand side of (11.29). We prove the existence of minimizers, and exploy it to show that the inequality (11.39) in Theorem 11.16 is optimal. First it is convenient to write the analogue of $\mathcal{F}_l$ in a doubled rectangle, see (12.4).

We start by introducing some notation. We denote R_{2l} the open doubled rectangle, $R_{2l} := (0,2l)\times(-1,1)$, and define its Dirichlet boundary[58] $\partial_D R_{2l} \subset \partial R_{2l}$ as

$$\partial_D R_{2l} := \big(\{0,2l\}\times[-1,1]\big) \cup \big((0,2l)\times\{-1\}\big),$$

so that $\partial R_{2l} \setminus \partial_D R_{2l} = (0,2l)\times\{1\}$.

Definition 12.1. *We set*

$$\mathcal{H}_{2l} = \big\{h : [0,2l] \to [-1,1],\ h \text{ convex},\ h(w_1) = h(2l-w_1)\ \forall w_1 \in [0,2l]\big\}. \tag{12.1}$$

For each $h \in \mathcal{H}_{2l}$, we further define

$$G_h := \{(w_1,h(w_1)) : w_1 \in (0,2l)\}, \qquad SG_h := \{(w_1,w_2) \in R_{2l} : w_2 < h(w_1)\},$$

where $SG_h := \emptyset$ in the case $h \equiv -1$. We set

$$L_h := \Big(\{0\}\times(h(0),1)\Big) \cup \Big(\{2l\}\times(h(2l),1)\Big), \tag{12.2}$$

which is either empty, or the union of two equal intervals, see Fig. 17.

Define

$$\varphi : \partial_D R_{2l} \to [0,1], \qquad \varphi(w_1,w_2) := \begin{cases} \sqrt{1-{w_2}^2} & \text{if } (w_1,w_2) \in \{0,2l\}\times[-1,1], \\ 0 & \text{if } (w_1,w_2) \in (0,2l)\times\{-1\}. \end{cases} \tag{12.3}$$

The graph of φ on $\{0,2l\}\times[-1,1]$ consists of two half-circles of radius 1 centered at $(0,0)$ and $(2l,0)$ respectively, see Fig. 18. We notice that such a φ extends definition (11.21).

[58]Note that $\partial_D R_{2l}$ consists of three edges of ∂R_{2l}, while $\partial_D R_l$ (see (11.1)) consists of two edges of ∂R_l.

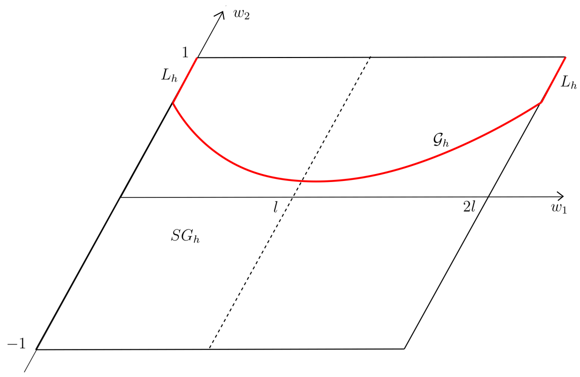

Figure 17: the graph of a convex symmetric function $h \in \mathcal{H}_{2l}$; L_h, defined in (12.2), consists of the two vertical segments over the boundary of $(0, 2l)$, from $h(0) = h(2l)$ to 1.

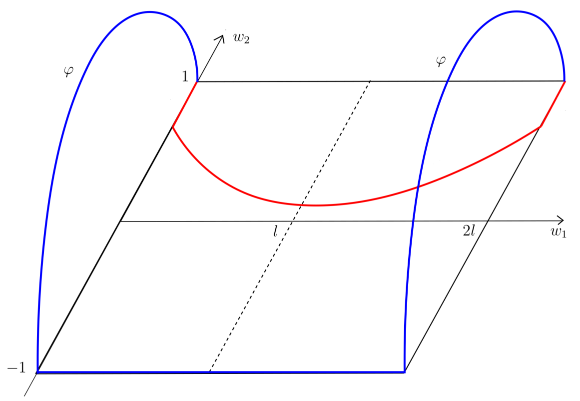

Figure 18: the graph of the boundary condition function φ in (12.3) on the Dirichlet boundary of R_{2l}. We also draw the graph of a function $h \in \mathcal{H}_{2l}$, and the two segments L_h.

Definition 12.2 (The functional $\mathcal{F}_{2l}$). *We define*

$$X_{2l}^{\text{conv}} := \{(h,\psi) : h \in \mathcal{H}_{2l}, \psi \in BV(R_{2l},[0,1]), \psi = 0 \text{ on } R_{2l} \setminus SG_h\},$$

and for any $(h,\psi) \in X_{2l}^{\text{conv}}$,

$$\mathcal{F}_{2l}(h,\psi) := \mathbb{A}(\psi; R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi|\, d\mathcal{H}^1. \tag{12.4}$$

Remark 12.3. (i) The only case in which the last addendum on the right-hand side of (12.4) may be positive is when h is identically 1 on $\partial R_{2l} \setminus \partial_D R_{2l}$;

(ii) We have

$$\mathcal{F}_{2l}(h,\psi) = \mathbb{A}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi|\, d\mathcal{H}^1 + \int_{G_h \setminus \{w_2 = -1\}} |\psi^-|\, d\mathcal{H}^1 + \int_{L_h} \varphi\, d\mathcal{H}^1, \tag{12.5}$$

where

$$\partial_D SG_h := (\partial_D R_{2l}) \cap \partial SG_h, \tag{12.6}$$

and ψ^- denotes the trace of ψ from the side of SG_h. To show (12.5), we start to observe that, using that $\psi = 0$ on $R_{2l} \setminus SG_h$, it follows $\int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi|\, d\mathcal{H}^1 = \int_{G_h \cap \{w_2 = 1\}} |\psi|\, d\mathcal{H}^1$. This last term is nonzero only if $h \equiv 1$, in which case L_h is empty, and the equivalence between (12.4) and (12.5) easily follows. If instead h is not identically 1, then, using again that $\psi = 0$ on $R_{2l} \setminus SG_h$, we see that the last term on the right-hand side of (12.4) is null, and

$$\mathbb{A}(\psi, SG_h) = \mathbb{A}(\psi, R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) - \int_{G_h \cap R_{2l}} |\psi^-| d\mathcal{H}^1. \tag{12.7}$$

Hence, if h is not identically 1, inserting (12.7) into (12.4), we obtain, splitting $\partial_D R_{2l} = (\partial_D SG_h) \cup L_h \cup (G_h \cap \{w_2 = -1\})$, and using that $\varphi = 0$ on $(0,2l) \times \{-1\}$,

$$\begin{aligned}
\mathcal{F}_{2l}(h,\psi) =& \mathbb{A}(\psi, R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 \\
=& \mathbb{A}(\psi, SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{G_h \cap R_{2l}} |\psi^-| d\mathcal{H}^1 \\
=& \mathbb{A}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| d\mathcal{H}^1 + \int_{L_h} |\varphi| d\mathcal{H}^1 + \int_{G_h \cap \{w_2 = -1\}} |\varphi| d\mathcal{H}^1 \\
&+ \int_{G_h \cap R_{2l}} |\psi^-| d\mathcal{H}^1 \\
=& \mathbb{A}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| d\mathcal{H}^1 + \int_{G_h \setminus \{w_2 = -1\}} |\psi^-| d\mathcal{H}^1 + \int_{L_h} \varphi d\mathcal{H}^1.
\end{aligned}$$

(iii) We have

$$\begin{aligned}
&\inf_{(h,\psi) \in X_{2l}^{\text{conv}}} \mathcal{F}_{2l}(h,\psi) \\
&= \inf \Big\{ \mathbb{A}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi|\, d\mathcal{H}^1 + \int_{G_h \setminus \{w_2 = -1\}} |\psi^-|\, d\mathcal{H}^1 + \int_{L_h} \varphi\, d\mathcal{H}^1 \\
&\qquad : h \in \mathcal{H}_{2l} \setminus \{h \equiv -1\},\ \psi \in \mathrm{BV}(SG_h, [0,1]) \Big\}.
\end{aligned} \tag{12.8}$$

(iv) If $h > -1$ everywhere, then SG_h is connected, $\partial_D SG_h = \partial_D R_{2l} \setminus L_h$, and the sum of the first three terms on the right-hand side of (12.5) gives the area of the graph of ψ on $\overline{SG_h}$, with the boundary condition φ set to be 0 on G_h.

(v) Our aim is to have a surface in $\overline{R}_{2l} \times \mathbb{R} \subset \mathbb{R}^3 = \mathbb{R}^2_{(w_1,w_2)} \times \mathbb{R}$, of graph type, whose boundary consists of the union of the graph of φ and the graph of a convex function $h \in \mathcal{H}_{2l}$. The last three terms in (12.5) are an area penalization to force the solution to attain these boundary conditions by filling, with vertical walls, the gap between the boundary of any competitor surface (the generalized graph of ψ) and the required boundary conditions. In particular the presence of the last term of (12.5) is explained as follows: assume that $h(0) < 1$, *i.e.*, $L_h \neq \emptyset$; the graph of any $\psi \in \mathrm{BV}(SG_h, [0,1])$ does not reach the graph of $\varphi|_{L_h}$ (simply because $L_h \cap \overline{SG_h} = \emptyset$). To overcome this, the graph of ψ is glued to the wall consisting of the subgraph of $\varphi|_{L_h}$ (inside $\overline{R}_{2l}$).

(vi) Take $h_n := -1+\frac{1}{n}$, and $\psi_n := c > 0$ on SG_{h_n}, then $\lim_{n\to+\infty} \mathbb{A}(\psi_n, SG_{h_n}) = 0$, $\lim_{n\to+\infty} \int_{\partial_D SG_{h_n}} |\psi_n - \varphi| \, d\mathcal{H}^1 = 2cl$, and $\lim_{n\to+\infty} \int_{G_{h_n}\setminus\{h_n=-1\}} |\psi| \, d\mathcal{H}^1 = 2cl$, $\lim_{n\to+\infty} \int_{L_{h_n}} \varphi \, d\mathcal{H}^1 = \pi$, hence

$$\mathcal{F}_{2l}(-1,0) = \pi < \lim_{n\to+\infty} \mathcal{F}_{2l}(h_n,\psi_n) = 4cl + \pi,$$

that is the functional $\mathcal{F}_{2l}$ in some sense forces a minimizing sequence to attain the boundary conditions as much as possible.

By symmetry, we easily infer

$$2 \inf_{(h,\psi)\in X_l^{\mathrm{conv}}} \mathcal{F}_l(h,\psi) = \inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}_{2l}(h,\psi), \tag{12.9}$$

therefore we can now restate the content of Theorem 11.16 as follows:

$$\overline{\mathcal{A}}(u,\Omega) \geq \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}_{2l}(h,\psi). \tag{12.10}$$

Remark 12.4 (Two explicit estimates from above). Let $h \equiv 1$ and $\psi(w_1,w_2) := \sqrt{1-w_2{}^2} =: \psi_s(w_1,w_2)$ for any $(w_1,w_2) \in R_{2l}$. Then (h,ψ) is one of the competitors in (12.9) and therefore

$$\inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}_{2l}(h,\psi) \leq \mathcal{F}_{2l}(1,\psi_s) = 2\pi l \qquad \forall l > 0,$$

which is the lateral area of the cylinder $(0,2l) \times B_1$. Also, $\mathcal{F}_{2l}$ is well defined for $h \equiv -1$, in which case $SG_h = \emptyset$, $\psi \equiv 0$ in R_{2l}, and therefore

$$\mathcal{F}_{2l}(-1,0) = \int_{\{0,2l\}\times(-1,1)} \varphi \, d\mathcal{H}^1 = \pi, \tag{12.11}$$

which is the area of the two half-disks joined by the segment $(0,2l)\times\{-1\}$, see Fig. 18. In particular

$$\inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}(h,\psi) \leq \pi \qquad \forall l > 0. \tag{12.12}$$

In the next section we shall prove the existence and regularity of minimizers for the minimum problem on the right-hand side of (12.10).

12.1 Existence of a minimizer of $\mathcal{F}_{2l}$

The construction of a suitable recovery sequence for the relaxed area of the graph of the vortex map u depends on the existence and regularity of one-codimensional area minimizing surfaces of graph type for problem (12.9). We start by analysing the features of the space $\mathcal{H}_{2l}$ in (12.1). Clearly the graph of $h \in \mathcal{H}_{2l}$ is symmetric with respect to $\{w_1 = l\}$; also, the convexity of h implies $h \in \mathrm{Lip}_{\mathrm{loc}}((0, 2l))$, and h has a continuous extension on $[0, 2l]$.

Lemma 12.5 (**Compactness of** $\mathcal{H}_{2l}$)**.** *Every sequence $(h_k) \subset \mathcal{H}_{2l}$ has a subsequence converging uniformly on compact subsets of $(0, 2l)$ to some element of $\mathcal{H}_{2l}$.*

Proof. See for instance [22, Sec. 1.1]. □

It is convenient to extend φ in the doubled rectangle $\overline{R}_{2l}$ by defining the extension $\widehat{\varphi}$ as:

$$\widehat{\varphi}(w_1, w_2) = \widehat{\varphi}(0, w_2) := \sqrt{1 - w_2^2} \qquad \forall (w_1, w_2) \in \overline{R}_{2l}. \tag{12.13}$$

In the rest of this section we want to prove the following result.

Theorem 12.6 (**Minimizing pairs**)**.** *There exists $(h^\star, \psi^\star) \in X_{2l}^{\mathrm{conv}}$ such that*

$$\mathcal{F}_{2l}(h^\star, \psi^\star) = \min \big\{ \mathcal{F}_{2l}(h, \psi) : (h, \psi) \in X_{2l}^{\mathrm{conv}} \big\}, \tag{12.14}$$

and $\psi^\star$ is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$. Moreover, if $h^\star$ is not identically -1, then

(i) $h^\star(0) = 1 = h^\star(2l)$*, and* $h^\star > -1$ *in* $(0, 2l)$*;*

(ii) $\psi^\star$ *is locally Lipschitz (hence analytic) and strictly positive in* $SG_{h^\star}$*;*

(iii) $\psi^\star$ *is continuous up to the boundary of* $SG_{h^\star}$*, and attains the boundary conditions, i.e., for* $(w_1, w_2) \in \partial SG_{h^*}$*,*

$$\psi^\star(w_1, w_2) = \begin{cases} 0 & \text{if} \quad w_2 = -1 \text{ or } w_2 = h^\star(w_1), \\ \sqrt{1 - w_2{}^2} & \text{if} \quad w_1 = 0 \text{ or } w_1 = 2l, \end{cases} \tag{12.15}$$

hence

$$\mathcal{F}_{2l}(h^\star, \psi^\star) = \mathbb{A}(\psi^\star, SG_{h^\star}); \tag{12.16}$$

(iv) we have

$$\psi^\star < \widehat{\varphi} \text{ in } R_{2l}. \tag{12.17}$$

The rest of this section is devoted to prove this theorem; we start with some preparation.

Definition 12.7 (**Convergence in** X_{2l}^{conv})**.** *We say that a sequence $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$ converges to $(h, \psi) \in X_{2l}^{\mathrm{conv}}$, if*

- *(h_n) converges to h uniformly on compact subsets of $(0, 2l)$;*
- *(ψ_n) converges to ψ in $L^1(R_{2l})$.*

Lemma 12.8 (**Closedness of** X_{2l}^{conv})**.** *Let $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$ be a sequence such that (h_n) converges to $h \in \mathcal{H}_{2l}$ uniformly on compact subsets of $(0, 2l)$, and (ψ_n) converges to $\psi \in \mathrm{BV}(R_{2l})$ in $L^1(R_{2l})$. Then $(h, \psi) \in X_{2l}^{\mathrm{conv}}$.*

Proof. Possibly passing to a (not relabelled) subsequence, we can assume that (ψ_n) converges to ψ pointwise in $A \subseteq R_{2l}$, with $\mathcal{H}^2(R_{2l} \setminus A) = 0$, and $\psi_n = 0$ in $A \cap (R_{2l} \setminus SG_{h_n})$ for all $n \in \mathbb{N}$. We only have to show that $\psi = 0$ in $A \cap (R_{2l} \setminus SG_h)$. We can also assume that A does not intersect the graph of h. If $(w_1, w_2) \in A \cap (R_{2l} \setminus SG_h)$, then $w_2 > h(w_1)$. From the local uniform convergence of (h_n) to h in $(0, 2l)$ it follows that $w_2 > h_n(w_1)$ for n large enough, *i.e.*, $(w_1, w_2) \in A \cap (R_{2l} \setminus SG_{h_n})$, and the assertion follows. $\square$

Lemma 12.9 (**Lower semicontinuity of $\mathcal{F}_{2l}$**)**.** *Let $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$ be a sequence converging to $(h, \psi) \in X_{2l}^{\mathrm{conv}}$ in the sense of Definition 12.7. Then*

$$\mathcal{F}_{2l}(h, \psi) \leq \liminf_{n \to +\infty} \mathcal{F}_{2l}(h_n, \psi_n). \tag{12.18}$$

Proof. It is standard[59] to show that the functional

$$\psi \in BV(R_{2l}, [0,1]) \to \mathbb{A}(\psi, R_{2l}) + \int_{\partial_D R_{2l}} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi| \, d\mathcal{H}^1 \tag{12.19}$$

is $L^1(R_{2l})$-lower semicontinuous. Since (h_n) converges to h pointwise in $(0, 2l)$, we also have $\lim_{n \to +\infty} \mathcal{H}^2(R_{2l} \setminus SG_{h_n}) = \mathcal{H}^2(R_{2l} \setminus SG_h)$. The assertion follows. $\square$

Proposition 12.10 (**Existence of a minimizer of** (12.14))**.** *There exists $(h^\star, \psi^\star) \in X_{2l}^{\mathrm{conv}}$ satisfying* (12.14)*.*

Proof. Note that

$$h(w_1) := -1, \quad \psi(w_1, w_2) := 0, \qquad (w_1, w_2) \in R_{2l},$$

is a competitor in (12.14). Hence, for a minimizing sequence $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$, recalling (12.11) we have

$$\lim_{n \to +\infty} \mathcal{F}_{2l}(h_n, \psi_n) = \inf \left\{ \mathcal{F}_{2l}(h, \psi) : (h, \psi) \in X_{2l}^{\mathrm{conv}} \right\} \leq \pi. \tag{12.20}$$

Thus $\sup_{n \in \mathbb{N}} |D\psi_n|(R_{2l}) < +\infty$, and there exists $\psi^\star \in BV(R_{2l}, [0,1])$ such that, up to a (not relabelled) subsequence, (ψ_n) converges to $\psi^\star$ in $L^1(\Omega)$.

