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Willmore-type Energies of Curves and Surfaces

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Abstract

In this thesis we discuss variational problems concerning Willmore-type energies of curves and surfaces. By Willmore-type energy of an immersed manifold we mean a functional depending on the volume (length or area) of the manifold and on some L^p-norm of the mean curvature of the manifold. The functionals we consider are the p-elastic energy of an immersed curve, defined as the sum of length and L^p-norm of the curvature vector, and the Willmore energy of an immersed surface, which is the L^2-norm of the mean curvature of the surface. We shall consider problems of a variational nature both in the smooth setting of manifolds and in the context of geometric measure theoretic objects. The necessary definitions and preliminaries are collected and discussed in Chapter 1.

We then address the following problems.

- In Chapter 2 we consider a gradient flow of the p-elastic energy of immersed curves into complete Riemannian manifolds. We investigate the smooth convergence of the flow to critical points of the functional, proving that suitable hypotheses on the sub-convergence of the flow imply the existence of the full limit of the evolving solution.

- In Chapter 3 we address the problem of finding a generalized weak definition of p-elastic energy of subsets of the plane satisfying some meaningful variational requirement. We find such a definition by characterizing a suitable relaxed functional, of which we then discuss qualitative properties and applications.

- In Chapter 4 we study the minimization of the Willmore energy of surfaces with boundary under different boundary conditions and constraints. We focus on the existence theory for such minimization problems, proving both existence and non-existence theorems, and some functional inequalities.

The results are obtained by the author partly in collaboration with Matteo Novaga, and they comprise contributions from the research papers [NP20; Poz20a; Poz20b; Poz20c] as well as some unpublished results.

Contents

NOTATION AND SYMBOLS

$\sharp X$	cardinality of a set X
$\langle \cdot, \cdot \rangle$	a given scalar product
$\langle \cdot, \cdot \rangle_{V^\star, V}$	duality pairing between a Banach space V and its dual $V^\star$
$\lvert \cdot \rvert$	a norm induced by a given scalar product
$\mathring{E}$	interior of a set E
$A^\star$	adjoint of a linear map A
$B_r(x)$	open ball of center x and radius r in a given metric space
$\overline{B_r(x)}$	closed ball of center x and radius r in a given metric space
B_r	open ball of center 0 and radius r in $\mathbb{R}^n$
$C^k(U)$	real valued C^k-functions on an open set U in a manifold
$C^k(U, N)$	C^k-functions on an open set U in a manifold with values in a manifold N
$C_c(X)$	continuous functions with compact support on a metric space X
$C_0(X)$	closure of $C_c(X)$ with respect to the supremum norm
$\mathrm{diam}(Y)$	diameter of a set Y in a metric space
$d_{\mathcal{H}}$	Hausdorff distance in a given metric space
df_x	differential of a differentiable function $f : U \subset M \to N$ between manifolds at x
$d^M f_x$	tangential differential of f at a point x along $T_x M$ for a rectifiable set M
$E \Delta F$	symmetric difference between two sets E and F
(γ)	support of a rectifiable curve γ
$\mathcal{H}^s$	s-dimensional Hausdorff measure in a given metric space, for any $s \geq 0$
id	identity/inclusion map between given sets
$\mathrm{Lip}(f)$	Lipschitz constant of a function f between metric spaces
$L^p_{loc}(\mu)$	real valued locally p-integrable functions with respect to the measure μ
$L^p(\mu)$	real valued p-integrable functions with respect to the measure μ
$L^p_{loc}(\mu, \mathbb{R}^n)$	$\mathbb{R}^n$-valued functions whose components belong to $L^p_{loc}(\mu)$
$L^p(\mu, \mathbb{R}^n)$	$\mathbb{R}^n$-valued functions whose components belong to $L^p(\mu)$
$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$
$\mathcal{L}^n(E) \equiv \lvert E \rvert$	Lebesgue measure of a set $E \subset \mathbb{R}^n$
$L(\gamma)$	length of a rectifiable curve γ
$M_{\mathfrak{g}}$	closed orientable surface of genus $\mathfrak{g}$
$\mathcal{N}_\varepsilon(Y)$	open tubular neighborhood of width ε of a subset Y of a metric space
$\mathbb{N}$	natural numbers
$\mathbb{Q}$	rational numbers
$\mathbb{R}^n$	Euclidean n-dimensional space
$\mathbb{S}^n$	unit n-dimensional sphere
$\mathrm{spt}\mu$	support of a measure μ
$T_x M$	tangent space of M at x, or approximate tangent space of a rectifiable set M at x
χ_E	characteristic function of a set E
$\mathfrak{X}(M)$	space of vector fields on a manifold M
$V \subset\subset U$	the closure of the set V is compact and it is contained in U
$W^{k,p}(\Omega)$	Sobolev space on an open set $\Omega \subset \mathbb{R}^n$, for $k \in \mathbb{N}$, $k \geq 1$, and $p \in [1, +\infty]$
$W^{k,p}(\mathbb{S}^1)$	2π-periodic functions f on $\mathbb{R}$ s.t. $f\rvert_I \in W^{k,p}(I)$, for any $I \subset \mathbb{R}$ bounded interval
$W^{k,p}(\Omega, \mathbb{R}^n)$	$\mathbb{R}^n$-valued functions whose components belong to $W^{k,p}(\Omega)$
$W^{k,p}(\mathbb{S}^1, \mathbb{R}^n)$	$\mathbb{R}^n$-valued functions whose components belong to $W^{k,p}(\mathbb{S}^1)$
ω_n	volume of the n-dimensional unit ball in $\mathbb{R}^n$

Introduction

In this thesis we study problems of a variational nature regarding Willmore-type energies of curves and surfaces. By Willmore-type energies we mean functionals associating to an immersed manifold a number depending on its volume (length or area) and on some L^p-norm of the mean curvature. All the basic definitions and results needed in the sequel are recalled and discussed in Chapter 1, and we will give them for granted in this introduction.

The class of functionals we are interested in owes its name to the celebrated *Willmore energy*, that is one of the geometric energies we will study in this thesis. If $\varphi : \Sigma \hookrightarrow (M^m, g)$ is a smooth immersion of a 2-dimensional manifold Σ in a Riemannian manifold of dimension $m \geq 3$, its Willmore energy is defined by

$$\mathcal{W}(\varphi) := \int_\Sigma |H|^2 \, d\mu,$$

where H is the mean curvature vector of the immersion, integration is understood with respect to the induced area measure (see Section 1.1.2), and the integral might diverge. In the last fifty years a lot of interest has been devoted to such a functional, starting from the work of Thomas Willmore [Wil65] in 1965, that is the reason why today the functional is named after him.

The functional enjoys several interesting geometric properties. The most important one is probably its *conformal invariance*, that is, conformal transformations of the ambient metric do not change the value of the Willmore energy (see Theorem 4.1.4 and Remark 4.1.5).

In [Wil65], the author initiated the study of minimization problems on $\mathcal{W}$ by proving that round spheres in $\mathbb{R}^3$, having energy equal to 4π, are the only absolute minimizers of $\mathcal{W}$ among closed immersed surfaces in $\mathbb{R}^3$ (Theorem 4.1.1). The result was then extended to higher codimensions by Chen in [Che71a; Che71b]. Such an existence theorem of global minimizers can be also proved by using the conformal invariance of the functional [Che74; Whi73; Wei78]. Actually, the conformal invariance of $\mathcal{W}$ in the Euclidean space leads to the celebrated *Li–Yau inequality*, which is an estimate that not only identifies the value 4π of the infimum of $\mathcal{W}$ among immersions of closed surfaces, but also implies that closed immersed surfaces in $\mathbb{R}^n$ with self-intersections have energy greater or equal than 8π (Corollary 4.1.12, Corollary 1.3.8). It is then clear that the Li–Yau inequality has a great importance in the study of variational problems.

Moreover, in [Wil65] the author stated his celebrated conjecture about the minimization of the Willmore energy among tori (Conjecture 4.1.6). The *Willmore conjecture* states that the infimum of $\mathcal{W}$ among immersions of tori in $\mathbb{R}^3$ equals $2\pi^2$ and that it is achieved only by the torus of revolution given by a circle of radius 1 whose center is located at distance $\sqrt{2}$ from the axis of revolution, up to conformal transformation of the ambient. The conjecture then motivated the study of the minimization of $\mathcal{W}$ among surfaces of a fixed topology, that is, closed surfaces of given genus in $\mathbb{R}^n$. By the works of Simon [Sim93] and Bauer–Kuwert [BK03] we now know that for any genus $\mathfrak{g} \in \mathbb{N}$ there is a minimizer of $\mathcal{W}$ among immersed closed surfaces of genus $\mathfrak{g}$ in $\mathbb{R}^n$, and such a minimizer is actually an embedding with Willmore energy strictly less than 8π (Theorem 4.1.8). Also, in [Sim93] the author introduced the use of *varifolds* and, more generally, of measure theoretic tools that allow the use of direct methods in Calculus of Variations in the study of the minimization of $\mathcal{W}$. We will be particularly interested in such methods and in measure theoretic objects in the sequel.

The Willmore conjecture has been eventually proved by Marques–Neves in [MN14]. In this paper, as well as in other very recent works, the authors develop deep variational tools useful for the study of *minimal surfaces* in Riemannian manifolds. Their results (see Theorem 4.1.9) are actually even stronger than the statement of the Willmore conjecture and their methods involve both techniques from Riemannian Geometry, Topology, and Geometric Measure Theory.

All the above mentioned results, as well as some others related to the existence theory of minimization problems on the Willmore energy, are recalled more in detail in Section 4.1.

The second class of Willmore-type energies we want to consider are defined on curves. If $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is an immersed smooth closed curve, then for $p \in [1, +\infty)$ we define its *p-elastic energy* by

$$\mathcal{E}_p(\gamma) := L(\gamma) + \frac{1}{p} \int_{\mathbb{S}^1} |k|^p \, ds,$$

where $L(\gamma)$ is the length of γ, k is the curvature vector of γ, and integration is understood with respect to the induced length measure. The same definition will be also used for curves immersed in Riemannian manifolds. The presence of the constant $\frac{1}{p}$ in front of the integral is just the choice of a normalization and it is clearly meaningless from a variational point of view, and in fact in the literature one can find different choices of constant weights between the length and the curvature terms. The p-elastic energy can be seen as a natural generalization of the more common *elastic energy*, which is just $\mathcal{E}_2$. A great interest toward the elastic energies has grown since the seminal papers of Langer–Singer [LS84c; LS84a; LS84b; LS85; LS87]. In these works the authors studied and classified the critical points of the energy functional $\int |k|^2$, which are called *elastic curves* or *elasticae*, both under constraints of fixed length or not, both in Euclidean space or Riemannian manifolds of constant sectional curvature. The main motivation stated in [LS84c] was to investigate closed geodesics in constant sectional curvature Riemannian manifolds by means of the study of a *gradient flow* of the energy $\int |k|^2$ of curves with fixed constrained length. In fact, geodesics are absolute minimizers of $\int |k|^2$, as they have zero curvature.

A gradient flow of a geometric energy E defined on curves is an evolution equation of the form $\partial_t \gamma = -\nabla_\gamma E$ that prescribes the motion of the solution $\gamma = \gamma(t, x)$, for $(t, x) \in [0, T) \times \mathbb{S}^1$, in such a way that the driving velocity is the opposite of the gradient $\nabla_\gamma E$ of the energy evaluated at γ, defined with respect to some duality or scalar product (see Section 2.1 for a more detailed introduction). In such a way, the energy $E(\gamma(t, \cdot))$ decreases in time. Gradient flows are fundamental in the investigation of geometric functionals, both for the study of their own properties and for the understanding of variational features of the functionals. A gradient flow for the functional $\int |k|^2$ has been introduced, indeed, in [LS85; LS87], under the name of *curve straightening flow*. The authors proved the convergence of this flow, that is, the existence of a limit for the evolving curve asymptotically as time increases, by finding remarkable functional analytic properties on the considered energy, namely the fact that the functional satisfies a Palais–Smale condition.

Since the works of Langer–Singer, many flows of geometric energies of curves have been widely investigated. In particular, we will be interested in gradient flows of the p-elastic energy $\mathcal{E}_p$ of curves in Riemannian manifolds. The study of the L^2-gradient flow of $\mathcal{E}_2$ in $\mathbb{R}^2$, that is, the flow defined with respect to L^2-duality, has been firstly studied by Polden in [Pol96] both for what concerns the *short time* behavior of the flow, i.e., local existence and uniqueness, and the *long time* behavior, proving that the solution exists for all times. The systematic study of this flow in $\mathbb{R}^n$ is due to Dziuk–Kuwert–Schätzle [DKS02].

In the literature authors sometimes refer to the functional $\mathcal{E}_2$ as the *Willmore energy of curves*, for the obvious analogy with the Willmore energy of surfaces. However, the similarities between the elastic energies and the Willmore one should not be sought in the mathematical properties of these functionals considered independently. Indeed their are geometrically quite different as, for example, none of the summands defining $\mathcal{E}_p$ for $p > 1$ is scaling invariant, and then not at all conformally invariant. On the other hand, the functionals $\mathcal{E}_p$ and $\mathcal{W}$ are very often considered good integral quantities representing the total *bending energy* of immersed curves or surfaces respectively. Actually, the use of $\mathcal{E}_2$ for modelling the bending energy of a rod goes back to Daniel Bernoulli; for a nice survey on the history of elastic and bending energies one can see [Tru83]. The Willmore energy, or suitable variants involving also the surface area, appears as a model of the bending energy of biological membranes [Can70; Hel73; GGS10; EFH17].

Even if the elastic energies and the Willmore energy satisfy different geometric properties, we have to mention a perhaps surprising fact that relates these functionals. Let $\gamma : \mathbb{S}^1 \to \mathbb{H}^2$ be a regular closed curve in the hyperbolic plane represented by the upper half-plane $\{y > 0\} \subset \mathbb{R}^2$ with the hyperbolic metric, and assume $\mathbb{R}^2 = \{z = 0\} \subset \mathbb{R}^3$ with the standard choice of orthonormal axes in $\mathbb{R}^3$. Then, if $\Sigma \subset \mathbb{R}^3$ is the torus of revolution in the Euclidean $\mathbb{R}^3$ obtained by the rotation of the support of γ about the x-axis, it holds

$$\mathcal{W}(\Sigma) = \frac{\pi}{2} \int_{\mathbb{S}^1} |k|^2 \, ds,$$

where k and ds here are the curvature and the length measure of γ as a curve in $\mathbb{H}^2$. This equality has been proved in [LS84a, pp. 532-533] and independently in [BG86]. Studying the elastic energy of a closed curve in $\mathbb{H}^2$ it is then possible to prove that among tori of revolution in $\mathbb{R}^3$ the Willmore energy achieves its minimum as predicted by the Willmore conjecture. This was at that time a first evidence supporting the conjecture.

As already mentioned, besides the properties of Willmore-type energies as defined on smooth manifolds, we will be interested in generalizations of those in order to solve minimization problems by means of direct methods in Calculus of Variations in suitable weak formulations. In particular, we will exploit the concept of *varifold* as a geometric measure theoretic generalization of immersed manifold. We shall recall the basic theory of varifolds in Section 1.2, but for the time being let us consider a k-dimensional rectifiable set $M \subset \mathbb{R}^n$ with $k \in \{1, 2\}$. For a given positive function $\theta : M \to \mathbb{N}$ that is $\mathcal{H}^k$-locally integrable on M we will see that the measure

$$\mu = \theta \, \mathcal{H}^k \llcorner M$$

identifies a varifold V_μ belonging to the class of *integer rectifiable varifolds*. In case there exists a function $H : M \to \mathbb{R}^n$ which is μ-locally integrable such that for every $X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ one has

$$\int_M \mathrm{div}_M X \, d\mu = -k \int \langle X, H \rangle \, d\mu + \int X \, d\sigma,$$

where $\mathrm{div}_M X$ is the tangential divergence of X on M and σ is a vector valued Radon measure σ which is singular with respect to μ, then H is said to be the *generalized mean curvature of* V_μ and σ_V is the *generalized boundary of* V_μ. Hence it is possible to define the Willmore energy of such a varifold by

$$\mathcal{W}(V_\mu) := \int |H|^2 \, d\mu \ \in [0, +\infty].$$

In the framework of measures, one can then hope to exploit compactness properties to solve minimization problems.

The concept of varifold was firstly introduced by Young in [You51], but the modern definition and theory are due to the seminal works of Almgren (see for example [Alm66]), in which the author uses the theory of varifolds in order to solve the Plateau's problem. The word varifold was, indeed, coined by Almgren as a shortcut for *vari*ational mani*folds*, underlying the role of varifolds in solving minimization problems. We have to mention also the celebrated works of Allard [All72; All75], which have been equally fundamental in the development of the theory.

As already anticipated, we can attribute the first application of varifolds to the study of minimization problems on the Willmore energy to Simon in [Sim93], in which he proved the existence of a minimizer of $\mathcal{W}$ among smooth immersed tori in $\mathbb{R}^n$, taking a first step towards the Willmore conjecture. The method of Simon has then been widely used in the last years for solving different minimization problems. In his arguments a fundamental role is played by *monotonicity formulas*, that is, the existence of some monotone function involving the area and the Willmore energy of a varifold (see Section 1.3).

In this thesis we shall use the theory of varifolds not only for solving minimization problems, but also for defining new notions of generalized elastic energies. Our general scope is to deepen our knowledge of Willmore-type energies in the aspects concerning existence theory of generalized minimization problems and properties of gradient flows of these functionals.

Now we briefly introduce the content of the thesis and the main results collected in the sequel. For further details, bibliographical references, and motivations about each problem we refer the reader to the corresponding chapters.

Elastic flows of curves

Chapter 2 is devoted to the study of the long time behavior of the $(L^p, L^{p'})$-gradient flow of the p-elastic energy of curves in complete Riemannian manifolds. If $\gamma : \mathbb{S}^1 \to M$ is a smooth immersed curve (with non-vanishing curvature if $p > 2$) in a complete Riemannian manifold (M^m, g) and $p \geq 2$, the first variation of the p-elastic energy at γ turns out to be

$$\delta\mathcal{E}_p[\varphi] = \int_{\mathbb{S}^1} \left\langle \nabla^2 |k|^{p-2}k + \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k, \tau)\tau, \varphi \right\rangle ds,$$

for any smooth vector field φ along γ, where R is the Riemann tensor of (M, g) and ∇ is the normal connection along γ (see Section 1.1 and Proposition 2.3.12). The same formula holds if γ is an immersion of class $W^{4,p}$ (with non-vanishing curvature if $p > 2$). It follows that we can define the $(L^p, L^{p'})$-gradient flow of $\mathcal{E}_p$ starting from a smooth immersed curve $\gamma_0 : \mathbb{S}^1 \to M$ to be a smooth solution $\gamma : [0, T) \times \mathbb{S}^1 \to M$ of the evolution equation

$$\begin{cases} \partial_t \gamma = -\Big(\nabla^2 |k|^{p-2}k + \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k, \tau)\tau\Big) & \text{on } [0,T) \times \mathbb{S}^1, \\ \gamma(0, \cdot) = \gamma_0(\cdot) & \text{on } \mathbb{S}^1. \end{cases}$$

The concept of gradient flow is discussed more in detail in Section 2.1. The energy $\mathcal{E}_p$ clearly decreases along the flow and, in case the maximal time of existence $T = +\infty$, we are interested in investigating whether there exists the limit $\lim_{t\to+\infty} \gamma(t, \cdot)$ in C^m for suitable $m \in \mathbb{N}$.

Geometric evolution equations like the one above turn out to be parabolic in the unknown given by the parametrization γ. Therefore, one can hope to apply useful parabolic estimates implying bounds on the Sobolev norm of the curvature vector of γ uniformly in time (up to reparametrizations). This leads not only to the fact that the maximal time of existence T equals $+\infty$, but also, by Sobolev embeddings, to the *sub-convergence* of the flow. With the term sub-convergence we mean the existence of a sequence of times $t_j \to +\infty$ such that there exists the limit $\lim_j \gamma(t_j, \cdot)$ in C^m for any $m \in \mathbb{N}$, up to reparametrization and isometry of the ambient, and the limit is a critical point of the energy (see Theorem 2.2.8 for example). Isometries of the ambient manifold (or an analogous family of maps) are a priori necessary, indeed the above mentioned Sobolev bounds on the curvature cannot imply that the flow remains in a compact subset of the ambient for any time. As the evolution equation is of fourth order in the parametrization γ, the uniform boundedness cannot be argued by means of a maximum principle either. Therefore, in order to promote the sub-convergence of a flow to its *full convergence*, i.e., the existence of the limit $\lim_{t\to+\infty} \gamma(t, \cdot)$, one needs further results.

In Chapter 2 we employ a strategy based on the use of a so-called *Łojasiewicz–Simon gradient inequality*. Such a method is inspired by the seminal paper [Sim83a] in which Simon firstly introduced the use of such inequalities in the study of the long time behavior of parabolic equations having the form of a gradient flow. This kind of inequalities have been introduced by Łojasiewicz in [Ło63] in the setting of functions defined on *finite* dimensional vector spaces, and the form of such inequalities is as follows. An *analytic* function $f : U \to \mathbb{R}$ defined on an open set $U \subset \mathbb{R}^m$ is said to satisfy a Łojasiewicz–Simon gradient inequality at $x_0 \in U$ if there exist constants $\sigma, C > 0$ and $\theta \in (0, \frac{1}{2}]$ such that

$$|f(x) - f(x_0)|^{1-\theta} \leq C|\nabla f(x)|,$$

for any $x \in B_\sigma(x_0)$. Any analytic function f as above actually satisfies such an inequality in a neighborhood of any point x_0. This fact was firstly proved by Łojasiewicz in [Ło63; Ło65]. The inequality is particularly relevant in case $\nabla f(x_0) = 0$, i.e., x_0 is a critical point, as on the left hand side we find an exponent $1 - \theta \in (0, 1)$. Such an inequality clearly does not hold in case f is merely smooth. The inequality can be extended to analytic functions on *infinite* dimensional Banach spaces [Sim83a] and, as observed also in [Ło84], it can be used to study the evolution of a gradient flow in a neighborhood of a critical point. As proved by Simon in [Sim83a], one may hope that, once the solution of a gradient flow of an energy satisfying a Łojasiewicz–Simon gradient inequality "passes sufficiently close" to a critical point, then it never leaves some neighborhood of the critical point and eventually converges.

