

# A UNIFIED DIVERGENT APPROACH TO HARDY–POINCARÉ INEQUALITIES IN CLASSICAL AND VARIABLE SOBOLEV SPACES

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ABSTRACT. We present a unified strategy to derive Hardy–Poincaré inequalities on bounded and unbounded domains. The approach allows proving a general Hardy–Poincaré inequality from which the classical Poincaré and Hardy inequalities immediately follow. The idea also applies to the more general context of variable exponent Sobolev spaces. The argument, concise and constructive, does not require a priori knowledge of compactness results and retrieves geometric information on the best constants.

## 1. INTRODUCTION

In its classical form, Poincaré inequality states that if  $\Omega$  is an open and *bounded* subset of  $\mathbb{R}^N$ ,  $N \geq 1$ , and  $p \in [1, \infty)$ , then there exists a positive constant  $c_{\Omega,p}$ , depending only on  $p$  and  $\Omega$  (in particular, on  $N$ ), such that

$$\left( \int_{\Omega} |u(x)|^p dx \right)^{1/p} \leq c_{\Omega,p} \left( \int_{\Omega} |\nabla u(x)|^p dx \right)^{1/p} \quad \forall u \in C_c^\infty(\Omega), \quad (1.1)$$

where  $C_c^\infty(\Omega)$  is the space of infinitely differentiable functions with compact support in  $\Omega$ , and  $|\nabla u|^p = \sum_{i=1}^N |\partial_i u|^p$ .

The multidimensional Hardy inequality states that if  $\Omega$  is an open subset of  $\mathbb{R}^N$  (possibly *unbounded*),  $p \in [1, \infty)$ , and  $N > p$  (therefore, necessarily  $N \geq 2$ ), then there exists a positive constant  $c_{\Omega,p}$ , depending only on  $\Omega$  and  $p$ , such that

$$\left( \int_{\Omega} \frac{|u(x)|^p}{|x|^p} dx \right)^{\frac{1}{p}} \leq c_{\Omega,p} \left( \int_{\Omega} |\nabla u(x)|^p dx \right)^{\frac{1}{p}} \quad \forall u \in C_c^\infty(\Omega). \quad (1.2)$$

The version of (1.1) reported here dates back to Steklov [41] but originates in the work of Poincaré [52], where the inequality is established in the class of smooth functions that have zero mean in  $\Omega$ . Likewise, (1.2) dates back to Leray [44] and is the multidimensional analog of the one-dimensional Hardy inequality [35] (see [41, 39, 50, 53] for some historical details).

Over the years, (1.1) and (1.2) have been intensively investigated, and various extended and refined versions have been derived to cover different functional settings (see, e.g., [49, 11, 12, 25, 1]). Also, the knowledge of sharp constants in such functional inequalities (e.g., the minimal constants  $c_{\Omega,p}$  for which (1.1) or (1.2) holds), or even upper bounds on the optimal constants which explicitly highlight their dependence on the geometry of the domain, have remarkable applications in the Analysis of PDEs and Numerics (see, e.g., [48, 17, 13, 33, 57, 56, 34, 9, 22, 42, 23]). The literature on Hardy and Poincaré inequalities is endless; for further sources and developments, we refer the reader to the comprehensive presentations in [40, 53, 26] for Hardy inequalities and [59, Chap. 4] for Poincaré-type inequalities.

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This paper aims to present a unified strategy to derive Hardy–Poincaré inequalities through a concise argument based on the divergence theorem. Our approach extends to the more general context of variable exponent Sobolev spaces [24, 16]. Unified frameworks to treat such inequalities are of some interest. In [59] a whole chapter is devoted to a unified approach to Poincaré inequalities. The underlying argument proceeds by contradiction and is based on Rellich–Kondrachov compactness theorem; while the approach is both elegant and simple, it relies on a compactness argument that requires workarounds to deal with unbounded domains where, in general, one lacks compactness. Moreover, the nonconstructive nature of the approach produces a loss of information on the geometric content of the Poincaré constant, which can be only partially recovered by scaling arguments. Instead, our approach, more in the spirit of [7, 8, 34, 55], is constructive: it does not require a priori knowledge of compactness results and returns geometric information on the Hardy–Poincaré constants. In some cases, it also provides the best possible constant (cf. Remark 3.3)

Our arguments’ main idea came when the authors were analyzing some aspects of the so-called *demagnetizing field*, which is the primary source of nonlocal interactions in the variational theory of micromagnetism [20, 21]. For a magnetization  $m \in L^2(\mathbb{R}^3, \mathbb{R}^3)$  the demagnetizing field  $h[m]$  induced by  $m$  is given by  $h[m] := \nabla u_m$  with  $u_m$  the demagnetizing potential which solves

$$-\Delta u_m = \operatorname{div} m \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \quad (1.3)$$

Lax–Milgram lemma guarantees that equation (1.3) possesses a unique solution in the Beppo Levi space  $BL_0^1(\mathbb{R}^3)$ , where for every open subset  $\Omega \subseteq \mathbb{R}^3$  (cf. [17, Chap. XI, Part B], see also [18, 19, 45])

$$BL^1(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) : \frac{u(\cdot)}{\sqrt{1 + |\cdot|^2}} \in L^2(\Omega) \quad \text{and} \quad \nabla u \in L^2(\Omega, \mathbb{R}^3) \right\},$$

and  $BL_0^1(\Omega)$  is the closure of  $C_c^\infty(\Omega)$  in  $BL^1(\Omega)$ . These spaces are Hilbert spaces when endowed with the norm:

$$\|u\|_{BL^1(\Omega)}^2 = \int_{\Omega} \frac{|u(x)|^2}{1 + |x|^2} dx + \int_{\Omega} |\nabla u(x)|^2 dx. \quad (1.4)$$

The only (a priori) nontrivial point in the application of the Lax–Milgram lemma to the weak formulation of (1.3) is in the observation that the gradient seminorm  $\|\nabla u\|_{L^2(\Omega)}$  is actually a norm on  $BL_0^1(\Omega)$  equivalent to (1.4). For that, one has to prove something less than Hardy inequality (1.2), namely that

$$\int_{\Omega} \frac{|u(x)|^2}{1 + |x|^2} dx \leq c_{\Omega, p}^2 \int_{\Omega} |\nabla u(x)|^2 dx \quad \forall u \in C_c^\infty(\Omega). \quad (1.5)$$

A proof of (1.5) is given in [17, Thm. 1, p. 114] where these weighted Sobolev spaces are covered in the three-dimensional case and applied to the study of integral equations associated with elliptic boundary value problems in exterior domains of  $\mathbb{R}^3$ . Again, the proof is by contradiction and based on a localization argument that allows for compactness results.

The attempt to find a simplified proof of (1.5) led us to Theorem 3.1. An accurate bibliographic search reveals that, when  $p = 2$ , divergence theorem is adopted in [54, Thm 7.61, p. 466] to give a proof of the classical Poincaré inequality (1.1) in  $H_0^1(\Omega)$  and is also recognizable behind some computations in [51] (again in the case  $p = 2$ ). Our paper moves forward in this direction and shows how the idea can be used to derive Hardy–Poincaré inequalities in more general settings: Poincaré inequalities on domains bounded in one direction, weighted

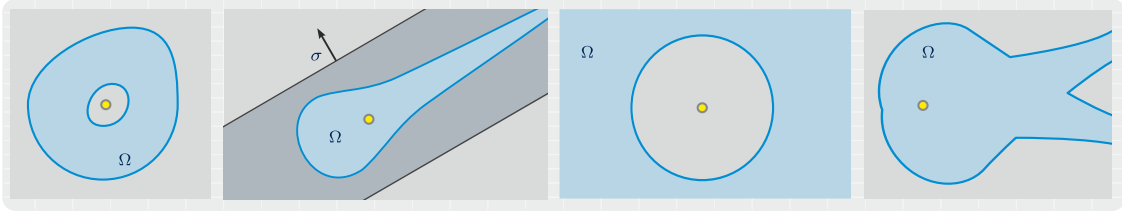


FIGURE 1. Geometrically, a domain is bounded along the  $\sigma$ -direction if it is absorbed by a strip whose boundary (consisting of two parallel hyperplane) is perpendicular to  $\sigma$ . From left to right, examples of domains in  $\mathbb{R}^2$ :  $\Omega$  bounded (in all directions);  $\Omega$  unbounded but bounded in the  $\sigma$  direction;  $\Omega$  unbounded (and not bounded in any direction);  $\Omega$  unbounded, but representable as the union of two open sets that are bounded in one direction.