Using Lemmas 12.5 and 12.8, we may assume that (h_n) converges locally uniformly to some $h^\star \in \mathcal{H}_{2l}$, and $\psi^\star = 0$ in $R_{2l} \setminus SG_{h^\star}$. The assertion then follows from Lemma 12.9. $\square$

We now turn to the regularity of minimizers.

Proposition 12.11 (**Analyticity and positivity of a minimizer**)**.** *Suppose that (h, ψ) is a minimizer of* (12.14)*, and that h is not identically -1. Then ψ is analytic in SG_h, and solves the equation*

$$\operatorname{div}\Big(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \Big) = 0 \qquad \text{in } SG_h. \tag{12.21}$$

Moreover $\psi > 0$ in SG_h.

[59]Indeed, let $\tilde{\varphi} : \partial R_{2l} \to [0,1]$ be defined as $\tilde{\varphi} := \varphi$ on $\partial_D R_{2l}$, and $\tilde{\varphi} := 0$ on $\partial R_{2l} \setminus \partial_D R_{2l}$. Let $B \subset \mathbb{R}^2$ be an open disc containing $\overline{R_{2l}}$. We extend $\tilde{\varphi}$ to a $W^{1,1}$ function in $B \setminus \overline{R_{2l}}$, [19, Thm. 2.16], and we still denote by $\tilde{\varphi}$ such an extension. For every $\psi \in BV(R_{2l})$, define $\widehat{\psi} := \psi$ in R_{2l} and $\widehat{\psi} := \tilde{\varphi}$ in $B \setminus R_{2l}$. We have

$$\mathbb{A}(\psi, R_{2l}) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi| d\mathcal{H}^1 = \mathbb{A}(\psi, R_{2l}) + \int_{\partial R_{2l}} |\psi - \tilde{\varphi}| d\mathcal{H}^1 = \mathbb{A}(\widehat{\psi}, B) - \mathbb{A}(\tilde{\varphi}, B \setminus \overline{R_{2l}}),$$

where the last equality follows from [19, (2.15)]. Thus the lower semicontinuity of the functional in (12.19) follows from the $L^1(B)$-lower semicontinuity of the area functional.

Proof. Since by assumption h is not identically -1, we have that SG_h is nonempty. Moreover minimality ensures

$$\int_{SG_h} \sqrt{1+|D\psi|^2} \leq \int_{SG_h} \sqrt{1+|D\psi_1|^2}$$

for any $\psi_1 \in BV(SG_h)$, $\mathrm{spt}(\psi - \psi_1) \subset\subset SG_h$. Thus, by [19, Thm 14.13], ψ is locally Lipschitz, and hence analytic, in SG_h, and (12.21) follows. Now, let $z \in SG_h$ and take a disc $B_\eta(z) \subset\subset SG_h$. Since $\psi \geq 0$ on $\partial B_\eta(z)$ we find, by the strong maximum principle [19, Thm. C.4], that either ψ is identically zero in $B_\eta(z)$, or $\psi > 0$ in $B_\eta(z)$. Hence from the analyticity of ψ and the arbitrariness of z, we have that either ψ is identically zero in SG_h or $\psi > 0$ in SG_h. Now $\mathcal{F}_{2l}(h,0) = |SG_h| + \pi > \mathcal{F}_{2l}(-1,0) = \pi$, see (12.11). Thus $(h,0)$ is not a minimizer, and the positivity of ψ in SG_h is achieved.

□

Lemma 12.12 (Symmetric minimizers). *There exists a minimizer (h,ψ) of (12.14) such that $h(\cdot) = h(2l - \cdot)$ and ψ is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$.*

Proof. Let (h,ψ) be a minimizer of (12.14). Let $I \subset (0,2l)$ be an open interval; consistently with (12.4), and since ψ is continuous in SG_h, we set

$$\begin{aligned}\mathcal{F}_{2l}(h,\psi;I) :=& \mathbb{A}(\psi, I\times(-1,1)) - \mathcal{H}^2\Big(I\times(-1,1)\setminus SG_h\Big) + \int_{(\partial_D R_{2l})\cap(\overline{I}\times[-1,1))} |\psi-\varphi|\, d\mathcal{H}^1 \\ &+ \int_{(\partial R_{2l}\setminus\partial_D R_{2l})\cap(\overline{I}\times(-1,1])} |\psi|\, d\mathcal{H}^1.\end{aligned}$$

Recall that $h \in \mathcal{H}_{2l}$, hence its graph is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$. Define $\tilde{\psi} := \psi$ on $(0,l)\times(-1,1)$ and $\tilde{\psi}(w_1,w_2) := \psi(2l - w_1, w_2)$ for $(w_1,w_2) \in (l,2l)\times(-1,1)$, in particular the graph of $\tilde{\psi}$ is symmetric with respect to $\{w_1 = l\}\cap R_{2l}$. Since $\mathcal{F}_{2l}(h,\psi;(0,l)) = \mathcal{F}_{2l}(h,\psi;(l,2l))$, it follows $\mathcal{F}_{2l}(h,\tilde{\psi}) = \mathcal{F}_{2l}(h,\psi)$ for, if $\mathcal{F}_{2l}(h,\psi;(0,l)) < \mathcal{F}_{2l}(h,\psi;(l,2l))$, then $\mathcal{F}_{2l}(h,\tilde{\psi}) < \mathcal{F}_{2l}(h,\psi)$ which contradicts the minimality of (h,ψ). □

Lemma 12.13. *Suppose that (h,ψ) is a minimizer of (12.14) such that:*

(i) $h(\cdot) = h(2l - \cdot)$;

(ii) ψ *is symmetric with respect to* $\{w_1 = l\}\cap R_{2l}$;

(iii) h *is not identically* -1.

Then

$$h(w_1) > -1 \qquad \forall w_1 \in [0,2l].$$

Proof. By the symmetry of h and ψ with respect to $\{w_1 = l\}\cap R_{2l}$ (Lemma 12.12) we may restrict our argument to $[0,l]$. Assume by contradiction that there exists $\overline{w}_1 \in (0,l]$ such that $h(\overline{w}_1) = -1$. Recall that h is convex, nonincreasing in $[0,l]$ and continuous at l. Let

$$w_1^0 := \min\{w_1 \in (0,l] : h(w_1) = -1\}.$$

Since h is not identically -1, we have $w_1^0 > 0$, and h is strictly decreasing in $(0,w_1^0)$. Let

$$h^{-1} : [-1,h(0)] \to [0,w_1^0]$$

be the inverse of $h|_{[0,w_1^0]}$. We have, using (12.5) and (12.6),

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{2l}(h,\psi) &= \frac{1}{2}\left[\mathbb{A}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi-\varphi| \, d\mathcal{H}^1 + \int_{G_h\setminus\{w_2=-1\}} |\psi^-| \, d\mathcal{H}^1 + \int_{L_h} \varphi \, d\mathcal{H}^1\right] \\
=&\frac{1}{2}\mathbb{A}(\psi, SG_h) + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)| \, dw_2 + \int_{(0,w_1^0)} |\psi(w_1,-1)-\varphi(w_1,-1)| \, dw_1 \\
&+ \int_{G_{h\llcorner(0,w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2)dw_2.
\end{aligned}$$

Now, we argue by slicing the rectangle $R_l = (0,l)\times(-1,1)$ with lines $\{w_1=\tau\}, \tau\in(0,l)$. Recalling the expression of SG_h (which is non empty by assumption (iii)), and neglecting the third addendum,

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{2l}(h,\psi) =& \int_0^{w_1^0}\int_{-1}^{h(w_1)} \sqrt{1+|\nabla\psi|^2} \, dw_2dw_1 \\
&+ \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)|dw_2 + \int_{(0,w_1^0)} |\psi(w_1,-1)-\varphi(w_1,-1)| \, dw_1 \\
&+ \int_{G_{h\llcorner(0,w_1^0)}} |\psi| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2)dw_2 \\
\geq& \int_0^{w_1^0}\int_{-1}^{h(w_1)} \sqrt{1+|\nabla\psi|^2} dw_2dw_1 + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)|dw_2 \\
&+ \int_{G_{h\llcorner(0,w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2)dw_2 \\
>& \int_0^{w_{10}}\int_{-1}^{h(w_1)} |D_{w_1}\psi(w_1,w_2)|dw_2dw_1 + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)|dw_2 \\
&+ \int_{G_{h\llcorner(0,w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2)dw_2,
\end{aligned}$$

where D_{w_1} stands for the partial derivative with respect to w_1. Neglecting $\sqrt{1+(\frac{d}{ds}h^{-1})^2}$ in the third addendum on the right-hand side, we deduce

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{2l}(h,\psi) >& \int_{-1}^{h(0)}\int_0^{h^{-1}(w_2)} |D_{w_1}\psi(w_1,w_2)| \, dw_1dw_2 + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)|dw_2 \\
&+ \int_{(-1,h(0))} \psi(h^{-1}(w_2),w_2)dw_2 + \int_{(h(0),1)} \varphi(0,w_2)dw_2 \\
\geq& \int_{-1}^{h(0)} \left|\int_0^{h^{-1}(w_2)} D_{w_1}\psi(w_1,w_2)dw_1\right| dw_2 - \int_{(-1,h(0))} \psi(0,w_2)dw_2 \\
&+ \int_{(-1,h(0))} \psi(h^{-1}(w_2),w_2)dw_2 + \int_{(-1,1)} \varphi(0,w_2)dw_2 \\
\geq& \int_{(-1,h(0))} |\psi(h^{-1}(w_2),w_2)-\psi(0,w_2)|dw_2 - \int_{(-1,h(0))} \psi(0,w_2)dw_2 \\
&+ \int_{(-1,h(0))} \psi(h^{-1}(w_2),w_2)dw_2 + \int_{(-1,1)} \varphi(0,w_2)dw_2 \\
\geq& \int_{(-1,1)} \varphi(0,w_2)dw_2 = \frac{1}{2}\mathcal{F}_{2l}(-1,0).
\end{aligned}$$

Hence the value of $\mathcal{F}_{2l}$ on the pair $h \equiv -1$, $\psi \equiv 0$ is smaller than $\mathcal{F}_{2l}(h, \psi)$, thus contradicting the minimality of (h, ψ). □

We now prove point (iii) of Theorem 12.6: this will be a consequence of the next lemma and Theorem 12.16.

Lemma 12.14. *Let (h, ψ) be as in Lemma 12.13. Then ψ attains the boundary condition on $\partial_D SG_h$.*

Proof. The result follows from [19, Theorem 15.9], since $\partial_D SG_h$ is union of three segments. □

Remark 12.15. In the hypotheses of Lemma 12.13, if $h \equiv 1$ then the graph of h is a segment and, as in Lemma 12.14, $\psi = 0$ on G_h.

The conclusion of the proof of Theorem 12.6 (iii) is given by the following delicate result.

Theorem 12.16. *Let (h, ψ) be as in Lemma 12.13. Then there exists a solution $(\widetilde{h}, \widetilde{\psi}) \in X_{2l}^{\rm conv}$ of the minimum problem* (12.14) *such that $L_{\widetilde{h}}$ is empty, $\widetilde{\psi}$ is continuous up to $G_{\widetilde{h}}$, and*

$$\widetilde{\psi} = 0 \quad \text{on } G_{\widetilde{h}}.$$

Proof. By Remark 12.15, we can assume that h is not identically 1 and, by Lemma 12.13, also that $h(w_1) \geq h(l) > -1$ for any $w_1 \in [0, 2l]$. Therefore, fix a number $\bar{s} \in (-1, h(l))$ and set

$$K := (0, 2l) \times (\bar{s}, 1) \subset R_{2l}.$$

We extend ψ in $\mathbb{R}^2 \setminus R_{2l}$ as follows: we define $\widehat{\psi} : \mathbb{R}^2 \to [0, 1]$, $\widehat{\psi} := \psi$ in R_{2l}, and

$$\widehat{\psi}(w_1, w_2) := \begin{cases} \varphi(w_2) & \text{if } w_1 < 0 \text{ or } w_1 > 2l, \text{ and } |w_2| \leq 1, \\ 0 & \text{if } |w_2| > 1. \end{cases}$$

In this way $\widehat{\psi}$ is continuous in $\mathbb{R}^2 \setminus \overline{R}_{2l}$.

Now, we divide the proof into six steps. We start by regularizing $\widehat{\psi}$ (step 1) in order that the regularized functions have smooth graphs (hence of disc-type[60]). Next (step 2), we will compare these graphs with the solution of a suitable disc-type Plateau problem.

Step 1: Approximation of $\widehat{\psi}$. Let $n > 0$ be a natural number (that will be sent to $+\infty$ later) such that $\bar{s} + \frac{1}{n} < h(l)$, and consider the enlarged rectangle

$$K_n := \left(-\frac{1}{n}, 2l + \frac{1}{n} \right) \times \left(\bar{s}, 1 + \frac{1}{n} \right), \tag{12.22}$$

see Fig. 19. Note that

$$\widehat{\psi} \text{ is continuous on } \partial K_n.$$

Given $n \in \mathbb{N}$, we claim that we can build a sequence $(\psi_k^n)_{k \in \mathbb{N}}$ (depending on n) which satisfies the following properties:

$$\begin{aligned} &\psi_k^n \in C^\infty(K_n, [0, 1]) \cap C(\overline{K}_n, [0, 1]) \qquad \forall k > 0, \\ &\psi_k^n \rightharpoonup \widehat{\psi} \text{ weakly}^\star \text{ in } BV(K_n) \text{ as } k \to +\infty, \\ &\int_{K_n} |\nabla \psi_k^n| \, dw \to |D\widehat{\psi}|(K_n) \text{ as } k \to +\infty, \\ &\psi_k^n = \widehat{\psi} \text{ on } \partial K_n \qquad \forall k > 0. \end{aligned} \tag{12.23}$$

[60] We expect the graph of $\widehat{\psi}$, considering also a possible vertical part over the graph of h, to be a surface of disc-type; however, we miss the proof of this fact, mainly due to possible high degree of irregularity of the trace of $\widehat{\psi}$ over G_h.

In order to obtain these features for ψ_k^n we use standard arguments (details can be found in [2, Thm. 3.9] or [18, Thm. 1, Section 4.1.1]). To the aim of our discussion, we just recall that we proceed by constructing an increasing sequence $(U_i)_{i\geq 1}$ of subsets of K_n, $U_i = U_{i,n}$, $U_i \subset\subset U_{i+1} \subset\subset K_n$, $\cup_i U_i = K_n$ (for $i \geq 1$ we take $U_i := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathbb{R}^2 \setminus K_n) > \frac{1}{i+n}\}$ for definitiveness) and with the aid of a partition of unity (η_i) associated to $V_1 := U_2$, $V_i = V_{i,n} := U_{i+1} \setminus \overline{U}_{i-1}$ for $i \geq 2$, we mollify $\widehat{\psi}$ accordingly in V_i. For our purpose we choose[61] η_i in such a way that

$$\mathrm{supp}\ (\eta_i) = \overline{V}_i. \tag{12.24}$$

Since ψ_k^n is obtained by mollification we have $\psi_k^n \in C^\infty(K_n)$ and moreover $\psi_k^n \in C(\overline{K}_n)$ because it attains the continuous boundary datum $\widehat{\psi}$ on ∂K_n. Notice that we use the same mollifier $\rho \in C_c^\infty(B_1)$ in each V_i, choosing $\rho_i(w) = \rho_{i,k}(w) := \rho(w/r_{i,k})$ with $r_{i,k} := r_i/k > 0$, r_i decreasing with respect to $i \geq 1$, with[62] $r_i \to 0^+$ as $i \to +\infty$. Finally, $[0, 2l] \times [\bar{s} + \frac{1}{n}, 1] \subset U_1 \subset V_1$, and $V_i \cap \big([0, 2l] \times [\bar{s} + \frac{1}{n}, 1]\big) = \emptyset$ for $i \geq 2$. It follows

$$\psi_k^n = \widehat{\psi} \star \rho_{1,k} \qquad \text{in } [0, 2l] \times \left[\bar{s} + \frac{1}{n}, 1\right] \qquad \forall n \in \mathbb{N}. \tag{12.25}$$

Using [18, Prop. 3 Sec. 4.2.4 pag. 408, and Th. 1 Sec. 4.1.5 pag. 331] we infer

$$\mathbb{A}(\psi_k^n, K_n) \to \mathbb{A}(\widehat{\psi}, K_n) \qquad \text{as } k \to +\infty. \tag{12.26}$$

Now that properties (12.23) are achieved, by a diagonal argument we select functions $\psi_n := \psi_{k_n}^n \in (\psi_k^n)$ such that

$$\begin{aligned} &\psi_n \rightharpoonup \widehat{\psi} \text{ weakly}^* \text{ in } BV(K) \text{ as } n \to +\infty, \\ &\int_{K_n} |\nabla \psi_n|\ dw \to |D\widehat{\psi}|(\overline{K}) \text{ as } n \to +\infty, \\ &\psi_n = \widehat{\psi} \text{ on } \partial K_n \qquad \forall n \in \mathbb{N}. \end{aligned} \tag{12.27}$$

On the basis of (12.26) and (12.27), we can also ensure[63] that

$$\mathbb{A}(\psi_n, K_n) \to \mathbb{A}(\widehat{\psi}, \overline{K}) \qquad \text{as } n \to +\infty. \tag{12.28}$$

Here, by $\mathbb{A}(\widehat{\psi}, \overline{K})$ we mean the area of the graph of $\widehat{\psi}$ relative to the closed rectangle $\overline{K}$, which, recalling also Proposition 12.11, reads as

$$\mathbb{A}(\widehat{\psi}, \overline{K}) = \mathbb{A}(\widehat{\psi}, K) + \int_{\{0\}\times(\bar{s},1)} |\widehat{\psi}^- - \varphi|\ d\mathcal{H}^1 + \int_{\{2l\}\times(\bar{s},1)} |\widehat{\psi}^- - \varphi|\ d\mathcal{H}^1, \tag{12.29}$$

where $\widehat{\psi}^-$ denotes the trace of $\widehat{\psi}$ on ∂K.

