

# Completeness of Reidemeister-type moves for surfaces embedded in three-dimensional space

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## Abstract

In this paper we are concerned with labelled apparent contours, namely with apparent contours of generic orthogonal projections of embedded surfaces in  $\mathbb{R}^3$ , endowed with a suitable depth information. We show that there exists a finite set of elementary moves (i.e. local topological changes) on labelled apparent contours such that the following theorem holds: two generic embeddings of a closed surface  $M$  in  $\mathbb{R}^3$  are isotopic if and only if their apparent contours can be connected using only smooth isotopies and a finite sequence of moves.

## 1 Introduction

In knot theory it is well known that two *link diagrams* represent ambiently isotopic knots or links, if and only if they can be related by a finite number of local Reidemeister moves or their inverses. The diagram of a link is simply the orthogonal projection of the image of the link onto some plane, with the addition of the knowledge of which strand *goes over* at each crossing. Up to a small perturbation of the link, transversal crossings are the only possible singularities of the diagram.

A similar question can be formulated in the setting of two-dimensional closed (i.e. compact without boundary) manifolds  $M$  embedded in  $\mathbb{R}^3$ ; namely we are interested in understanding whether two embedded surfaces can be deformed into each other by a smooth path of embeddings.

In analogy with the link diagram, we can rely on the *apparent contour*  $\text{AppCon}(\varphi)$  of  $\varphi := \pi \circ e : M \rightarrow \mathbb{R}^2$ , i.e. the image of the singular points of a generic orthogonal projection  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  of an embedding  $e : M \rightarrow \mathbb{R}^3$ . The apparent contour carries a natural orientation and leads to a graph similar to a link diagram, with possibly the addition of *cusp points* (see e.g. [22]). If we also add a so-called *Huffman labelling* (see [13], [23], [6], [3]) which is a nonnegative integer attached to each arc, giving information about the relative *depth* of the corresponding fold with respect to the remaining preimages of the arc, an apparent contour provides complete information on the 3D embedding (up to compactly supported

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deformations of  $\mathbb{R}^3$  in the projection direction) [3]. In Section 3 we recall the main properties of the apparent contours needed in our paper.

It turns out that there are six basic *moves* (local change of topology) on the apparent contour that correspond to a general deformation of the corresponding embedded surface; they can be used in exactly the same way as the Reidemeister moves on link diagrams. We show that this set of moves, namely K (from the russian word *kasanie* = *tangency*), L (lips), B (beak-to-beak), C (cusp-fold), S (swallow tail) and T (triple point) is complete. This means that two embedded surfaces in generic position with respect to the projection, that can be deformed into each other, have apparent contours that can be connected using only smooth isotopies and a finite sequence of such moves.

This list of moves (see Figure 2 for a graphical representation) is essentially the same found in the literature for the related subject of maps from two-manifolds into  $\mathbb{R}^2$  (see e.g. [16]), however the addition of the Huffman labelling entails a different classification of the list of moves belonging to each of the six aforementioned types: see Section 4. We remark that the problem of finding a complete set of Reidemeister moves relating two equivalent knotted surfaces in  $\mathbb{R}^4$  has been solved. We refer in particular to the set of moves found by Roseman [19], to the papers of Carter and Saito [4], [5] where generic embedded surfaces in  $\mathbb{R}^4$  are considered, projected in  $\mathbb{R}^3$  (diagram) and projected further in  $\mathbb{R}^2$  to construct a *chart*, and to the paper [11] of Goryunov. Similar classifications appear in various contexts, in particular in Thom's catastrophe theory [20] and in Cerf's theory [8], in the paper of Wassermann [21], in the papers of Mancini and Ruas [14], and of Rieger [18]. We refer to the books [6], [7], [1] and [2] for further information.

Our proof, similar in spirit to the one described in [6] for the embedding of surfaces in  $\mathbb{R}^4$ , relies on the classification (reported very quickly in Section 2) of the singularities of stable maps between 3-manifolds: see [9] and references therein. Roughly, the idea of the proof is the following. Given an orthogonal projection  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and a smooth closed surface  $M$ , we consider an isotopy  $\gamma \in C^\infty(M \times [0, 1]; \mathbb{R}^3)$  between an initial embedding ( $t = 0$ ) and a final embedding ( $t = 1$ ) of  $M$  in  $\mathbb{R}^3$ , in generic positions with respect to  $\pi$ . Since it is convenient to deal with closed source manifolds, we extend in a smooth periodic way  $\gamma$  to a map (still denoted by  $\gamma$ ) defined on  $\mathcal{X} := M \times \mathbb{S}^1$ . Let us interpret as time the last coordinate  $t \in \mathbb{S}^1$ , and denote by  $(x, t)$  the points of  $\mathcal{X}$ . Let us now consider the level preserving map  $F_\gamma : \mathcal{X} \rightarrow \mathcal{Y} := \mathbb{R}^2 \times \mathbb{S}^1$  obtained as the composition of the track  $(x, t) \in M \times \mathbb{S}^1 \rightarrow (\gamma(x, t), t) \in \mathbb{R}^3 \times \mathbb{S}^1$  of  $\gamma$  with the projection map  $(y, z, t) \in \mathbb{R}^3 \times \mathbb{S}^1 \rightarrow (y, t) \in \mathbb{R}^2 \times \mathbb{S}^1$ , i.e.,

$$F_\gamma(x, t) = (\gamma_1(x, t), \gamma_2(x, t), t).$$

Provided  $F_\gamma$  is *stable*, its singular locus gives a stratification of  $\mathcal{X}$  into smooth submanifolds and, in a natural way, also a stratification  $\{Y_0, Y_1, Y_2, Y_3\}$  of  $\mathcal{Y}$ . The family of apparent contours relating the two embeddings then satisfies

$$\bigcup_{t \in \mathbb{S}^1} (\text{AppCon}(\varphi_t) \times \{t\}) = Y_1 \cup Y_2 \cup Y_3, \quad (1.1)$$

where  $\varphi_t(x) := (\gamma_1(x, t), \gamma_2(x, t))$ ,  $Y_1$  being the stratum of fold surfaces,  $Y_2$  the stratum of cusp curves and double curves, and  $Y_3$  the discrete stratum of cusp-fold points, swallow tails and triple points. Let us now consider the projection  $p : \mathcal{Y} \rightarrow \mathbb{S}^1$ ,  $p(y, t) = t$ . It turns out

that, provided  $p$  is a *stratified Morse function*, the union of  $Y_3$  and the critical points of the restriction  $p|_{Y_2}$  determine the complete list of moves on the apparent contours (see Corollary 6.1 and Figure 3). The main technical points (see Lemmata 5.6, 5.8 and Corollary 5.7) rely in showing that the original map  $F_\gamma$  can be slightly deformed and made stable, and then perturbed once more in order to make  $p$  stratified: this is the content of Theorem 5.5.

## 2 Preliminaries on stable maps and their singularities

We shall recover here briefly a few well known facts about singularity theory (see for instance [9], [2], [1] and references therein).

### 2.1 Density of stable maps

Let  $\mathcal{X}$  denote a closed smooth manifold of dimension  $m$ , and let  $\mathcal{Y}$  be a smooth manifold of dimension  $n$  without boundary. We denote by  $C^\infty(\mathcal{X}, \mathcal{Y})$  the set of smooth maps from  $\mathcal{X}$  to  $\mathcal{Y}$  endowed with the Whitney topology;  $C^\infty(\mathcal{X}, \mathcal{Y})$  turns out to be a Baire space.