Hardy–Poincaré inequalities on general domains, and Poincaré inequalities in variable exponent Sobolev spaces. In particular, in Theorem 4.1, we derive modular Hardy–Poincaré inequalities suited for variable exponent Sobolev spaces. Our results complement the remarkable findings obtained in [28, Thm. 3.1] which, roughly speaking, claim that Poincaré inequality cannot hold in a bounded domain  $\Omega$  of  $\mathbb{R}^N$  if the variable exponent  $p(\cdot)$  is radial with respect to a point  $x_0 \in \mathbb{R}^N$  (i.e.,  $p(x) = p(|x - x_0|)$ ) and the profile of  $p$  is decreasing. Instead, we show that this is the case provided that one restricts the class of competitors to the space of functions  $u \in C_c^\infty(\Omega)$  such that  $|u| \leq 1$  in  $\Omega$ . Moreover, our result holds even if  $\Omega$  is unbounded, and returns explicit dependence of the modular Poincaré constant in terms of the geometry of the domain and  $p(\cdot)$ .

The paper is organized as follows. In Section 2, we give a short proof of the Poincaré inequality on domains that are bounded in one direction. The result is well-known, but the proof we present allows us to show the main ideas in a clear way, and also to discuss some issues on the optimality of the Poincaré constant. In Section 3, we present a general Hardy–Poincaré inequality from which well-known functional inequalities follow as corollaries. In particular, the sharp multidimensional inequality in  $\mathbb{R}^N$ . Finally, in Section 4, we derive modular Hardy–Poincaré inequalities suited for variable exponent Sobolev spaces.

## 2. POINCARÉ INEQUALITY ON DOMAINS BOUNDED IN ONE DIRECTION

Let  $\sigma \in \mathbb{S}^{N-1}$ . We say that an open subset  $\Omega \subseteq \mathbb{R}^N$  is bounded along the  $\sigma$ -direction if

$$\sup_{x \in \Omega} |x \cdot \sigma| < +\infty. \quad (2.1)$$

We say that  $\Omega$  is bounded in *one* direction if there exists a direction  $\sigma \in \mathbb{S}^{N-1}$  along which  $\Omega$  is bounded. Geometrically, a domain is bounded along the  $\sigma$ -direction if it is absorbed by a strip whose boundary (consisting of two parallel hyperplane) is perpendicular to  $\sigma$  (see Figure 1).

It is well-known that if  $\Omega$  is bounded along a direction, then the Poincaré inequality holds in  $W_0^{1,p}(\Omega)$  (see [6, Theorem 5.3.1, p. 161]). Precisely, if the open subset  $\Omega \subseteq \mathbb{R}^N$  is bounded in the  $\sigma$ -direction for some  $\sigma \in \mathbb{S}^{N-1}$ , then there exists a constant  $c_\Omega(p) > 0$ , depending only on  $p$  and  $\Omega$  (in particular, on  $N$ ), such that

$$\|u\|_{L^p(\Omega)} \leq c_{p,\Omega} \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega) \quad (2.2)$$

with

$$\|\nabla u\|_{L^p(\Omega)}^p = \sum_{i \in \mathbb{N}_N} \int_{\Omega} |\partial_i u|^p dx. \quad (2.3)$$

A common proof of (2.2) consists of an argument by contradiction and is based on Rellich–Kondrachov compactness theorem (see, e.g., [10, 59]). Another way to infer (2.2) is by aligning the  $\sigma$ -direction to one of the coordinates axes, and then applying the one-dimensional fundamental theorem of calculus to get estimates of  $u$  in terms of its first-order derivatives (see, e.g., [6, 43]). From this perspective, we can say that our approach is based on the fundamental theorem of calculus in  $N$ -dimensions and applied to a vector-weighted version of  $u$ , which allows simplifying the computations and giving unified arguments regardless of  $\Omega$  being bounded or not. To clarify what we mean, we give a concise proof of a more general version of (2.2), which will clarify our arguments common strategy.

**Theorem 2.1.** *Suppose that  $\Omega \subseteq \mathbb{R}^N$  is open and bounded in the  $\sigma$ -direction for some  $\sigma \in \mathbb{S}^{N-1}$ . Then, for every  $p \in [1, +\infty)$  we have*

$$\|u\|_{L^p(\Omega)} \leq pc_{\Omega, \sigma} \|\partial_{\sigma} u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega), \quad (2.4)$$

with

$$c_{\Omega, \sigma} := \inf_{x_0 \in \mathbb{R}^N} \sup_{x \in \Omega} |(x - x_0) \cdot \sigma| \quad (2.5)$$

depending only on the projection of  $\Omega$  onto the  $\sigma$ -direction.

**Remark 2.1.** The domain on the right in Figure 1 is not bounded in one direction. However, it is representable as a finite union of open sets that are bounded in one direction. For these domains, something can still be said, and we refer the reader to Agmon’s book for details (cf. [3, Lemma 7.4, p. 75]).

**Remark 2.2.** In particular, if  $\Omega$  is bounded, then given any (not necessarily orthogonal) unit basis  $(\sigma_i)_{i \in \mathbb{N}_N}$  of  $\mathbb{R}^N$ , we get that

$$\|u\|_{L^p(\Omega)} \leq pc_{\Omega, \sigma_*} \|\partial_{\sigma_*} u\|_{L^p(\Omega)} \leq pc_{\Omega, \sigma_*} \|\nabla u\|_{L^p(\Omega)}, \quad (2.6)$$

with  $\sigma_* := \operatorname{argmin}_{i \in \mathbb{N}_N} c_{\Omega, \sigma_i}$ , as well as

$$\|u\|_{L^p(\Omega)} \leq \frac{p}{N^{1/p}} \left( \sum_{i \in \mathbb{N}_N} c_{\Omega, \sigma_i}^p \|\partial_{\sigma_i} u\|_{L^p(\Omega)}^p \right)^{1/p} \leq \frac{p}{N^{1/p}} \left( \max_{i \in \mathbb{N}_N} c_{\Omega, \sigma_i} \right) \|\nabla u\|_{L^p(\Omega)}, \quad (2.7)$$

with  $\nabla u = (\partial_{\sigma_1} u, \dots, \partial_{\sigma_N} u)$ . Note that if  $\Omega$  is *connected* then for every  $i \in \mathbb{N}_N$  the projection of  $\Omega$  onto  $\sigma_i$  defined by

$$\Pi_{\sigma_i}(\Omega) = \{t \in \mathbb{R} : t = x \cdot \sigma_i \text{ for some } x \in \Omega\} \quad (2.8)$$

is an interval and, since  $\Pi_{\sigma_i}$  is a nonexpansive map, we have that

$$\operatorname{diam} \Pi_{\sigma_i}(\Omega) \leq \operatorname{diam} \Omega. \quad (2.9)$$

