A zero sum differential game with correlated informations on the initial position. A case with a continuum of initial positions.

C. Jimenez

Univ Brest, Laboratoire de Mathématiques de Bretagne Atlantique, CNRS-UMR 6205,

6, avenue Victor Le Gorgeu, CS 93837, 29238 Brest cedex 3, France.

chloe.jimenez@univ-brest.fr.

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Abstract

We study a two player zero sum game where the initial position z_0 is not communicated to any player. The initial position is a function of a couple (x_0, y_0) where x_0 is communicated to player I while y_0 is communicated to player II. The couple (x_0, y_0) is chosen according a probability measure $dm(x, y) = h(x, y)d\mu(x)d\nu(y)$. We show that the game has a value and, under additional regularity assumptions, that the value is a solution of Hamilton Jacobi Isaacs equation in a dual sense.

Key words. Differential game; symmetric information; Isaacs condition; continuous initial distribution; Wasserstein distance; Functional on measures.

AMS subject classifications. 49N70, 49L25, 91A23, 49Q20.

Introduction

In this paper we study a zero-sum two players differential game with symmetric information on the initial position. The game starts at a fixed time $t_0 \in [0, T]$, an initial position $z_0 \in \mathbb{R}^d$ is chosen (unknown for the players), the actions are controls denoted by $u(\cdot)$ for Player I and $v(\cdot)$ for player II and taking their values on compact subsets U and V of some finite dimensional spaces. Then a trajectory $t \mapsto X_t^{t_0,z_0,u(\cdot),v(\cdot)}$ is generated via the following dynamic system in $\mathbb{R}^d$:

$$z'(t) = f(z(t), u(t), v(t)), \tag{1}$$

($f : \mathbb{R}^d \times U \times V \to \mathbb{R}^d$) together with the initial condition $z(t_0) = z_0$. The payoff is given by $g(z(T))$ with $g : \mathbb{R}^d \to \mathbb{R}^+$. Player I wants to minimize the payoff, while Player II wants to maximize it. Both players observe the actions of her/his opponent. The crucial point is that z_0 is not communicated to any player, $z_0 = \Phi(x_0, y_0)$ depends on the private informations $x_0 \in \mathbb{R}^N$ of player I and $y_0 \in \mathbb{R}^M$ of Player II. Moreover (x_0, y_0) is chosen randomly according to some bounded non-negative measure $m \in \mathcal{M}_b(\mathbb{R}^{N+M})$. In order to get regular upper and lower values, we make the following additional assumptions on m:

$$dm(x,y) = h(x,y)\ d\mu(x)d\nu(x) \text{ with } h \in L^1_{\mu\times\nu}(\mathbb{R}^{N+M}, \mathbb{R}^+), \quad \mu \in \mathcal{P}_1(\mathbb{R}^N),\ \nu \in \mathcal{P}_1(\mathbb{R}^M).$$

Note that the informations of the players are correlated by h. Both know the measure m and the dynamic and the final cost g. They also know the function Φ which links their informations x_0 and y_0 to the initial position $z_0 = \Phi(x_0, y_0)$. Moreover, they will play randomly in order to hide they private information. It is important to notice that none of the players know really what payoff she/he is actually optimizing.

As usually, an Isaac's condition (4) will be required. Proving the existence of the value without Issac's condition might be done in a different setting where the upper and lower values are obtained as limits of sequences of values corresponding to a sequence of games where players have the same delay (see for instance [2] and [14]). The crucial point in these papers is the knowledge they have of their opponent's delay.

The game previously defined is a generalization of the game studied by P. Cardaliaguet in [5], section 6. In his setting, before the game starts, a couple of indices $(i,j) \in \{1,\dots,I\} \times \{1,\dots,J\}$ is chosen on a finite set with some (uncorrelated probability) $p_i \times q_j$. The initial position is then given by some $x_{ij} \in \mathbb{R}^d$ which is not communicated to any player while i is communicated to player I and j to player II. They both know the family of points (x_{ij}) and the probabilities $(p_i)_i$ $(q_j)_j$. This problem is clearly contained in the case considered here by setting $h = 1$, $\mu = \sum_i p_i \delta_i$, $\nu = \sum_j p_i \delta_j$, $\Phi(i,j) = x_{i,j}$. In [5], it is proved that the game has a value which is characterized as the unique viscosity solution of a Hamilton-Jacobi equation in $\mathbb{R}^d$. Due to the stucture of the information, the Hamilton Jacobi equation is satisfied in a dual sense. More precisely, the convex conjugate on the $(p_i)_i$ variable of the lower value is proved to satisfy a subdynamic principle while the concave conjugate on the $(q_j)_j$ variable of the upper value satisfies a superdynamic principle. The seminal work of P. Cardaliaguet has been widely generalized by M. Oliu-Barton in [15]. In particular, his setting allows correlated information on the initial position, the probabilities $(p_i)_i$ and $(q_j)_j$ being replaced by some $(p_{ij})_{i,j}$. In our case, this can be obtained by taking a non-constant function h. For an interesting case with signals that we do not fit in our case, see [19]. In [5] and [15] (also in [19] with a different setting), the values are functions of x_{ij} and p_{ij} which are finite dimensional, in particular, it is quite easy to show that these functions are continuous with respect to the Euclidian norm.

In the case we are considering, we no longer assume that the initial position should be taken in a finite set. The upper and lower value are then functionals depending on $\Phi \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d)$ and $m \in \mathcal{M}_b(\mathbb{R}^{N+M})$ which are infinite dimensional. The space $\mathcal{M}_b(\mathbb{R}^{N+M})$ may naturally be equipped with the weak star topology. Unfortunately, for a general m, the values don't seem to be continuous for this topology. Moreover, the computations of the convex conjugate of the lower value appears quite difficult due to the lack of regularity of the disintegration $x \mapsto \rho^x \in \mathcal{P}(\mathbb{R}^M)$ of m with respect to its first marginal. For these reasons, we restrict ourselves to measures m that are continuous densities with respect to a product measure. The main difficulty is then to define a proper notion of dual viscosity solution in $\mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d)$ which is a non reflexive space. A good notion of such solution should provide a comparison principle, which, in this setting, gives a characterization of the value of the game. This might be the more important contribution of the present paper.

A case with a continuum of initial positions and no correlation (h constant) was considered in [7] (see also [10] and [11] for a different approach), the authors proved the regularity of the values. Then the finitely supported measures being dense in the probability measures, they got the existence of the value by passing to the limit on the result of [5]. They also proved that the value exists in pure strategy providing the probability on the initial position has no atom. We will prove the existence of the value in random and pure strategy using the same type of arguments. In [13], in the same setting as [7], a definition of viscosity solution is introduced, unfortunately, the proof of the comparison principle happens to be false, we will give an erratum.
Note that a case with a continuum of initial positions is also considered in [4] but in the case where players have no information on the initial position. This leads to a completely different notion of viscosity solution.

The paper is organized as follows, in section 1, we give the definitions of the objects, the assumptions and recall some useful results. In section 2, we study the regularity of the upper and the lower value. Then, in section 3, we prove that the value satisfies some dual subdynamic and superdynamic principles. In section 5, we introduce the Hamilton Jacobi Isaac equation as well as the notions of dual viscosity sub and supersolution, we show a comparison principle. In section 6, we characterize the value of our game as the unique dual solution the Hamilton Jacobi Isaac equation. Finally, in section 7, we consider the case of the article [13] as an example and give an erratum.

1 Preliminaries and Assumptions

Throughout the paper, finite dimensional spaces are equipped with the euclidean norm denoted $|x|$ associated with the scalar product denoted by $x.x'$, the closed ball of center x and of radius $r > 0$ is denoted by $B(x, r)$. The Lebesgue measure on $\mathbb{R}^N$ is denoted by $\mathcal{L}^N$. The notation $\mathcal{C}_b(\mathbb{R}^N \times \mathbb{R}^M, \mathbb{R}^d)$ stands for the space of bounded continuous functions from $\mathbb{R}^N \times \mathbb{R}^M$ to $\mathbb{R}^d$ while $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^M)$ is the space of real valued continuous functions which vanish at the infinity. We will take $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ two compact sets and consider $\mathcal{C}(X \times Y, \mathbb{R}^d)$ and $\mathcal{C}(X \times Y, \mathbb{R}^+)$ the spaces of continuous functions on $X \times Y$ with values on $\mathbb{R}^d$ and $\mathbb{R}$.

1.1 Dynamics and payment

We denote by U and V two compact subsets of two finite dimensional spaces. The final time $T > 0$ is fixed, the set $\mathcal{U}(t_0)$ denotes the set of all measurable controls from $[t_0, T]$ to U. Similarly the set of measurable controls from $[t_0, T]$ to V is denoted by $\mathcal{V}(t_0)$.

The function $f : \mathbb{R}^N \times \mathbb{R}^M \times U \times V$ which appears in the dynamics (1) satisfies the following assumptions:

$$\begin{cases} f \text{ is continuous with respect to all variables,} \\ f \text{ is Lipschitz continuous in the first variable uniformly with respect to } (u,v). \end{cases}$$

Then, it is well-known that for any $u(\cdot) \in \mathcal{U}(t_0)$ and $v(\cdot) \in \mathcal{V}(t_0)$, associated with the initial condition $z(t_0) = z_0$ there is a unique absolutely continuous solution to (1) denoted by $t \mapsto X_t^{t_0,z_0,u(\cdot),v(\cdot)}$ which is defined on $[t_0,T]$. Standard estimates show that there exists a constant $C(f) > 0$ such that for all $z, z' \in \mathbb{R}^d$ and all $s, s' \in [t_0, T]$,

$$(2) \qquad \begin{cases} \left|X_s^{t_0,z,u(\cdot),v(\cdot)} - X_{s'}^{t_0,z,u(\cdot),v(\cdot)}\right| & \leq & C(f)\ |s-s'|, \\ \\ \left|X_s^{t_0,z,u(\cdot),v(\cdot)} - X_s^{t_0,z',u(\cdot),v(\cdot)}\right| & \leq & C(f)\,|z-z'| \end{cases}$$

where $C(f)$ is a constant depending only of f. The cost function $g : \mathbb{R}^N \times \mathbb{R}^M \mapsto \mathbb{R}$ satisfies

$$\begin{cases} g \text{ is bounded and Lipschitz continuous,} \\ g \text{ is non-negative.} \end{cases}$$

If the second assumption is not satisfied, just replace g by $(g - \inf_{x\in R^N} g(x))$.
We will denote $C(f,g) = \text{Lip}(g) \times C(f)$ so that for all $z, z' \in \mathbb{R}^d$ and all $s, s' \in [0,T]$,

$$(3) \qquad \begin{cases} \left|g(X_s^{t_0,z,u(\cdot),v(\cdot)}) - g(X_{s'}^{t_0,z,u(\cdot),v(\cdot)})\right| & \leq & C(f,g)\ |s-s'|, \\ \\ \left|g(X_s^{t_0,z,u(\cdot),v(\cdot)}) - g(X_s^{t_0,z',u(\cdot),v(\cdot)})\right| & \leq & C(f,g)\ |z-z'|. \end{cases}$$

1.2 Isaac's condition

We will assume the following Isaac's condition:

(4)

$$\forall (\Phi, p, \mu, \nu) \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d) \times \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^M),$$

$$\inf_{u\in U} \sup_{v\in V} \int_{\mathbb{R}^{N+M}} f(\Phi(x,y),u,v)\cdot p(x,y)\ d\mu(x)d\nu(y) = \sup_{v\in V} \inf_{u\in U} \int_{\mathbb{R}^{N+M}} f(\Phi(x,y),u,v)\cdot p(x,y)\ d\mu(x)d\nu(y)$$

where $\mathcal{P}(\mathbb{R}^N)$ and $\mathcal{P}(\mathbb{R}^M)$ denote the probability measures on $\mathbb{R}^N$ and $\mathbb{R}^M$.

The following equivalence result is similar to Proposition 1 in [13].

Proposition 1. *The conditions below is equivalent to the Isaacs' condition (4): For all $k \in \mathbb{N}$ and $(\xi_l)_{l=1,\dots,k} \in \mathbb{R}^d$, $(z_l)_{l=1,\dots,k} \in \mathbb{R}^d$:*

$$\inf_{u\in U} \sup_{v\in V} \sum_{l=1}^{k} f(z_l,u,v)\cdot \xi_l = \sup_{v\in V} \inf_{u\in U} \sum_{l=1}^{k} f(z_l,u,v)\cdot \xi_l.$$

The proof of this result is very close to the one appearing in [13]. Note that the condition appearing in the proposition implies the Isaac's conditions required in [5] and [15].

1.3 Spaces of Measures

In this subsection, we introduce some notations and recall some useful definitions and results about measures and optimal transport.

Let E be a subset of some $\mathbb{R}^k$. The space of probability measures on E is denoted by $\mathcal{P}(E)$, we also define $\mathcal{P}_1(E)$ the spaces of probability measures with finite moment of order 1:

$$\mathcal{P}_1(E) := \left\{ \mu \in \mathcal{P}(E) : \quad \int_E |x| \, d\mu(x) < +\infty \right\}.$$

Both spaces will be equipped with the Kantorovich norm (see again [18] or [17] for more details) defined for all $\mu_1, \mu_2 \in \mathcal{P}_1(E)$:

$$\|\mu_1 - \mu_2\|_{MK} = \sup_{\varphi \in Lip_1(E)} \left\{ \int_X \varphi(x) d\mu_1(x) - \int_X \varphi(x) d\mu_2(x) \right\}$$

where $Lip_1(E)$ denotes the space of 1-Lipschitz real valued functions. Note that $\mathcal{P}_1(E)$ is closed for the topology induced by $\| \cdot \|_{MK}$. The following results are well known:

Theorem 1.1. *By duality, we have the equality with the 1-Wassertein distance:*

$$\|\mu_1 - \mu_2\|_{MK} = W_1(\mu_1, \mu_2) := \min_{\gamma \in \Pi(\mu_1, \mu_2)} \left\{ \int_{E^2} |x_1 - x_2| \, d\gamma(x_1, x_2) \right\}$$

where $\Pi(\mu_1, \mu_2)$ is the set of probability measures γ on E^2 which has μ_1 as first marginal and μ_2 as second one.
A transport plan $\gamma \in \Pi(\mu, \nu)$ achieving the above minimum is called an optimal plan from μ_1 to μ_2. Denote by $\Pi_0(\mu_1, \mu_2)$ the set of optimal transport plans from μ_1 to μ_2.

Let $\Phi : E \to E$ be a Bored measurable map, we denote by $\Phi \sharp \mu_1$ the push-forward of μ_1 by Φ namely the measure in $\mathcal{P}(E)$ such that

$$\phi \sharp \mu_1(A) = \mu_1 \left(\phi^{-1}(A) \right) \qquad \text{for any Borel set } A \subset E.$$

If $\Phi \sharp \mu_1 = \mu_2$, Φ is called a transport map from μ_1 to μ_2. If μ_1 has no atom, such a transport map always exists (see for instance [16]).

We also denote by $\mathcal{M}_b(E)$ the space of bounded Borelian measures. We recall that, when E is compact, $\mathcal{M}_b(E)$ is topological dual of $\mathcal{C}(E)$ the space of continuous functions on E, moreover a sequence $(\mu_n)_n$ in $\mathcal{M}_b(E)$ converges for the weak star topology to $\mu \in \mathcal{M}_b(E)$ if:

$$\lim_{n \to +\infty} \int_X \varphi(x) \, d(\mu_n - \mu)(x) = 0, \forall \varphi \in \mathcal{C}(E).$$

We recall the following result (see for instance [18], Theorem 7.12 p 212):

Theorem 1.2. *When E is compact, the topology of the Wasserstein distance is the weak star topology of measures.*

When E is compact, we will also use the space of zero total mass measures on E, namely:

$$\mathcal{M}_0(E) := \left\{\eta \in \mathcal{M}_b(E) : \ \eta^+(E) = \eta^-(E)\right\},$$

where $\eta = \eta^+ - \eta^-$ is the Hahn decomposition of η. This space is naturally equipped with the Kantorovich norm, and the corresponding topological dual is $Lip_0(E) := Lip(E)/\mathbb{R}$ the space of Lipschitz functions defined up to a constant (see [12]) . This last space is equipped with its usual norm

$$Lip(\varphi) = \sup_{x,y\in E,\ x\neq y} \frac{\varphi(x) - \varphi(y)}{|x-y|}.$$

When E is not compact, $Lip_0(E)$ is the dual of the following slightly different space (equipped again with the Kantorovich norm):

$$\mathcal{M}_{0,b}(E) := \left\{ \eta \in \mathcal{M}_b(E) : \begin{array}{c} \eta^+(E) = \eta^-(E), \\ \exists \gamma \in \Pi(\eta^+, \eta^-), \ \displaystyle\int_{E^2} |x-y| d\gamma(x,y) < +\infty \end{array} \right\}. \tag{5}$$

We will need some classic definitions of convex analysis on $\mathcal{M}_b(E)$ with E compact (see for instance [1] or [9]). Let $V : \mathcal{M}_b(E) \to \mathbb{R}$ be a measure functional, we call convex conjugate and bi-conjugate the following functionals:

$$V^*(\varphi) := \sup_{\mu\in\mathcal{M}_b(E)} \left\{ \int_E \varphi(x)\ d\mu(x) - V(\mu) \right\},\ \forall \varphi \in \mathcal{C}(E),$$

$$V^{**}(\mu_0) := \sup_{\varphi\in\mathcal{C}(E)} \left\{ \int_E \varphi(x)\ d\mu_0(x) - V^*(\varphi) \right\},\ \forall \varphi \in \mathcal{M}_b(E).$$

We recall the crucial result (see for instance Theorem 9.3.4. in [1]):

Theorem 1.3. *If V is convex l.s.c. for the weak star topology of measures then $V^{**} = V$.*

We will also use the concave conjugate of V:

$$V^\sharp(\varphi) := \inf_{\nu\in\mathcal{M}_b(E)} \left\{ \int_E \varphi(y)\ d\nu(y) - V(\nu) \right\}.$$

Finally we recall the definitions of the convex subdifferential and superdifferential (possibly empty) of V at $\mu_0 \in \mathcal{M}_b(E)$:

$$\begin{aligned} \partial^- V(\mu_0) &:= \operatorname{argmax}_{\varphi\in\mathcal{C}(E)} \left\{ \int_E \varphi(x)\ d\mu_0(x) - V^*(\varphi) \right\} \\ \partial^+ V(\mu_0) &:= \operatorname{argmin}_{\varphi\in\mathcal{C}(E)} \left\{ \int_E \varphi(y)\ d\nu_0(y) - V^*(\varphi) \right\}. \end{aligned}$$

1.4 Strategies

The strategies of the players should involve only their available information. This leads to the following notion of random strategies (comp. [5, 6, 7]). The sets $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ are endowed with the Borel σ-field associated with $L^1_U[t_0,T]$ and $L^1_V[t_0,T]$. As they are symmetric for Player I and II we only give a definition for player I:

Definitions 1. *(i) Let S be the set of triples $(\Omega, \mathcal{F}, P)$ such that $\Omega = [0,1]^m$ for some m, $\mathcal{F}$ is a σ-field contained in the class of Borel sets $B([0,1]^m)$ and P a probability measure on $(\Omega, \mathcal{F})$. For any $t_0 \in [0,T[$, we denote by $A_r(t_0)$ the set of random strategies for Player I starting from t_0.*

A random strategy in $A_r(t_0)$ is a pair $((\Omega, \mathcal{F}, P), \alpha)$ where $\alpha : \mathbb{R}^N \times \Omega \times \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ is a Borel measurable map and there exists a delay $\tau > 0$ such that for all $\omega \in \Omega$: $\alpha(x, \omega, \cdot) : \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ is nonanticipative with delay τ. Namely for any $v_1, v_2 \in \mathcal{V}(t_0)$, for any $t \in [t_0, T[$, if $v_1 = v_2$ a.e. on $[t_0, t]$, then $\alpha(x, \omega, v_1) = \alpha(x, \omega, v_2)$ a.e. on $[t_0, (t+\tau) \wedge T]$.