**Definition 2.1.** *Two maps  $F, G \in C^\infty(\mathcal{X}, \mathcal{Y})$  are smoothly left-right equivalent (briefly, equivalent) if there exist two diffeomorphisms  $\phi : \mathcal{X} \rightarrow \mathcal{X}$  and  $\psi : \mathcal{Y} \rightarrow \mathcal{Y}$  such that  $G \circ \phi = \psi \circ F$ .*

**Definition 2.2.** *A map  $F \in C^\infty(\mathcal{X}, \mathcal{Y})$  is smoothly stable (briefly, stable) if there exists a neighbourhood  $U_F \subset C^\infty(\mathcal{X}, \mathcal{Y})$  of  $F$  such that any map in  $U_F$  is equivalent to  $F$ .*

We set

$$\text{Stab}(\mathcal{X}, \mathcal{Y}) := \{F \in C^\infty(\mathcal{X}, \mathcal{Y}) : F \text{ is stable}\},$$

which is an open subset of  $C^\infty(\mathcal{X}, \mathcal{Y})$ . We recall that if  $F$  is stable, then any map equivalent to  $F$  is stable. We also recall that the set  $\text{Emb}(\mathcal{X}; \mathcal{Y})$  of embeddings of  $\mathcal{X}$  into  $\mathcal{Y}$  is open in  $C^\infty(\mathcal{X}, \mathcal{Y})$ , and  $\text{Emb}(\mathcal{X}; \mathcal{Y}) \subseteq \text{Stab}(\mathcal{X}, \mathcal{Y})$ .

### 2.2 Nonremovable singularities of maps for $m = n = 3$

Assume in this section that  $m = n = 3$ . These dimensions are nice, so that  $\text{Stab}(\mathcal{X}, \mathcal{Y})$  is dense in  $C^\infty(\mathcal{X}, \mathcal{Y})$ .

#### 2.2.1 Stratification of $\mathcal{X}$ associated with a stable map

We recall (see [9, Chap. 7, Sec. 6]) that a map  $F \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  has a singular locus in the source manifold  $\mathcal{X}$  of the following types:

- **folds.** Denoted by  $S_1(F)$ , it is a smooth submanifold of  $\mathcal{X}$  of codimension 1, consisting in the points where the differential  $dF_\xi$  of  $F$  at  $\xi \in \mathcal{X}$  has corank 1;
- **pleats.** Denoted by  $S_{1_2}(F) \subset S_1(F)$ , it is a smooth submanifold of  $\mathcal{X}$  of codimension 2 in  $\mathcal{X}$ ;
- **swallow-tails in  $\mathcal{X}$ .** Denoted by  $S_{1_3}(F) \subset S_{1_2}(F)$ , it is a smooth submanifold of  $\mathcal{X}$  of codimension 3, hence it is a finite set of points.

Define  $X_0$  as the set of the regular points of  $F$  and

$$X_1 := S_1(F) \setminus S_{1_2}(F), \quad X_2 := S_{1_2}(F) \setminus S_{1_3}(F), \quad X_3 := S_{1_3}(F),$$

the index denoting the codimension in  $\mathcal{X}$ . Then  $\{X_0, X_1, X_2, X_3\}$  forms a stratification of  $\mathcal{X}$ , in the sense that  $\mathcal{X}$  is the union of the mutually disjoint smooth submanifolds  $X_j$  (the strata), such that  $\overline{X_j} = \bigcup_{j \leq h \leq 3} X_h$  for any  $j = 0, 1, 2, 3$ . We call such a stratification the stratification of  $\mathcal{X}$  associated with the stable map  $F$ .

## 2.2.2 Stratification of $\mathcal{Y}$ induced by a stable map

By [9, Chap. 7, Th. 6.3], if  $F \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  the images of the strata  $X_i$  must intersect transversally. On  $F(X_0)$  there are no conditions. The set  $F(X_1)$  must selfintersect transversally, the resulting intersection is a set of **double curves** of codimension 2 in  $\mathcal{Y}$  (points having two singular preimages, i.e. two preimages in  $S_1(F)$ ) and a set of **triple points** (codimension 3, points with three singular preimages) in  $\mathcal{Y}$ ; moreover  $F(X_1)$  must intersect  $F(X_2)$  transversally, giving a finite set of **cusp-fold** points in  $\mathcal{Y}$ . The remaining cases have dimension that is too low to give rise to any intersection set. Therefore, we define the following subsets of the target manifold  $\mathcal{Y}$  (the index denoting the codimension in  $\mathcal{Y}$  and the superscript denoting the number of singular preimages):

- $Y_0$  is the set of all  $\eta \in \mathcal{Y}$  such that no element in  $F^{-1}(\eta)$  belongs to  $X_1 \cup X_2 \cup X_3$ ; hence  $Y_0 \cap F(\mathcal{X}) \subseteq F(X_0)$ , namely  $Y_0$  is contained in the set of regular values of  $F$ ;
- $Y_1$  is the set of all  $\eta \in \mathcal{Y}$  such that  $F^{-1}(\eta)$  has one element in  $X_1$  and the other elements in  $X_0$ . Hence  $Y_1$  is contained in  $F(X_1)$ ; we call  $Y_1$  the set of **fold surfaces**. It carries a natural orientation since it separates points where the number of preimages of  $F$  jumps of two units;
- $Y_2^1$  is the set of all  $\eta \in \mathcal{Y}$  such that  $F^{-1}(\eta)$  has one element in  $X_2$ , and the other elements in  $X_0$ . We call  $Y_2^1$  the set of **cusp curves**;
- $Y_2^2$  is the set of all  $\eta \in \mathcal{Y}$  such that  $F^{-1}(\eta)$  has two elements in  $X_1$  and the other elements in  $X_0$ . We call  $Y_2^2$  the set of **double curves**;
- $Y_3^1$  is the set of all  $\eta \in \mathcal{Y}$  such that  $F^{-1}(\eta)$  has one element in  $X_3$ , and the other elements in  $X_0$ . We call  $Y_3^1$  the set of **swallow tails**;
- $Y_3^2$  is the set of all  $\eta \in \mathcal{Y}$  such that  $F^{-1}(\eta)$  has one element in  $X_2$ , one element in  $X_1$  and the other elements in  $X_0$ . We call  $Y_3^2$  the set of **cusp-fold points**;
- $Y_3^3$  is the set of all  $\eta \in \mathcal{Y}$  such that  $F^{-1}(\eta)$  has three elements in  $X_1$  and the other elements in  $X_0$ . We call  $Y_3^3$  the set of **triple points**.

Such a description allows to define a natural stratification of the target manifold  $\mathcal{Y}$  in the smooth submanifolds  $Y_0, Y_1, Y_2, Y_3$  (the strata), where

$$Y_2 := Y_2^1 \cup Y_2^2, \quad Y_3 := Y_3^1 \cup Y_3^2 \cup Y_3^3.$$

This stratification will be denoted by  $\{Y_j\}_F$ , and will be called the stratification of  $\mathcal{Y}$  induced by the stable map  $F$ . For simplicity of notation, unless otherwise specified we drop the dependence on  $F$  of each strata  $Y_j$ .

We conclude this section by recalling the definition of a stratified Morse function defined on the stratified space  $(\mathcal{Y}, \{Y_j\}_F)$  induced by the stable map  $F$  (see [17], [10]).

**Definition 2.3.** *Let  $F \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  and let  $u : \mathcal{Y} \rightarrow \mathbb{R}$  be a smooth function. We say that  $u$  is a stratified Morse function on  $(\mathcal{Y}, \{Y_j\}_F)$  if the following three conditions hold:*

- for any  $j \in \{0, 1, 2\}$  the restriction  $u|_{Y_j}$  of  $u$  to stratum  $Y_j$  is a Morse function, and the set  $\text{crit}(u|_{Y_j})$  of its critical points is finite;
- the critical values  $\bigcup_{j \in \{0, 1, 2\}} u|_{Y_j}(\text{crit}(u|_{Y_j})) \cup u(Y_3)$  are distinct, where points of  $Y_3$  are considered as critical points;
- if  $j \in \{1, 2, 3\}$  and  $\bar{\eta} \in Y_j$ , then  $\ker(du_{\bar{\eta}})$  does not contain any limit of a sequence of tangent spaces to  $Y_h$  at  $\eta_k \in Y_h$ , where  $0 \leq h < j$  and  $\lim_{k \rightarrow +\infty} \eta_k = \bar{\eta}$ .

### 3 Apparent contours

In this section we briefly recall the notions of apparent contour and labelling of an apparent contour. In the sequel  $M$  is a two-dimensional smooth closed orientable manifold, and  $\varphi : M \rightarrow \mathbb{R}^2$  is a smooth stable map.