Therefore, if we set  $c_{\Omega, \sigma_*} := \max_{i \in \mathbb{N}_N} c_{\Omega, \sigma_i}$ , denote by  $\eta_* \in \Pi_{\sigma_*}(\Omega)$  the center of the interval  $\Pi_{\sigma_*}(\Omega)$ , and by  $y_*$  one of the elements of  $\mathbb{R}^N$  such that  $\Pi_{\sigma_*}(y_*) = \eta_*$ , i.e., such that  $y_* \cdot \sigma_* = \eta_*$ , by (2.9) we infer that

$$c_{\Omega, \sigma_*} \leq \sup_{x \in \Omega} |(x - y_*) \cdot \sigma_*| = \sup_{x \in \Omega} |x \cdot \sigma_* - \eta_*| = \frac{\operatorname{diam} \Pi_{\sigma_*}(\Omega)}{2} \leq \frac{\operatorname{diam} \Omega}{2}. \quad (2.10)$$

Overall, by (2.6) and (2.9) we get that if  $\Omega$  is *connected* then, for every  $u \in C_c^\infty(\Omega)$  there holds

$$\|u\|_{L^p(\Omega)} \leq \frac{p}{N^{1/p}} \frac{\operatorname{diam} \Omega}{2} \|\nabla u\|_{L^p(\Omega)}. \quad (2.11)$$

The previous relation is the analog of the well-known sharp estimate obtained for  $p = 1$  in [2] in the class of *convex* domains having a prescribed diameter. Note, however, that in [2] the optimal estimate concerns the class of functions with zero mean in  $\Omega$ ; here, we consider the class of functions vanishing on  $\partial\Omega$ . Nevertheless, the main difficulty in [2] is the sharpening of the Poincaré constant from  $\text{diam } \Omega$  to  $\frac{1}{2}(\text{diam } \Omega)$ ; this is due to the well-known obstruction that, in general, in dimension  $N \geq 2$ , it is not true that a convex subset of  $\mathbb{R}^N$  is contained in a ball of radius half its diameter. Although our result overcomes this issue, there is no pretense of optimality in (2.11).

*Proof.* By density, it is sufficient to prove (2.4) for every  $u \in C_c^\infty(\Omega)$ . First, assume that  $p > 1$  so that  $|u|^p \in C_c^1(\Omega)$  and (with  $\text{sign } t = 1$  if  $t \geq 0$  and  $\text{sign } t = -1$  otherwise) we have

$$\nabla |u|^p = p(\text{sign } u)|u|^{p-1}\nabla u. \quad (2.12)$$

For any unit vector  $\sigma \in \mathbb{S}^{N-1}$ , and any  $x_0 \in \mathbb{R}^N$ , we set  $\tau_{x_0}(x) := (x - x_0)$ . Observe that  $\text{div}_x((\tau_{x_0} \cdot \sigma)\sigma) = \text{div}_x((x \cdot \sigma)\sigma) = |\sigma|^2 = 1$ , therefore, pointwise in  $\Omega$  we have that

$$\text{div}[(\tau_{x_0} \cdot \sigma)\sigma |u|^p] = |u|^p + p(\text{sign } u)(\tau_{x_0} \cdot \sigma)|u|^{p-1} \cdot \partial_\sigma u. \quad (2.13)$$

After that, integrating both sides of the previous relation on  $\Omega$ , using the divergence theorem and then Hölder inequality, we get that

$$\int_\Omega |u|^p dx = -p \int_\Omega (\text{sign } u)|u|^{p-1}(\tau_{x_0} \cdot \sigma)\partial_\sigma u dx \quad (2.14)$$

$$\leq p \sup_{x \in \Omega} |\tau_{x_0} \cdot \sigma| \left( \int_\Omega |u|^p dx \right)^{1/q} \left( \int_\Omega |\partial_\sigma u|^p dx \right)^{1/p}, \quad (2.15)$$

where we denoted by  $q$  the conjugate exponent of  $p$ , i.e.,  $1/p + 1/q = 1$ . By the arbitrariness of  $x_0 \in \mathbb{R}^N$ , we get (2.4) for every  $p > 1$ . Eventually, since  $c_{\Omega, \sigma}$  does not depend on  $p$ , if  $u \in C_c^\infty(\Omega)$ , passing to the limit for  $p \rightarrow 1^+$  in (2.4) we conclude.  $\square$

### 3. HARDY–POINCARÉ INEQUALITIES IN ARBITRARY OPEN SETS

In this section, we prove a weighted inequality that applies to general open subsets of  $\mathbb{R}^N$ . The argument is short and constructive. Moreover, it is sharp in the sense that, as a particular case, we get the classical Hardy inequality in  $C_c^\infty(\mathbb{R}^N)$  with optimal constant (see Remark 3.3). To the best of the authors' knowledge, the inequality we present is new, but in any case, the point we want to emphasize here resides in the short argument used to derive it.

**Theorem 3.1.** *Let  $\Omega$  be an open subset (bounded or not) of  $\mathbb{R}^N$ ,  $N \geq 1$ , and  $p \in [1, +\infty)$ . For any  $\lambda > 0, \alpha > 0, \beta \neq 0$  in the set of real numbers and such that  $\alpha\beta < N$ , and for arbitrary  $x_0 \in \mathbb{R}^N$ , we have that*

$$\int_\Omega \frac{|u(x)|^p}{(\lambda + |x - x_0|^\alpha)^\beta} dx \leq \left( \frac{p}{\kappa_+} \right)^p \int_\Omega |\nabla u(x)|^p \frac{|x - x_0|^p}{(\lambda + |x - x_0|^\alpha)^\beta} dx \quad \forall u \in C_c^\infty(\Omega). \quad (3.1)$$

Here, the strictly positive constant  $\kappa_+$  is given by  $\kappa_+ := (N - \alpha\beta)$  if  $\beta > 0$  and  $\kappa_+ = N$  if  $\beta < 0$ .

**Remark 3.1.** For  $p = \alpha = 2$ ,  $\lambda = \beta = 1$ ,  $x_0 = 0$ , and in the three-dimensional case  $N = 3$ , (3.1) reduces to the inequality

$$\int_\Omega \frac{|u(x)|^2}{1 + |x|^2} dx \leq 4 \int_\Omega |\nabla u(x)|^2 \frac{|x|^2}{1 + |x|^2} dx \quad \forall u \in C_c^\infty(\Omega), \quad (3.2)$$

which is a sharper version of the Hardy–Poincaré inequality (1.5) in the Beppo Levi space  $BL_0^1(\Omega)$  which plays an important role in the analysis of elliptic problems in exterior domains. As already recalled in the introduction, the proof of (1.5) given in [17, Thm. 1, p. 114] is by contradiction and based on a localization and compactness argument.

**Remark 3.2.** Note that  $\kappa_+$  is a *universal constant* in the sense that it depends only on the dimension  $N$  of the ambient space if  $\beta < 0$ , as well as on the decay-at-infinity rate  $\alpha\beta$  when  $\beta > 0$ . In particular,  $\kappa_+$  does not depend on  $\Omega$ . Also, we point out that a closer look at the proof of Theorem 3.1 (cf. (3.4)) shows that when  $\beta < 0$ , one can be more accurate and set

$$\kappa_+ := \begin{cases} N - \alpha|\beta| & \text{if } \beta > 0, \\ N + \alpha|\beta| \min_{x \in \Omega} \frac{|x-x_0|^\alpha}{\lambda + |x-x_0|^\alpha} & \text{if } \beta < 0. \end{cases}$$

However, this sharper definition of  $\kappa_+$ , makes the constant no more universal when  $\beta < 0$  and  $\Omega$  is such that  $\min_{x \in \Omega} |x - x_0| > 0$  for some  $x_0 \in \mathbb{R}^N$  (in particular, if  $\Omega$  is bounded).