(ii) A strategy in $A_r(t_0)$ which does not depend on the random variable ω will be called a pure strategy. We denote by $A(t_0)$ the set of pure strategies.

(iii) A strategy in $A(t_0)$ which does not depend on the space variable x will be called constant in space. We denote by $A_c(t_0)$ the set of strategies that are constant in space.

We denote by $B_r(t_0)$, $B(t_0)$ and $B_c(t_0)$ the symmetric sets for player II.

Now we associate to any pair of random strategies a trajectory thanks to the Lemma below (we don't prove it as it is very similar to Lemma 2.4. in ([7]). This enables us to write the game in a normal form.

Lemma 1. *Let $((\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha), \tau_\alpha, \alpha)$ and $((\Omega_\beta, \mathcal{F}_\beta, P_\beta), \tau_\beta, \beta)$ be two strategies in $A_r(t_0)$ and $B_r(t_0)$.*

For any $\omega := (\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$ and for any couple of types $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, there is a unique pair $(u_{\omega,x,y}, v_{\omega,x,y}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, such that

$$\alpha(x, \omega_\alpha, v_{\omega,x,y}) = u_{\omega,x,y} \text{ and } \beta(y, \omega_\beta, u_{\omega,x,y}) = v_{\omega,x,y}\ . \tag{6}$$

Furthermore the map $(\omega, x, y) \mapsto (u_{\omega,x,y}, v_{\omega,x,y}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ is Borel measurable.

Consequently to $(\alpha, \beta) \in A_r(t_0) \times B_r(t_0)$, $\Phi \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d)$, $(x,y) \in \mathbb{R}^{N+M}$ we may associate a trajectory defined by

$$t \in [t_0, T] \mapsto X_t^{t_0, \Phi(x,y), \alpha(x,\omega_\alpha,\cdot), \beta(y,\omega_\beta,\cdot)} := X_t^{t_0, \Phi(x,y), u_{\omega,x,y}, v_{\omega,x,y}}$$

where $u_{\omega,x,y}$ and $v_{\omega,x,y}$ are associated to (α, β) by the Lemma 1.

1.5 Definitions of several Values

Definitions 2. *Fix* $t_0 \in [0,T]$, $(\Phi, h, \mu, \nu) \in L^1_{\mu\times\nu}(\mathbb{R}^{N+M}, \mathbb{R}^d) \times L^1_{\mu\times\nu}(\mathbb{R}^{N+M}) \times \mathcal{P}_1(\mathbb{R}^N) \times \mathcal{P}_1(\mathbb{R}^M)$.

- *We define the upper and lower random values:*

$$\mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu) = \inf_{\alpha \in A_r(t_0)} \sup_{\beta \in B(t_0)} \int_\Omega \int_{\mathbb{R}^{N+M}} g(X_T^{t_0, \Phi(x,y), \alpha(\omega,x,\cdot), \beta(y,\cdot)}) h(x,y)\ d\mu(x) d\nu(y) dP(\omega),$$

$$\mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu) = \sup_{\beta \in B_r(t_0)} \inf_{\alpha \in A(t_0)} \int_\Omega \int_{\mathbb{R}^{N+M}} g(X_T^{t_0, \Phi(x,y), \alpha(x,\cdot), \beta(\omega,y,\cdot)}) h(x,y)\ d\mu(x) d\nu(y) dP(\omega).$$

- *We call upper and lower value in pure strategy the following functionals:*

$$\mathcal{V}^+(t_0, \Phi, h, \mu, \nu) := \inf_{\alpha \in A(t_0)} \sup_{\beta \in B(t_0)} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha(x,\cdot), \beta(y,\cdot)}\right) h(x,y)\ d\mu(x) d\nu(y),$$

$$\mathcal{V}^-(t_0, \Phi, h, \mu, \nu) := \sup_{\beta \in B(t_0)} \inf_{\alpha \in A(t_0)} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha(x,\cdot), \beta(y,\cdot)}\right) h(x,y)\ d\mu(x) d\nu(y).$$

We also introduce the following definitions of values (we will see in section 3 that they coincides with the random values):

Proposition and Definition 1. *Fix* $t_0 \in [0,T]$, $(\Phi, h, \mu, \nu) \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d) \times \mathcal{C}_0(\mathbb{R}^{N+M}) \times \mathcal{P}_1(\mathbb{R}^N) \times \mathcal{P}_1(\mathbb{R}^M)$. *The following upper and lower values are well defined:*

$$C_r^+(t_0, \Phi, h, \mu, \nu) := \inf_{\alpha \in A_r(t_0)} \int_{\mathbb{R}^M} \sup_{\beta \in B_c(t_0)} \left[\int_{\Omega \times \mathbb{R}^N} g(X_T^{t_0, \Phi(x,y), \alpha(\omega,x,\cdot) \beta(\cdot)}) h(x,y)\ d\mu(x) dP(\omega) \right] d\nu(y),$$

$$C_r^-(t_0, \Phi, h, \mu, \nu) := \sup_{\beta \in B_r(t_0)} \int_{\mathbb{R}^N} \inf_{\alpha \in A_c(t_0)} \left[\int_{\Omega \times \mathbb{R}^M} g(X_T^{t_0, \Phi(x,y), \alpha(\cdot) \beta(\omega,y,\cdot)}) h(x,y)\ d\nu(y) dP(\omega) \right]\ d\mu(x).$$

The proof of this result is an immediate consequence of the following lemma:

Lemma 2. *Fix* $t_0 \in [0,T]$, $(\Phi, h, \mu, \nu) \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d) \times \mathcal{C}_0(\mathbb{R}^{N+M}) \times \mathcal{P}_1(\mathbb{R}^N) \times \mathcal{P}_1(\mathbb{R}^M)$. *For any* $\beta \in B_r(t_0)$ *and any* $\alpha \in A_c(t_0)$, *set:*

$$\varphi_{\alpha,\beta}(x) := \int_{\mathbb{R}^M \times \Omega} g\left(X_T^{t_0, \Phi(x,y), \alpha(\cdot), \beta(\omega,y,\cdot)}\right) h(x,y)\ d\nu(y) dP(\omega),$$

For any $\beta \in B_r(t_0)$, *the application* $x \mapsto \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x)$ *is in* $\mathcal{C}_0(\mathbb{R}^N)$.
In the same way, for any $\alpha \in A_r(t_0)$ *the following application is in* $\mathcal{C}_0(\mathbb{R}^M)$:

$$y \mapsto \sup_{\beta \in B_c(t_0)} \left[\int_{\Omega \times \mathbb{R}^N} g(X_T^{t_0, \Phi(x,y), \alpha(\omega,x,\cdot) \beta(\cdot)}) h(x,y)\ d\mu(x) dP(\omega) \right]$$

Proof: We focus on the application $x \mapsto \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x)$.

- We first show the continuity. Let $x_n \to x$ in $\mathbb{R}^N$ and $\varepsilon > 0$. It exists α_n such that:

$$\begin{aligned} & \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x) - \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x_n) \\ \leq\ & \varphi_{\alpha_n,\beta}(x) - \varphi_{\alpha_n,\beta}(x_n) + \varepsilon \\ \leq\ & \int_{\mathbb{R}^M \times \Omega} \left(g\left(X_T^{t_0,\Phi(x,y),\alpha_n(\cdot),\beta(\omega,y,\cdot)}\right) - g\left(X_T^{t_0,\Phi(x_n,y),\alpha_n(\cdot),\beta(\omega,y,\cdot)}\right)\right) h(x,y)\ d\nu(y)dP(\omega) \\ & + \int_{\mathbb{R}^M \times \Omega} g\left(X_T^{t_0,\Phi(x_n,y),\alpha_n(\cdot),\beta(\omega,y,\cdot)}\right)(h(x,y) - h(x_n,y))\ d\nu(y)dP(\omega) + \varepsilon. \end{aligned}$$

  Then, by (3) we have:

$$\left| g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right) - g\left(X_T^{t_0,\Phi(x_n,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right)\right| \leq C(f,g)|\Phi(x_n,y) - \Phi(x,y)|$$

  and using dominated convergence theorem, we get:

$$\limsup_{n \to +\infty} \left(\inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x) - \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x_n) \right) \leq \varepsilon.$$

  In a symmetric way:

$$\liminf_{n \to +\infty} \left(\inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x) - \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x_n) \right) \geq \varepsilon.$$

  As this is true for any $\varepsilon > 0$, we get :

$$\lim_{n \to +\infty} \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x_n) = \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x).$$

- Now we show that $\inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}$ vanishes at the infinity. As h is in $\mathcal{C}_0(\mathbb{R}^N)$, for all $\varepsilon > 0$, it exists R_ε such that with $C_\varepsilon = B(0_{\mathbb{R}^N}, R_\varepsilon) \times B(0_{\mathbb{R}^M}, R_\varepsilon)$, we have:

$$\sup_{x \notin B(0_{\mathbb{R}^N}, R_\varepsilon) \text{ or } y \notin B(0_{\mathbb{R}^M}, R_\varepsilon)} |h(x,y)| = \sup_{(x,y) \notin C_\varepsilon} |h(x,y)| \leq \varepsilon.$$

  So that for any $\beta \in B_r(t_0)$ and $x \notin B(0_{\mathbb{R}^N}, R_\varepsilon)$:

$$\forall \alpha \in A_c(t_0): \quad \varphi_{\alpha,\beta}(x) = \int_{\mathbb{R}^M \times \Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right) h(x,y)\ d\nu dP \leq \|g\|_\infty \varepsilon$$

  and $\inf_{x \notin B(0_{\mathbb{R}^N}, R_\varepsilon)} \inf_{\alpha \in A_c(t_0)} \varphi_{\alpha,\beta}(x) \leq \|g\|_\infty \varepsilon$.

QED

Remark 1. *1) When Φ and h are regular, we have the following inequalities:*

$$C_r^-(t_0, \Phi, h, \mu, \nu) \leq \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu) \leq \mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu) \leq C_r^+(t_0, \Phi, h, \mu, \nu).$$

If μ is finitely supported, the first inequality is an equality; in a symmetric way, if ν is finitely supported, the last inequality is an equality.

2) Both C_r^+ and C_r^- are meaningful. For instance, as Player I knows the exact $x_0 \in \mathbb{R}^N$ initially chosen, why should he minimize the average payment

$$\int_{\Omega\times\mathbb{R}^{N+M}} g(X_T^{t_0,\Phi(x,y),\alpha(x,\cdot)\beta(\omega,y,\cdot)})h(x,y)\ d\mu(x)d\nu(y)dP(\omega)?$$

It seems more reasonable for him to minimize

$$\inf_{\alpha\in A_c(t_0)} \int_{\Omega\times\mathbb{R}^{M}} g(X_T^{t_0,\Phi(x_0,y),\alpha(\cdot)\beta(y,\cdot)})h(x_0,y)\ d\nu(y)dP(\omega).$$

3) When Φ and h are in some $L^p_{\mu\times\nu}$, the map $x \mapsto \inf_{\alpha\in A_c(t_0)} \varphi_{\alpha,\beta}(x)$ may be not measurable so that C_r^- may not be well defined. The same holds for C_r^+.

2 Regularity of the values

In this subsection, we study the regularity of $\mathcal{V}_r^\pm$, $C_r^\pm$ and $\mathcal{V}^\pm$.

The following result is classic (see for instance [3]):

Lemma 3. *For any $(\Phi,h,\nu,\mu) \in \times L^1_{\mu\times\nu}(\mathbb{R}^{N+M},\mathbb{R}^d)\times L^1_{\mu\times\nu}(\mathbb{R}^{N+M})\times\mathcal{P}_1(\mathbb{R}^M)\times\mathcal{P}_1(\mathbb{R}^N)$ and for any $t,s\in[t_0,T]$,*

$$|\mathcal{V}_r^\pm(t,\Phi,h,\mu,\nu)-\mathcal{V}_r^\pm(s,\Phi,h,\mu,\nu)| \le C(f,g)\|h\|_{L^1_{\mu\times\nu}}|t-s|.$$

The same property holds for $\mathcal{V}^\pm$ and for $C_r^\pm$ if $(\Phi,h)\in[0,T]\times\mathcal{C}_b(\mathbb{R}^{N+M},\mathbb{R}^d)\times\mathcal{C}_0(\mathbb{R}^{N+M})$.

Lemma 4. *(i) Let (t_0,Φ,μ,ν) be an element of $[t_0,T]\times L^1_{\mu\times\nu}(\mathbb{R}^{N+M},\mathbb{R}^d)\times\mathcal{P}_1(\mathbb{R}^N)\times\mathcal{P}_1(\mathbb{R}^M)$. For any h_1,$h_2\in L^1_{\mu\times\nu}(\mathbb{R}^{N+M})$, we have:*

$$|\mathcal{V}_r^\pm(t_0,\Phi,h_1,\mu,\nu)-\mathcal{V}_r^\pm(t_0,\Phi,h_2,\mu,\nu)| \le \|g\|_\infty\|h_1-h_2\|_{L^1_{\mu\times\nu}},$$

(ii) Let (t_0,μ,ν) be an element of $[t_0,T]\times\mathcal{P}_1(\mathbb{R}^N)\times\mathcal{P}_1(\mathbb{R}^M)$. Let $h\in L^q_{\mu\times\nu}(\mathbb{R}^{N+M})$ and Φ_1,$\Phi_2\in L^p_{\mu\times\nu}(\mathbb{R}^{N+M},\mathbb{R}^d)$, with $p\in[1,+\infty]$ and $1/p+1/q=1$, we have:

$$|\mathcal{V}_r^\pm(t_0,\Phi_1,h,\mu,\nu)-\mathcal{V}_r^\pm(t_0,\Phi_2,h,\mu,\nu)| \le C(f,g)\|h\|_{L^q_{\mu\times\nu}}\|\Phi_1-\Phi_2\|_{L^p_{\mu\times\nu}},$$

in particular, $\mathcal{V}_r^\pm$ is Lipschitz in Φ for the norm $\|\cdot\|_{L^p_{\mu\times\nu}}$ with Lipschitz constant $C(f,g)\|h\|_{L^q_{\mu\times\nu}}$.

The same properties hold for $\mathcal{V}^\pm$ and for $C_r^\pm$ if $(\Phi,h)\in[0,T]\times\mathcal{C}_b(\mathbb{R}^{N+M},\mathbb{R}^d)\times\mathcal{C}_0(\mathbb{R}^{N+M})$.

Proof: We make only the proofs for $\mathcal{V}_r^-$, the proofs for the other values being similar (note that here the randomness of the strategies does not play any role).

(i) Take $\varepsilon > 0$ and $((\Omega_1, \mathcal{F}_1, P_1), \beta_1) \in B_r(t_0)$ such that:

$$\mathcal{V}_r^-(t_0, \Phi, h_1, \mu, \nu) \leq \varepsilon + \inf_{\alpha \in A(t_0)} \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h_1(x,y) d\mu d\nu dP_1(\omega).$$

Then choose $\alpha_2 \in A(t_0)$ such that:

$$\begin{aligned} \varepsilon \; &+ \; \inf_{\alpha \in A(t_0)} \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h_2(x,y) d\mu d\nu dP_1(\omega) \\ &\geq \; \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h_2(x,y) d\mu d\nu dP_1(\omega). \end{aligned}$$

Then we have:

$$\begin{aligned} & \mathcal{V}_r^-(t_0, \Phi, h_1, \mu, \nu) - \mathcal{V}_r^-(t_0, \Phi, h_2, \mu, \nu) \\ \leq \; & 2\varepsilon \; + \; \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h_1(x,y) d\mu d\nu dP_1(\omega) \\ & - \; \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h_2(x,y) d\mu d\nu dP_1(\omega) \\ = \; & 2\varepsilon \; + \; \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; (h_1 - h_2)(x,y) d\mu d\nu dP_1 \\ \leq \; & 2\varepsilon \; + \; \|g\|_\infty \|h_1 - h_2\|_{L^1_{\mu\times\nu}} \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} \; d\mu(x) d\nu(y) dP_1(\omega) \end{aligned}$$

so that (i) holds (recall that P_1, μ and ν are probability measures).
In a similar way we get the proof for C_r^-.

(ii) Repeating the same arguments as above, it exists $((\Omega_1, \mathcal{F}_1, P_1), \beta_1) \in B_r(t_0)$ and $\alpha_2 \in A(t_0)$ such that:

$$\begin{aligned} & \mathcal{V}_r^-(t_0, \Phi_1, h, \mu, \nu) - \mathcal{V}_r^-(t_0, \Phi_2, h, \mu, \nu) \\ \leq \; & 2\varepsilon \; + \; \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi_1(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h(x,y) d\mu d\nu dP_1(\omega) \\ & - \; \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0, \Phi_2(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \; h(x,y) d\mu d\nu dP_1(\omega). \end{aligned}$$

Then by (3):

$$\begin{aligned} & \mathcal{V}_r^-(t_0, \Phi_1, h, \mu, \nu) - \mathcal{V}_r^-(t_0, \Phi_2, h, \mu, \nu) \\ \leq \; & 2\varepsilon + \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} \left| g\left(X_T^{t_0, \Phi_1(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) - g\left(X_T^{t_0, \Phi_2(x,y), \alpha_2(x,\cdot), \beta_1(\omega,y,\cdot)}\right) \right| \; h(x,y) d\mu d\nu dP_1 \\ \leq \; & 2\varepsilon + C(f,g) \int_{\Omega_1} \int_{\mathbb{R}^{N+M}} |\Phi_1(x,y) - \Phi_2(x,y)| \; h(x,y) d\mu d\nu dP_1 \\ \leq \; & 2\varepsilon + C(f,g) \|h\|_{L^q_{\mu\times\nu}} \|\Phi_1 - \Phi_2\|_{L^p_{\mu\times\nu}}. \end{aligned}$$

QED

For any fixed $\Phi\in\mathcal{C}_b(\mathbb{R}^{N+M},\mathbb{R}^d)$, $h\in\mathcal{C}_0(\mathbb{R}^{N+M})$, $\nu\in\mathcal{P}_1(\mathbb{R}^M)$ the study of the regularity in μ and ν of $\mathcal{V}_r^{\pm}$ make appear the following objects from optimal transport, similar to the usual Monge Kantorovich norm and defined for all $\mu_0,\mu_1\in\mathcal{M}_b^+(\mathbb{R}^N)$ with $\mu_0(\mathbb{R}^N)=\mu_1(\mathbb{R}^N)$:

$$
\begin{aligned}
W_{\Phi,h,\nu}(\mu_0,\mu_1) &:= \inf_{\gamma\in\Pi(\mu_0,\mu_1)}\int_{\mathbb{R}^{2N}} c_{\Phi,h,\nu}(x_0,x_1)\ d\gamma(x_0,x_1),\\
\text{with } c_{\Phi,h,\nu}(x_0,x_1) &:= \int_{\mathbb{R}^M} |\Phi(x_0,y)-\Phi(x_1,y)|+|h(x_0,y)-h(x_1,y)|\ d\nu(y).
\end{aligned}
\tag{7}
$$

We define $W_{\phi,h,\mu}$ with $\mu\in\mathcal{P}_1(\mathbb{R}^N)$ in the same way:

$$
W_{\Phi,h,\mu}(\nu_0,\nu_1) := \inf_{\gamma\in\Pi(\nu_0,\nu_1)}\int_{\mathbb{R}^{N+2M}} |\Phi(x,y_0)-\Phi(x,y_1)|+|h(x,y_0)-h(x,y_1)|\ d\mu(x)d\gamma(y_0,y_1)
$$

for all $\nu_0,\nu_1\in\mathcal{M}_b^+(\mathbb{R}^M)$ such that $\nu_0(\mathbb{R}^M)=\nu_1(\mathbb{R}^M)$.

