Crack occurrence in bodies with gradient polyconvex energies

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Abstract

In a set of infinitely many reference configurations differing from a chosen fit region $\mathscr{B}$ in the three-dimensional space and from each other only by possible crack paths, a set parameterized by special measures, namely curvature varifolds, energy minimality selects among possible configurations of a continuous body those that are compatible with assigned boundary conditions of Dirichlet-type. The use of varifolds allows us to consider both "material phase" (cracked or non-cracked) and crack orientation. The energy considered is gradient polyconvex: it accounts for relative variations of second-neighbor surfaces and pressure-confinement effects. We prove existence of minimizers for such an energy. They are pairs of deformations and curvature varifolds. The former ones are taken to be SBV maps satisfying an impenetrability condition. Their jump set is constrained to be in the varifold support.

Key words: Fracture, Varifolds, Ground States, Second-neighbor interactions, Curvature effects, Calculus of Variations, Geometric Measure Theory

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1 Introduction

Deformation-induced material effects involving interactions beyond those of first-neighbor-type can be accounted for by considering, among the fields defining states, higher-order deformation gradients. In short, we can say that these effects emerge from *latent* microstructures, intending those which do not strictly require to be represented by independent (observable) variables accounting for small-spatial-scale degrees of freedom. Rather they are such that 'though its effects are felt in the balance equations, all relevant quantities can be expressed in terms of geometric quantities pertaining to apparent placements' [11, p. 49]. A classical example is the one of Korteweg's fluid: the presence of menisci in capillarity phenomena implies curvature influence on the overall motion; it is (say) measured by second gradients [34] (see also [17] for pertinent generalizations). In solids length scale effects appear to be non-negligible for sufficiently small test specimens in various geometries and loading programs; in particular, when plasticity occurs in poly-crystalline materials, such effects are associated with grain size and accumulation of both randomly stored and geometrically necessary dislocations [22], [20], [29].

These higher-order effects influence possible nucleation and growth of cracks because the corresponding hyperstresses enter the expression of Hamilton-Eshelby's configurational stress [43], [45], i.e., they influence the laws of crack evolution.

Here we look at energy minimization and consider a variational description of crack nucleation in a body with second-gradient energy dependence. We do not refer to higher order theories in abstract sense (see [17], [43], [55] for a general setting, [11] for a physical explanations in terms of microstructural effects, [45] for a generalization of [17] to higher-order complex bodies), rather we consider an energy the bulk term of which accounts for the gradient of surface variations (e.g., between neighboring staking faults in the case of crystalline bodies with dislocations) and confinement effects due to the spatial variation of volumetric strain. Specifically, the energy we consider reads as

$$\begin{aligned}\mathscr{F}(y,V;\mathscr{B}) := &\int_{\mathscr{B}} \hat{W}\Big(\nabla y(x), \nabla[\mathrm{cof}\nabla y(x)], \nabla[\det \nabla y(x)]\Big)\, dx \\ &+ \bar{a}\mu_V(\mathscr{B}) + \int_{\mathscr{G}_2(\mathscr{B})} a_1 \|A\|^{\overline{p}}\, dV + a_2\|\partial V\|\,,\end{aligned} \tag{1.1}$$

with $\mathscr{B}$ a fit region in the three-dimensional real space, $\bar{a}$, a_1, and a_2 positive constants, $y : \mathscr{B} \longrightarrow \tilde{\mathbb{R}}^3$, a special map of bounded variation, a deformation that preserves the local orientation and is such that its jump set is contained in the support over $\mathscr{B}$ of a two-dimensional varifold V (a special kind of measure, indeed), with boundary ∂V and generalized curvature tensor A. Such a support is a 2-rectifiable subset of $\mathscr{B}$ with measure $\mu_V(\mathscr{B})$, meaning that the chosen set has Hausdorff dimension equal to 2 and there is a countable family of Lipschitz's maps $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that their images cover the set up to a subset with null two-dimensional Hausdorff measure. We consider such set as a possible crack path. The choice to consider it as a rectifiable set allows us to include in our treatment highly (piecewise) irregular cracks. The terms

$$\bar{a}\mu_V(\mathscr{B}) + \int_{\mathscr{G}_2(\mathscr{B})} a_1 \|A\|^{\bar{p}}\, dV + a_2 \|\partial V\|$$

introduce a modification of the traditional Griffith energy [28], which is just $\bar{a}\mu_V(\mathscr{B})$ (i.e., it is proportional to the lateral surface area of the crack), so they have a configurational nature. The energy density $\hat{W}$ is assumed to be gradient polyconvex, according to the definition introduced in reference [9].

We presume that a minimality requirement for $\mathscr{F}(y, V; \mathscr{B})$ selects among cracked and free-of-crack configurations. We prove an existence theorem for such minima under Dirichlet-type boundary conditions; we also impose a condition allowing contact of distant body boundary pieces but avoiding self-penetration. This is the main result of this paper.

As an admissible class of deformation and varifold pairs, we take a set of curvature varifolds supported by 2-rectifiable sets, already mentioned, and orientation preserving deformations that are special maps of bounded variation with jump set contained in the support of the varifold. Our choice allows us to consider cases in which, after deformation, crack margins are in contact at least partially but the across-margin bonds are broken. Furthermore, for technical needs, which will be clear below, we presume that cofactor and determinant of the deformation gradient are taken to be generalized special maps of bounded variation admitting gradients in L^q and L^r spaces, with appropriate values of q and r. We adopt the symbol $\mathscr{A}_{\bar{p},p,q,r,s,K,C}$ for such a class of curvature varifold and deformation pairs. With respect to it we state our main result:

Theorem 1.1 *If the class $\mathscr{A} := \mathscr{A}_{\bar{p},p,q,r,s,K,C}$ of admissible couples (y, V) is not empty and* $\inf\{\mathscr{F}(y, V; \mathscr{B}) \mid (y, V) \in \mathscr{A}\} < \infty$*, the functional $(y, V) \mapsto \mathscr{F}(y, V; \mathscr{B})$ attains a minimum in $\mathscr{A}$.*

For a proof, the main difficulty to overcome is a control of the weak convergence of the deformation gradient minors. To do it, we consider currents associated with deformation graphs. In physical terms, each current can be intended as a functional that represents an internal (deformation) work accounting even for possible incompatible strain. The space of currents admits a closure theorem due to H. Federer and W. H. Fleming [18]. It allows us to obtain the desired convergence of deformation gradient minors.

In the minimization process, the sequences associated with y, namely those of deformation maps, their first and second gradients, minors of the gradient matrices are in principle independent, but we recover (reciprocal) compatibility to the limit. For this reason, the physical interpretation of currents as internal work (indeed, a rather evident interpretation) involves also the work associated with strain that can be even incompatible.

Also, if we prescribe that the deformation is a special map of bounded variation, as we do here because we want that the deformation might jump over some set, we get that the bound $\|\mathrm{cof}\nabla y\|_\infty < \infty$ is not granted. It means that we could have unbounded surface strain, a circumstance conflicting with physical plausibility. For this reason we restrict ourselves to deformations y such that $\mathrm{cof}\nabla y$ is in the class of generalized special maps of bounded variation, a circumstance assuring us to avoid meeting unbounded surface strain. Also, such a choice allows us to recover in the minimization process the weak continuity of the approximate gradients $\nabla[\mathrm{cof}\nabla y]$, a necessary ingredient to grant existence of minimizers, together with properties of compactness.

We provide below motivations for the energy (1.1) and analytical details pertaining to the scenario above summarized. We essentially refer to the three-dimensional setting because we are analyzing a concrete specific class of physical phenomena. However, the definition of some tools adopted in the analysis holds generically in n-dimensional spaces. Then, for the sake of completeness and to avoid distracting the reader from the consciousness that our work is at all not restricted to the three-dimensional case, we maintain generic the dimension in that definitions.

2 Physical insight

2.1 Energy depending on $\nabla[\mathrm{cof}(\cdot)]$*: a significant case*

The choice of allowing a dependence of the energy density $\hat{W}$ on $\nabla[\mathrm{cof}\nabla y]$ has physical ground: we consider an effect due to relative variations of neighboring surfaces. Such a situation occurs, for example, in gradient plasticity. We do not tackle directly its analysis here, but in this section we explain just its geometric reasons.

At first, however, we fix a scenario in which, to set our analysis, we consider two isomorphic but distinct copies of the three-dimensional real point space, namely $\mathbb{R}^3$ and $\tilde{\mathbb{R}}^3$ with bases $\{\mathbf{e}_A\}$, $\{\tilde{\mathbf{e}}_i\}$, i, $A = 1, 2, 3$, and metrics g and $\tilde{g}$, respectively. The isomorphism $\iota : \mathbb{R}^3 \longrightarrow \tilde{\mathbb{R}}^3$ distinguishing these two copies of the real $3D$ space is simply the identification.