The apparent contour  $\text{AppCon}(\varphi) \subset \mathbb{R}^2$  of  $\varphi$  (also called the discriminant set of  $\varphi$ ) is the image through  $\varphi$  of the singular locus of  $\varphi$  in  $M$ . We set

$$f(y) := \#\{\varphi^{-1}(y)\}, \quad y = (y_1, y_2) \in \mathbb{R}^2. \quad (3.1)$$

The function  $f$  is constant on each connected component (called region) of  $\mathbb{R}^2 \setminus \text{AppCon}(\varphi)$ , and it jumps of two units along an arc of  $\text{AppCon}(\varphi)$ . The apparent contour carries a natural orientation, so that the highest value of  $f$  lies locally on the left, in this way  $f$  can also be recovered as twice the winding number of the apparent contour.

It is well known (see for instance [24] and references therein) that the stability of  $\varphi$  implies that  $\text{AppCon}(\varphi)$  has the following structure: any point of  $\text{AppCon}(\varphi)$  has a neighbourhood in  $\mathbb{R}^2$  which is diffeomorphic to one of the pictures in Figure 1 ( $\text{AppCon}(\varphi) = \text{Arcs}(\varphi) \cup \text{Cusps}(\varphi) \cup \text{Crossings}(\varphi)$ ), with matching orientations and appropriate choice of the values of  $f$  and  $d$  (see below). Because of the depth information coming from the embedding we distinguish cusps where the  $d$  value is decreasing from cusps where  $d$  is increasing; a similar distinction is made for crossings. The gap shown in the pictures for arcs at a crossing is just added for visual convenience to help distinguish the arc with larger values of  $d$  (where  $d$  jumps by two, broken arc) from the arc with smaller value of  $d$  (unbroken).

#### 3.1 Labelling of apparent contours

A Huffman labelling (labelling for short) of  $\text{AppCon}(\varphi)$  (see [13], [23], [6, p. 19], [7, pp. 19,20], [3]) is the pair  $(f, d)$ , where the function  $d : \text{Arcs}(\varphi) \rightarrow \mathbb{N}$  has the following properties:

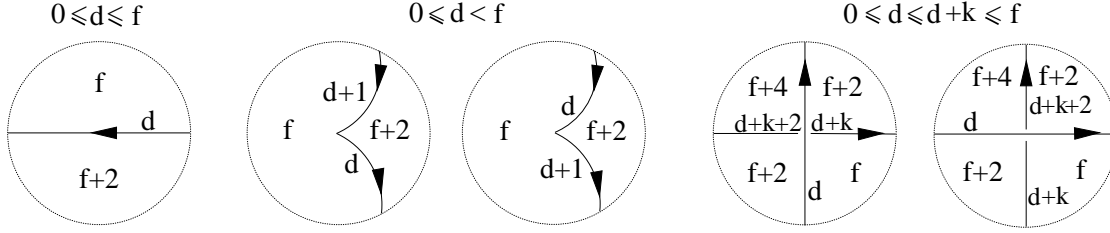


Figure 1: (b), (c): the canonical cusp of  $\text{AppCon}(\varphi)$ , i.e. the semicubic curve  $(y_1^2, y_1^3)$ . (d), (e): the canonical crossings of  $\text{AppCon}(\varphi)$ . (a)-(e): compatibility conditions between  $f$  (defined in (3.1)) and  $d$  (defined in Section 3.1).

- $d$  is locally constant on  $\text{Arcs}(\varphi)$ ;
- $0 \leq d(\bar{y}) \leq \liminf_{y \rightarrow \bar{y}} f(y)$ , for all  $\bar{y} \in \text{Arcs}(\varphi)$ ;
- the compatibility conditions between  $f$  and  $d$  depicted in Fig. 1 must be satisfied.

### 3.2 Factorization of $\varphi$ through an embedding and a projection

**Definition 3.1.** Let  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be an orthogonal projection and let  $\Sigma$  be a smooth closed surface embedded in  $\mathbb{R}^3$ . We say that  $\Sigma$  is in generic position with respect to  $\pi$  if  $\pi|_{\Sigma} : \Sigma \rightarrow \mathbb{R}^2$  is stable.

**Remark 3.2.** As shown in [3], the existence of a labelling (as defined in Section 3.1) on a planar graph with crossings and cusps, is a necessary and sufficient condition for the existence of a smooth closed two-manifold  $S$  and of a smooth map  $\varphi : S \rightarrow \mathbb{R}^2$  factorizing as

$$\varphi = \pi \circ e,$$

where  $e : S \rightarrow \mathbb{R}^3$  is a smooth embedding,  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is an orthogonal projection such that  $\Sigma := e(S)$  is in generic position with respect to  $\pi$ , and such that the planar graph is  $\text{AppCon}(\varphi)$ . Moreover such an embedding is unique up to reparametrizations of  $S$  and a compactly supported deformation of  $\mathbb{R}^3$  invariant with respect to  $\pi$ .

The meanings of  $f$  and  $d$  then become the following:

- $f(y)$  is the number of points of  $\Sigma$  that project on  $y \in \mathbb{R}^2$ ;
- $d(y)$  counts the number of layers of  $\Sigma$  in front of the point of the singular locus of  $\varphi$  that projects on  $y$ .

The following simple observation is concerned with the stability of  $\pi \circ e$  (see [15] for general stability theorems for composite mappings), and shows that, in the statement of Theorem 5.5, there is no loss of generality in assuming the initial and final surfaces  $\Sigma_0$  and  $\Sigma_1$  to be in generic position with respect to  $\pi$ .

**Remark 3.3.** Given  $e \in \text{Emb}(M, \mathbb{R}^3)$  and a neighbourhood  $U_e \subset C^\infty(M, \mathbb{R}^3)$  of  $e$ , there exists  $\hat{e} \in \text{Emb}(M, \mathbb{R}^3) \cap U_e$  such that  $\hat{\Sigma} := \hat{e}(M)$  is in generic position with respect to

$\pi$ . Indeed, by identifying  $M$  with  $\Sigma := e(M)$  we can suppose that  $e : \Sigma \rightarrow \mathbb{R}^3$  is the identity. Then  $\kappa := \pi \circ e \in C^\infty(\Sigma, \mathbb{R}^2)$  is a map between two 2-manifolds, with  $\Sigma$  closed. From the density of  $\text{Stab}(\Sigma, \mathbb{R}^2)$  in  $C^\infty(\Sigma, \mathbb{R}^2)$  it follows that, given any neighbourhood  $U_\kappa \subset C^\infty(\Sigma, \mathbb{R}^2)$  of  $\kappa$  there exists a map  $\varphi \in \text{Stab}(\Sigma, \mathbb{R}^2) \cap U_\kappa$ . Define  $\widehat{\Sigma} := \{(\varphi(y, z), z) : (y, z) \in \Sigma\}$ . Taking  $U_\kappa$  small enough and recalling that  $\text{Emb}(\Sigma, \mathbb{R}^3)$  is open in  $C^\infty(\Sigma, \mathbb{R}^3)$ , we obtain that there exists  $\widehat{e} \in \text{Emb}(M, \mathbb{R}^3) \cap U_e$  such that  $\widehat{e}(M) = \widehat{\Sigma}$ . Moreover  $\pi|_{\widehat{\Sigma}} = \varphi$ , so that the stability of  $\varphi$  implies that  $\widehat{\Sigma}$  is in generic position with respect to  $\pi$ .

## 4 The moves on apparent contours

In this section we list the moves on the apparent contour; in view of Theorem 5.5, and as explained in the introduction, this list is complete (see Corollary 6.1).

**Definition 4.1.** *The moves on an apparent contour are given as K, L, B, C, S, T, by identifying a box in  $\mathbb{R}^2$  diffeomorphic to the box on the left side of the picture and replacing it with a box diffeomorphic to the box on the right, see Figure 2.*

**Remark 4.2.** We require that the moves leave unchanged a (small) neighbourhood of the boundary of the box. The same definition as Definition 4.1 can be given, but for the move T, by switching the role of the two boxes: this is equivalent to reverse the orientation of the  $t$ -axis, and to consider the inverse moves as temporal inverse moves. The corresponding moves will be denoted by  $K^{-1}$ ,  $L^{-1}$ ,  $B^{-1}$ ,  $C^{-1}$ ,  $S^{-1}$ . The (direct) moves K, L, B, C, S, T are chosen in such a way that they simplify the local topology of the apparent contour (i.e., they decrease the number of crossings/cusps). There is no distinction between direct and inverse moves of type T as will be explained in the sequel.