In this thesis we will derive a Łojasiewicz–Simon gradient inequality out of a general result of Chill [Chi03], which does not even assume the full analyticity of the functional. Then we apply a strategy inspired by [CFS09] used in the study of the L^2-gradient flow of the Willmore energy of closed surfaces. However, the Łojasiewicz–Simon gradient inequality we derive is stated at a purely functional analytic level and it can be possibly applied to different geometric flows. Such an inequality will be stated as a corollary of Chill's results in Section 2.2.2 as follows.

Corollary 2.2.7 (Abstract Łojasiewicz–Simon gradient inequality). *Let V be a Banach space and let $E : B_{\rho_0}(0) \subset V \to \mathbb{R}$ be an analytic map. Suppose that 0 is a critical point of E. Let $\mathscr{M} : U \to V^\star$ be the Fréchet first derivative and $\mathscr{L} : U \to L(V; V^\star)$ the Fréchet second derivative. Let $\mathcal{L} := \mathscr{L}(0) \in L(V; V^\star)$.*

Suppose that $W = Z^\star \hookrightarrow V^\star$ is a Banach space with $V \hookrightarrow Z$, and that $\mathscr{M} : B_\rho(0) \to W$ is W-valued and analytic. Suppose also that $\mathcal{L} \in L(V, W)$ and $\mathcal{L} : V \to W$ is Fredholm of index zero.

Then there exist C, $\rho > 0$ and $\theta \in (0, \frac{1}{2}]$ such that

$$|E(\psi) - E(0)|^{1-\theta} \leq C\|\mathscr{M}(\psi)\|_W,$$

for any $\psi \in B_\rho(0)$.

In the above corollary we remark that a key hypothesis is the Fredholmness assumption on the energy functional (see Remark 2.2.4 for the basic definitions). As we shall prove, this is ultimately a consequence of the parabolicity of the gradient flow, that is, the fact that critical points satisfy an elliptic equation. Therefore, one may hope to apply Corollary 2.2.7 to a huge family of energy functionals. Also, we believe that the method we will present can be applied to many high order geometric flows, leading to a unified point of view on the long time behavior of such a family of flows (see also [MP20]).

By means of the Łojasiewicz–Simon gradient inequality, the main result we will prove in Chapter 2 is the following theorem, which promotes sub-convergence to full convergence.

Theorem 2.3.33 (Smooth convergence)**.** *Suppose that (M, g) is an analytic complete Riemannian manifold endowed with an analytic metric tensor g. Let $p \geq 2$ and suppose that $\gamma : [0, +\infty) \times \mathbb{S}^1 \to M$ is a smooth solution of*

$$\begin{cases} \partial_t \gamma = -\left(\nabla^2 |k|^{p-2} k + \frac{1}{p'} |k|^p k - k + R(|k|^{p-2} k, \tau)\tau \right) & \text{on } [0, +\infty) \times \mathbb{S}^1, \\ \gamma(0, \cdot) = \gamma_0(\cdot) & \text{on } \mathbb{S}^1. \end{cases}$$

Suppose that there exist a sequence of isometries $I_n : M \to M$, a sequence of times $t_n \nearrow +\infty$, and a smooth critical point $\gamma_\infty : \mathbb{S}^1 \to M$ of $\mathcal{E}_p$ such that

$$I_n \circ \overline{\gamma}(t_n, \cdot) - \gamma_\infty(\cdot) \xrightarrow[n\to\infty]{} 0 \qquad \text{in } C^m(\mathbb{S}^1),$$

for any $m \in \mathbb{N}$, where $\overline{\gamma}(t_n, \cdot)$ is some reparametrization of $\gamma(t_n, \cdot)$. If $p > 2$ assume also that $|k_{\gamma_\infty}(x)| \neq 0$ for any x.

Then the flow $\gamma(t, \cdot)$ converges in $C^m(\mathbb{S}^1)$ to a critical point as $t \to +\infty$, for any $m \in \mathbb{N}$ and up to reparametrization.

As stated in Corollary 2.3.34 and Corollary 2.3.35, the above theorem can be applied to the remarkable case of $p = 2$ taking (M, g) to be the Euclidean space, the hyperbolic plane, or a compact manifold.

It remains an open problem to quantify, if possible, the size of the compact set spanned by a converging gradient flow. In particular, Huisken's conjecture on the elastic flow of curves in $\mathbb{R}^2$ for $p = 2$ is still unproved: if the initial datum γ_0 is contained in a halfplane H, does the flow γ intersects H for any time?

Elastic energy of planar sets

In Chapter 3 we study a weak definition of p-elastic energy for subsets of $\mathbb{R}^2$. If E is a smooth bounded open set in $\mathbb{R}^2$, its boundary ∂E is the union of finitely many disjoint smooth embeddings $\gamma_1, ..., \gamma_N$ of $\mathbb{S}^1$. It is then reasonable to define the p-elastic energy of E by $\mathcal{E}_p(E) \coloneqq \sum_1^N \mathcal{E}_p(\gamma_i)$. It is clear that for variational purposes one needs some weak definition that extends the former one on possibly less regular sets. A classical method for defining such an extension of a functional is argue by *relaxation*. This consists in defining an initial energy functional $\mathcal{F} : X \to [0, +\infty]$ on a metric space (X, d) so that the subset $\{\mathcal{F} < +\infty\}$ identifies the *"regular objects"*, that is, the elements of X having finite energy even before we generalize the definition of $\mathcal{F}$. Then one defines the *relaxed functional* $\overline{\mathcal{F}} : X \to [0, +\infty]$ by setting

$$\overline{\mathcal{F}}(x) \coloneqq \inf \left\{ \liminf_n \mathcal{F}(x_n) \mid d(x_n, x) \to 0 \right\}.$$

This canonical definition yields a new functional $\overline{\mathcal{F}} \leq \mathcal{F}$ that is automatically lower semicontinuous and attributes finite energy to possibly new elements of X. In such a way, the relaxed functional can be seen as a generalization of the initial energy. It is clear that $\overline{\mathcal{F}}$ strongly depends on the choice of the distance on X and, more importantly, on the definition of $\mathcal{F}$, that is, on the choice of the set of regular objects $\{\mathcal{F} < +\infty\}$.

The p-elastic energy $\mathcal{E}_p$ can be defined on the metric space of characteristic functions of essentially bounded sets of finite perimeter endowed with the L^1-distance (see Section 1.4). Choosing smooth bounded open sets as the sets with finite $\mathcal{E}_p$-energy yields to the classical notions of relaxation widely studied in [BDMP93] and also formulated in terms of varifolds in [BM04; BM07]. As we shall discuss in Section 3.1, the resulting relaxed functional still assigns infinite energy to sets whose boundary is given by an immersed curve with transversal self-intersections. However, curves of this type perfectly support the definition of elastic energy, and self-intersections of any kind might occur during a gradient flow a priori. In would be significant, indeed, to study generalized notions of the gradient flow of the p-elastic energy, that is, weak formulations of the flow. For instance, one could try to define a weak notion of the gradient flow of $\mathcal{E}_p$ by means of *minimizing movements* in the spirit of [ATW93] and [LS95]. But in order to study such a problem, a good generalized and lower semicontinuous notion of the energy functional on weaker objects is needed.

Let us also mention that, more generally, it is still absent in the literature a satisfactory study of the relaxed definition of Willmore-type energies in higher dimension, which can also be extended to BV functions that not necessarily identify finite perimeter sets (see [MN13a; MN13b], [BM05], and [BMO15]).

For the above reasons we study the L^1-relaxation of the p-elastic energy introducing the following definition of $\mathcal{E}_p$ from which to start. By the theory of sets of finite perimeter (Section 1.4), if $E \subset \mathbb{R}^2$ is an essentially bounded set of finite perimeter, we can define the associated varifold

$$V_E \coloneqq \mathbf{v}(\mathcal{F}E, 1),$$

where the notation is as in Section 1.2.2 and $\mathcal{F}E$ is the reduced boundary of E. For $p > 1$ we then define

$$\mathcal{E}_p(E) = \begin{cases} \mathcal{E}_p(V_E) & \text{if } \ V_E = \sum_{i\in I}(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)), \quad \gamma_i : \mathbb{S}^1 \to \mathbb{R}^2 \ C^2\text{-immersion}, \quad \sharp I < +\infty, \\ +\infty & \text{otherwise}, \end{cases}$$

for any measurable essentially bounded set $E \subset \mathbb{R}^2$, where $(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$ denotes the image varifold defined by γ_i (Section 1.2.2) and $\mathcal{E}_p$ is defined on a varifold V by

$$\mathcal{E}_p(V) \coloneqq \int 1 + \frac{1}{p}|k|^p \, d\mu_V,$$

where k is the generalized mean curvature of V and μ_V is the weight of V (see Section 1.2.2). Therefore, sets with finite $\mathcal{E}_p$-energy are those whose boundary is “covered” by finitely many closed curves, which are not necessarily embedded.

If $E \subset \mathbb{R}^2$ is measurable, we further define

$$
\begin{aligned}
\mathcal{A}(E) = \Bigg\{ V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i \in I} (\gamma_i)_\sharp (\mathbf{v}(\mathbb{S}^1, 1)) \ \Bigg| \ & \gamma_i : \mathbb{S}^1 \to \mathbb{R}^2 \ W^{2,p}\text{-immersion}, \ \sharp I < +\infty, \\
& \sum_{i \in I} \mathcal{E}_p(\gamma_i) < +\infty, \\
& \partial E \subset \Gamma, \ V_E \le V, \\
& \mathcal{F}E \subset \{x \in \mathbb{R}^2 \,|\, \theta_V(x) \text{ is odd}\}, \\
& \mathcal{H}^1(\{x \,|\, \theta_V(x) \text{ is odd}\} \setminus \mathcal{F}E) = 0 \Bigg\},
\end{aligned}
$$

which is well defined as if the p-elastic energy of a varifold is finite, then the multiplicity function θ_V can be defined pointwise at any point as the 1-dimensional density of μ_V (Theorem 1.3.1). Roughly speaking, an element $V \in \mathcal{A}(E)$ is a varifold identified by finitely many closed curves "passing an odd number of times" exactly on the boundary of E.

The main result of Chapter 3 is the following characterization of $\overline{\mathcal{E}_p}$.

Theorem 3.1.7 (Relaxation). *For any measurable set $E \subset \mathbb{R}^2$ we have that the following holds.*

1. *If $\mathcal{A}(E) \neq \emptyset$ and E is essentially bounded, then the minimum*

$$
\min \left\{ \mathcal{E}_p(V) \mid V \in \mathcal{A}(E) \right\}
$$

   exists.

2. *It holds that*

$$
\overline{\mathcal{E}_p}(E) = \begin{cases} +\infty & \text{if } \ \mathcal{A}(E) = \emptyset \ \text{ or } \ E \text{ is ess. unbounded}, \\ \min \left\{ \mathcal{E}_p(V) \mid V \in \mathcal{A}(E) \right\} & \text{otherwise.} \end{cases}
$$

In Section 3.2 we then study qualitative properties and applications of the relaxed energy $\overline{\mathcal{E}_p}$, comparing the functional with the classical notions of relaxation above mentioned.

Willmore energy of surfaces with boundary

Chapter 4 contains some results about the Willmore energy of surfaces with boundary or varifolds with boundary (see Section 1.2). We are interested in the existence theory for minimization problems on the Willmore energy of such geometric objects. We refer to Section 4.1 for a collection of some fundamental related results that motivate our study. In particular, as a result of the works of Simon [Sim93] and Bauer–Kuwert [BK03], recall that we already have a good description of the minimization of the Willmore energy among closed surfaces of a given genus. Indeed, the minimization problem

$$
\min \left\{ \mathcal{W}(\varphi) \mid \varphi : M_{\mathfrak{g}} \to \mathbb{R}^3 \text{ smooth immersion} \right\},
$$

where $M_{\mathfrak{g}}$ is the orientable closed surface of genus $\mathfrak{g}$ and $\mathfrak{g} \ge 0$ is a fixed integer, has minimizers, which are smooth embedded surfaces. The infimum of the problem, that is denoted by $\beta_{\mathfrak{g}}$, satisfies $\beta_{\mathfrak{g}} \in [4\pi, 8\pi)$. Moreover $\mathcal{W}(\varphi) = 4\pi$ if and only if φ is the embedding of a round sphere.

The first problem we consider is the direct generalization of the above in the case of surfaces with boundary. We will refer to such a problem as the *Plateau–Douglas problem for the Willmore energy*. The classical Plateau–Douglas problem [DHT10] is formulated as the minimization of the area functional among immersed surfaces spanning a given boundary and having fixed topology. The problem for the Willmore energy therefore reads as follows.

Let $\Gamma_1, ..., \Gamma_k \subset \mathbb{R}^n$ be finitely many smooth embedded curves, and $n \geq 3$. Fix an integer $\mathfrak{g} \geq 0$ and let $\Sigma_{\mathfrak{g}}$ be the 2-dimensional closed surface of genus $\mathfrak{g}$ with k disks removed. The scope is to characterize existence of minimizers and infimum for the minimization problem

$$\min \left\{ \mathcal{W}(\varphi) \mid \varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^n \text{ smooth immersion, } \varphi|_{\partial\Sigma_{\mathfrak{g}}} : \partial\Sigma_{\mathfrak{g}} \to \sqcup_i \Gamma_i \text{ smooth embedding} \right\}.$$

We started the study of such a problem in $\mathbb{R}^3$ considering one assigned planar boundary curve in [Poz20c]. In this special case the problem already appears quite rich. In Section 4.2 we consider the case in which the assigned boundary is a circle of unit radius, and we prove the following non-existence result.

Theorem 4.2.1 (Non-existence of minimizers)**.** *Denote by $\mathbb{S}^1$ a unit circle in $\mathbb{R}^3$. For any genus $\mathfrak{g} \geq 1$, the minimization problem*

$$\min\{\mathcal{W}(\varphi) \mid \varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3 \text{ smooth immersion, } \varphi|_{\partial\Sigma_{\mathfrak{g}}} \to \mathbb{S}^1 \text{ smooth embedding}\},$$

has no minimizers and the infimum equals $\beta_{\mathfrak{g}} - 4\pi$.

The analysis of the Plateau–Douglas problem for the Willmore energy, as well as of the minimization problems already appeared in the literature, is strongly related to the so-called *Li–Yau inequality*, as already mentioned. Roughly speaking, recall that the classical Li–Yau inequality [LY82] states that if a smooth immersion $\varphi : \Sigma \to \mathbb{R}^n$ of a closed surface Σ is not an embedding, i.e., self-intersections occur, then $\mathcal{W}(\varphi) \geq 8\pi$. We will prove this inequality even at the level of varifolds in Corollary 1.3.8. It is clear that an estimate of such a kind can be very helpful in the study of a minimization problem among immersed surfaces, allowing the reduction to embedded ones. The next result is a generalization with circle boundary of the Li–Yau inequality.

Theorem 4.2.6 (Li–Yau-type inequality with circle boundary)**.** *Denote by $\mathbb{S}^1$ a unit circle in $\mathbb{R}^3$ and let $\mathfrak{g} \geq 0$ be a fixed integer. If $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ is an immersion such that $\varphi : \partial\Sigma_{\mathfrak{g}} \to \mathbb{S}^1$ is an embedding and there is $p_0 \in \mathbb{R}^3$ such that $\sharp\varphi^{-1}(p_0) \geq 2$, then $\mathcal{W}(\varphi) \geq 4\pi$.*

As a consequence of the above theorem, a non-existence result like the one in Theorem 4.2.1 holds also for the problem analogously defined on embedded surfaces (Corollary 4.2.7).

In the second part of Chapter 4 we focus on another family of problems, which are defined at the level of varifolds. Once again, we formulate minimization problems on $\mathcal{W}$ by analogy with the classical problems on the area functional.

The well known Plateau's problem aims at finding minimizers of the area functional among competitors spanning a given boundary. Many weak formulations of such a problem are known, with the suitable generalizations of the concept of *area* and *spanned boundary* (see [Mor09], [Sim83b]), and existence of minimizers in such generalized setting is today quite understood.

As minimal surfaces have zero Willmore energy, minimization of $\mathcal{W}$ among surfaces spanning a given boundary may recover solutions to the Plateau's problem with the given boundary. However, the presence of additional constraints may force the existence of minimizers with non-vanishing mean curvature. In particular, we will be interested in finding minimizers of $\mathcal{W}$ spanning a given boundary under the constraint that the support of the minimizers *connects* all the boundary components assigned. A motivating example deeply discussed in Section 4.3 is the case in which the assigned boundary consists of two coaxial circles in $\mathbb{R}^3$. Indeed in such a case, if the two circles are located too far, no connected minimal surfaces spanning the two circles exists (Section 4.3.3).

We can understand the minimization problem we are going to define as a problem which recovers the optimal *connected* elastic surface spanning a given boundary. The problems we will study in this context are of two kinds and they are stated at the level of varifolds, thus the boundary conditions are imposed on the *generalized boundary* of the varifolds (see Section 1.2.2), which, we recall, is a vector valued measure (see Section 1.2.1).

The first general existence theorem we prove is the following result, in which *clamped* conditions are imposed at the boundary.

Theorem 4.3.8. *Let $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ be a disjoint union of smooth embedded closed curves in $\mathbb{R}^3$ with $\alpha \in \mathbb{N}_{\geq 2}$. Let*

$$\sigma_0 = m\,\nu_0\,\mathcal{H}^1 \llcorner \gamma$$

be a vector valued Radon measure, where $m : \gamma \to \mathbb{N}_{\geq 1}$ and $\nu_0 : \gamma \to (T\gamma)^\perp$ are $\mathcal{H}^1$-measurable functions with $m \in L^\infty(\mathcal{H}^1 \llcorner \gamma)$ and $|\nu_0| = 1$ $\mathcal{H}^1$-ae. Let $\mathcal{P}$ be the minimization problem

$$\mathcal{P} \coloneqq \min\left\{\mathcal{W}(V) \quad | \quad V = \mathbf{v}(M, \theta_V): \quad \sigma_V = \sigma_0, \quad \mathrm{spt}V \cup \gamma \;\; \text{compact, connected}\right\}.$$

If $\inf \mathcal{P} < 4\pi$*, then $\mathcal{P}$ has minimizers.*

The second existence theorem we prove is the following one, which prescribes a bound on the generalized boundary of competitors, but no conditions on the generalized conormal are imposed.

Theorem 4.3.9. *Let $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ be a disjoint union of smooth embedded closed curves in $\mathbb{R}^3$ with $\alpha \in \mathbb{N}_{\geq 2}$. Let $m : \gamma \to \mathbb{N}_{\geq 1}$ be $\mathcal{H}^1$-measurable with $m \in L^\infty(\mathcal{H}^1 \llcorner \gamma)$. Let $\mathcal{Q}$ be the minimization problem*

$$\mathcal{Q} \coloneqq \min\Big\{\mathcal{W}(V) \quad | \quad V = \mathbf{v}(M, \theta_V): \quad |\sigma_V| \leq m\mathcal{H}^1 \llcorner \gamma, \quad \mathrm{spt}V \cup \gamma \;\; \text{compact, connected}\Big\}.$$

If $\inf \mathcal{Q} < 4\pi$*, then $\mathcal{Q}$ has minimizers.*

Both the above theorems are proved by means of a direct method. Hence the main issue is to prove that the connectedness constraint passes to the limit. This is done by proving that, in the setting of these problems, convergence in the sense of varifolds implies that the supports of the varifolds of a sequence converge in *Hausdorff distance*. This key result is proved in Theorem 4.3.4.

We remark that the only tool we will employ in the proof of almost every result of Chapter 4 is a *monotonicity formula* for varifolds with boundary, which is recalled and proved in Section 1.3.2.

Chapter 1

Preliminaries in Geometric Analysis and Geometric Measure Theory

Contents

In this chapter we recall the main definitions and tools we will need. We first present the concept of extrinsic curvature of submanifolds, and we discuss some notions from Geometric Measure Theory. In particular we focus on the theory of rectifiable varifolds and the definition of Willmore energy for such objects. Then we present some results from [Poz20a] and [NP20], that consider the consequences of monotonicity formulas on varifolds with bounded Willmore-type energies. Finally we recall a few definitions and facts that we will need about sets of finite perimeter and in the theory of currents.

1.1 Submanifolds

This section contains basic definitions and results in the theory of submanifolds of Riemannian manifolds. Even if we will deal with low dimensional objects, namely curves and surfaces, we recall here definitions and facts in their full generality.

1.1.1 Extrinsic curvature

Throughout this section we adopt the following notation. We assume that

$$\varphi : M^k \hookrightarrow (\overline{M}^n, \overline{g}),$$

is a smooth immersion of a k-dimensional manifold into a complete Riemannian manifold $(\overline{M}^n, \overline{g})$, with $1 \le k < n$. We endow M with the pull back metric $g = \varphi^* \overline{g}$. We denote by D and $\overline{D}$ the corresponding Levi-Civita connections on M and $\overline{M}$ respectively. We denote by

$$d\varphi_x : T_x M \to T_{\varphi(x)} \overline{M}$$

the differential, or push forward, of φ at point $x \in M$. For basic Differential and Riemannian Geometry we refer to [Car92; AT11; Lee13; Pet16].