The distinction between the two spaces is at the ground of the standard statement that different observers relatively moving one with respect to the other (a process in which reference frames on the *whole* ambient space change) evaluate the *same* reference configuration $\mathscr{B}$, which we select in $\mathbb{R}^3$. We consider $\mathscr{B}$ to be bounded and connected, endowed with a piecewise Lipschitz boundary. Those macroscopic shapes considered to be deformed with respect to $\mathscr{B}$ are detected in $\tilde{\mathbb{R}}^3$ by orientation preserving differentiable maps $x \longmapsto y(x) \in \tilde{\mathbb{R}}^3$ already mentioned above and considered here to be of bounded variation. As usual, we indicate by F the derivative $Dy(x)$. In components we have $F = \frac{\partial y(x)^i}{\partial x^A}\tilde{\mathbf{e}}_i \otimes \mathbf{e}^A$, where $\mathbf{e}^A$ is the A-th vector of the dual basis $\left\{\mathbf{e}^A\right\}$ of $\{\mathbf{e}_A\}$, defined to be such that $\mathbf{e}^A \cdot \mathbf{e}_B = \delta^A_B$, where the dot indicates dual pairing (precisely, $\mathbf{e}^A \cdot \mathbf{e}_B$ is $\mathbf{e}^A(\mathbf{e}_B)$, i.e., the value of the linear map $\mathbf{e}^A$ over the vector $\mathbf{e}_B$), and δ^A_B is Kronecker's delta. At every x where it is defined, F is a linear operator that maps the tangent space of $\mathscr{B}$ at x onto the linear space tangent to the deformed configuration $y(\mathscr{B})$ at $y(x)$. F brings naturally with it two other linear operators: its transpose F^{T} and the formal adjoint F^*. The formula $F^{\mathsf{T}} = g^{-1}F^*\tilde{g}$ connects the two (see [47] for the proof). More precisely, F^{T} is of the form $F^A_i\mathbf{e}_A \otimes \mathbf{e}^i$, so it maps the tangent space to $y(\mathscr{B})$ at $y(x)$ onto the analogous space to $\mathscr{B}$ at x. On its side, F^* maps the cotangent space (i.e. the one of linear maps over the tangent space) of $y(\mathscr{B})$ at $y(x)$ onto the analogous space to $\mathscr{B}$ at x. Of course, when the chosen metrics are flat, i.e., they refer to orthonormal frames, F^{T} and F^* coincide.

In periodic and quasi-periodic crystals, plastic strain emerges from dislocation motion through the lattice [52]. Such phenomenon includes meta-dislocations and their approximants in quasi-periodic lattices [19], [46]. In poly-crystalline materials, dislocation clusters at granular interstices obstruct or favor the re-organization of matter, while in amorphous materials other microstructural rearrangements determining plastic (irreversible) strain occur. Examples are creation of voids, entanglement and disentanglement of polymers.

At macroscopic scale, the one of large wavelength approximation, a traditional way to account indirectly for the cooperative effects of irreversible microscopic mutations is to accept a multiplicative decomposition of the deformation gradient, commonly indicated by F, into so-called "elastic", F^e, and "plastic", F^p factors [35], [38], namely $F = F^eF^p$, which we commonly name the *Kröner-Lee decomposition.* The plastic factor F^p describes rearrangements of matter at a low scale, while F^e accounts for macroscopic strain and rotation.

At every point $x \in \mathscr{B}$, the plastic factor F^p maps the tangent space of $\mathscr{B}$ at x into a linear space, say $\mathfrak{L}_{F^p}$, not otherwise specified, except assigning a metric $g_{\mathfrak{L}}$ to it. Then, F^e transforms such a space into the tangent space of the deformed configuration.

In general, the plastic factor F^p allows us to describe an incompatible strain, so its curl does not vanish, i.e., curl$F^p \neq 0$, unless we consider just a single crystal in which irrecoverable strain emerges from slips along crystalline planes. So, the condition curl$F^p \neq 0$, which may hold notwithstanding curl$F = 0$, does not allow us to sew up one with the other linear spaces $\mathfrak{L}_{F^p}$, varying $x \in \mathscr{B}$, so we cannot reconstruct an intermediate configuration, unless in the case of a single crystal behaving as a deck of cards, parts of which can move along slip planes (of course, curl$F^p = 0$ when F^e reduces to the identity). In other words, the union of all intermediate spaces, each associated with a single x, does not necessarily determine the tangent bundle of a set that is a fit region as $\mathscr{B}$, a set that we could consider as an intermediate configuration obtained by rearranging the inner structure of the matter composing the body under analysis. So, in general, we can appropriately speak of *intermediate spaces* rather than thinking of intermediate configurations.

At this stage, F^p is no further specified. Its values emerge from appropriate flow rules describing the evolution of F^p (see, e.g., [56]). However, we need to remind that we do not have a theory of plasticity, rather we have theories. In particular, when we look at crystals and accept as admissible deformations special functions of bounded variation, i.e., those jumping on a two-dimensional set in $3D$ space, the multiplicative decomposition emerges naturally and the plastic factor F^p appears to be a measure (see [53] and [54] for the pertinent analyses). We may also have another type of multiplicative decomposition when we look directly at crystal lattices, as shown in reference [51].

In the view offered by the multiplicative decomposition, the plastic factor F^p indicates through its time-variation how much (locally) the material goes far from thermodynamic equilibrium transiting from an energetic well to another, along a path in which the matter rearranges irreversibly. In the presence of quasi-periodic atomic arrangements, as in quasicrystals, such a viewpoint requires extension to the phason field gradient, a vector field describing local relative rearrangements of atoms that grant the quasi-periodic structure when boundary conditions vary [39], [42].

Here, we restrict the view to cases in which just F and its decomposition play a significant role: they include periodic crystals, polycrystals, even amorphous materials like cement or polymeric bodies, in this last case at least when we neglect at a first glance direct representation of the material microstructure in terms of appropriate morphological descriptors to be involved in Laundau-type descriptions coupled with strain.

In modeling elastic-perfectly-plastic materials in a large strain regime, we usually consider first-neighbor effects (those associated with the deformation gradient only) and assume that the free energy density ψ has a functional dependence on state variables of the type $\psi := \tilde{\psi}(x, F, F^p)$. It is formally equivalent to the choice $\psi := \tilde{\psi}(x, F, F^p, g)$, because the metric g in the reference space is presumed not to vary, so it has only a parametric role. Different is the case, not treated here, in which instead of resorting to the multiplicative decomposition we accept to describe plastic phenomena by considering g as time varying, as suggested in reference [48].

We maintain an acceptance of the multiplicative decomposition; so, further assumptions are listed below.

- *Plastic indifference*, which is invariance under changes in the reference shape, leaving unaltered the material structure (*material isomorphims*); formally it reads as
$$\tilde{\psi}(x, F, F^p, g) = \tilde{\psi}(x, FG, F^pG, G^*gG),$$
for any orientation preserving unimodular second rank tensor G mapping at every x the tangent space $T_x\mathscr{B}$ of $\mathscr{B}$ at x onto itself (the requirement $\det G = 1$ ensures mass conservation along changes in reference configuration).
- *Objectivity*: invariance with respect to the action of $SO(3)$ on the physical space; it formally reads
$$\tilde{\psi}(x, F, F^p, g) = \tilde{\psi}(x, QF, F^p, g),$$
for any $Q \in SO(3)$. Indeed, by definition of objectivity, Q acts on the physical space, which is distinct from reference and intermediate spaces. The first component of F refers to the reference space, while the second, a contravariant one, to the actual space, i.e., the physical one, so F is sensitive to the action of Q. At variance, F^p has no components in the actual space, and g also, so they are both not affected by the action of Q.

Denoting by A^{-*} the adjoint of A^{-1}, with A any invertible second-rank linear operator, plastic indifference implies $\tilde{\psi}(x, F, F^p) = \hat{\psi}(x, F^e, \bar{g})$, where $\bar{g} := F^{p-*}gF^{p-1}$ is at each x the push-forward of g onto the pertinent intermediate space $\mathfrak{L}_{F^p}$ through F^p. Indeed, by the action of G over the reference space, the material metric g becomes G^*gG, so we get

$$\bar{g} = F^{p-*}gF^{p-1} \xrightarrow{G} (F^{p-*}G^{-*})G^*gG(G^{-1}F^{p-1}) = \bar{g}.$$

Then, objectivity requires $\hat{\psi}(x, F^e, \bar{g}) = \hat{\psi}(x, \tilde{C}^e, \bar{g})$, with $\tilde{C}^e$ the right Cauchy-Green tensor $\tilde{C}^e = F^{e\mathrm{T}}F^e$, where $\tilde{C}^e = g_{\mathfrak{L}}^{-1}C^e$, with $C^e := F^{e*}\tilde{g}F^e$ the pull-back in $\mathfrak{L}_{F^p}$ of the physical space metric $\tilde{g}$.

To account for second-neighbor effects, we commonly accept the free energy density to be like $\hat{\psi}(x, F^e, D_\alpha F^e)$ or $\hat{\psi}(x, F^e, \bar{g}, D_\alpha F^e)$, with α indicating that the derivative is computed with respect to coordinates over $\mathfrak{L}_{F^p(x)}$.

We claim here that at least for crystalline materials *this choice—i.e., the presence of* $D_\alpha F^e$ *in the list of state variables—is related to the possibility of assigning energy to the variations of oriented areas between neighboring staking faults when* $\det F^p = 1$.

To prove such a statement we start considering that, since $\det F^p > 0$, linear algebra tells us that $\mathrm{cof} F^p = (\det F^p) F^{p-*}$. Specifically, $\mathrm{cof} F^p$ governs at each point x the variations of oriented areas from the reference shape to the linear intermediate space associated with the same point. In the case of crystals, neighboring staking faults determine such variations in the microstructural arrangements collected in what we call plastic flows.