Then we can state the regularity property in μ and ν of $\mathcal{V}_r^{\pm}$:

Lemma 5. *(i) Let (t_0,ν) be in $[t_0,T]\times\mathcal{P}_1(\mathbb{R}^M)$ and $\Phi\in\mathcal{C}_b(\mathbb{R}^{N\times M},\mathbb{R}^d)$, $h\in C_0(\mathbb{R}^{N\times M},\mathbb{R}^+)$. For any μ_0, μ_1 in $\mathcal{P}_1(\mathbb{R}^N)$, we have:*

$$
|\mathcal{V}_r^{\pm}(t_0,\Phi,h,\mu_0,\nu)-\mathcal{V}_r^{\pm}(t_0,\Phi,h,\mu_1,\nu)|\le (C(f,g)\|h\|_\infty+\|g\|_\infty)W_{\Phi,h,\nu}(\mu_0,\mu_1),
$$
$$
|C_r^{-}(t_0,\Phi,h,\mu_0,\nu)-C_r^{-}(t_0,\Phi,h,\mu_1,\nu)|\le (C(f,g)\|h\|_\infty+\|g\|_\infty)W_{\Phi,h,\nu}(\mu_0,\mu_1),
$$

(ii) Let (t_0,μ) be in $[t_0,T]\times\mathcal{P}_1(\mathbb{R}^N)$ and $\Phi\in\mathcal{C}_b(\mathbb{R}^{N\times M},\mathbb{R}^d)$, $h\in C_0(\mathbb{R}^{N\times M},\mathbb{R}^+)$. For any ν_0, ν_1 in $\mathcal{P}_1(\mathbb{R}^M)$, we have:

$$
|\mathcal{V}_r^{\pm}(t_0,\Phi,h,\mu,\nu_0)-\mathcal{V}_r^{\pm}(t_0,\Phi,h,\mu,\nu_0)|\le (C(f,g)\|h\|_\infty+\|g\|_\infty)W_{\Phi,h,\mu}(\nu_0,\nu_1),
$$
$$
|C_r^{+}(t_0,\Phi,h,\mu,\nu_0)-C_r^{+}(t_0,\Phi,h,\mu,\nu_0)|\le (C(f,g)\|h\|_\infty+\|g\|_\infty)W_{\Phi,h,\mu}(\nu_0,\nu_1).
$$

In the proof below, the randomness of the strategies is crucial, the proof will not work if we replace $\mathcal{V}_r^+$ by $\mathcal{V}^+$.

Proof: We will only prove the result (i) for $\mathcal{V}_r^+$. The other statements being similar. We mimic the proof contained in [13]. Fix $\varepsilon>0$ and take $((\Omega,\mathcal{F},P),\alpha_0)$ an ε-optimal random strategy for $\mathcal{V}_r^+(t_0,\Phi,h,\mu_0,\nu)$ namely

$$
\sup_{\beta\in B(t_0)}\int_\Omega\int_{\mathbb{R}^{N\times M}} g(X_T^{t_0,\Phi(x_0,y),\alpha(x_0,\omega,\cdot)\beta(y,\cdot)})\, h(x_0,y)d\mu_0(x_0)d\nu(y)dP(\omega)\le \mathcal{V}_r^+(t_0,\Phi,h,\mu_0,\nu)+\varepsilon.
\tag{8}
$$

Take γ be any element of $\Pi(\mu_0,\mu_1)$. Then we disintegrate the measure γ with respect to μ_1 as follows

$$
d\gamma(x_0,x_1)=d\gamma_{x_1}(x_0)d\mu_1(x_1).
$$

It has been proven in [7] there exists a measurable map $\xi:\ (x_1,\omega')\in\mathbb{R}^N\times[0,1]^N\mapsto\xi(y,\omega')\in\mathbb{R}^N$ such that

$$
\xi(x_1,\cdot)\sharp\mathcal{L}^N\lfloor[0,1]^N=\gamma_{x_1}\text{ for }\mu_1\text{-almost all }x_1.
$$

This enables us to define the following random strategy for the first player

$$\alpha_1 : (x_1, \omega, \omega', v) \in \mathbb{R}^N \times \Omega \times [0,1]^N \times \mathcal{V}(t_0) \mapsto \alpha_0(\xi(x_1, \omega'), \omega, v) \in \mathcal{U}(t_0).$$

Then for any $\beta \in B(t_0)$ we have

$$\int_{\Omega\times[0,1]^N}\int_{\mathbb{R}^{N+M}} g(X_T^{t_0,\Phi(x_1,y),\alpha_1(x_1,\omega,\omega',\cdot)\beta(y,\cdot)})\ h(x_1,y)d\mu_1(y)d\nu(y)dP(\omega)d\omega'$$

$$= \int_\Omega\int_{\mathbb{R}^{N+M}} \left(\int_{[0,1]^N} g(X_T^{t_0,\Phi(x_1,y),\alpha_0(\xi(x_1,\omega'),\omega,\cdot)\beta(y,\cdot)})\ h(x_1,y)d\omega'\right)\ d\mu_1(y)d\nu(y)dP(\omega)$$

$$= \int_{\Omega\times\mathbb{R}^{2N}\times\mathbb{R}^M} g(X_T^{t_0,\Phi(x_1,y),\alpha_0(x_0,\omega,\cdot)\beta(y,\cdot)})h(x_1,y)\ d\nu(y)d\gamma_{x_1}(x_0)d\mu_1(x_1)dP(\omega)$$

(Using Fubini Theorem and the definition of α_1)

$$= \int_{\Omega\times\mathbb{R}^{2N}\times\mathbb{R}^M} g(X_T^{t_0,\Phi(x_1,y),\alpha_0(x_0,\omega,\cdot)\beta(y,\cdot)})h(x_1,y)\ d\nu(y)d\gamma(x_0,x_1)dP(\omega)$$

$$\le \int_{\Omega\times\mathbb{R}^{2N}\times\mathbb{R}^M} g(X_T^{t_0,\Phi(x_0,y),\alpha_0(x_0,\omega,\cdot)\beta(\cdot)})\ h(x_1,y)\ d\nu(y)dP(\omega)d\gamma(x_0,x_1)$$
$$+C(f,g)\|h\|_\infty \int_{\mathbb{R}^{2N}} |\Phi(x_0,y)-\Phi(x_1,y)|\ d\nu(y)d\gamma(x_0,x_1)$$

(Using inequality (3))

$$\le \int_{\Omega\times\mathbb{R}^{2N+M}} g(X_T^{t_0,\Phi(x_0,y),\alpha_0(x_0,\omega,\cdot)\beta(\cdot)})\ h(x_0,y)\ d\nu(y)dP(\omega)d\gamma(x_0,x_1)$$
$$+\|g\|_\infty \int_{\mathbb{R}^{2N}} |h(x_0,y)-h(x_1,y)|\ d\nu(y)d\gamma(x_0,x_1)$$
$$+C(f,g)\|h\|_\infty \int_{\mathbb{R}^{2N}} |\Phi(x_0,y)-\Phi(x_1,y)|\ d\nu(y)d\gamma(x_0,x_1).$$

Then taking the infimum in $\gamma \in \Pi(\mu_0,\mu_1)$ leads:

$$\int_{\Omega\times[0,1]^N}\int_{\mathbb{R}^{N+M}} g(X_T^{t_0,\Phi(x_1,y),\alpha_1(x_1,\omega,\omega',\cdot)\beta(y,\cdot)})\ h(x_1,y)d\mu_1(y)d\nu(y)dP(\omega)d\omega'$$
$$\le \int_{\Omega\times\mathbb{R}^{2N}\times\mathbb{R}^M} g(X_T^{t_0,\Phi(x_0,y),\alpha_0(x_0,\omega,\cdot)\beta(\cdot)})\ h(x_0,y)\ d\nu(y)dP(\omega)d\mu_0(x_0)$$
$$+(C(f,g)\|h\|_\infty + \|g\|_\infty)W_{\Phi,h,\nu}(\mu_0,\mu_1).$$

We conclude by taking the supremum in $\beta \in B(t_0)$ and recalling (8) :

$$\begin{aligned}
&\mathcal{V}_r^+(t_0,\Phi,h,\mu_1,\nu)\\
\le\ & \sup_\beta \int_{\Omega\times[0,1]^N}\int_{\mathbb{R}^{N+M}} g(X_T^{t_0,\Phi(x_1,y),\alpha_1(x_1,\omega,\omega',\cdot)\beta(y,\cdot)})\ h(x_1,y)d\mu_1(y)d\nu(y)dP(\omega)d\omega'\\
\le\ & \sup_\beta \int_{\Omega\times\mathbb{R}^{2N}\times\mathbb{R}^M} g(X_T^{t_0,\Phi(x_0,y),\alpha_0(x_0,\omega,\cdot)\beta(y,\cdot)})\ h(x_0,y)\ d\nu(y)dP(\omega)d\mu_0(x_0)\\
&\quad +(C(f,g)\|h\|_\infty + \|g\|_\infty)W_{\Phi,h,\nu}(\mu_0,\mu_1)\\
\le\ & \mathcal{V}_r^+(t_0,\Phi,h,\mu_0,\nu) + \varepsilon + (C(f,g)\|h\|_\infty + \|g\|_\infty)W_{\Phi,h,\nu}(\mu_0,\mu_1).
\end{aligned}$$

Sending ε to zero we finally have:

$$\mathcal{V}_r^+(t_0,\Phi,h,\mu_1,\nu) \le \mathcal{V}_r^+(t_0,\Phi,h,\mu_0,\nu) + (C(f,g)\|h\|_\infty + \|g\|_\infty) W_{\Phi,h,\nu}(\mu_0,\mu_1).$$

Interchanging μ_0 and μ_1, the proof is complete.

QED

Lemma 6. *Let Φ be in $\mathcal{C}_b(\mathbb{R}^{N+M},\mathbb{R}^d)$, $h \in \mathcal{C}_0(\mathbb{R}^{N+M},\mathbb{R}^+)$, and $\nu \in \mathcal{P}_1(\mathbb{R}^M)$. The following properties hold:*

(i) $W_{\Phi,h,\nu}(\mu_0,\mu_1) = W_{\Phi,h,\nu}(\mu_1,\mu_0)$, *for all* μ_0, μ_1 *in* $\mathcal{M}_b^+(\mathbb{R}^N)$ *with* $\mu_0(\mathbb{R}^N) = \mu_1(\mathbb{R}^N)$,

(ii) $W_{\Phi,h,\nu}(\mu_0,\mu_1) \ge 0$ *and* $W_{\Phi,h,\nu}(\mu_0,\mu_0) = 0$, *for all* μ_0, μ_1 *in* $\mathcal{M}_b^+(\mathbb{R}^N)$ *with* $\mu_0(\mathbb{R}^N) = \mu_1(\mathbb{R}^N)$,

(iii) $\forall \mu_0,\ \mu_1,\ \mu_2 \in \mathcal{M}_b^+(\mathbb{R}^N)$ *with* $\mu_0(\mathbb{R}^N) = \mu_1(\mathbb{R}^N) = \mu_2(\mathbb{R}^N)$, *the triangle inequality is satisfied*

$$W_{\Phi,h,\nu}(\mu_0,\mu_2) \le W_{\Phi,h,\nu}(\mu_0,\mu_1) + W_{\Phi,h,\nu}(\mu_1,\mu_2),$$

(iv) let μ_0, μ_1 *be in* $\mathcal{M}_b^+(\mathbb{R}^N)$ *with* $\mu_0(\mathbb{R}^N) = \mu_1(\mathbb{R}^N)$, *assume* $W_{\Phi,h,\mu}(\mu_0,\mu_1) < +\infty$ *then*

$$\begin{aligned} & W_{\Phi,h,\nu}(\mu_0,\mu_1) \\ = & \sup_{\varphi\in\mathcal{C}_b(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \varphi d\mu_1 - \int_{\mathbb{R}^N} \varphi d\mu_0 :\ \varphi(x_1) - \varphi(x_0) \le c_{\Phi,h,\nu}(x_0,x_1)\ \forall x_0, x_1 \in \mathbb{R}^N \right\} \end{aligned}$$

where $c_{\Phi,h,\nu}$ *is the cost defined in (7).*

As a consequence it exists $N_{\Phi,h,\nu}$ *a semi-norm on* $\mathcal{M}_{0,b}(\mathbb{R}^N)$ *(see (5)) such that:*

$$N_{\phi,h,\nu}(\eta) = W_{\Phi,h,\nu}(\eta^+,\eta^-).$$

(v) let Φ_1, Φ_2 *be in* $\mathcal{C}_b(\mathbb{R}^{N+M},\mathbb{R}^d)$, $h_1, h_2 \in \mathcal{C}_0(\mathbb{R}^{N+M})$, *assuming all the quantities are finite, we have:*

$$W_{\Phi_1,h,\nu}(\mu_0,\mu_1) - W_{\Phi_2,h,\nu}(\mu_0,\mu_1) \le 2\|\Phi_1 - \Phi_2\|_\infty,$$
$$W_{\Phi,h_1,\nu}(\mu_0,\mu_1) - W_{\Phi,h_2,\nu}(\mu_0,\mu_1) \le 2\|h_1 - h_2\|_\infty.$$

Of course same properties can be shown for $W_{\Phi,h,\mu}$.

Proof: We only show (iii), (iv), the other properties being straightforward.
Proof of (iii): We assume that $W_{\Phi,h,\nu}(\mu_0,\mu_1) < +\infty$ and $W_{\Phi,h,\nu}(\mu_1,\mu_2) < +\infty$. We use a classic argument (see for instance [17]). Fix $\varepsilon > 0$ and let $\gamma_{0,1} \in \Pi(\mu_0,\mu_1)$ and $\gamma_{1,2} \in \Pi(\mu_1,\mu_2)$ be two ε-optimal transport plans that is:

$$\varepsilon + W_{\Phi,h,\nu}(\mu_0,\mu_1) \ge \int_{\mathbb{R}^{2N}} c_{\Phi,h,\nu}(x_0,x_1)\ d\gamma_{0,1}(x_0,x_1),$$

$$\varepsilon + W_{\Phi,h,\nu}(\mu_1,\mu_2) \ge \int_{\mathbb{R}^{2N}} c_{\Phi,h,\nu}(x_1,x_2)\ d\gamma_{1,2}(x_1,x_2).$$

We disintegrate $\gamma_{0,1}$ and $\gamma_{1,2}$ as follows:

$$d\gamma_{0,1}(x_0,x_1) := d\gamma_{0,1}^{x_1}(x_0)d\mu_1(x_1), \quad d\gamma_{1,2}(x_1,x_2) := d\gamma_{1,2}^{x_1}(x_2)d\mu_1(x_1).$$

We build and admissible transport plan $\gamma_{0,1,2} \in \Pi(\mu_0,\mu_2)$ for $W_{\Phi,h,\nu}(\mu_0,\mu_2)$ by setting:

$$d\gamma_{0,1,2}(x_0,x_2) := \int_{\mathbb{R}^N} d\gamma_{0,1}^{x_1}(x_0)d\gamma_{1,2}^{x_1}(x_2)d\mu_1(x_1).$$

Note that $c_{\Phi,h,\nu}$ satisfies a triangular inequality that is:

$$c_{\Phi,h,\nu}(x_0,x_2) \le c_{\Phi,h,\nu}(x_0,x_1) + c_{\Phi,h,\nu}(x_1,x_2).$$

Frome this inequality, we get:

$$\begin{aligned}
W_{\Phi,h,\nu}(\mu_0,\mu_2) &\le \int_{\mathbb{R}^{2N}} c_{\Phi,h,\nu}(x_0,x_2)\ d\gamma_{0,1,2}(x_0,x_2) \le \int_{\mathbb{R}^{3N}} c_{\Phi,h,\nu}(x_0,x_2)\ d\gamma_{0,1}^{x_1}(x_0)d\gamma_{1,2}^{x_1}(x_2)d\mu_1(x_1)\\
&\le \int_{\mathbb{R}^{3N}} c_{\Phi,h,\nu}(x_0,x_1) + c_{\Phi,h,\nu}(x_1,x_2)\ d\gamma_{1,2}^{x_1}(x_2)d\mu_1(x_1)\\
&\le \int_{\mathbb{R}^{2N}} c_{\Phi,h,\nu}(x_0,x_1)\ d\gamma_{0,1}(x_0,x_1) + \int_{\mathbb{R}^{2N}} c_{\Phi,h,\nu}(x_1,x_2)\ d\gamma_{1,2}(x_1,x_2)\\
&\le 2\varepsilon + W_{\Phi,h,\nu}(\mu_0,\mu_1) + W_{\Phi,h,\nu}(\mu_1,\mu_2).
\end{aligned}$$

Sending ε to zero gives the result.

Proof of (iv): For simplicity we denote by c the cost $c_{\Phi,h,\nu}$. It is symmetric, satisfies $c(x,x)=0$ for all $x \in \mathbb{R}^N$ and the triangular inequality holds. It is non-negative and continuous, so by classic Kantorovich duality (see for instance [17]):

$$W_{\Phi,h,\nu}(\mu_0,\mu_1) = \sup_{\varphi\in\mathcal{C}_0(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \varphi^{cc}(x_0)d\mu_0(x_0) + \int_{\mathbb{R}^N} \varphi^{c}(x_1)d\mu_1(x_1) \right\}$$

where $\psi^c(x) := \inf_{z\in\mathbb{R}^N}\{c(x,z)-\psi(z)\}$.
Note that, for all x_0, $x_1 \in \mathbb{R}^N$,

$$\varphi^c(x_1) - \varphi^c(x_0) \le c(x_0,x_1). \tag{9}$$

Indeed, for any $\varepsilon > 0$ it exists z_ε such that

$$\varphi^c(x_0) + \varepsilon \ge c(x_0,z_\varepsilon) - \varphi(z_\varepsilon),$$

and using z_ε as candidate for $\inf_{z\in\mathbb{R}^N}\{c(x_1,z)-\varphi(z)\}$:

$$\varphi^c(x_1) - \varphi^c(x_0) \le c(x_1,z_\varepsilon) - \varphi(z_\varepsilon) - c(x_0,z_\varepsilon) + \varphi(z_\varepsilon) + \varepsilon \le c(x_0,x_1) + \varepsilon.$$

Let us show $\varphi^{cc} = -\varphi^c$. Choosing $x_1 = x_0$, we get:

$$\varphi^{cc}(x_0) := \inf_{x_1\in\mathbb{R}^N}\{c(x_0,x_1) - \varphi^c(x_1)\} \le -\varphi^c(x_0),$$

the opposite inequality comes from (9). We get that:

$$W_{\Phi,h,\nu}(\mu_0,\mu_1) := \sup_{\varphi\in\mathcal{C}_0(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} -\varphi^c(x_0)d\mu_0(x_0) + \int_{\mathbb{R}^N} \varphi^c(x_1)d\mu_1(x_1) \right\}.$$

Observe that by (9) and the continuity of c, φ^c is continuous for all $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$. Moreover φ^c is bounded because, as c is non-negative and $c(x,x)=0$:

$$\varphi^c(x) = \inf_{z\in\mathbb{R}^N}\{c(x,z)-\varphi(z)\} \le -\varphi(x) \le \|\varphi\|_\infty,$$

$$\varphi^c(x) = \inf_{z\in\mathbb{R}^N}\{c(x,z)-\varphi(z)\} \ge \inf_{z\in\mathbb{R}^N}\{-\varphi(z)\} \ge -\|\varphi\|_\infty.$$

So that:

$$\begin{aligned} & W_{\Phi,h,\mu}(\mu_0,\mu_1) \\ \le \; & \sup_{\varphi\in\mathcal{C}_b(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \varphi(x)\ d\mu_1(x) - \int_{\mathbb{R}^N} \varphi(x)\ d\mu_0(x) : \quad \varphi(x_1)-\varphi(x_0) \le c(x_0,x_1)\ \forall x_0,x_1\in\mathbb{R}^N \right\}. \end{aligned}$$

The other inequality is obvious.