We recall that all apparent contours that we consider are oriented, and different orientations determine different moves: for simplicity of notation in Figure 2 often the orientations will not be depicted; moreover, we do not indicate the values of  $f$  before and after the moves (the values of  $f$  can be inferred from the orientation of the apparent contour).

- *The four moves of type K.* We divide the moves of type K into four different moves as follows. Up to a rotation of 180 degrees, we can assume that the arc with the two extremal points on the left is in front of the other arc. The four moves therefore are classified on the basis of the four possible orientations of the two arcs. In addition they are parameterized by two nonnegative integers  $d$  and  $k$  as explained in Section 4.1 below.

- *The moves L and B.* Up to a rotation of 180 degrees, we can assume that the highest value of  $d$  is on the upper arc. Then there is one move L and one move B, as depicted in Figure 2.

- *The eight moves of type C.* We divide the moves of type C into two groups of four different types. First we distinguish the case when the cusp is in front of the (vertical) arc. The first of the figures for C is in turn divided into four cases, depending on whether the value of  $d$  decreases or increases when parameterizing the cusp, and on the orientation of the vertical arc. In the second figure the cusp is behind the vertical arc: similarly as before, we have four cases. The set of values taken by  $d$  along the cusped arc are  $\{d+k, d+k+1, d+k+2, d+k+3\}$ . Again the meaning of the two parameters  $d$  and  $k$  is explained in Section 4.1.

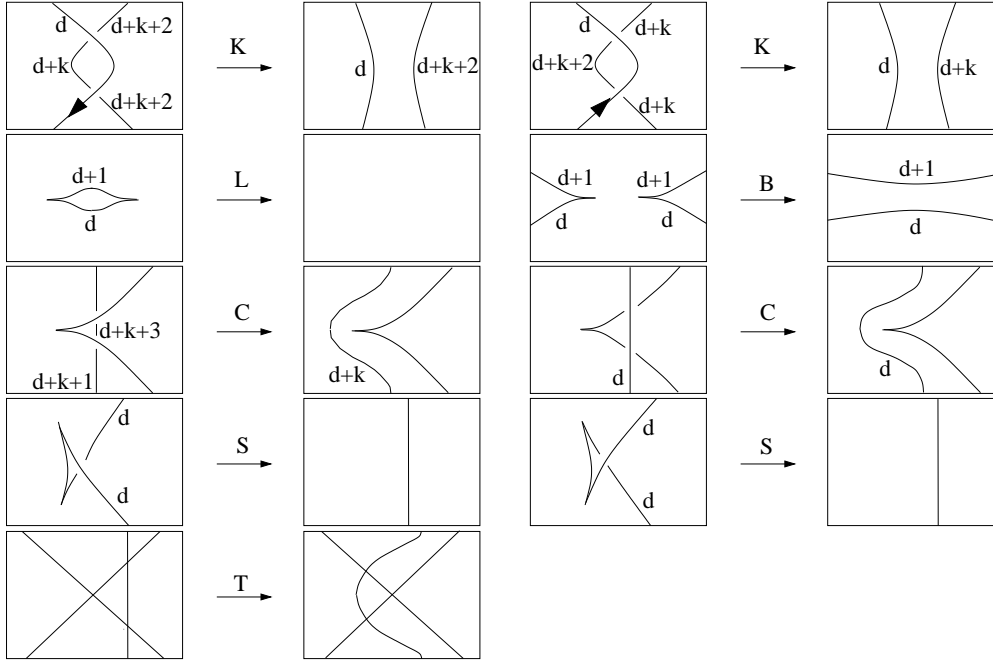


Figure 2: List of moves on the apparent contour

- *The two moves of type S.* We divide the moves of type S into two groups: in the first picture the value of  $d$  jumps up by two at the crossing and is decreasing at (both) cusps (the arcs are traversed according to their natural orientation), whereas in the second picture  $d$  is increasing at the cusps.

- *The sixteen moves of type T.* The three arcs carry a natural ordering according to their relative depth (increasing values of  $d$ ); we can always rotate the picture so that the *nearest* arc (lowest  $d$ ), is the vertical one. We then have two different possibilities for the position of the intermediate and of the furthest arcs. Each of the three arcs can be oriented in two ways: the internal triangular region can lie on the left or on the right. In the end we have sixteen different possibilities which however also account for the corresponding *time reversed* moves, in the sense that the inverse of a T move is still a T move. This is in contrast to what happens for all the other moves. If  $d_1$ ,  $k_1$  and  $k_2$  denote respectively the number of layers in front of the first fold (nearest arc), the number of layers interposed between the first and second fold and the number of layers interposed between the second and the third fold, then  $d_1$  is the (constant) value of  $d$  on the first arc,  $d_1 + k_1$  and  $d_1 + k_1 + 2$  are the two values taken by  $d$  on the second arc, and the values of  $d$  on the third arc are contained in the set  $\{d_1 + k_1 + k_2 + i : i = 0, 2, 4\}$ , the precise values depending on the orientation of the first and second arcs.



## 4.1 Number of layers involved

In order to make a complete classification of the moves, the number of layers, at different depths, of the corresponding three dimensional embedded surface must be taken into account: this introduces further degrees of freedom in the list of different moves, as follows. Moves L, B and S have one nonnegative integer parameter  $d$ , counting the number of layers in front of the fold. Moves of type K and C have two nonnegative integer parameters  $d$  and  $k$ , counting the number of layers in front of the first fold, and the number of layers in between the two folds. Moves of type T have three nonnegative integer parameters, given respectively by the number of layers in front of the first fold surface, by the number of layers between the first and the second fold surface, and by the number of layers between the second and the third fold surface.

## 5 Stability of $F_\gamma$ and stratifiability of $p$

In order to state the main result (Theorem 5.5) we need some preparation. In this section  $M$  denotes a smooth two-dimensional closed manifold.

Recall [12] that an isotopy from  $M$  to  $\mathbb{R}^3$  is a map  $\gamma \in C^\infty(M \times [0, 1], \mathbb{R}^3)$  such that for any  $t \in [0, 1]$  the map  $\gamma(\cdot, t) : M \rightarrow \mathbb{R}^3$  is an embedding.

**Definition 5.1.** *Let  $\Sigma_0$  and  $\Sigma_1$  be the images in  $\mathbb{R}^3$  of two smooth embeddings of  $M$ . We say that  $\Sigma_0$  and  $\Sigma_1$  are isotopic if there exists an isotopy  $\gamma$  such that  $\gamma(M, 0) = \Sigma_0$  and  $\gamma(M, 1) = \Sigma_1$ .*

In the sequel it is convenient to consider maps  $\gamma$  defined on a closed smooth manifold. Therefore, we perform the following operations. We first reparametrize the map  $\gamma(x, \cdot)$  by composing it with a strictly increasing  $C^\infty([0, 1], [0, 1])$  function having vanishing derivatives of all order at 0 and 1. We still denote by  $t$  the new variable, so that

$$\frac{\partial^k \gamma(x, t)}{\partial t^k} \Big|_{t=0} = \frac{\partial^k \gamma(x, t)}{\partial t^k} \Big|_{t=1} = 0, \quad k \in \mathbb{N}, k \geq 1. \quad (5.1)$$

We next extend  $\gamma$  on the whole of  $M \times \mathbb{R}$  by reflecting it about 0 and 1, resulting in a smooth periodic function of period 2 in the variable  $t$ . If we identify  $\mathbb{R}/[0, 2]$  with  $\mathbb{S}^1$  we obtain a smooth function, still denoted by  $\gamma$ , defined on the closed smooth manifold  $M \times \mathbb{S}^1$  with values in  $\mathbb{R}^3$ . In this way 0 and 1 are two distinct points in the oriented circle  $\mathbb{S}^1$ .