Consequently, assigning energy to area variations due to first-neighbor staking faults, we may take a structure for the free energy as

$$\psi := \tilde{\psi}(x, F, F^p, g, {}^{\urcorner} D\mathrm{cof} F^p),$$

where D indicates the spatial derivative with respect to x, and the apex $^{\urcorner}$ indicates minor left adjoint operation of the first two indexes of a third order tensor (it corresponds to the minor left transposition when the metric is flat or the first two tensor components are both covariant or contravariant). At least in the case of volume-preserving crystal slips over planes, we have $\det F^p = 1$ so that $\mathrm{cof} F^p = F^{p-*}$, whence we can write in operational form $D\mathrm{cof} F^p = F^{p-*} \otimes D$, which implies ${}^{\urcorner} D\mathrm{cof} F^p = F^{p-1} \otimes D$. When we impose plastic indifference as above, under the action of G, describing a change in the reference shape, we have $F^{p-*} \otimes D \xrightarrow{G} ((GF^{p-1})^* \otimes D)G$. Consequently, for volume-preserving plastic flows, the requirement of *plastic invariance* reads

$$\tilde{\psi}(x, F, F^p, g, F^{p-1} \otimes D) = \tilde{\psi}(x, FG, F^p G, G^* g G, ((G^{-1} F^{p-1})) \otimes DG)$$

for any choice of G with $\det G = 1$. The latter condition implies

$$\begin{aligned} \tilde{\psi}(x, F, F^p, g, F^{p-*} \otimes D) &= \tilde{\psi}(x, FF^{p-1}, \bar{g}, ((FF^{p-1}) \otimes D)F^{p-1}) \\ &= \tilde{\psi}(x, FF^{p-1}, \bar{g}, (DF^e)F^{p-1}) = \hat{\psi}(x, F^e, \bar{g}, D_\alpha F^e), \end{aligned}$$

which concludes the proof.

Alternatively, if we choose

$$\psi := \tilde{\psi}(x, F, F^p, g, D\mathrm{cof} F^p),$$

with the same argument as above we get

$$\tilde{\psi}(x, F, F^p, g, D\mathrm{cof}F^p) = \hat{\psi}(x, F^e, \bar{g}, D_\alpha F^{e*}).$$

In our analysis below the density $\hat{W}$ is less intricate than $\tilde{\psi}(x, F, F^p, g, D\mathrm{cof}F^p)$, however, the analysis of its structure indicates a fruitful path for dealing with more complex situations.

Also, the dependence of $\tilde{W}$ on $\nabla[\det F]$ is a way of accounting for confinement effects due to non-homogeneous volume variations (see [10] for a pertinent analysis in small strain regime).

Finally, from now on we just assume flat metrics so that we write ∇ instead of D, which appears to indicate the weak derivative of special functions of bounded variation, a measure indeed. Also, we refer just to F and do not consider the plasticity setting depicted by the multiplicative decomposition. Despite this, our choice of considering the gradient of $\mathrm{cof}F$ among the entries of $\tilde{W}$ is intended as an indicator of relative surface variation effects.

Although motivated by plasticity, the minimization problem that we analyze does not involve a representation and an analysis of plastic flows. So, in essence we refer to an elastic initial phase (always foreseen; see the proof in reference [41]), or to an elastic trial in a return mapping algorithm. Then a question is whether any existing material may admit such a bulk energy. We have two cases in mind:

- Consider a body made of a soft matrix reinforced by polymer chains scattered throughout the volume. In this case non-local (second-gradient-type) effects would appear when the molecules would be so dense in the matrix to be entangled into complex nets.
- Analogous circumstances may occur for metamaterials. Imagine to have homogenized at continuum scale a metamaterial made of the superposition of two lattices. The first one is a sort of mosquito net: in it just first-neighbor interactions between nodes occur, exerted by springs. The second (superposed) lattice provides second-neighbor interactions on nodes of the first net.

The choice to consider only $\nabla[\mathrm{cof}F]$ and $\nabla[\det F]$ in the list of state variables entering the bulk energy, instead of the full $\nabla^2 y$ means only that we give prominence to area and volume variations when we refer to second-neighbor effects.

We then consider the formation of a crack in such a kind of materials. To tackle the issue we need further analytical tools, those we use to describe and parameterize crack paths.

A primary assumption in continuum mechanics is that the deformation is a one-to-one mapping. When a crack occurs after deformation and the crack margins come off, the deformation itself is no more one-to-one over the set in $\mathscr{B}$ of points that have two images over the crack, because the margins were originally connected on a single surface in $\mathscr{B}$. However, the cracked shape is indeed in one-to-one correspondence with another reference configuration differing from $\mathscr{B}$ just by a set that is a pre-image in $\mathscr{B}$ of the crack in $y(\mathscr{B})$. For this reason, we can depict the possible occurrence of cracks in the reference space by taking infinitely many copies of $\mathscr{B}$ that are different from each other only by a possible crack path, each taken to be a $\mathscr{H}^2$-rectifiable set. In this reference picture, each crack path can be considered fictitious, i.e., the projection over $\mathscr{B}$ of the real crack occurring in the deformed shape, a sort of shadow over a wall. This set of reference configurations includes also $\mathscr{B}$: the uncracked configuration. Assigned boundary conditions, we presume that a requirement of energy minimality selects a reference shape in the set just described (i.e., a potential crack path in essence) and a deformation putting it in one-to-one correspondence with the actual configuration of the body. Minimization of Griffith's energy [28] as a first step to approximate a cracking process has been proposed in 1998 by G. Francfort and J. J. Marigo [23]. The path starts selecting a finite partition of the time interval presuming to go from the state at instant t_k to the one at t_{k+1} by minimizing the energy. In principle, the subsequent step should be to compute a limit as the partition interval goes to zero. This program rests on De Giorgi's notion of minimizing movements [15], motivated by problems of image segmentation.

In the minimum problem suggested in reference [23], deformation and crack paths are the unknowns. A non-trivial difficulty emerges: in three dimensions we cannot control minimizing sequences of surfaces. A way of overcoming the difficulty is to consider as unknown just the deformation taken in the space of those special functions of bounded variations, which are orientation preserving. We give their formal definition in the next section. Here, we just need to know that in $3D$ space they admit a jump set with non-zero $\mathscr{H}^2$ measure. Once minima of such a type are found, the crack path is identified with the deformation jump set [14]. In this sense the two unknown recalled above reduce to one: the deformation. Although such a view is source of nontrivial analytical problems and pertinent results [14], it does not cover cases in which portions of the crack margins are in contact but material bonds across them are broken. To account for these phenomena, we need to recover the original proposal in reference [23], taking once again separately deformations and crack paths. However, the problem of controlling minimizing sequences of surfaces or more irregular crack paths reappears. A way of overcoming it is to select minimizing sequences with bounded curvature because this restriction would avoid surface blow up. This is the idea leading to the representation of cracks in terms of varifolds, which parameterize the set of infinitely many reference configurations differing from each other by the (fictitious, i.e., immaterial) pre-image of possible crack paths, as already described above. Such a representation emerges when we take $x \in \mathscr{B}$ and realize that the question to be considered is not only whether x belongs to a potential crack path but also, in the affirmative case, what is the crack orientation across x, i.e., the tangent (even in approximate sense) to the crack at x, among all planes Π crossing x. Each pair (x, Π) can be viewed as a typical point of a fiber bundle $\mathscr{G}_k(\mathscr{B})$, $k = 1, 2$, with natural projector $\pi : \mathscr{G}_k(\mathscr{B}) \longrightarrow \mathscr{B}$ and typical fiber $\pi^{-1}(x) = \mathscr{G}_{k,3}$ the Grassmanian of 2D-planes or straight lines in $3D$ space, associated with $\mathscr{B}$. A k-varifold over $\mathscr{B}$ is a non-negative Radon measure V over the bundle $\mathscr{G}_k(\mathscr{B})$ [3], [1], [2], [40].

For the sake of simplicity, here we consider just $\mathscr{G}_2(\mathscr{B})$, avoiding $1D$ cracks in a $3D$-body. The generalization to include $1D$ cracks is straightforward. Itself, V has a projection $\pi_{\#}V$ over $\mathscr{B}$, which is a Radon measure over $\mathscr{B}$, indicated for short by μ_V. Specifically, we may consider varifolds supported by $\mathscr{H}^2$-rectifiable subsets of $\mathscr{B}$, i.e., by potential crack paths. We look at those varifolds admitting a certain notion of generalized curvature (its formal definition is in the next section). Consequently, rather than sequences of cracks, we consider sequences of varifolds. The choice allows us to avoid the problem of controlling sequences of surfaces but forces us to include the varifold and its curvature in the energy, leading (at least in the simplest case) to a variant of Griffith's energy augmented by

$$\int_{\mathscr{G}_2(\mathscr{B})} a_1 \|A\|^{\overline{p}}\, dV + a_2 \|\partial V\|$$

with respect to the traditional term just proportional to the surface crack area, namely $\bar{a}\mu_V(\mathscr{B})$ in the formula (1.1). Such a view point has been introduced first in references [25] and [44] (see also [24]).

The discussion in this section justifies a choice of a energy functional like $\mathscr{F}(y, V; \mathscr{B})$, indicated in the formula (1.1), which we analyze in the next sections.

3 Background analytical material

We collect here some notions that are necessary tools to prove our results. They are not restricted to the three-dimensional ambient space that we consider here. For this reason and the sake of completeness, we present them in n-dimensional space, coming back to the specific physical ambient considered here in the next sections.

3.1 A few notations

For $G : \mathbb{R}^n \to \mathbb{R}^N$ a linear map, where $n \geq 2$ and $N \geq 1$, we indicate also by $G = (G_i^j)$, $j = 1, \ldots, N$, $i = 1, \ldots n$, the $(N \times n)$-matrix representing G once we have assigned bases $(e_1, \ldots, e_n)$ and $(\epsilon_1, \ldots, \epsilon_N)$ in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively.