QED

Corollary 1. *Let t_0 be in $[0,T]$, $\Phi \in \mathcal{C}_b(\mathbb{R}^{N\times M},\mathbb{R}^d)$, $h \in \mathcal{C}_0(\mathbb{R}^{N\times M})$.*

- *For any fixed $\nu_0 \in \mathcal{P}_1(\mathbb{R}^M)$, the maps $\mu \in \mathcal{P}_1(\mathbb{R}^N) \mapsto \mathcal{V}_r^{\pm}(t_0,\Phi,h,\mu,\nu_0)$ are uniformly continuous with respect to $\|\cdot\|_{MK}$.*
- *For any fixed $\mu_0 \in \mathcal{P}_1(\mathbb{R}^N)$, the maps $\nu \in \mathcal{P}_1(\mathbb{R}^N) \mapsto C_r^{-}(t_0,\Phi,h,\mu_0,\nu)$ are uniformly continuous with respect to $\|\cdot\|_{MK}$.*

The same properties hold for $C_r^{\pm}$.

Proof: We fix $\nu \in \mathcal{P}_1(\mathbb{R}^M)$ and prove that $\mu \mapsto \mathcal{V}_r^{\pm}(t_0,\Phi,h,\mu,\nu)$ is uniformly continuous, the remaining being similar. We denote by C the constant $(C(f,g)\ \|h\|_\infty + \|g\|_\infty)$. We fix $\varepsilon > 0$ some $\Phi_\varepsilon \in Lip(\mathbb{R}^{N+M},\mathbb{R}^d)$, $h_\varepsilon \in Lip(\mathbb{R}^{N+M},\mathbb{R}^+)$ such that:

$$\|\Phi_\varepsilon - \Phi\|_\infty \le \frac{\varepsilon}{6C}, \quad \|h_\varepsilon - h\|_\infty \le \frac{\varepsilon}{6C}.$$

Then, for all $\mu_0,\mu_1 \in \mathcal{P}_1(\mathbb{R}^N)$ such that $\|\mu_0-\mu_1\|_{MK} \le \frac{\varepsilon}{3C(Lip(h_\varepsilon)+Lip(\Phi_\varepsilon))}$, we have for any optimal plan $\gamma \in \Pi_0(\mu_0,\mu_1)$, by Lemma 5:

$$\begin{aligned} & |\mathcal{V}_r^{+}(t_0,\Phi,h,\mu_1,\nu) - \mathcal{V}_r^{+}(t_0,\Phi,h,\mu_0,\nu)| \le C W_{\Phi,h,\nu}(\mu_0,\mu_1) \\ & \le C\left[\int_{\mathbb{R}^{2N+M}} |\Phi(x_0,y)-\Phi(x_1,y)| + |h(x_0,y)-h(x_1,y)| d\gamma(x_0,x_1)d\nu(y)\right] \\ & \le 2C\|h-h_\varepsilon\|_\infty + 2C\|\Phi-\Phi_\varepsilon\|_\infty + C\left[\int |\Phi_\varepsilon(x_0,y)-\Phi_\varepsilon(x_1,y)| + |h_\varepsilon(x_0,y)-h_\varepsilon(x_1,y)| d\gamma(x_0,x_1)d\nu(y)\right] \\ & \le 2\frac{\varepsilon}{3} + C(Lip(h_\varepsilon)+Lip(\Phi_\varepsilon))\|\mu_0-\mu_1\|_{MK} d\nu(y) \le \varepsilon. \end{aligned}$$

The proof is complete.

QED

3 Values in Pure and Mixed Strategies, equalities between several definitions of values

In view of the regularity of the upper and lower values and the results proved in [19], we get that the game has a value:

Theorem 1. *Assume that Isaac's condition (4) holds. Then, for any* $t_0 \in [0,T]$, $\nu \in \mathcal{P}_1(\mathbb{R}^M)$ *and* $\mu \in \mathcal{P}_1(\mathbb{R}^N)$, $(\Phi, h) \in L^p_{\mu\times\nu}(\mathbb{R}^{N+M}, \mathbb{R}^d) \times L^q_{\mu\times\nu}(\mathbb{R}^{N+M})$ *with* $p \in [1, +\infty[$ *and* $q = \frac{p}{p-1}$, *it holds:*

$$\mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu).$$

We denote by $\mathcal{V}_r(t_0, \Phi, h, \mu, \nu)$ *the value above.*

Proof: Step 1: Assume for a while that Φ and h are regular: $\Phi \in \mathcal{C}_b(\mathbb{R}^{N\times M}, \mathbb{R}^d)$, $h \in \mathcal{C}_0(\mathbb{R}^{N\times M}, \mathbb{R}^+)$. For any $k \in \mathbb{N}$ let μ_k be in $\mathcal{P}_1(\mathbb{R}^N)$ and $\nu_k \in \mathcal{P}_1(\mathbb{R}^M)$ a pair of probability measures with finite support

$$\mu_k = \sum_{i=1}^{n_k} a_i^k \delta_{x_i^k}, \quad \nu_k = \sum_{j=1}^{m_k} b_j^k \delta_{y_j^k},$$

such that

$$\lim_{k\to+\infty} \|\mu - \mu_k\|_{MK} = \lim_{k\to+\infty} \|\nu - \nu_k\|_{MK} = 0.$$

Set for any $k \in \mathbb{N}$, any $i = 1, \dots, n_k$ and any $j = 1, \dots, m_k$:

$$X_{ij}^k := \Phi(x_i^k, y_j^k), \quad q_{i,j}^k = h(x_i^k, y_j^k) a_i^k b_j^k, \quad p_{i,j}^k = \frac{q_{i,j}^k}{\sum_{l=1}^{n_k} \sum_{l'=1}^{m_k} q_{l,l'}^k}.$$

By construction, for any $k \in \mathbb{N}$, $(p_{i,j}^k)_{i,j}$ is non negative and satisfies $\sum_{i,j} p_{i,j}^k = 1$ so it belongs to the simplex $\Delta(n_k m_k)$. Then using Corollary 1 and [19], we get:

$$\begin{aligned}
&\mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu) = \lim_{k\to+\infty} \mathcal{V}_r^+(t_0, \Phi, h, \mu_k, \nu_k)\\
=& \lim_{k\to+\infty} \inf_{\alpha\in A_r(t_0)} \sup_{\beta\in B(t_0)} \int_\Omega \sum_{i=1}^{n_k} \sum_{j=1}^{m_k} g\left(X_T^{t_0, X_{i,j}^k, \alpha(\omega, x_i, \cdot), \beta(y_j, \cdot)}\right) q_{i,j}^k \, dP(\omega)\\
=& \lim_{k\to+\infty} \left(\sum_{l=1}^{n_k} \sum_{l'=1}^{m_k} q_{l,l'}^k\right) \inf_{\alpha\in A_r(t_0)} \sup_{\beta\in B(t_0)} \int_\Omega \sum_{i=1}^{n_k} \sum_{j=1}^{m_k} g\left(X_T^{t_0, X_{i,j}^k, \alpha(\omega, x_i, \cdot), \beta(y_j, \cdot)}\right) p_{i,j}^k \, dP(\omega)\\
=& \lim_{k\to+\infty} \left(\sum_{l=1}^{n_k} \sum_{l'=1}^{m_k} q_{l,l'}^k\right) \sup_{\beta\in B_r(t_0)} \inf_{\alpha\in A(t_0)} \int_\Omega \sum_{i=1}^{n_k} \sum_{j=1}^{m_k} g\left(X_T^{t_0, X_{i,j}^k, \alpha(\omega, x_i, \cdot), \beta(y_j, \cdot)}\right) p_{i,j}^k \, dP(\omega)\\
=& \lim_{k\to+\infty} \mathcal{V}_r^-(t_0, \Phi, h, \mu_k, \nu_k) = \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu).
\end{aligned}$$

Step 2: Let $\Phi \in \mathcal{C}_b(\mathbb{R}^{N\times M}, \mathbb{R}^d)$, $h \in L^1_{\mu\times\nu}(\mathbb{R}^{N+M})$. Taking a sequence $h_n \in \mathcal{C}_c(\mathbb{R}^{N+M})$ such that $h_n \to h$ in $L^1_{\mu\times\nu}(\mathbb{R}^{N+M})$ and applying Lemma 4, we get:

$$\mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu).$$

Repeating the same argument with $\Phi \in L^p(\mathbb{R}^{N\times M}, \mathbb{R}^d)$, $h \in L^q_{\mu\times\nu}(\mathbb{R}^{N+M})$, $\Phi_n \in C_c(\mathbb{R}^{N+M}, \mathbb{R}^d)$ converging to Φ in $L^p(\mathbb{R}^{N+M}, \mathbb{R}^d)$ and using again Lemma 4, we get the result.

QED

Corollary 2. *Let* (t_0, μ, ν) *be in* $[t_0, T] \times \mathcal{P}_1(\mathbb{R}^N) \times \mathcal{P}_1(\mathbb{R}^M)$ *and* $\Phi \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d)$, $h \in \mathcal{C}_0(\mathbb{R}^{N+M}, \mathbb{R}^+)$.

(i) We have: $C_r^-(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu), \quad C_r^+(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu)$.

(ii) If Isaac's condition (4) holds then:

$$C_r^-(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu) = C_r^+(t_0, \Phi, h, \mu, \nu).$$

Proof: We only show (i). It is easily seen that the first (resp. the second) inequality holds for any μ (resp. ν) with finite support. Then, as any $\mu \in \mathcal{P}_1(\mathbb{R}^N)$ (resp. any $\nu \in \mathcal{P}_1(\mathbb{R}^M)$) can be approximate for the $\| \cdot \|_{MK}$-norm by a sequence of $(\mu_n)_n$ (resp. $(\nu_n)_n$) with finite support, considering the regularity of both sides of the equality with respect to the $\| \cdot \|_{MK}$-norm, we get the desired result.

QED

We now show that if μ and ν has no atom, the values don't change if we consider only pure strategies (cf [7]).

Proposition 2. *For any* $t_0 \in [0, T]$, $\nu \in \mathcal{P}_1(\mathbb{R}^M)$ *and* $\mu \in \mathcal{P}_1(\mathbb{R}^N)$, $(\Phi, h) \in L^p_{\mu\times\nu}(\mathbb{R}^{N+M}, \mathbb{R}^d) \times L^q_{\mu\times\nu}(\mathbb{R}^{N+M}, \mathbb{R}^+)$ *with* $p \in [1, +\infty[$ *and* $q = \frac{p}{p-1}$, *it holds:*

(i) If μ *has no atom, the following equality holds:* $\mathcal{V}^+(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu)$.

(ii) If ν *has no atom, the following equality holds:* $\mathcal{V}^-(t_0, \Phi, h, \mu, \nu) = \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu)$.

(iv) Assume that Isaac's condition (4) holds, then, if μ *and* ν *has no atom:*
$\mathcal{V}^+(t_0, \Phi, h, \mu, \nu) = \mathcal{V}^-(t_0, \Phi, h, \mu, \nu)$.

Proof of (i) : Arguing as in the proof of Theorem 1, it is enough to show the result for $\Phi \in \mathcal{C}_b(\mathbb{R}^{N+M}, \mathbb{R}^d)$, $h \in \mathcal{C}_0(\mathbb{R}^{N+M})$. It is easily seen that Lemma 4 (ii) is also satisfied for $\mathcal{V}^+$. As a consequence, it is enough to prove (i) when Φ is uniformly continuous, indeed any Φ in $\mathcal{C}_b$ can be approximate for the norm $L^2_{\mu\times\nu}$ by a sequence of functions in C_c^∞. Assume Φ is uniformly continuous and μ has no atom. Take $\varepsilon > 0$, as Φ and h are uniformly continuous, it exists N_ε such that for all $n \geq N_\varepsilon$ we have:

$$(10) \qquad |x - x'| \leq \frac{1}{n} \Rightarrow \int_{\mathbb{R}^M} |\Phi(x, y) - \Phi(x', y)| + |h(x, y) - h(x', y)| d\nu(y) \leq \varepsilon.$$

Take $n := n_\varepsilon \geq N_\varepsilon$, we consider a partition $(A_i^n)_{i\in\mathbb{N}}$ of $\mathbb{R}^N$ where all the A_i^n are Borel and have diameter less than $\frac{1}{n}$ and choose $x_i^n \in A_i^n$ for all $i \in \mathbb{N}$. We consider the following discrete probability measure:

$$\mu_n := \mu_{n_\varepsilon} = \sum_{i\in\mathbb{N}} \mu(A_i^n) \delta_{x_i^n}.$$

It satisfies:

$$W_{\Phi,h,\nu}(\mu,\mu_{n_\varepsilon}) \leq \sum_{i\in\mathbb{N}} \int_{A_i^n}\int_{\mathbb{R}^M} |\Phi(x,y)-\Phi(x_i^n,y)| + |h(x,y)-h(x_i^n,y)| d\nu(y)\ d\mu(x) \leq \varepsilon.$$

By Lemma 5, this leads:

$$\lim_{\varepsilon\to 0} \mathcal{V}_r^+(t_0,\Phi,h,\mu_{n_\varepsilon},k) = \mathcal{V}_r^+(t_0,\Phi,h,\mu,k). \tag{11}$$

Let $(([0,1]^k,\mathcal{F},\mathcal{L}^k\lfloor[0,1]^k),\alpha_\varepsilon^n)$ be a mixed strategy for player 1 such that:

$$\mathcal{V}_r^+(t_0,\Phi,h,\mu_n,\nu)+\varepsilon \geq \sup_{\beta\in B(t_0)} \sum_{i\in\mathbb{N}} \int_{[0,1]^k}\int_{\mathbb{R}^M} g\left(X_T^{t_0,\Phi(x_i^n,y),\alpha_\varepsilon^n(\omega,x_i^n,\cdot),\beta(y,\cdot)}\right) h(x_i^n,y)\mu(A_i^n)\ d\omega d\nu(y).$$

For any $i\in\mathbb{N}$ we consider a map $T_i^n:\mathbb{R}^N\to[0,1]^k$ such that:

$$T_i^n\sharp\left(\frac{\mu\lfloor A_i^n}{\mu(A_i^n)}\right) = \mathcal{L}^k\lfloor[0,1]^k \text{ that is } \frac{1}{\mu(A_i^n)}\int_{A_i^n}\varphi(T_i^n(x))d\mu(x) = \int_{[0,1]^k}\varphi(z)dz\ \forall\varphi\in\mathcal{C}_0(\mathbb{R}^k).$$

As μ has no atoms, such transport map always exists (see for instance [16]). Then we build a pure strategy for Player 1 by setting:

$$\hat{\alpha}_\varepsilon^n(x,\cdot) = \sum_{i\in\mathbb{N}} 1_{A_i^n}(x)\ \alpha_\varepsilon^n(T_i^n(x),x_i^n,\cdot).$$

Using this strategy as a candidate for $\mathcal{V}^+(t_0,\Phi,h,\mu,\nu)$ leads:

$$\begin{aligned}
&\mathcal{V}^+(t_0,\Phi,h,\mu,\nu)\\
\leq\ & \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^{N+M}} g\left(X_T^{t_0,\Phi(x,y),\hat{\alpha}_i^n(x,\cdot),\beta(y,\cdot)}\right) h(x,y)\ d\mu(x)d\nu(y)\\
\leq\ & \sup_{\beta\in B(t_0)} \sum_{i\in\mathbb{N}} \int_{A_i^n\times\mathbb{R}^M} g\left(X_T^{t_0,\Phi(x,y),\alpha_i^n(T_i^n(x),x_i^n,\cdot),\beta(y,\cdot)}\right) h(x,y)\ d\mu(x)d\nu(y)\\
\leq\ & (C(f,g)\|h\|_\infty+\|g\|_\infty)\sum_{i\in\mathbb{N}}\int_{A_i^n\times\mathbb{R}^M} |\Phi(x,y)-\Phi(x_i^n,y)|+|h(x,y)-h(x_i^n,y)|d\nu(y)\\
&+\sup_{\beta\in B(t_0)} \sum_{i\in\mathbb{N}} \int_{A_i^n\times\mathbb{R}^M} g\left(X_T^{t_0,\Phi(x_i^n,y),\alpha_i^n(T_i^n(x),x_i^n,\cdot),\beta(y,\cdot)}\right) h(x_i^n,y)\ d\mu(x)d\nu(y).
\end{aligned}$$

Then, setting $C=(C(f,g)\|h\|_\infty+\|g\|_\infty)$, using (10), the definition of T_i^n and α_ε^n we get:

$$\begin{aligned}
&\mathcal{V}^+(t_0,\Phi,h,\mu,\nu)\\
\leq\ & (C(f,g)\|h\|_\infty+\|g\|_\infty)\times\varepsilon\\
&+\sup_{\beta\in B(t_0)} \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^M\times[0,1]^k} \mu(A_i^n) g\left(X_T^{t_0,\Phi(x_i^n,y),\alpha_i^n(\omega,x_i^n,\cdot),\beta(y,\cdot)}\right) h(x_i^n,y)\ T_i^n\sharp\left(\frac{\mu\lfloor A_i^n}{\mu(A_i^n)}\right)(\omega)d\nu(y)\\
\leq\ & C\varepsilon+\sup_{\beta\in B(t_0)} \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^M\times[0,1]^k} g\left(X_T^{t_0,\Phi(x_i^n,y),\alpha_\varepsilon^n(\omega,x_i^n,\cdot),\beta(y,\cdot)}\right) h(x_i^n,y)\mu(A_i^n)\ d\omega d\nu(y)\\
\leq\ & C\varepsilon+\varepsilon+\mathcal{V}_r^+(t_0,\Phi,h,\mu_n,\nu).
\end{aligned}$$

We have proved that for any $n \geq N_\varepsilon$ we have:

$$\mathcal{V}^+(t_0, \Phi, h, \mu, \nu) \leq (C+1)\varepsilon + \mathcal{V}_r^+(t_0, \Phi, h, \mu_{n_\varepsilon}, \nu).$$

This inequality being true for any ε, by (11), making ε go to zero, we have:

$$\mathcal{V}^+(t_0, \Phi, h, \mu, \nu) \leq \mathcal{V}_r^+(t_0, \Phi, h, \mu, \nu).$$

The other inequality is straightforward.

QED

4 Subdynamic and superdynamic Programming Principles

From now on, we take $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ two compact sets and we will consider only probability measures μ supported inside X and ν supported inside Y. Namely $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$. Moreover we will require some regularity for Φ and $h \geq 0$:

$$\Phi \in \mathcal{C}(X \times Y, \mathbb{R}^d), \quad h \in \mathcal{C}(X \times Y, \mathbb{R}^+).$$

Because of the lack of information, both the subdynamic and superdynamic principles will be dual. So, as we get into convex analysis, we will need the following properties.

4.1 Convexity/concavity properties

Lemma 7. *For any $(t, \Phi, h, \nu) \in [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X \times Y, \mathbb{R}^+) \times \mathcal{P}(Y)$, the map $\mu_0 \in \mathcal{P}(X) \mapsto \mathcal{V}_r^\pm(t, \Phi, h, \mu_0, \nu)$ is convex.*
For any $(t, \Phi, h, \mu) \in [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X \times Y, \mathbb{R}^+) \times \mathcal{P}(X)$, the map $\nu_0 \in \mathcal{P}(Y) \mapsto \mathcal{V}_r^\pm(t_0, \Phi, h, \mu, \nu_0)$ is concave.

This lemma can be proved similarly to Lemma 3 in [13]. Nevertheless, note that the convexity of $\mathcal{V}_r^-$ in the μ variable is obvious from the equality with C_r^-, indeed, this last functional is a supremum of a linear application in μ. We can get the concavity of $\mathcal{V}_r^+$ in a symmetric way.

In the next sections, we will need the following lemma:

Lemma 8. *Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$ be two compact subsets.*

(i) Let t_0 be in $[0,T]$, $(\Phi, h) \in C(X \times Y, \mathbb{R}^d) \times \mathcal{C}_0(X \times Y, \mathbb{R}^+)$, $\nu \in \mathcal{P}(Y)$. For all $\mu_0 \in \mathcal{P}(X)$, the convex subdifferential $\partial^- \mathcal{V}_r^-(t_0, \Phi, h, \mu_0, \nu)$ of $\mathcal{V}_r^-$ at μ_0 is not empty.