*Proof of Theorem 3.1.* In what follows,  $x_0 \in \mathbb{R}^N$  denotes an arbitrary point. With  $\lambda > 0$ ,  $\alpha > 0$ ,  $\beta \neq 0$  in the set of real numbers, and  $\alpha\beta < N$ , we define the weight

$$\omega(x) := \frac{1}{(\lambda + |\tau_{x_0}(x)|^\alpha)^\beta}, \quad \tau_{x_0}(x) := x - x_0. \quad (3.3)$$

We note that, pointwise in  $\Omega$ ,

$$\operatorname{div}_x[\omega\tau_{x_0}](x) = \left( N - \alpha\beta \frac{|\tau_{x_0}(x)|^\alpha}{\lambda + |\tau_{x_0}(x)|^\alpha} \right) \omega(x) \geq \kappa_+ \omega(x), \quad (3.4)$$

with  $\kappa_+ = (N - \alpha\beta)$  if  $\beta > 0$  and  $\kappa_+ = N$  if  $\beta < 0$ . Note that, by assumption,  $N > \alpha\beta$  and, therefore,  $\kappa_+ > 0$  regardless of the value of the parameters  $\alpha > 0$  and  $\beta \neq 0$ . Also, note that for easier readability, we do not report in the notation the dependences of  $\omega$  and  $\kappa_+$  on  $\lambda, \alpha, \beta, N$ , and  $x_0$ .

First, assume that  $p > 1$  so that  $|u|^p \in C_c^1(\Omega)$  and (with  $\operatorname{sign} t = 1$  if  $t \geq 0$  and  $\operatorname{sign} t = -1$  otherwise) we have  $\nabla|u|^p = p(\operatorname{sign} u)|u|^{p-1}\nabla u$ . Then, we lower bound the divergence of  $\omega|u|^p\tau_{x_0}$  as follows:

$$\begin{aligned} \operatorname{div}_x[\omega|u|^p\tau_{x_0}] &= |u|^p \operatorname{div}_x[\omega\tau_{x_0}] + p(\operatorname{sign} u)|u|^{p-1}\omega\nabla u \cdot \tau_{x_0} \\ &\stackrel{(3.4)}{\geq} \kappa_+ \omega|u|^p + p(\operatorname{sign} u)|u|^{p/q}\omega\nabla u \cdot \tau_{x_0}, \end{aligned}$$

where we denoted by  $q$  the conjugate exponent of  $p$ , i.e.,  $1/p + 1/q = 1$ . By the divergence theorem, integrating over  $\Omega$  the first and last member of the previous expression, we obtain

$$\kappa_+ \int_{\Omega} |u|^p \omega \, dx \leq p \int_{\Omega} \left( |u|^{p/q} \omega^{1/q} \right) \left( \omega^{1/p} |\nabla u| \cdot |\tau_{x_0}| \right) \, dx. \quad (3.5)$$

By Hölder inequality, we get

$$\kappa_+ \int_{\Omega} |u|^p \omega \, dx \leq p \left( \int_{\Omega} |u|^p \omega \, dx \right)^{\frac{1}{q}} \left( \int_{\Omega} |\nabla u|^p |\tau_{x_0}|^p \omega \, dx \right)^{\frac{1}{p}}, \quad (3.6)$$

that is, in more explicit terms,

$$\int_{\Omega} \frac{|u(x)|^p}{(\lambda + |x - x_0|^\alpha)^\beta} \, dx \leq \left( \frac{p}{\kappa_+} \right)^p \int_{\Omega} |\nabla u(x)|^p \frac{|x - x_0|^p}{(\lambda + |x - x_0|^\alpha)^\beta} \, dx. \quad (3.7)$$

This concludes the proof under the assumption that  $p > 1$ . The case  $p = 1$  can be obtained by noting that since  $u \in C_c^\infty(\Omega)$ , Lebesgue dominated convergence theorem permits to pass to the limit for  $p \rightarrow 1^+$  in the previous expression.  $\square$

**3.1. Corollaries.** The general Poincaré-type inequality in Theorem 3.1 unifies many well-known inequalities on bounded and unbounded domains. The first consequence is a generalized version of the classical multidimensional Hardy inequality [36, 40, 53], which follows as a particular case of Theorem 3.1 when  $\beta > 0$ .

**Corollary 3.2.** *Let  $\Omega$  be an open subset (bounded or not) of  $\mathbb{R}^N$ ,  $N \geq 1$ , and  $p \in [1, +\infty)$ . For any  $0 < \gamma < N$  and for arbitrary  $x_0 \in \mathbb{R}^N$ , the following Hardy-type inequality holds:*

$$\int_{\Omega} \frac{|u(x)|^p}{|x - x_0|^\gamma} dx \leq \left( \frac{p}{N - \gamma} \right)^p \int_{\Omega} \frac{|\nabla u(x)|^p}{|x - x_0|^{\gamma-p}} dx \quad \forall u \in C_c^\infty(\Omega). \quad (3.8)$$

In particular, for  $\gamma := p < N$  we get a generalized version of Hardy inequality in  $\mathbb{R}^N$ :

$$\sup_{x_0 \in \mathbb{R}^N} \int_{\Omega} \frac{|u(x)|^p}{|x - x_0|^p} dx \leq \left( \frac{p}{N - p} \right)^p \int_{\Omega} |\nabla u(x)|^p dx \quad \forall u \in C_c^\infty(\Omega), \quad (3.9)$$

which reduces to the classical (sharp) Hardy inequality when  $\Omega := \mathbb{R}^N$ :

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx \leq \left( \frac{p}{N - p} \right)^p \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \quad \forall u \in C_c^\infty(\Omega). \quad (3.10)$$

**Remark 3.3.** We stress that the constant  $p/(N - p)$  in (3.10) is known to be sharp (see, e.g., [53, Chap. 2]). Also, note that while (3.8) makes sense in any spatial dimension  $N$ , the condition  $N > p \geq 1$  in (3.9) and (3.10) forces space dimensions  $N \geq 2$ .

*Proof.* Consider (3.1) with  $\beta > 0$ . Taking the  $\liminf$  for  $\lambda \rightarrow 0^+$  of both sides of (3.1) and invoking Beppo Levi's monotone convergence theorem, we get (3.8) with  $\gamma := \alpha\beta$ . Finally, recall that  $\kappa_+ = N - \alpha|\beta|$  when  $\beta > 0$ .  $\square$

The second consequence we want to establish is a dual version of Hardy inequality. It follows as a particular case of Theorem 3.1 when  $\beta < 0$ .

**Corollary 3.3.** *Let  $\Omega \subseteq \mathbb{R}^N$  be an open set (bounded or not),  $p \in [1, +\infty)$ . For any  $\gamma > 0$  and for arbitrary  $x_0 \in \mathbb{R}^N$ , the following Hardy-type inequality holds:*

$$\int_{\Omega} |u(x)|^p |x - x_0|^\gamma dx \leq \left( \frac{p}{N} \right)^p \int_{\Omega} |\nabla u(x)|^p |x - x_0|^{p+\gamma} dx \quad \forall u \in C_c^\infty(\Omega). \quad (3.11)$$

In particular,

$$\int_{\Omega} |u(x)|^p dx \leq \left( \frac{p}{N} \right)^p \int_{\Omega} |\nabla u(x)|^p |x - x_0|^p dx \quad \forall u \in C_c^\infty(\Omega). \quad (3.12)$$

*Proof.* Consider (3.1) with  $\beta < 0$ . We then have, for every  $u \in C_c^\infty(\Omega)$ ,

$$\int_{\Omega} |u(x)|^p (\lambda + |x - x_0|^\alpha)^{|\beta|} dx \leq \left( \frac{p}{N} \right)^p \int_{\Omega} |\nabla u(x)|^p |x - x_0|^p (\lambda + |x - x_0|^\alpha)^{|\beta|} dx,$$

because  $\kappa_+ := N$  when  $\beta < 0$ . With  $\gamma := \alpha|\beta|$ , taking the limit for  $\lambda \rightarrow 0^+$  of both sides of the previous inequality we get

$$\int_{\Omega} |u(x)|^p |x - x_0|^\gamma dx \leq \left( \frac{p}{N} \right)^p \int_{\Omega} |\nabla u(x)|^p |x - x_0|^{p+\gamma} dx. \quad (3.13)$$

Since  $u \in C_c^\infty(\Omega)$ , by Lebesgue's dominated convergence theorem we can pass to the limit for  $\gamma \rightarrow 0^+$  to obtain (3.12).  $\square$

For completeness, we observe that another consequence of Theorem 3.1 is the classical Poincaré inequality in bounded domains of  $\mathbb{R}^N$  which, as already pointed out in Remark 2.2, is also a consequence of Theorem 2.1.