For any ordered multi-indices α in $\{1, \ldots, n\}$ and β in $\{1, \ldots, N\}$ with length $|\alpha| = n - k$ and $|\beta| = k$, we denote by $G_{\overline{\alpha}}^{\beta}$ the $(k \times k)$-submatrix of G with rows $\beta = (\beta_1, \ldots, \beta_k)$ and columns $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_k)$, where $\overline{\alpha}$ is the element which complements α in $\{1, \ldots, n\}$, and $0 \leq k \leq \overline{n} := \min\{n, N\}$. We also denote by

$$M_{\overline{\alpha}}^{\beta}(G) := \det G_{\overline{\alpha}}^{\beta}$$

the determinant of $G_{\overline{\alpha}}^{\beta}$, set $M_0^0(G) := 1$, and indicate by $M(G)$ the fully skew-symmetric third-rank tensor with $\alpha\beta$-th component given by $M_{\overline{\alpha}}^{\beta}(G)$. Also, the Jacobian $|M(G)|$ of the graph map $x \mapsto (Id \bowtie G)(x) := (x, G(x))$ from $\mathbb{R}^n$ into $\mathbb{R}^n \times \mathbb{R}^N$ satisfies

$$|M(G)|^2 := \sum_{|\alpha|+|\beta|=n} M_{\overline{\alpha}}^{\beta}(G)^2. \tag{3.1}$$

If $G : \mathbb{R}^3 \to \tilde{\mathbb{R}}^3$, $M(G)$ is a fully skew-symmetric third-rank tensor with components all the entries of G, $\mathrm{cof}G$, and $\det G$.

3.2 Currents carried by approximately differentiable maps

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with $\mathscr{L}^n$ the pertinent Lebesgue measure. For being $u : \Omega \to \mathbb{R}^N$ an $\mathscr{L}^n$-a.e. approximately differentiable map, we denote by $\nabla u(x) \in \mathbb{R}^{N\times n}$ its approximate gradient at a.e. $x \in \Omega$. The map u has a *Lusin representative* on the subset $\widetilde{\Omega}$ of Lebesgue points pertaining to both u and ∇u. Also, we have $\mathscr{L}^n(\Omega \setminus \widetilde{\Omega}) = 0$. [1]

In this setting, we write $u \in \mathscr{A}^1(\Omega, \mathbb{R}^N)$ if

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[1] By Lusin's theorem, measurable functions f into topological spaces with a countable basis can be approximated by continuous functions on arbitrarily large portions of their domain. Also, if $f : \Omega \to \mathbb{R}^N$ is locally summable in Lebesgue's sense, by the Lebesgue differentiation theorem almost every x in Ω is a Lebesgue point of f, i.e., a point such that for some $\lambda \in \mathbb{R}^N$

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z) - \lambda| \, \mathrm{d}x = 0$$

with $B(x,r)$ a ball of radius r, centered at x, which Lebesgue measure is $|B(x,r)|$. The number $\lambda = f(x)$ is called Lebesgue value of f at x.

- $\nabla u \in L^1(\Omega, \mathbb{M}^{N\times n})$ and
- $M_\alpha^\beta(\nabla u) \in L^1(\Omega)$ for any ordered multi-indices α and β with $|\alpha| + |\beta| = n$.

The *graph* $\mathscr{G}_u$ of a map $u \in \mathscr{A}^1(\Omega, \mathbb{R}^N)$ is defined by

$$\mathscr{G}_u := \Big\{(x,y) \in \Omega \times \mathbb{R}^N \mid x \in \widetilde{\Omega}\,,\ y = \widetilde{u}(x)\Big\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u. It turns out that $\mathscr{G}_u$ is a countably n-dimensional rectifiable set of $\Omega \times \mathbb{R}^N$, with $\mathscr{H}^n(\mathscr{G}_u) < \infty$. The approximate tangent n-plane at $(x, \widetilde{u}(x))$ is generated by the vectors $\mathbf{t}_i(x) = (e_i, \partial_i u(x)) \in \mathbb{R}^n \times \mathbb{R}^N$, for $i = 1, \ldots, n$, where the partial derivatives are the column vectors of the gradient matrix ∇u, and we take $\nabla u(x)$ as the Lebesgue value of ∇u at $x \in \widetilde{\Omega}$.

The unit n-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \cdots \wedge \mathbf{t}_n(x)}{|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \cdots \wedge \mathbf{t}_n(x)|}$$

provides an orientation to the graph $\mathscr{G}_u$. In the previous formula, $\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)$ is the skew-component of $\mathbf{t}_1(x) \otimes \mathbf{t}_2(x)$, so $\xi(x)$ is a fully skew-symmetric contravariant tensor of rank n. For being $\mathscr{D}^n(\Omega \times \mathbb{R}^N)$ *the vector space of compactly supported smooth n-forms in* $\Omega \times \mathbb{R}^N$ (they are maps with values that are fully skew-symmetric covariant tensors of rank n), and $\mathscr{H}^n$ the n-dimensional Hausdorff measure, one defines the current G_u carried by the graph of u through the integration of n-form on $\mathscr{G}_u$, namely

$$\langle G_u, \omega\rangle := \int_{\mathscr{G}_u} \langle \omega, \xi\rangle \, d\mathscr{H}^n\,, \qquad \omega \in \mathscr{D}^n(\Omega \times \mathbb{R}^N),$$

where $\langle , \rangle$ indicates the duality pairing. Consequently, since G_u is a linear functional over $\mathscr{D}^n(\Omega \times \mathbb{R}^N)$, it is an element of the (strong) dual of the space $\mathscr{D}^n(\Omega \times \mathbb{R}^N)$. Write $\mathscr{D}_n(\Omega \times \mathbb{R}^N)$ for such a dual space. Any element of it is properly a current.

By writing U for an open set in $\mathbb{R}^n \times \mathbb{R}^N$, we define *mass* of $T \in \mathscr{D}_k(U)$ the number

$$\mathbf{M}(T) := \sup\{\langle T, \omega\rangle \mid \omega \in \mathscr{D}^k(U)\,,\ \|\omega\| \le 1\}$$

and call a *boundary* of T the $(k-1)$-current ∂T defined by

$$\langle \partial T, \eta\rangle := \langle T, d\eta\rangle, \qquad \eta \in \mathscr{D}^{k-1}(U),$$

where $d\eta$ is the differential of η.

A *weak convergence* $T_h \rightharpoonup T$ of currents in $\mathscr{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h, \omega\rangle = \langle T, \omega\rangle \qquad \forall\, \omega \in \mathscr{D}^k(U)\,.$$

If $T_h \rightharpoonup T$, by lower semicontinuity we also have

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h)\,.$$

With these notions in mind, we say that G_u is an *integer multiplicity* (in short i.m.) *rectifiable current* in $\mathscr{D}_n(\Omega \times \mathbb{R}^N)$, with finite mass $\mathbf{M}(G_u)$ equal to the area $\mathscr{H}^n(\mathscr{G}_u)$ of the u-graph. According to (3.1), since the Jacobian $|M(\nabla u)|$ of the graph map $x \mapsto (Id \bowtie u)(x) := (x, u(x))$ is equal to $|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x) \wedge \cdots \wedge \mathbf{t}_n(x)|$, by the area formula

$$\langle G_u, \omega\rangle = \int_\Omega (Id \bowtie u)^{\#}\omega = \int_\Omega \langle \omega(x, u(x)), M(\nabla u(x))\rangle\, dx$$

for any $\omega \in \mathscr{D}^n(\Omega \times \mathbb{R}^N)$, so that

$$\mathbf{M}(G_u) = \mathscr{H}^n(\mathscr{G}_u) = \int_\Omega |M(\nabla u)|\, dx < \infty\ .$$

If u is of class C^2, the Stokes theorem implies

$$\langle \partial G_u, \eta\rangle = \langle G_u, d\eta\rangle = \int_{\mathscr{G}_u} d\eta = \int_{\partial\mathscr{G}_u} \eta = 0$$

for every $\eta \in \mathscr{D}^{n-1}(\Omega \times \mathbb{R}^N)$, i.e., the null-boundary condition

$$(\partial G_u) \llcorner \Omega \times \mathbb{R}^N = 0\,. \tag{3.2}$$

Such a property (3.2) holds true also for Sobolev maps $u \in W^{1,\overline{n}}(\Omega, \mathbb{R}^N)$, by approximation. However, in general, the boundary ∂G_u does not vanish and may not have finite mass in $\Omega \times \mathbb{R}^N$. On the other hand, if ∂G_u has finite mass, the boundary rectifiability theorem states that ∂G_u is an i.m. rectifiable current in $\mathscr{R}_{n-1}(\Omega \times \mathbb{R}^N)$. An extended treatment of currents is in the two-volume treatise [27].

Remark 3.1 *Consider the case $n = N = 3$, which is under analysis in the next sections. As already mentioned, $M(\nabla u)$ collects as its entries the value* $\det \nabla u$ *and those of all components of ∇u and* $\operatorname{cof} \nabla u$*. The product $\langle \omega(x, u(x)), M(\nabla u(x))\rangle$ (a duality pairing, indeed, also indicated above by a dot) is a sum of the ω components that multiply those of $M(\nabla u)$, which describe line, oriented surface, and volume variations, as it emerges from the list* $(\nabla u, \operatorname{cof} \nabla u, \det \nabla u)$*. Consequently, $\langle \omega, M(\nabla u)\rangle$ is a way of writing in terms of forms the internal (deformational) work associated with u. When we consider a generic (smooth) map $G : \Omega \to \mathbb{R}^{N\times n}$, the product $\langle \omega, M(G)\rangle$ maintains the same physical meaning but now the works associated with volume, oriented area, and line changes are in principle independent from each other unless G is compatible with some u, i.e.,* $\operatorname{curl} G = 0$.

Let $\{u_h\}$ be a sequence in $\mathscr{A}^1(\Omega, \mathbb{R}^N)$, a space of approximately differentiable maps above defined.