(ii) Let t_0 be in $[0,T]$, $(\Phi, h) \in C(X \times Y, \mathbb{R}^d) \times \mathcal{C}_0(X \times Y, \mathbb{R}^+)$, $\mu \in \mathcal{P}(X)$. For all $\nu_0 \in \mathcal{P}(Y)$, the convex superdifferential $\partial^+ \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu_0)$ of $\mathcal{V}_r^-$ at ν_0 is not empty.

Proof: We prove only (i).
Fix $t_0 \in [0,T]$, $(\Phi,h) \in \mathcal{C}_b(X\times Y,\mathbb{R}^d)\times\mathcal{C}_0(X\times Y,\mathbb{R}^+)$, $\nu\in\mathcal{P}(Y)$. Take x_0 any point in X, we introduce the following convex subset of $\mathcal{M}_0(X)$:

$$Z_{x_0} := \{\mu - \delta_{x_0} :\ \mu \in \mathcal{P}(X)\}\,.$$

We set, for any $\eta \in \mathcal{M}_0(X)$:

$$G(\eta) := \inf_{\eta_0=\mu_0-\delta_{x_0}\in Z_{x_0}} \left\{(C(f,g)\|h\|_\infty + \|g\|_\infty)N_{\Phi,h,\nu}(\eta-\eta_0) + \mathcal{V}_r^-(t_0,\Phi,h,\mu_0,\nu)\right\}.$$

Remember that, by Lemma 6, $N_{\Phi,h,\nu}$ is a semi-norm.

Step 1: We show that $G(\eta) = \mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu)$ for all $\eta=\mu-\delta_{x_0}\in Z_{x_0}$.
Indeed, on the one hand, it is easy to see, by choosing $\eta_0=\eta$ that:

$$G(\eta) \le \mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu).$$

On the other hand, by Proposition 5 (i), the functional $\mathcal{V}_r^-$ is Lipchitz with respect to the semi-norm $N_{\Phi,h,\nu}$ so, for any $\eta_0=\mu_0-\delta_{x_0}$:

$$\mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu) \le (C(f,g)\|h\|_\infty + \|g\|_\infty)N_{\Phi,h,\nu}(\mu-\mu_0) + \mathcal{V}_r^-(t_0,\Phi,h,\mu_0,\nu)$$

As this last inequality is true for any $\eta_0=\mu_0-\delta_{x_0}\in Z_{x_0}$, taking the infimum on all $\eta_0\in Z_0$ gives the desired result.

Step 2: It can be easily proved that G is convex and Lipschitz continuous with respect to the semi-norm $N_{\Phi,h,\nu}$. Moreover, arguing as in the proofs of Corollary 1, G is uniformly continuous for the topology induced by $\|\cdot\|_{MK}$.

Step 3: The functional G is convex, continuous on the normed vectorial space $\mathcal{M}_0(X)$, so its convex subdifferential is non-empty at any η (see [9], Proposition 5.2. p 22). So for any $\mu_0\in\mathcal{P}(X)$, it exists $\varphi_0\in Lip_0(X)$ such that:

$$G(\mu-\delta_{x_0}) \ge G(\mu_0-\delta_{x_0}) + \int_{\mathbb{R}^N}\varphi_0(x)d(\mu-\mu_0),\ \forall \mu\in\mathcal{P}(X).$$

Recalling Step 1, then:

$$\mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu) \ge \mathcal{V}_r^-(t_0,\Phi,h,\mu_0,\nu) + \int_{\mathbb{R}^N}\varphi_0(x)d(\mu-\mu_0),\ \forall\mu\in\mathcal{P}(X).$$

QED

4.2 Convex conjugate of $\mathcal{V}_r^-$:

Fix $(t_0,\nu)\in[0,T]\times\mathcal{P}(Y)$ and $\Phi\in\mathcal{C}(X\times Y,\mathbb{R}^d)$, $h\in\mathcal{C}(X\times Y,\mathbb{R}^+)$. We extend $\mu\in P(X)\mapsto\mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu)$ by $+\infty$ inside $\mathcal{M}_b(X)\backslash\mathcal{P}(X)$. We still denote the extension by $\mathcal{V}_r^-$. We are going to compute the convex conjugate of $\mathcal{V}_r^-$ in the variable μ. Note that, as $\mathcal{V}_r^-$ is convex, l.s.c. with respect to the weak star topology of $\mathcal{M}_b(X)$, we have:

$$(\mathcal{V}_r^-)^{**}(t_0,\Phi,h,\mu,\nu) = \mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu).$$

Lemma 9. *Fix $(t_0,\nu)\in[0,T]\times\mathcal{P}(Y)$ and $\Phi\in\mathcal{C}(X\times Y,\mathbb{R}^d)$, $h\in\mathcal{C}(X\times Y,\mathbb{R}^+)$. For any $\varphi\in\mathcal{C}(X)$, it holds:*

$$
\begin{aligned}
&(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu)\\
&\le \inf_{\beta\in B_r(t_0)}\sup_{\alpha\in A_c(t_0)}\sup_{x\in X}\left\{\varphi(x)-\int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right)h(x,y)d\nu(y)dP(\omega)\right\}.
\end{aligned}
$$

In the sequel, we set for all $\varphi\in\mathcal{C}(X)$,

$$
z(\varphi)=\inf_{\beta\in B_r(t_0)}\sup_{\alpha\in A_c(t_0)}\sup_{x\in X}\left\{\varphi(x)-\int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right)h(x,y)d\nu(y)dP(\omega)\right\}.
$$

Proof: By Corollary 2:

$$
\begin{aligned}
&(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu)\\
&=\sup_{\mu\in\mathcal{P}(X)}\left\{\int_X\varphi(x)d\mu(x)-\mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu)\right\}\\
=&\sup_{\mu\in\mathcal{P}(X)}\left\{\int\varphi d\mu-\sup_{\beta\in B_r(t_0)}\int_X\inf_{\alpha\in A_c(t_0)}\left[\int g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right)h(x,y)d\nu(y)dP(\omega)\right]\ d\mu(x)\right\}\\
=&\sup_{\mu\in\mathcal{P}(X)}\inf_{\beta\in B_r(t_0)}\left\{\int_X\left[\varphi(x)-\inf_{\alpha\in A_c(t_0)}\int g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right)h(x,y)d\nu(y)dP(\omega)\right]d\mu(x)\right\}\\
&\le\inf_{\beta\in B_r(t_0)}\sup_{\alpha\in A_c(t_0)}\sup_{x\in X}\left\{\varphi(x)-\int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)}\right)h(x,y)d\nu(y)dP(\omega)\right\}.
\end{aligned}
$$

The inequality follows.

QED

Lemma 10. *The functional $z:\mathcal{C}(X)\to\mathbb{R}$ is convex and l.s.c. As a consequence $z^{**}=z$.*

The proof of this lemma is very similar to the proof of Lemma 9 in [13], therefore it is omitted.

Proposition 3. *Fix $(t_0,\nu)\in[0,T]\times\mathcal{P}(Y)$ and $\Phi\in\mathcal{C}(X\times Y,\mathbb{R}^d)$, $h\in\mathcal{C}(X\times Y,\mathbb{R}^+)$. For any $\varphi\in\mathcal{C}(X)$, it holds:*

$$
(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu)=z(\varphi).
$$

Proof:

Step 1: Let us show that $z^*(\mu) = +\infty$ if $\mu \in \mathcal{M}_b(X)\backslash\mathcal{P}(X)$. Indeed:

$$z^*(\mu) := \sup_{\varphi \in \mathcal{C}(X)} \left\{ \int_X \varphi(x) d\mu(x) - z(\varphi) \right\}$$
$$= \sup_{\varphi \in \mathcal{C}(X)} \left\{ \int_X \varphi(x) d\mu(x) - \right.$$
$$\left. \inf_{\beta \in B_r(t_0)} \sup_{\alpha \in A_c(t_0)} \sup_{x \in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\beta(\omega,y,\cdot)} \right) h(x,y)\ d\nu(y) dP(\omega) \right\} \right\}$$
$$\geq -\|g\|_\infty \|h\|_\infty + \sup_{\varphi \in \mathcal{C}(X)} \left\{ \int_X \varphi d\mu - \sup_{x\in X} \varphi(x) \right\}$$

We have on the one hand:

$$\sup_{\varphi \leq 0} \left\{ \int_X \varphi d\mu - \sup_{x\in X} \varphi(x) \right\} \ \geq \ \sup_{\varphi\leq 0} \int \varphi d\mu = \begin{cases} 0 \text{ if } \mu \geq 0 \\ +\infty \text{ otherwise,} \end{cases}$$

and on the other hand for any $\mu \geq 0$:

$$\sup_{\varphi \geq 0} \left\{ \int_X \varphi d\mu - \sup_{x\in X} \varphi(x) \right\} = \sup_{\varphi\geq 0} \left\{ \int_X \varphi d\mu - \|\varphi\|_\infty \right\} = \begin{cases} 0 \text{ if } \mu(X) = 1 \\ +\infty \text{ otherwise.} \end{cases}$$

We conclude that $z^*(\mu) = +\infty$ if μ is not a probability.

Step 2: We show:

$$z^*(\mu) \geq \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu) \quad \forall \mu \in \mathcal{P}(X).$$

Indeed, recalling the notations of Lemma 2:

$$\begin{aligned} z^*(\mu) &= \sup_{\varphi\in\mathcal{C}(X)} \left\{ \int_X \varphi\ d\mu - \inf_{\beta\in B_r(t_0)} \sup_{\alpha\in A_c(t_0),\ x\in X} [\varphi(x) - \varphi_{\alpha,\beta}(x)] \right\} \\ &= \sup_{\beta\in B_r(t_0)} \sup_{\varphi\in\mathcal{C}(X)} \inf_{x\in X} \left\{ \int_X \varphi\ d\mu - \left[\varphi(x) - \inf_{\alpha\in A_c(t_0)} \varphi_{\alpha,\beta}(x) \right] \right\} \end{aligned}$$

By lemma 2, for any fixed β, the map $x \mapsto \inf_{\alpha\in A_c(t_0)} \varphi_{\alpha,\beta}(x)$ is in $\mathcal{C}(X)$ and choosing $\varphi = \inf_{\alpha\in A_c(t_0)} \varphi_{\alpha,\beta}$ we get:

$$z^*(\mu) \geq \sup_{\beta\in B_r(t_0)} \left\{ \int_X \inf_{\alpha\in A_c(t_0)} \varphi_{\alpha,\beta}\ d\mu \right\}$$

We conclude by using Corollary 2.

Conclusion: Putting together Step 1 and 2 we have:

$$z^*(\mu) \geq \mathcal{V}_r^-(t_0, \Phi, h, \mu, \nu) \quad \forall \mu \in \mathcal{M}_b(X).$$

Now, using Lemma 10, we get for all $\varphi \in \mathcal{C}(X)$:

$$
\begin{aligned}
(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu) &= \sup_{\mu\in\mathcal{M}_b(X)} \left\{ \int \varphi d\mu - \mathcal{V}_r^-(t_0,\Phi,h,\mu,\nu) \right\} \\
&\geq \sup_{\mu\in\mathcal{M}_b(X)} \left\{ \int \varphi d\mu - z^*(\mu) \right\} = z^{**}(\varphi) = z(\varphi).
\end{aligned}
$$

QED

4.3 Subdynamic and superdynamic principles:

Proposition 4. *Let t_0, t_1 be such that $0 \leq t_0 < t_1 \leq T$, $\Phi \in \mathcal{C}(X \times Y, \mathbb{R}^d)$, $h \in \mathcal{C}_0(\mathbb{R}^{N+M}, \mathbb{R}^+)$, $\varphi \in \mathcal{C}(X)$, $\nu \in \mathcal{P}(X)$:*

$$
(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu) \leq \inf_{\beta\in B_c(t_0)} \sup_{u\in\mathcal{U}(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0,\Phi(\cdot,\cdot),u,\beta(u)}, h, \varphi, \nu).
$$

Proof: We follow [13]. Take $\varepsilon > 0$ and β_ε be an ε-optimal strategy such that:
(12)

$$
\inf_{\beta\in B_c(t_0)} \sup_{u\in\mathcal{U}(t_0)} (\mathcal{V}_r^-)^*(t_1, X_{t_1}^{t_0,\Phi(\cdot,\cdot),u,\beta(u)}, h, \varphi, \nu) + \varepsilon \geq \sup_{u\in\mathcal{U}(t_0)} (\mathcal{V}_r^-)^* \left(t_1, X_{t_1}^{t_0,\Phi,u,\beta_\varepsilon(u)}, \varphi, \nu \right).
$$

Let also $((\Omega_1, \mathcal{F}_1, P_1), \beta_1)$ be an element of $B_r(t_1)$. We glue together β_ε and β_1 to get a new element of $B_r(t_1)$:
(13)

$$
\forall (\omega, y, u, s) \in \Omega_1 \times Y \times \mathcal{U}(t_0) \times [t_0, T], \quad \bar{\beta}(\omega, y, u)(s) := \begin{cases} \beta_\varepsilon(u)(s) & \text{if } s \in [t_0, t_1[\\ \beta_1(\omega, y, u)(s) & \text{else.} \end{cases}
$$

Let α_0 be any element of $A_c(t_0)$ such that:

$$
\begin{aligned}
&\sup_{\alpha\in A_c(t_0)} \sup_{x\in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\bar{\beta}(\omega,y,\cdot)} \right) h(x,y) d\nu(y) dP(\omega) \right\} \\
&\leq \sup_{x\in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha_0(\cdot),\bar{\beta}(\omega,y,\cdot)} \right) h(x,y) d\nu(y) dP(\omega) \right\} + \varepsilon.
\end{aligned}
$$

Then take $(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ associated to $(\alpha_0, \beta_\varepsilon)$ by Lemma 1 and set:

$$
\forall v \in \mathcal{V}(t_1), \quad \alpha_1(v) = \alpha_0(v_0\lfloor[t_0, t_1] + v\lfloor[t_1, T]).
$$

We get:

$$
\begin{aligned}
&\sup_{\alpha\in A_c(t_0)} \sup_{x\in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha(\cdot),\bar{\beta}(\omega,y,\cdot)} \right) h(x,y) d\nu(y) dP(\omega) \right\} \\
\leq\ & \sup_{x\in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_0,\Phi(x,y),\alpha_0(\cdot),\bar{\beta}(\omega,y,\cdot)} \right) h(x,y) d\nu(y) dP(\omega) \right\} + \varepsilon \\
=\ & \sup_{x\in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_1,X_{t_1}^{t_0,\Phi(x,y),u_0,\beta_\varepsilon(u_0)},\alpha_1(\cdot),\beta_1(\omega,y,\cdot)} \right) h(x,y) d\nu(y) dP(\omega) \right\} + \varepsilon \\
\leq\ & \sup_{\alpha\in A_c(t_1)} \sup_{x\in X} \left\{ \varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_1,X_{t_1}^{t_0,\Phi(x,y),u_0,\beta_\varepsilon(u_0)},\alpha(\cdot),\beta_1(\omega,y,\cdot)} \right) h(x,y) d\nu(y) dP(\omega) \right\} + \varepsilon.
\end{aligned}
$$

And by the definition of z:

$$
\begin{aligned}
&(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu) = z(t_0,\Phi,h,\varphi,\nu)\\
\leq\ & \sup_{\alpha\in A_c(t_1)}\sup_{x\in X}\left\{\varphi(x) - \int_{Y\times\Omega} g\left(X_T^{t_1,X_{t_1}^{t_0,\Phi(x,y),u_0,\beta_\varepsilon(u_0)},\alpha(\cdot),\beta_1(\omega,y,\cdot)}\right)h(x,y)d\nu(y)dP(\omega)\right\}+\varepsilon
\end{aligned}
$$

As this true for any $\beta_1\in B_r(t_1)$, we have by 12:

$$
\begin{aligned}
(\mathcal{V}_r^-)^*(t_0,\Phi,h,\varphi,\nu) &\leq (\mathcal{V}_r^-)^*(t_1,X_{t_1}^{t_0,\Phi(\cdot,\cdot),u_0,\beta_\varepsilon(u_0)},h,\varphi,\nu)+\varepsilon\\
&\leq \sup_{u\in U(t_0)}(\mathcal{V}_r^-)^*(t_1,X_{t_1}^{t_0,\Phi(\cdot,\cdot),u,\beta_\varepsilon(u)},h,\varphi,\nu)+\varepsilon\\
&\leq \inf_{\beta\in B_c(t_0)}\sup_{u\in\mathcal{U}(t_0)}(\mathcal{V}_r^-)^*(t_1,X_{t_1}^{t_0,\Phi(\cdot,\cdot),u,\beta(u)},h,\varphi,\nu)+2\varepsilon.
\end{aligned}
$$

QED

The following can be proved similarly to Proposition 4:

Proposition 5. *For any $0\leq t_0<t_1\leq T$, $\Phi\in\mathcal{C}(X\times Y,\mathbb{R}^d)$, $h\in\mathcal{C}_0(\mathbb{R}^{N+M},\mathbb{R}^+)$, $\phi\in\mathcal{C}(Y)$, $\mu\in\mathcal{P}(Y)$, it holds:*

$$
(\mathcal{V}_r^+)^\sharp(t_0,\Phi,h,\mu,\phi)\geq \sup_{\alpha\in A_c(t_0)}\inf_{v\in\mathcal{V}(t_0)}(\mathcal{V}_r^+)^\sharp(t_1,X_{t_1}^{t_0,\Phi(\cdot,\cdot),\alpha(v),v},h,\mu,\varphi).
$$

5 Hamilton Jacobi Isaacs equations

We introduce the following Hamiltonian defined for any $(\mu_0,\nu_0,\Phi_0,p_\Phi)$ in $\mathcal{P}(X)\times\mathcal{P}(Y)\times\mathcal{C}(X\times Y,\mathbb{R}^d)^2$ by:

$$
\mathcal{H}(\mu_0,\nu_0,\Phi_0,p_\Phi) := \inf_{u\in U}\sup_{v\in V}\int_{X\times Y} f(\Phi_0(x,y),u,v)\cdot p_\Phi(x,y)\ d\mu_0(x)d\nu_0(y)
$$

$$
= \sup_{v\in V}\inf_{u\in U}\int_{X\times Y} f(\Phi_0(x,y),u,v)\cdot p_\Phi(x,y)\ d\mu_0(x)d\nu_0(y)
$$

(where Isaac's condition (4) is assumed) and the Hamilton Jacobi Isaacs equation:

$$
\partial_t W(t_0,\Phi_0,\mu_0,\nu_0)+\mathcal{H}(\mu_0,\nu_0,\Phi_0,D_\Phi W(t_0,\Phi_0,\mu_0,\nu_0))=0. \tag{14}
$$

In our case this equation will be considered with the terminal condition:

$$
W(T,\Phi_0,\mu_0,\nu_0)=\int_{X\times Y} g\left(\Phi_0(x,y)\right)h(x,y)\ d\mu_0(x)d\nu_0(y).
$$

We also set:

$$
\widehat{\mathcal{H}}(\mu_0,\nu_0,\Phi_0,p_\Phi) := -\mathcal{H}(\mu_0,\nu_0,\Phi_0,-p_\Phi)
$$

$$
= \inf_{v\in V}\sup_{u\in U}\int_{X\times Y} f(\Phi_0(x,y),u,v)\cdot p_\Phi(x,y)\ d\mu_0(x)d\nu_0(y)
$$

$$= \sup_{u\in U}\inf_{v\in V}\int_{X\times Y} f(\Phi_0(x,y),u,v)\cdot p_\Phi(x,y)\ d\mu_0(x)d\nu_0(y)$$