From now on we set

$$\mathcal{X} := M \times \mathbb{S}^1. \quad (5.2)$$

Variables in  $\mathcal{X}$  will be denoted by  $(x, t)$  with  $x \in M$ ,  $x = (x_1, x_2)$  (locally) and  $t \in \mathbb{S}^1$ . Moreover, we shall denote by  $(y, z)$  a point of  $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$  where  $y = (y_1, y_2) \in \mathbb{R}^2$  and  $z \in \mathbb{R}$ .

### 5.1 The map $F_\gamma$

From now on we set

$$\mathcal{Y} := \mathbb{R}^2 \times \mathbb{S}^1. \quad (5.3)$$

Variables in  $\mathcal{Y}$  will be denoted by  $(y, t)$  with  $y = (y_1, y_2) \in \mathbb{R}^2$  and  $t \in \mathbb{S}^1$ . We let  $\pi : \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  denote the orthogonal projection defined by  $\pi(y, z) := y$ .

**Definition 5.2.** Let  $\gamma \in C^\infty(\mathcal{X}, \mathbb{R}^3)$ . We define  $F_\gamma \in C^\infty(\mathcal{X}, \mathcal{Y})$  as

$$F_\gamma(x, t) := (\pi(\gamma(x, t)), t), \quad (x, t) \in \mathcal{X}. \quad (5.4)$$

The map  $\gamma \in C^\infty(\mathcal{X}, \mathbb{R}^3) \rightarrow F_\gamma \in C^\infty(\mathcal{X}, \mathcal{Y})$  is continuous.

It is immediately seen that the differential of  $F_\gamma$  has rank 2 at  $(\bar{x}, \bar{t}) \in \mathcal{X}$  if and only if the map

$$\varphi_{\bar{t}} := \pi(\gamma(\cdot, \bar{t})) : x \in M \rightarrow \varphi_{\bar{t}}(x) := (\gamma_1(x, \bar{t}), \gamma_2(x, \bar{t})) \in \mathbb{R}^2 \quad (5.5)$$

has differential with rank one at  $\bar{x} \in M$ . This in particular implies (1.1). Note that, defining  $f_t(y)$  as in (3.1) with  $\varphi_t$  in place of  $\varphi$ , we have  $f_t(y) = \#\{F_\gamma^{-1}(y, t)\}$  for any  $(y, t) \in \mathcal{Y}$ .

## 5.2 Statement of the theorem and remarks

We denote by  $p : \mathcal{Y} \rightarrow \mathbb{S}^1$  the projection

$$p(y, t) := t, \quad y \in \mathbb{R}^2, t \in \mathbb{S}^1, \quad (5.6)$$

defined on the target manifold  $\mathcal{Y}$ .

Let  $\Sigma_0$  and  $\Sigma_1$  be two isotopic embedded surfaces in  $\mathbb{R}^3$ , and let  $\gamma$  be the isotopy. In order to prove the completeness of the set of moves (Section 6) we need to have that  $F_\gamma \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  and, at the same time, that  $p : \mathcal{Y} \rightarrow \mathbb{S}^1$  is a stratified Morse function on the stratification  $(\mathcal{Y}, \{Y_j\}_{F_\gamma})$  (the definition of stratified Morse function with values in  $\mathbb{S}^1$  is as in Definition 2.3 with obvious modifications). Corollary 5.7 below shows that it is possible to approximate  $F_\gamma$  by stable maps of the special form  $F_{\bar{\gamma}}$  for suitable  $\bar{\gamma}$ . The stability of  $F_{\bar{\gamma}}$  implies that  $p|_{Y_0}$  and  $p|_{Y_1}$  have no critical points (Remark 5.3) but does not imply, in general, that  $p$  is a stratified Morse function (Remark 5.4), namely it may happen that a curve in  $Y_2$  having an endpoint in  $Y_3$  has there a tangent line contained in the plane  $\{t = \text{const}\}$ .

**Remark 5.3.** Let  $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^3)$  be such that  $F_\alpha \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ , and let  $(\mathcal{Y}, \{Y_j\}_{F_\alpha})$  be the stratification induced by  $F_\alpha$ . Recalling definition (5.6) of  $p$  it is immediate to check that  $p|_{Y_0}$  does not have any critical point. Moreover:

- $p|_{Y_1}$  does not have any critical point, or equivalently the tangent plane to  $Y_1$  at a point is never orthogonal to  $(0, 0, 1)$ . Indeed, if the rank of the differential of  $F_\alpha$  at  $(\bar{x}, \bar{t}) \in \mathcal{X}$  is two and  $F_\alpha(\bar{x}, \bar{t}) \in Y_1$ , necessarily the tangent space at  $F_\alpha(\bar{x}, \bar{t})$  contains a vector of the form  $(c_1, c_2, 1)$  for some  $c_1, c_2 \in \mathbb{R}$ . In particular the tangent plane to  $Y_1$  at  $F_\alpha(\bar{x}, \bar{t})$  is transversal to  $\{t = \bar{t}\}$ , and this holds uniformly with respect to the points in  $Y_1$ .

- Let  $(\bar{y}, \bar{t}) \in Y_2 \cup Y_3$  be a point which is limit of points  $(y_k, t_k)$  belonging to  $Y_1$ . Assume that the limit of the sequence of tangent planes to  $Y_1$  at  $(y_k, t_k) \in Y_1$  exists, and denote by  $T$  such a limit. Then, by the previous item and by continuity, still  $(c_1, c_2, 1)$  is one of the two vectors spanning  $T$ . Therefore  $T$  is transverse to  $\{t = \bar{t}\}$  at  $(\bar{y}, \bar{t})$ .

**Remark 5.4.** Let us adopt the notation of Remark 5.3. Then the function  $p$  could not be a stratified Morse function on  $(\mathcal{Y}, \{Y_j\}_{F_\alpha})$ , since the third condition of Definition 2.3 may fail, when  $u = p$  and  $j = 3$ . For instance:

- It is not difficult to find an example of a map  $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^3)$  with  $F_\alpha \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ , having a triple point at  $(\bar{y}, \bar{t}) = (0, 0) \in Y_3^3$  with one of the double curves in  $Y_2^2$  parallel to  $\{t = 0\}$ . This is the case for instance of a map  $F_\alpha \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  having, locally around  $(0, 0)$ , the fold surfaces of the form  $\{y_1 = \pm t\}$  and  $\{y_2 = 0\}$ . These folds are obviously mutually transverse,  $\{y_1 = t = 0\}$  is locally one of the double curves and it is parallel to the plane  $\{t = 0\}$ .

- Up to a change of variables in  $\mathcal{X}$  and  $\mathcal{Y}$ , a swallow tail singularity at  $(0, 0)$  has the local description  $\eta_1 = \xi_1 \xi_2 + \xi_1^2 \xi_3 + \xi_1^4$ ,  $\eta_2 = \xi_2$ ,  $\eta_3 = \xi_3$ . There are two cusp curves and one double curve originating at the singularity with a common tangent vector  $(0, 0, -1)$ ; moreover all the fold surfaces are locally tangent at the singularity to the plane  $\{\eta_1 = 0\}$ . We can provide two simple realizations in our context of the canonical representation above. The choice  $x = (\xi_1, \xi_2)$ ,  $y = (\eta_1, \eta_2)$ ,  $t = \xi_3 = \eta_3$  corresponds to the move S, whereas the choice  $x = (\xi_1, \xi_3)$ ,  $y = (\eta_1, \eta_3)$ ,  $t = \xi_2 = \eta_2$  (whence  $\pi(\alpha(x, t)) = (tx_1 + x_2 x_1^2 + x_1^4, x_2)$ ) corresponds to an evolution that is degenerate at  $t = 0$ : indeed the corresponding apparent contour has a cusp with one of the two departing arcs that is (locally) completely contained in another arc of the contour.

In Lemma 5.8 we perform a further perturbation of a stable map  $F_\beta$  in order to get a new map making the function  $p$  stratified. Eventually, stability of  $F_\gamma$  and stratifiability of  $p$  can be achieved at the same time, and this is one of the by-products of the following theorem.