---

[61] We need the full set $\overline{V}_i$ as support in order that the argument to detect the behaviour of h_n (defined in (12.32)) in $[-\frac{1}{n}, 0]$ applies.

[62] $\rho_{i,k}$ and $r_{i,k}$ depend on n. We could take $r_i = \frac{1}{i+2+n}$.

[63] To prove claim (12.28), fix $m \in \mathbb{N}$, and set $\tilde{\psi}_n := \widehat{\psi}$ outside K_n and $\tilde{\psi}_n = \psi_n$ in K_n, so that

$$\begin{aligned} &\tilde{\psi}_n \rightharpoonup \widehat{\psi} \text{ weakly}^* \text{ in } BV(K_m) \text{ as } n \to +\infty, \\ &|\nabla \tilde{\psi}_n|(K_m) \to |D\widehat{\psi}|(K_m) = |D\widehat{\psi}|(\overline{K}) + |D\widehat{\psi}|(K_m \setminus \overline{K}) \text{ as } n \to +\infty. \end{aligned}$$

Then $\limsup_{n\to+\infty} \mathbb{A}(\psi_n, K_n) \leq \limsup_{n\to+\infty} \mathbb{A}(\tilde{\psi}_n, K_m) = \mathbb{A}(\widehat{\psi}, K_m) = \mathbb{A}(\widehat{\psi}, \overline{K}) + \mathbb{A}(\widehat{\psi}, K_m \setminus \overline{K})$, the first equality following from the strict convergence of $\tilde{\psi}_n$ to $\widehat{\psi}$ [18, Prop. 3 Sec. 4.2.4 pag. 408 and Thm. 1 Sec. 4.1.5 pag. 371]. Taking the limit as $m \to +\infty$, since $\widehat{\psi} \in W^{1,1}(K_m \setminus \overline{K})$ we conclude $\limsup_{n\to+\infty} \mathbb{A}(\psi_n, K_n) \leq \mathbb{A}(\psi, \overline{K})$. Then (12.28) follows by lower-semicontinuity.

We now construct functions $h_n : (-\frac{1}{n}, 2l + \frac{1}{n}) \to (\bar{s}, 1 + \frac{1}{n})$ such that

$$
\begin{aligned}
& h_n(\cdot) = h_n(2l - \cdot), \\
& \psi_n = 0 \quad K_n \setminus SG_{h_n}, \\
& h_n \text{ is nondecreasing in } \Big[-\frac{1}{n}, l\Big], \\
& \mathcal{H}^2(S_{h_n}) \to \mathcal{H}^2(SG_h \cap K),
\end{aligned} \tag{12.30}
$$

where

$$
S_{h_n} := \left\{ (w_1, w_2) : w_1 \in \left(-\frac{1}{n}, 2l + \frac{1}{n}\right),\ w_2 \in (\bar{s}, h_n(w_1)) \right\}, \tag{12.31}
$$

and finally

$$
\lim_{n \to +\infty} \mathbb{A}(\psi_n, S_{h_n}) = \mathcal{F}_{2l}(h, \psi) - \mathbb{A}(\psi, R_{2l} \setminus K).
$$

For any $n \in \mathbb{N}$ we define

$$
\begin{aligned}
h_n(w_1) &:= \sup \left\{ w_2 \in \left(\bar{s}, 1 + \frac{1}{n}\right) : \psi_n(w_1, w_2) > 0 \right\} \qquad \forall w_1 \in \Big(-\frac{1}{n}, 2l + \frac{1}{n}\Big). \\
\widehat{h}(w_1) &:= \sup \left\{ w_2 \in \left(\bar{s}, 1 + \frac{1}{n}\right) : \widehat{\psi}(w_1, w_2) > 0 \right\} \qquad \forall w_1 \in \Big(-\frac{1}{n}, 2l + \frac{1}{n}\Big).
\end{aligned} \tag{12.32}
$$

Since (see Proposition 12.11) $\widehat{\psi}$ is positive in $SG_h \cup ((-\frac{1}{n}, 0) \times (\bar{s}, 1)) \cup ((2l, 2l + \frac{1}{n}) \times (\bar{s}, 1))$ it turns out, recalling also that ψ_n is obtained by mollification, that

$$
\begin{aligned}
& -1 < h(w_1) < h_n(w_1) < 1 + \frac{1}{n} \qquad \forall w_1 \in (0, 2l), \\
& 1 < h_n(w_1) < 1 + \frac{1}{n} \qquad \forall w_1 \in \Big(-\frac{1}{n}, 0\Big] \cup \Big[2l, 2l + \frac{1}{n}\Big).
\end{aligned} \tag{12.33}
$$

Also, $\widehat{h} = h$ in $[0, 2l]$, and $\widehat{h} = 1$ in $(-1/n, 0) \cup (2l, 2l + 1/n)$. Moreover, again the positivity of $\widehat{\psi}$ implies that

$$
\psi_n > 0 \quad \text{in} \quad S_{h_n} \subset K_n, \tag{12.34}
$$

whereas

$$
\psi_n(w_1, w_2) = 0 \qquad \text{if } w_1 \in \Big(-\frac{1}{n}, 2l + \frac{1}{n}\Big),\ w_2 \in \Big[h_n(w_1), 1 + \frac{1}{n}\Big), \tag{12.35}
$$

because $\widehat{\psi}(w_1, w_2) = 0$ if $w_1 \in [0, 2l]$, $w_2 > h(w_1)$ and if $w_2 > 1$. Exploiting (12.25), and the fact that h is nonincreasing (resp. nondecreasing) in $[0, l]$ (resp. in $[l, 2l]$), one checks[64] that also h_n is nonincreasing in $[0, l]$ (and nondecreasing in $[l, 2l]$). Concerning the behaviour of h_n in $(-\frac{1}{n}, 0]$ (and similarly in $[2l, 2l + \frac{1}{n})$), we see that in $V_i = V_{i,n}$ $(i > 1)$, we are mollifying with ρ_{i,k_n} whose radius of mollification is r_i/k_n, so that $\widehat{\psi} \star \rho_{i,k_n}$ equals 0 on the line $\{w_2 = 1 + \frac{r_i}{k_n}\}$, and nonzero below inside

[64]Let us show for instance that h_n is decreasing in $[0, l]$. Recall that the function $\widehat{\psi}$ vanishes above the graph of h, which is decreasing in $[0, l]$. Now, take a point $(w_1, w_2) \in K_n$, $w_1 \in [0, l)$, $w_2 > h(w_1)$; suppose first that $w_1 \geq r_1$. If $\mathrm{dist}((w_1, w_2), \mathrm{graph}(h)) > r_1$, then $\psi_n(w_1, w_2) = \widehat{\psi} \star \rho_1(w_1, w_2) = 0$, and if $\mathrm{dist}((w_1, w_2), \mathrm{graph}(h)) < r_1$, then $\psi_n(w_1, w_2) = \widehat{\psi} \star \rho_1(w_1, w_2) > 0$. Hence, if $\widehat{\psi} \star \rho_1(w_1, w_2) = 0$ then also $\widehat{\psi} \star \rho_1(w_1 + \varepsilon, w_2) = 0$ for $\varepsilon > 0$ small enough, because $\mathrm{dist}((w_1 + \varepsilon, w_2), \mathrm{graph}(h)) > \mathrm{dist}((w_1, w_2), \mathrm{graph}(h))$, being h decreasing in $[0, l]$. This argument applies also when $w_1 \in [0, r_1)$, by (12.25), since $\overline{h}$ is nonincreasing also in $(-1/n, l)$.

K_n (this follows from the fact that $\widehat{\psi}$ is 0 on the line $\{w_2 = 1\} \cap K$ and nonzero below). We have defined the radii r_i in such a way that they are decreasing with respect to i, so that, ψ_n being the sum of $\widehat{\psi} \star \rho_{i,k_n}$ (whose support is $\overline{V}_{i,n}$ by (12.24)), it turns out that ψ_n is 0 on $\{w_2 = 1 + \frac{r_i}{k_n}\}$ and nonzero below[65] in $V_{i,n} \setminus V_{i-1,n}$ (from this it follows that $h_n = 1 + \frac{r_i}{k_n}$ in $(-\frac{1}{n} + \frac{1}{i+n+1}, -\frac{1}{n} + \frac{1}{i+n}]$). In particular h_n is piecewise constant and nondecreasing in $(-\frac{1}{n}, 0]$.

Therefore we have

$$h_n \in BV\Big((-\frac{1}{n}, 2l + \frac{1}{n})\Big). \tag{12.36}$$

Finally, it is not difficult to see that the functions h_n converge to h in $L^1((0, 2l))$ as $n \to \infty$, and

$$\mathcal{H}^2(S_{h_n}) \to \mathcal{H}^2(SG_h \cap K). \tag{12.37}$$

From this, (12.28), Lemma 12.14, (12.29) and (12.4) we deduce

$$\mathbb{A}(\psi_n, S_{h_n}) = \mathbb{A}(\psi_n, K_n) - \mathcal{H}^2(K_n \setminus S_{h_n}) \to \mathcal{F}_{2l}(h, \psi) - \mathbb{A}(\psi, R_{2l} \setminus K). \tag{12.38}$$

Step 2: Comparison with a Plateau problem. In this step we want to compare the graph of ψ_n with the solution of a disc-type Plateau problem. In particular we will obtain a disc-type surface Σ_n^+ whose area is smaller than or equal to the area of the graph of ψ_n, see (12.39). In step 3 (see (12.41)) we will compare this surface with the graph of ψ on K.

We recall that ψ_n is continuous in $\overline{K}_n$, it is positive on the bottom edge $[-\frac{1}{n}, 2l + \frac{1}{n}] \times \{\bar{s}\}$ of K_n (see (12.27)), it is zero on the top edge $[-\frac{1}{n}, 2l + \frac{1}{n}] \times \{1 + \frac{1}{n}\}$ by (12.33), and on the lateral edges of K_n it coincides with $\widehat{\psi}$; more specifically

$$\begin{aligned}
&\psi_n\Big(-\frac{1}{n}, w_2\Big) = \psi_n\Big(2l + \frac{1}{n}, w_2\Big) = \varphi(0, w_2) > 0 \qquad \text{for } w_2 \in [\bar{s}, 1),\\
&\psi_n\Big(-\frac{1}{n}, w_2\Big) = \psi_n\Big(2l + \frac{1}{n}, w_2\Big) = 0 \qquad \text{for } w_2 \in \Big[1, 1 + \frac{1}{n}\Big).
\end{aligned}$$

Define

$$\partial_D K_n := \Big(\Big[-\frac{1}{n}, 2l + \frac{1}{n}\Big] \times \{\bar{s}\}\Big) \cup \Big(\Big\{-\frac{1}{n}, 2l + \frac{1}{n}\Big\} \times [\bar{s}, 1]\Big).$$

From (12.27), we see that ψ_n coincides with $\widehat{\psi}$ over $\partial_D K_n$, and its graph over this set is a curve, that we denote by Γ_n^+. This curve, excluding its endpoints $P_n = (-\frac{1}{n}, 1, 0)$ and $Q_n = (2l + \frac{1}{n}, 1, 0)$, is contained in the half-space $\{w_3 > 0\}$, while $P_n, Q_n \in \{w_3 = 0\}$. We further denote by Γ_n^- the symmetric of Γ_n^+ with respect to the plane $\{w_3 = 0\}$, so that

$$\Gamma_n := \Gamma_n^+ \cup \Gamma_n^-$$

is a Jordan curve in $\mathbb{R}^3$, see Fig. 19. We can now solve the disc-type Plateau problem with boundary Γ_n [14] and call $\Sigma_n \subset \mathbb{R}^3$ one of its solutions[66]. Finally, we can assume that Σ_n is symmetric with respect to $\{w_3 = 0\}$ and that

$$\mathcal{H}^2(\Sigma_n^+) = \mathcal{H}^2(\Sigma_n^-),$$

with $\Sigma_n^\pm := \Sigma_n \cap \{w_3 \gtrless 0\}$, respectively (see Fig. 19).

We now want to compare the area of the graph of ψ_n in S_{h_n} with $\mathcal{H}^2(\Sigma_n^+)$. To this aim we start by observing that ψ_n, being smooth in K_n and continuous in $\overline{K}_n$, is such that its graph over S_{h_n}

[65] Notice that in $V_{i,n} \setminus V_{i-1,n}$ only $\widehat{\psi} \star \rho_{i,k_n}$ and $\widehat{\psi} \star \rho_{i+1,k_n}$, are nonzero.

[66] Σ_n is the image of an area-minimizing map from the unit disc into $\mathbb{R}^3$.

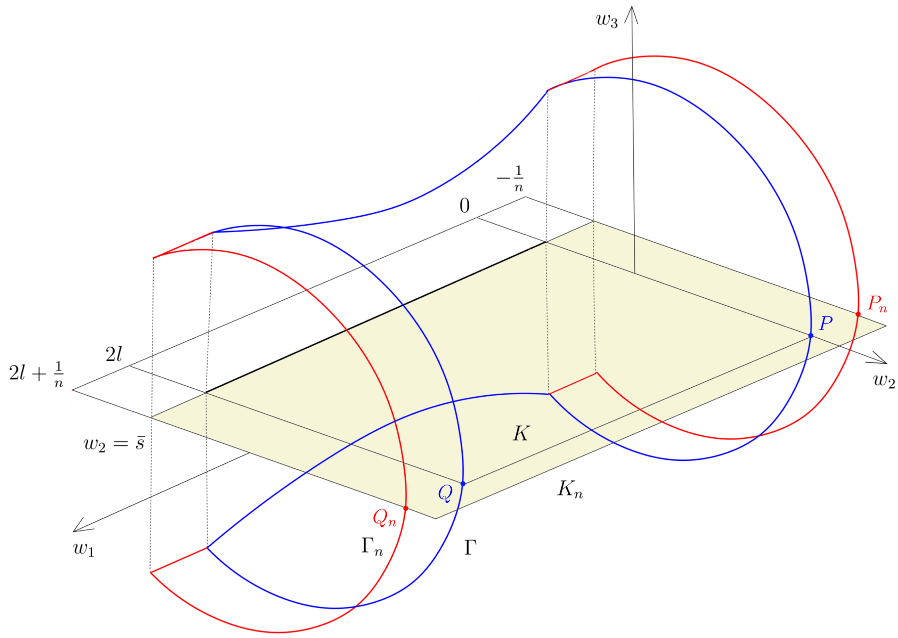

Figure 19: The rectangle K_n in (12.22) is colored, and the rectangle inside is K. Γ is the curve passing through Q and P, the curves Γ_n (which pass through Q_n and P_n) approach the curve Γ (Γ and Γ_n coincide and overlap on the graph of ψ over the bold segment $\{w_2 = \bar{s}\} \cap K$).

has the topology of S_{h_n}, that is the topology of the disc[67]. Denoting by $\mathcal{G}^+_{\psi_n}$ the graph of ψ_n over S_{h_n}, we consider the graph $\mathcal{G}^-_{\psi_n}$ of $-\psi_n$ over S_{h_n}, and observe that the closure of $\mathcal{G}^+_{\psi_n} \cup \mathcal{G}^-_{\psi_n}$ is a disc-type surface with boundary Γ_n. Therefore, by minimality,

$$\mathbb{A}(\psi_n, S_{h_n}) = \mathcal{H}^2(\mathcal{G}^+_{\psi_n}) \geq \mathcal{H}^2(\Sigma^+_n). \tag{12.39}$$