Note that $\widehat{\mathcal{H}}$ satisfy the following Lipschitz condition for all $(\mu_0,\nu_0,p_\Phi)\in\mathcal{P}(X)\times\mathcal{P}(Y)\times\mathcal{C}(X\times Y,\mathbb{R}^d)$, (Φ_0,Φ_1) in $\mathcal{C}(X\times Y,\mathbb{R}^d)^2$:

$$(15)\qquad |\widehat{\mathcal{H}}(\mu_0,\nu_0,\Phi_0,p_\Phi)-\widehat{\mathcal{H}}(\mu_0,\nu_0,\Phi_1,p_\Phi)|\le Lip(f)\ \|p_\Phi\|_{L^2_{\mu_0\times\nu_0}}\|\Phi_0-\Phi_1\|_{L^2_{\mu_0\times\nu_0}}.$$

5.1 Dual subsolution and Dual supersolution

Consider the functional $w:[0,T]\times\mathcal{C}(X\times Y,\mathbb{R}^d)\times\mathcal{P}(X)\times\mathcal{P}(Y)\mapsto\mathbb{R}$ and its convex and concave conjugate resp. on the μ and ν variable:

$$w^*:[0,T]\times\mathcal{C}(X\times Y,\mathbb{R}^d)\times\mathcal{C}(X)\times\mathcal{P}(Y)\mapsto\mathbb{R}$$

$$w^\sharp:[0,T]\times\mathcal{C}(X\times Y,\mathbb{R}^d)\times\mathcal{P}(X)\times\mathcal{C}(Y)\mapsto\mathbb{R}.$$

Slightly abusing, we will use the following notations:

$$\begin{aligned}&\partial^- w^*(t_0,\Phi_0,\varphi_0,\nu_0)\\ :=\ &\left\{\mu_0\in\mathcal{P}(X):\ \forall\varphi\in\mathcal{C}(X),\ \int_X\varphi-\varphi_0\ d\mu_0\le w^*(t_0,\Phi_0,\varphi,\nu_0)-w^*(t_0,\Phi_0,\varphi_0,\nu_0)\right\},\end{aligned}$$

$$\begin{aligned}&\partial^+ w^\sharp(t_0,\Phi_0,\mu_0,\psi_0)\\ :=\ &\left\{\nu_0\in\mathcal{P}(Y):\ \forall\psi\in\mathcal{C}(Y),\ \int_Y\psi-\psi_0\ d\nu_0\ge w^\sharp(t_0,\Phi_0,\mu_0,\psi)-w^\sharp(t_0,\Phi_0,\mu_0,\psi_0)\right\}.\end{aligned}$$

To define the δ-superdifferential $D^+_\delta w^*(t_0,\Phi_0,\varphi_0,\nu_0)$, we will need the following lemma:

Lemma 11. *Let $w:(t,\Phi,\mu,\nu)\in[0,T]\times\mathcal{C}(X,X)\times\mathcal{P}(X)\times\mathcal{P}(Y)\to\mathbb{R}$ be some functional. Then the Fenchel conjugate $(t,\Phi,\varphi,\nu)\in[0,T]\times\mathcal{C}(X,X)\times\mathcal{C}(X)\times\mathcal{P}(Y)\mapsto w^*(t,\Phi,\varphi,\nu)$ is Lipschitz-continuous in φ, uniformly in (t,Φ,ν). As a consequence, we have, for all $(t,\Phi,\varphi,\nu)\in[0,T]\times\mathcal{C}(X,X)\times\mathcal{C}(X)\times\mathcal{P}(Y)$: $\partial^- w^*(t,\Phi,\varphi,\nu)\neq\emptyset$.*

A symmetric resul holds for $w^\sharp$: $\partial^- w^\sharp(t,\Phi,\mu,\psi)\neq\emptyset$.

Proof of the lemma: Let $\varphi_0,\varphi_1\in\mathcal{C}(X)$, we have:

$$\begin{aligned}w^*(t,\Phi,\varphi_0,\nu)\ &:=\ \sup_{\mu\in\mathcal{P}(X)}\left\{\int_X\varphi_0(x)\ d\mu(x)-w(t,\Phi,\mu,\nu)\right\}\\ &=\ \sup_{\mu\in\mathcal{P}(X)}\left\{\int_X(\varphi_0-\varphi_1)(x)\ d\mu(x)+\int_X\varphi_1(x)\ d\mu(x)-w(t,\Phi,\mu,\nu)\right\}\\ &\le\ \|\varphi_0-\varphi_1\|_\infty+\sup_{\mu\in\mathcal{P}(X)}\left\{\int_X\varphi_1(x)\ d\mu(x)-w(t,\Phi,\mu,\nu)\right\}\\ &\le\ \|\varphi_0-\varphi_1\|_\infty+w^*(t,\Phi,\varphi_1,\nu).\end{aligned}$$

QED

Hereafter, we give the appropriate definitions of viscosity subdifferential and superdifferential:

Definition 1. • *Take $\delta > 0$ and $(t_0, \Phi_0, \varphi_0, \nu_0) \in [0, T[\times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X) \times \mathcal{P}(Y)$. Assume moreover that*

$$\partial^- w^*(t_0, \Phi_0, \varphi_0, \nu_0) = \{\mu_0\}.$$

We say that $(p_t, p_\Phi) \in \mathbb{R}\times\mathcal{C}(X\times Y, \mathbb{R}^d)$ belongs to the δ-superdifferential $D^+_\delta w^*(t_0, \Phi_0, \varphi_0, \nu_0)$ to w^* iff

$$\forall \Phi \in \mathcal{C}(X \times Y, \mathbb{R}^d), \quad \forall t \in [0, T[,$$

$$w^*(t, \Phi, \varphi_0, \nu_0) - w^*(t_0, \Phi_0, \varphi_0, \nu_0) - p_t(t - t_0)$$
$$+ \sup_{\mu \in \partial_- w^*(t,\Phi,\varphi_0,\nu_0)} \left\{ - \int_{X\times Y} (\Phi - \Phi_0)(x, y) \cdot p_\Phi(x, y) d\mu(x) d\nu_0(y) \right\}$$
$$-\delta(\|\Phi - \Phi_0\|_\infty + |t - t_0|) + o(\|\Phi - \Phi_0\|_\infty + |t - t_0|) \leq 0$$

where $o(\tau) = \tau\varepsilon(\tau)$ with $\varepsilon(\tau) \to 0$ as $\tau \to 0$.

- *Take $\delta > 0$ and $(t_0, \Phi_0, \mu_0, \psi_0) \in [0, T[\times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{P}(X) \times \mathcal{C}(Y)$. Assume moreover that*

$$\partial^+ w^\sharp(t_0, \Phi_0, \mu_0, \psi_0) = \{\nu_0\}.$$

We say that $(p_t, p_\Phi) \in \mathbb{R}\times\mathcal{C}(X\times Y, \mathbb{R}^d)$ belongs to the δ-subdifferential $D^-_\delta w(t_0, \Phi_0, \mu_0, \psi_0)$ to $w^\sharp$ iff

$$\forall \Phi \in \mathcal{C}(X \times Y, \mathbb{R}^d), \quad \forall t \in [0, T[,$$

$$w^\sharp(t, \Phi, \mu_0, \psi_0) - w^\sharp(t_0, \Phi_0, \mu_0, \psi_0) - p_t(t - t_0)$$
$$+ \inf_{\nu \in \partial^+ w^\sharp(t,\Phi,\mu_0,\psi_0)} \left\{ - \int_{X\times Y} (\Phi - \Phi_0)(x, y) \cdot p_\Phi(x, y) \; d\mu_0(x) d\nu(y) \right\}$$
$$+\delta(\|\Phi - \Phi_0\|_\infty + |t - t_0|) + o(\|\Phi - \Phi_0\|_\infty + |t - t_0|) \geq 0$$

where $o(\tau) = \tau\varepsilon(\tau)$ with $\varepsilon(\tau) \to 0$ as $\tau \to 0$.

Remark 2. *We give some comments and explanation on the definition of the δ-superdifferential above, same remarks can be made on the δ-subdifferential.*

- *The definition of the δ-superdifferential has to be understood in the following sense. For any sequence $(\Phi_n)_n$ in $\mathcal{C}(X, X)$ and $(t_n)_n \in [0, T]$ such that $\|\Phi_n - \Phi_0\|_\infty \to 0$ and $t_n \to t_0$ as $n \to +\infty$, and any $\mu_n \in \partial_- w^*(t_n, \Phi_n, \varphi_0, \nu_0)$ (which is not empty by Lemma 11), it holds:*

$$\limsup_{n\to+\infty} \quad \frac{w^*(t_n, \Phi_n, \varphi_0, \nu_0) - w^*(t_0, \Phi_0, \varphi_0, \nu_0) - p_t(t - t_0)}{\|\Phi_n - \Phi_0\|_\infty + |t_n - t_0|}$$
$$- \frac{\int_{X\times Y} (\Phi_n - \Phi_0)(x, y) \cdot p_\Phi(x, y) \; d\mu_n(x) d\nu_0(y)}{\|\Phi_n - \Phi_0\|_\infty + |t_n - t_0|} \leq \delta.$$

- *The assumption $\partial_- w^*(t_0,\Phi_0,\varphi_0,\nu_0)=\{\mu_0\}$ can be seen as a local strict convexity of $w(t_0,\Phi_0,\cdot,\nu_0)$ at μ_0. This type of hypothesis also appears in the finite dimensional case (see [6]).*

The previous definition is unusual as one would expect to have the scalar product $-\int_{X\times Y}(\Phi-\Phi_0)(x,y)\cdot p_\Phi(x,y)\, d\mu_0(x)d\nu_0(y)$ instead of

$$\sup_{\mu\in\partial_- w^*(t,\Phi,\varphi_0,\nu_0)}\left\{-\int_{X\times Y}(\Phi-\Phi_0)(x,y)\cdot p_\Phi(x,y)d\mu(x)d\nu_0(y)\right\}$$

The following lemma shows that, at the limit, both coincide:

Lemma 12. *Consider $w:\ (t,\Phi,\mu,\nu)\in[0,T]\times\mathcal{C}(X,X)\times\mathcal{P}(X)\times\mathcal{P}(Y)\to\mathbb{R}$ a continuous application, Lipschitz in (t,Φ) uniformly in μ, ν. Take $\xi\in\mathcal{C}(X)$ and a sequence $(t_n,\Phi_n,\mu_n)\in[0,T]\times\mathcal{C}(X,X)\times\mathcal{P}(X)$ such that when $n\to+\infty$:*

$$t_n\to t_0,\quad \|\Phi_n-\Phi_0\|_\infty\to 0,\quad \mu_n\in\partial_- w^*(t_n,\Phi_n,\xi,\nu_0).$$

Moreover assume that $\{\mu_0\}=\partial^- w^*(t_0,\Phi_0,\xi,\nu_0)$. Then $W_2(\mu_n,\mu_0)\to 0$ when $n\to+\infty$.

Proof: First note that, due to the Lipschitz assumption on w, it exists $C>0$ such that for any $(s,t)\in[0,T]^2$, $\Phi,\Psi\in\mathcal{C}(X,X)$, we have:

$$\begin{aligned} w^*(t,\Phi,\xi,\nu_0) &:= \sup_{\mu\in\mathcal{P}(X)}\left\{\int_X\xi(x)d\mu(x)-w(t,\Phi,\mu,\nu_0)\right\}\\ &\le \sup_{\mu\in\mathcal{P}(X)}\left\{\int_X\xi(x)d\mu(x)-w(s,\Psi,\mu,\nu_0)\right\}+C(|t-s|+\|\Phi-\Psi\|_\infty)\\ &\le w^*(s,\Psi,\xi,\nu_0)+C(|t-s|+\|\Phi-\Psi\|_\infty). \end{aligned}$$

So that: $\lim_{k\to+\infty}w^*(t_{n_k},\Phi_{n_k},\xi,\nu_0)=w^*(t_0,\Phi_0,\xi,\nu_0)$. As $(\mu_n)_n$ is a sequence of probability measures on a compact set X, we can extract $(\mu_{n_k})_k$ converging to some $\mu\in\mathcal{P}(X)$. Then, using the continuity of w:

$$\begin{aligned} \int_X\xi d\mu = \lim_{k\to+\infty}\int_X\xi d\mu_{n_k} &= \lim_{k\to+\infty}w^*(t_{n_k},\Phi_{n_k},\xi,\nu_0)+w(t_{n_k},\Phi_{n_k},\mu_{n_k},\nu_0)\\ &= w^*(t_0,\Phi_0,\xi,\nu_0)+w(t_0,\Phi_0,\mu,\nu_0) \end{aligned}$$

which means $\mu_{n_k}\overset{*}{\rightharpoonup}\mu=\mu_0\in\partial_- w^*(t_0,\Phi_0,\xi,\nu_0)$. As this is true for any converging subsequence of $(\mu_n)_n$ we get $W_2(\mu_n,\mu_0)\to 0$.

QED

Hereafter, we give the appropriate definitions of solutions:

Definition 2. • *The functional $w:[0,T]\times\mathcal{C}(X\times Y,\mathbb{R}^d)\times\mathcal{P}(X)\times\mathcal{P}(Y)\mapsto\mathbb{R}$ is a* **viscosity dual subsolution** *to (3) iff it exists $C>0$ such that for all $\delta>0$ and all $(t_0,\Phi_0,\mu_0,\psi_0)\in[0,T[\times\mathcal{C}(X\times Y,\mathbb{R}^d)\times\mathcal{P}(X)\times\mathcal{C}(Y)$ and $(p_t,p_\Phi)\in D^-_\delta w^\sharp(t_0,\Phi_0,\mu_0,\psi_0)$ with $\partial^+ w^\sharp(t_0,\Phi_0,\mu_0,\psi_0)=\{\nu_0\}$, we have :*

$$p_t+\widehat{\mathcal{H}}(\mu_0,\nu_0,\Phi_0,p_\Phi)\le C\delta.$$

- *The functional* $w : [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{P}(X) \times \mathcal{P}(Y) \mapsto \mathbb{R}$ *is a* **viscosity dual supersolution** *to (3) iff it exists* $C > 0$ *such that for all* $\delta > 0$ *and all* $(t_0, \Phi_0, \varphi_0, \nu_0) \in [0,T[\times C(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X) \times \mathcal{P}(Y)$ *and* $(p_t, p_\Phi) \in D^+_\delta w^*(t_0, \Phi_0, \varphi_0, \nu_0)$ *with* $\partial^- w^*(t_0, \Phi_0, \varphi_0, \nu_0) = \{\mu_0\}$*, we have:*

$$p_t + \widehat{\mathcal{H}}(\mu_0, \nu_0, \Phi_0, p_\Phi) \geq -C\delta.$$

We state now a comparison principle for the Hamilton Jacobi Isaacs equation:

Theorem 2. *For* $i = 1, 2$*, let* $w_i : [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R}$ *be a continuous bounded maps when* $\mathcal{C}(X \times Y, \mathbb{R}^d)$ *is equipped with the infinity norm and both* $\mathcal{P}(X)$ *and* $\mathcal{P}(Y)$ *are equipped with the Kantorovich norm. The Hamiltonian* $\widehat{\mathcal{H}}$ *is supposed to be Lipschitz in* Φ *that is for all* $(\mu_0, \nu_0, p_\Phi) \in \mathcal{P}(X) \times \mathcal{P}(Y) \times \mathcal{C}(X \times Y, \mathbb{R}^d)$, (Φ_0, Φ_1) *in* $\mathcal{C}(X \times Y, \mathbb{R}^d)^2$ *:*

$$|\widehat{\mathcal{H}}(\mu_0, \nu_0, \Phi_0, p_\Phi) - \widehat{\mathcal{H}}(\mu_0, \nu_0, \Phi_1, p_\Phi)| \leq k \|p_\Phi\|_{L^2_{\mu_0 \times \mu_1}} \|\Phi_0 - \Phi_1\|_{L^2_{\mu_0 \times \mu_1}}. \tag{16}$$

Moreover, we assume that:

(H1) for any fixed $(\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(Y)$, $w_i(\cdot, \cdot, \nu, \mu)$ *is* k*-Lipschitz continuous with* $k > 0$ *i.e. for all* (Φ, Ψ) *in* $\mathcal{C}(X \times Y, \mathbb{R}^d)^2$*:*

$$|w_i(t, \Phi, \mu, \nu) - w_i(s, \Psi, \mu, \nu)| \leq k \left(|s - t| + \|\Phi - \Psi\|_{L^2_{\mu \times \nu}} \right).$$

(H2) the map w_i *is convex in the* μ*-variable, concave in the* ν*-variable. Moreover, for all* $(t_0, \Phi_0, \mu_0, \nu_0) \in [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{P}(X) \times \mathcal{P}(Y)$*, it exists* $\xi_0 \in \partial^- w_i(t_0, \Phi_0, \mu_0, \nu_0)$ *and* $\zeta_0 \in \partial^+ w_i(t_0, \Phi_0, \mu_0, \nu_0)$ *i.e. such that:*

$$w_i^*(t_0, \Phi_0, \xi_0, \nu_0) + w_i(t_0, \Phi_0, \mu_0, \nu_0) = \int_X \xi_0(x)\ d\mu_0(x),$$

$$w_i^\sharp(t_0, \Phi_0, \mu_0, \zeta_0) + w_i(t_0, \Phi_0, \mu_0, \nu_0) = \int_Y \zeta_0(y)\ d\nu_0(y).$$

(H3) w_1 *is a dual subsolution of (3) and* w_2 *is a dual supersolution of (3);*

(H4) the following equality holds: $w_1(T, \cdot, \cdot, \cdot) \leq w_2(T, \cdot, \cdot, \cdot)$.

Then for all $(t, \Phi, \mu, \nu) \in [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{P}(X) \times \mathcal{P}(Y)$*:*

$$w_1(t, \Phi, \mu, \nu) \leq w_2(t, \Phi, \mu, \nu).$$

As we need some bounded distance on $\mathcal{C}(X \times Y, \mathbb{R}^d)$, we set:

$$d_1(\Phi, \Psi) := \min\{\|\Phi - \Psi\|_\infty, 1\} \text{ for any } \Phi, \Psi \in \mathcal{C}(X \times Y, \mathbb{R}^d).$$

Note that d_1 metrizes the uniform topology.

Proof of the theorem: By contradiction, assume it exists some $\alpha > 0$ and $(t_0, \Phi_0, \mu_0, \nu_0)$ such that:

$$(w_2 - w_1)(t_0, \Phi_0, \mu_0, \nu_0) \le -\frac{\alpha}{2}. \tag{17}$$

Denote by C the constant appearing in the definition of the dual subsolution and supersolution. We choose some $\eta > 0$ small enough such that

$$T\eta < \frac{\alpha}{4}, \tag{18}$$

and take $\varepsilon \in]0,1[$ that satisfies:

$$2\varepsilon(k(k+1)^2 + C) < \eta, \quad \varepsilon(k(k+2) + 2c^2) < \frac{\alpha}{4} \tag{19}$$

where

$$c = \max\{\max_{x_0 \in X, x_1 \in [0,1]^N} |x_0 - x_1|, \max_{y_0 \in Y, y_1 \in [0,1]^M} |y_0 - y_1|\}. \tag{20}$$

We introduce the following functional defined for all $(s,t) \in [0,T]^2$, $(\Phi, \Psi) \in \mathcal{C}(X \times Y, \mathbb{R}^d)^2$, $(\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(Y)$:

$$\theta(t,s,\Phi,\Psi,\mu,\nu) := w_2(s,\Psi,\mu,\nu) - w_1(t,\Phi,\mu,\nu) - F(\mu) - G(\nu) + \frac{1}{\varepsilon}\left(\|\Psi - \Phi\|^2_{L^2_{\mu\times\nu}} + |t-s|^2\right) - \eta s$$

where F and G are defined by:

$$F(\mu) = \varepsilon W_2^2(\mathcal{L}^N_{[0,1]^N}, \mu), \quad G(\nu) = \varepsilon W_2^2(\mathcal{L}^M_{[0,1]^M}, \nu)$$

where $\mathcal{L}^N_{[0,1]^N}$ and $\mathcal{L}^M_{[0,1]^M}$ denote the Lebesgue measures on $\mathbb{R}^N$ and on $\mathbb{R}^M$ restricted to $[0,1]^N$ and $[0,1]^M$. Note that both F and G are bounded by $c^2\varepsilon$.