**Corollary 3.4.** *Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded open set,  $p \in [1, +\infty)$ . The following Poincaré-type inequality*

$$\left( \int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} \leq \frac{p}{N} c_{\Omega} \left( \int_{\Omega} |\nabla u(x)|^p dx \right)^{\frac{1}{p}} \quad \forall u \in W_0^{1,p}(\Omega) \quad (3.14)$$

with

$$c_{\Omega} := \inf_{x_0 \in \mathbb{R}^N} \sup_{x \in \Omega} |x - x_0| \leq \text{diam } \Omega. \quad (3.15)$$

*Proof.* By density, it is sufficient to assume that  $u \in C_c^{\infty}(\Omega)$ . Since now  $\Omega$  is assumed bounded, from (3.8), we get that when  $0 < \gamma < N$ , for every  $x_0 \in \mathbb{R}^N$  and every  $u \in C_c^{\infty}(\Omega)$  there holds

$$\frac{1}{\sup_{x \in \Omega} |x - x_0|^{\gamma}} \int_{\Omega} |u(x)|^p dx \leq \left( \frac{p}{N - \gamma} \right)^p \sup_{x \in \Omega} |x - x_0|^{p-\gamma} \int_{\Omega} |\nabla u(x)|^p dx. \quad (3.16)$$

Therefore, for  $0 < \gamma < \min\{p, N\}$ , we get

$$\int_{\Omega} |u(x)|^p dx \leq \left( \frac{p}{N - \gamma} \right)^p \left( \sup_{x \in \Omega} |x - x_0| \right)^p \int_{\Omega} |\nabla u(x)|^p dx. \quad (3.17)$$

Eventually, taking the limit for  $\gamma \rightarrow 0^+$  we conclude.  $\square$

#### 4. HARDY–POINCARÉ INEQUALITY IN VARIABLE EXPONENT SOBOLEV SPACES

In the variable exponents' framework, the classical Poincaré inequality (1.1) can be seen either as a special case of the norm inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \forall u \in C_c^{\infty}(\Omega), \quad (4.1)$$

or as a particular case of the modular inequality

$$\int_{\Omega} |u(x)|^{p(\cdot)} dx \leq c \int_{\Omega} |\nabla u(x)|^{p(\cdot)} dx \quad \forall u \in C_c^{\infty}(\Omega). \quad (4.2)$$

A straightforward consequence of the definition of the norm of variable exponent Lebesgue spaces (see, e.g., [16, Def. 2.16, p. 20]) is that modular inequalities imply the corresponding norm inequalities; therefore, in general, the validity of a modular inequality requires the same or stronger assumptions on the exponent function.

The norm form (4.1) of the Poincaré inequality is known to hold in bounded domains provided that uniform continuity or small local oscillation on the exponent is assumed (see, e.g., [38, Theorem 3.10], [27, Theorem 2.11, p. 69], [24, Theorem 8.2.18, p. 263], [37, Theorem 6.2.8, p. 130], [47, Proposition 2.4], [14, Section 4], [4, Theorem 1.1]). The norm form (4.1) also holds under certain regularity assumptions expressed in terms of the boundedness of the maximal operator ([16, Theorem 6.21, p. 249], [24, Theorem 8.2.4, p. 255]). Finally, we mention [32, Example 3.5], where the classical way to get the Poincaré inequality as a consequence of the Sobolev inequality is stated in terms of modulars, and [58, Lemma 13.7] where the same inequality is obtained under a compactness assumption.

The modular form (4.2) of the Poincaré is a broader issue. The interest in it, supported by a general grown attention to modular inequalities (see, e.g., [31, 15]), is mainly because of various examples of its invalidity, which prompted for sufficient conditions for its validity (see, e.g., the counterexamples in [29, Example, p. 444], [24, Example 8.2.7, p. 257], [46, Section 2]; see also [30] for an overview about typical problems in the variable exponents framework). Positive results are, e.g., [46, Theorem 4.1], [28, Theorem 3.3], [5, Theorem 1],



(see also [24, Proposition 8.2.8(a), p. 257], [37, Theorem 6.2.10, p. 131]). Note that all the cited results assume the boundedness of the domain.

In this Section, we derive modular Hardy–Poincaré inequalities suited for variable exponent Sobolev spaces, which complement the remarkable results obtained in [28]. Indeed, as a consequence of [28, Thm. 3.1], one obtains that there is no hope for a Poincaré inequality in a bounded domain  $\Omega$  of  $\mathbb{R}^N$  if the variable exponent  $p(\cdot)$  is radial with respect to a point  $x_0 \in \mathbb{R}^N$ , i.e.,  $p(x) = p(|x - x_0|)$ , and the profile of  $p$  is decreasing. Instead, quite surprisingly, statement *ii.* of Theorem 4.1 below shows that this is the case provided that one restricts the class of competitors to the space of functions  $u \in C_c^\infty(\Omega)$  such that  $|u| \leq 1$  in  $\Omega$  (see next Remark 4.2). Moreover, our result holds even if  $\Omega$  is unbounded, and returns explicit dependence of the modular Poincaré constant in terms of the geometry of the domain and  $p(\cdot)$ .

**Theorem 4.1.** *Let  $\Omega \subseteq \mathbb{R}^N$  be an open set (bounded or not), and suppose that the variable exponent*

$$p(\cdot) : \Omega \rightarrow [1, +\infty)$$

*is in  $L^\infty(\Omega)$ . The following assertions hold:*

- i.* *If  $p(\cdot)$  is constant in the  $\sigma$  direction, for some  $\sigma \in \mathbb{S}^{N-1}$ , then for every  $u \in C_c^\infty(\Omega)$  there holds*

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa(p) \inf_{x_0 \in \mathbb{R}^N} \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} |(x - x_0) \cdot \sigma|^{p(x)} dx. \quad (4.3)$$

*for some positive constant which depends on  $p(\cdot)$ , given by*

$$\kappa(p) := \sup_{x \in \Omega} \left[ 2^{p(x)} (p(x) - 1)^{p(x)-1} \right]. \quad (4.4)$$

*In particular, if  $\Omega$  is also bounded in the  $\sigma$ -direction, then for every  $u \in C_c^\infty(\Omega)$  there holds*

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa_{\Omega, \sigma}(p) \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} dx \quad (4.5)$$

*for some positive constant  $\kappa_{\Omega, \sigma}(p)$  which depends on the projection of  $\Omega$  on  $\sigma$ , and  $p(\cdot)$ :*

$$\kappa_{\Omega, \sigma}(p) := \kappa(p) \inf_{x_0 \in \mathbb{R}^N} \sup_{x \in \Omega} |(x - x_0) \cdot \sigma|^{p(x)}. \quad (4.6)$$

- ii.* *If  $p(\cdot)$  is radial with respect to a point  $x_0 \in \mathbb{R}^N$ , i.e.,  $p(x) = p(|x - x_0|)$ , and the profile of  $p$  is decreasing, then, for every  $u \in C_c^\infty(\Omega)$  such that  $|u| \leq 1$  in  $\Omega$  there holds*

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa(p) \inf_{(x_0, \sigma) \in \mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} |(x - x_0) \cdot \sigma|^{p(x)} dx \quad (4.7)$$

*with  $\kappa(p)$  given by (4.4). In particular, if  $\Omega$  is bounded in the  $\sigma$  direction, then for every  $u \in C_c^\infty(\Omega)$  such that  $|u| \leq 1$  in  $\Omega$  there holds*

$$\int_{\Omega} |u|^{p(x)} dx \leq \kappa_{\Omega, \sigma}(p) \int_{\Omega} |\partial_{\sigma} u|^{p(x)} dx \quad (4.8)$$

*with  $\kappa_{\Omega, \sigma}(p)$  given by (4.6).*

**Remark 4.1.** Clearly, in the hypotheses of *ii.*, if  $\Omega$  is bounded (in any direction) then given any (not necessarily orthogonal) unit basis  $(\sigma_i)_{i \in \mathbb{N}_N}$  of  $\mathbb{R}^N$ , we get, by (4.8), the following Poincaré inequality

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa_{\Omega}(p) \int_{\Omega} |\nabla u(x)|^{p(x)} dx \quad (4.9)$$

where now  $|\nabla u|^{p(x)} := \sum_{i=1}^N |\partial_{\sigma_i} u|^{p(x)}$  and  $\kappa_\Omega(p) > 0$  is a positive constant linked to the constants  $\kappa_{\Omega, \sigma_i}$ , e.g., in the way already discussed in Remark 2.2.