In the above setting, the map φ is an isometric immersion of a manifold (M, g) into an ambient $(\overline{M}, \overline{g})$. We are interested in recalling the basic notions that measure how the immersed manifold $\varphi(M)$ is "curved inside" the ambient $\overline{M}$. This is why we talk about *extrinsic curvature*, that is, a notion of curvature depending on how φ "immerses" M into $\overline{M}$, as opposed to *intrinsic curvatures*, that is, notions of curvature depending only on a chosen metric on a manifold without any reference to an immersion or an ambient space. However, when a metric is the pull back metric defined by an isometric immersion, as in our setting, there exist relations between extrinsic and intrinsic curvatures, but we will come back on this later.

We denote by $\mathfrak{X}(M)$ the space of vector fields on M and by $(TM)^\perp$ the normal bundle of M in $\overline{M}$, that is,

$$(TM)^\perp := \bigcup_{x \in M} (d\varphi_x(T_x M))^\perp,$$

where $(\cdot)^\perp$ denotes orthogonality in $T_{\varphi(x)}\overline{M}$ with respect to the scalar product $\overline{g}$. More precisely, we have the orthogonal splitting $T_{\varphi(x)}\overline{M} = d\varphi_x(T_x M) \oplus_{\perp_{\overline{g}}} (d\varphi_x(T_x M))^\perp$, and $(\cdot)^\perp$ is the projection onto $(d\varphi_x(T_x M))^\perp$. Similarly, we will denote by $(\cdot)^\top$ the projection onto $d\varphi_x(T_x M)$.

Moreover, recall that since φ is an immersion, it is locally an embedding. Then for any field $X \in \mathfrak{X}(M)$ and any point $x \in M$ there exists a neighborhood U of x in M such that if $\widetilde{X}$ is the restriction of X to U, then $d\varphi(\widetilde{X})$ can be extended to a vector field on $\overline{M}$ locally defined in a neighborhood of $\varphi(x)$. We will denote such an extension by $\overline{X}$.

We can then define the second fundamental form as follows.

Definition 1.1.1. The *second fundamental form* $B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to (TM)^\perp$ is the operator

$$B_x(X, Y) := \left(\overline{D}_{\overline{X}} \overline{Y}\right)^\perp (\varphi(x)),$$

where $\overline{X}, \overline{Y}$ are local extensions of $d\varphi(X), d\varphi(Y)$ in a neighborhood of $\varphi(x)$ for any $x \in M$.

The following lemma states that the definition of B is well posed and that it is a tensor.

Lemma 1.1.2 ([Car92, Chapter 6, Section 2])**.** *The second fundamental form B satisfies the following properties.*

1. *The definition of B is well posed, that is, for any $X, Y \in \mathfrak{X}(M)$ and any point $x \in M$ the vector $\left(\overline{D}_{\overline{X}} \overline{Y}\right)^\perp (\varphi(x))$ is independent of the specific extensions $\overline{X}, \overline{Y}$.*

2. *B is a symmetric tensor with values in the normal bundle of M.*

We will also need to consider the following operator, which is somehow equivalent to the notion of second fundamental form.

Definition 1.1.3. Let $N : U \subset M \to (TM)^\perp$ be a locally defined smooth normal vector field along φ, that is, $N(x) \in (d\varphi_x(T_x M))^\perp$ for any $x \in U$ and U is open. The *shape operator* associated to N is the endomorphism

$$S_N : \mathfrak{X}(U) \to \mathfrak{X}(U) \qquad\qquad S_N(X) = -\left(\overline{D}_{\overline{X}} N\right)^\top,$$

where $\overline{X}$ is a local extension of $d\varphi(X)$ in a neighborhood of $\varphi(x)$ for any $x \in U$.

If $S_N : \mathfrak{X}(U) \to \mathfrak{X}(U)$ is a shape operator as in Definition 1.1.3, we remark that S_N is a vector valued tensor by Lemma 1.1.2. In fact, it holds that

$$\overline{g}(B(X,Y), N) = g(X, S_N(Y)) = g(Y, S_N(X)),$$

on U for any $X, Y \in \mathfrak{X}(U)$, where we are identifying TU with $d\varphi(TU) \subset T\overline{M}$. Indeed $\overline{g}(X,N) \equiv \overline{g}(Y,N) \equiv 0$, and the identity follows by compatibility of $\overline{D}$ with $\overline{g}$, i.e., it holds that $0 \equiv \overline{D}_{\overline{X}}(\overline{g}(Y,N)) = \overline{g}(B(X,Y),N) + \overline{g}(Y, \overline{D}_{\overline{X}} N)$.

We can now define the mean curvature vector of an isometric immersion as the following section of the normal bundle.

Definition 1.1.4. The *mean curvature vector* of φ is the normal vector $H : M \to (TM)^\perp$ defined by

$$H(x) := \frac{1}{k} \mathrm{tr}_g B_x,$$

where tr_g denotes trace with respect to the metric g. More explicitly, we have

$$H(x) = \frac{1}{k} \sum_{i=1}^{k} B_x(e_i, e_i),$$

where $\{e_i\}_{i=1}^k$ is a g-orthonormal basis of $T_x M$.

We say that the immersion φ is *minimal* if H identically vanishes. If also M has dimension 2, we say that $\varphi : M \hookrightarrow \overline{M}$ is a *minimal surface*.

Finally we recall the definition of Riemann curvature tensor, specifying the convention we employ in this thesis. This is the first intrinsic notion of curvature we encounter.

Definition 1.1.5. Let (M^m, g) be a Riemannian manifold of dimension $m \geq 2$ with Levi-Civita connection D. The *Riemann tensor* R is defined by

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R} \qquad\qquad R(X,Y,Z,W) := g(R(Z,W)Y, X),$$

where

$$R(Z,W)Y := D_Z D_W Y - D_W D_Z Y - D_{[Z,W]} Y,$$

and $[Z,W]$ denotes the Lie bracket of vector fields. The Riemann tensor is, indeed, a tensor (see [Car92, Chapter 4, Section 2]).

If $e_1, e_2 \in T_p M$ are two orthonormal tangent vectors and $\Pi = \mathrm{span}(e_1, e_2)$, the *sectional curvature* of M on Π is given by

$$K(\Pi) := R(e_1, e_2, e_1, e_2).$$

It can be directly checked that the sectional curvature $K(\Pi)$ does not depend on the choice of the orthonormal basis of the plane Π.

In case the dimension of M is $m = 2$, then the sectional curvature is always evaluated on the whole tangent space $T_p M$ and the resulting function $K : M \to \mathbb{R}$ is called *Gaussian curvature*.

We recall that the Riemann tensor R of a Riemannian manifold (M, g) satisfies the symmetries

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y),$$

for any fields X, Y, Z, W on M (see [Car92, Chapter 4, Proposition 2.5]).

In particular, if (M^2, g) is a 2-dimensional Riemannian manifold, the Gaussian curvature K completely determines the Riemann tensor R as

$$R(X,Y)Z = K\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right). \tag{1.1}$$

Indeed letting $\{e_1, e_2\}$ be an orthonormal basis of T_pM, by the symmetries of R we have that $R(e_i, e_j, e_k, e_l) \neq 0$ if and only if $i \neq j$ and $k \neq l$, and then

$$R(e_1,e_2)e_2 = Ke_1, \qquad R(e_1,e_2)e_1 = -Ke_2,$$

so that $R(e_1, e_2)Z = K(\langle e_2, Z\rangle e_1 - \langle e_1, Z\rangle e_2)$, and (1.1) follows by linearity.

The following fundamental theorem states the above mentioned relation between intrinsic and extrinsic curvature, when the metric on a manifold is the pull back metric defined by an isometric immersion.

Theorem 1.1.6 (Gauss equation, [Car92, Chapter 6, Proposition 3.1])**.** *Let $\varphi : (M, g) \hookrightarrow (\overline{M}, \overline{g})$ be an isometric immersion. Denote by B, R, and $\overline{R}$ the second fundamental form of φ and the Riemann tensor on M and $\overline{M}$ respectively. Then*

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + \overline{g}(B(X,Z), B(Y,W)) - \overline{g}(B(X,W), B(Y,Z)),$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$, where we identified T_xM with $d\varphi(T_xM) \subset T_{\varphi(x)}\overline{M}$.

In Chapter 2 we will deal with immersed curves into manifolds. In this case the notions of second fundamental form and mean curvature reduce to the concept of *curvature vector* of a curve. More precisely, if $\gamma : (0,1) \to M$ is an (isometric) smooth immersion of a curve into a smooth Riemannian manifold (M^m, g) of dimension $m \geq 2$, we define its tangent vector as

$$\tau_\gamma := |\gamma'|^{-1}\gamma',$$

and its curvature vector as

$$k_\gamma := D_\tau \tau,$$

if D is the Levi-Civita connection of (M, g). We will usually refer to the curvature vector k_γ simply as the curvature of γ. We recall that γ is a *geodesic* if k_γ identically vanishes.

If $\gamma : \mathbb{S}^1 \to M$ is a smooth immersion of a closed curve in a Riemannian manifold (M, g) and $p \in [1, +\infty)$, we define the *p-elastic energy* of γ as

$$\mathcal{E}_p := L(\gamma) + \frac{1}{p}\int_{\mathbb{S}^1} |k_\gamma|^p \, ds,$$

where $ds := |\gamma'|dx$ denotes integration with respect to length measure and $L(\gamma)$ is the length of γ. The support of a given curve γ will be denoted by (γ).

By Nash Theorem [Nas56] we will usually assume without loss of generality that a Riemannian manifold (M, g) is isometrically embedded in the Euclidean space $\mathbb{R}^n$ for some n sufficiently big, that is, $M \hookrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. In such a case we shall look at the manifold M as a subset of $\mathbb{R}^n$, and then the metric on M will be simply the restriction of the Euclidean scalar product $\langle\cdot,\cdot\rangle$ on the tangent space of M. Moreover, still denoting by $\overline{D}$ the connection of the ambient, that in this case is the Euclidean connection, we recall that the connection D on M turns out to be the projection of $\overline{D}$ on the tangent space of M, that is

$$D_X Y = (\overline{D}_X Y)^\top,$$

for fields $X, Y \in \mathfrak{X}(M)$, where $(\cdot)^\top$ denotes projection onto TM.

We finally observe that if $M \hookrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, a smooth curve $\gamma : (0,1) \to M$ can be seen as a smooth curve in $\mathbb{R}^n$ with image contained in M. In such a case the curvature of γ as a curve in $\mathbb{R}^n$ is $\partial_s^2\gamma$, where

$$\partial_s := |\gamma'|^{-1}\partial_x,$$

denotes the standard arclength derivative, and then the curvature of γ as a curve in M is

$$k_\gamma = (\partial_s^2\gamma)^\top.$$

In this setting the curvature k_γ is sometimes called *geodesic curvature* of γ along M.

For any curve $\gamma : (0,1) \to \mathbb{R}^n$ or vector field $X : (0,1) \to \mathbb{R}^n$ along γ, we will denote arclength derivatives also by $\dot\gamma := \partial_s\gamma$ and $\dot X := \partial_s X$ in order to simplify the notation.

1.1.2 Surfaces in the Euclidean space and Willmore energy

Let us turn our attention to submanifolds of the Euclidean space. We are now interested in recalling some definitions and facts about surfaces with boundary in $\mathbb{R}^n$.

Throughout this section we will always consider a smooth isometric immersion

$$\varphi : \Sigma \hookrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle),$$

of a 2-dimensional manifold Σ with boundary in the Euclidean space $\mathbb{R}^n$ with $n \geq 3$, and the induced metric on Σ is $g = \varphi^*\langle\cdot,\cdot\rangle$. Without loss of generality, we assume that Σ is connected. We denote by $\partial\Sigma \subset \Sigma$ the boundary of Σ, that is diffeomorphic to a disjoint union of copies of $\mathbb{S}^1$ or $\mathbb{R}$. By saying that φ is an immersion, we are also assuming that the restriction of φ on $\partial\Sigma$ is an immersion of each component of $\partial\Sigma$.

We define the *conormal* $co_\varphi : \partial\Sigma \to \mathbb{R}^n$ at any point $x \in \partial\Sigma$ to be the unique unit vector $co_\varphi(x) \in \mathbb{R}^n$ that is tangent along φ and normal along $\varphi|_{\partial\Sigma}$, and that points outwards of Σ.

The second fundamental form and the mean curvature vector of φ are pointwise defined on $\Sigma \setminus \partial\Sigma$. Moreover, we denote by μ_φ the volume measure induced by φ on Σ, that is, the 2-dimensional Hausdorff measure defined by the geodesic distance on Σ. Recall that on orientable manifolds, integration with respect to Hausdorff measure is equivalent to the classical integration on manifolds with respect to the Riemannian volume form (see [Lee13, Chapter 16, Densities] and [Lee13, Exercise 16.46]).

Therefore we can give the following definition.

Definition 1.1.7. If $\varphi : \Sigma \to \mathbb{R}^n$ is a smooth immersion as above and H is the mean curvature of φ, we define the *Willmore energy* of φ as

$$\mathcal{W}(\varphi) := \int_\Sigma |H|^2 \, d\mu_\varphi,$$

where the integral defining $\mathcal{W}$ is understood in the sense of Lebesgue and it may equal $+\infty$. In particular, we do not assume orientability of Σ in this definition.

If $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ is a C^1 compactly supported vector field and S is a k-dimensional subspace, we recall that the *tangential divergence* of X on S at x is

$$\mathrm{div}_S X(x) \coloneqq \sum_{i=1}^{k} \langle dX_x(e_i), e_i \rangle,$$

where $\{e_i\}_{i=1}^k$ is any orthonormal basis of S. If $\varphi : \Sigma \to \mathbb{R}^n$ is an immersed surface as above and $x \in \Sigma \setminus \partial\Sigma$, we shall denote

$$\mathrm{div}_{T_x\Sigma} X \coloneqq (\mathrm{div}_{d\varphi_x(T_x\Sigma)} X)(\varphi(x)),$$

in order to simplify the notation.

We can now state the following classical integration by parts formula for the tangential divergence.

Proposition 1.1.8. *Let $\varphi : \Sigma \to \mathbb{R}^n$ be a smooth isometric proper immersion of a 2-dimensional manifold Σ with boundary $\partial\Sigma$. Let $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\int_\Sigma \mathrm{div}_{T_x\Sigma} X \, d\mu_\varphi(x) = -2 \int_\Sigma \langle X \circ \varphi, H \rangle \, d\mu_\varphi + \int_{\partial\Sigma} \langle X \circ \varphi, co_\varphi \rangle \, ds_\varphi,$$

where ds_φ is the length measure induced by φ on $\partial\Sigma$.

Proof. For any $x \in \Sigma \setminus \partial\Sigma$ we can define

$$Y(x) = (X(\varphi(x)))^\top, \qquad\qquad Z(x) = (X(\varphi(x)))^\perp,$$

where $(\cdot)^\top$ and $(\cdot)^\perp$ denote projection onto $d\varphi_x(T_x\Sigma)$ and $(d\varphi_x(T_x\Sigma))^\perp$ respectively. Since φ is proper, it follows that Y defines a C^1_c tangent field on Σ, while Z defines a C^1_c normal field along φ.

Fix $x \in \Sigma \setminus \partial\Sigma$ and let $\{e_1(x), e_2(x)\}$ be an orthonormal basis of $d\varphi_x(T_x\Sigma)$. As φ is a local embedding in a neighborhood U of x, we can extend $\{e_1(x), e_2(x)\}$ to a local orthonormal basis of $d\varphi_y(T_y\Sigma)$ for $y \in U$. Denoting $p = \varphi(x)$, as $X^\perp$ is well defined on $\varphi(U)$, by linearity of the tangential divergence we have

$$\mathrm{div}_{T_x\Sigma} X = \mathrm{div}_\Sigma Y(x) + \sum_{i=1}^{2} \langle (\partial_{e_i(x)} X^\perp)(p), e_i(x) \rangle, \tag{1.2}$$

where $(\partial_{e_i(x)} X^\perp)(p)$ is the partial derivative of $X^\perp$ in direction $e_i(x)$ evaluated at p, and $\mathrm{div}_\Sigma Y(x)$ is the divergence of Y as a vector field on Σ. Hence, since $\langle Z(y), e_i(y) \rangle \equiv 0$ on U, we have

$$\begin{aligned} \sum_{i=1}^{2} \langle \partial_{e_i(x)} Z(x), e_i(x) \rangle &= \sum_{i=1}^{2} -\langle Z(x), \partial_{e_i} e_i(x) \rangle = -\sum_{i=1}^{2} \langle (X(\varphi(x)))^\perp, (\partial_{e_i} e_i(x))^\perp \rangle \\ &= -\left\langle X(\varphi(x)), \sum_{i=1}^{2} (\partial_{e_i} e_i(x))^\perp \right\rangle = -2\langle X(\varphi(x)), H(x) \rangle. \end{aligned}$$

Therefore the thesis follows by integrating (1.2) on both sides over Σ, using that

$$\int_\Sigma \mathrm{div}_\Sigma Y \, d\mu_\varphi = \int_{\partial\Sigma} \langle Y, co_\varphi \rangle \, ds_\varphi.$$

by the Divergence Theorem on (possibly unorientable) manifolds [Lee13, Theorem 16.48] (see also [Lee13, Theorem 16.32] for the classical Divergence Theorem). ☐

Remark 1.1.9 (First variation of the area)**.** We remark that the surface integral of the tangential divergence of a vector field is the result of the first variation of the area functional. More precisely, let φ be as in Proposition 1.1.8 and assume that the area $Area(\varphi) := \int_\Sigma \mu_\varphi$ is finite. Let $X \in C^1_c(U, \mathbb{R}^n)$ be a vector field defined on an open set $U \subset \mathbb{R}^n$. Let $\phi : (-t_0, t_0) \times \mathbb{R}^n \to \mathbb{R}^n$ be a map of class C^1 such that $\phi(t, \cdot)$ is a diffeomorphism for any $|t| < t_0$. Then $\phi(t, \varphi(\cdot)) : \Sigma \to \mathbb{R}^n$ is a C^1 immersion for any t sufficiently small, and it holds that

$$\left.\frac{d}{dt}\right|_0 Area(\phi(t, \varphi(\cdot))) = \int_\Sigma \operatorname{div}_{T_x\Sigma} X \, d\mu_\varphi(x).$$

For a proof of this fact see for example [Sim83b, Chapter 2, Section 5].

Let us conclude by recalling the classical Gauss–Bonnet Theorem. Recall that if $\gamma, \sigma : [a, b] \to (M, g)$ are two smooth regular curves in a Riemannian manifold such that $\gamma(b) = \sigma(a)$, denoting by $\tau_\gamma(b) = \lim_{x\to b^-} \tau_\gamma(x)$ and $\tau_\sigma(a) = \lim_{x\to a^+} \tau_\sigma(x)$, assuming $\tau_\gamma(b), \tau_\sigma(a)$ are linearly independent, given an orientation on the plane $\Pi = \operatorname{span}(\tau_\gamma(b), \tau_\sigma(a))$, the *oriented angle* $\measuredangle(\tau_\gamma(b), \tau_\sigma(a)) \in (-\pi, \pi)$ is the angle between the two vectors, taken with positive (resp. negative) sign if the oriented couple $[\tau_\gamma(b), \tau_\sigma(a)]$ is a positive (resp. negative) basis of Π.

Theorem 1.1.10 (Gauss–Bonnet Theorem, [AT12, Theorem 6.3.9])**.** *Let $\varphi : \Sigma \to \mathbb{R}^n$ be an isometric immersion of an oriented smooth Riemannian 2-dimensional compact manifold (Σ, g) with piecewise smooth boundary, that is, there exists finitely many smooth curves $\gamma_1, ..., \gamma_N : [0, 1] \to \partial\Sigma$ touching only at the endpoints such that $\gamma_i(1) = \gamma_{i+1}(0)$ for any i, understanding $N + 1 = 1$, and $\partial\Sigma = \cup_i \gamma([0, 1])$. Assume that each γ_i positively orients a piece of the boundary, that is, for any $t \in (0, 1)$ the oriented couple $[d\varphi_{\gamma_i(t)}(\tau_{\gamma_i}(t)), -co_\varphi(\gamma_i(t))]$ is an oriented basis on Σ. Assume that $\tau_{\gamma_i}(1), \tau_{\gamma_{i+1}}(0)$ are linearly independent for any i, and denote by $\alpha_i = \measuredangle(\tau_{\gamma_i}(1), \tau_{\gamma_{i+1}}(0)) \in (-\pi, \pi)$.*

Then

$$\int_\Sigma K \, d\mu_\varphi + \sum_i \int_0^1 \langle k_{\varphi\circ\gamma_i}, -co_\varphi \circ \gamma_i\rangle \, ds_{\varphi\circ\gamma_i} + \sum_i \alpha_i = 2\pi\chi(\Sigma),$$

where K is the Gaussian curvature of Σ, $k_{\varphi\circ\gamma_i}$ is the curvature of $\varphi \circ \gamma_i$, and $\chi(\Sigma)$ is the Euler characteristic of Σ. In particular, the left hand side of the above identity is a topological invariant.

Remark 1.1.11. Using Gauss–Bonnet Theorem 1.1.10, we derive an important identity between the Willmore energy of a surface and the integral of the squared norm of its second fundamental form.