Step 3: Passing to the limit as $n \to +\infty$: the surface Σ. The graph of ψ over the segment $[0, 2l] \times \{\bar{s}\}$ and the graph of φ over the two segments $\{0, 2l\} \times [\bar{s}, 1]$ form a simple continuous curve Γ^+ which, excluding the two endpoints $P = (0, 1, 0)$ and $Q = (2l, 1, 0)$, is contained in the half-space $\{w_3 > 0\}$, while $P, Q \in \{w_3 = 0\}$ (see Fig. 19). If we consider

$$\Gamma := \Gamma^+ \cup \Gamma^-,$$

with Γ^- the symmetric of Γ^+ with respect to $\{w_3 = 0\}$, a direct check shows that the curves Γ_n converge to the curve Γ in the sense of Frechet [27], as $n \to +\infty$. As a consequence, the area-minimizing disc-type surfaces Σ_n satisfy $\mathcal{H}^2(\Sigma_n) \to \mathcal{H}^2(\Sigma)$ (see [27, Paragraphs 301, 305]), with Σ a disc-type area-minimizing surface spanned by Γ. It follows

$$\mathcal{H}^2(\Sigma^+_n) \to \mathcal{H}^2(\Sigma^+) \qquad \text{as } n \to +\infty, \tag{12.40}$$

[67] S_{h_n} is bounded by construction, and it is open from (12.34), (12.35). In addition, it is connected and simply connected. Indeed, take any continuous curve $\gamma : S^1 \to S_{h_n}$. Using (12.33), let $\widehat{s} \in (\bar{s}, 1)$ be such that $\{w_2 = \widehat{s}\} \cap K_n \subset S_{h_n}$; hence we can (vertically) contract γ continuously to its projection on the line $\{w_2 = \widehat{s}\}$, and then contract it continuously to the middle point of $\{w_2 = \widehat{s}\} \cap K_n$, showing that γ is homotopic to the constant curve. Hence, by the Riemann mapping theorem, S_{h_n} is biholomorphic to the open unit disc, and $\overline{S}_{h_h}$ is homeomorphic to the closure of the disc, thanks to the fact that ∂S_{h_n} is a Jordan curve, due to the BV-regularity of h_n.

where $\Sigma^+ := \Sigma \cap \{w_3 > 0\}$. From (12.40), (12.39), (12.38) we deduce

$$\begin{aligned}\mathcal{H}^2(\Sigma^+) &= \lim_{n\to+\infty} \mathcal{H}^2(\Sigma_n^+) \le \lim_{n\to+\infty} \mathbb{A}(\psi_n, S_{h_n}) \\ &= \lim_{n\to+\infty} \big(\mathbb{A}(\psi_n, K_n) - \mathcal{H}^2(K_n \setminus S_{h_n})\big) = \mathcal{F}_{2l}(h,\psi) - \mathbb{A}(\psi, R_{2l} \setminus K).\end{aligned}$$

Since $\psi_n = \psi$ on $R_{2l} \setminus K$, we get

$$\lim_{n\to+\infty} (\mathbb{A}(\psi_n, S_{h_n}) + \mathbb{A}(\psi, R_{2l} \setminus K)) = \mathcal{F}_{2l}(h,\psi) \ge \mathcal{H}^2(\Sigma^+) + \mathbb{A}(\psi, R_{2l} \setminus K). \tag{12.41}$$

Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) : \overline{B}_1 \to \Sigma \subset \mathbb{R}^3$ be a parametrization of Σ, which is analytic and conformal in the open unit disc B_1 and continuous up to ∂B_1 with $\Phi(\partial B_1) = \Gamma$. Exploiting the results in [25] (see also [14, pag. 343]) we know that

$$\Phi \text{ is an embedding,} \tag{12.42}$$

since Γ is a simple curve on the boundary of the convex set $K \times \mathbb{R}$.

Now, we need to prove several qualitative properties of Σ.

Step 4: $\Sigma \cap \{w_3 = 0\}$ is a simple curve connecting P and Q. This can be seen as follows: Assume $\Phi(p_0) = P$ and $\Phi(q_0) = Q$ for two distinct points $p_0, q_0 \in \partial B_1$. By standard arguments[68], the disc B_1 is splitted into two connected components $\{x \in B_1 : \Phi_3(x) \ge 0\}$ and $\{x \in B_1 : \Phi_3(x) < 0\}$ and the set $\{\Phi_3 = 0\}$ must be a simple curve in B_1 connecting p_0 and q_0 (here we use that the points p_0 and q_0 are the unique points in ∂B_1 where $\Phi_3 = 0$ and that the two arcs in ∂B_1 with extreme points p_0 and q_0 are mapped in $\{w_3 > 0\}$ and $\{w_3 < 0\}$ respectively). By the injectivity of Φ (property (12.42)) we conclude that

$$\Gamma_0 := \Phi(\{\Phi_3 = 0\}) \tag{12.43}$$

is a simple curve connecting P and Q on the plane $\{w_3 = 0\}$, and more specifically $\Gamma_0 \subset K$.

In the next step we show that, due to the particular form of Γ, the surface Σ admits a semicartesian parametrization [8], namely that if we slice Σ with a plane orthogonal to the first coordinate (in $(0, 2l)$) then the intersection is a curve connecting the two corresponding points on Γ; in addition, in this present case, this curve turns out to be simple. We will also show that the free part of Σ, *i.e.*, Γ_0, leaves a trace on R_{2l} which is the graph of a convex function (of one variable).

Step 5: The projection $p(\Sigma)$ of Σ on the plane $\{w_3 = 0\}$ is the subgraph of a convex function $\widetilde{h} \in \mathcal{H}_{2l}$.

We first show that $p(\Sigma)$ is the subgraph of a function $\widetilde{h}$, and then we prove that $\widetilde{h} \in \mathcal{H}_{2l}$. Take a point $W = (W_1, W_2, W_3) \in \Sigma \setminus \Gamma$; by the strong maximum principle, $p(W) \notin \partial K$ (this follows since points in $\Sigma \setminus \Gamma$ are in the interior of the convex envelope of Γ, see [14]). Consider the (unique) point $x \in B_1$ such that $\Phi(x) = W$. Due to the particular structure of Γ, one easily checks that $\partial B_1 = \Phi^{-1}(\Gamma)$ splits into two connected components, $\Phi_1^{-1}((W_1, 2l]) \cap \partial B_1$ and $\Phi_1^{-1}([0, W_1]) \cap \partial B_1$, since $\Phi_1^{-1}(\{W_1\}) \cap \partial B_1$ consists of two points q_1, q_2 in ∂B_1. In particular, the continuous function $\Phi_1(\cdot) - W_1$ changes sign only twice on ∂B_1, namely at q_1 and q_2. From Rado's lemma [14, Lemma 2, pag. 295] it follows that there are no points on $\Sigma \cap \{w_1 = W_1\}$ where the two area-minimizing surfaces Σ and the plane $\{w_1 = W_1\}$ are tangent to each other[69]. It follows that, if $\mathcal{P} \in (\Sigma \setminus \Gamma) \cap \{w_1 = W_1\}$, then the set $(\Sigma \setminus \Gamma) \cap \{w_1 = W_1\}$ is, in a neighbourhood of $\mathcal{P}$, an

[68] See also step 5 where a similar statement is proved.

[69] If $\mathcal{P}$ is a tangence point, then the differential of Φ_1 must vanish at $\Phi^{-1}(\mathcal{P}) \in B_1$.

analytic curve, see again [14, Lemma 2, pag. 295]. Hence, $\{\Phi_1 = W_1\} \cap B_1$ is, in a neighbourhood of $\Phi^{-1}(\mathcal{P})$, an analytic curve. If $\gamma_I : I \to B_1$ is a parametrization of this curve, $I = (a, b)$ a bounded open interval, we see that the limits as $t \to a^+$ and $t \to b^-$ of $\gamma_I(t)$ exist[70] and are points in $\overline{B_1}$. If $\lim_{t\to a^+} \gamma_I(t)$ belongs to ∂B_1, it must be either q_1 or q_2, if instead it is in B_1, then we can always extend γ_I in a neighbourhood of a and find a larger interval $J \supset I$ on which γ_I can be extended. Similarly, for $\lim_{t\to b^-} \gamma_I(t)$. Let now $I_m = (a_m, b_m)$ be a maximal interval on which γ_I is defined, so that, by maximality, the limits as $t \to a_m^+$ and $t \to b_m^-$ are q_1 and q_2, respectively. We can then consider the closure $\overline{I}_m$ of I_m and we have that $\gamma_{\overline{I}_m}(\overline{I}_m)$ is a curve in $\overline{B}_1$ joining q_1 and q_2. Thus we have proved that $\sigma_W := \Sigma \cap \{w_1 = W_1\}$ equals $\Phi(\gamma_{\overline{I}_m}(\overline{I}_m))$. In particular σ_W is a curve in $\mathbb{R}^3$ contained in the plane $\{w_1 = W_1\}$ and connecting the points $Q_1 := \Phi(q_1) \in \Gamma$ and $Q_2 := \Phi(q_2) \in \Gamma$. But we know that $p(Q_1) = p(Q_2) = (W_1, \bar{s}, 0)$, so $p(\sigma_W)$ is a segment in R_{2l} with endpoints $(W_1, \bar{s}, 0)$ and $(W_1, s^+, 0)$ for some $s^+ > \bar{s}$, and $s^+ \geq W_2$. In particular the whole segment "below" $p(W)$, namely the one with endpoints $(W_1, \bar{s}, 0)$ and $(W_1, W_2, 0)$, belongs to $p(\Sigma)$, and $p(\Sigma)$ is then the subgraph of some function $\widetilde{h}$. As a remark, due to the symmetry of the curve Γ, we can assume $\widetilde{h}$ is symmetric with respect to $\{w_1 = l\}$, namely $\widetilde{h}(\cdot) = \widetilde{h}(2l - \cdot)$.

Now we show that $\widetilde{h}$ is convex. Assume it is not, and take two points $(t_1, \widetilde{h}(t_1), 0), (t_2, \widetilde{h}(t_2), 0) \in R_{2l}$, such that there is a third point $(t_3, \widetilde{h}(t_3), 0)$, with $t_1 < t_3 < t_2$, which is strictly above the segment l_{12} in R_{2l} joining $(t_1, \widetilde{h}(t_1), 0)$ and $(t_2, \widetilde{h}(t_2), 0)$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a nonzero affine function[71] vanishing on the plane passing through l_{12} and orthogonal to $\{w_3 = 0\}$, and assume that f is positive at $(t_3, \widetilde{h}(t_3), 0)$. Let $\mathcal{Q} \in \Sigma$ be such that $p(\mathcal{Q}) = (t_3, \widetilde{h}(t_3), 0)$. Then $f \circ \Phi : B_1 \to \mathbb{R}$ is harmonic, and by the maximum principle there is a continuous curve $\gamma_{\mathcal{Q}}$ in B_1 joining $\Phi^{-1}(\mathcal{Q})$ to ∂B_1 such that $f \circ \Phi$ is always positive on $\gamma_{\mathcal{Q}}$. But now, the continuous curve $p \circ \Phi(\gamma_{\mathcal{Q}})$ joins $(t_3, \widetilde{h}(t_3), 0)$ to $p(\Gamma)$ and remains, in R_{2l}, strictly above the segment l_{12}. This is a contradiction, because $p \circ \Phi(\gamma_{\mathcal{Q}})$ must be in the interior of the subgraph of $\widetilde{h}$.

Before passing to step 6, recall the definition of Γ_0 in (12.43), and observe that the Jordan curve $\Gamma^+ \cup \Gamma_0$ is the boundary of the disc-type surface Σ^+. Let us denote by $U \subset K$ the connected component of $K \setminus \Gamma_0$ with boundary $\Gamma_0 \cup (\{0\} \times [\bar{s}, 1]) \cup ([0, 2l] \times \{\bar{s}\}) \cup (\{2l\} \times [\bar{s}, 1])$.

We are now in a position to show that Σ^+ admits a non-parametric description over the plane $\{w_3 = 0\}$.

Step 6: Graphicality of Σ^+: the disc-type surface Σ^+ can be written as a graph over the plane $\{w_3 = 0\}$ of a $W^{1,1}$ function $\widetilde{\psi} : U \to [0, +\infty)$. At first we observe that if Σ^+ is not Cartesian with respect to $\{w_3 = 0\}$, then there is some point $\mathcal{P} \in \Sigma^+ \setminus \partial\Sigma^+$ where the tangent plane to Σ^+ is vertical[72], that is, it contains the line $\{\mathcal{P} + (0, 0, w_3) : \ w_3 \in \mathbb{R}\}$. We will show, with an argument similar to the one needed to prove Rado's Lemma [14, Lemma 2, pag. 295], that any vertical plane is tangent to Σ in at most one point.

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[70] $\overline{B}_1$ is compact, hence $\gamma_I(t)$ has some accumulation point as $t \to a^+$. Notice that I and $\gamma_I(I)$ are homeomorphic by contruction; in turn $\gamma_I(I)$ is homeomorphic to the analytic curve $\Phi \circ \gamma_I(I)$. Assume x is an accumulation point for $\gamma_I(t)$ as $t \to a^+$. If $x \in B_1$, there is a small neighborhood U of x such that $\sigma := \Phi(U) \cap \{w_1 = W_1\}$ is an analytic curve. Then γ_I, in a right neighbourhood J of a, is homeomorphic to the analytic curve $\Phi \circ \gamma_I(J) \in \mathbb{R}^3$ emanating from $\Phi(x)$, which in turn is the restriction of σ. In particular $\gamma_I(I)$ is a curve emanating from x and the limit as $t \to a^+$ of $\gamma_I(t)$ is x. If instead $x \in \partial B_1$ then x must be the unique accumulation point. Indeed, $\lim_{t\to a^+} \Phi_1 \circ \gamma_I(t) = W_1$, and then $x = q_1$ or $x = q_2$ (say $x = q_1$). Assume there is another accumulation point y as $t \to a^+$; then $y \notin B_1$, otherwise we fall in the previous case, and therefore necessarily $y = q_2$. But in this case, we see that there must be another accumulation point $z \in B_1$ (as $t \to a^+$, we move between a neighbourhood U of x and a neighbourhood V of y frequently, so that there should be some other accumulation point in $\overline{B}_1 \setminus (U \cup V)$) leading us to the previous case again.

[71] Take the signed distance from the plane.

[72] This can be seen as follows: as shown in step 5, the intersection between Σ^+ and any plane $\{w_1 = \text{cost}\}$, $\text{cost} \in (0, 2l)$, is a simple curve with endpoints in $\partial\Sigma^+$. If Σ^+ is not Cartesian, one of these curves γ is not Cartesian, and then there is a point where the tangent vector to γ is vertical. At such a point the tangent plane to Σ^+ is vertical.

Claim: If Π is a vertical plane which is tangent to Σ, then there is at most one point where Π and Σ are tangent.

Assume Π intersects the relative interior of Σ. It is easy to see that the intersection between Π and the Jordan curve Γ consists at most of four points[73] p_i, $i = 1, 2, 3, 4$. Let f be a linear function on $\mathbb{R}^3$ vanishing on Π. Then $f \circ \Phi$ is harmonic in B_1 and continuous in $\overline{B}_1$; in addition, it vanishes at $\{p_i,\ i = 1, 2, 3, 4\}$, and alternates its sign on the four arcs $\overline{p_i p_{i+1}}$ on ∂B_1 with endpoints p_i. With no loss of generality, we may assume $f \circ \Phi > 0$ on $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$. By harmonicity of $f \circ \Phi$, any connected component of the region $\{x \in \overline{B}_1 : f \circ \Phi(x) > 0\}$ must contain part of $\overline{p_1 p_2}$ or $\overline{p_3 p_4}$, so that we deduce that these connected components are at most two.

Assume now by contradiction that there are two distinct points $\mathcal{P}$ and $\mathcal{Q}$ of Σ such that Π is tangent to Σ at $\mathcal{P}$ and $\mathcal{Q}$. Since $f \circ \Phi$ has null differential at $\Phi^{-1}(\mathcal{P})$ and $\Phi^{-1}(\mathcal{Q})$, the set $\{f \circ \Phi = 0\}$, in a neighbourhood of $\Phi^{-1}(\mathcal{P})$, consists of $2m_p$ analytic curves crossing at $\Phi^{-1}(\mathcal{P})$, whereas in a neighbourhood of $\Phi^{-1}(\mathcal{Q})$, it consists of $2m_q$ analytic curves crossing at $\Phi^{-1}(\mathcal{Q})$. Therefore, in a neighbourhood of $\Phi^{-1}(\mathcal{P})$, the set $\{f \circ \Phi > 0\}$ counts at least 2 open regions (and similarly at $\Phi^{-1}(\mathcal{Q})$). Let us call the two of these regions A_i, $i = 1, 2$ and B_i, $i = 1, 2$ (A_i's around $\Phi^{-1}(\mathcal{P})$ and B_i's around $\Phi^{-1}(\mathcal{Q})$). By harmonicity each A_i and B_i must be connected to one of the arcs $\overline{p_1 p_2}$ or $\overline{p_3 p_4}$. Hence some of these regions must belong to the same connected component of $\{f \circ \Phi > 0\}$. Then we are reduced to two following cases (see Fig. (20)):

(Case A) A_1 and A_2 belong to the same connected component (say the one containing $\overline{p_1 p_2}$). Hence we can construct two disjoint curves in $\{f \circ \Phi > 0\}$, both joining $\Phi^{-1}(\mathcal{P})$ to a point in $\overline{p_1 p_2}$, emanating from $\Phi^{-1}(\mathcal{P})$, one in region A_1 and one in region A_2. This contradicts the maximum principle, because these two curves would enclose a region where $f \circ \Phi$ takes also negative values, whereas its boundary is in $\{f \circ \Phi > 0\}$.