*Proof.* We first show the results under the additional regularity assumption that  $p(\cdot) \in C_b^1(\Omega)$ , where  $C_b^1(\Omega)$  is the space of continuously differentiable functions on  $\Omega$  whose partial derivatives are, again, continuous and bounded in  $\Omega$ . Later we use a density argument to extend the results to the general case of  $p(\cdot) \in L^\infty(\Omega)$ .

Let  $u \in C_c^\infty(\Omega)$  and  $p \in C_b^1(\Omega)$ . First, assume that  $p(x) > 1$  for every  $x \in \Omega$ . This assures that  $|u|^{p(\cdot)}$  is in  $C_c^1(\Omega)$ . For any  $\sigma \in \mathbb{S}^{N-1}$  we have  $\operatorname{div}_x((x \cdot \sigma)\sigma) = |\sigma|^2 = 1$  and, therefore,

$$\begin{aligned} \operatorname{div}_x \left[ ((x - x_0) \cdot \sigma)\sigma |u(x)|^{p(x)} \right] &= |u(x)|^{p(x)} + \nabla \left[ |u(x)|^{p(x)} \right] \cdot ((x - x_0) \cdot \sigma)\sigma \\ &= [1 + (\partial_\sigma p(x)(x - x_0) \cdot \sigma) \log |u(x)|] |u(x)|^{p(x)} \\ &\quad + p(x) |u(x)|^{p(x)-1} (\operatorname{sign} u(x)) ((x - x_0) \cdot \sigma) \partial_\sigma u(x) \\ &= [1 + (\partial_\sigma p(x)(x - x_0) \cdot \sigma) \log |u(x)|] |u(x)|^{p(x)} \\ &\quad + p(x) |u(x)|^{\frac{p(x)}{q(x)}} (\operatorname{sign} u(x)) ((x - x_0) \cdot \sigma) \partial_\sigma u(x), \end{aligned} \quad (4.10)$$

where  $q(x)$  stands for the conjugate exponent of  $p(x)$ , i.e.,  $1/p(x) + 1/q(x) = 1$ . Integrating both sides of (4.10) and invoking divergence theorem, we get

$$\begin{aligned} \int_\Omega [1 + (\partial_\sigma p(x)(x - x_0) \cdot \sigma) \log |u(x)|] |u(x)|^{p(x)} dx \\ \leq \int_\Omega p(x) |u(x)|^{\frac{p(x)}{q(x)}} |\partial_\sigma u(x)| |(x - x_0) \cdot \sigma| dx. \end{aligned} \quad (4.11)$$

By Young's inequality for products we can estimate the integrand on the right-hand side of (4.11) as follows (with  $\delta : \Omega \rightarrow \mathbb{R}_+$  a positive measurable function to be determined afterward)

$$\begin{aligned} |u(x)|^{\frac{p(x)}{q(x)}} |\partial_\sigma u(x)| |(x - x_0) \cdot \sigma| &= |u(x)|^{\frac{p(x)}{q(x)}} |\delta(x)|^{\frac{1}{q(x)}} |\partial_\sigma u(x)| |\delta(x)|^{-\frac{1}{q(x)}} |(x - x_0) \cdot \sigma| \\ &\leq \delta(x) \frac{|u(x)|^{p(x)}}{q(x)} + \\ &\quad + \frac{|\partial_\sigma u(x)|^{p(x)}}{p(x)} \left( \frac{1}{\delta(x)} \right)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)}. \end{aligned}$$

Therefore,

$$\begin{aligned} p(x) \left( |u(x)|^{\frac{p(x)}{q(x)}} |\partial_\sigma u(x)| |(x - x_0) \cdot \sigma| \right) &\leq \delta(x)(p(x) - 1) |u(x)|^{p(x)} + \\ &\quad + |\partial_\sigma u(x)|^{p(x)} \left( \frac{1}{\delta(x)} \right)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)}. \end{aligned} \quad (4.12)$$

From (4.11) and (4.12) it follows that

$$\begin{aligned} \int_\Omega [1 - \delta(x)(p(x) - 1) + (\partial_\sigma p(x)(x - x_0) \cdot \sigma) \log |u(x)|] |u(x)|^{p(x)} dx \\ \leq \int_\Omega |\partial_\sigma u(x)|^{p(x)} \left( \frac{1}{\delta(x)} \right)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)} dx. \end{aligned} \quad (4.13)$$

*Proof of i.* Assume the existence of  $\sigma \in \mathbb{S}^{N-1}$  such that  $p(\cdot)$  is constant along  $\sigma$  and that  $p(x) > 1$  for every  $x \in \Omega$ . From the previous relation (4.13) we infer that

$$\int_{\Omega} [1 - \delta(x)(p(x) - 1)] |u(x)|^{p(x)} dx \leq \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} \left( \frac{1}{\delta(x)} \right)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)} dx. \quad (4.14)$$

Therefore, if we define  $\delta(x) := \frac{1}{2} \frac{1}{p(x)-1}$ , then  $[1 - \delta(x)(p(x) - 1)] = \frac{1}{2}$  in  $\Omega$ , and the previous inequality (4.14) specializes to

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} 2^{p(x)} (p(x) - 1)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)} dx. \quad (4.15)$$

Next, we note that if we remove the hypothesis that  $p(\cdot) > 1$ , i.e., if we assume that  $p(\cdot)$  can also assume the value 1, then for every  $\varepsilon > 0$ , the previous inequality (4.15) holds with  $p(\cdot)$  replaced by  $p_{\varepsilon}(\cdot) := p(\cdot) + \varepsilon$ , and gives

$$\int_{\Omega} |u(x)|^{p_{\varepsilon}(x)} dx \leq \int_{\Omega} |\partial_{\sigma} u(x)|^{p_{\varepsilon}(x)} 2^{p_{\varepsilon}(x)} (p_{\varepsilon}(x) - 1)^{p_{\varepsilon}(x)-1} |(x - x_0) \cdot \sigma|^{p_{\varepsilon}(x)} dx. \quad (4.16)$$

Since  $u \in C_c^{\infty}(\Omega)$ , we can pass to the limit for  $\varepsilon \rightarrow 0^+$  to conclude that (4.15) holds even if  $p(\cdot)$  is allowed to assume the value 1 somewhere in  $\Omega$ , with the understanding that  $(p(x) - 1)^{p(x)-1} = 1$  whenever  $p(\cdot)$  assumes the value 1. Overall, if  $p(\cdot) \in C_b^1(\Omega)$  is constant along the  $\sigma$ -direction,  $\sigma \in \mathbb{S}^{N-1}$ , then

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa(p) \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} |(x - x_0) \cdot \sigma|^{p(x)} dx. \quad (4.17)$$

with  $\kappa(p)$  given by (4.4). Finally, passing to the infimum over the point  $x_0 \in \mathbb{R}^N$  we get (4.3). This concludes the proof of *i* under the additional regularity assumption that  $p(\cdot) \in C_b^1(\Omega)$ .