Set for any $(\Phi, \Psi) \in \mathcal{C}(X \times Y, \mathbb{R}^d)$:

$$\Theta(\Phi, \Psi) := \inf_{(t,s)\in[0,T]^2, (\mu,\nu)\in\mathcal{P}(X)\times\mathcal{P}(Y)} \theta(t,s,\Phi,\Psi,\mu,\nu),$$

note that the infimum is actually a minimum since $[0,T]^2 \times \mathcal{P}(X) \times \mathcal{P}(Y)$ is a compact set and θ is continuous.
As Θ is lower-semicontinuous and $(\mathcal{C}(X \times Y, \mathbb{R}^d), d_1)$ is a complete metric space, from Ekeland's variationnal principle (see [8]), it exists some $(\bar{\Phi}, \bar{\Psi}) \in \mathcal{C}(X \times Y, \mathbb{R}^d)^2$ such that

- $\Theta(\bar{\Phi}, \bar{\Psi}) \le \Theta(\Phi_0, \Phi_0)$ where Φ_0 is the function appearing in (17),
- $\forall (\Phi, \Psi) \in \mathcal{C}(X \times Y, \mathbb{R}^d)^2$:

$$\Theta(\bar{\Phi}, \bar{\Psi}) \le \Theta(\Phi, \Psi) + \varepsilon\left(d_1(\Phi, \bar{\Phi}) + d_1(\Psi, \bar{\Psi})\right).$$

Then, taking $(\bar{t}, \bar{s}, \bar{\mu}, \bar{\nu}) \in [0,T]^2 \times \mathcal{P}(X) \times \mathcal{P}(Y)$ such that

$$\Theta(\bar{\Phi}, \bar{\Psi}) = \theta(\bar{t}, \bar{s}, \bar{\Phi}, \bar{\Psi}, \bar{\mu}, \bar{\nu})$$

we get:

(E1) $\theta(\bar{t}, \bar{s}, \bar{\Phi}, \bar{\Psi}, \bar{\mu}, \bar{\nu}) \le \theta(t_0, t_0, \Phi_0, \Phi_0, \mu_0, \nu_0)$,

(E2) $\forall (t, s, \Phi, \Psi, \mu, \nu) \in [0, T]^2 \times \mathcal{C}(X \times Y, \mathbb{R}^d)^2 \times \mathcal{P}(X) \times \mathcal{P}(Y)$:

$$\theta(\bar{t}, \bar{s}, \bar{\Phi}, \bar{\Psi}, \bar{\mu}, \bar{\nu}) \le \theta(t, s, \Phi, \Psi, \mu, \nu) + \varepsilon \left(d_1(\Phi, \bar{\Phi}) + d_1(\Psi, \bar{\Psi}) \right).$$

Step 1 : We prove some estimates on $|\bar{t} - \bar{s}|$ and $\|\bar{\Phi} - \bar{\Psi}\|_{L^2_{\bar{\mu}\times\bar{\nu}}}$.

- Applying (E2) with $(t, s, \Phi, \Psi, \mu, \nu) = (\bar{t}, \bar{s}, \bar{\Psi}, \bar{\Psi}, \bar{\mu}, \bar{\nu})$, we get:

$$-w_1(\bar{t}, \bar{\Phi}, \bar{\mu}, \bar{\nu}) + \frac{1}{\varepsilon}\|\bar{\Psi} - \bar{\Phi}\|^2_{L^2_{\bar{\mu}\times\bar{\nu}}} \le -w_1(\bar{t}, \bar{\Psi}, \bar{\mu}, \bar{\nu}) + \varepsilon d_1(\bar{\Psi}, \bar{\Phi}).$$

Then, using the Lipschitz property of w_1 and the definition of d_1:

$$\frac{1}{\varepsilon}\|\bar{\Psi} - \bar{\Phi}\|^2_{L^2_{\bar{\mu}\times\bar{\nu}}} \le w_1(\bar{t}, \bar{\Phi}, \bar{\mu}, \bar{\nu}) - w_1(\bar{t}, \bar{\Psi}, \bar{\mu}, \bar{\nu}) + \varepsilon \le k\|\bar{\Psi} - \bar{\Phi}\|_{L^2_{\bar{\mu}\times\bar{\nu}}} + \varepsilon.$$

So that $\rho := \|\bar{\Psi} - \bar{\Phi}\|_{L^2_{\bar{\mu}\times\bar{\nu}}}$ satisfies:

$$\frac{1}{\varepsilon}\rho^2 - k\rho - \varepsilon \le 0$$

and $\rho \le \frac{\varepsilon}{2}(k + \sqrt{k^2 + 4}) \le \varepsilon(k + 1)$. Finally, we get the estimation:

$$\|\bar{\Psi} - \bar{\Phi}\|_{L^2_{\bar{\mu}\times\bar{\nu}}} \le \varepsilon(k + 1). \tag{21}$$

- Now applying (E2) with $(t, s, \Phi, \Psi, \mu, \nu) = (\bar{s}, \bar{s}, \bar{\Phi}, \bar{\Psi}, \bar{\mu}, \bar{\nu})$, we get:

$$-w_1(\bar{t}, \bar{\Phi}, \bar{\mu}, \bar{\nu}) + \frac{1}{\varepsilon}|\bar{t} - \bar{s}|^2 \le -w_1(\bar{s}, \bar{\Phi}, \bar{\mu}, \bar{\nu}).$$

The using the Lipschitz property of w_1 given by $(H1)$, we have:

$$\frac{1}{\varepsilon}|\bar{t} - \bar{s}|^2 \le w_1(\bar{t}, \bar{\Phi}, \bar{\mu}, \bar{\nu}) - w_1(\bar{s}, \bar{\Phi}, \bar{\mu}, \bar{\nu}) \le k|t - s|.$$

Finally, we get:

$$|\ \bar{t} - \bar{s}\ | \le k\varepsilon. \tag{22}$$

Step 2 : Assume $\bar{s}, \bar{t} \in [0, T[$ and get a contradiction.

- We build some ξ such that $\partial^- w_2^*(\bar{s}, \bar{\Psi}, \xi, \bar{\nu}) = \{\bar{\mu}\}$.
We apply again $(E2)$ with $(t, s, \Phi, \Psi, \mu, \nu) = (\bar{t}, \bar{s}, \bar{\Phi}, \bar{\Psi}, \mu, \bar{\nu})$, we get:

$$w_2(\bar{s}, \bar{\Psi}, \bar{\mu}, \bar{\nu}) - w_1(\bar{t}, \bar{\Phi}, \bar{\mu}, \bar{\nu}) - F(\bar{\mu}) + \frac{1}{\varepsilon}\int |\bar{\Phi} - \bar{\Psi}|^2 d(\bar{\mu} \times \bar{\nu})$$

$$\le w_2(\bar{s}, \bar{\Psi}, \mu, \bar{\nu}) - w_1(\bar{t}, \bar{\Phi}, \mu, \bar{\nu}) - F(\mu) + \frac{1}{\varepsilon}\int |\bar{\Phi} - \bar{\Psi}|^2 d(\mu \times \bar{\nu})$$

which rewrites as

$$(23)\qquad \begin{aligned}&[w_1(\bar{t},\bar{\Phi},\mu,\bar{\nu}) + F(\mu) - \frac{1}{\varepsilon}\int |\bar{\Phi}-\bar{\Psi}|^2 d(\mu\times\bar{\nu})]\\ &-[w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu}) + F(\bar{\mu}) - \frac{1}{\varepsilon}\int |\bar{\Phi}-\bar{\Psi}|^2 d(\bar{\mu}\times\bar{\nu})]\\ &\leq w_2(\bar{s},\bar{\Psi},\mu,\bar{\nu}) - w_2(\bar{s},\bar{\Psi},\bar{\mu},\bar{\nu}).\end{aligned}$$

We introduce the following map:

$$(24)\qquad \mathcal{F}(\mu) = [w_1(\bar{t},\bar{\Phi},\mu,\bar{\nu}) + F(\mu) - \frac{1}{\varepsilon}\int |\bar{\Phi}-\bar{\Psi}|^2 d(\mu\times\bar{\nu})].$$

By (H2), we know $\partial^- w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu}) \neq \emptyset$, moreover, by definition of F, we have also $\partial^- F(\bar{\mu}) \neq \emptyset$ (see for instance [17]). We then can choose $\xi \in \partial^-\mathcal{F}(\bar{\mu})$. As F is strictly convex (see again [17]), w_1 is convex and $\mu \mapsto -\frac{1}{\varepsilon}\int |\bar{\Phi}-\bar{\Psi}|^2 d(\mu\times\bar{\nu})$ being linear, $\mathcal{F}$ is strictly convex so we have:

$$(25)\qquad \partial^-\mathcal{F}^*(\xi) = \{\bar{\mu}\}.$$

By (23), we have $\xi \in \partial_- w_2(\bar{s},\bar{\Psi},\bar{\mu},\bar{\nu})$ and in a symmetric way $\bar{\mu} \in \partial_-(w_2^*)(\bar{s},\bar{\Psi},\xi,\bar{\nu})$. We are going to prove that $\bar{\mu}$ is indeed the unique element of $\partial_-(w_2^*)(\bar{s},\bar{\Psi},\xi,\bar{\nu})$. Indeed assume $\hat{\mu} \in \partial_-(w_2^*)(\bar{s},\bar{\Psi},\xi,\bar{\nu})$ and $\hat{\mu} \neq \bar{\mu}$, then, by (23) we have:

$$\int \xi d(\hat{\mu}-\bar{\mu}) \geq w_2(\bar{s},\bar{\Psi},\hat{\mu},\bar{\nu}) - w_2(\bar{s},\bar{\Psi},\bar{\mu},\bar{\nu}) \geq \mathcal{F}(\hat{\mu}) - \mathcal{F}(\bar{\mu}).$$

But, as $\xi \in \partial_-\mathcal{F}(\bar{\mu})$ we also have

$$\int \xi d(\hat{\mu}-\bar{\mu}) \leq \mathcal{F}(\hat{\mu}) - \mathcal{F}(\bar{\mu}).$$

From these two inequalities we deduce:

$$\mathcal{F}^*(\xi) = \int \xi d\bar{\mu} - \mathcal{F}(\bar{\mu}) = \int \xi d\hat{\mu} - \mathcal{F}(\hat{\mu})$$

that is $\hat{\mu} \in \partial_-\mathcal{F}^*(\xi)$ which is in contradiction with (25). So we can conclude:

$$(26)\qquad \partial^- w_2^*(\bar{s},\bar{\Psi},\xi,\bar{\nu}) = \{\bar{\mu}\}, \quad \xi \in \partial^-\mathcal{F}(\bar{\mu}).$$

- In the same way, we can build some $\zeta \in \mathcal{C}(Y)$ such that:

$$(27)\qquad \partial^+ w_1^\sharp(\bar{s},\bar{\Psi},\bar{\mu},\zeta) = \{\bar{\nu}\}, \quad \zeta \in \partial^+\mathcal{G}(\bar{\nu})$$

with $\mathcal{G}$ is the strictly concave functional defined by:

$$\mathcal{G}(\nu) := [w_2(\bar{s},\bar{\Psi},\bar{\mu},\nu) - G(\nu) + \frac{1}{\varepsilon}\int |\bar{\Phi}-\bar{\Psi}|^2\ d(\bar{\mu}\times\nu)].$$

- We now show that $\frac{2}{\varepsilon}(\bar{s}-\bar{t},\bar{\Psi}-\bar{\Phi}) \in D^-_\varepsilon w_1^\sharp(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu})$. We apply $(E2)$ with $(t,s,\Phi,\Psi,\mu,\nu) = (t,\bar{s},\Phi,\bar{\Psi},\bar{\mu},\nu)$ and get:

$$\theta(\bar{t},\bar{s},\bar{\Phi},\bar{\Psi},\bar{\mu},\bar{\nu}) \leq \theta(t,\bar{s},\Phi,\bar{\Psi},\bar{\mu},\nu) + \varepsilon d_1(\Psi,\bar{\Psi}).$$

So that:

$$w_2(\bar{s},\bar{\Psi},\bar{\mu},\bar{\nu}) - w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu}) - G(\bar{\nu}) + \frac{1}{\varepsilon}\left(\int |\bar{\Psi}-\bar{\Phi}|^2 d(\bar{\mu}\times\bar{\nu}) + |\bar{t}-\bar{s}|^2\right)$$

$$\leq w_2(\bar{s},\bar{\Psi},\bar{\mu},\nu) - w_1(t,\Phi,\bar{\mu},\nu) - G(\nu) + \frac{1}{\varepsilon}\left(\int |\bar{\Psi}-\Phi|^2 d(\bar{\mu}\times\nu) + |t-\bar{s}|^2\right) + \varepsilon d_1(\Phi,\bar{\Phi}).$$

Note that:

$$\int |\bar{\Psi}-\Phi|^2 d(\bar{\mu}\times\nu) - \int |\bar{\Psi}-\bar{\Phi}|^2 d(\bar{\mu}\times\bar{\nu})$$

$$= \int |\bar{\Psi}-\bar{\Phi}|^2 d\bar{\mu}\ d(\nu-\bar{\nu}) + 2\int (\bar{\Phi}-\bar{\Psi})\cdot(\Phi-\bar{\Phi})\ d(\bar{\mu}\times\nu) + \int |\Phi-\bar{\Phi}|^2\ d(\bar{\mu}\times\nu)$$

and

$$|t-\bar{s}|^2 - |\bar{t}-\bar{s}|^2 = 2(t-\bar{t})(\bar{t}-\bar{s}) + |t-\bar{t}|^2.$$

So we get:

$$\begin{aligned} &w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu}) - w_1(t,\Phi,\bar{\mu},\nu) + \mathcal{G}(\nu) - \mathcal{G}(\bar{\nu}) \\ &+\frac{2}{\varepsilon}\int (\bar{\Phi}-\bar{\Psi})\cdot(\Phi-\bar{\Phi})\ d(\bar{\mu}\times\nu) + \frac{2}{\varepsilon}(t-\bar{t})(\bar{t}-\bar{s}) \\ &+\varepsilon d_1(\Phi,\bar{\Phi}) + \frac{1}{\varepsilon}\left(\int |\Phi-\bar{\Phi}|^2\ d(\bar{\mu}\times\nu) + |t-\bar{t}|^2\right) \geq 0. \end{aligned}$$

Then, as $\zeta \in \partial^+\mathcal{G}(\bar{\nu})$:

$$\begin{aligned} &w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu}) - w_1(t,\Phi,\bar{\mu},\nu) + \int \zeta d(\nu-\bar{\nu}) \\ &-\frac{2}{\varepsilon}\int (\bar{\Psi}-\bar{\Phi})\cdot(\Phi-\bar{\Phi})\ d(\bar{\mu}\times\nu) - \frac{2}{\varepsilon}(t-\bar{t})(\bar{s}-\bar{t}) \\ &+\varepsilon\|\Phi-\bar{\Phi}\|_\infty + \frac{1}{\varepsilon}(\|\Phi-\bar{\Phi}\|_\infty + |t-\bar{t}|)^2 \geq 0, \end{aligned}$$

as this inequality is true for any $\nu \in \partial^+ w_1^\sharp(t,\Phi,\bar{\mu},\zeta)$ and $\zeta \in \partial^+ w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu})$:

$$\begin{aligned} &w_1^\sharp(t,\Phi,\bar{\mu},\zeta) - w_1^\sharp(\bar{t},\bar{\Phi},\bar{\mu},\zeta) - \frac{2}{\varepsilon}(t-\bar{t})(\bar{s}-\bar{t}) \\ &+ \inf_{\nu\in\partial^+ w_1^\sharp(t,\Phi,\bar{\mu},\zeta)} \left\{-\frac{2}{\varepsilon}\int (\bar{\Psi}-\bar{\Phi})\cdot(\Phi-\bar{\Phi})\ d\bar{\mu}\ d\nu\right\} \\ &+\varepsilon(\|\Phi-\bar{\Phi}\|_\infty + |t-\bar{t}|) + o(\|\Phi-\bar{\Phi}\|_\infty + |t-\bar{t}|) \geq 0. \end{aligned}$$

By definition, as (27) holds, this is $\frac{2}{\varepsilon}(\bar{s}-\bar{t},\bar{\Psi}-\bar{\Phi}) \in D^-_\varepsilon w_1^\sharp(\bar{t},\bar{\Phi},\bar{\mu},\zeta)$. Then as w_1 is a viscosity dual subsolution to (3), we get:

$$\frac{2}{\varepsilon}(\bar{s}-\bar{t}) + \widehat{\mathcal{H}}(\bar{\mu},\bar{\nu},\bar{\Phi},\frac{2}{\varepsilon}(\bar{\Psi}-\bar{\Phi})) \leq C\varepsilon. \tag{28}$$

- In the same way, we can prove that $(\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta, \frac{2}{\varepsilon}(\bar{\Psi}-\bar{\Phi})) \in D_\varepsilon^+ w_2^*(\bar{s},\bar{\Psi},\xi,\bar{\nu})$. As w_2 is a dual supersolution to (3), we get:

$$\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta+\widehat{\mathcal{H}}(\bar{\mu},\bar{\nu},\bar{\Psi},\frac{2}{\varepsilon}(\bar{\Psi}-\bar{\Phi})) \geq -C\varepsilon. \tag{29}$$

Then, (31) and (32) give:

$$-2C\varepsilon \leq -\eta+\widehat{\mathcal{H}}(\bar{\mu},\bar{\nu},\bar{\Psi},\frac{2}{\varepsilon}(\bar{\Psi}-\bar{\Phi}))-\widehat{\mathcal{H}}(\bar{\mu},\bar{\nu},\bar{\Phi},\frac{2}{\varepsilon}(\bar{\Psi}-\bar{\Phi}))$$

and by (16), this implies:

$$-2C\varepsilon-\frac{2k}{\varepsilon}\|\bar{\Psi}-\bar{\Phi}\|^2_{L^2_{\bar{\mu}\times\bar{\nu}}} \leq -\eta$$

and by estimation (21) we get:

$$-2C\varepsilon-2k\varepsilon(k+1)^2 \leq \eta.$$

This last inequality is in contradiction with (19).

Step 3 : Let us now prove that $\bar{s}$, $\bar{t}$ are different from T. Indeed, assume $\bar{s}=T$ then by (E1) and (17):

$$\begin{aligned}
\theta(\bar{t},T,\bar{\Phi},\bar{\Psi},\bar{\mu},\bar{\nu}) &\leq \theta(t_0,t_0,\Phi_0,\Phi_0,\mu_0,\nu_0)\\
&:= w_2(t_0,\Phi_0,\mu_0,\nu_0)-w_1(t_0,\Phi_0,\mu_0,\nu_0)-F(\mu_0)-G(\nu_0)-\eta t_0\\
&\leq w_2(t_0,\Phi_0,\mu_0,\nu_0)-w_1(t_0,\Phi_0,\mu_0,\nu_0) \leq \frac{-\alpha}{2}.
\end{aligned}$$

Which rewrites as:

$$w_2(T,\bar{\Psi},\bar{\mu},\bar{\nu})-w_1(\bar{t},\bar{\Phi},\bar{\mu},\bar{\nu})-F(\bar{\mu})-G(\bar{\nu})+\frac{1}{\varepsilon}\left(\|\bar{\Phi}-\bar{\Psi}\|^2_{L^2_{\bar{\mu}\times\bar{\nu}}}+|\bar{t}-T|^2\right)-\eta T \leq \frac{-\alpha}{2},$$

then by (H1), as $\left(\|\bar{\Phi}-\bar{\Psi}\|_{L^2_{\bar{\mu}\times\bar{\nu}}}+|\bar{t}-T|^2\right) \geq 0$, we have:

$$w_2(T,\bar{\Psi},\bar{\mu},\bar{\nu})-w_1(T,\bar{\Psi},\bar{\mu},\bar{\nu})-k\left(\|\bar{\Phi}-\bar{\Psi}\|_{L^2_{\bar{\mu}\times\bar{\nu}}}+|\bar{t}-T|\right)-F(\bar{\mu})-G(\bar{\nu})-\eta T \leq \frac{-\alpha}{2}.$$

Recall that, by (20) F and G are bounded by $c\varepsilon$. By use of (H4), (21) and (22), we get:

$$-2\varepsilon c^2-k(\varepsilon(k+1)+k\varepsilon)-\eta T \leq \frac{-\alpha}{2}$$

which, by (19) and (18) rewrites as

$$\frac{\alpha}{2} \leq \varepsilon(k(k+2)+2c^2)+\eta T < \frac{\alpha}{2}$$

and we get a contradiction.