(Case B) A_1 and B_1 are joined to $\overline{p_1 p_2}$ and A_2 and B_2 are joined to $\overline{p_3 p_4}$. In this case we can construct four curves in $\{f \circ \Phi > 0\}$: σ_1 and σ_2 emanating from $\Phi^{-1}(\mathcal{P})$ in regions A_1 and A_2 and reaching $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, respectively; β_1 and β_2 emanating from $\Phi^{-1}(\mathcal{Q})$ in regions B_1 and B_2 and reaching $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, respectively. The region enclosed between these 4 curves has boundary contained in $\{f \circ \Phi > 0\}$ and necessarily inside it the function $f \circ \Phi$ takes also negative values, again in contrast with the maximum principle.

From the above discussion our claim follows.

We are now ready to conclude the proof of step 6: suppose by contradiction that Σ^+ is not Cartesian with respect to $\{w_3 = 0\}$, and take a point $P^+ \in \Sigma^+ \setminus \Gamma$ where the tangent plane Π to Σ^+ at P^+ is vertical. By symmetry of Σ, the point P^-, defined as the symmetric of P^+ with respect to the rectangle R_{2l}, belongs to Σ^-, and the tangent plane to Σ^- at P^- is the same plane Π. This contradicts the claim. We eventually observe that $\widetilde{\psi}$ is analytic on the subgraph of $\widetilde{h}$, since its graph is Σ^+. We conclude that $\widetilde{\psi}$ belongs to $W^{1,1}(SG_{\widetilde{h}})$, since also its total variation is bounded by the area of its graph, which is finite.

Step 7: the pair $(\widetilde{h}, \widetilde{\psi})$ *is an admissible competitor for* $\mathcal{F}_{2l}$. To see this, we recall that in step 5 we proved that $\widetilde{h}$ is convex and $\widetilde{h}(\cdot) = \widetilde{h}(2l - \cdot)$, *i.e.* $\widetilde{h} \in \mathcal{H}_{2l}$. Furthermore Σ^+ is the graph of $\widetilde{\psi}$,

[73] A vertical plane intersects K on a straight segment. In turn, this segment intersects ∂K in two points. If a vertical plane intersects Γ in a point (W_1, W_2, W_3), then $(W_1, W_2, 0) \in \partial K$. Moreover this plane intersects Γ also at $(W_1, W_2, -W_3)$. Thus, the points of intersection are at most four. The degenerate cases in which Π contains a full $\mathcal{H}^1$-measured part of Γ are excluded by this analysis, because in these cases Π does not intersect the interior of Σ. Instead, the cases in which the intersection consists of 2 or 3 points are easier to treat, and we detail only the 4-points case (notice that by the geometry of Γ, the case of 3 points occurs when this plane is tangent to Γ at one of the points $(0, 1, 0)$ or $(2l, 1, 0)$).

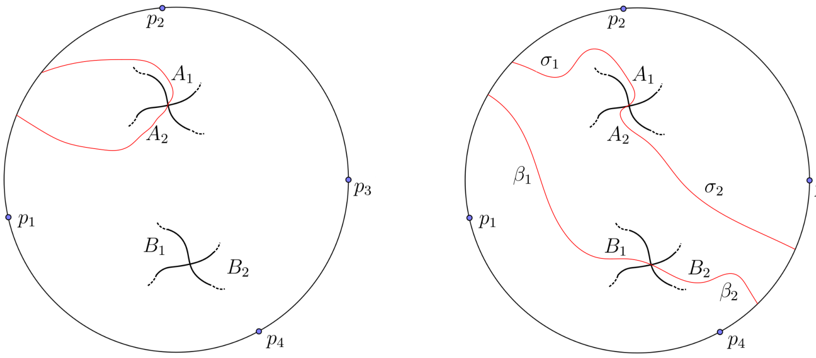

Figure 20: On the left it is represented case A in step 6 of the proof of Theorem 12.16. The point $\Phi^{-1}(\mathcal{P})$ is in the cross where the two emphasized paths start from. These curves stand in the region $\{f \circ \Phi > 0\}$ and join $\Phi^{-1}(\mathcal{P})$ with the boundary arc $\overline{p_1 p_2}$. The picture on the right represents instead case B. The two cross points are $\Phi^{-1}(\mathcal{P})$ and $\Phi^{-1}(\mathcal{Q})$ and the paths σ_1, σ_2, β_1, β_2 are depicted.

and its projection on the plane $\{w_3 = 0\}$ is the subgraph of $\widetilde{h}$. It follows that the area of the graph of $\widetilde{\psi}$ is exactly the area of Σ^+ upon $SG_{\widetilde{h}}$. Let us also recall the $W^{1,1}$ regularity of $\widetilde{\psi}$ proved in step 6. Setting $\widetilde{\psi} := \psi$ in $R \setminus K$ we infer the admissibility of $(\widetilde{h}, \widetilde{\psi})$.

Step 8: Conclusion. From (12.41) we deduce

$$\mathcal{F}_{2l}(h, \psi) \geq \mathcal{H}^2(\Sigma^+) + \mathbb{A}(\psi, R \setminus K) = \mathcal{F}_{2l}(\widetilde{h}, \widetilde{\psi}), \tag{12.44}$$

where the last equality follows from the fact that $\widetilde{\psi}$ is continuous on $\partial_D R_{2l}$. Hence, also $(\widetilde{h}, \widetilde{\psi})$ is a minimizer for $\mathcal{F}_{2l}$. We now show that $\widetilde{\psi}$ is continuous and equals 0 on $G_{\widetilde{h}}$. Indeed $\Sigma = \Sigma^+ \cup \Sigma^-$ is analytic, hence also $\widetilde{h}$ is smooth (and convex). Moreover we know that $\widetilde{\psi}$ is smooth in $SG_{\widetilde{h}}$. If its trace $\widetilde{\psi}^+$ on $G_{\widetilde{h}}$ is strictly positive somewhere, we infer that the vertical subset of Σ^+ defined as

$$\{(w_1, w_2, w_3) : (w_1, w_2) \in G_{\widetilde{h}},\ w_3 \in (0, \widetilde{\psi}^+(w_1, w_2))\},$$

has positive $\mathcal{H}^2$-measure and cannot have zero mean curvature (the only case in which its mean curvature vanishes is when $\widetilde{h}$ is linear, but in this case Σ^+ must be contained in a plane containing $G_{\widetilde{h}}$ which is impossible, since Γ^+ is not). We conclude $\widetilde{\psi}^+ = 0$ on $G_{\widetilde{h}}$. Finally, $L_{\widetilde{h}} = \emptyset$ for, if not, the vertical part of Σ^+ obtained on $L_{\widetilde{h}}$ is flat and then, by analyticity, also Σ^+ is, a contradiction. The thesis of the theorem, and hence of Theorem 12.6 (iii), is achieved. $\square$

To conclude the proof of Theorem 12.6, it remains to show (iv). The pair $(h \equiv 1, \widehat{\varphi})$, where the function $\widehat{\varphi}$ is as in (12.13), is one of the competitors for problem (12.14) (notice that $\widehat{\varphi}$ attains the boundary condition); in addition, its subgraph is strictly convex (see Fig. 18), hence[74] necessarily $\psi^\star \leq \widehat{\varphi}$ in $\overline{R_{2l}}$ (where we have taken $\psi^\star = \widetilde{\psi}$, the solution given by Theorem 12.16).

Eventually, the strict inequality in (12.17) is a consequence of the strong maximum principle: indeed, the internal points of a minimal surface are always strictly inside the convex hull of its boundary, with the only exception in the case of part of a plane (see [27, pag 63, section 70]);

[74] As observed, the minimal surface Σ^+ is the graph of $\psi^\star = \widetilde{\psi}$, and it must be contained in the convex envelope of Γ, *i.e.*, inside the subgraph of $\widehat{\varphi}$.

so that internal points of Σ^+ are strictly inside the the graph $G_{\widehat{\varphi}}$ of $\widehat{\varphi}$ (that is half of the lateral boundary of a cylinder). $\square$

We conclude this section by observing a consequence of Theorem 12.16: Let G_w be the graph in R_{2l} of a function $w \in C([0,2l],(-1,1])$ such that $w(0) = w(2l) = 1$, and consider the curve Γ_w obtained by concatenation of G_w with the graph of φ over $\partial_D R_{2l}$.

Corollary 12.17. *We have*

$$\mathcal{F}_{2l}(h,\psi) = \inf \mathcal{P}_{\Gamma_w}(X_{\min}), \tag{12.45}$$

where $(h,\psi) \in X_{2l}^{\mathrm{conv}}$ is a minimizer of $\mathcal{F}_{2l}$, $X_{\min}$ is a parametrization of a disc-type area-mininizing solution of the Plateau problem spanning Γ_w (see (2.9)*), and the infimum is computed over all functions w as above.*

The proof of this corollary can be achieved by adapting the proof of Theorem 12.16, which shows that the solution to the Plateau problem in (12.45) is Cartesian and the optimal w is convex.

13 Upper bound

A minimizer $(h^\star,\psi^\star)$ of (12.14) needs to be used for constructing a recovery sequence $(u_k) \subset \mathrm{Lip}(\Omega,\mathbb{R}^2)$, see formulas (13.25) and (13.27): we know that $\psi^\star$ is locally Lipschitz, but not Lipschitz, in R_{2l}; therefore we need first a regularization procedure.

Let $(h^\star,\psi^\star)$ be a minimizer provided by Theorem 12.6, and assume that $h^\star$ is not identically -1. We fix an integer $m > 0$ and, recalling the definition of $\widehat{\varphi}$ in (12.13), define

$$\varphi_m := \Big(\widehat{\varphi} - \frac{2}{m}\Big) \vee 0 \qquad \text{in } \overline{R}_{2l}. \tag{13.1}$$

We observe that φ_m is Lipschitz continuous in $\overline{R}_{2l}$. We then set

$$\psi_m^\star := \Big(\big(\psi^\star - \frac{1}{m}\big) \vee 0\big)\Big) \wedge \varphi_m \qquad \text{in } R_{2l}. \tag{13.2}$$

Since $\psi^\star$ is locally Lipschitz in R_{2l}, an easy check shows that $\psi_m^\star$ is Lipschitz continuous in R_{2l} for any m (with an unbounded Lipschitz constant as $m \to +\infty$). This follows from the fact that $\psi^\star$ is continuous up to the boundary of R_{2l} (see Theorem 12.6 (iii)) and hence $\psi_m^\star$ coincides with either 0 or φ_m in a neighbourhood of $(\partial_D R_{2l}) \cup G_{h^\star}$ in R_{2l}. Furthermore still $\psi_m^\star = 0$ on the upper graph $\overline{R_{2l}} \setminus SG_{h^\star} = \{(w_1,w_2) \in \overline{R_{2l}} : w_2 \geq h^\star(w_1)\}$ of $h^\star$.

Lemma 13.1 (Properties of $\psi_m^\star$). *Let $(h^\star,\psi^\star)$ be a minimizer of $\mathcal{F}_{2l}$ as in Theorem 12.6 and assume $h^\star$ is not identically -1. For all $m > 0$ let $\psi_m^\star$ be defined as in* (13.2)*. Then:*

- *(i) $\psi_m^\star$ is Lipschitz continuous in $\overline{SG_{h^\star}}$, $\psi_m^\star = 0$ on $([0,2l]\times\{-1\}) \cup (\overline{R_{2l}} \setminus SG_{h^\star})$, and $\psi_m^\star(0,\cdot) = \varphi_m(0,\cdot)$, so that $|\partial_{w_2}\psi_m^\star(0,\cdot)| \leq |\partial_{w_2}\varphi(0,\cdot)| = |\partial_{w_2}\psi^\star(0,\cdot)|$ a.e. in $[-1,1]$;*
- *(ii) $(\psi_m^\star)$ converges to $\psi^\star$ uniformly on $\{0,2l\}\times[-1,1]$ as $m \to +\infty$;*
- *(iii) we have*

$$\lim_{m\to+\infty} \mathbb{A}(\psi_m^\star, SG_{h^\star}) = \mathbb{A}(\psi^\star, SG_{h^\star}). \tag{13.3}$$

As a consequence $\mathcal{F}_{2l}(h^\star,\psi_m^\star) \to \mathcal{F}_{2l}(h^\star,\psi^\star)$ as $m \to +\infty$.

Proof. (i) and (ii) are direct consequences of the definitions. To show (iii) we start to observe that $\psi^\star_m \to \psi^\star$ pointwise in R_{2l}: indeed, this follows from the definitions of $\varphi^\star_m$ and $\psi^\star_m$ up to noticing that $\varphi_m \to \widehat{\varphi}$ pointwise in R_{2l} as $m \to +\infty$, and $\psi^\star \leq \widehat{\varphi}$ on R_{2l}. From Theorem 12.6 (iv) it follows that, at any point $(w_1, w_2) \in R_{2l}$, for m large enough $\varphi_m(w_1, w_2) > \psi^\star(w_1, w_2)$ (since $\widehat{\varphi}(w_1, w_2) > \psi^\star(w_1, w_2)$), so that $\psi^\star_m(w_1, w_2) = \psi^\star(w_1, w_2) - \frac{1}{m}$. As a consequence the set $A_m := \{0 < \psi^\star - \frac{1}{m} < \varphi_m\}$ satisfies

$$\lim_{m \to +\infty} \mathcal{H}^2(SG_{h^\star} \setminus A_m) = 0,$$

and on A_m it holds $\psi^\star_m = \psi^\star - \frac{1}{m}$ and $\nabla\psi^\star_m = \nabla\psi^\star$. Moreover, on $SG_{h^\star} \setminus A_m$, either $\psi^\star_m = 0$ (and hence $\nabla\psi^\star_m = 0$) or $\psi^\star_m = \varphi_m$ (and hence $\nabla\psi^\star_m = \nabla\varphi_m$). Therefore

$$\int_{SG_{h^\star} \setminus A_m} \sqrt{1 + |\nabla\psi^\star_m|^2}\; dx \leq \int_{SG_{h^\star} \setminus A_m} \sqrt{1 + |\nabla\varphi_m|^2}\; dx$$

and

$$\lim_{m \to +\infty} \int_{SG_{h^\star} \setminus A_m} \sqrt{1 + |\nabla\psi^\star_m|^2}\; dx \leq \lim_{m \to +\infty} \int_{SG_{h^\star} \setminus A_m} \sqrt{1 + |\nabla\varphi_m|^2}\; dx = 0,$$

because $|\nabla\varphi_m|$ are uniformly bounded in $L^1(R_{2l})$. Also

$$\mathbb{A}(\psi^\star_m, SG_{h^\star}) = \int_{A_m} \sqrt{1 + |\nabla\psi^\star|^2}\; dx + \int_{SG_{h^\star} \setminus A_m} \sqrt{1 + |\nabla\psi^\star_m|^2}\; dx,$$

and (13.3) follows. □

The main result of this section reads as follows.

Theorem 13.2 (Upper bound for the area of the vortex map). *The relaxed area of the graph of the vortex map u satisfies*

$$\overline{\mathcal{A}}(u, \Omega) \leq \int_\Omega |\mathcal{M}(\nabla u)|\; dx + 2 \min\left\{\mathcal{F}_l(h, \psi) : (h, \psi) \in X_l^{\text{conv}}\right\}. \tag{13.4}$$

Proof of theorem 13.2. To prove the theorem, we need to construct a sequence $(u_k) \subset \mathrm{Lip}(\Omega, \mathbb{R}^2)$ converging to u in $L^1(\Omega, \mathbb{R}^2)$ such that

$$\lim_{k \to +\infty} \mathcal{A}(u_k, \Omega) \leq \int_\Omega |\mathcal{M}(\nabla u)| dx + \mathcal{F}_{2l}(h^\star, \psi^\star),$$

where $(h^\star, \psi^\star)$ is a pair minimizing $\mathcal{F}_{2l}$ as in Theorem 12.6. We can assume that $h^\star$ is not identically -1, otherwise the result follows from [1].

We will specify various subsets of Ω and define the sequence (u_k) on each of these sets (see Fig. 21). More precisely, we will define u_k as a map into $\mathbb{S}^1$ in the largest sector (step 1). This construction is similar to the one in [1] (see also Remark 13.3 below). The contribution of the area in this sector will equal, as $k \to \infty$, the first term in (13.4). The second term will be instead provided by the contribution of u_k in region $C_k \setminus B_{r_k}$ (step 2), where we will need the aid of the functions $(h^\star, \psi^\star)$ (suitably regularized, in order to render u_k Lipschitz continuous). The other regions surrounding $C_k \setminus B_{r_k}$ are needed to glue u_k between the aforementioned regions. This is done in steps 3, 4 and 5, where it is also proven that the corresponding area contribution is negligible. Finally, in steps 6 and 7 we show the crucial estimates to prove (13.4). In Fig. 21 this subdivion of the domain Ω is drawn.

Remark 13.3. Our construction differs from the one in [1], even when in place of $(h^\star, \psi^\star)$ we use $(1, \sqrt{1-s^2})$ (*i.e.*, the one in Section 10.1) in the following sense. We use the full graph of $\pm\psi^\star$ to construct u_k (and therefore, in the case when $(h^\star, \psi^\star)$ is replaced by $(1, \sqrt{1-s^2})$, the image of u_k covers the whole cylinder and not only a part of it). Since $h^\star$ *may be not identically* 1 (and actually is not explicit in general), the presence of a new set T_k is now needed, as an intermediate region to glue the trace of u_k along two segments $\{\theta = \pm\overline{\theta}_k\}$. The image set $u_k(T_k)$ covers a small part of the unit circle. See Fig. 21.