Take $N = 1$, i.e., consider u to be real-valued. Suppose also to have in hands sequences $\{u_h\}$ and $\{\nabla u_h\}$ such that $u_h \to u$ strongly in $L^1(\Omega)$ and $\nabla u_h \rightharpoonup v$ weakly in $L^1(\Omega, \mathbb{R}^n)$, where $u \in L^1(\Omega)$ is approximately differentiable almost everywhere (a.e.) and $v \in L^1(\Omega, \mathbb{R}^n)$. In general, *we cannot conclude* that $v = \nabla u$ a.e. in Ω. The question has a positive answer provided that $\{u_h\}$ is a sequence in $W^{1,1}(\Omega)$. Notice that, when $N = 1$, affirming that a function $u \in \mathscr{A}^1(\Omega, \mathbb{R})$ belongs to the Sobolev space $W^{1,1}(\Omega)$ is equivalent to say that it admits the null-boundary condition (3.2).

When $N \geq 2$, assume that $u_h \to u$ strongly in $L^1(\Omega, \mathbb{R}^N)$, with u some a.e. approximately differentiable $L^1(\Omega, \mathbb{R}^N)$ map. Presume also that $M_{\overline{\alpha}}^{\beta}(\nabla u_h) \rightharpoonup v_{\overline{\alpha}}^{\beta}$ weakly in $L^1(\Omega)$, with $v_{\overline{\alpha}}^{\beta} \in L^1(\Omega)$, for every multi-indices α and β, with $|\alpha| + |\beta| = n$. A *sufficient condition* ensuring that $v_{\overline{\alpha}}^{\beta} = M_{\overline{\alpha}}^{\beta}(\nabla u)$ a.e. is again the validity of equation (3.2) for each u_h.

We can weaken such a condition by requiring a control on the G_{u_h} boundaries of the type

$$\sup_h \mathbf{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^N) < \infty \,, \tag{3.3}$$

as stated by Federer-Fleming's closure theorem [18], which refers to sequences of graphs G_{u_h} which have equi-bounded masses, i.e., $\sup_h \mathbf{M}(G_{u_h}) < \infty$, and satisfy the condition (3.3) [27, Vol. I, Sec. 3.3.2].

Theorem 3.1 (Closure theorem). *Let $\{u_h\}$ be a sequence in $\mathscr{A}^1(\Omega, \mathbb{R}^N)$ such that $u_h \to u$ strongly in $L^1(\Omega, \mathbb{R}^N)$ to an a.e. approximately differentiable map $u \in L^1(\Omega, \mathbb{R}^N)$. For any multi-indices α and β with $|\alpha| + |\beta| = n$, assume*

$$M_{\overline{\alpha}}^{\beta}(\nabla u_h) \rightharpoonup v_{\overline{\alpha}}^{\beta} \qquad \text{weakly in } L^1(\Omega),$$

with $v_{\overline{\alpha}}^{\beta} \in L^1(\Omega)$. If the bound (3.3) holds, the inclusion $u \in \mathscr{A}^1(\Omega, \mathbb{R}^N)$ also holds and, for every α and β,

$$v_{\overline{\alpha}}^{\beta}(x) = M_{\overline{\alpha}}^{\beta}(\nabla u(x)) \qquad \mathscr{L}^n\text{-a.e in } \Omega \,. \tag{3.4}$$

Moreover, $G_{u_h} \rightharpoonup G_u$ weakly in $\mathscr{D}_n(\Omega \times \mathbb{R}^N)$ and the inequalities

$$\mathbf{M}(G_u) \leq \liminf_{h \to \infty} \mathbf{M}(G_{u_h}) < \infty$$

and

$$\mathbf{M}((\partial G_u) \llcorner \Omega \times \mathbb{R}^N) \leq \liminf_{h \to \infty} \mathbf{M}((\partial G_{u_h}) \llcorner \Omega \times \mathbb{R}^N) < \infty$$

hold true.

A summable function $u \in L^1(\Omega)$ is said to be *of bounded variation* if its distributional derivative Du is a finite measure in Ω. Also, u is approximately differentiable $\mathscr{L}^n$-a.e. in Ω and its approximate gradient ∇u agrees with the Radon-Nikodym derivative density of Du with respect to $\mathscr{L}^n$. Precisely, the decomposition $Du = \nabla u \, \mathscr{L}^n + D^s u$ holds true, where $D^s u$ is singular with respect to $\mathscr{L}^n$.

The function u jumps; its *jump set* $S(u)$ is a countably $(n-1)$-rectifiable subset of Ω that agrees $\mathscr{H}^{n-1}$-essentially (i.e., to within a set of $\mathscr{H}^{n-1}$ measure) with the complement of Lebesgue's set of u. If, in addition, the singular component $D^s u$ is concentrated on the jump set $S(u)$, we say that u is a *special function of bounded variation*, and write in short $u \in SBV(\Omega)$.

A vector valued function $u : \Omega \to \mathbb{R}^N$ belongs to the class $SBV(\Omega, \mathbb{R}^N)$ if all its components u^j are in $SBV(\Omega)$. In this case, $Du = \nabla u \, \mathscr{L}^n + D^s u$, where the approximate gradient ∇u belongs to $L^1(\Omega, \mathbb{R}^{N\times n})$, and the jump set $S(u)$ is defined component-wise as in the scalar case, so that $D^s u = (u^+ - u^-) \otimes \nu \mathscr{H}^{n-1} \llcorner S(u)$, where ν is an unit normal to $S(u)$ and $u^\pm$ are the one-sided limits at $x \in S(u)$. Therefore, for each Borel set $B \subset \Omega$ we get

$$|Du|(B) = \int_B |\nabla u| \, dx + \int_{B \cap S(u)} |u^+ - u^-| \, d\mathscr{H}^{n-1} \, .$$

Compactness and lower semicontinuity results hold in the space of SBV maps. The treatise [6] offers an accurate analysis of the SBV scenario. Here, we just recall that the compactness theorem in reference [4] relies on a generalization of the following characterization of SBV functions with $\mathscr{H}^{n-1}$-rectifiable jump sets.

According to reference [5], we denote by $\mathscr{T}(\Omega \times \mathbb{R})$ the class of C^1-functions $\varphi(x, y)$ such that $|\varphi| + |D\varphi|$ is bounded and the support of φ is contained in $K \times \mathbb{R}$ for some compact set $K \subset \Omega$.

Proposition 3.1 *Take $u \in BV(\Omega)$. Then, $u \in SBV(\Omega)$, with $\mathscr{H}^{n-1}(S(u)) < \infty$, if and only if for every $i = 1, \dots, n$ there exists a Radon measure μ_i on $\Omega \times \mathbb{R}$ such that*

$$\int_\Omega \Big(\frac{\partial \varphi}{\partial x_i}(x, u(x)) + \frac{\partial \varphi}{\partial y}(x, u(x)) \, \partial_i u(x) \Big) \, dx = \int_{\Omega \times \mathbb{R}} \varphi \, d\mu_i$$

for any $\varphi \in \mathscr{T}(\Omega \times \mathbb{R})$. In this case, we have

$$\mu_i = -(Id \bowtie u^+)_\#(\nu_i \mathscr{H}^{n-1} \llcorner S(u)) + (Id \bowtie u^-)_\#(\nu_i \mathscr{H}^{n-1} \llcorner S(u)) \, .$$

As a consequence, we infer that if a sequence $\{u_h\} \in \mathscr{A}^1(\Omega, \mathbb{R}^N)$ satisfies

$$\sup_h \Big(\|u_h\|_\infty + \int_\Omega |M(\nabla u_h)|^p \, dx \Big) < \infty \,, \qquad p > 1,$$

and the boundary mass bound (3.3), the inclusion $\{u_h\} \subset SBV(\Omega, \mathbb{R}^N)$ and the SBV compactness theorem hold. In fact, by Proposition 3.1 we get

$$\mathscr{H}^{n-1} \llcorner S(u_h) \leq \pi_\# |\partial G_{u_h}|(\Omega) \qquad \forall\, h$$

where $\pi : \Omega \times \mathbb{R}^N \to \Omega$ is a projection onto the first n coordinates, the subscript $\sharp$ indicates that the symbol it decorates is intended as a measure, and $|\cdot|$ denotes total variation, so that $\pi_\# |\partial G_u|(B) = |\partial G_u|(B \times \mathbb{R}^N)$ for each Borel set $B \subset \Omega$.

3.5 Generalized functions of bounded variation

When the bound $\sup_h \|u_h\|_\infty < \infty$ fails, the SBV compactness theorem cannot be applied. This happens, e.g., if $u_h = \nabla y_h$ for some sequence $\{y_h\} \subset W^{1,p}(\Omega)$. When such sequences play a role in the problems analyzed, we find it convenient to call upon *generalized special functions of bounded variation*, the class of which is commonly denoted by $GSBV$.

To define them, first write $SBV_{\text{loc}}(\Omega)$ for functions $v : \Omega \to \mathbb{R}$ that are SBV on every compact set $K \subset \Omega$.

Definition 3.1 *A function $u : \Omega \to \mathbb{R}^N$ belongs to the class $GSBV(\Omega, \mathbb{R}^N)$ if $\phi \circ u \in SBV_{loc}(\Omega)$ for every $\phi \in C^1(\mathbb{R}^N)$ with the support of $\nabla\phi$ to be a compact set.*

The following compactness theorem holds.

Theorem 3.2 *Let $\{u_h\} \subset GSBV(\Omega, \mathbb{R}^N)$ be such that*

$$\sup_h \Big(\int_\Omega \big(|u_h|^p + |\nabla u_h|^p \big) \, dx + \mathscr{H}^{n-1}(S_{u_h}) \Big) < \infty$$

for some real exponent $p > 1$. Then, there exists a function $u \in GSBV(\Omega, \mathbb{R}^N)$ and a (not relabeled) subsequence of $\{u_h\}$ such that $u_h \to u$ in $L^p(\Omega, \mathbb{R}^N)$, $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{N\times n})$, and $\mathscr{H}^{n-1} \llcorner S(u_h)$ weakly converges in Ω to a measure μ greater than $\mathscr{H}^{n-1} \llcorner S(u)$.