Let $\varphi : \Sigma \to \mathbb{R}^n$ be an isometric immersion of an oriented smooth Riemannian 2-dimensional compact manifold (Σ, g) with smooth boundary $\partial\Sigma$. Then

$$4\mathcal{W}(\varphi) = \int_\Sigma |B|^2 \, d\mu_\varphi + 4\pi\chi(\Sigma) + 2\int_{\partial\Sigma} \langle k_{\varphi|_{\partial\Sigma}}, co_\varphi\rangle \, ds_\varphi.$$

Indeed, if $\{e_1, e_2\}$ is an orthonormal basis of $T_x\Sigma$, we have that

$$\begin{aligned} 4|H(x)|^2 = \left|\sum_{i=1}^2 B_x(e_i, e_i)\right|^2 &= |B_x|^2 + 2\langle B_x(e_1, e_1), B_x(e_2, e_2)\rangle - 2|B_x(e_1, e_2)|^2 \\ &= |B_x|^2 + 2K(x), \end{aligned}$$

where $K(x)$ is the Gaussian curvature at x and we used Gauss equation (Theorem 1.1.6) in the last equality. Integrating on both sides over Σ and using Theorem 1.1.10 yields the desired identity.

We observe that the importance of the above identity, at least from a variational perspective, is the fact that if the topology of Σ is chosen and $\int_{\partial\Sigma} \langle k_{\varphi|_{\partial\Sigma}}, co_\varphi \rangle \, ds_\varphi$ is bounded, then a bound on the Willmore energy $\mathcal{W}(\varphi)$ implies a bound on the L^2-norm of the second fundamental form.

1.2 Varifolds

In this section we introduce a measure theoretic object called varifold. We shall think of a varifold as a generalization of the notion of immersed submanifold.

1.2.1 Basic measure theory and general varifolds

Let us first recall a few definitions in measure theory. We refer to [AFP00] and [Sim83b] for basic measure theory and the theory of varifolds.

Definition 1.2.1. Let X be a locally compact separable metric space. A *σ-algebra* on X is a subset $\mathcal{Q}$ of the power set of X such that it contains the empty set and it is closed under complement and countable union. We denote by $\mathcal{B}(X)$ the *Borel σ-algebra* of X, that is, the smallest σ-algebra on X containing every open set. A set in $\mathcal{B}(X)$ is said to be a *Borel set*.

In the definition of σ-algebra, it is clearly not necessary any assumption on the topology of X in order to define the concept of σ-algebra. However, we will be interested here only in cases where X is a locally compact separable metric space.

Definition 1.2.2. Let X be a locally compact separable metric space.

1. If $\mathcal{Q}$ is a σ-algebra on X, a map $\mu : \mathcal{Q} \to [0, +\infty]$ is a *positive measure* if $\mu(\emptyset) = 0$ and μ is σ-additive on $\mathcal{Q}$, that is, for any countable family $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}$ of pairwise disjoint sets it holds that
$$\mu\left(\bigcup_{n=0}^{+\infty} E_n\right) = \sum_{n=0}^{+\infty} \mu(E_n).$$
If also $\mu(X) < +\infty$, we say that μ is also *finite*.

2. If $\mathcal{Q}$ is a σ-algebra on X, a map $\mu : \mathcal{Q} \to \mathbb{R}^m$ is a *measure* if $\mu(\emptyset) = 0$ and for any countable family $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}$ of pairwise disjoint sets it holds that
$$\mu\left(\bigcup_{n=0}^{+\infty} E_n\right) = \sum_{n=0}^{+\infty} \mu(E_n).$$
In case $m = 1$, we say that μ is a *signed measure*. In case $m > 1$, we say that μ is a *vector valued measure*.

3. If $\mathcal{Q}$ is a σ-algebra on X and $\mu : \mathcal{Q} \to \mathbb{R}^m$ is a measure, we define the *total variation* $|\mu| : \mathcal{Q} \to [0, +\infty]$ by
$$|\mu|(E) := \sup \left\{ \sum_{n=0}^{+\infty} |\mu(E_n)| \;\; : \;\; E_n \in \mathcal{Q} \text{ pairwise disjoint}, \; E = \bigcup_{n=0}^{+\infty} E_n \right\}.$$
By [AFP00, Theorem 1.6] it follows that $|\mu|$ is a positive finite measure.

4. A *Borel measure* is a positive measure μ on X defined on the Borel σ-algebra $\mathcal{B}(X)$. If it is finite on compact sets, we say that μ is a *positive Radon measure.*

5. A map $\mu : \{A \in \mathcal{B}(X) : A \text{ relatively compact}\} \to \mathbb{R}^m$ such that for any compact set $K \subset X$ the restriction of μ to $\mathcal{B}(K)$ is a measure, is a *(signed or vector valued) Radon measure.* If also μ is defined on $\mathcal{B}(X)$ and it is a measure on $\mathcal{B}(X)$, then we say that μ is a *finite (signed or vector valued) Radon measure.*

6. A function $f : X \to Y$, where Y is a metric space, is *Borel measurable* if $f^{-1}(A) \in \mathcal{B}(X)$ for any open set $A \subset Y$. If μ is a Borel or a Radon measure on X, the function f is *μ-measurable* if for any open set $A \subset Y$ it holds that $f^{-1}(A) \in \mathcal{B}(X)$ up to a μ-negligible set.

   If $f : X \to Y$ is continuous and proper, and μ is a Radon measure, we define the *push forward* $f_\sharp\mu$ as the map on $\mathcal{B}(Y)$ defined by

   $$f_\sharp\mu(E) \coloneqq \mu(f^{-1}(E)).$$

   By [AFP00, Remark 1.71], it follows that $f_\sharp\mu$ is a Radon measure on Y.

7. If μ is a measure on a σ-algebra $\mathcal{Q}$ and $E \in \mathcal{Q}$, we define the *restriction* of μ on E as the measure $\mu \llcorner E$ defined by

   $$\mu \llcorner E\,(F) \coloneqq \mu(E \cap F),$$

   for any $F \in \mathcal{Q}$. It follows that $\mu \llcorner E$ is a measure on $\mathcal{Q}$ as well. If μ is a Radon (resp. Borel, positive, finite) measure, then so is $\mu \llcorner E$.

We remark here that for any $n \in \mathbb{N}$ with $n \geq 1$ we denote by

$$\mathcal{L}^n$$

the Lebesgue measure on $\mathbb{R}^n$. Also, in a given metric space (X, d), for any $s \geq 0$, we denote by

$$\mathcal{H}^s$$

the s-dimensional Hausdorff measure on X. For definition and properties of Lebesgue and Hausdorff measures we refer to [Sim83b, Chapter 1]. In particular, we recall that Hausdorff measures are positive Borel measures (see [Sim83b, Chapter 1, Equation (2.6)]).

Whenever μ is a Radon measure and $u \in C_c(X)$ is a continuous compactly supported function, the integral

$$\int_X u \, d\mu,$$

is understood in the sense of Lebesgue with respect to μ (see [AFP00, Definition 1.14]).

A sequence μ_n of Radon measures *locally (weakly*) converges* to a Radon measure μ if

$$\lim_n \int_X u \, d\mu_n = \int_X u \, d\mu \qquad\qquad \forall\, u \in C_c(X).$$

If the measures are also finite, we say that the sequence *(weakly*) converges* if the above holds for any $u \in C_0(X)$, where $C_0(X)$ is the closure of $C_c(X)$ with respect to the supremum norm. In such cases we shall write $\mu_n \overset{\star}{\rightharpoonup} \mu$ (locally) on X. Recall that a Radon measure μ is uniquely defined by duality with $C_c(X)$ functions. On the other hand, by Riesz Theorem (see [AFP00, Theorem 1.54] and [AFP00, Corollary 1.55]), Radon measures identify the dual space of $C_0(X)$, and that is the reason why we speak about weak* convergence.

We recall that by [AFP00, Remark 1.63] if μ_n, μ are positive Radon measures and μ_n locally converges to μ, then

$$\mu(K) \geq \limsup_n \mu_n(K) \qquad \forall\, K \subset X \text{ compact},$$

$$\mu(A) \leq \liminf_n \mu_n(A) \qquad \forall\, A \subset X \text{ open}.$$

Also, if μ_n is a sequence of Radon measures, and

$$\sup_n |\mu_n|(K) \leq C(K) < +\infty \qquad \forall\, K \subset X \text{ compact},$$

then there exists a locally converging subsequence. Moreover, if μ is the limit measure, it holds that $|\mu|(A) \leq \liminf_n |\mu_n|(A)$ for any open set $A \subset X$.

In order to define a varifold we need the following definition.

Definition 1.2.3. Let $1 \leq k < n$ be integers. We define the *Grassmannian* of (unoriented) k-subspaces in $\mathbb{R}^n$ the set $G(k,n)$ of the k-dimensional subspaces of $\mathbb{R}^n$. This set is equipped with the following distance ρ. Considering the canonical basis on $\mathbb{R}^n$, we can identify an element $S \in G(k,n)$ with the orthogonal projection matrix (S_{ij}) onto the subspace S. The distance $\rho(S,P)$ between $S \in G(k,n)$ and $P \in G(k,n)$ is the Frobenius distance between the two corresponding projection matrices, that is

$$\rho(S,T) := \left(\sum_{i,j} (S_{ij} - P_{ij})^2 \right)^{\frac{1}{2}}.$$

By identifying elements $S \in G(k,n)$ with the corresponding projection matrices, one immediately has that the metric space $(G(k,n),\rho)$ is sequentially compact, and thus compact, since it is a metric space. Moreover, let $GL^+(n)$ be the connected topological space of $n \times n$-matrices with strictly positive determinant endowed with the Frobenius distance, and $X_{k,n}$ be the subspace of $n \times k$-matrices with rank equal to k equipped with the same distance. We can consider the map

$$GL^+(n) \xrightarrow{f} X_{k,n} \xrightarrow{g} G(k,n),$$

where f is the projection onto the first k columns, and g associates to $A \in X_{k,n}$ the subspace generated by its columns. Since both f and g are continuous and surjective, we deduce that $G(k,n)$ is also connected.

Definition 1.2.4. Let $1 \leq k < n$ be integers and let $U \subset \mathbb{R}^n$ be a (non-empty) set. We define

$$G_k(U) := U \times G(k,n),$$

and such set is equipped with the product topology. A k-dimensional *varifold* V *on* U is a positive Radon measure on $G_k(U)$.

The *weight* measure of a k-varifold V on U is the positive Radon measure on U defined by $\mu_V := \pi_\sharp V$, where $\pi : G_k(U) \to U$ is the projection onto the first entry.

If U is open, we say that a sequence V_n of k-dimensional varifolds on U converges to a k-dimensional varifold V on U *in the sense of varifolds* if V_n locally converges to V on $G_k(U)$, that is, locally weakly* on $G_k(U)$.

Comparing with the notion of (embedded) submanifold, that is a set with a precise localization in the space and a unique well defined tangent space at any point, we can look at a varifold as an object describing a possibly diffused distribution of points and "tangent spaces" of a certain fixed dimension.

However, throughout this thesis we will only need a special type of varifolds, that are introduced in the next section.

1.2.2 Integer rectifiable varifolds

In this thesis we will always encounter a particular class of varifolds, that we present in this section. We need to start from the concept of rectifiable set in $\mathbb{R}^n$.

Definition 1.2.5. Let $1 \leq k < n$ be integers. A k-dimensional *rectifiable set* M is a subset of $\mathbb{R}^n$ such that

$$M \subset M_0 \cup \bigcup_{j=0}^{+\infty} F_j(\mathbb{R}^k),$$

where $F_j : \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz for any j and $\mathcal{H}^k(M_0) = 0$.

Some equivalent definitions of rectifiable set are recalled in the next result.

Proposition 1.2.6 (Rectifiable sets, [Sim83b, Chapter 3], [Mag12, Theorem 10.1])**.** *Let $1 \leq k < n$ be integers. Let $M \subset \mathbb{R}^n$ be a k-dimensional rectifiable set. Then the following holds.*

1. *There exist countably many sets $A_j \subset \mathbb{R}^k$ and Lipschitz functions $F_j : \mathbb{R}^k \to \mathbb{R}^n$ such that*

$$M = M_0 \cup \bigcup_{j=0}^{+\infty} F_j(A_j),$$

where $\mathcal{H}^k(M_0) = 0$.

Moreover, for any $t > 1$ it is possible to find F_j, A_j as above such that F_j is differentiable at every point of A_j with differential dF_j, each point of A_j is a Lebesgue point for dF_j,

$$\lim_{r \searrow 0} \frac{\mathcal{L}^k\left(A_j \cap B_r(0)\right)}{r^k} = 1,$$

and $\operatorname{Lip} F_j \leq t$ *for any j; also, for any j, any $x, y \in A_j$, and any $v \in \mathbb{R}^k$ it holds that*

$$\frac{1}{t}|x-y| \leq |F_j(x) - F_j(y)| \leq t|x-y|, \qquad \frac{1}{t}|v| \leq |d(F_j)_x(v)| \leq t|v|, \qquad t^{-k} \leq JF_j(x) \leq t^k,$$

where $JF_j(x) \coloneqq \sqrt{(dF_j(x))^\star dF_j(x)}$.

2. *There exist countably many embedded k-dimensional manifolds $N_j \subset \mathbb{R}^n$ of class C^1 such that*

$$M = M_0 \cup \bigcup_{j=0}^{+\infty} M_j,$$

where $\mathcal{H}^k(M_0) = 0$ and $M_j \subset N_j$ for any j.

3. *There exist countably many affine k-planes L_j in $\mathbb{R}^n$, sets $B_j \subset L_j$, vectors $v_j \in \mathbb{R}^n$, and Lipschitz functions $G_j : B_j \to \mathbb{R}$ such that v_j is normal to L_j and*

$$M = M_0 \cup \bigcup_{j=0}^{+\infty} \{x + G_j(x)v_j \mid x \in B_j\},$$

where $\mathcal{H}^k(M_0) = 0$.

The following theorem states that rectifiable sets support a weak notion of tangent space.

Theorem 1.2.7 (Approximate tangent spaces, [Sim83b, Chapter 3, Theorem 1.6])**.** *Let $1 \le k < n$ be integers. Let $M \subset \mathbb{R}^n$ be $\mathcal{H}^k$-measurable and such that $\mathcal{H}^k(M \cap K) < +\infty$ for any $K \subset \mathbb{R}^n$ compact.*

Then M is k-dimensional rectifiable if and only if for $\mathcal{H}^k$-ae point $x \in M$ there exists a k-dimensional subspace T_xM in $\mathbb{R}^n$ such that

$$\lim_{\varepsilon \searrow 0} \int_{\frac{1}{\varepsilon}(M-x)} \varphi(x)\, d\mathcal{H}^k(x) = \int_{T_xM} \varphi(x)\, d\mathcal{H}^k(x), \tag{1.3}$$

for any $\varphi \in C_c(\mathbb{R}^n)$.

Whenever a set $M \subset \mathbb{R}^n$ satisfies (1.3) at some point $x \in M$ for some k-subspace T_xM, we say that T_xM is the *approximate tangent space* of M at x. A smooth embedded k-dimensional manifold clearly satisfies (1.3), where the approximate tangent space is just the classical one. Also, observe that a k-subspace T_xM is the approximate tangent space of M at x if and only if

$$\mathcal{H}^k \llcorner \left(\frac{M-x}{\varepsilon}\right) \overset{\star}{\rightharpoonup} \mathcal{H}^k \llcorner T_xM,$$

locally weakly* as measures on $\mathbb{R}^n$.

Next we recall the area formula on rectifiable sets, which generalizes the change of variables formula on open subsets of the Euclidean space.

Theorem 1.2.8 (Area formula, [AFP00, Theorem 2.91], [Sim83b, Chapter 3, Section 2])**.** *Let $M \subset \mathbb{R}^n$ be a k-dimensional rectifiable set, and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map with $m \ge k$. Let $\varphi : \mathbb{R}^n \to [0, +\infty)$ be $\mathcal{H}^k$-measurable.*

Then the tangential differential *$d^M f_x : T_xM \to \mathbb{R}^m$ of f on T_xM, that is the differential of the restricted map $f|_{x+T_xM} : x + T_xM \to \mathbb{R}^m$, is well defined $\mathcal{H}^k$-almost everywhere on M, and the* area formula *holds:*

$$\int_M \varphi(x) J^M f(x)\, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} \varphi\, d\mathcal{H}^0\, d\mathcal{H}^k(y),$$

where

$$J^M f(x) := \left(\det((d^M f_x)^\star \circ d^M f_x)\right)^{\frac{1}{2}},$$

is the tangential Jacobian *of f on M, which is defined $\mathcal{H}^k$-almost everywhere on M.*

Now we turn our attention to the definition and properties of integer rectifiable varifolds. Many of the following definitions and facts generalize to general varifolds, but we will not need to treat arbitrary varifolds in the following. From now on, whenever we speak about varifolds, we assume that we are dealing with integer rectifiable varifolds, that are defined in the next definition.

Definition 1.2.9 (Integer rectifiable varifold)**.** Let $1 \le k < n$ be integers and let $M \subset U$ be a k-rectifiable set, where $U \subset \mathbb{R}^n$ is open. Let also $\theta : \mathbb{R}^n \to \mathbb{N}$ be $\mathcal{H}^k$-measurable and such that $\mathcal{H}^k(\{\theta > 0\}\Delta M) = 0$. Assume also that $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner M)$, that is, $\int_{K\cap M} \theta \, d\mathcal{H}^k < +\infty$ for any $K \subset U$ compact.

An *integer rectifiable varifold* V is identified by the equivalence class $\mathbf{v}(M,\theta)$, where two couples (M,θ), $(\widetilde{M},\widetilde{\theta})$ as above are equivalent if and only if $\mathcal{H}^k(M\Delta\widetilde{M}) = 0$ and $\mathcal{H}^k(\{\theta \neq \widetilde{\theta}\}) = 0$, by setting that

$$\int_{G_k(U)} \varphi(x,S)\, dV(x,S) := \int_M \varphi(x,T_xM)\,\theta(x)\, d\mathcal{H}^k(x),$$

for any $\varphi \in C_c(G_k(U))$, where T_xM is the ($\mathcal{H}^k$-ae defined) approximate tangent space of M at x. The ($\mathcal{H}^k$-ae defined) function θ is called *multiplicity*, or *density*, of $\mathbf{v}(M,\theta)$.

Whenever an integer rectifiable varifold V is given by a class $\mathbf{v}(M,\theta)$, we shall write $V = \mathbf{v}(M,\theta)$ and tacitly take arbitrary representatives M, θ in calculations. If V is such a varifold, we notice that its weight is just $\mu_V = \theta\mathcal{H}^k \llcorner M$.

It is clear that usual k-dimensional smooth isometrically immersed submanifolds of $\mathbb{R}^n$ define k-dimensional varifolds. More precisely, a smooth proper isometric immersion $\varphi : N \hookrightarrow \mathbb{R}^n$ of a k-dimensional manifold N induces an *image varifold* $\operatorname{Im}\varphi := \mathbf{v}(M,\theta)$ by taking $M = \varphi(N)$ and $\theta(x) = \sharp\varphi^{-1}(x)$. We observe that in such a case we have that the k-dimensional Riemannian volume of $\varphi^{-1}(K)$ equals $\mu_{\operatorname{Im}\varphi}(K)$ for any compact set $K \subset \mathbb{R}^n$.

Also, if $\varphi : N \to \mathbb{R}^n$ is a Lipschitz proper map defined on a k-rectifiable set $N \subset \mathbb{R}^{n'}$ and an integer rectifiable varifold $W = \mathbf{v}(N,\theta_W)$ is given, we define the image varifold $\operatorname{Im}\varphi := \mathbf{v}(M,\theta_V)$ by setting $M = \varphi(N)$ and $\theta_V(x) = \sum_{y\in\varphi^{-1}(x)} \theta_W(y)$. Observe that we are naming image varifolds two slightly different constructions, as in the first case N is an abstract manifold, while in the second case N is a subset of $\mathbb{R}^{n'}$. However, we believe that there is no risk of confusion.

In the literature this definition sometimes goes under the name of *push forward varifold*, but, as this operation differs from the usual push forward of measures, we prefer to adopt the name of image varifolds.

Recalling Proposition 1.1.8 and Remark 1.1.9, we give the following definitions.

Definition 1.2.10. Let $V = \mathbf{v}(M,\theta)$ be a k-dimensional integer rectifiable varifold on an open set $U \subset \mathbb{R}^n$. The *first variation* of V is the functional $\delta V : C^1_c(U,\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\delta V(X) := \int \operatorname{div}_{T_xM} X(x)\, d\mu_V(x).$$

It is interesting to know that, as in Remark 1.1.9, the expression for the first variation of a varifold is the result of the first variation of a suitably defined notion of area for varifolds, called *mass*, which is just the total mass of the weight μ_V (see [Sim83b, Chapter 4, Section 2] and [Sim83b, Chapter 8, Section 2]). Among many important results, let us mention Allard regularity theorems in [All72; All75], that deeply improved the comprehension of the properties of the first variation of varifolds.

We remark that if V_n is a sequence of integer rectifiable varifolds converging to a varifold V, then

$$\lim_n \delta V_n(X) = \delta V(X), \tag{1.4}$$

for any field $X \in C^1_c(U,\mathbb{R}^n)$. Indeed

$$\delta V_n(X) = \int_{G_k(U)} \varphi_X(x,S)\, dV_n(x,S),$$

where $\varphi_X(x,S) = \sum_{i=1}^k \langle dX_x(e_i), e_i\rangle$ and $\{e_i\}$ is any orthonormal basis of S. Since $\varphi_X \in C_c(G_k(U))$, Equation (1.4) follows by the definition of convergence of varifolds.