Let $k \in \mathbb{N}$ and let $(r_k), (\theta_k), (\overline{\theta}_k)$ be infinitesimal sequences of positive numbers such that $\overline{\theta}_k - \theta_k =: \delta_k > 0$. We shall suppose[75]

$$\lim_{k\to+\infty} (\theta_k k) = 0. \tag{13.5}$$

Let B_{r_k} be the open disc centered at the origin with radius r_k, and

$$C_k := \{(r,\theta) \in [0,l) \times [0,2\pi) : \theta \in [0,\theta_k] \cup [2\pi - \theta_k, 2\pi)\}, \tag{13.6}$$

be the half-cone in Ω, with vertex at the origin and aperture equal to $2\theta_k$, see Fig. 21. Let

$$T_k := \{(r,\theta) \in [0,l) \times [0,2\pi) : \theta \in [\theta_k, \overline{\theta}_k] \cup [2\pi - \overline{\theta}_k, 2\pi - \theta_k]\}. \tag{13.7}$$

We set

$$C_k^+ := C_k \cap \{\theta \in [0,\theta_k]\}, \quad C_k^- := C_k \cap \{\theta \in [2\pi - \theta_k, 2\pi]\},$$

and divide $C_k \cap (\Omega \setminus \mathrm{B}_{r_k})$ into two sets

$$C_k \setminus \mathrm{B}_{r_k} := \big(C_k^+ \setminus \mathrm{B}_{r_k}\big) \cup \big(C_k^- \setminus \mathrm{B}_{r_k}\big). \tag{13.8}$$

Step 1. Definition of u_k in $\overline{\Omega \setminus (C_k \cup T_k)}$.

In this step our construction is similar to the one in [1, Lem. 5.3], see also (10.1); in order to define u_k, in the source we use polar coordinates (r,θ) and in the target Cartesian coordinates. Define

$$u_k(r,\theta) := \begin{cases} u(r,\theta) = (\cos\theta, \sin\theta), & (r,\theta) \in (\overline{\Omega \setminus (C_k \cup T_k)}) \setminus \mathrm{B}_{r_k/2}, \\ \Big(\cos(\frac{2r}{r_k}(\theta - \pi) + \pi), \sin(\frac{2r}{r_k}(\theta-\pi) + \pi)\Big), & (r,\theta) \in \overline{\mathrm{B}_{r_k/2} \setminus (C_k \cup T_k)}. \end{cases} \tag{13.9}$$

Observe that

$$\begin{aligned} & u_k(0,0) = (-1,0) = u_k(r,\pi), \qquad r \in [0,l); \\ & u_k(r,\overline{\theta}_k) = (\cos\overline{\theta}_k, \sin\overline{\theta}_k), \qquad u_k(r, 2\pi - \overline{\theta}_k) = (\cos\overline{\theta}_k, \sin(-\overline{\theta}_k)), \qquad r \in (r_k/2, l); \\ & u_k(r,\overline{\theta}_k) = \Big(\cos(\frac{2r}{r_k}(\overline{\theta}_k - \pi) + \pi)\ ,\ \sin(\frac{2r}{r_k}(\overline{\theta}_k - \pi) + \pi)\Big), \qquad r \in [0, r_k/2], \\ & u_k(r, 2\pi - \overline{\theta}_k) = \Big(\cos(\frac{2r}{r_k}(\pi - \overline{\theta}_k) + \pi)\ ,\ \sin(\frac{2r}{r_k}(\pi - \overline{\theta}_k) + \pi)\Big), \qquad r \in [0, r_k/2]. \end{aligned} \tag{13.10}$$

The relevant contribution to the area of the graph of u_k is the one in region C_k, and more specifically in $C_k \setminus B_{r_k}$; it is in this region that we need to use a minimizing pair of $\mathcal{F}_{2l}$.

Step 2. Definition of u_k on $C_k \setminus \mathrm{B}_{r_k}$.

We first need a regularization of $h^\star$: assuming without loss of generality $1/k < l$, we define

$$h_k^\star(w_1) := \begin{cases} h^\star(w_1) & \text{for } w_1 \in [\frac{1}{k}, l], \\ k\,\big(h^\star(\frac{1}{k}) - h^\star(0)\big)\, w_1 + h^\star(0) & \text{for } w_1 \in [0, \frac{1}{k}), \end{cases} \tag{13.11}$$

[75] This assumption will be used only in step 7.

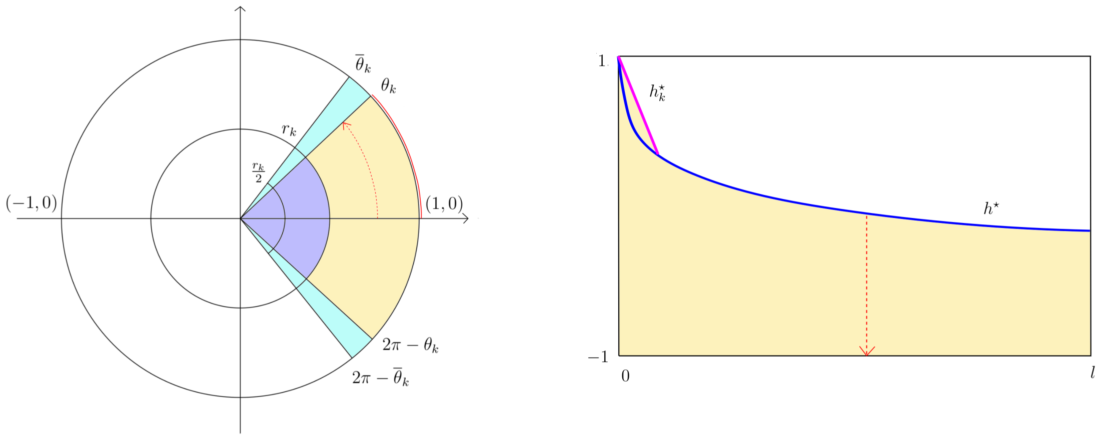

Figure 21: On the left the subdivision of B_l in sectors. Specifically, the sectors $C_k^+ \setminus \mathrm{B}_{r_k}$ and $C_k^- \setminus \mathrm{B}_{r_k}$ are emphasized in light grey. The map $\mathcal{T}_k$ defined in (13.22) sends $C_k^+ \setminus \mathrm{B}_{r_k}$ in the (reflected) subgraph of $h_k^\star$ in R_l, depicted on the right. This parametrization maps the segment joining $(r_k, 0)$ to $(1,0)$ onto the graph of $h_k^\star$, and the radius corresponding to $\theta = \theta_k$ to the basis of R_l, following the orientation emphasized by the dashed arrow. The graph of $h_k^\star$ starts linearly from the point $(0,1)$ with negative derivative, then joins and next coincides with the graph of $h^\star$. The definition of u_k in $C_k^+ \setminus \mathrm{B}_{r_k}$ makes use of this parametrization of $SG_{h_k^\star}$ (see (13.25)). This parametrization needs a reflection, in order to glue u_k on the horizontal segment $\{\theta = 0\}$ with the definition of u_k in $C_k^- \setminus \mathrm{B}_{r_k}$. Note that $\Psi_k(C_k^+ \setminus \mathrm{B}_{r_k}) \subset \mathbb{R}^3$, with Ψ_k defined in (5.1), is the graph of $\psi_k^*(w_1, -w_2)$ above the reflection of the two-dimensional region in between the graph of h_k^* and the horizontal segment $\{w_2 = -1\}$.

where we recall that $h^\star(0) = 1$ (see Theorem 12.6), and we set $h_k^\star(w_1) := h_k^\star(2l - w_1)$ for $w_1 \in [l, 2l]$ (see Fig. 21, right). Notice that $h_k^\star(0) = 1$, $h_k^\star \in \mathrm{Lip}([0, 2l])$ and the convexity of $h^\star$ implies that also $h_k^\star$ is convex, $h_k^\star \geq h^\star$, and therefore by Lemma 13.1 (i) we see that $(h_k^\star, \psi_k^\star) \in X_{2l}^{\mathrm{conv}}$, where $\psi_k^\star$ is the approximation of $\psi^\star$ in Lemma 13.1 (with $k = m$), see formula (13.2). Again by Lemma 13.1, $\mathcal{F}_{2l}(h_k^\star, \psi_k^\star) = \mathcal{F}_{2l}(h^\star, \psi_k^\star) + \int_0^{2l} (h_k^\star(w_1) - h^\star(w_1))\ dw_1 \to \mathcal{F}_{2l}(h^\star, \psi^\star)$ as $k \to +\infty$.

We start with the construction of u_k on $C_k^+ \setminus \mathrm{B}_{r_k}$. Set

$$\tau_k : [r_k, l] \to [0, l], \qquad \tau_k(r) := \frac{l}{l - r_k}(r - r_k), \tag{13.12}$$

$$s_k : [r_k, l] \times [0, \theta_k] \to [-1, 1], \qquad s_k(r, \theta) := \frac{1 + h_k^\star(\tau_k(r))}{\theta_k}\theta - h_k^\star(\tau_k(r)). \tag{13.13}$$

Note that $s_k(r, \cdot) : [0, \theta_k] \to [-h_k^\star(\tau_k(r)), 1]$ is a bijective increasing function, for any $r \in [r_k, l]$. Thus

$$s_k(r, 0) = -h_k^\star(\tau_k(r)) \quad \text{for any } r \in [r_k, l], \text{ in particular } s_k(r_k, 0) = -1, \tag{13.14}$$

$$s_k(r, \theta_k) = 1, \qquad r \in [r_k, l], \tag{13.15}$$

$$s_k(r_k, \theta) = \frac{2\theta}{\theta_k} - 1, \qquad \theta \in [0, \theta_k]. \tag{13.16}$$

We have, for all $r \in [r_k, l]$ and $\theta \in [0, \theta_k]$,

$$\tau_k'(r) = \frac{l}{l - r_k}, \tag{13.17}$$

$$\partial_\theta s_k(r, \theta) = \frac{1 + h_k^\star(\tau_k(r))}{\theta_k}, \tag{13.18}$$

and, for almost every $r \in [r_k, l]$ and all $\theta \in [0, \theta_k]$,

$$\partial_r s_k(r, \theta) = \left(\frac{\theta}{\theta_k} - 1\right) \tau_k'(r) h_k^{\star\prime}(\tau_k(r)) = \frac{l}{l - r_k}\left(\frac{\theta}{\theta_k} - 1\right) h_k^{\star\prime}(\tau_k(r)). \tag{13.19}$$

Moreover we define

$$R_k : [0, l] \to [r_k, l], \qquad R_k(w_1) := \frac{l - r_k}{l} w_1 + r_k \tag{13.20}$$

to be the inverse of τ_k and, recalling that $\overline{R}_l = [0, l] \times [-1, 1]$,

$$\Theta_k : SG_{h_k^\star} \cap \overline{R}_l \to [0, \theta_k], \qquad \Theta_k(w_1, w_2) := \frac{\theta_k}{1 + h_k^\star(w_1)}(h_k^\star(w_1) - w_2). \tag{13.21}$$

Notice that $\Theta_k(w_1, \cdot) : [-1, h_k^\star(w_1)] \to [0, \theta_k]$ is a linearly decreasing bijective function.

The map

$$\mathcal{T}_k : C_k^+ \setminus \mathrm{B}_{r_k} \to SG_{h_k^\star} \cap \overline{R}_l, \qquad \mathcal{T}_k(r, \theta) := (\tau_k(r), -s_k(r, \theta)), \tag{13.22}$$

is invertible, and its inverse is the map

$$\mathcal{T}_k^{-1} : SG_{h_k^\star} \cap \overline{R}_l \to C_k^+ \setminus \mathrm{B}_{r_k}, \qquad \mathcal{T}_k^{-1}(w_1, w_2) := (R_k(w_1), \Theta_k(w_1, w_2)). \tag{13.23}$$

The modulus of the determinant of the Jacobian of $\mathcal{T}_k^{-1}$ is given by

$$|J_{\mathcal{T}_k^{-1}}| = \left(\frac{l - r_k}{l}\right) \frac{\theta_k}{1 + h_k^\star(w_1)}. \tag{13.24}$$

We set

$$u_k(r,\theta) := \Big(s_k(r,\theta), \psi_k^\star\big(\mathcal{T}_k(r,\theta)\big)\Big) = \Big(u_{k1}(r,\theta), u_{k2}(r,\theta)\Big), \qquad r \in [r_k, l], \theta \in [0,\theta_k]. \tag{13.25}$$

Observe that, using the definition of $\psi_k^\star$,

$$\begin{aligned}
&u_k \in \mathrm{Lip}(C_k^+ \setminus \mathrm{B}_{r_k}, \mathbb{R}^2),\\
&u_k(r,\theta_k) = (s_k(r,\theta_k), \psi_k^\star(\mathcal{T}_k(r,\theta_k))) = (1,0),\\
&u_k(r,0) = (-h_k^\star(\tau_k(r)), \psi_k^\star(\tau_k(r), h_k^\star(\tau_k(r)))) = (-h_k^\star(\tau_k(r)), 0),\\
&u_k(r_k,\theta) = (s_k(r_k,\theta), \psi_k^\star(0, -s_k(r_k,\theta))) = (s_k(r_k,\theta), \varphi_k(0, -s_k(r_k,\theta))),
\end{aligned} \tag{13.26}$$

for $r \in [r_k, l]$ and $\theta \in [0, \theta_k]$, as it follows from (13.12), (13.14), (13.15), and (12.15), where φ_k is defined in (13.1) (with $k = m$).

Eventually we define u_k on $C_k^- \setminus \mathrm{B}_{r_k}$ as

$$u_k(r,\theta) := (u_{k1}(r, 2\pi-\theta), -u_{k2}(r, 2\pi-\theta)), \qquad r \in [r_k, l], \theta \in [2\pi-\theta_k, 2\pi). \tag{13.27}$$

It turns out

$$\begin{aligned}
&u_k \in \mathrm{Lip}(C_k^- \setminus \mathrm{B}_{r_k}, \mathbb{R}^2),\\
&u_k(r, 2\pi-\theta_k) = (1,0),\\
&u_k(r, 2\pi) = (-h_k^\star(\tau_k(r)), -\psi_k^\star(\tau_k(r), h_k^\star(\tau_k(r)))) = (-h_k^\star(\tau_k(r)), 0),\\
&u_k(r_k,\theta) = (s_k(r_k, 2\pi-\theta), -\psi_k^\star(0, -s_k(r_k, 2\pi-\theta))),
\end{aligned}$$

for $r \in [r_k, l]$, $\theta \in [2\pi-\theta_k, 2\pi)$.

The area of the graph of u_k on $C_k \setminus B_{r_k/2}$ will be computed in step 7.

Step 3. Definition of u_k on $C_k \cap (\overline{B}_{r_k} \setminus \mathrm{B}_{r_k/2})$ and its area contribution.

Let $G_{\psi_k^\star(0,\cdot)} \subset \mathbb{R}^2$ (resp. $G_{\psi^\star(0,\cdot)} \subset \mathbb{R}^2$) denote the graph of $\psi_k^\star(0,\cdot)$ (resp. of $\psi^\star(0,\cdot)$) on $[-1,1]$. We introduce the retraction map $\Upsilon : (\mathbb{R} \times [0,+\infty)) \setminus O \subset \mathbb{R}^2_{\rm target} \to G_{\psi^\star(0,\cdot)} \subset \mathbb{R}^2_{\rm target}$, $O = (0,0)$, defined by

$$\Upsilon(p) = q := G_{\psi^\star(0,\cdot)} \cap \ell_{Op} \qquad \forall p \in (\mathbb{R} \times [0,+\infty)) \setminus O,$$

where ℓ_{Op} is the line passing through O and p. Then Υ is well-defined and it is Lipschitz continuous in a neighbourhood of $G_{\psi^\star(0,\cdot)}$ in $\mathbb{R} \times [0,+\infty)$. We also define

$$\Upsilon_k : G_{\psi_k^\star(0,\cdot)} \to G_{\psi^\star(0,\cdot)}$$

as the restriction of Υ to $G_{\psi_k^\star(0,\cdot)}$; see Fig. 22. As a consequence, since for $k \in \mathbb{N}$ large enough $G_{\psi_k^\star(0,\cdot)}$ is contained in a neighbourhood of $G_{\psi^\star(0,\cdot)}$, we have that Υ_k is Lipschitz continuous with Lipschitz constant independent of k. Notice also that $\Upsilon_k((-1,0)) = (-1,0)$ and $\Upsilon_k((1,0)) = (1,0)$.