4 Crack in $3D$ bodies through curvature varifolds with boundary

We now turn to the physical dimension $n = N = 3$ focusing on the problem that we tackle here: cracks in bodies showing second-neighbor interaction effects. The reference configuration $\mathscr{B}$ is already defined in Section 2.1. The idea above discussed of representing cracks in terms of varifolds requires some precise formalization.

Definition 4.1 *A general* 2*-varifold in* $\mathscr{B}$ *is a non-negative Radon measure on the trivial bundle* $\mathscr{G}_2(\mathscr{B}) := \mathscr{B} \times \mathscr{G}_{2,3}$*, where* $\mathscr{G}_{2,3}$ *is the Grassmanian manifold of* 2*-planes* Π *through the origin in* $\mathbb{R}^3$*.*

If $\mathfrak{C}$ is a 2-rectifiable subset of $\mathscr{B}$, for $\mathscr{H}^2 \llcorner \mathfrak{C}$ a.e. $x \in \mathscr{B}$ there exists the approximate tangent 2-space $T_x\mathfrak{C}$ to $\mathfrak{C}$ at x. We thus denote by $\Pi(x)$ the 3×3 matrix that identifies the orthogonal projection of $\mathbb{R}^3$ onto $T_x\mathfrak{C}$ and define

$$V_{\mathfrak{C},\theta}(\varphi) := \int_{\mathscr{G}_2(\mathscr{B})} \varphi(x, \Pi)\, dV_{\mathfrak{C},\theta}(x, \Pi) := \int_{\mathfrak{C}} \theta(x)\varphi(x, \Pi(x))\, d\mathscr{H}^2(x) \qquad (4.1)$$

for any $\varphi \in C_c^0(\mathscr{G}_2(\mathscr{B}))$, where $\theta \in L^1(\mathfrak{C}, \mathscr{H}^2)$ is a nonnegative density function. If θ is integer valued, then $V = V_{\mathfrak{C},\theta}$ is said to be the *integer rectifiable varifold* associated with $(\mathfrak{C}, \theta, \mathscr{H}^2)$.

The *weight measure* of V is the Radon measure in $\mathscr{B}$ given by $\mu_V := \pi_\# V$, where $\pi : \mathscr{G}_2(\mathscr{B}) \to \mathscr{B}$ is the canonical projection. Then, we have $\mu_V = \theta\, \mathscr{H}^2 \llcorner \mathfrak{C}$ and call

$$\|V\| := V(\mathscr{G}_2(\mathscr{B})) = \mu_V(\mathscr{B}) = \int_{\mathfrak{C}} \theta\, d\mathscr{H}^2$$

a *mass* of V.

Definition 4.2 *An integer rectifiable* 2*-varifold* $V = V_{\mathfrak{C},\theta}$ *is called a* curvature 2-varifold with boundary *if there exist a function* $A \in L^1(\mathscr{G}_2(\mathscr{B}), \mathbb{R}^{3*} \otimes \mathbb{R}^3 \otimes \mathbb{R}^{3*})$*,* $A = (A_j^{\ell i})$*, and a* $\mathbb{R}^3$*-valued measure* ∂V *in* $\mathscr{G}_2(\mathscr{B})$ *with finite mass* $\|\partial V\|$*, such that*

$$\int_{\mathscr{G}_2(\mathscr{B})} (\Pi D_x\varphi + AD_\Pi\varphi + \varphi AI)\, dV(x, \Pi) = -\int_{\mathscr{G}_2(\mathscr{B})} \varphi\, d\partial V(x, \Pi)$$

for every $\varphi \in C_c^\infty(\mathscr{G}_2(\mathscr{B}))$*, where* I *is the 1-contravariant, 1-covariant identity so that* AI *is a vector with component* $(AI)^\ell = A_J^{\ell H}\delta_H^J$*, where, as usual, summation over repeated indices is understood. Also, for some real exponent* $\overline{p} > 1$*, the subclass of curvature* 2*-varifolds with boundary such that* $|A| \in L^{\overline{p}}(\mathscr{G}_2(\mathscr{B}))$ *is indicated by* $CV_2^{\overline{p}}(\mathscr{B})$*.*

Varifolds in $CV_2^{\overline{p}}(\mathscr{B})$ have generalized curvature in $L^{\overline{p}}$ [40]. Therefore, Allard's compactness theorem applies (see [1], [2], but also [3]):

Theorem 4.1 *For $1 < \overline{p} < \infty$, let $\{V^{(h)}\} \subset CV_2^{\overline{p}}(\mathscr{B})$ be a sequence of curvature 2-varifolds $V^{(h)} = V_{\mathfrak{C}_h,\theta_h}$ with boundary. The corresponding curvatures and boundaries are indicated by $A^{(h)}$ and $\partial V^{(h)}$, respectively. Assume that there exists a real constant $c > 0$ such that for every h*

$$\mu_{V^{(h)}}(\mathscr{B}) + \|\partial V^{(h)}\| + \int_{\mathscr{G}_2(\mathscr{B})} |A^{(h)}|^{\overline{p}} \, dV^{(h)} \leq c.$$

Then, there exists a (not relabeled) subsequence of $\{V^{(h)}\}$ and a 2-varifold $V = V_{\mathfrak{C},\theta} \in CV_2^{\overline{p}}(\mathscr{B})$, with curvature A and boundary ∂V, such that

$$V^{(h)} \rightharpoonup V, \quad A^{(h)} \, dV^{(h)} \rightharpoonup A \, dV, \qquad \partial V^{(h)} \rightharpoonup \partial V,$$

in the sense of measures. Moreover, for any convex and lower semicontinuous function $f : \mathbb{R}^{3*} \otimes \mathbb{R}^3 \otimes \mathbb{R}^{3*} \to [0, +\infty]$, we get

$$\int_{\mathscr{G}_2(\mathscr{B})} f(A) \, dV \leq \liminf_{h \to \infty} \int_{\mathscr{G}_2(\mathscr{B})} f(A^{(h)}) \, dV^{(h)}.$$

5 Gradient polyconvexity

According to references [9], [36], and [37], we take a continuous function

$$\hat{W} : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3\times 3} \times \mathbb{R}^3 \to (-\infty, +\infty],$$

and we set $\hat{W} = \hat{W}(G, \Delta_1, \Delta_2)$. We assume also existence of four real exponents p, q, r, s satisfying the inequalities

$$p > 2\,, \quad q \geq \frac{p}{p-1}\,, \quad r > 1\,, \quad s > 0 \tag{5.1}$$

and a positive real constant c such that for every $(G, \Delta_1, \Delta_2) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3\times 3} \times \mathbb{R}^3$ the following estimates holds:

$$\hat{W}(G, \Delta_1, \Delta_2) \geq c\left(|G|^p + |\mathrm{cof} G|^q + (\det G)^r + (\det G)^{-s} + |\Delta_1|^q + |\Delta_2|^r\right)$$

if $\det G > 0$, and $\hat{W}(G, \Delta_1, \Delta_2) = +\infty$ if $\det G \leq 0$.

Definition 5.1 *With $\mathscr{B} \subset \mathbb{R}^3$ the domain already described, consider the functional*

$$J(F; \mathscr{B}) := \int_{\mathscr{B}} \hat{W}\Big(F(x), \nabla[\mathrm{cof} F(x)], \nabla[\det F(x)]\Big) \, dx$$

defined on the class of integrable functions $F : \mathscr{B} \to \mathbb{R}^{3\times 3}$ for which the approximate derivatives $\nabla[\mathrm{cof} F(x)]$, $\nabla[\det F(x)]$ exist for $\mathscr{L}^3$-a.e. $x \in \mathscr{B}$ and *are both integrable functions in $\mathscr{B}$. Then, $J(F;\mathscr{B})$ is called* gradient polyconvex *if the integrand $\hat{W}(G,\cdot,\cdot)$ is convex in $\mathbb{R}^{3\times3\times3}\times\mathbb{R}^3$ for every $G\in\mathbb{R}^{3\times3}$.*

We complement J with a Dirichlet condition. Specifically, we assume that $\Gamma_0\cup\Gamma_1$ is an $\mathscr{H}^2$-measurable partition of the $\mathscr{B}$ boundary such that $\mathscr{H}^2(\Gamma_0)>0$. For some given measurable function $y_0:\Gamma_0\to\mathbb{R}$, we consider the class

$$\hat{\mathscr{A}}_{p,q,r,s}:=\{y\in W^{1,p}(\mathscr{B},\mathbb{R}^3)\mid \mathrm{cof}\nabla y\in W^{1,q}(\mathscr{B},\mathbb{R}^{3\times3})\,,\ \det\nabla y\in W^{1,r}(\mathscr{B})\,,\\ \det\nabla y>0 \text{ a.e. in } \mathscr{B}\,,\ (\det\nabla y)^{-1}\in L^s(\mathscr{B})\,,\ y=y_0 \text{ on } \Gamma_0\}\,,$$

where p,q,r,s satisfy the inequalities (5.1).

The following existence result has been proven in reference [9] (see also [36]).