Definition 1.2.11. Let $V = \mathbf{v}(M,\theta)$ be a k-dimensional integer rectifiable varifold on an open set $U \subset \mathbb{R}^n$. A function $H \in L^1_{loc}(\mu_V, \mathbb{R}^n)$ and a $\mathbb{R}^n$-valued Radon measure σ_V which is singular with respect to μ_V are called the *generalized mean curvature of* V and the *generalized boundary of* V if

$$\int \operatorname{div}_{T_xM} X(x)\, d\mu_V(x) = -k \int \langle X, H\rangle\, d\mu_V + \int X\, d\sigma_V,$$

for any field $X \in C^1_c(U, \mathbb{R}^n)$. Writing $\sigma_V = \nu_V|\sigma_V|$ with $\nu_V \in L^1_{loc}(|\sigma_V|, \mathbb{R}^n)$ in polar decomposition [AFP00, Corollary 1.29], the vector ν_V is called *generalized conormal of* V.

It is clear that if V is the image varifold induced by a proper embedding $\varphi : N \hookrightarrow \mathbb{R}^n$, generalized mean curvature and conormal coincide with the classical ones.

Remark 1.2.12 (Orthogonality of the generalized mean curvature)**.** If $V = \mathbf{v}(M,\theta)$ is a k-dimensional integer rectifiable varifold on an open set $U \subset \mathbb{R}^n$ and it has generalized mean curvature H, then

$$H(x) \perp T_xM \qquad \text{at } \mu_V\text{-a.e. } x \in U,$$

that is, the vector $H(x)$ is orthogonal to the approximate tangent space at μ_V-a.e. point $x \in U$ such that T_xM exists and x is a Lebesgue point of H (w.r.t μ_V). This fundamental orthogonality property, which is true by definition in the smooth setting, has been proved in [Bra78, Section 5.8]. It is important to remember that the hypothesis on V of being *integer* rectifiable is essential. More precisely, if V is just a *rectifiable varifold* on some $U \subset \mathbb{R}^n$, i.e.,

$$\int_{G_k(U)} \varphi(x,S)\, dV(x,S) := \int_M \varphi(x, T_xM)\, \theta(x)\, d\mathcal{H}^k(x),$$

for any $\varphi \in C_c(G_k(U))$, where T_xM is the ($\mathcal{H}^k$-ae defined) approximate tangent space of a rectifiable set $M \subset \mathbb{R}^n$ at x and $\theta : U \to (0,+\infty)$ is in $L^1_{loc}(\mathcal{H}^k \llcorner M)$, defining generalized mean curvature and boundary as in Definition 1.2.11, it is false that H is μ_V-a.e. orthogonal to the approximate tangent space of M. A simple example is given in [Ton19, pp. 20-21], in which it is shown a 1-dimensional rectifiable varifold concentrated on a rectifible set Γ in $\mathbb{R}^2$ having generalized mean curvature that is even tangent to $T_x\Gamma$ at *every* point of Γ. In this thesis we will, however, only deal with integer rectifiable varifolds.

By analogy with surfaces, given a 2-dimensional varifold $V = \mathbf{v}(M,\theta)$ on some open set U, if there exists its generalized mean curvature H, we define its *Willmore energy* by

$$\mathcal{W}(V) = \int |H|^2\, d\mu_V \quad \in [0,+\infty],$$

and we set $\mathcal{W}(V) = +\infty$ in case the generalized mean curvature does not exists. We shall see in Section 1.3.2 the consequences implied by the fact that a varifold has finite Willmore energy.

It is a classical result in the theory the fact that integrability bounds on the generalized mean curvature imply good structural properties on the varifold, as stated in the next result.

Proposition 1.2.13 ([Sim83b, Chapter 4, Remark 4.10])**.** *Let $V = \mathbf{v}(M,\theta)$ be a k-dimensional integer rectifiable varifold on an open set $U \subset \mathbb{R}^n$. Suppose that V has mean curvature H and $\sigma_V = 0$. Assume that $H \in L^p_{loc}(\mu_V)$ for some $p > n$.*

Then the limit $\Theta(x) := \lim_{\rho\searrow 0} \frac{\mu_V(B_\rho(x))}{\omega_k \rho^k}$ exists at any point $x \in U$, and $V = \mathbf{v}(\mathrm{spt}\mu_V \cap U, \Theta)$. Moreover $\Theta(x) \geq 1$ at every point $x \in \mathrm{spt}\mu_V \cap U$ and it is upper semicontinuous on U.

Observe that Proposition 1.2.13 cannot be applied to 2-dimensional varifolds with bounded Willmore energy. Nevertheless, we shall derive a similar thesis also in such a case, but this will require a further study (Section 1.3.2).

We remark that Proposition 1.2.13 allows to represent a varifold satisfying the hypotheses by means of a canonical couple $\mathbf{v}(M,\theta)$ where $M = \mathrm{spt}\mu_V \cap U$ is relatively closed in U, and θ is upper semicontinuous and coincide with the density $\lim_{\rho\searrow 0} \frac{\mu_V(B_\rho(x))}{\omega_k \rho^k}$ at any point of M.

We finally recall the classical compactness theorem of integer rectifiable varifolds.

Theorem 1.2.14 (Compactness of varifolds, [Sim83b, Chapter 8, Theorem 5.8])**.** *Let V_n be a sequence of integer rectifiable k-dimensional varifolds on an open set $U \subset \mathbb{R}^n$ with $1 \le k < n$. Suppose that*

$$\sup_n \mu_{V_n}(W) + \sup_n \sup_{\substack{X\in C^1_c(W) \\ \|X\|_\infty \le 1}} |\delta V_n(X)| \le C(W) < +\infty,$$

for any open set $W \subset\subset U$.

Then there exist an integer rectifiable k-varifold V on U and a subsequence n_k such that V_{n_k} converges to V in the sense of varifolds.

Let us conclude with some comments on the hypotheses of Theorem 1.2.14. Let V be a k-dimensional integer rectifiable varifold on an open set $U \subset \mathbb{R}^n$. If

$$\sup_{\substack{X\in C^1_c(W) \\ \|X\|_\infty \le 1}} |\delta V(X)| \le C(W) < +\infty, \tag{1.5}$$

for any open set $W \subset\subset U$, then δV can be naturally extended to a continuous linear functional on continuous fields $X \in C_c(U, \mathbb{R}^n)$. By Riesz Theorem [AFP00, Theorem 1.54, Corollary 1.55], the functional $\delta V : C_c(U, \mathbb{R}^n) \to \mathbb{R}$ is represented by a vector valued Radon measure on U. Recalling Definition 1.2.11, by Besicovitch Derivation Theorem [AFP00, Theorem 2.22], we see that

$$\delta V = -kH\mu_V + \sigma_V \tag{1.6}$$

is just the Radon–Nikodým decomposition of δV with respect to μ_V. Such a decomposition exists if and only if (1.5) holds, and in such a case we say that V has *locally bounded first variation.*

Therefore we have the following corollary of Theorem 1.2.14, that we will use several times.

Corollary 1.2.15. *Let V_n be a sequence of integer rectifiable k-dimensional varifolds on an open set $U \subset \mathbb{R}^n$ with $1 \le k < n$. Suppose that V_n has mean curvature H_n and generalized boundary σ_{V_n}. If*

$$\sup_n \mu_{V_n}(U) + \int_U |H_n|\, d\mu_{V_n} + |\sigma_{V_n}|(U) < +\infty,$$

where $|\sigma_{V_n}|$ is the total variation of σ_{V_n}, then V_n converges to an integer rectifiable k-varifold V in the sense of varifolds (up to subsequence).

Moreover, the above observations imply the following consequence on the lower semicontinuity of the Willmore energy with respect to the convergence of varifolds.

Corollary 1.2.16 (Lower semicontinuity of the Willmore energy)**.** *Let V_n be a sequence of 2-dimensional integer rectifiable varifolds on an open set $U \subset \mathbb{R}^n$ with $n \ge 3$ converging to an integer rectifiable varifold V in the sense of varifolds. Suppose that V_n has mean curvature H_n and $\sup_n \mathcal{W}(V_n) < +\infty$. Then the following holds.*

1. *If $\sigma_{V_n} = 0$ on U for any n, then $\sigma_V = 0$ on U and*

$$\mathcal{W}(V) \le \liminf_n \mathcal{W}(V_n).$$

2. *If $\sup_n |\delta V_n|(U) < +\infty$, then V has generalized mean curvature H and generalized boundary σ_V on U. If also $\mathcal{H}^2(\mathrm{spt}\sigma_V) = \mathcal{H}^2\left(\overline{\cup_n \mathrm{spt}\sigma_{V_n}}\right) = 0$, then*

$$\mathcal{W}(V) \le \liminf_n \mathcal{W}(V_n).$$

Proof. We can assume without loss of generality that $\sup_n \mu_{V_n}(U) < +\infty$. Indeed, since $V_n \to V$, there is an increasing sequence of open sets $U_l \subset U_{l+1} \subset U$ such that $\cup_l U_l = U$ and $\sup_n \mu_{V_n}(U_l) < +\infty$ for any l. Hence, once the thesis is proved on U_l, passing to the supremum over l yields the result.

We now prove the items separately.

1. By (1.6) we have that $\delta V_n = -2H_n \mu_{V_n}$, that is, $-2H_n = \frac{\delta V_n}{\mu_{V_n}}$ in the sense of Radon–Nikodým derivatives. Since $V_n \to V$ as varifolds, then $\mu_{V_n} = \pi_\sharp V_n \overset{\star}{\rightharpoonup} \pi_\sharp V = \mu_V$ locally weakly* on U. Moreover $\delta V_n \overset{\star}{\rightharpoonup} \delta V$ by (1.4). Hence we can apply [AFP00, Theorem 2.34] with $f(p) = |p|^2$, that yields exactly the desired lower semicontinuity result for integral functionals whose integrand is a convex function of the derivative $\frac{\delta V_n}{\mu_{V_n}}$. It also follows that $\sigma_V = 0$ on U (see also [AFP00, Example 2.36]).

2. Since $|\delta V|(U) \le \liminf_n |\delta V_n|(U) < +\infty$, the limit varifold has mean curvature H and generalized boundary σ_V on U. Let

$$U' = U \setminus \left(\mathrm{spt}\sigma_V \cup \overline{\cup_n \mathrm{spt}\sigma_{V_n}}\right),$$

which is open. By construction none of the varifolds V, V_n have generalized boundary as varifolds on U'. Hence by Item 1 we have

$$\int_{U'} |H|^2 \, d\mu_V \le \liminf_n \int_{U'} |H_n|^2 \, d\mu_{V_n} \le \liminf_n \mathcal{W}(V_n).$$

By assumption $\mathcal{H}^2(U \Delta U') = 0$, and then $\mu_V \llcorner U' = \mu_V$, and thus the left hand side coincides with $\mathcal{W}(V)$.

□

We mention that the lower semicontinuity of the Willmore energy has been also studied in the setting of the convergence of currents (see Section 1.5) in [Sch09].

1.2.3 Curvature varifolds with boundary

In this part we consider a further class of varifolds, introducing also a generalized definition of second fundamental form and another notion of boundary of a varifold. These concepts have been introduced by Hutchinson [Hut86]. However, we will refer to the more recent treatise of Mantegazza [Man96], who also introduced some differences and improvements on the theory developed in [Hut86].

We restrict the discussion to the case of integer rectifiable varifolds.

Definition 1.2.17. Let $1 \leq k < n$. Let $V = \mathbf{v}(M, \theta)$ be a k-dimensional integer rectifiable varifold on an open set $U \subset \mathbb{R}^n$. We say that V is a *curvature varifold with boundary* on U if there exist functions $A_{ijk} \in L^1_{loc}(V)$ and a $\mathbb{R}^n$-valued Radon measure $\partial V \equiv (\partial V_1, ..., \partial V_n)$ on $G_k(U)$ such that

$$\int_{G_k(U)} P_{ij}\partial_{x_j}\varphi(x,P) + A_{ijk}(x,P)\partial_{P_{jk}}\varphi(x,P)\, dV(x,P)$$
$$= -\int_{G_k(U)} \varphi(x,P)A_{jij}(x,P)\, dV(x,P) + \int_{G_k(U)} \varphi(x,P)\, d\partial V_i(x,P),$$

for any $i = 1, ..., k$ for any $\varphi \in C^1_c(G_k(U))$. In the above equation summation over repeated indices is understood and elements of the Grassmannian are identified by projection matrices represented on the canonical basis of $\mathbb{R}^n$. In such a case the measure ∂V is called *boundary*, or *boundary measure*, of V.

Notice that a curvature varifold with boundary satisfies an integration by parts formula at the level of $G_k(U)$. This has to be compared with the integration by parts formula of Definition 1.2.11, which takes place in U. In fact, the following result implies that the notion of curvature varifold with boundary is stronger than the existence of generalized mean curvature and generalized boundary.

Theorem 1.2.18 (Curvature varifolds, [Man96, Section 3])**.** *Let $1 \leq k < n$. Let $V = \mathbf{v}(M, \theta)$ be a k-dimensional curvature varifold with boundary on $U \subset \mathbb{R}^n$. In the notation of Definition 1.2.17, understanding summation over repeated indices, the following holds true.*

1. *$A_{ijk} = A_{ikj}$, $A_{ijj} = 0$, and $A_{ijk} = P_{jr}A_{irk} + P_{rk}A_{ijr}$ at V-ae $(x, P) \in G_k(U)$.*
2. *$P_{il}\partial V_l = \partial V_i$ as measures on $G_k(U)$.*
3. *$P_{il}A_{ljk} = A_{ijk}$ at V-ae $(x, P) \in G_k(U)$.*
4. *V has generalized mean curvature H, and the i-th component H_i of H is given by $H_i(x) = \frac{1}{k}A_{jij}(x, T_xM)$ at μ_V-ae $x \in U$, where T_xM is the approximate tangent space of M at x. Moreover $P(x)_{ij}H_j(x) = 0$ at μ_V-ae $x \in U$, where $P(x)$ is the matrix representing the orthogonal projection on T_xM.*
5. *V has generalized boundary $\sigma_V = \pi_\sharp(\partial V)$, where $\pi : G_k(U) \to U$ is the natural projection.*

Definition 1.2.19. Let $1 \leq k < n$. Let $V = \mathbf{v}(M, \theta)$ be a k-dimensional curvature varifold with boundary on $U \subset \mathbb{R}^n$. We define the *generalized second fundamental form* of V to be a function $B \in L^1_{loc}(\mu_V, \mathbb{R}^{n^3})$ whose components are given by $B^m_{ij}(x) \coloneqq P(x)_{jl}A_{iml}(x, T_xM)$, where $P(x)$ is the matrix representing the orthogonal projection on T_xM.

Using the symmetries stated in Theorem 1.2.18, we have that, if V is a curvature varifold, then

$$B^m_{jj} = P_{jl}A_{jlm} = A_{jjm} - P_{lm}A_{jjl} = A_{jmj} - P_{ml}A_{jlj} = kH_m - kP_{ml}H_l = kH_m,$$

and

$$A_{ijk} = B^k_{ij} + B^j_{ki},$$

at μ_V-ae $x \in U$. Therefore, it is important to notice that a bound on the functions A_{ijk} is equivalent to a bound on each component of B.

As in the case of the generalized mean curvature and of the generalized boundary, one can check that the generalized second fundamental form and the boundary of a curvature varifold induced by a smooth immersion can be brought back to classical quantities. In fact, the integration formula in Definition 1.2.17 is first known to hold in the smooth setting with classical notions of boundary and second fundamental form, just as in the case of the identity in Proposition 1.1.8. If $\varphi : N \hookrightarrow \mathbb{R}^n$ is a smooth proper isometric embedding of a k-dimensional manifold N and $V = \operatorname{Im}\varphi = \mathbf{v}(M, \theta_V)$ and $\{E_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$, identifying T_xM with $d\varphi_x(T_xM)$ we have that

$$B_{ij}^m(\varphi(x)) = \langle B_x(E_i^\top, E_j^\top), E_m \rangle,$$

where $(\cdot)^\top$ denotes projection onto T_xM, B is the second fundamental form of φ, and B_{ij}^m is the generalized second fundamental form of the varifold $\operatorname{Im}\varphi$. For what concerns the boundary of $\operatorname{Im}\varphi$, one can similarly derive that it only depends on the tangent space of $\varphi|_{\partial N}$ and the conormal co_φ determined by the embedding, and $|\partial \operatorname{Im}\varphi|(G_k(U)) = \mathcal{H}^{k-1}(\varphi(\partial N))$. We just mention that analogous relations can be argued if φ is a smooth proper isometric immersion, but we will not need further details.

We now conclude with the compactness theorem for curvature varifolds.

Theorem 1.2.20 (Compactness of curvature varifolds, [Man96, Theorem 6.1]). *Let $p > 1$ and V_l a sequence of k-dimensional curvature varifolds with boundary in $U \subset \mathbb{R}^n$, with $1 \le k < n$. Denote $A_{ijk}^{(l)}$ the functions A_{ijk} of V_l. Suppose that $A_{ijk}^{(l)} \in L^p(V)$ and*

$$\sup_l \quad \left\{ \mu_{V_l}(W) + \int_{G_k(W)} \left| \sum_{i,j,k} |A_{ijk}^{(l)}| \right|^p dV_l + |\partial V_l|(G_n(W)) \right\} \quad \le C(W) < +\infty$$

for any $W \subset\subset U$, where $|\partial V_l|$ is the total variation of ∂V_l. Then:

1. *V_l converges (up to subsequence) to a curvature varifold with boundary V in the sense of varifolds. Moreover $A_{ijk}^{(l)} V_l \stackrel{\star}{\rightharpoonup} A_{ijk} V$ and $\partial V_l \stackrel{\star}{\rightharpoonup} \partial V$ locally weakly* as measures on $G_k(U)$.*

2. *For every lower semicontinuous function $f : \mathbb{R}^{n^3} \to [0, +\infty]$ it holds that*

$$\int_{G_k(U)} f(A_{ijk})\, dV \le \liminf_l \int_{G_k(U)} f(A_{ijk}^{(l)})\, dV_l.$$

It follows from the above theorem that the Willmore energy of 2-dimensional curvature varifolds with boundary is lower semicontinuous with respect to varifold convergence of curvature varifolds with boundary satisfying the hypotheses of Theorem 1.2.20. Indeed we have that

$$\mathcal{W}(V) = \int_U |H|^2 \, d\mu_V = \int_{G_2(U)} f(A_{ijk})\, dV,$$

for any curvature varifold $V = \mathbf{v}(M, \theta)$, where $f(A_{ijk}) = \frac{1}{4} \sum_m \left(\sum_j A_{jmj} \right)^2$.

1.3 Monotonicity and Willmore-type energies of varifolds

In the theory of varifolds, as well as in the theory of minimal surfaces, it is very common to speak about monotonicity formulas. With this term we refer to the existence of integral quantities, which are usually functions of one variable $\rho > 0$, that are monotone in ρ and involve geometric or measure theoretic terms. As we shall see, a monotonicity formula for a measure μ may have important consequences on the structure of μ, that is, it can give useful information about the support and the density of μ. For a general introduction to monotonicity formulas see [Sim83b, Chapter 4, Chapter 8].

1.3.1 A monotonicity formula for 1-dimensional varifolds

As a first case we consider 1-dimensional integer rectifiable varifolds in $\mathbb{R}^n$ with $n \geq 2$. We will typically denote such a varifold as $V = \mathbf{v}(\Gamma, \theta)$, where Γ is 1-dimensional rectifiable set. As in the case of Willmore energy of surfaces, we define the p-elastic energy of a varifold $V = \mathbf{v}(\Gamma, \theta)$ in $\mathbb{R}^n$ by analogy with the p-elastic energy of curves. Hence we set

$$\mathcal{E}_p(V) := \mu_V(\mathbb{R}^n) + \frac{1}{p}\int |k|^p \, d\mu_V \qquad \in [0, +\infty],$$

for any $p \in [1, +\infty)$. In the above definition, we denoted by k the generalized (mean) curvature of V. In case k does not exist, then we set $\mathcal{E}_p(V) = +\infty$.

Throughout this part, let us fix $x_0 \in \mathbb{R}^n$, and let $V = \mathbf{v}(\Gamma, \theta) \neq 0$ be an integer 1-dimensional rectifiable varifold in $\mathbb{R}^n$ with curvature k. For $0 < r < +\infty$, we define

$$M_{V,x_0}(r) := \frac{\mu_V(B_r(x_0))}{r} - \int_{B_r(x_0)} \left\langle k, \frac{x - x_0}{|x - x_0|} \right\rangle d\mu_V(x) + \frac{1}{r}\int_{B_r(x_0)} \langle k, x - x_0 \rangle \, d\mu_V(x).$$

For $0 < \sigma < \rho < +\infty$, we consider the vector field

$$X(x) = \left(\frac{1}{|x - x_0|_\sigma} - \frac{1}{\rho} \right)_+ (x - x_0),$$

where $(\cdot)_+$ denotes the non-negative part and $|\cdot|_\sigma := \max\{|\cdot|, \sigma\}$. For any set E, let $E_\sigma := E \cap B_\sigma$, $E_\rho := E \cap B_\rho$, and $E_{\rho,\sigma} := E_\rho \setminus \overline{B_\sigma}$. A direct computation shows that the tangential divergence of X on the rectifiable set Γ is

$$\operatorname{div}_{T_x\Gamma} X(x) = \begin{cases} \frac{1}{\sigma} - \frac{1}{\rho} & \mathcal{H}^1\text{-ae on } \Gamma_\sigma, \\ \frac{|(x-x_0)^\perp|^2}{|x-x_0|^3} - \frac{1}{\rho} & \mathcal{H}^1\text{-ae on } \Gamma_{\rho,\sigma}, \\ 0 & \mathcal{H}^1\text{-ae on } \Gamma \setminus B_\rho(x_0), \end{cases}$$

for almost every $0 < \sigma < \rho < +\infty$, where $(\cdot)^\perp$ denotes projection onto $T_x\Gamma^\perp$.