*Proof of ii.* By assumption,  $p(x) := p(|x - x_0|)$  is radial with respect to the point  $x_0 \in \mathbb{R}^N$ , therefore

$$\nabla p(x) = p'(|x - x_0|) \frac{x - x_0}{|x - x_0|}.$$

Note that, to shorten notation, we identify  $p(\cdot)$  with the associated one-dimensional profile of  $p(\cdot)$  formally defined by  $t \in \mathbb{R}_+ \mapsto p(te_N)$  with  $e_N \in \mathbb{S}^{N-1}$ . Its derivative is here and hereafter denoted by the prime symbol. It follows that

$$\partial_{\sigma} p(x) (x - x_0) \cdot \sigma = p'(|x - x_0|) \frac{((x - x_0) \cdot \sigma)^2}{|x - x_0|}.$$

First, we assume that  $p(x) > 1$  for every  $x \in \Omega$ . If the radial profile of  $p(\cdot)$  is decreasing and  $|u| \leq 1$  then

$$(\partial_{\sigma} p(x) (x - x_0) \cdot \sigma) (\log |u(x)|) |u(x)|^{p(x)} \geq 0.$$

Therefore, from (4.13), we infer that

$$\int_{\Omega} [1 - \delta(x)(p(x) - 1)] |u(x)|^{p(x)} dx \leq \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} \left( \frac{1}{\delta(x)} \right)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)} dx. \quad (4.18)$$

As in the proof of *i*, if we choose  $\delta(x) := \frac{1}{2} \frac{1}{p(x)-1}$ , then the previous inequality (4.18) specializes to

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} 2^{p(x)} (p(x) - 1)^{p(x)-1} |(x - x_0) \cdot \sigma|^{p(x)} dx. \quad (4.19)$$

As in the proof of *i*, we can remove the hypothesis that  $p(\cdot) > 1$ , noting that if  $p(\cdot)$  also assumes the value 1, then for every  $\varepsilon > 0$ , the previous inequality  $p(\cdot)$  replaced by  $p_{\varepsilon}(\cdot) :=$

$p(\cdot) + \varepsilon$ , and a limit process gives the validity of (4.19) in the general case  $p(\cdot) \geq 1$  in  $\Omega$ . Eventually, from (4.19) we get

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa(p) \inf_{(x_0, \sigma) \in \mathbb{R}^N \times \mathbb{S}^{N-1}} \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} |(x - x_0) \cdot \sigma|^{p(x)} dx \quad (4.20)$$

with  $\kappa(p)$  given by (4.4). Finally, if  $\Omega$  is bounded in the  $\sigma$ -direction, then from (4.19) we get

$$\int_{\Omega} |u(x)|^{p(x)} dx \leq \kappa(p) \inf_{x_0 \in \mathbb{R}^N} \left( \sup_{x \in \Omega} |(x - x_0) \cdot \sigma|^{p(x)} \right) \int_{\Omega} |\partial_{\sigma} u(x)|^{p(x)} dx. \quad (4.21)$$

This concludes the proof of *ii.* under the additional regularity assumption that  $p(\cdot) \in C_b^1(\Omega)$ .

*Proof of i. for  $p(\cdot) \in L^{\infty}(\Omega)$ .* We want to show how the  $C_b^1(\Omega)$  hypothesis of regularity on  $p(\cdot)$  can be weakened to  $p(\cdot) \in L^{\infty}(\Omega)$ .

We start by showing that if  $p(\cdot) \in L^{\infty}(\Omega)$  is constant in the  $\sigma$  direction for some  $\sigma \in \mathbb{S}^{N-1}$ , then (4.3) still holds. For that, it is sufficient to prove that if (4.15) holds for any  $p(\cdot) \in C_b^1(\Omega)$  which is constant in the  $\sigma$  direction, then it still holds for every  $p(\cdot) \in L^{\infty}(\Omega)$  which is constant along  $\sigma$ .

We argue as follows. Since  $u \in C_c^{\infty}(\Omega)$ , the integrals on  $\Omega$  in (4.15) can be replaced by integrals over a bounded open subset  $\mathcal{O}$  of  $\Omega$  such that  $\text{supp } u \subset \mathcal{O} \subset \bar{\mathcal{O}} \subset \Omega$ . In other words, for a given  $u \in C_c^{\infty}(\Omega)$  and for every  $p_{\varepsilon}(\cdot) \in C_b^1(\mathcal{O})$  which is constant along  $\sigma$ , there holds (cf. (4.15)):

$$\int_{\mathcal{O}} |u(x)|^{p_{\varepsilon}(x)} dx \leq \int_{\mathcal{O}} |\partial_{\sigma} u(x)|^{p_{\varepsilon}} 2^{p_{\varepsilon}(x)} (p(x) - 1)^{p_{\varepsilon}(x) - 1} |(x - x_0) \cdot \sigma|^{p_{\varepsilon}(x)} dx. \quad (4.22)$$

Now, for any sufficiently small  $\delta > 0$ , the open  $\delta$ -neighborhood of  $\mathcal{O}$  defined by  $\mathcal{O}_{\delta} := \{x \in \Omega : d(x, \mathcal{O}) < \delta\}$  is still included in  $\Omega$ . Therefore, if we set

$$\tilde{p}(x) = \begin{cases} p(x) & \text{if } x \in \mathcal{O}_{\delta}, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \mathcal{O}_{\delta}, \end{cases}$$

then  $\tilde{p}(\cdot)$  is constant in  $\mathcal{O}_{\delta}$  along  $\sigma$ , and  $\tilde{p}(\cdot) \in L^1(\mathbb{R}^N)$ .

Next, we regularize  $\tilde{p}(\cdot)$ . We consider a positive and symmetric mollifier  $\eta \in C_c^{\infty}(\mathbb{R}^N)$ ,  $\text{supp } \eta \subseteq B_1$ ,  $0 \leq \eta \leq 1$ . As usual, for every sufficiently small  $\varepsilon > 0$ , we set  $\eta_{\varepsilon}(y) := \varepsilon^{-N} \eta(\varepsilon^{-N} y)$ . Also, for every sufficiently small  $\varepsilon > 0$ , we define  $\tilde{p}_{\varepsilon}(x) := (\tilde{p} * \eta_{\varepsilon})(x)$ . The regularized exponent  $\tilde{p}_{\varepsilon}(\cdot)$  is in  $C_b^1(\mathcal{O})$  and is constant in the  $\sigma$  direction (in  $\mathcal{O}$ ). Indeed, for every  $x \in \mathcal{O}$  we have,

$$\begin{aligned} \tilde{p}_{\varepsilon}(x + (\delta/2)\sigma) &= \int_{\mathbb{R}^N} \eta_{\varepsilon}(y - (x + (\delta/2)\sigma)) \tilde{p}(y) dy \\ &= \int_{\mathbb{R}^N} \eta(y) \tilde{p}((x + \varepsilon y) + (\delta/2)\sigma) dy \\ &= \int_{\mathbb{R}^N} \eta(y) \tilde{p}((x + \varepsilon y)) dy \\ &= \tilde{p}_{\varepsilon}(x), \end{aligned}$$

at least for every  $\varepsilon < (\delta/2)$  so that  $x + \varepsilon y \in \mathcal{O}_{\delta}$  and  $(x + \varepsilon y) + (\delta/2)\sigma \in \mathcal{O}_{\delta}$  for every  $y \in B_1$ . Also,  $\tilde{p}_{\varepsilon}(\cdot) \geq 1$  in  $\mathcal{O}$  because  $\tilde{p}(\cdot) \geq 1$  in  $\mathcal{O}_{\delta}$  and  $\|\eta\|_{L^1(\mathbb{R}^N)} = 1$ .