Theorem 5.1 *Under the previous assumptions, if the class $\hat{\mathscr{A}}_{p,q,r,s}$ is non-empty and $\inf\{J(\nabla y;\mathscr{B})\mid y\in\hat{\mathscr{A}}_{p,q,r,s}\}<\infty$, the functional $y\mapsto J(\nabla y;\mathscr{B})$ attains a minimum in $\hat{\mathscr{A}}_{p,q,r,s}$.*

6 Gradient polyconvex bodies with fractures

We now look at an energy modified by the introduction of a varifold, through which we parametrize possible fractured configurations with respect to the reference one. Specifically, we consider a curvature varifold with boundary: $V\in CV_2^{\overline{p}}(\mathscr{B})$. The choice implies a fracture energy modified with respect to the Griffith one. In fact, the latter is just proportional to the crack area, which implies considering material bonds of spring-like type. The additional presence in our case of the generalized curvature tensor implies, instead, considering beam-like material bonds for which bending effects play a role. In a certain sense, the energy we propose is a regularization of the Griffith one, since we require that the coefficient in front of the curvature tensor term does not vanish.

To have a concrete idea of how these curvature terms in the energy play a role, consider the metamaterial already mentioned and imagine to have homogenized it at continuum scale.

- If it is made by a single lattice of beams connecting first-neighbor nodes, we do not have second-neighbor (non-local) effects in the bulk while curvature contributions due to beam bending appear along the crack margins.
- If the metamaterial is made of two superposed beam-type lattices, the first as above, the second connecting second-neighbor nodes, we have bulk non-local effects and the curvature ones along the crack margins.

Besides this example, in general, by considering the energy proposed here we look for minimizing deformations that are bounded and may admit a jump set contained in the varifold support. We cannot assume the deformation y to be a Sobolev map, as usual in classical elasticity. More generally we require $y \in SBV(\mathscr{B}, \mathbb{R}^3)$.

As above mentioned, the main issue in proving existence is recovering the weak convergence of minors. To achieve it we look at the approximate gradient and exploit Federer-Fleming's closure theorem as in Theorem 3.1. On the other hand, since some properties as the bound $\|\mathrm{cof}\nabla y\|_\infty < \infty$ fail to hold, we assume $\mathrm{cof}\nabla y$ to be in the class $GSBV$, with jump set controlled by the varifold support. In this way we recover the weak continuity of the approximate gradients $\nabla[\mathrm{cof}\nabla y_h]$ along minimizing sequences.

Our existence result below could be generalized to the case in which the crack path is described by a stratified family of varifolds in the sense introduced in references [25] and [44] (see also [24]). In this way, we could assign curvature-type energy to the crack tip, taking possibly into account energy concentrations at tip corners, when the tip is not smooth. Also, we could describe the formation of linear defects in front of the crack tip; in the case of crystalline materials, they are dislocations nucleating in front of the tip (see the proof in reference [24] for the case in which second-neighbor interactions are not included). However, for the sake of simplicity, we restrict ourselves to the choice of a single varifold, avoiding to foresee an additional tip energy and also corner energies.

Consequently, we consider the energy functional

$$\mathscr{F}(y, V; \mathscr{B}) := J(\nabla y; \mathscr{B}) + \mathscr{E}(V; \mathscr{B}) ,$$

where $F \mapsto J(F; \mathscr{B})$ is the functional in Definition 5.1, and

$$\mathscr{E}(V; \mathscr{B}) := \bar{a}\mu_V(\mathscr{B}) + \int_{\mathscr{G}_2(\mathscr{B})} a_1 \|A\|^{\overline{p}} \, dV + a_2 \|\partial V\| ,$$

with $\bar{a}$, a_1, and a_2 positive constants.

The couples deformation-varifold are in the class $\mathscr{A}_{\overline{p},p,q,r,s,K,C}$ defined below.

Definition 6.1 *Let $\overline{p} > 1$ and p, q, r, s be real exponents satisfying* (5.1)*, and let K, C be two positive constants. We say that a couple (y, V) belongs to the class $\mathscr{A}_{\overline{p},p,q,r,s,K,C}$ if the following properties hold:*

(1) $V = V_{\mathfrak{C},\theta}$ is a curvature 2-varifold with boundary in $CV_2^{\overline{p}}(\mathscr{B})$;
(2) $y \in \mathscr{A}^1(\mathscr{B}, \mathbb{R}^3)$, with $\|y\|_\infty \leq K$;
(3) $\pi_\# |\partial G_y| \leq C \cdot \mu_V$;

(4) *the approximate gradient* $\nabla y \in L^p(\mathscr{B},\mathbb{R}^{3\times3})$, $\mathrm{cof}\nabla y \in L^q(\mathscr{B},\mathbb{R}^{3\times3})$, *and* $\det\nabla y \in L^r(\mathscr{B})$*;*
(5) $\det\nabla y > 0$ *a.e. in* $\mathscr{B}$*, and* $(\det\nabla y)^{-1} \in L^s(\mathscr{B})$*;*
(6) $\mathrm{cof}\nabla y \in GSBV(\mathscr{B},\mathbb{R}^{3\times3})$*, with* $|\nabla[\mathrm{cof}\nabla y]| \in L^q(\Omega)$*;*
(7) $\det\nabla y \in GSBV(\mathscr{B},\mathbb{R})$*, with* $|\nabla[\det\nabla y]| \in L^r(\Omega)$*;*
(8) $\mathscr{H}^2 \llcorner S(\mathrm{cof}\nabla y) \le \mu_V$ *and* $\mathscr{H}^2 \llcorner S(\det\nabla y) \le \mu_V$*.*

Assumptions (2) and (3) imply $y \in SBV(\mathscr{B},\mathbb{R}^3)$, with jump set contained in the varifold support, namely $\mathscr{H}^2 \llcorner S(y) \le \mu_V$. Moreover, if $y \in \hat{\mathscr{A}}_{p,q,r,s}$, the graph current G_y has null boundary $(\partial G_y) \llcorner \mathscr{B}\times\mathbb{R}^3 = 0$, see [27, Vol. I, Sec. 3.2.4]. Therefore, taking $V = 0$, i.e., in the absence of fractures, it turns out that the couple $(y,0)$ belongs to the class $\mathscr{A}_{\overline{p},p,q,r,s,K,C}(\mathscr{B})$, provided that $\|y\|_\infty \le K$, independently from the choice of $\overline{p}$ and C.

For reader's convenience, we repeat the theorem stated in the Introduction.

Theorem 6.1 *If the class* $\mathscr{A} := \mathscr{A}_{\overline{p},p,q,r,s,K,C}$ *of admissible couples* (y,V) *is not empty and* $\inf\{\mathscr{F}(y,V;\mathscr{B}) \mid (y,V) \in \mathscr{A}\} < \infty$*, the functional* $(y,V) \mapsto \mathscr{F}(y,V;\mathscr{B})$ *attains a minimum in* $\mathscr{A}$*.*

Proof. Let $\{(y_h, V^{(h)})\}$ be a minimizing sequence in $\mathscr{A}$. By Theorem 4.1, since $\sup_h \mathscr{E}(V^{(h)};\mathscr{B}) < \infty$ we can find a (not relabeled) subsequence of $\{V^{(h)}\}$ and a 2-varifold $V = V_{\mathfrak{C},\theta} \in CV_2^{\overline{p}}(\mathscr{B})$, with curvature A and boundary ∂V, such that $V^{(h)} \rightharpoonup V$, $A^{(h)}\,dV^{(h)} \rightharpoonup A\,dV$, and $\partial V^{(h)} \rightharpoonup \partial V$ in the sense of measures, so that by lower semicontinuity

$$\mathscr{E}(V;\mathscr{B}) \le \liminf_{h\to\infty} \mathscr{E}(V^{(h)};\mathscr{B}) < \infty\,.$$

The domain $\mathscr{B}$ being bounded, in terms of a (not relabeled) subsequence $\{y_h\} \subset \mathscr{A}^1(\mathscr{B},\mathbb{R}^3)$ we find an a.e. approximately differentiable map $y \in L^1(\mathscr{B},\mathbb{R}^3)$ such that $y_h \to y$ strongly in $L^1(\mathscr{B},\mathbb{R}^3)$ and functions $v_{\overline{\alpha}}^{\beta} \in L^1(\mathscr{B})$, for any choice of multi-indices α and β, with $|\alpha|+|\beta| = 3$, such that

$$M_{\overline{\alpha}}^{\beta}(\nabla y_h(x)) \rightharpoonup v_{\overline{\alpha}}^{\beta}(x) \qquad \text{weakly in } L^1(\mathscr{B})\,.$$

Moreover, we get the bound $\sup_h \mathbf{M}(G_{y_h}) < \infty$ on the mass of the i.m. rectifiable currents G_{y_h} in $\mathscr{R}_3(\mathscr{B}\times\mathbb{R}^3)$ carried by the y_h graphs, whereas the inequalities $\pi_\#|\partial G_{y_h}| \le C\cdot\mu_{V^{(h)}}$ imply the bound $\sup_h \mathbf{M}((\partial G_{y_h}) \llcorner \mathscr{B}\times\mathbb{R}^3) < \infty$ on the boundary current masses. Therefore, Theorem 3.1 yields $y \in \mathscr{A}^1(\mathscr{B},\mathbb{R}^3)$ and $v_{\overline{\alpha}}^{\beta}(x) = M_{\overline{\alpha}}^{\beta}(\nabla y(x))$ a.e in $\mathscr{B}$, for every α and β, whereas $G_{y_h} \rightharpoonup G_y$ weakly in $\mathscr{D}_3(\mathscr{B}\times\mathbb{R}^3)$; the current G_y is i.m. rectifiable in $\mathscr{R}_3(\mathscr{B}\times\mathbb{R}^3)$, and the inequality $\pi_\#|\partial G_y| \le C\cdot\mu_V$ holds true.

By taking into account that $\mathscr{H}^2 \llcorner S(y_h) \le \mu_{V^{(h)}}$ and $\sup_h \|y_h\|_\infty \le K$, the compactness theorem in SBV applies to the sequence $\{y_h\} \subset SBV(\mathscr{B},\mathbb{R}^3)$, yielding the convergence $Dy_h \rightharpoonup Dy$ as measures, whereas $\mathscr{H}^2 \llcorner S(y) \leq \mu_V$ and $\|y\|_\infty \leq K$, by lower semicontinuity.