In this setting we can prove the following monotonicity formula.

Theorem 1.3.1 (Monotonicity formula for 1-dimensional varifolds)**.** *Let $V = \mathbf{v}(\Gamma, \theta)$ be a 1-dimensional varifold in $\mathbb{R}^n$ with $n \geq 2$ and with generalized curvature k. Suppose that $\sigma_V = 0$. Then*

$$M_{V,x_0}(\sigma) + \int_{B_\rho(x_0) \setminus B_\sigma(x_0)} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^3} \, d\mu_V(x) = M_{V,x_0}(\rho), \tag{1.7}$$

for almost every $0 < \sigma < \rho < +\infty$. In particular $r \mapsto M_{V,x_0}(r)$ is essentially non-decreasing. Moreover, the following holds.

1. *Assuming in addition that $\int |k|^p \, d\mu_V < +\infty$ for some $p \in [1, \infty)$, then*

$$\limsup_{\sigma \searrow 0} \frac{\mu_V(B_\sigma(x_0))}{\sigma} \leq \liminf_{\rho \nearrow \infty} \frac{\mu_V(B_\rho(x_0))}{\rho} + \int_{B_\rho(x_0)} |k| \, d\mu_V(x). \tag{1.8}$$

2. Assuming in addition that $\mathcal{E}_p(V) < +\infty$ for some $p \in [1,+\infty)$, then

$$2 \leq \int |k|\, d\mu_V \leq \mu_V(\mathbb{R}^n)^{\frac{1}{p'}} \|k\|_{L^p(\mu_V)}, \tag{1.9}$$

and we have the following bounds on the density of μ_V:

$$p > 1 \quad \Rightarrow \quad \exists \lim_{\sigma \searrow 0} \frac{\mu_V(B_\sigma(x))}{2\sigma} \leq \frac{1}{2}\int |k|\, d\mu_V \qquad \forall\, x \in \mathbb{R}^n,$$

$$p = 1 \quad \Rightarrow \quad \exists \lim_{\sigma \searrow 0} \frac{\mu_V(B_\sigma(x))}{2\sigma} \leq \frac{1}{2}\int |k|\, d\mu_V \qquad \text{at } \mathcal{H}^1\text{-ae } x \in \mathbb{R}^n.$$

3. Assuming in addition that $\mathcal{E}_1(V) < +\infty$ and that Γ is essentially bounded, i.e., $\mathcal{H}^1(\Gamma \setminus B_R(0)) = 0$ for R large enough, then

$$p = 1 \quad \Rightarrow \quad \exists \lim_{r \searrow 0} \frac{\mu_V(B_r(x))}{2r} \leq \frac{1}{2}\int |k|\, d\mu_V \qquad \forall\, x \in \mathbb{R}^n.$$

In cases 2 and 3, the multiplicity function $\theta(x)$ equals $\lim_{\sigma\searrow 0} \frac{\mu_V(B_\sigma(x))}{2\sigma}$ at $\mathcal{H}^1$-ae $x \in \mathbb{R}^n$.

Proof. We adopt the notation introduced above. Integrating the divergence $\operatorname{div}_{T_x\Gamma} X$ above with respect to μ_V and using the first variation formula we get

$$\begin{aligned}
&\frac{\mu_V(B_\sigma(x_0))}{\sigma} + \frac{1}{\sigma}\int_{B_\sigma(x_0)} \langle k, x - x_0\rangle\, d\mu_V(x) + \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}\, d\mu_V(x) \\
&= \frac{\mu_V(B_\rho(x_0))}{\rho} + \frac{1}{\rho}\int_{B_\rho(x_0)} \langle k, x - x_0\rangle\, d\mu_V(x) - \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \left\langle k, \frac{x-x_0}{|x-x_0|}\right\rangle d\mu_V(x),
\end{aligned} \tag{1.10}$$

which is exactly (1.7). Now we deal with the different cases separately.

1. Dropping the positive term on the left of (1.10), we obtain

$$\begin{aligned}
&\frac{\mu_V(B_\sigma(x_0))}{\sigma} + \frac{1}{\sigma}\int_{B_\sigma(x_0)} \langle k, x - x_0\rangle\, d\mu_V(x) \\
&\qquad \leq \frac{\mu_V(B_\rho(x_0))}{\rho} + \int_{B_\rho(x_0)} \left\langle k, \frac{x-x_0}{\rho} - \frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\right\rangle d\mu_V(x).
\end{aligned}$$

Since

$$\left|\frac{1}{\sigma}\int_{B_\sigma(x_0)} \langle k, x - x_0\rangle\, d\mu_V(x)\right| \leq \left(\int_{B_\sigma(x_0)} |k|^p\, d\mu_V\right)^{\frac{1}{p}} \left(\mu_V(B_\sigma(x_0))\right)^{\frac{1}{p'}} \xrightarrow[\sigma\to 0]{} 0,$$

and

$$\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)} \xrightarrow[\sigma\to 0]{} \frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)} \qquad \text{in } L^{p'}(\mu_V),$$

letting $\sigma \searrow 0$ and then $\rho \nearrow \infty$ we get the inequality

$$\begin{aligned}
\limsup_{\sigma\searrow 0} \frac{\mu_V(B_\sigma(x_0))}{\sigma} &\leq \liminf_{\rho\nearrow\infty} \frac{\mu_V(B_\rho(x_0))}{\rho} + \int_{B_\rho(x_0)} \left\langle k, \left(\frac{1}{\rho} - \frac{1}{|x-x_0|}\right)(x-x_0)\right\rangle d\mu_V(x) \\
&\leq \liminf_{\rho\nearrow\infty} \frac{\mu_V(B_\rho(x_0))}{\rho} + \int_{B_\rho(x_0)} |k| \left|\frac{|x-x_0|}{\rho} - 1\right| d\mu_V(x) \\
&\leq \liminf_{\rho\nearrow\infty} \frac{\mu_V(B_\rho(x_0))}{\rho} + \int_{B_\rho(x_0)} |k|\, d\mu_V(x),
\end{aligned}$$

that is (1.8).

2. From (1.8) we get
$$\limsup_{\sigma\searrow 0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}\le\int|k|\,d\mu_V. \tag{1.11}$$
Equation (1.11) gives us the desired bounds.

If $p>1$ we can apply Proposition 1.2.13 and use (1.11). Suppose then that $p=1$. In this case (by [AFP00, Theorem 2.22]) we clearly have
$$\theta(x)=\lim_{\sigma\searrow 0}\frac{\mu_V(B_\sigma(x_0))}{2\sigma}\le\frac{1}{2}\int|k|\,d\mu_V\qquad\text{for } \mathcal{H}^1\text{-ae } x\in\Gamma.$$
Therefore, since $V\neq 0$, then $\theta(x)\ge 1$ at some point x, and we conclude that inequality (1.9) holds for any $p\in[1,\infty)$.

3. We need to show that the limit $\lim_{\sigma\searrow 0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}$ does exist for any $x_0\in\mathbb{R}^n$. In fact we have
$$\begin{aligned}
&\left|\frac{1}{\sigma}\int_{B_\sigma(x_0)}\langle k,x-x_0\rangle\,d\mu_V(x)\right|\to 0 &&\text{as } \sigma\to 0,\\
&\left|\frac{1}{\rho}\int_{B_\rho(x_0)}\langle k,x-x_0\rangle\,d\mu_V(x)\right|\le\frac{R_0}{\rho}\int_{B_{R_0}(x_0)}|k|\,d\mu_V\to 0 &&\text{as } \rho\to\infty,\\
&\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\to\frac{x-x_0}{|x-x_0|} &&\text{in } L^1(\mathbb{R}^n,\mu_V),
\end{aligned}$$
where the last limit is for $\sigma\to 0$ and $\rho\to+\infty$ and follows by Dominated Convergence. Therefore there exists finite the limit
$$\lim_{\sigma\searrow 0}\lim_{\rho\nearrow\infty}\int\left\langle k,\frac{x-x_0}{|x-x_0|}\chi_{B_\rho(x_0)\setminus B_\sigma(x_0)}\right\rangle d\mu_V(x),$$
Hence (1.10) implies that
$$\sup_{\sigma,\rho>0}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}\,d\mu_V(x)<+\infty,$$
thus, by monotonicity, the limit
$$\lim_{\sigma\searrow 0}\lim_{\rho\nearrow\infty}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3}\,d\mu_V(x)$$
exists finite. Since $\lim_{\rho\to\infty}\frac{\mu_V(B_\rho(x_0))}{\rho}\to 0$, Equation (1.10) implies that
$$\exists\lim_{\sigma\searrow 0}\frac{\mu_V(B_\sigma(x_0))}{\sigma}<+\infty\qquad\forall x_0\in\mathbb{R}^n,$$
which completes the proof.

The last assertion follows from Proposition 1.2.13 if $p>1$. While, in case $p=1$, it follows as usual by Derivation Theorem [AFP00, Theorem 2.22]. □

Remark 1.3.2. Theorem 1.3.1 also follows from [Men16, Corollary 4.8] (see also [MS18, Theorem 3.5]), which also includes additional technical details that we will not need in the following.

Let us add an additional comment about the case $p = 2$. In such a case, we can identify another useful monotone quantity.

Remark 1.3.3. Let $p = 2$, and adopt the above notation. Suppose, as in Theorem 1.3.1, that $\sigma_V = 0$. For $r > 0$ let

$$B_{V,x_0}(r) := \left(\frac{1}{2} + \frac{1}{r}\right)\mu_V(B_r(x_0)) + \frac{1}{r}\int_{B_r(x_0)} \langle k, x - x_0\rangle \, d\mu_V(x) + \frac{1}{2}\int_{B_r(x_0)} |k|^2 \, d\mu_V.$$

Then

$$B_{V,x_0}(\sigma) + \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \left(\frac{|(x-x_0)^\perp|^2}{|x-x_0|^3} + \frac{1}{2}\left|k + \frac{x-x_0}{|x-x_0|}\right|^2\right) d\mu_V(x) = B_{V,x_0}(\rho), \tag{1.12}$$

for almost every $0 < \sigma < \rho < +\infty$. In particular $r \mapsto B_{V,x_0}(r)$ is essentially non-decreasing.

Indeed, to prove (1.12), it is enough to insert the identity $\langle k, \frac{x}{|x|}\rangle = \frac{1}{2}\big(\big|k + \frac{x}{|x|}\big|^2 - |k|^2 - 1\big)$ in (1.10).

Moreover, if we additionally require that $\mu_V(\mathbb{R}^n) < +\infty$, then

$$\begin{aligned}\frac{1}{R}\int_{B_R(x_0)} \langle k, x - x_0\rangle \, d\mu_V(x) &= \frac{1}{R}\int_{B_r(x_0)} \langle k, x - x_0\rangle \, d\mu_V(x) + \frac{1}{R}\int_{B_R(x_0)\setminus B_r(x_0)} \langle k, x - x_0\rangle \, d\mu_V(x) \\ &\le \frac{1}{R}\int_{B_r(x_0)} \langle k, x - x_0\rangle \, d\mu_V(x) + \left(\int_{B_R(x_0)\setminus B_r(x_0)} |k|^2\right)^{\frac{1}{2}} \big(\mu_V(B_R(x_0)\setminus B_r(x_0))\big)^{\frac{1}{2}}\end{aligned}$$

for any $r < R$. So, letting first $R \to \infty$ and then $r \to \infty$, we get that $\frac{1}{R}\int_{B_R(x_0)} \langle k, x - x_0\rangle \, d\mu_V(x) \to 0$ as $R \to \infty$. Thus we obtain that

$$\lim_{r\to\infty} B_{V,x_0}(r) = \frac{1}{2}\Big(\mu_V(\mathbb{R}^n) + \mathcal{E}_2(V)\Big),$$

for any choice of $x_0 \in \mathbb{R}^n$.

We conclude by proving the following fundamental estimate concerning curves in $\mathbb{R}^2$, that has to be compared with Equation (1.9) of Theorem 1.3.1.

Lemma 1.3.4. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a regular curve in $W^{2,p}$ for some $p \in [1, \infty)$. Then*

$$2\pi \le \int_{\mathbb{S}^1} |k_\gamma| \, ds_\gamma \le \left(\int_{\mathbb{S}^1} |k_\gamma|^p \, ds_\gamma\right)^{\frac{1}{p}} (L(\gamma))^{\frac{1}{p'}}, \tag{1.13}$$

where $L(\gamma)$ denotes the length of the curve. Moreover, in the first inequality, equality holds if and only if γ parametrizes the boundary of a convex set and it is injective.

Proof. By approximation it is enough to prove the statement for $\gamma \in C^\infty$. Then the closed convex envelop of the support (γ) is a $C^{1,1}$-smooth set, and its boundary can be parametrized by an embedded curve $\sigma : \mathbb{S}^1 \to \mathbb{R}^2$ of class H^2. Then $\int |k_\sigma| \, ds_\sigma \le \int |k_\gamma| \, ds_\gamma$. Hence, if we can prove that

$$\int_{\mathbb{S}^1} |k_\sigma| \, ds_\sigma \ge 2\pi, \tag{1.14}$$

the thesis follows by Hölder inequality. By approximation, we can assume without loss of generality that $\sigma : \mathbb{S}^1 \to \mathbb{R}^2$ is an embedded curve of class C^∞ that positively parametrizes the boundary of the bounded set it encloses. Moreover, as $\int |k_\sigma| \, ds_\sigma$ is scaling invariant, we can assume that σ is parametrized by arclength. If τ_σ is the tangent vector along σ, we clearly have that $\tau_\sigma(\mathbb{S}^1) = \mathbb{S}^1$, that is, it is surjective. Indeed, σ has index equal to 1 (see [AT12, Theorem 2.4.7]), hence the degree of τ_σ equals 1, and then $\tau_\sigma : \mathbb{S}^1 \to \mathbb{S}^1$ has to be surjective. Therefore we estimate

$$\int_{\mathbb{S}^1} |k_\sigma|\, ds_\sigma = \int_{\mathbb{S}^1} |\partial_x \tau_\sigma(x)|\, dx = \int_{\mathbb{S}^1} \sharp\tau_\sigma^{-1}(v)\, d\mathcal{H}^1(v) \geq \mathcal{H}^1(\mathbb{S}^1) = 2\pi,$$

where we used the area formula (Theorem 1.2.8). Letting $\tau_\sigma(x) = (\cos\theta(x), \sin\theta(x))$ for a smooth angle function $\theta : [0, 2\pi] \to \mathbb{R}$, we have that $|k_\sigma| = |\partial_x\theta|$. Hence the equality case in the above estimate implies that $\partial_x\theta$ has a sign. As $\partial_x\theta = \langle k_\sigma, \nu_\sigma\rangle$, this means that σ is convex. $\square$

Let us conclude with a few comments on Lemma 1.3.4 and Theorem 1.3.1.

Remark 1.3.5. We mention that inequality (1.14) is already present in [DNP18], proved with a different method in the setting of networks. However, we will need the approach used in the proof of Lemma 1.3.4 in the following, namely the use of the area formula in relation with the "angle spanned" by the tangent vector, i.e., its image in $\mathbb{S}^1$.

Remark 1.3.6. Let us compare (1.13) with (1.9). It is clear that (1.13) is sharp, as equality holds on unit circles. Similarly, it would be interesting to understand whether the first inequality in Equation (1.9) is sharp. However, this is still not known, up to our knowledge.

1.3.2 A monotonicity formula for 2-dimensional varifolds with boundary

In the context of 2-dimensional varifolds the use of a monotonicity formula in the study of the Willmore energy goes back to [Sim93], in which varifolds without boundary are considered. The important consequences of such monotonicity formula are collected and revised, for example, in [KS04]. Here we are interested in extending these results to the case of varifolds with possibly non-vanishing boundary.

From now on and for the rest of the section let $V = \mathbf{v}(M, \theta_V)$ be a 2-dimensional varifold on $\mathbb{R}^n$, with $n \geq 3$, having mean curvature H and generalized boundary σ_V. We assume that

$$\mathcal{W}(V) < +\infty.$$

For a fixed $x_0 \in \mathbb{R}^n$ and $0 < r < +\infty$ we define

$$A_{V,x_0}(r) := \frac{\mu_V(B_r(x_0))}{r^2} + \frac{1}{4}\int_{B_r(x_0)} |H|^2\, d\mu_V + R_{V,x_0}(r),$$

with

$$\begin{aligned} R_{V,x_0}(r) &:= \int_{B_r(x_0)} \frac{\langle H, x - x_0\rangle}{r^2}\, d\mu_V(x) + \frac{1}{2}\int_{B_r(x_0)} \left(\frac{1}{|x-x_0|^2} - \frac{1}{r^2}\right)(x - x_0)\, d\sigma_V(x) \\ &=: \int_{B_r(x_0)} \frac{\langle H, x - x_0\rangle}{r^2}\, d\mu_V(x) + T_{V,x_0}(r). \end{aligned}$$

In the following, the symbol $(\cdot)^\perp$ will denote projection onto $T_xM^\perp$, for any x such that the approximate tangent space T_xM exists.

Theorem 1.3.7 (Monotonicity formula for 2-dimensional varifolds)**.** *Let $V = \mathbf{v}(M, \theta_V)$ be a 2-dimensional integer rectifiable varifold on $\mathbb{R}^n$, with $n \geq 3$, having mean curvature H and generalized boundary σ_V. Suppose that $\mathcal{W}(V) < +\infty$. Then*

$$A_{V,x_0}(\sigma) + \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \left| \frac{H}{2} + \frac{(x-x_0)^\perp}{|x-x_0|^2} \right|^2 d\mu_V(x) = A_{V,x_0}(\rho), \tag{1.15}$$

for almost every $0 < \sigma < \rho < +\infty$. In particular, the map $r \mapsto A_{V,x_0}(r)$ is essentially non-decreasing.

Moreover, if $S \subset \mathbb{R}^n$ is compact with $\mathcal{H}^2(S) = 0$, $\mathrm{spt}\sigma_V \subset S$, and

$$\limsup_{R\to\infty} \frac{\mu_V(B_R(0))}{R^2} \leq K < +\infty,$$

then the limit

$$\lim_{\rho\searrow 0} \frac{\mu_V(B_\rho(x))}{\rho^2}$$

exists at any point $x \in \mathbb{R}^n \setminus S$, the density function $\widetilde{\theta}_V(x) := \lim_{\rho\searrow 0} \frac{\mu_V(B_\rho(x))}{\pi\rho^2}$ is upper semi-continuous on $\mathbb{R}^n \setminus S$ and bounded by a constant $C = C(d(x,S), |\sigma_V|(S), K, \mathcal{W}(V))$ depending only on the distance $d(x,S)$, $|\sigma_V|(S)$, K and $\mathcal{W}(V)$. Moreover $V = \mathbf{v}(\widetilde{M}, \widetilde{\theta}_V)$ where $\widetilde{M} = \{x \in \mathbb{R}^n \setminus S \,|\, \widetilde{\theta}_V(x) \geq \frac{1}{2}\} \cup S$ is closed.

Proof. Integrating the tangential divergence of the field $X(x) = \left(\frac{1}{|x-x_0|_\sigma^2} - \frac{1}{\rho^2}\right)_+ (p - x_0)$, where $|x-x_0|_\sigma^2 = \max\{\sigma^2, |x-x_0|^2\}$, with respect to the measure μ_V and using the integration by parts formula of the tangential divergence and Remark 1.2.12, one immediately obtains (1.15).