We define u_k on $C_k^+ \cap (\overline{\mathrm{B}}_{r_k} \setminus \mathrm{B}_{r_k/2})$ setting, for $r \in [\frac{r_k}{2}, r_k]$ and $\theta \in [0, \theta_k]$,

$$u_k(r,\theta) := \Big(2 - \frac{2r}{r_k}\Big)\Upsilon_k\big(s_k(r_k,\theta), \psi_k^\star(0, -s_k(r_k,\theta))\big) + \Big(\frac{2r}{r_k} - 1\Big)\big(s_k(r_k,\theta), \psi_k^\star(0, -s_k(r_k,\theta))\big).$$

We have

$$u_k(r_k,\theta) = (s_k(r_k,\theta), \psi_k^\star(0, -s_k(r_k,\theta))),$$

so that u_k glues, on $C_k^+ \cap \partial\mathrm{B}_{r_k}$, with the values obtained in step 2 (last formula in (13.26)), and

$$u_k(r,\theta_k) = (1,0), \qquad u_k(r,0) = (-1,0).$$

This formula shows that u_k also glues, on $C_k^+ \cap \{(r,\theta) : r \in [r_k/2, r_k], \theta \in \{0, \theta_k\}\}$, with the values obtained in step 2 (second and third formula in (13.26)). Moreover

$$u_k(r_k/2,\theta) = \Upsilon_k\big(s_k(r_k,\theta), \psi_k^\star(0,-s_k(r_k,\theta))\big), \qquad \theta \in [0,\theta_k]. \tag{13.28}$$

In addition, the derivatives of u_k satisfy, for $r \in (\frac{r_k}{2}, r_k)$ and $\theta \in (0,\theta_k)$, using (13.16),

$$\begin{aligned}
\partial_r u_k(r,\theta) &= -\frac{2}{r_k}\Upsilon_k\big(s_k(r_k,\theta), \psi_k^\star(0,-s_k(r_k,\theta))\big) + \frac{2}{r_k}\big(s_k(r_k,\theta), \psi_k^\star(0,-s_k(r_k,\theta))\big),\\
\partial_\theta u_k(r,\theta) &= \Big(2-\frac{2r}{r_k}\Big)\nabla\Upsilon_k\big(s_k(r_k,\theta), \psi_k^\star(0,-s_k(r_k,\theta))\big)\cdot\Big(\frac{2}{\theta_k}, -\frac{2}{\theta_k}\partial_{w_2}\psi_k^\star(0,-s_k(r_k,\theta))\Big)\\
&\quad + \Big(\frac{2r}{r_k}-1\Big)\Big(\frac{2}{\theta_k}, -\frac{2}{\theta_k}\partial_{w_2}\psi_k^\star(0,-s_k(r_k,\theta))\Big),
\end{aligned}$$

so that

$$\begin{aligned}
|\partial_r u_k(r,\theta)| &\le \frac{4}{r_k},\\
|\partial_\theta u_k(r,\theta)| &\le \frac{2(\widehat{C}+1)}{\theta_k}(|\partial_{w_2}\psi_k^\star(0,-s_k(r_k,\theta))|+1),
\end{aligned}$$

where $\widehat{C}$ is a positive constant independent of k, which bounds the gradient of Υ_k. Since $\psi_k^\star$ is Lipschitz, we deduce that u_k is Lipschitz continuous[76] on $C_k^+ \cap (\mathrm{B}_{r_k} \setminus \mathrm{B}_{r_k/2})$.

Furthermore the image of $(\frac{r_k}{2}, r_k)\times(0,\theta_k)$ through the map $(r,\theta)\mapsto u_k(r,\theta)$ is the region enclosed by $G_{\psi_k^\star}$ and $G_{\psi^\star}$ (with multiplicity 1). The area of this region is infinitesimal as $k\to+\infty$, so that, by the area formula,

$$\int_{r_k/2}^{r_k}\int_0^{\theta_k} r|Ju_k(r,\theta)|d\theta dr = o(1) \qquad \text{as } k\to+\infty.$$

Hence, using the fact that the gradient in polar coordinates is $(\partial_r, \frac{1}{r}\partial_\theta)$, we eventually estimate (see also (2.3))

$$\begin{aligned}
\int_{r_k/2}^{r_k}\int_0^{\theta_k} r|\mathcal{M}(\nabla u_k)|d\theta dr &\le \int_{r_k/2}^{r_k}\int_0^{\theta_k}\Big(r+\frac{4r}{r_k}+\frac{C}{\theta_k}|\partial_{w_2}\psi_k^\star(0,1-\frac{2\theta}{\theta_k})|+\frac{C}{\theta_k}\Big)\,d\theta dr + o(1),\\
&= o(1) + C\frac{r_k}{2\theta_k}\int_0^{\theta_k}|\partial_{w_2}\psi_k^\star(0,1-\frac{2\theta}{\theta_k})|d\theta = o(1)
\end{aligned}\tag{13.29}$$

as $k\to+\infty$. In the last equality we use that $|\partial_{w_2}\psi_k^\star(0,\cdot)| \le |\partial_{w_2}\psi^\star(0,\cdot)|$, which is integrable via the change of variables $w_2 = 1-\frac{2\theta}{\theta_k}$ (it also makes θ_k disappear at the denominator in front of the integral in (13.29)).

This proves that the contribution of area of the graph of u_k over $C_k^+ \cap (\mathrm{B}_{r_k}\setminus\mathrm{B}_{r_k/2})$ is infinitesimal as $k\to+\infty$.

Eventually, for $r\in[r_k/2, r_k]$, $\theta\in[2\pi-\theta_k, 2\pi)$, we set

$$u_k(r,\theta) := (u_{k1}(r,2\pi-\theta), -u_{k2}(r,2\pi-\theta)). \tag{13.30}$$

Observe that, thanks to (13.27), u_k is continuous on $\partial\mathrm{B}_{r_k}$, and similar estimates as in (13.29) for the area hold on $(\mathrm{B}_{r_k}\setminus\mathrm{B}_{r_k/2})\cap C_k^-$.

Step 4. Definition of u_k on $C_k\cap\mathrm{B}_{r_k/2}$ and its area contribution.

[76] The Lipschitz constant of u_k on this set turns out to be unbounded with respect to k.

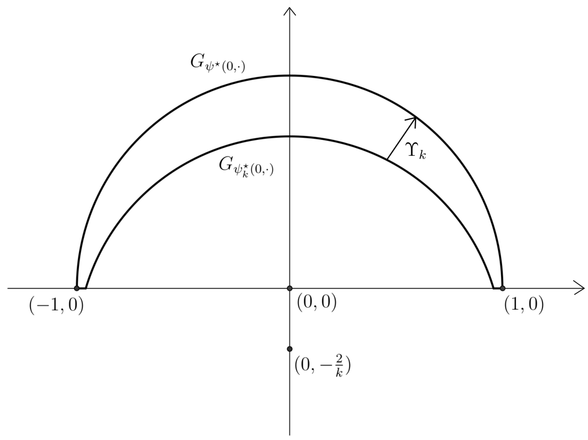

Figure 22: the graphs of the functions $\psi^\star_k(0,\cdot)$ and $\psi^\star(0,\cdot)$; these contain arcs of circle centered at $(0,0)$ and $(0,-\frac{2}{k})$ respectively. The map Υ_k is emphasized. This turns out to be the restriction of $x \mapsto \frac{x}{|x|}$ on $\psi^\star_k(0,\cdot)$.

We start with the construction of u_k on $C^+_k \cap \mathrm{B}_{r_k/2}$. For $r \in [0, r_k/2)$ and $\theta \in [0,\theta_k]$ we set

$$u_k(r,\theta) := \Upsilon_k\Big(\frac{4r\theta}{r_k\theta_k} - 1, \psi^\star_k\big(0, 1 - \frac{4r\theta}{r_k\theta_k}\big)\Big). \tag{13.31}$$

First we observe that

$$u_k\Big(\frac{r_k}{2},\theta\Big) := \Upsilon_k\Big(\frac{2\theta}{\theta_k} - 1, \psi^\star_k\big(0, 1 - \frac{2\theta}{\theta_k}\big)\Big), \qquad \theta \in (0,\theta_k),$$

so that u_k is continuous on $C^+_k \cap \partial\mathrm{B}_{r_k/2}$ (see (13.28) and (13.16)), and

$$u_k(r,\theta_k) = \Upsilon_k\Big(\frac{4r}{r_k} - 1, \psi^\star_k\big(0, 1 - \frac{4r}{r_k}\big)\Big), \tag{13.32}$$

$$u_k(r,0) = (-1, \psi^\star_k(0,1)) = (-1,0). \tag{13.33}$$

Direct computations lead to the following estimates:

$$|\partial_r u_k(r,\theta)| \leq \widehat{C}\frac{4\theta}{r_k\theta_k}\Big(1 + |\partial_{w_2}\psi^\star_k(0, 1 - \frac{4\theta r}{r_k\theta_k})|\Big), \tag{13.34}$$

$$|\partial_\theta u_k(r,\theta)| \leq \widehat{C}\frac{4r}{r_k\theta_k}\Big(1 + |\partial_{w_2}\psi^\star_k(0, 1 - \frac{4\theta r}{r_k\theta_k})|\Big), \tag{13.35}$$

where $\widehat{C}$ is the constant bounding the gradient of Υ_k as in step 3. Finally, since by (13.31) u_k takes values in $\mathbb{S}^1 \subset \mathbb{R}^2$, we have $Ju_k(r,\theta) = 0$ for all $r \in (0, r_k/2)$, $\theta \in [0,\theta_k]$. Hence, the area of the graph of u_k on $C^+_k \cap \mathrm{B}_{r_k/2}$ is

$$\int_0^{r_k/2}\int_0^{\theta_k} r|\mathcal{M}(\nabla u_k)(r,\theta)|\, d\theta dr \leq \int_0^{r_k/2}\int_0^{\theta_k} (r+C) + \frac{C}{\theta_k} + \frac{Cr}{r_k}(1+\frac{1}{\theta_k})|\partial_{w_2}\psi^\star_k(0, 1 - \frac{4\theta r}{r_k\theta_k})| d\theta dr,$$

where C is a positive constant independent of k. Exploiting that $|\partial_{w_2}\psi^\star_k(0,\cdot)| \leq |\partial_{w_2}\psi^\star(0,\cdot)|$, we can estimate the right-hand side of the previous formula as follows:

$$\begin{aligned}
&C\int_0^{r_k/2}\int_0^{\theta_k}\frac{r}{r_k}\Big(1+\frac{1}{\theta_k}\Big)|\partial_{w_2}\psi^\star(0,1-\frac{4\theta r}{r_k\theta_k})|d\theta dr+o(1)\\
\le &C\int_0^{r_k/2}\int_{-1}^{1}\theta_k\Big(1+\frac{1}{\theta_k}\Big)|\partial_{w_2}\psi^\star(0,w_2)|dw_2dr+o(1)\\
\le &C\int_0^{r_k/2}(\theta_k+1)dr+o(1)=o(1),
\end{aligned} \tag{13.36}$$

where $o(1)\to 0$ as $k\to+\infty$, and C is a positive constant independent of k which might change from line to line.

In $C_k^-\cap \mathrm{B}_{r_k/2}$ we set, for $r\in[0,r_k/2)$, $\theta\in[2\pi-\theta_k,2\pi)$,

$$u_k(r,\theta):=(u_{k1}(r,2\pi-\theta),-u_{k2}(r,2\pi-\theta)).$$

Similar estimates as in (13.36) for the area hold on $C_k^-\cap \mathrm{B}_{r_k/2}$.

Step 5. Definition of u_k on T_k and its area contribution.

We first construct u_k on $T_k\cap\{(r,\theta):r\in[0,r_k/2],\theta\in[\theta_k,\overline{\theta}_k]\}$. We define $\beta_k:[0,r_k/2]\times[\theta_k,\overline{\theta}_k]\to[0,\pi]$ as

$$\beta_k(r,\theta):=\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k}\alpha_k(r)+(1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\big(\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi\big),$$

where

$$\alpha_k(r):=\arccos\big(\Upsilon_{k1}(\frac{4r}{r_k}-1,\psi_k^\star(0,1-\frac{4r}{r_k}))\big),\qquad r\in[0,r_k/2].$$

Notice that α_k is decreasing and takes values in $[0,\pi]$. Therefore we set

$$u_k(r,\theta):=\big(\cos(\beta_k(r,\theta)),\sin(\beta_k(r,\theta))\big),\qquad (r,\theta)\in[0,r_k/2]\times[\theta_k,\overline{\theta}_k].$$

One checks that $\beta_k(r,\theta_k)=\alpha_k(r)$, $\beta_k(r,\overline{\theta}_k)=\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi$ (see also (13.9)), and

$$\begin{aligned}
&\alpha_k(r_k/2)=0,\\
&u_k(r_k/2,\theta)=\big(\cos((1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\overline{\theta}_k),\sin((1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\overline{\theta}_k)\big),\\
&u_k(r,\theta_k)=\big(\cos(\alpha_k(r)),\sin(\alpha_k(r))\big)=\Upsilon_k(\frac{4r}{r_k}-1,\psi_k^\star(0,1-\frac{4r}{r_k})),\\
&u_k(r,\overline{\theta}_k)=\big(\cos(\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi),\sin(\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi)\big),
\end{aligned}$$

so that u_k is continuous on $\{\theta\in\{\theta_k,\overline{\theta}_k\},\ r\in[0,r_k/2]\}\cap\Omega$, see (13.10) and (13.32).

Notice also that u_k is continuous at $(0,0)\in\mathbb{R}^2$ and $u_k(0,0)=(-1,0)$. Finally, since u_k takes values in $\mathbb{S}^1$, the determinant of its Jacobian vanishes, so that in order to estimate the area contribution of the graph of u_k in $T_k\cap\{(r,\theta):r\in[0,r_k/2],\ \theta\in[\theta_k,\overline{\theta}_k]\}$ it is sufficient to estimate the derivatives of u_k. We have

$$\begin{aligned}
&|\partial_r u_k(r,\theta)|=|\partial_r\beta_k(r,\theta)|\le|\partial_r\alpha_k(r)|+\frac{2\pi}{r_k},\\
&|\partial_\theta u_k(r,\theta)|=|\partial_\theta\beta_k(r,\theta)|\le\frac{|\alpha_k(r)|}{\overline{\theta}_k-\theta_k}+\frac{\pi}{\overline{\theta}_k-\theta_k}\le\frac{2\pi}{\overline{\theta}_k-\theta_k}.
\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^{r_k/2}\int_{\theta_k}^{\overline{\theta}_k} r|\mathcal{M}(\nabla u_k)(r,\theta)|d\theta dr &\le \int_0^{r_k/2}\int_{\theta_k}^{\overline{\theta}_k} \Big[\frac{r_k}{2}(1+|\partial_r\beta_k(r,\theta)|)+|\partial_\theta\beta_k(r,\theta)|\Big]\, d\theta dr \\ \le o(1)+\int_0^{r_k/2}\int_{\theta_k}^{\overline{\theta}_k}\Big(\frac{r_k}{2}|\partial_r\alpha_k(r)|+\pi+\frac{2\pi}{\overline{\theta}_k-\theta_k}\Big)\, d\theta dr = o(1), \qquad (13.37)\end{aligned}$$

with $o(1)\to 0$ as $k\to+\infty$. Notice that the integral of $|\partial_r\alpha_k(r)|$ with respect to r can be computed via the fundamental integration theorem, since α_k is monotone.

In $T_k\cap\{(r,\theta): r\in[0,r_k/2],\theta\in[2\pi-\overline{\theta}_k,2\pi-\theta_k]\}$ we set

$$u_k(r,\theta) := (u_{k1}(r,2\pi-\theta),-u_{k2}(r,2\pi-\theta)).$$

We now define u_k on $T_k\cap\{(r,\theta): r\in(r_k/2,l),\theta\in[\theta_k,\overline{\theta}_k]\}$. We set

$$u_k(r,\theta) := \big(\cos((1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\overline{\theta}_k),\sin((1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\overline{\theta}_k)\big).$$

Then $u_k\in \mathrm{Lip}(T_k,\mathbb{S}^1)$, and

$$\begin{aligned}&u_k(r,\theta_k)=(1,0), \qquad u_k(r,\overline{\theta}_k)=(\cos\overline{\theta}_k,\sin\overline{\theta}_k) \quad \text{for } r\in(r_k/2,l),\\ &\partial_r u_k(r,\theta)=0,\\ &\partial_\theta u_k(r,\theta)=\frac{\overline{\theta}_k}{\overline{\theta}_k-\theta_k}\Big(-\sin((1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\overline{\theta}_k),\cos((1-\frac{\overline{\theta}_k-\theta}{\overline{\theta}_k-\theta_k})\overline{\theta}_k)\Big).\end{aligned}$$

Hence

$$\int_{r_k/2}^{l}\int_{\theta_k}^{\overline{\theta}_k} r|\mathcal{M}(\nabla u_k)(r,\theta)|d\theta dr \le \int_{r_k/2}^{l}\int_{\theta_k}^{\overline{\theta}_k}\Big(r+\frac{\overline{\theta}_k}{\overline{\theta}_k-\theta_k}\Big)\, d\theta dr = o(1) \qquad (13.38)$$

as $k\to+\infty$.

Finally in $T_k\cap\{(r,\theta): r\in(r_k/2,l),\theta\in[2\pi-\overline{\theta}_k,2\pi-\theta_k]\}$ we set

$$u_k(r,\theta) := (u_{k1}(r,2\pi-\theta),-u_{k2}(r,2\pi-\theta)).$$

Similar estimates as in (13.37), (13.38) for the area hold on $T_k\cap\{(r,\theta): r\in(0,r_k/2),\theta\in[2\pi-\overline{\theta}_k,2\pi-\theta_k]\}$, $T_k\cap\{(r,\theta): r\in(r_k/2,l),\theta\in[2\pi-\overline{\theta}_k,2\pi-\theta_k]\}$, respectively.