**Theorem 5.5.** *Let  $\pi : \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  be the orthogonal projection, and for  $j = 0, 1$  let  $e_j \in \text{Emb}(M, \mathbb{R}^3)$  be such that  $\Sigma_0 := e_0(M)$  and  $\Sigma_1 := e_1(M)$  are in generic position with respect to  $\pi$ . Assume that  $\gamma \in C^\infty(\mathcal{X}, \mathbb{R}^3)$  is an isotopy between  $\Sigma_0$  and  $\Sigma_1$ . Then for any neighbourhood  $U_{e_j} \subset C^\infty(M, \mathbb{R}^3)$  of  $e_j$ ,  $j = 0, 1$ , and for any neighbourhood  $U_\gamma \subset C^\infty(\mathcal{X}, \mathbb{R}^3)$  of  $\gamma$  there exists a map  $\tilde{\gamma} \in U_\gamma$  such that*

- (i)  $\tilde{\Sigma}_0 := \tilde{\gamma}(M, 0)$  and  $\tilde{\Sigma}_1 := \tilde{\gamma}(M, 1)$  are in generic position with respect to  $\pi$ ,  $\tilde{\gamma}(\cdot, 0) \in U_{e_0}$ ,  $\tilde{\gamma}(\cdot, 1) \in U_{e_1}$ , and  $\tilde{\gamma}$  is an isotopy between  $\tilde{\Sigma}_0$  and  $\tilde{\Sigma}_1$ ;
- (ii)  $F_{\tilde{\gamma}} \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ , where  $F_{\tilde{\gamma}}(x, t) := (\pi \circ \tilde{\gamma}(x, t), t)$  for any  $(x, t) \in \mathcal{X}$ ;
- (iii)  $p : \mathcal{Y} \rightarrow \mathbb{S}^1$  is a stratified Morse function on the stratification  $(\mathcal{Y}, \{Y_j\}_{F_{\tilde{\gamma}}})$  of  $\mathcal{Y}$  induced by  $F_{\tilde{\gamma}}$ .

### 5.3 Proof of the theorem

We split the proof into various steps.

**Lemma 5.6.** *Let  $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^3)$ . For any neighbourhood  $N_\alpha$  of  $\alpha$  in  $C^\infty(\mathcal{X}, \mathbb{R}^3)$  we can find a neighbourhood  $V_{F_\alpha}$  of  $F_\alpha$  in  $C^\infty(\mathcal{X}, \mathcal{Y})$  such that for any  $G \in V_{F_\alpha}$  there exists  $\bar{\alpha} \in N_\alpha$  so that*

$$F_{\bar{\alpha}} \text{ is equivalent to } G.$$

*Proof.* Let  $N_\alpha$  be a neighbourhood of  $\alpha$ . Let also  $V_{F_\alpha}$  be a neighbourhood of  $F_\alpha$  in  $C^\infty(\mathcal{X}, \mathcal{Y})$ . We want to show that, reducing  $V_{F_\alpha}$ , any  $G \in V_{F_\alpha}$  is equivalent to a map having the third component equal to  $t$ . Write  $G$  in components as  $G = (G_1, G_2, G_3) : \mathcal{X} \rightarrow \mathbb{R}^2 \times \mathbb{S}^1$ . Then, provided  $V_{F_\alpha}$  is sufficiently small, for any  $x \in M$  the function  $t \in \mathbb{S}^1 \rightarrow G_3^x(t) := G_3(x, t) \in \mathbb{S}^1$

is close to the identity in  $C^\infty(\mathbb{S}^1, \mathbb{S}^1)$ , and therefore it is invertible. Let  $g^x(\cdot) : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be its inverse; note that the map  $(x, t) \in \mathcal{X} \rightarrow g^x(t) \in \mathbb{S}^1$  is smooth. Moreover  $g^x(G_3^x(t)) = t$  and  $G_3^x(g^x(s)) = s$ . Setting  $\alpha = (\alpha_x, \alpha_z) \in \mathbb{R}^2 \times \mathbb{R}$ , we define  $\bar{\alpha} : \mathcal{X} \rightarrow \mathbb{R}^3$  (writing  $t$  in place of  $s$ ) as

$$\bar{\alpha}(x, t) := (G_1(x, g^x(t)), G_2(x, g^x(t)), \alpha_z(x, t)), \quad (x, t) \in \mathcal{X}. \quad (5.7)$$

Since  $\bar{\alpha}$  depends continuously on  $G$ , possibly restricting the neighbourhood  $V_{F_\alpha}$  we can ensure that  $\bar{\alpha} \in N_\alpha$ . Then the map  $(x, t) \in \mathcal{X} \rightarrow F_{\bar{\alpha}}(x, t) := (\pi \circ \bar{\alpha}(x, t), t) \in \mathcal{Y}$  satisfies

$$F_{\bar{\alpha}}(x, t) = (G_1(x, g^x(t)), G_2(x, g^x(t)), t), \quad (x, t) \in \mathcal{X}.$$

The map  $\phi : (x, t) \in \mathcal{X} \rightarrow \phi(x, t) := (x, g^x(t)) \in \mathcal{X}$  is a diffeomorphism of  $\mathcal{X}$ . Since  $G \circ \phi = F_{\bar{\alpha}} = \text{id}_{\mathcal{Y}} \circ F_{\bar{\alpha}}$ , it follows that  $F_{\bar{\alpha}}$  is equivalent to  $G$ .  $\square$

**Corollary 5.7.** *Let  $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^3)$ . For any neighbourhood  $N_\alpha$  of  $\alpha$  in  $C^\infty(\mathcal{X}, \mathbb{R}^3)$  there exists  $\bar{\alpha} \in N_\alpha$  such that*

$$F_{\bar{\alpha}} \in \text{Stab}(\mathcal{X}, \mathcal{Y}).$$

*Proof.* Let  $V_{F_\alpha}$  and  $\bar{\alpha}$  be as in Lemma 5.6. Since  $\text{Stab}(\mathcal{X}, \mathcal{Y})$  is dense in  $C^\infty(\mathcal{X}, \mathcal{Y})$ , there exists  $H \in V_{F_\alpha} \cap \text{Stab}(\mathcal{X}, \mathcal{Y})$ . Therefore, choosing  $G = H$  in Lemma 5.6 we deduce that  $F_{\bar{\alpha}}$  is equivalent to  $H \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ . As a consequence, also  $F_{\bar{\alpha}} \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ .  $\square$

**Lemma 5.8.** *Let  $\beta \in C^\infty(\mathcal{X}, \mathbb{R}^3)$  be such that  $F_\beta \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ . For any neighbourhood  $W_\beta$  of  $\beta$  in  $C^\infty(\mathcal{X}, \mathbb{R}^3)$  there exists a map  $\hat{\beta} \in W_\beta$  such that  $F_{\hat{\beta}} \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  and*

$$p : \mathcal{Y} \rightarrow \mathbb{S}^1 \text{ is a stratified Morse function on } (\mathcal{Y}, \{Y_j\}_{F_{\hat{\beta}}}). \quad (5.8)$$

*Proof.* Since  $F_\beta \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ , from Remark 5.3 it follows that  $p|_{Y_0}$  and  $p|_{Y_1}$  have no critical points. The examples in Remark 5.4 show however that  $F_\beta$  must be slightly perturbed in order the function  $p$  to be a stratified Morse function. Let  $W_\beta$  be a neighbourhood of  $\beta$  in  $C^\infty(\mathcal{X}, \mathcal{Y})$ . Recalling Definition 2.3, in order to prove (5.8) it remains to show that there exists  $\hat{\beta} \in W_\beta$  such that the function  $p : (\mathcal{Y}, \{Y_j\}_{F_{\hat{\beta}}}) \rightarrow \mathbb{R}$  satisfies the following three properties:

- (1) if  $(\bar{y}, \bar{t}) \in Y_3$  then all curves in  $Y_2$  having  $(\bar{y}, \bar{t})$  as an end point cannot have a limit tangent line at  $(\bar{y}, \bar{t})$  contained in  $\{t = \bar{t}\}$ ;
- (2) all critical values of  $p|_{Y_2}$  are distinct, and distinct from  $p(Y_3)$ , in turn consisting of distinct points of  $\mathbb{S}^1$ ;
- (3) the critical points of  $p|_{Y_2}$  are nondegenerate.