From now on assume that $\mathrm{spt}\sigma_V \subset S$, where S is compact, and that

$$\limsup_{R\to\infty} \frac{\mu_V(B_R(0))}{R^2} \leq K < +\infty.$$

Since S is compact, it follows that $|\sigma_V|(S) < +\infty$. We have

$$\begin{aligned} \left| \int_{B_\rho(x_0)} \frac{\langle H, x-x_0\rangle}{\rho^2} \, d\mu_V(x) \right| &\leq \left(\frac{\mu_V(B_\rho(x_0))}{\rho^2} \right)^{\frac{1}{2}} \left(\int_{B_\rho(x_0)} |H|^2 \, d\mu_V \right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon}{2} \frac{\mu_V(B_\rho(x_0))}{\rho^2} + \frac{2}{\varepsilon} \int_{B_\rho(x_0)} |H|^2 \, d\mu_V. \end{aligned} \tag{1.16}$$

Also, if $d(x_0, S) \geq \delta$ we have that

$$\left| \int_{B_\rho(x_0)} \left(\frac{1}{|x-x_0|^2} - \frac{1}{\rho^2} \right) (x-x_0) \, d\sigma_V(x) \right| \leq \left(\frac{1}{\delta} + \frac{1}{\rho} \right) |\sigma_V|(S \cap B_\rho(x_0)).$$

In particular, as $|\sigma_V|(S \cap B_\rho(x_0)) = 0$ for $\rho < \delta$, the monotone function $A_{V,x_0}(\rho)$ is bounded below and there exists finite the limit $\lim_{\rho\searrow 0} A_{V,x_0}(\rho)$. Keeping $x_0 \notin S$, Equation (1.15) implies that

$$\begin{aligned} \frac{\mu_V(B_\sigma(x_0))}{\sigma^2} &\leq \frac{\mu_V(B_\rho(x_0))}{\rho^2} + \frac{1}{4} \int_{B_\rho(x_0)} |H|^2 \, d\mu_V(x) + R_{x_0,\rho} - R_{x_0,\sigma} \\ &\leq \frac{\mu_V(B_\rho(x_0))}{\rho^2} + \frac{1}{4}\mathcal{W}(V) + \left(\frac{\mu_V(B_\rho(x_0))}{\rho^2} \right)^{\frac{1}{2}} \mathcal{W}(V)^{\frac{1}{2}} - T_{x_0,\sigma} \\ &\quad + \left(\frac{1}{\delta} + \frac{1}{\rho} \right) |\sigma_V|(S \cap B_\rho(x_0)) + \frac{\varepsilon}{2} \frac{\mu_V(B_\sigma(x_0))}{\sigma^2} + \frac{2}{\varepsilon}\mathcal{W}(V) \end{aligned} \tag{1.17}$$

Letting $\rho \to \infty$ and $\sigma < \delta$ in (1.17) we get that $T_{x_0,\sigma} = 0$ and

$$\frac{\mu_V(B_\sigma(x_0))}{\sigma^2} \le C(\delta, K, |\sigma_V|(S), \mathcal{W}(V)) < +\infty \qquad \forall\, 0 < \sigma < \delta. \tag{1.18}$$

Letting $\rho \to 0$ in (1.16) and using the first inequality in (1.18) we get that

$$\lim_{\rho \to 0} \left| \int_{B_\rho(x_0)} \frac{\langle H, x - x_0 \rangle}{\rho^2} \, d\mu_V(p) \right| = 0. \tag{1.19}$$

Therefore we see that if $x_0 \in \mathbb{R}^3 \setminus S$, then

$$\exists\ \widetilde{\theta}_V(x_0) := \lim_{\sigma \searrow 0} \frac{\mu_V(B_\sigma(x_0))}{\pi \sigma^2} \le C(\delta, |\sigma_V|(S), K, \mathcal{W}(V)).$$

Now we prove that $\widetilde{\theta}_V$ is upper semicontinuous on $\mathbb{R}^n \setminus S$. Let $x_0 \in \mathbb{R}^3 \setminus S$ and consider a sequence $x_k \to x_0$. Let $\rho \in (0, d(x_0, S)/2)$ and denote $\rho_0 = d(x_0, S)/2$, then by (1.15) we have that

$$\begin{aligned}
\frac{\mu_V(\overline{B_\rho(x_0)})}{\rho^2} &\ge \limsup_k \frac{\mu_V(B_\rho(x_k))}{\rho^2} \ge \limsup_k \pi \widetilde{\theta}_V(x_k) - R_{x_k,\rho} - \frac{1}{4} \int_{B_\rho(x_k)} |H|^2 \, d\mu_V \\
&\ge \limsup_k \pi \widetilde{\theta}_V(x_k) - \int_{B_{2\rho}(x_0)} \frac{|H|}{\rho} \, d\mu_V - \frac{1}{4} \int_{B_{2\rho}(x_0)} |H|^2 \, d\mu_V \\
&\ge \limsup_k \pi \widetilde{\theta}_V(x_k) - \left(\frac{\mu_V(B_{2\rho}(x_0))}{\rho^2} \right)^{\frac{1}{2}} \left(\int_{B_{2\rho}(x_0)} |H|^2 \, d\mu_V \right)^{\frac{1}{2}} - \frac{1}{4} \int_{B_{2\rho}(x_0)} |H|^2 \, d\mu_V \\
&\ge \limsup_k \pi \widetilde{\theta}_V(x_k) - \left(C(2\rho_0, |\sigma_V|(S), K, \mathcal{W}(V)) + \frac{1}{4} \right) \left(\int_{B_{2\rho}(x_0)} |H|^2 \, d\mu_V \right)^{\frac{1}{2}},
\end{aligned}$$

and thus letting $\rho \searrow 0$ suitably we get

$$\widetilde{\theta}_V(x_0) \ge \limsup_k \widetilde{\theta}_V(x_k),$$

i.e., the multiplicity function $\widetilde{\theta}_V$ is upper semicontinuous on $\mathbb{R}^3 \setminus S$. Since θ_V is integer valued, the set $\{x \in \mathbb{R}^3 \setminus S \,|\, \widetilde{\theta}_V(x) \ge \frac{1}{2}\}$ is closed in $\mathbb{R}^3 \setminus S$, and the thesis follows using that $\mathcal{H}^2(S) = 0$. $\square$

From Theorem 1.3.7, it follows that whenever an integer rectifiable varifold $V = \mathbf{v}(M, \theta_V)$ has finite Willmore energy and generalized boundary such that $\mathrm{spt}\sigma_V \subset S$, where S is compact and $\mathcal{H}^2$-negligible, and also

$$\limsup_{R \to \infty} \frac{\mu_V(B_R(0))}{R^2} \le K < +\infty,$$

then we can always assume that

$$M = \left\{ x \in \mathbb{R}^n \setminus S \mid \theta_V(x) \ge \frac{1}{2} \right\} \cup S,$$

is closed in $\mathbb{R}^n$, and that θ_V coincides at any point $x \in \mathbb{R}^n \setminus S$ with the density $\lim_{\rho \searrow 0} \frac{\mu_V(B_\rho(x))}{\pi \rho^2}$, which exists and is bounded as stated in Theorem 1.3.7. When the above conditions on a varifold $V = \mathbf{v}(M, \theta_V)$ are satisfied, we shall always assume that M and θ_V are as above.

Moreover, the following classical optimal bound holds.

Corollary 1.3.8 (Li–Yau inequality for varifolds)**.** *Let $V = \mathbf{v}(M, \theta_V) \neq 0$ be a 2-dimensional integer rectifiable varifold on $\mathbb{R}^n$, with $n \geq 3$. Suppose that V has mean curvature H and generalized boundary $\sigma_V = 0$. Assume also that M is essentially bounded, that is, there is R such that $\mu_V(\mathbb{R}^n \setminus B_R(0)) = 0$. Then*

$$\mathcal{W}(V) \geq 4\pi \limsup_{\rho \searrow 0} \frac{\mu_V(B_\rho(x))}{\pi \rho^2},$$

for any $x \in \mathbb{R}^n$. In particular $\mathcal{W}(V) \geq 4\pi$.

Proof. The proof immediately follows passing to the limits $\rho \to \infty$ and $\sigma \searrow 0$ in Equation (1.15), using (1.19), (1.16), and the fact that $\mu_V(\mathbb{R}^n) < +\infty$. □

The above estimate first appeared in the literature in [LY82] in the context of smooth surfaces. Then the inequality has been extended to the setting of varifolds by means of monotonicity formulas in [Sim93]. It is common to refer to such a result as the Li–Yau inequality. In Chapter 4 we prove some generalizations of this type of inequality in the setting of surfaces with boundary. We also show that the lower bound of 4π on the Willmore energy of varifolds without boundary and compact support in $\mathbb{R}^3$ is only achieved by spheres (Proposition 4.3.16), just as happens in the context of closed smooth surfaces (Theorem 4.1.1).

Also, we can prove the following consequence, which clearly resembles what one expects in the smooth realm.

Corollary 1.3.9 (Unbounded varifolds)**.** *Let $V = \mathbf{v}(M, \theta_V)$ be a 2-dimensional integer rectifiable curvature varifold with boundary with $\mathcal{W}(V) < +\infty$. Denote by σ_V the generalized boundary and by S a compact set containing the support $\mathrm{spt}\sigma_V$ such that $\mathcal{H}^2(S) = 0$. Then*

$$M \text{ is ess. unbounded} \qquad \Leftrightarrow \qquad \limsup_{\rho \to \infty} \frac{\mu_V(B_\rho(0))}{\pi \rho^2} \geq 1,$$

where M essentially unbounded means that for every $R > 0$ there is $B_r(x) \subset \mathbb{R}^3 \setminus B_R(0)$ such that $\mu_V(B_r(x)) > 0$.
Moreover, in any of the above cases the limit $\lim_{\rho \to \infty} \frac{\mu_V(B_\rho(0))}{\rho^2}$ exists.

Proof. Suppose that M is essentially unbounded. We can assume that $\limsup_{\rho \to \infty} \frac{\mu_V(B_\rho(0))}{\rho^2} \leq K < +\infty$. Then

$$\begin{aligned}\left| \int_{B_\rho(0)} \frac{1}{\rho^2} \langle H, x \rangle \, d\mu_V \right| &\leq \frac{1}{\rho^2} \left(\int_{B_\sigma(0)} |H||x| \, d\mu_V(x) + \int_{B_\rho(0) \setminus B\sigma(0)} |H||x| \, d\mu_V(x) \right) \\ &\leq \frac{\sigma}{\rho^2} \sqrt{\int_{B_\sigma(0)} |H|^2 \, d\mu_V} \sqrt{\mu_V(B_\sigma(0))} + \sqrt{\frac{\mu_V(B_\rho(0))}{\rho^2}} \sqrt{\int_{B_\rho(0) \setminus B_\sigma(0)} |H|^2 \, d\mu_V}\end{aligned}$$

for any $0 < \sigma < \rho < +\infty$. Passing to the $\limsup_{\rho \to \infty}$ and then to $\sigma \to \infty$, we conclude that

$$\lim_{\rho \to \infty} \left| \int_{B_\rho(0)} \frac{1}{\rho^2} \langle H, x \rangle \, d\mu_V(x) \right| = 0. \tag{1.20}$$

Hence, assuming without loss of generality that $0 \notin S$, the monotone quantity $A_{V,0}(\rho)$ gives that

$$\exists \lim_{\rho \to \infty} A_{V,0}(\rho) = \mathcal{W}(V) + \frac{1}{2} \int \frac{x}{|x|^2} \, d\sigma_V(x) + \limsup_{\rho \to \infty} \frac{\mu_V(B_\rho(0))}{\rho^2},$$

and thus $\exists \lim_{\rho\to\infty} \frac{\mu_V(B_\rho(0))}{\rho^2} \leq K < +\infty$. Also, by Theorem 1.3.7, we can assume that M is closed.

We now prove that M has at least one unbounded connected component. Indeed any compact connected component N of M defines a varifold $\mathbf{v}(N, \theta_V|_N)$ with generalized mean curvature. If $\mathrm{spt}\sigma_V \cap N = \emptyset$ then $\mathcal{W}(N) \geq 4\pi$ by Corollary 1.3.8, and thus there are finitely many compact connected components without boundary. If instead $\mathrm{spt}\sigma_V \cap N \neq \emptyset$, R_0 is such that $S \subset B_{R_0}(0)$ by compactness, and there is $x_0 \in N \setminus B_r(0)$ for $r > R_0$, since N is compact, then the monotonicity formula applied on $\mathbf{v}(N, \theta_V|_N)$ at point x_0 gives

$$\pi \leq \lim_{\sigma\to 0} A_{\mathbf{v}(N,\theta_V|_N),x_0}(\sigma) \leq \lim_{\rho\to\infty} A_{\mathbf{v}(N,\theta_V|_N),x_0}(\rho) \leq \frac{1}{4}\mathcal{W}(\mathbf{v}(N,\theta_V|_N)) + \frac{1}{2}\frac{|\sigma_V|(S)}{r - R_0}. \tag{1.21}$$

Since M is essentially unbounded, if by contradiction any connected component of M is compact we would find infinitely many compact connected components N, points $p_0 \in N$, and r arbitrarily big in (1.21) so that the Willmore energy of any such N is greater than 2π, implying that $\mathcal{W}(V) = +\infty$.

As M has a connected unbounded component, for any ρ sufficiently large there is $x_\rho \in M \cap \partial B_{2\rho}(0)$. Applying the monotonicity formula on V at x_ρ for ρ sufficiently big so that $S \subset B_\rho(0)$ we get that

$$\begin{aligned}\pi \leq \lim_{\sigma\to 0} A_{V,x_\rho}(\sigma) &\leq \frac{\mu_V(B_\rho(x_\rho))}{\rho^2} + \frac{1}{4}\int_{B_\rho(x_\rho)} |H|^2\, d\mu_V + \frac{1}{\rho}\int_{B_\rho(x_\rho)} |H|\, d\mu_V \\ &\leq 9\frac{\mu_V(B_{3\rho}(0))}{(3\rho)^2} + \frac{1}{4}\int_{\mathbb{R}^3\setminus B_\rho(0)} |H|^2\, d\mu_V + \varepsilon\frac{\mu_V(B_\rho(x_\rho))}{\rho^2} + C_\varepsilon \int_{B_\rho(x_\rho)} |H|^2\, d\mu_V,\end{aligned}$$

that implies that

$$\lim_{\rho\to\infty} \frac{\mu_V(B_\rho(0))}{\rho^2} \geq \frac{\pi}{9+\varepsilon},$$

for any $\varepsilon > 0$.

Consider now any sequence $R_n \to \infty$ and the sequence of blow-in varifolds given by

$$V_n = \mathbf{v}\left(\frac{M}{R_n}, \theta_n\right),$$

where $\theta_n(x) = \theta_V(R_n x)$. Since $0 \notin S$, by (1.15) we have

$$\mu_{V_n}(B_R(0)) = \frac{1}{R_n^2}\mu_V(B_{R_nR}(0)) = \frac{1}{(RR_n)^2}\mu_V(B_{RR_n}(0))R^2 \leq K'R^2$$

is bounded for any $R > 0$. Moreover $\mathcal{W}(V_n) = \mathcal{W}(V)$ and $|\sigma_{V_n}|(\mathbb{R}^3) \to 0$, thus by Theorem 1.2.14 we get that V_n converges to an integer rectifiable varifold W (up to subsequence). We have that $W \neq 0$, indeed $0 \in \mathrm{spt}W$ by the fact that

$$\mu_W(\overline{B_1(0)}) \geq \liminf_n \mu_{V_n}(B_1(0)) = \liminf_n \frac{\mu_V(B_{R_n}(0))}{R_n^2} \geq \frac{\pi}{9}.$$

Since $|\delta V_n|(B_R)$ is bounded for any R, the limit varifold W has mean curvature and generalized boundary in $\mathbb{R}^n$. By (1.4) we have that $\mathrm{spt}\sigma_W \subset \{0\}$, hence we can apply Corollary 1.2.16, and we get that W is stationary, indeed for any $r > 0$ we have that

$$\int_{\mathbb{R}^3\setminus\overline{B_r(0)}} |H_W|^2\, d\mu_W \leq \liminf_n \int_{\mathbb{R}^3\setminus\overline{B_r(0)}} |H_{V_n}|^2\, d\mu_{V_n} = \liminf_n \int_{\mathbb{R}^3\setminus\overline{B_{R_nr}(0)}} |H_V|^2\, d\mu_V = 0.$$

Also we can prove that $\sigma_W = 0$. In fact for any $X \in C^0_c(\mathbb{R}^3)$ the convergence of the first variation (see (1.4)) reads

$$\lim_n -2\int \langle H_{V_n}, X\rangle \, d\mu_{V_n} + \int X \, d\sigma_{V_n} = \lim_n -2\int \langle H_{V_n}, X\rangle \, d\mu_{V_n} = \int X \, d\sigma_W,$$

and $\mathrm{spt}\sigma_W \subset \{0\}$. Taking $X = \Lambda_m Y$ for $Y \in C^0_c(\mathbb{R}^3)$ and

$$\Lambda_m(x) = \begin{cases} 1 - m|x| & |x| \le \frac{1}{m}, \\ 0 & |x| > \frac{1}{m}, \end{cases}$$

we see that

$$\lim_n \left| \int \langle H_{V_n}, X\rangle \, d\mu_{V_n} \right| = \lim_n \left| \int_{B_{\frac{1}{m}}(0)} \langle H_{V_n}, \Lambda_m Y\rangle \, d\mu_{V_n} \right| \le \|Y\|_\infty \mathcal{W}(V)^{\frac{1}{2}} \left(K' \frac{1}{m^2} \right)^{\frac{1}{2}},$$

for any $m \ge 1$, and thus $\int Y \, d\sigma_W = 0$ for any $Y \in C^0_c(\mathbb{R}^3)$. Finally, the monotonicity formula applied on W gives

$$\lim_n \frac{\mu_V(R_n(0))}{R_n^2} \ge \liminf_n \mu_{V_n}(B_1(0)) \ge \mu_W(B_1(0)) = A_{W,0}(1) \ge \lim_{\sigma \to 0} A_{W,0}(\sigma) \ge \pi,$$

and the proof is complete. □

We can conclude by stating some further consequences on surfaces with boundary that we will use in the following.

Lemma 1.3.10. *Let $\Sigma \subset \mathbb{R}^3$ be a compact connected immersed surface with boundary. Then*

$$4 \lim_{\sigma \searrow 0} \frac{|\Sigma \cap B_\sigma(x_0)|}{\sigma^2} + 4 \int_\Sigma \left| \frac{H}{2} + \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 = \mathcal{W}(\Sigma) + 2 \int_{\partial\Sigma} \left\langle \frac{x - x_0}{|x - x_0|^2}, co \right\rangle, \tag{1.22}$$

for any $x_0 \in \mathbb{R}^3$. In particular

$$\forall x_0 \in \mathbb{R}^3 \setminus \partial\Sigma : \quad 4 \lim_{\sigma \searrow 0} \frac{|\Sigma \cap B_\sigma(x_0)|}{\sigma^2} + 4 \int_\Sigma \left| \frac{H}{2} + \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 \le \mathcal{W}(\Sigma) + 2 \frac{\mathcal{H}^1(\partial\Sigma)}{d(x_0, \partial\Sigma)}. \tag{1.23}$$

Moreover, writing $d_{\mathcal{H}}(\Sigma, \partial\Sigma) = d(\overline{x_0}, \partial\Sigma)$ for some $\overline{x_0} \in \Sigma \setminus \partial\Sigma$, it holds that

$$4 \lim_{\sigma \searrow 0} \frac{|\Sigma \cap B_\sigma(\overline{x_0})|}{\sigma^2} + 4 \int_\Sigma \left| \frac{H}{2} + \frac{(x - \overline{x_0})^\perp}{|x - \overline{x_0}|^2} \right|^2 \le \mathcal{W}(\Sigma) + 2 \frac{\mathcal{H}^1(\partial\Sigma)}{d_{\mathcal{H}}(\Sigma, \partial\Sigma)}. \tag{1.24}$$

Proof. It suffices to prove (1.22). Since Σ is smooth we have that

$$\left| \int_{B_\rho(x_0)} \left(\frac{1}{|x - x_0|^2} - \frac{1}{\rho^2} \right) \langle x - x_0, co\rangle \, d\mathcal{H}^1(x) \right| \le \int_{B_\rho(x_0)} \left| \frac{1}{|x - x_0|^2} - \frac{1}{\rho^2} \right| O_{x_0}(|x - x_0|^2) \, d\mathcal{H}^1(x)$$

and the right hand side goes to zero as $\rho \to 0$. The surface Σ is smooth and immersed, i.e. there is a smooth immersion $\varphi : M^2 \hookrightarrow \mathbb{R}^3$ of a 2-dimensional compact connected manifold with boundary M with $\Sigma = \varphi(M)$. Denoting by Σ also the varifold induced by φ, by Theorem 1.3.7 we have that

$$A_{\Sigma,x_0}(\sigma) \xrightarrow[\sigma \to 0]{} \theta_\Sigma(x_0) := \lim_{\sigma \searrow 0} \frac{|\Sigma \cap B_\sigma(x_0)|}{\sigma^2},$$

while by compactness we get

$$A_{\Sigma,x_0}(\rho) \xrightarrow[\rho \to \infty]{} \frac{1}{4}\mathcal{W}(\Sigma) + \frac{1}{2} \int_{\partial\Sigma} \langle \frac{x - x_0}{|x - x_0|^2}, co \rangle.$$

Hence (1.22) follows from (1.15). □

We mention that (1.23) already appeared in [Riv13].

1.4 Sets of finite perimeter

In this section we introduce basic definitions and results about sets of finite perimeter. For the complete theory we refer to [AFP00].

Definition 1.4.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $u \in L^1(\Omega)$ is a *function of bounded variation* if there exists a vector valued Radon measure $Du = (D_1 u, ..., D_n u)$ on Ω such that

$$\int_\Omega u \,\mathrm{div} X \, dx = -\int_\Omega X \, dDu := -\sum_i \int_\Omega X^i \, dD_i u,$$

for every vector field $X = (X^1, ..., X^n) \in C_c^\infty(\Omega, \mathbb{R}^n)$. In such a case we write $u \in BV(\Omega)$. We write that $u \in BV_{loc}(\Omega)$ if $u \in BV(A)$ for any open set $A \subset\subset \Omega$.

Definition 1.4.2. Let $\Omega \subset \mathbb{R}^n$ be an open set. A Lebesgue-measurable set $E \subset \Omega$ is said to be *of finite perimeter in* Ω if the characteristic function χ_E is of bounded variation in Ω. Equivalently, there exists a vector valued Radon measure $D\chi_E$ such that

$$\int_E \mathrm{div} X \, dx = -\int_\Omega \langle X, \nu \rangle \, d|D\chi_E|,$$

where $D\chi_E = \nu |D\chi_E|$ is the polar decomposition of $D\chi_E$. We define the *perimeter* $P(E, \Omega)$ of E in Ω as the supremum

$$P(E, \Omega) := \sup \left\{ \int_E \mathrm{div} X \, dx \mid X \in C_c^\infty(\Omega, \mathbb{R}^n), \ |X| \leq 1 \right\},$$

which is finite if and only if E is of finite perimeter in Ω, and in such a case $P(E, \Omega) = |D\chi_E|(\Omega)$. If $\chi_E \in BV_{loc}(\Omega)$, we say that E is a set of *locally finite perimeter in* Ω.

Roughly speaking, a finite perimeter set verifies the Divergence Theorem in a weak sense. In the setting of functions of bounded variation we can introduce the following notion of convergence.