Step 6. We claim that

$$\int_{\Omega\setminus(C_k\cup T_k)}|\mathcal{M}(\nabla u_k)|dx \longrightarrow \int_\Omega|\mathcal{M}(\nabla u)|\, dx \qquad \text{as } k\to+\infty, \qquad (13.39)$$

where we recall that $C_k\cup T_k=\{(r,\theta)\in\Omega: r\in[0,l),\theta\in[0,\overline{\theta}_k]\cup[2\pi-\overline{\theta}_k,2\pi)\}$.

Indeed, on $\Omega\setminus(C_k\cup T_k)$ the maps u_k and u take values in the circle $\mathbb{S}^1$, hence

$$\det(\nabla u_k)=0, \qquad \det(\nabla u)=0, \qquad \text{in } \Omega\setminus(C_k\cup T_k).$$

Thus

$$\int_{\Omega\setminus(C_k\cup T_k)}|\mathcal{M}(\nabla u_k)-\mathcal{M}(\nabla u)|\, dx \le \sum_{i=1,2}\int_{\Omega\setminus(C_k\cup T_k)}|\nabla(u_{ki}-u_i)|\, dx.$$

From (13.9), we have

$$
\begin{aligned}
|\partial_r(u_k - u)| &= 0 && \text{in} \quad \Omega \setminus (\mathrm{B}_{r_k} \cup C_k \cup T_k),\\
|\partial_r(u_k - u)| &\le \frac{\pi}{r_k} && \text{in} \quad \mathrm{B}_{r_k} \setminus (C_k \cup T_k),\\
|\partial_\theta(u_k - u)| &= 0 && \text{in} \quad \Omega \setminus (\mathrm{B}_{r_k} \cup C_k \cup T_k),\\
|\partial_\theta(u_k - u)| &\le 2 && \text{in} \quad \mathrm{B}_{r_k} \setminus (C_k \cup T_k).
\end{aligned}
\tag{13.40}
$$

Our previous remarks and the fact that $r_k, \theta_k, (\overline{\theta}_k - \theta_k) \to 0^+$ as $k \to +\infty$, imply (13.39).

Step 7. We know from (13.29), (13.36), (13.37), and (13.38), that the integral of $|\mathcal{M}(\nabla u_k)|$ is infinitesimal as $k \to +\infty$, on the region $(\mathrm{B}_{r_k} \cap C_k) \cup T_k$. Therefore it remains to compute the area of the graphs of u_k in the region $C_k \setminus \mathrm{B}_{r_k}$. We claim that this contribution is

$$
\lim_{k\to+\infty} \int_{C_k\setminus \mathrm{B}_{r_k}} |\mathcal{M}(\nabla u_k)|\, dx \le 2\mathcal{F}_l(h^\star, \psi^\star) = \mathbb{A}(\psi^\star, SG_{h^\star}).
\tag{13.41}
$$

To prove this, we start to compute the area of the graph of u_k restricted to $C_k^+ \setminus \mathrm{B}_{r_k}$. From (13.25), (13.17), (13.19) and (13.18), we have

$$
\begin{aligned}
\partial_r u_{k1} &= \Big(\frac{\theta}{\theta_k} - 1\Big)\tau_k' h_k^{\star\prime} = \frac{l}{l - r_k}\Big(\frac{\theta}{\theta_k} - 1\Big) h_k^{\star\prime},\\
\partial_\theta u_{k1} &= \frac{1 + h_k^\star}{\theta_k},\\
\partial_r u_{k2} &= \tau_k'\Big[\Big(1 - \frac{\theta}{\theta_k}\Big) h_k^{\star\prime}\partial_{w_2}\psi_k^\star + \partial_{w_1}\psi_k^\star\Big] = \frac{l}{l-r_k}\Big[\Big(1 - \frac{\theta}{\theta_k}\Big) h_k^{\star\prime}\partial_{w_2}\psi_k^\star + \partial_{w_1}\psi_k^\star\Big],\\
\partial_\theta u_{k2} &= -\Big[\frac{1 + h_k^\star}{\theta_k}\Big]\partial_{w_2}\psi_k^\star,\\
\partial_r u_{k1}\partial_\theta u_{k2} - \partial_\theta u_{k1}\partial_r u_{k2} &= -\left(\frac{1 + h_k^\star}{\theta_k}\right)\frac{l}{l - r_k}\partial_{w_1}\psi_k^\star,
\end{aligned}
\tag{13.42}
$$

where $h_k^{\star\prime}$ denotes the derivative of $h_k^\star$ with respect to w_1, $h_k^\star, h_k^{\star\prime}$ are evaluated at $\tau_k(r)$, and the two partial derivatives $\partial_{w_2}\psi_k^\star$, $\partial_{w_1}\psi_k^\star$ of $\psi_k^\star$ with respect to w_2, w_1 are evaluated at $(\tau_k(r), -s_k(r,\theta))$. Note carefully that, in the computation of the Jacobian, the terms containing $\partial_{w_2}\psi_k^\star$ cancel each other.

Notice that, since $h_k^\star$ is convex, its derivative is nonincreasing, and therefore $\int_{r_k}^l |h_k^{\star\prime}|\, dr < +\infty$. As a consequence of (13.42), from (2.3), we have

$$
\begin{aligned}
&\mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k})\\
&= \int_{r_k}^{l}\int_0^{\theta_k} r\Bigg\{1 + \left(\frac{l}{l - r_k}\right)^2\left(\frac{\theta}{\theta_k} - 1\right)^2 (h_k^{\star\prime})^2\\
&\quad + \left(\frac{l}{l - r_k}\right)^2\Big[\big(\frac{\theta}{\theta_k} - 1\big)^2 (h_k^{\star\prime})^2(\partial_{w_2}\psi_k^\star)^2 + 2\big(1 - \frac{\theta}{\theta_k}\big) h_k^{\star\prime}\partial_{w_2}\psi_k^\star\partial_{w_1}\psi_k^\star + (\partial_{w_1}\psi_k^\star)^2\Big]\\
&\quad + \frac{1}{r^2}\left(\frac{1 + h_k^\star}{\theta_k}\right)^2\left(1 + (\partial_{w_2}\psi_k^\star)^2 + \left(\frac{l}{l - r_k}\right)^2(\partial_{w_1}\psi_k^\star)^2\right)\Bigg\}^{\frac{1}{2}} drd\theta,
\end{aligned}
$$

where $\partial_{w_2}\psi_k^\star$, $\partial_{w_1}\psi_k^\star$ are evaluated at $(\tau_k(r), -s_k(r,\theta))$, and $h_k^\star$, $h_k^{\star\prime}$ are evaluated at $\tau_k(r)$. Now we use the change of variable (13.22): from (13.24), we have

$$
\begin{aligned}
&\mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) \\
=& \int_0^l \int_{-1}^{h_k^\star(w_1)} \left(\frac{l-r_k}{l}\right)\left(\frac{\theta_k}{1+h_k^\star}\right) R_k(w_1) \Bigg\{ 1 + (\frac{l}{l-r_k})^2 \left(\frac{\Theta_k(w_1,w_2)}{\theta_k} - 1\right)^2 ({h_k^\star}')^2 \\
&+ \left(\frac{l}{l-r_k}\right)^2 \Big[(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k})^2 ({h_k^\star}')^2 (\partial_{w_2}\psi_k^\star)^2 + 2(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}) {h_k^\star}' \partial_{w_2}\psi_k^\star \partial_{w_1}\psi_k^\star + (\partial_{w_1}\psi_k^\star)^2 \Big] \\
&+ \frac{1}{(R_k(w_1))^2} (\frac{1+h_k^\star}{\theta_k})^2 \Big(1 + (\partial_{w_2}\psi_k^\star)^2 + (\frac{l}{l-r_k})^2 (\partial_{w_1}\psi_k^\star)^2 \Big) \Bigg\}^{\frac12} dw_2 dw_1,
\end{aligned}
$$

where $R_k(w_1)$, $\Theta_k(w_1, w_2)$ are defined in (13.20), (13.21), ${h_k^\star}'$ is evaluated at w_1, and $\partial_{w_1}\psi_k^\star$ and $\partial_{w_2}\psi_k^\star$ are evaluated at (w_1, w_2). Therefore

$$
\mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) = \int_0^l \int_{-1}^{h_k^\star(w_1)} \Big\{ \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k + \mathrm{IV}_k + \mathrm{V}_k + \mathrm{VI}_k \Big\}^{\frac12} dw_2 dw_1, \tag{13.43}
$$

where

$$
\begin{cases}
\mathrm{I}_k = \left(\frac{l-r_k}{l}\right)^2 \left(\frac{\theta_k}{1+h_k^\star}\right)^2 (R_k(w_1))^2, \\[2ex]
\mathrm{II}_k = \left(\frac{\theta_k}{1+h_k^\star}\right)^2 \left(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}\right)^2 (R_k(w_1))^2 ({h_k^\star}')^2, \\[2ex]
\mathrm{III}_k = \left(\frac{\theta_k}{1+h_k^\star}\right)^2 (R_k(w_1))^2 \Big[(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k})^2 ({h_k^\star}')^2 (\partial_{w_2}\psi_k^\star)^2 \\[2ex]
\qquad\qquad + 2(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}) {h_k^\star}' \partial_{w_2}\psi_k^\star \partial_{w_1}\psi_k^\star + (\partial_{w_1}\psi_k^\star)^2 \Big], \\[2ex]
\mathrm{IV}_k = (\frac{l-r_k}{l})^2, \\[2ex]
\mathrm{V}_k = (\frac{l-r_k}{l})^2 (\partial_{w_2}\psi_k^\star)^2, \\[2ex]
\mathrm{VI}_k = (\partial_{w_1}\psi_k^\star)^2.
\end{cases}
$$

Since $\lim_{k\to\infty} \frac{l-r_k}{l} = 1$ and $\lim_{k\to+\infty} \theta_k = 0$, we deduce from (13.20), (13.21),

$$
\lim_{k\to+\infty} R_k(w_1) = w_1, \qquad \lim_{k\to+\infty} \frac{\Theta_k(w_1,w_2)}{\theta_k} = \frac{h^\star(w_1) - w_2}{1 + h^\star(w_1)}.
$$

Therefore we see that

$$
\int_0^l \int_{-1}^{h_k^\star(w_1)} (\mathrm{I}_k)^{\frac12} + (\mathrm{II}_k)^{\frac12} dw_2 dw_1 = o(1),
$$

as $k \to +\infty$. Moreover

$$
\int_0^l \int_{-1}^{h_k^\star(w_1)} (\mathrm{III}_k)^{\frac12} dw_2 dw_1 = o(1) \tag{13.44}
$$

as $k \to +\infty$. Indeed we may estimate

$$\int_0^l \int_{-1}^{h_k^\star(w_1)} (\mathrm{III}_k)^{\frac{1}{2}} dw_2 dw_1 \leq C\theta_k \int_0^l \int_{-1}^{h_k^\star(w_1)} |h_k^{\star\prime}(w_1)||\partial_{w_2}\psi_k^\star(w_1,w_2)| + |\partial_{w_2}\psi_k^\star(w_1,w_2)| dw_2 dw_1,$$

and using that $|h_k^{\star\prime}(w_1)| \leq 2k$ (see (13.11)), if we assume (13.5), *i.e.*, $\theta_k k \to 0$, then (13.44) follows, since the BV-norm of $\psi_k^\star$ is bounded uniformly with respect to k.

Hence, from (13.43),

$$\begin{aligned} \mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) &\leq \int_0^l \int_{-1}^{h_k^\star(w_1)} \Big\{ \mathrm{IV}_k + \mathrm{V}_k + \mathrm{VI}_k \Big\}^{\frac{1}{2}} \, dw_2 dw_1 + o(1) \\ &\leq \int_0^l \int_{-1}^{h_k^\star(w_1)} \sqrt{1 + (\partial_{w_1}\psi_k^\star)^2 + (\partial_{w_2}\psi_k^\star)^2} \, dw_2 dw_1 + o(1) \\ &= \mathbb{A}(\psi_k^\star, SG_{h^\star} \cap R_l) + o(1) = \frac{1}{2}\mathbb{A}(\psi_k^\star, SG_{h^\star}) + o(1) \end{aligned} \tag{13.45}$$

as $k \to +\infty$. Then taking the limit as $k \to +\infty$ in (13.45), and using Lemma 13.1 (iii), we get

$$\lim_{k \to +\infty} \mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) \leq \mathbb{A}(\psi^\star, SG_{h^\star}) = \mathcal{F}_{2l}(h^\star, \psi^\star), \tag{13.46}$$

where the last equality follows from (12.16).

Step 8. Conclusion. Notice that $u_k \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$, and $u_k \to u$ in $L^1(\Omega, \mathbb{R}^2)$. Inequality (13.4) follows from (13.39) (which gives the term $\int_\Omega |\mathcal{M}(\nabla u)| dx$), from (13.41) (which gives the second term in (13.4)), and from estimates (13.29), (13.36), (13.37), and (13.38), showing that all the other contributions are negligible. □

Corollary 13.4 (**Minimizers for l large enough**). *For l large enough, a solution to the minimum problem on the right-hand side of* (12.9) *is given by $h \equiv -1$ and $\psi \equiv 0$.*

Proof. Recall that for l large enough, we have

$$\overline{\mathcal{A}}(u, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| dx + \pi,$$

see [1]. The assertion then follows from Theorem 11.16 and (12.10). □

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References

[1] E. Acerbi and G. Dal Maso, *New lower semicontinuity results for polyconvex integrals*, Calc. Var. Partial Differential Equations **2** (1994), 329–371.

[2] L. Ambrosio, N. Fusco and D. Pallara, "Functions of Bounded Variation and Free Discontinuity Problems", Mathematical Monographs, Oxford Univ. Press, 2000.

[3] J.M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal. **63** (1977), 337-403.

[4] J.M. Ball and F. Murat, *$W^{1,p}$-Quasi-convexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225-253.

[5] G. Bellettini, A. Elshorbagy, M. Paolini and R. Scala, *On the relaxed area of the graph of discontinuous maps from the plane to the plane taking three values with no symmetry assumptions*, Ann. Mat. Pura Appl. **199** (2019), 445–477.

[6] G. Bellettini and M. Paolini, *On the area of the graph of a singluar map from the plane to the plane taking three values*, Adv. Calc. Var. **3** (2010), 371–386.

[7] G. Bellettini, M. Paolini and L. Tealdi, *On the area of the graph of a piecewise smooth map from the plane to the plane with a curve discontinuity*, ESAIM: Control, Optimization and Calculus of Variations **22** (2015), 29–63.

[8] G. Bellettini, M. Paolini and L. Tealdi, *Semicartesian surfaces and the relaxed area of maps from the plane to the plane with a line discontinuity*, Ann. Mat. Pura Appl. **195** (2016), 2131-2170.

[9] F. Cagnetti, M. Perugini and D. Stöger, *Rigidity for perimeter inequality under spherical symmetrisation*, Calc. Var. Partial Differential Equations **59** (2020), 59-139.

[10] P. Creutz, *Plateau's problem for singular curves*, arXiv:1904.12567.

[11] G. Dacorogna, "Direct Methods in the Calculus of Variations", Springer, Berlin-Heidelberg-New York, 1989.

[12] G. Dal Maso, *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals*, Manuscripta Math. (1980), 387-416.

[13] E. De Giorgi, *On the relaxation of functionals defined on cartesian manifolds*, In "Developments in Partial Differential Equations and Applications in Mathematical Physics" (Ferrara 1992), Plenum Press, New York, 1992.

[14] U. Dierkes, S. Hildebrandt and F. Sauvigny, "Minimal Surfaces", Grundlehren der mathematischen Wissenschaften, Vol. 339, Springer-Verlag, Berlin-Heidelberg, 2010.

[15] H. Federer, "Geometric Measure Theory", Die Grundlehren der mathematischen Wissenschaften, Vol. 153, Springer-Verlag, New York Inc., New York, 1969.

[16] R. Finn, "Equilibrium Capillary Surfaces", Die Grundlehren der mathematischen Wissenschaften, Vol. 284, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1986.

[17] N. Fusco and J.E. Hutchinson, *A direct proof for lower semicontinuity of polyconvex functionals*, Manuscripta Mat. **87** (1995), 35-30.

[18] M. Giaquinta, G. Modica and J. Souček, "Cartesian Currents in the Calculus of Variations I. Cartesian Currents", Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 37, Springer-Verlag, Berlin-Heidelberg, 1998.

[19] E. Giusti, "Minimal Surfaces and Functions of Bounded Variation", Birkhäuser, Boston, 1984.

[20] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J. **31** (1964), 159–178.

[21] J. Hass, *Singular curves and the Plateau problem*, Internat. J. Math. **2** (1991), 1–16.

[22] L. Hoṙmander, "Notions of Convexity", Birkhäuser, Boston, 1994.

[23] G. Krantz and R. Parks, "Geometric Integration Theory", Cornerstones, Birkhäuser Boston, Inc., Boston, MA, 2008.

[24] F. Maggi, "Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory", Cambridge Univ. Press, Cambridge, 2012.

[25] W. H. Meeks and S. T. Yau, *The classical Plateau problem and the topology of three-dimensional manifolds*, Topology **21** (1982), 409-440.

[26] C.B. Morrey, "Multiple Integrals in the Calculus of Variations", Grundlehren der mathematischen Wissenschaften, Vol. 130, Springer-Verlag, New York, 1966.

[27] J. C. C. Nitsche, "Lectures on Minimal Surfaces", Vol. I, Cambridge University Press, Cambridge, 1989.

[28] R. Scala, *Optimal estimates for the triple junction function and other surprising aspects of the area functional*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **XX** (2020), 491-564.