QED

6 Characterization of the value

Theorem 3. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be two compact sets. The value $\mathcal{V}_r := \mathcal{V}_r^+ = \mathcal{V}_r^-$ is the unique bounded continuous functional from $[0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X \times Y, \mathbb{R}^+) \times \mathcal{P}(X) \times \mathcal{P}(Y)$ to $\mathbb{R}$ satisfying the following properties:*

(i) $\mathcal{V}_r$ is Lipschitz in (t, Φ), convex in μ, concave in ν, moreover for all $(t, \Phi, h, \mu, \nu) \in [0,T] \times \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X \times Y, \mathbb{R}^+) \times \mathcal{P}(X) \times \mathcal{P}(Y)$, we have:

$$\partial^+ \mathcal{V}_r(t, \Phi, h, \mu, \nu) \neq \emptyset, \quad \partial^- \mathcal{V}_r(t, \Phi, h, \mu, \nu) \neq \emptyset.$$

(ii) $\mathcal{V}_r$ is a dual subsolution and a dual supersolution of the following Hamilton-Jacobi-Isaac equation:

$$\partial_t W(t, \Phi_0, \mu_0, \nu_0) + \mathcal{H}(\mu_0, \nu_0, \Phi_0, D_\Phi W) = 0.$$

with

$$\mathcal{H}(\mu, \nu, \Phi_0, p_\Phi) := \inf_{u \in U} \sup_{v \in V} \int_{X \times Y} f(\Phi_0(x,y), u, v) \cdot p_\Phi(x,y) d\mu(x) d\nu(y)$$

$$= \sup_{v \in V} \inf_{u \in U} \int_{X \times Y} f(\Phi_0(x,y), u, v) \cdot p_\Phi(x,y) d\mu(x) d\nu(y),$$

(iii) for all $(\Phi_0, h, \mu_0, \nu_0) \in \mathcal{C}(X \times Y, \mathbb{R}^d) \times \mathcal{C}(X \times Y, \mathbb{R}^+) \times \mathcal{P}(X) \times \mathcal{P}(Y)$:

$$\mathcal{V}_r(T, \Phi_0, h, \mu_0, \nu_0) = \int_{X \times Y} g\left(\Phi_0(x,y)\right) h(x,y) \; d\mu_0(x) d\nu_0(y).$$

To prove $\mathcal{V}_r^-$ is a dual supersolution of (3), we need the following lemma:

Lemma 13. *Let $\Phi \in \mathcal{C}(X \times Y, \mathbb{R}^d)$ and $p_\Phi \in \mathcal{C}(X \times Y, \mathbb{R}^d)$. We consider the following application:*

$$(u, v, \mu, \nu) \in U \times V \times \mathcal{P}(X) \times \mathcal{P}(Y) \mapsto \theta(u, v, \mu, \nu) = \int_{X \times Y} f(\Phi(x,y), u, v) \cdot p_\Phi(x,y) \; d\mu(x) d\nu(y).$$

Then θ is continuous in the $\mu \times \nu$-variable uniformly in (u, v, μ, ν), more precisely, for all $\varepsilon > 0$ it exists $C(f, p_\phi)$ and $C(\varepsilon)$ such that for all $(u, v, \mu_0, \mu_1, \nu_0, \nu_1) \in U \times V \times \mathcal{P}(X)^2 \times \mathcal{P}(Y)^2)$, it holds:

$$|\theta(u, v, \mu_0, \nu_0) - \theta(u, v, \mu_1, \nu_1)| \leq C(f, p_\Phi)[C(\varepsilon)(W_2(\mu_0, \nu_0) + W_2(\mu_1, \nu_1)) + \varepsilon].$$

Proof: Let (x_0, y_0), $(x_1, y_1) \in X \times Y$, then for any $u, v \in U \times V$:

$$|f(\Phi(x_0, y_0), u, v) \cdot p_\Phi(x_0, y_0) - f(\Phi(x_1, y_1), u, v) \cdot p_\Phi(x_1, y_1)|$$

$$\leq |(f(\Phi(x_0, y_0), u, v) - f(\Phi(x_1, y_1), u, v)) \cdot p_\Phi(x_0, y_0)| + |f(\Phi(x_1, y_1), u, v) \cdot (p_\Phi(x_0, y_0) - p_\Phi(x_1, y_1))|$$

$$\leq Lip(f) \times \|p_\Phi\|_\infty |\Phi(x_0, y_0) - \Phi(x_1, y_1)| + \|f\|_\infty |p_\Phi(x_0, y_0) - p_\Phi(x_1, y_1)|.$$

Choosing $p_\varepsilon,\ \Phi_\varepsilon \in Lip(X \times Y, \mathbb{R}^d)$ such that

$$\|p_\Phi - p_\varepsilon\| \leq \frac{\varepsilon}{2}, \quad \|\Phi - \Phi_\varepsilon\| \leq \frac{\varepsilon}{2}$$

leads to:

$$|f(\Phi(x_0,y_0),u,v)\cdot p_\Phi(x_0,y_0) - f(\Phi(x_1,y_1),u,v)\cdot p_\Phi(x_1,y_1)|$$
$$\leq Lip(f)\times\|p_\Phi\|_\infty(\varepsilon + Lip(\Phi_\varepsilon)|(x_0,y_0)-(x_1,y_1)|) + \|f\|_\infty(\varepsilon + Lip(p_\varepsilon)|(x_0,y_0)-(x_1,y_1)|).$$

The sequel is straightforward

QED

Proof of the Theorem: The equality of the upper and lower value has already been stated in Theorem 1.
Property (iii) is straightforward, the convexity/concavity is classic (Lemma 7), see also Lemma 8.
Let us focus on property (ii), we show that $\mathcal{V}_r$ is a dual supersolution (the other part being very similar). Let $(t_0,\Phi_0,\mu_0,\nu_0,\varphi) \in]0,T[\times\mathcal{C}(X\times Y,\mathbb{R}^d)\times\mathcal{P}(X)\times\mathcal{P}(Y)\times\mathcal{C}(X)$ and $(p_t,p_\Phi) \in D^+_\delta(\mathcal{V}_r^-)^*(t_0,\Phi_0,\varphi,\nu_0)$, with $\partial^-(\mathcal{V}_r^-)^*(t_0,\Phi_0,\varphi,\nu_0) = \{\mu_0\}$.
Take $t \in]t_0,T[$, for any $v \in V$, $u \in \mathcal{U}(t_0)$, by Lemma 11, it exists some

$$\mu_t^{u,v} \in \partial^-(\mathcal{V}_r^-)^*(t, X_t^{t_0,\Phi_0(\cdot),u,v},\varphi,\nu_0).$$

Then, we have:

$$\begin{aligned}
& (\mathcal{V}_r^-)^*(t_0,\Phi_0,\varphi,\nu_0) - (\mathcal{V}_r^-)^*(t,X_t^{t_0,\Phi_0(\cdot),u,v},\varphi,\nu_0) + p_t(t-t_0) + \int (X_t^{t_0,\Phi_0,u,v} - \Phi_0)\cdot p_\Phi \, d\mu_t^{u,v} d\nu_0 \\
\geq\ & -\left(\|X_t^{t_0,\Phi_0(\cdot),u,v} - \Phi_0\|_\infty + |t-t_0|\right)\left(\delta + \varepsilon\left(\|X_t^{t_0,\Phi_0(\cdot),u,v} - \Phi_0\|_\infty + |t-t_0|\right)\right)
\end{aligned}$$

where $\varepsilon(t) \to 0$ when $t \to 0$.
As we have $X_t^{t_0,\Phi_0(x,y),u,v} = \Phi_0(x,y) + \int_{t_0}^t f(X_s^{t_0,\Phi_0(x,y),u,v}, u(s), v)\ ds$, the previous expression rewrites as:

$$\begin{aligned}
& p_t(t-t_0) + \int_{X\times Y}\int_{t_0}^t f(X_s^{t_0,\Phi_0,u,v},u(s),v)\cdot p_\Phi \ ds d\mu_t^{u,v} d\nu_0 \\
\geq\ & -(\mathcal{V}_r^-)^*(t_0,\Phi_0,\varphi,\nu_0) + (\mathcal{V}_r^-)^*(t,X_t^{t_0,\Phi_0(\cdot),u,v},\varphi,\nu_0) \\
& \quad -\left(\|X_t^{t_0,\Phi_0(\cdot),u,v} - \Phi_0\|_\infty + |t-t_0|\right)\left(\delta + \varepsilon\left(\|X_t^{t_0,\Phi_0(\cdot),u,v} - \Phi_0\|_\infty + |t-t_0|\right)\right)
\end{aligned}$$

Once again we have

$$\|X_t^{t_0,\Phi_0(\cdot),u,v} - \Phi_0\|_\infty \leq C|t-t_0|. \tag{30}$$

Hence

$$\begin{aligned}
& p_t(t-t_0) + \int_{X\times Y}\int_{t_0}^t f(X_s^{t_0,\Phi_0,u,v},u(s),v)\cdot p_\Phi \ ds d\mu_t^{u,v} d\nu_0 \\
\geq\ & -(\mathcal{V}_r^-)^*(t_0,\Phi_0,\varphi,\nu_0) + (\mathcal{V}_r^-)^*(t,X_t^{t_0,\Phi_0(\cdot),u,v},\varphi,\nu_0) - (C+1)|t-t_0|\left(\delta + \varepsilon\left((C+1)|t-t_0|\right)\right).
\end{aligned}$$

Taking the supremum in u and the infimum in v, because by Proposition 4, $\mathcal{V}_r^-$ satisfy a dual subdynamic principle, we deduce that

$$\begin{aligned}
& p_t(t-t_0) + \inf_{v\in V}\sup_{u\in\mathcal{U}(t_0)} \int_{X\times Y}\int_{t_0}^{t} f(X_s^{t_0,\Phi_0,u,v},\alpha(v(s)),v(s))\cdot p_\Phi \; ds d\mu_t^{u,v} d\nu_0 \\
\geq\ & -(C+1)|t-t_0|\left(\delta+\varepsilon\left((C+1)|t-t_0|\right)\right).
\end{aligned}$$

Since f is bounded and Lipschitz and $X\times Y$ is compact, there exists a constant - denoted again by C - such that:

$$\begin{aligned}
& \int_{t_0}^{t}\left[\int_{X\times Y} f(X_s^{t_0,\Phi_0,u,v},u(s),v)\cdot p_\Phi \; d\mu_t^{u,v}d\nu_0\right]\; ds \\
\leq\ & \left[\sup_{u\in U}\int_{X\times Y} f(\Phi_0,u,v)\cdot p_\Phi \; d\mu_t^{u,v}d\nu_0\right]\; ds + C\int_{t_0}^{t}|s-t_0|\; ds \\
=\ & (t-t_0)\left(\sup_{u\in U}\int_{X\times Y} f(\Phi_0,u,v)\cdot p_\Phi \; d\mu_t^{u,v}d\nu_0 + C\frac{|t-t_0|}{2}\right).
\end{aligned}$$

So we get:

$$\begin{aligned}
& p_t(t-t_0) + (t-t_0)\inf_{v\in V}\sup_{u\in U}\left(\int_X f(\Phi_0(x),u,v)\cdot p_\Phi(x)\; d\mu_t^{u,v}(x) + C\frac{|t-t_0|}{2}\right) \\
\geq\ & -(C+1)|t-t_0|\left(\delta+\varepsilon\left((C+1)|t-t_0|\right)\right).
\end{aligned}$$

And, dividing by $(t-t_0)$, we get:

(31)

$$C\frac{|t-t_0|}{2} + p_t + \inf_{v\in V}\sup_{u\in U}\int_{X\times Y} f(\Phi_0,u,v)\cdot p_\Phi \; d\mu_t^{u,v}d\nu_0 \geq -(C+1)\left(\delta+\varepsilon\left((C+1)|t-t_0|\right)\right).$$

Let $\varepsilon_1,\varepsilon_2>0$, for all $t>0$, it exists $u_t^{\varepsilon_1}\in U$, $v_t^{\varepsilon_1}\in V$ such that:

$$\begin{aligned}
& \inf_{v\in V}\sup_{u\in U}\int_{X\times Y} f(\Phi_0,u,v)\cdot p_\Phi \; d\mu_t^{u,v}d\nu_0 - \inf_{v\in V}\sup_{u\in U}\int_{X\times Y} f(\Phi_0,u,v)\cdot p_\Phi \; d\mu_0 d\nu_0 \\
& \leq \int_{X\times Y} f(\Phi_0,u_t^{\varepsilon_1},v_t^{\varepsilon_1})\cdot p_\Phi \; d\mu_t^{u_t^{\varepsilon_1},v_t^{\varepsilon_2}}d\nu_0 - \int_{X\times Y} f(\Phi_0,u_t^{\varepsilon_1},v_t^{\varepsilon_1})\cdot p_\Phi \; d\mu_0 d\nu_0 + \varepsilon_1 \\
& \qquad \leq C(f,p_\Phi)(C(\varepsilon_2)W_2(\mu_t^{u_t^{\varepsilon_1},v_t^{\varepsilon_1}},\mu_0)+\varepsilon_1+\varepsilon_2),
\end{aligned}$$

the last inequality comes from the result of Lemma 13. Then (31) yields:

$$\begin{aligned}
(32)\qquad & C(f,p_\Phi)(C(\varepsilon_2)W_2(\mu_t^{u_t^{\varepsilon_1},v_t^{\varepsilon_1}},\mu_0)+\varepsilon_1+\varepsilon_2) + C\tfrac{|t-t_0|}{2} + p_t + \widehat{\mathcal{H}}(\mu_0,\nu_0,\Phi_0,p_\Phi) \\
& \geq -(C+1)\left(\delta+\varepsilon\left((C+1)|t-t_0|\right)\right).
\end{aligned}$$

By Lemma 12, we have $W_2(\mu_t^{u_t^{\varepsilon_1},v_t^{\varepsilon_1}},\mu_0)\to 0$ when $t\to t_0$, indeed recall that by (30)

$$\mu_t^{u_t^{\varepsilon_1},v_t^{\varepsilon_1}} \in \partial_-(\mathcal{V}_r^-)^*(t,X_t^{t_0,\Phi_0(\cdot),u_t^{\varepsilon_1},v_t^{\varepsilon_1}},\varphi,\nu_0) \quad \text{and } \|X_t^{t_0,\Phi_0(\cdot),u_t,v_t}-\Phi_0\|_\infty\to 0.$$

Finally make t tend to t_0 in (32) and get:

$$C(f,p_\Phi)(\varepsilon_1+\varepsilon_2) + p_t + \widehat{\mathcal{H}}(\mu_0,\nu_0,\Phi_0,p_\Phi) \geq -\delta(C+1).$$

As this is true for any $\varepsilon_1,\varepsilon_2>0$, we get the desired inequality.

QED

7 Exemple and Erratum to the article [13]

In C. Jimenez and M. Quincampoix ([13]), we have considered the particular case where Player II has no information on the initial position of the system and h is constantly equal to one. The lack of information can be expressed by choosing $\nu = \delta_{y_0}$ with y_0 any point of Y. Moreover, Φ belongs to $\mathcal{C}(X \times Y, X)$ (X is compact subset of $\mathbb{R}^N$ as previously) and depends only on x, the information of Player I. In this case the values write as:

$$\mathcal{V}_r^+(t_0, \Phi, \mu) := \mathcal{V}_r^+(t_0, \Phi, 1, \mu, \delta_{y_0}) = \inf_{\alpha \in A_r(t_0)} \sup_{\beta \in B(t_0)} \int_{\Omega_\alpha} \int_{\mathbb{R}^N} g(X_T^{t_0, \Phi(x), \alpha(x,\omega,\cdot)\beta(\cdot)}) \, d\mu(x) dP_\alpha(\omega),$$

$$\mathcal{V}_r^-(t_0, \Phi, \mu) := \mathcal{V}_r^-(t_0, \Phi, 1, \mu, \delta_{y_0}) = \sup_{\beta \in B_r(t_0)} \inf_{\alpha \in A(t_0)} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0, \Phi(x), \alpha(x,\cdot)\beta(\omega,\cdot)}) \, d\mu(x) dP_\beta(\omega).$$

Note that both functionals do not depend on the choice of y_0. Then, applying Proposition 3, we can compute the convex conjugate of $\mathcal{V}_r^-$ on the μ variable, the formula obtained is exactly the same as the one appearing in ([13]). The computation of the concave conjugate of $\mathcal{V}_r^+$ in the ν variable, is, in this case, very simple and leads to the following formula:

$$(\mathcal{V}_r^+)^\sharp(t_0, \Phi, 1, \mu, \varphi) = \inf_{y \in Y} \varphi(y) - \mathcal{V}_r^+(t_0, \Phi, 1, \mu, \delta_{y_0})$$

for any $\varphi \in \mathcal{C}(Y)$. The subdynamic and superdynamic principles of Proposition 4 and 5 then coincide with both subdynamics principles of ([13]).

In ([13]), a Hamilton Jacobi equation and some definitions of viscosity subsolution and dual supersolution are given. Then a comparison principle is stated (Theorem 1 p22). The proof of this theorem is false. More precisely, there is a mistake in second part of Step p24. Indeed, the functional θ is build with a L_2-norm depending on μ:

$$\theta(t, s, \Phi, \Psi, \mu) = w_2(s, \Psi, \mu) - w_1(t, \Phi, \mu) + \frac{1}{\varepsilon} \left(\|\Phi - \Psi\|_{L^2_\mu}^2 + |t - s|^2 \right) - \eta s.$$

This choice does not allow to get an element of the superdifferential of w_2^* as defined in [13], two different L^2-norms appearing in the computation.
The definition of the superdifferential should be changed:

Definition 3. *Let $\delta > 0$ and $(t_0, \Phi_0, \varphi) \in]0, T[\times \mathcal{C}(X, X) \times \mathcal{C}(X)$. Assume moreover that:*

$$\partial^- w^*(t_0, \Phi_0, \varphi) = \{\mu_0\}.$$

We say that $(p_t, p_\Phi) \in \mathbb{R} \times \mathcal{C}(X, \mathbb{R}^d)$ belongs to the δ-superdifferential $D_\delta^+ w^*(t_0, \Phi_0, \varphi)$ to w^* at (t_0, Φ_0, φ) iff

$$\limsup_{\substack{\|\Phi - \Phi_0\|_\infty \to 0, \\ t \to t_0}} \sup_{\mu \in \partial^- w^*(t, \Phi, \varphi)} \frac{w^*(t, \Phi, \varphi) - w^*(t_0, \Phi_0, \varphi) - p_t(t - t_0) - \int_X (\Phi - \Phi_0)(x) \cdot p_\Phi(x) \, d\mu(x)}{\|\Phi - \Phi_0\|_\infty + |t - t_0|} \le \delta.$$

Then $\mathcal{V}_r^-$ is a viscosity dual supersolution to (3). Moreover a comparison principle similar to Theorem 2 can be stated and proved slightly modifying the proof above.

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