Let us consider the stratification  $\{Y_j\}_{F_{\hat{\beta}}}$  induced by  $F_{\hat{\beta}}$ , and let  $(\bar{y}, \bar{t}) \in Y_3$ . Note that if  $(\bar{y}, \bar{t}) \in Y_3^3$  then only one of the three double curves concurring at  $(\bar{y}, \bar{t})$  may have the property of having a limit tangent line contained in  $\{t = \bar{t}\}$ : indeed, if two of them share this property, then there is a fold surface in  $Y_1$  having the tangent plane at  $(\bar{y}, \bar{t})$  parallel to  $\{t = \bar{t}\}$ , which is in contradiction with Remark 5.3. Recall also that, if  $(\bar{y}, \bar{t}) \in Y_3^1$  is a

swallow tail, then the two cusp curves and the double curve concurring at  $(\bar{y}, \bar{t})$  have the same tangent vector there.

Assume now that there is a curve  $c$  in  $Y_2 \cup \{(\bar{y}, \bar{t})\}$  having  $(\bar{y}, \bar{t})$  as an end point with a limit tangent line at  $(\bar{y}, \bar{t})$  contained in  $\{t = \bar{t}\}$ . Let  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in C^\infty([0, 1], \mathcal{Y})$  be a regular parameterization of  $c$  having  $(\bar{y}, \bar{t})$  as an end point, so that  $\lambda(0) = (\bar{y}, \bar{t})$  and  $\lambda'_3(0) = 0$ . Pick a smooth function  $a : \mathcal{Y} \rightarrow \mathbb{R}$  satisfying

$$a(\bar{y}, \bar{t}) = 0, \quad \frac{d}{d\sigma} a(\lambda(\sigma))|_{\sigma=0} \neq 0. \quad (5.9)$$

Let  $\Omega \subset \mathcal{Y}$  be a neighbourhood of  $(\bar{y}, \bar{t})$  small enough so that all points in  $Y_3 \setminus \{(\bar{y}, \bar{t})\}$  are not contained in  $\bar{\Omega}$ ; let also  $\chi$  be a smooth nonnegative function on  $\mathcal{Y}$  supported in  $\Omega$  which is constantly equal to one in a small neighbourhood of  $(\bar{y}, \bar{t})$  and let  $\varepsilon \in \mathbb{R}$ . Define the function  $j : \mathcal{Y} \rightarrow \mathbb{R}$  as

$$j(y, t) := t + \varepsilon \chi(y, t) a(y, t), \quad (y, t) \in \mathcal{Y}.$$

Then, provided  $|\varepsilon|$  is sufficiently small, we have that for any  $y \in \mathbb{R}^2$  the function  $t \in \mathbb{S}^1 \rightarrow j(y, t) \in \mathbb{S}^1$  is invertible. Therefore, the map  $\psi : \mathcal{Y} \rightarrow \mathcal{Y}$  defined as  $\psi(y, t) := (y, j(y, t))$  is a diffeomorphism of  $\mathcal{Y}$ . Define  $G : \mathcal{X} \rightarrow \mathcal{Y}$  as

$$G := \psi \circ F_\beta. \quad (5.10)$$

Then  $G$  is equivalent to  $F_\beta$ , and therefore since by assumption  $F_\beta \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ , it follows that also  $G \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ . Moreover

$$G(x, t) = (\pi \circ \beta(x, t), i^x(t)), \quad (x, t) \in \mathcal{X},$$

where

$$i^x(t) := t + \varepsilon \chi(F_\beta(x, t)) a(F_\beta(x, t)), \quad (x, t) \in \mathcal{X}.$$

Let  $V_{F_\beta}$  be the neighbourhood of  $F_\beta$  given by Lemma 5.6 (take  $\beta = \alpha$  and  $W_\beta = N_\alpha$ ). If  $|\varepsilon| \ll 1$  we have  $G \in V_{F_\beta}$ . Let us now consider the stratification  $\{Y_0^G, Y_1^G, Y_2^G, Y_3^G\}$  induced on  $\mathcal{Y}$  by  $G$ : by equality (5.10) it follows that such a stratification is the image through  $\psi$  of the stratification induced by  $F_\beta$ . The first relation in (5.9) implies that  $(\bar{y}, \bar{t}) \in Y_3^G$ ; moreover  $\psi(c) \subset Y_2^G \cup \{(\bar{y}, \bar{t})\}$  is regularly parameterized in a neighbourhood of  $(\bar{y}, \bar{t})$  by  $\sigma \in [0, 1] \rightarrow \psi(\lambda(\sigma)) = (\lambda_1(\sigma), \lambda_2(\sigma), j(\lambda(\sigma)))$ . Since

$$\frac{d}{d\sigma} j(\lambda(\sigma))|_{\sigma=0} = \lambda'_3(0) + \varepsilon \frac{d}{d\sigma} a(\lambda(\sigma))|_{\sigma=0} = \varepsilon \frac{d}{d\sigma} a(\lambda(\sigma))|_{\sigma=0}, \quad (5.11)$$

the second relation in (5.9) guarantees that the right hand side of (5.11) is nonzero. It follows that assertion (1) is satisfied for the stratification induced by  $G$ .

Denote by  $\phi : \mathcal{X} \rightarrow \mathcal{X}$  the diffeomorphism of  $\mathcal{X}$  defined by  $\phi(x, t) := (x, (i^x)^{-1}(t))$ . From the proof of Lemma 5.6, it follows that there exists  $\hat{\beta} \in W_\beta$  such that

$$G \circ \phi = F_{\hat{\beta}}. \quad (5.12)$$

Since  $G \in \text{Stab}(\mathcal{X}, \mathcal{Y})$  it follows that  $F_{\hat{\beta}} \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ . Moreover, (5.12) implies that the stratification of  $\mathcal{X}$  associated with  $G$  is the image through  $\phi$  of the stratification associated

with  $F_\beta$ , and that the stratification of  $\mathcal{Y}$  induced by  $G$  coincides with the the stratification induced by  $F_{\hat{\beta}}$ . We conclude therefore that the stratification induced by  $F_{\hat{\beta}}$  satisfies condition (1).

If we now replace the function  $a$  in the previous argument with the function  $a + b$ , where  $b : \mathcal{Y} \rightarrow \mathbb{R}$  is a smooth function satisfying  $b(\bar{y}, \bar{t}) \neq 0$  and  $db_{(\bar{y}, \bar{t})} = 0$ , we obtain that  $p(Y_3)$  consists of distinct points of  $\mathbb{S}^1$ .

Considering the stratification of  $\mathcal{Y}$  induced by  $F_{\hat{\beta}}$ , from condition (1) we deduce that  $p|_{Y_2}$  has no critical points on the boundary points of  $Y_2$ . Therefore, being all critical points of  $p|_{Y_2}$  interior to any arc of  $Y_2$  we can argue using one dimensional Morse theory, and obtain assertion (3). Also assertion (2) follows in a standard way.  $\square$

We are now in the position to conclude the proof of the theorem.

We apply Corollary 5.7 to  $\alpha = \gamma$  and to  $N_\alpha \subseteq U_\gamma$ , and we obtain a corresponding map  $\bar{\gamma} \in N_\alpha$  satisfying  $F_{\bar{\gamma}} \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ . Possibly reducing  $N_\alpha$ , we can assume that  $\bar{\gamma}(\cdot, 0) \in U_{e_0}$  and  $\bar{\gamma}(\cdot, 1) \in U_{e_1}$ . Since  $F_{\bar{\gamma}} \in \text{Stab}(\mathcal{X}, \mathcal{Y})$ , we can now apply Lemma 5.8 to  $\beta = \bar{\gamma}$  and to  $W_\beta$  a neighbourhood of  $\bar{\gamma}$  satisfying  $W_\beta \subseteq N_\alpha$ : if we set

$$\tilde{\gamma} := \hat{\beta},$$

it follows that  $\tilde{\gamma} \in W_\beta \subseteq N_\alpha \subseteq U_\gamma$ , and  $\tilde{\gamma}$  satisfies assertions (ii) and (iii). Moreover  $\tilde{\gamma}(\cdot, 0) \in U_{e_0}$  and  $\tilde{\gamma}(\cdot, 1) \in U_{e_1}$ .