After that, we know that  $\tilde{p}_{\varepsilon} \rightarrow \tilde{p}(x)$  in  $L^1(\mathbb{R}^N)$ . In particular, up to a subsequence,  $\tilde{p}_{\varepsilon} \rightarrow p$  a.e. in  $\mathcal{O}$ . Since (4.22) holds for every such  $\tilde{p}_{\varepsilon}(\cdot)$ , by Lebesgue dominated convergence

theorem, passing to the limit for  $\varepsilon \rightarrow 0$ , we conclude that (4.22) holds with  $p_\varepsilon(\cdot)$  replaced by  $p(\cdot)$ .

*Proof of ii. for  $p(\cdot) \in L^\infty(\Omega)$ .* We show that if  $p(\cdot) \in L^\infty(\Omega)$  is radial around  $x_0 \in \mathbb{R}^N$  then (4.7) still holds.

For that, it is sufficient to prove that if (4.19) holds for any  $p(\cdot) \in C_b^1(\Omega)$  which is radial around  $x_0 \in \mathbb{R}^N$ , then it holds also for every  $p(\cdot) \in L^\infty(\Omega)$  which is radial around  $x_0 \in \mathbb{R}^N$ . Note that (4.19) has the same expression of (4.15).

As before, since  $u \in C_c^\infty(\Omega)$ , the integrals on  $\Omega$  in (4.19) can be replaced by integrals over a bounded open subset  $\mathcal{O}$  of  $\Omega$  such that  $\text{supp } u \subset \mathcal{O} \subset \bar{\mathcal{O}} \subset \Omega$ . In other words, for a given  $u \in C_c^\infty(\Omega)$  we know that for every  $p_\varepsilon(\cdot) \in C_b^1(\Omega)$  which is radial with respect to  $x_0$  there holds:

$$\int_{\mathcal{O}} |u(x)|^{p_\varepsilon(x)} dx \leq \int_{\mathcal{O}} |\partial_\sigma u(x)|^{p_\varepsilon(x)} 2^{p_\varepsilon(x)} (p(x) - 1)^{p_\varepsilon(x)-1} |(x - x_0) \cdot \sigma|^{p_\varepsilon(x)} dx. \quad (4.23)$$

Let  $p(\cdot) \in L^\infty(\Omega)$  and denote by  $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  the one-dimensional decreasing profile of  $p(\cdot)$ , i.e.,  $p(x) = \pi(|x - x_0|)$  for each  $x \in \Omega$ . Without loss of generality, we assume that  $\pi \geq 1$  is defined on  $\mathbb{R}_+$ . We set  $\mathcal{O}(x_0) := \mathcal{O} \cup \{x_0\}$ , and introduce the one-dimensional profile  $\tilde{\pi} : \mathbb{R} \rightarrow \mathbb{R}_+$ , with compact support, obtained by redefining  $\pi(t)$  equal to 1 if  $t \in [\text{diam } \mathcal{O}(x_0), 2 \text{ diam } \mathcal{O}(x_0))$ , equal to zero if  $[2 \text{ diam } \mathcal{O}(x_0), +\infty)$ , and then extending it by zero to the whole of  $\mathbb{R}$ . We regularize  $\tilde{\pi}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$  by setting, for every sufficiently small  $\varepsilon > 0$ ,  $\tilde{\pi}_\varepsilon(x) := (\tilde{\pi} * \eta_\varepsilon)(x)$ , where now  $\eta_\varepsilon$  is the one-dimensional analog of the symmetric mollifier we introduced before. The regularized profile  $\tilde{\pi}_\varepsilon(\cdot)$  is in  $C_b^1(\mathbb{R})$  and still decreasing on  $\mathbb{R}_+$ . Indeed, for any  $0 \leq t_1 \leq t_2$  we have

$$\begin{aligned} \tilde{\pi}_\varepsilon(t_1) &= \int_{\mathbb{R}} \eta_\varepsilon(s - t_1) \tilde{\pi}(s) ds = \int_{\mathbb{R}} \eta(s) \tilde{\pi}(t_1 + \varepsilon s) ds \\ &= \int_{-t_1/\varepsilon}^{+\infty} \eta(s) \tilde{\pi}(t_1 + \varepsilon s) ds \geq \int_{-t_2/\varepsilon}^{+\infty} \eta(s) \tilde{\pi}(t_2 + \varepsilon s) ds \\ &= \tilde{\pi}_\varepsilon(t_2). \end{aligned}$$

After that, we know that  $\tilde{\pi}_\varepsilon \rightarrow \tilde{\pi}(x)$  in  $L^1(\mathbb{R})$ . In particular, up to a subsequence,  $\tilde{\pi}_\varepsilon \rightarrow \pi$  a.e. in  $\mathcal{O}$ . Note that, by construction,  $\tilde{\pi}_\varepsilon(\cdot) \geq 1$  in  $[0, \text{diam } \mathcal{O}]$  because of  $\tilde{\pi}(\cdot) \geq 1$  in  $[0, 2 \text{ diam } \mathcal{O}]$  and  $\|\eta\|_{L^1(\mathbb{R})} = 1$ . Therefore, if we define  $\tilde{p}_\varepsilon(x) = \tilde{\pi}_\varepsilon(|x - x_0|)$ , we get that  $\tilde{p}_\varepsilon \rightarrow p$  a.e. in  $\mathcal{O}$ . Since (4.23) holds for every such  $\tilde{p}_\varepsilon(\cdot)$ , by Lebesgue dominated convergence theorem, passing to the limit for  $\varepsilon \rightarrow 0$ , we conclude that (4.23) holds with  $p(\cdot)$  replaced by  $\tilde{p}(\cdot)$ .  $\square$

**Remark 4.2.** We want to highlight the reason why our statement *ii.* in Theorem 4.1 does not contradict the findings in [28, Thm. 3.1]. The key is in the logarithmic term in inequality (4.13) that, when  $p(\cdot)$  is radial with respect to  $x_0$ , reads under the form

$$\int_{\Omega} \left[ p'(|x - x_0|) \frac{((x - x_0) \cdot \sigma)^2}{|x - x_0|} \log |u(x)| \right] |u(x)|^{p(x)} dx. \quad (4.24)$$

To make this term positive, either one assumes that  $|u| \leq 1$  in  $\Omega$  and  $p'(\cdot)$  negative, or that  $p'(\cdot)$  is positive and  $|u| \geq 1$  in  $\Omega$ . Since we are working with functions with compact support in  $\Omega$  we are forced to restrict to the case  $|u| \leq 1$ .

Actually, a closer look at the proof of *ii.*, shows that for an arbitrarily radial exponent  $p(x) = p(|x - x_0|)$  in  $C_b^1(\Omega)$  (i.e., not necessarily decreasing), the result still holds in the subset of  $C_c^\infty(\Omega)$  consisting of functions that are less than or equal to 1 where  $p(\cdot)$  is decreasing, and greater than or equal to 1 in the points where  $p(\cdot)$  is increasing. In any case, in the

same spirit of logarithmic Sobolev inequalities, as soon as one is not interested in having an inequality in which the logarithmic term (4.24) is nonnegative, one can retain the term (4.24) to obtain an inequality that holds for every radial exponent  $u \in C_c^\infty(\Omega)$  (i.e., regardless of any monotonicity assumption on  $u$ ).

The term (4.24) also explains why, for the case of increasing  $p(\cdot)$ , the construction in [28, Thm. 3.1] relies on functions that take values in the interval  $[0, 1]$ . Indeed, this is the only possible choice if one wants to make the integrand in (4.24) negative in sign so as to invalidate the Poincaré inequality.

Finally, if one works in  $C^\infty(\bar{\Omega})$  then one can repeat the argument and focus on the case in which  $|u| \leq 1$  in  $\Omega$ . Of course, in this case, the divergence theorem produces a remainder term in the form of a surface integral. In fact, most of the results we presented still work in  $C^\infty(\bar{\Omega})$  and the resulting inequalities with a (surface) remainder have applications in the analysis of minimizers of energy functionals with boundary anisotropies.

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