From the uniform bound

$$\sup_h \int_{\mathscr{B}} \Big(|\nabla y_h|^p + |\mathrm{cof}\nabla y_h|^q + |\det \nabla y_h|^r \Big) dx < \infty \,,$$

which follows from the lower bound imposed on the density $\hat{W}$ of the functional $F \mapsto J(F; \mathscr{B})$, we obtain $\nabla y_h \rightharpoonup \nabla y$ in $L^p(\mathscr{B}, \mathbb{R}^{3\times 3})$, $\mathrm{cof}\nabla y_h \rightharpoonup \mathrm{cof}\nabla y$ in $L^q(\mathscr{B}, \mathbb{R}^{3\times 3})$, and $\det \nabla y_h \rightharpoonup \det \nabla y$ in $L^r(\mathscr{B})$.

Also, the inequalities $\mathscr{H}^2 \llcorner S(\mathrm{cof}\nabla y_h) \leq \mu_{V^{(h)}}$ and the lower bound on $\hat{W}$ imply that the sequence $\{\mathrm{cof}\nabla y_h\} \subset GSBV(\mathscr{B}, \mathbb{R}^{3\times 3})$ satisfies the inequality

$$\sup_h \Big(\int_{\mathscr{B}} \Big(|\mathrm{cof}\nabla y_h|^q + |\nabla[\mathrm{cof}\nabla y_h]|^q \Big)\, dx + \mathscr{H}^2(S(\mathrm{cof}\nabla y_h)) \Big) < \infty \,.$$

Therefore, by Theorem 3.2 we infer that

- $\mathrm{cof}\nabla y \in GSBV(\mathscr{B}, \mathbb{R}^{3\times 3})$,
- $\mathrm{cof}\nabla y_h \to \mathrm{cof}\nabla y$ in $L^q(\mathscr{B}, \mathbb{R}^{3\times 3})$,
- $\nabla[\mathrm{cof}\nabla y_h] \rightharpoonup \nabla[\mathrm{cof}\nabla y]$ weakly in $L^q(\mathscr{B}, \mathbb{R}^{3\times 3\times 3})$, and
- $\mathscr{H}^2 \llcorner S(\mathrm{cof}\nabla y) \leq \mu_V$.

Similarly, the inequalities $\mathscr{H}^2 \llcorner S(\det \nabla y_h) \leq \mu_{V^{(h)}}$ and the lower bound on $\hat{W}$ imply that the sequence $\{\det \nabla y_h\} \subset GSBV(\mathscr{B})$ satisfies the inequality

$$\sup_h \Big(\int_{\mathscr{B}} \Big(|\det \nabla y_h|^r + |\nabla[\det \nabla y_h]|^r \Big)\, dx + \mathscr{H}^2(S(\det \nabla y_h)) \Big) < \infty \,,$$

so that Theorem 3.2 entails that

- $\det \nabla y \in GSBV(\mathscr{B})$,
- $\det \nabla y_h \to \det \nabla y$ in $L^r(\mathscr{B})$,
- $\nabla[\det \nabla y_h] \rightharpoonup \nabla[\det \nabla y]$ weakly in $L^r(\mathscr{B}, \mathbb{R}^3)$, and
- $\mathscr{H}^2 \llcorner S(\det \nabla y) \leq \mu_V$.

Arguing as in the proof of Theorem 5.1, reported in reference [36], we obtain $\det \nabla y > 0$ a.e. in $\mathscr{B}$, and $(\det \nabla y)^{-1} \in L^s(\mathscr{B})$, whence we get $(y, V) \in \mathscr{A} = \mathscr{A}_{\overline{p},p,q,r,s,K,C}$.

Finally, on account of the previous convergences, the gradient polyconvexity assumption implies the lower semicontinuity inequality

$$J(\nabla y; \mathscr{B}) \leq \liminf_{h\to\infty} J(\nabla y_h; \mathscr{B}).$$

Then,

$$\mathscr{F}(y, V) \leq \liminf_{h\to\infty} \mathscr{F}(y_h, V^{(h)}),$$

which is the last step in the proof. ■

Remark 6.1 *Differently from what Theorem 5.1 refers to, a Dirichlet-type boundary condition—given by imposing that* $y = y_0$ $\mathscr{H}^2$*-a.e. on* Γ_0 *for some given summable function* $y_0 : \Gamma_0 \to \mathbb{R}$ *and some* $\mathscr{H}^2$*-measurable partition* $\Gamma_0 \cup \Gamma_1$ *of the boundary of* $\mathscr{B}$*—is not preserved by the weak convergence in the BV-sense. In Theorem 6.1, the circumstance could be avoided by imposing, e.g., a so called* confinement condition, *i.e., by requiring the existence of a compact set* $\mathscr{K}$ *well-contained in* $\mathscr{B}$ *such that* $\operatorname{spt}\mu_V \subset \mathscr{K}$. *In fact, by property (3) it turns out that the restriction* $y_{|\mathscr{B}\setminus\mathscr{K}}$ *is a Sobolev map in* $W^{1,p}$*, and the boundary condition holds in the sense of traces. Such a confinement constraint implies that the jump set* $S(y)$ *remains inside* $\mathscr{K}$. *Therefore, from a mechanical point of view, the constraint seems to be reasonable if we impose e.g. a homogeneous Dirichlet-type condition on the whole boundary* $\partial\mathscr{B}$*, allowing for possible cracks inside the body, not touching the boundary.*

6.1 By avoiding self-penetration

The restriction $\det \nabla y(x) > 0$ ensures that the deformation locally preserves orientation. However, we have also to allow possible self-contact between distant portions of the boundary preventing at the same time self-penetration of the matter. To this aim, in 1987 P. Ciarlet and J. Nečas proposed the introduction of an additional constraint, namely

$$\int_{\mathscr{B}'} \det \nabla y(x)\, dx \leq \mathscr{L}^3(\widetilde{y}(\widetilde{\mathscr{B}'}))$$

for any sub-domain $\mathscr{B}'$ of $\mathscr{B}$, where $\widetilde{\mathscr{B}'}$ is the intersection of $\mathscr{B}'$ with the domain $\widetilde{\mathscr{B}}$ of Lebesgue's representative $\widetilde{y}$ of y [12].

We adopt here a weaker constraint, introduced in 1989 by M. Giaquinta, G. Modica, and J. Souček [26] (see also [27, Vol. II, Sec. 2.3.2]). It reads

$$\int_{\mathscr{B}} f(x, y(x)) \det \nabla y(x)\, dx \leq \int_{\mathbb{R}^3} \sup_{x \in \mathscr{B}} f(x, y)\, dy\,,$$

for every compactly supported smooth function $f : \mathscr{B} \times \mathbb{R}^3 \to [0, +\infty)$.

We thus denote by $\widetilde{\mathscr{A}}_{\overline{p},p,q,r,s,K,C}$ the set of couples $(y, V) \in \mathscr{A}_{\overline{p},p,q,r,s,K,C}$ such that the deformation map y satisfies the previous inequality for every f.

Since that constraint is preserved by the weak convergence as currents $G_{y_h} \rightharpoonup G_y$ along minimizing sequences, arguing as in Theorem 6.1 we readily obtain the following existence result.

Corollary 6.2 *Under the previous assumptions, if the class* $\widetilde{\mathscr{A}} := \widetilde{\mathscr{A}_{\overline{p},p,q,r,s,K,C}}$ *of admissible couples* (y, V) *is not empty and* $\inf\{\mathscr{F}(y, V) \mid (y, V) \in \widetilde{\mathscr{A}}\} < \infty$*, the functional* $(y, V) \mapsto \mathscr{F}(y, V)$ *attains a minimum in* $\widetilde{\mathscr{A}}$*.*

7 Additional remarks

Remark 7.1 *Variational views on mechanical problems are at the ground of finite-element-based numerical schemes; paradigmatic is the case of linear elasticity. In terms of applications and with a view towards computations, an open issue in our work is the approximation in terms of a phase field (be it scalar or vector) of a varifold. If we look at the expression of the energy, we could construct an approximated form in terms of a phase field. In this case, however, we should also prove rigorously that such an approximate form converges in some sense (essentially via Gamma-convergence) to the full energy that we consider. Such a proof would give precise consistency to the results of numerical simulations, a matter of a possible future work.*

Remark 7.2 *Although motivated by plasticity, in the end we have considered an elastic-brittle energy. If we include plastic evolution, for rate-independent processes we should consider a dissipation distance, namely a convex and degree-1 positively homogeneous function of the "plastic" variables. It should be involved together with the energy into two inequalities: a stability condition and a dissipation inequality (as indicated in references [49], [50]). In other words, besides minimization of the energy (which comes from the first principle of thermodynamics), we should consider also the second law. However, such an analysis goes beyond our present work.*

Remark 7.3 *The choice of plastic variables mentioned in the previous remark can be variegate. We can choose* $\bar{g}$*, as we have above shown, slip velocity and its gradient [30], the Burgers vector [31] (and possibly its gradient), the Burgers tensor (which may be defined in different ways; compare [16] and [32]),* F^p *and its gradient [21], [20]. Plasticity can be intended (and it is per se) a history-dependent process. To enlighten this aspect, we could consider the cumulative plastic strain [13], [57], paying attention to the statement of flow rules, which could allow to some problems, too (see pertinent analyses in reference [33]). Our analysis, however, does not consider history dependent functionals. Eventually, we have to remind that in numerical simulations that involve crystals, we can look directly to the discrete structure of dislocations, making comparisons with the pertinent continuum modeling [7], [8], or looking in statistical sense to these discrete structures embedded in a material [58].*

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