Since  $\gamma(\cdot, t) \in \text{Emb}(M, \mathbb{R}^3)$  and  $\text{Emb}(M, \mathbb{R}^3)$  is open in  $C^\infty(M, \mathbb{R}^3)$  it follows (possibly reducing  $W_\beta$ ) that  $\tilde{\gamma}(M, 0)$  and  $\tilde{\gamma}(M, 1)$  are isotopic. Since by assumption  $\pi|_{\Sigma_j} \in \text{Stab}(\Sigma_j, \mathbb{R}^2)$ , possibly reducing  $W_\beta$  and recalling that  $\text{Stab}(\Sigma_j, \mathbb{R}^2)$  is open in  $C^\infty(\Sigma_j, \mathbb{R}^2)$  it follows that  $\pi|_{\tilde{\gamma}(M, 0)} \in \text{Stab}(\Sigma_0, \mathbb{R}^2)$  and  $\pi|_{\tilde{\gamma}(M, 1)} \in \text{Stab}(\Sigma_1, \mathbb{R}^2)$ . This proves (i) and concludes the proof of the theorem.

## 6 Completeness of moves

Let  $\pi$ ,  $e_0$ ,  $e_1$ ,  $\tilde{\gamma}$ ,  $(\mathcal{Y}, \{Y_j\}_{F_{\tilde{\gamma}}})$  and  $p$  be as in Theorem 5.5. By compactness, the set  $\text{crit}(p|_{Y_2})$  of critical points of  $p|_{Y_2}$  is finite. Since also  $Y_3$  consists of isolated points, it follows that  $p|_{Y_2}(\text{crit}(p|_{Y_2})) \cup p(Y_3)$  is a *finite* set of points of  $\mathbb{S}^1$ , that we call the set of critical times. If  $t_0$  is not a critical time, the apparent contour (given by a normal slice, i.e. the transversal intersection of  $Y_1 \cup Y_2 \cup Y_3$  with  $\{t = t_0\}$ ) varies smoothly, and its topology does not change. Hence we can find a smooth path of diffeomorphisms of  $\mathbb{R}^2$  connecting the apparent contours at times  $t_1$  and  $t_2$  whenever the interval  $[t_1, t_2]$  does not contain any critical time. Moreover, in view of the classification results on singularities of stable mappings between 3-manifolds (Section 2), we obtain the following corollary (compare with Figure 3).

**Corollary 6.1.** *A point  $(y, t) \in \text{crit}(p|_{Y_2}) \cup Y_3$  lies in one of the following classes, each determining a move in the list of Section 4:*

- $(y, t) \in Y_2^2$  is a local maximum (resp. a local minimum) of a double curve: moves of type K (resp. of type  $K^{-1}$ ).
- $(y, t) \in Y_2^1$  is a local maximum (resp. local minimum) of a cusp curve that bounds folds going downwards (resp. upwards): move L (resp. move  $L^{-1}$ ).
- $(y, t) \in Y_2^1$  is a local maximum (resp. local minimum) of a cusp curve that bounds folds going upwards (resp. downwards): move B (resp. move  $B^{-1}$ ).
- $(y, t) \in Y_3^1$ : moves of type S.
- $(y, t) \in Y_3^2$ : moves of type C.
- $(y, t) \in Y_3^3$ : moves of type T.

**Remark 6.2.** A more precise classification of each move can be obtained by looking at the orientation of the various folds involved and at the relative depth of the preimages in the singular set  $\bar{X}_1 = S_1(F_{\tilde{\gamma}})$  with respect to the regular preimages. For example, a points in  $Y_3^3$  has three distinct preimages in  $X_1$  which can be ordered according to the  $z$  coordinate (dropped by the projection  $\pi$ ). Each of the three involved folds, which are transversal with respect to “time”  $t$ , carries a natural orientation and hence contribute with a sign. The relative depth of the three singular preimages with respect to the remaining regular preimages provides for the three nonnegative integers parameters as explained in Section 4.1. Note that a cusp curve forces the orientation of the two adjacent folds. In this way we have a precise one-to-one correspondence between the list of moves of Section 4 and the list in Corollary 6.1.

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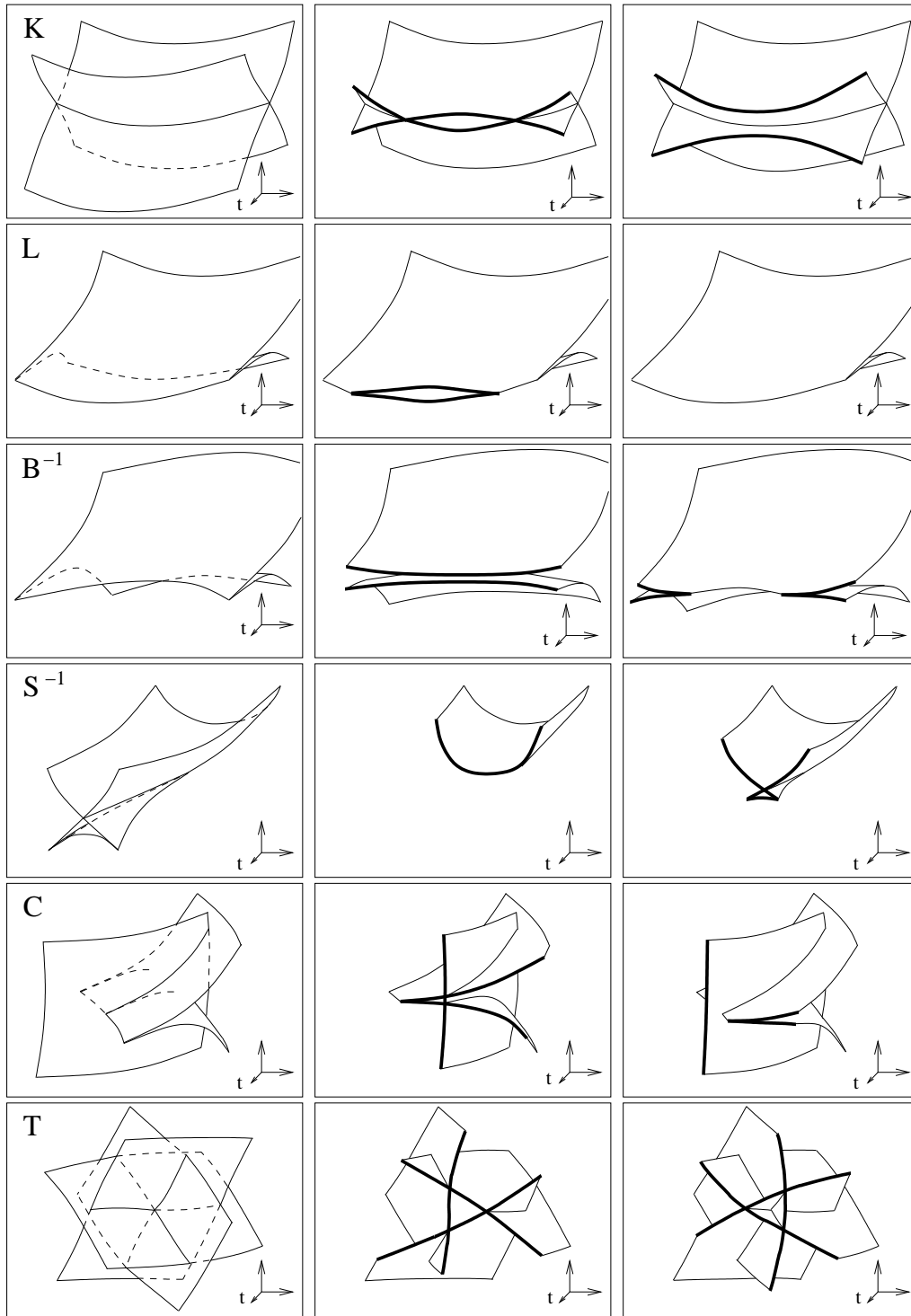


Figure 3: Each row represents the fold surfaces  $Y_1$  near different types of critical points for  $p$ : K, L, inverse of B, inverse of S, C, T. The second and third pictures on each row show the cutting with  $t = \text{const}$  (apparent contour) before and after the critical value.



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