Definition 1.4.3. Let $\Omega \subset \mathbb{R}^n$ be an open set. A sequence $u_n \in BV(\Omega)$ *(locally) weak* converges in* $BV(\Omega)$ to a function $u \in BV(\Omega)$ if $u_n \to u$ in $L^1(\Omega)$ and $Du_n \overset{\star}{\rightharpoonup} Du$ (locally) weakly* as measures on Ω. The notion of local weak* convergence is also extended to sequences $u_n \in BV_{loc}(\Omega)$.

We say that a sequence of finite perimeter sets E_n in Ω (locally) weakly* converges to a set of finite perimeter E in Ω, if the characteristic functions (locally) weakly* converge in $BV(\Omega)$. The notion of local weak* convergence is also extended to sequences of sets of locally finite perimeter in Ω.

The next proposition collects a few facts about the convergence of sets of finite perimeter.

Proposition 1.4.4 (Sets of finite perimeter)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $E, E_n \subset \Omega$ be Lebesgue measurable sets, for $n \in \mathbb{N}$. Then the following results hold.*

1. *Suppose E_n is of finite perimeter for any n. Then the sequence E_n weakly* converges to a finite perimeter set F if and only if $\chi_{E_n} \to \chi_F$ in $L^1(\Omega)$ and $\sup_n |D\chi_{E_n}|(\Omega) < +\infty$.*

2. *Suppose E_n is of locally finite perimeter for any n, and that $\chi_{E_n} \to \chi_E$ in $L^1_{loc}(\Omega)$. Then $\liminf_n P(E_n, A) \geq P(E, A)$ for any $A \subset\subset \Omega$.*

3. *If $\sup_n |E_n| + P(E_n, A) \leq C(A) < +\infty$ for any $A \subset\subset \Omega$, then there exists E of locally finite perimeter in Ω such that E_n locally weak* converges to E, up to subsequences.*

4. *If E is a set of finite perimeter in $\mathbb{R}^n$ and $n \geq 2$, then there exists a sequence F_k of smooth open sets such that $\chi_{F_k} \to \chi_E$ in L^1 and $\lim_k P(F_k, \mathbb{R}^n) = P(E, \mathbb{R}^n)$.*

Proof. The proof follows by Proposition 3.6, Proposition 3.13, Theorem 3.23 and Theorem 3.42 in [AFP00]. □

Finally, we recall the basic definitions and structure properties of sets of finite perimeter.

Definition 1.4.5. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \Omega$ be a set of finite perimeter in Ω. We call *generalized inner unit normal* of E at $x \in \Omega$ the vector

$$\nu_E(x) := \lim_{r \searrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$$

if the above limit exists and has unit norm. We define the *reduced boundary* $\mathcal{F}E$ of E the set

$$\mathcal{F}E := \{x \in \mathrm{spt} D\chi_E \cap \Omega \mid \exists\, \nu_E(x),\, |\nu_E(x)| = 1\}.$$

By Besicovitch Derivation Theorem [AFP00, Theorem 2.22], it follows that $D\chi_E$ is concentrated on $\mathcal{F}E$ and $\nu_E|D\chi_E|$ coincides with the polar decomposition of $D\chi_E$.

The notion of reduced boundary allows to state the following fundamental structure theorem.

Theorem 1.4.6 (De Giorgi, [AFP00, Theorem 3.59])**.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then $\mathcal{F}E$ is $(n-1)$-dimensional rectifiable and $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$.*

If $x \in \mathcal{F}E$, then $\frac{1}{\varepsilon}(E - x)$ converges in $L^1_{loc}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ to the halfspace orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$.

If $x \in \mathcal{F}E$, then the generalized tangent space $T_x\mathcal{F}E$ of $\mathcal{F}E$ at x is the subspace of $\mathbb{R}^n$ orthogonal to $\nu_E(x)$.

Given a set E, it is sometimes useful to consider the following classes distinguished by the density of E at those points.

Definition 1.4.7. Let $E \subset \mathbb{R}^n$ be a Lebesgue-measurable set. For $t \in [0, 1]$, we denote by

$$E^t := \left\{ x \in \mathbb{R}^n \mid \lim_{r \searrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\},$$

the set of points with density equal to t. The *essential boundary* of E is the set

$$\partial^* E := \mathbb{R}^n \setminus (E^0 \cup E^1),$$

that is, the set of points with density different from 0 and 1.

We can now conclude with the following structure theorem.

Theorem 1.4.8 (Federer, [AFP00, Theorem 3.61])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \Omega$ be a set of finite perimeter in Ω. Then*

$$\mathcal{F}E \cap \Omega \subset E^{\frac{1}{2}} \subset \partial^* E, \qquad \mathcal{H}^{n-1}\left(\Omega \setminus (E^0 \cup \mathcal{F}E \cup E^1)\right) = 0.$$

In particular, the set E has density equal to 0, $\frac{1}{2}$, or 1 at $\mathcal{H}^{n-1}$-ae point of Ω, and $\mathcal{H}^{n-1}$-ae point of $\partial^* E \cap \Omega$ belongs to $\mathcal{F}E$.

1.5 Integer rectifiable currents

In this section we recall some very basic definitions in the theory of currents, for which we refer to [Sim83b, Chapter 6]. We are only interested in the class of integer rectifiable currents, that we will define later.

First, we need to fix some notation in the context of multilinear algebra. Let $U \subset \mathbb{R}^n$ be an open (non-empty) set, and let $k \in \mathbb{N}$. The symbol $\Lambda^k(U)$ denotes the space of k-dimensional differential forms on U, which we canonically write as

$$\omega = \sum_{\alpha \in I_{k,n}} \omega_\alpha dx^\alpha,$$

for $\omega \in \Lambda^k(U)$, where

$$I_{k,n} := \{(i_1, ..., i_k) \mid 1 \leq i_1 < i_2 < ... < i_k \leq n\},$$

is the set of increasing multi-indices and $dx^\alpha := dx^{i_1} \wedge ... \wedge dx^{i_k}$.

Similarly, the symbol $\Lambda_k(U)$ denotes the space of k-vectors on U, whose elements are canonically written as

$$v = \sum_{\alpha \in I_{k,n}} v_\alpha e_\alpha,$$

where $e_\alpha := e_{i_1} \wedge ... \wedge e_{i_k}$. The duality between $\Lambda^k(U)$ and $\Lambda_k(U)$ is denoted by

$$\langle \omega, v \rangle := \sum_{\alpha \in I_{k,n}} \omega_\alpha v_\alpha.$$

Both $\Lambda^k(U)$ and $\Lambda_k(U)$ are endowed with the Euclidean scalar product defined by

$$\left\langle \sum_{\alpha \in I_{k,n}} \omega_\alpha dx^\alpha, \sum_{\alpha \in I_{k,n}} \eta_\alpha dx^\alpha \right\rangle := \sum_{\alpha \in I_{k,n}} \omega_\alpha \eta_\alpha,$$

for $\omega, \eta \in \Lambda^k(U)$, and similarly for k-vectors.

If V is a k-dimensional vector space endowed with a scalar product, the symbol $\Lambda_k(V)$ denotes the space of k-vectors on V, which is 1-dimensional. We recall that, if $\dim V = k$, if $\{E_i\}_{i=1}^k$ is an orthonormal basis of V, the choice of an orientation of V is the choice of an order on such a basis, which we write $[E_i]_{i=1}^k$, that corresponds to the choice of either the k-vector $E_1 \wedge ... \wedge E_k$ or $-E_1 \wedge ... \wedge E_k$. Indeed, if now $[e_i]_{i=1}^k$ is another ordered orthonormal basis, then $e_1 \wedge ... \wedge e_k = \det(A) E_1 \wedge ... \wedge E_k$, where A is the determinant of the matrix of change of basis, and the sign of $\det(A)$ distinguishes whether $[e_i]_{i=1}^k$ is oriented like $[E_i]_{i=1}^k$ or not.

The space of k-vectors on a smooth manifold M is given by the sections of the bundle $\Lambda_k(M) := \cup_{x \in M} \Lambda_k(T_x M)$.

If $M^k \subset \mathbb{R}^n$ is a smooth embedded complete k-dimensional submanifold of $\mathbb{R}^n$, if M is also orientable, the choice of an orientation on M corresponds to the choice of a continuous section $\tau : M \to \Lambda_k(M)$ such that $\tau(x) = \pm \tau_1(x) \wedge ... \wedge \tau_k(x)$ for any $x \in M$, where $\{\tau_i(x)\}_{i=1}^k$ is an orthonormal basis of $T_x M$. As we shall see, the notion of integer rectifiable current generalizes the one of oriented submanifold.

Definition 1.5.1 (Currents)**.** Let $U \subset \mathbb{R}^n$ be an open (non-empty) set, and let $k \in \mathbb{N}$.

1. The space $D^k(U)$ is the vector space of k-dimensional smooth differential forms on U with compact support in U. Such a space is endowed with the following notion of convergence. We say that $\omega_h \in D^k(U)$ converges to $\omega \in D^k(U)$ if there is a compact set $K \subset U$ such that the components $\omega_{\alpha,h}$ of ω_h, for $\alpha \in I_{k,n}$, converge in $C^m(K)$ to the components ω_α of ω for any $m \in \mathbb{N}$.

2. The space of *k-dimensional currents* $D_k(U)$ on U is the space of continuous linear functionals on $D^k(U)$, that is, its dual space. The duality between a current $T \in D_k(U)$ and a form $\omega \in D^k(U)$ is denoted by $\langle T, \omega\rangle$. We say that a sequence of currents $T_h \in D_k(U)$ converges to a current $T \in D_k(U)$ *in the sense of currents* if $\langle T_n, \omega\rangle \to \langle T, \omega\rangle$ for any $\omega \in D^k(U)$. The *mass* of a current $T \in D_k(U)$ is given by

$$\mathbf{M}(T) := \sup_{\substack{\omega\in D^k(U)\\ |\omega|\le 1}} \langle T, \omega\rangle \ \in [0, +\infty].$$

3. If $k \geq 1$ and $T \in D_k(U)$, the *boundary* of T is the current $\partial T \in D_{k-1}(U)$ defined by

$$\langle \partial T, \omega\rangle := \langle T, d\omega\rangle,$$

for any $\omega \in D^{k-1}(U)$.

4. If $T \in D_k(U)$, $V \subset \mathbb{R}^m$ is an open set, and $f : U \to V$ is smooth and proper, we define the *push forward current* $f_\sharp T \in D_k(V)$ by

$$\langle f_\sharp T, \omega\rangle := \langle T, f^\star \omega\rangle,$$

for any $\omega \in D^k(V)$, where $f^\star\omega$ is the pull-back of ω. The operations of boundary and push forward of currents commute, that is, $\partial f_\sharp T = f_\sharp \partial T$; indeed, pull-back and exterior derivative of forms commute.

5. A current $T \in D_k(U)$ is said to be *integer rectifiable* if the following is satisfied. There exist a k-dimensional rectifiable set $M \subset U$, a *multiplicity* function $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner M)$ which is integer valued, and an *orientation* function $\xi : M \to \Lambda_k(\mathbb{R}^n)$. At any point $x \in M$ admitting approximate tangent space T_xM, it holds that $\xi(x) \in \{\tau_1(x) \wedge ... \wedge \tau_k(x), -\tau_1(x) \wedge ... \wedge \tau_k(x)\}$, where $\{\tau_i(x)\}_{i=1}^k$ is an orthonormal basis of T_xM, and ξ is $\mathcal{H}^k$-measurable. The action of T on a form $\omega \in D^k(U)$ is given by

$$\langle T, \omega\rangle := \int_M \langle \omega(x), \xi(x)\rangle\, \theta(x) d\mathcal{H}^k(x).$$

In such a case we identify T with the triple $\boldsymbol{\tau}(M, \theta, \xi)$. Observe that the mass of such an integer rectifiable current is $\mathbf{M}(T) = \int_M |\theta| d\mathcal{H}^k$.

In case $M^k \subset \mathbb{R}^n$ is a smooth oriented embedded k-dimensional submanifold of $\mathbb{R}^n$, we denote by $[|M|]$ the integer rectifiable k-dimensional current $\boldsymbol{\tau}(M, 1, \xi)$, where ξ is the given orientation on M.

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function. If $S \subset \mathbb{R}^n$ is a k-dimensional subspace, we define $d^S f_x : S \to \mathbb{R}^m$ the *tangential differential* of f along S to be the differential of the restricted map $f|_{x+S} : x + S \to \mathbb{R}^m$, if it exists. In such a case, the *tangential Jacobian* of f on S is given by $J^S f(x) := \big(\det((d^S f_x)^\star \circ d^S f_x)\big)^{\frac{1}{2}}$. The function f is said to be *tangentially differentiable* at

x along S if $d^S f_x$ exists. As we wrote in Theorem 1.2.8, we shall denote by $d^M f_x$ the tangential differential of f along the generalized tangent space $T_x M$ of a rectifiable set M.

As in the case of vectors, if $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point x, we define the push forward of a k-vector $v_1 \wedge ... \wedge v_k$ by df_x as the k-vector in $\mathbb{R}^m$ given by

$$df_x(v_1 \wedge ... \wedge v_k) \coloneqq (df_x(v_1)) \wedge ... \wedge (df_x(v_k)).$$

Letting $S \coloneqq \operatorname{span}(v_1, ..., v_k)$, if f is tangentially differentiable at x along S, we define $d^S f_x(v_1 \wedge ... \wedge v_k)$ in the very same way as above.

Using the above definitions, in the case of integer rectifiable currents, we can extend the notion of push forward of currents as follows.

Definition 1.5.2. Let $U \subset \mathbb{R}^n$ be an open set, and let $k \in \mathbb{N}$. Let $T = \boldsymbol{\tau}(M, \theta, \xi) \in D_k(U)$ be an integer rectifiable current. Let $f : U \to V$ be a Lipschitz proper map, where $V \subset \mathbb{R}^m$ is an open set. We define the *push forward current* $f_\sharp T \in D_k(V)$ by

$$\langle f_\sharp T, \omega \rangle \coloneqq \int_{f(M)} \left\langle \omega(y), \sum_{x \in f^{-1}(y) \cap M_+} \theta(x) \frac{d^M f(\xi(x))}{|d^M f(\xi(x))|} \right\rangle d\mathcal{H}^k(y)$$

for any $\omega \in D^k(V)$, where $M_+ \coloneqq \left\{ x \in M \mid J^M f(x) > 0 \right\}$. Observe that $f_\sharp T$ is integer rectifiable. We remark that the definition is well-posed, as a Lipschitz function is tangentially differentiable $\mathcal{H}^k$-almost everywhere on the k-rectifiable set M [AFP00, Theorem 2.90]. Moreover, we observe that the equality

$$\langle f_\sharp T, \omega \rangle = \langle T, f^\star \omega \rangle,$$

now holds as a consequence of the area formula (Theorem 1.2.8) and not as a definition (see [Sim83b, Chapter 6, Lemma 3.9]). Also, in the above formula, as f is not smooth, we understood that $f^\star \omega \in D^k(U)$ is defined at $\mathcal{L}^n$-ae $x \in U$ by $(f^\star \omega)_x(v_1 \wedge ... \wedge v_k) \coloneqq \omega(df_x(v_1 \wedge ... \wedge v_k))$.

We observe that finite perimeter sets can be seen as integer rectifiable currents of dimension equal to the one of the ambient space. More precisely, let $E \subset U$ be a finite perimeter set E in an open set $U \subset \mathbb{R}^n$. We can canonically associate to E the current $[|E|] \in D_n(U)$ given by

$$\langle [|E|], f dx^1 \wedge ... \wedge dx^n \rangle \coloneqq \int_E f \, dx.$$

Therefore $[|E|] = \boldsymbol{\tau}(E, 1, \xi_0)$ is integer rectifiable, where ξ_0 is the standard constant orientation of $\mathbb{R}^n$. Moreover, if $X \in C_c^\infty(U, \mathbb{R}^n)$ is a vector field and we let

$$\omega \coloneqq \sum_{i=1}^n (-1)^{i-1} X^i dx^{\alpha_i} \in D^{n-1}(U),$$

where $\alpha_i \coloneqq (1, ..., i-1, i+1, ..., n)$, we have that

$$d\omega = \operatorname{div} X dx^1 \wedge ... \wedge dx^n.$$

Hence

$$\langle \partial [|E|], \omega \rangle = \langle [|E|], d\omega \rangle = \int_E \operatorname{div} X \, dx = -\int_{\mathcal{F}E} \langle X, \nu_E \rangle \, d\mathcal{H}^{n-1},$$

where we used the definition of boundary current and Theorem 1.4.6. It follows that $\partial [|E|]$ is integer rectifiable and

$$\partial [|E|] = \boldsymbol{\tau} (\mathcal{F}E, 1, \xi_E),$$

with

$$\xi_E(x) \coloneqq \sum_{i=1}^{n} (-1)^i \nu_E^i(x)\, e_{\alpha_i} \qquad\qquad \forall\, x \in \mathcal{F}E. \tag{1.25}$$

We further observe that if $E_n \to E$ as finite perimeter sets, that is, $\chi_{E_n} \to \chi_E$ weakly* in $BV(U)$, then $[|E_n|] \to [|E|]$, and then $\partial[|E_n|] \to \partial[|E|]$, in the sense of currents.

Remark 1.5.3. Observe that if E is a set of finite perimeter in $\mathbb{R}^2$ and $[e_1, e_2]$ is the canonical oriented basis of $\mathbb{R}^2$, then ξ_E is just the clockwise rotation of ν_E of an angle equal to $\frac{\pi}{2}$. Indeed, $(n-1)$-vectors in $\mathbb{R}^n = \mathbb{R}^2$ are just vectors and (1.25) gives

$$\xi_E = -\nu_E^1 e_2 + \nu_E^2 e_1.$$

This fact is quite intuitive recalling that, if E is smooth, then the normal ν_E points inside the set E.

Chapter 2

Smooth convergence of elastic flows of curves into manifolds

Contents

In this chapter we consider regular curves $\gamma : \mathbb{S}^1 \to M$ in complete Riemannian manifolds (M^m, g), where $m \geq 2$ is the dimension of M and will be usually omitted, and we are interested in the gradient flow of the p-elastic energy $\mathcal{E}_p$. As we will explain, a gradient flow is an evolution equation that prescribes the motion of a smooth family of curves $\gamma = \gamma(t, x)$ in such a way that the energy $\mathcal{E}_p$ decreases along the flow. We will investigate the *convergence* of the flow, that is, the possibility that $\gamma(t, \cdot)$ converges in some sense to a critical point of $\mathcal{E}_p$ as t tends to the maximal time of existence.

The evolution equations of these gradient flows turn out to be parabolic in the parametrization γ. Then, by means of parabolic estimates, it is usually possible to prove *sub-convergence* of the flow, that is, convergence to critical points up to reparametrizations and, more importantly, up to isometry of the ambient. Assuming that the flow sub-converges, we are interested in proving the smooth convergence of the flow, that is, the existence of the full limit of the evolving flow.

We first give an overview of the general strategy one can apply for proving such a statement. The crucial step is the application of a Łojasiewicz–Simon gradient inequality (Section 2.2.2). Then we apply such strategy to the flow of $\mathcal{E}_p$ of curves into manifolds, proving the desired improvement of sub-convergence to full smooth convergence of the flow to critical points. These results are contained in [Poz20b]. We also refer the reader to the related results contained in [MP20], in which we study the gradient flow of $\mathcal{E}_2$ in $\mathbb{R}^n$, which simplifies the proof. We also refer to [MPP20] for a survey on the elastic flow of curves in the plane.

Let us recall here some notation. For fixed $p \in [2, +\infty)$, if a regular curve γ is of class $W^{2,p}$, its p-elastic energy is defined by

$$\mathcal{E}_p(\gamma) := \int_{\mathbb{S}^1} 1 + \frac{1}{p}|k|^p \, ds,$$

where k is the curvature of γ in M and $ds = |\gamma'|dx$ is the length measure. We denote by D the Levi-Civita connection on M. In case of risk of confusion we will specify that a geometric quantity refers to a given curve γ by adding a subscript, like k_γ or ds_γ.

2.1 Elastic energies and geometric flows

In the last years a considerable interest has been devoted towards the elastic energy of curves. Here we are interested in studying some variational aspects of $\mathcal{E}_p$ and in particular we will investigate some properties of a gradient flow of this energy. In order to present this concept, let us assume for the moment that $M = \mathbb{R}^m$ with the Euclidean metric and everything is smooth. We define the first variation functional $\delta\mathcal{E}_p$ at a given curve γ to be the operator

$$\delta\mathcal{E}_p[\varphi] := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}_p(\gamma + \varepsilon\varphi),$$

where φ is a vector field along γ. Explicit calculations may lead to expressions like

$$\delta\mathcal{E}_p[\varphi] = \langle\langle V(\gamma), \varphi\rangle\rangle,$$

where $\langle\langle \cdot, \cdot\rangle\rangle$ is some duality defining the action of the vector $V(\gamma)$ on φ. Then the $\langle\langle \cdot, \cdot\rangle\rangle$*-gradient flow* of $\mathcal{E}_p$ is a function $\gamma : [0, T) \times \mathbb{S}^1 \to \mathbb{R}^m$ solving the evolution equation

$$\partial_t\gamma(t, x) = -V(\gamma(t, x)).$$

Moreover, an initial datum $\gamma(0, \cdot) = \gamma_0(\cdot)$ is given, and we understand that the equation is satisfied in the classical sense. It is clear that different representations of the first variation $\delta\mathcal{E}_p$ define different driving velocities $\partial_t\gamma$, and thus different gradient flows of $\mathcal{E}_p$.

Observe that the formal definition of gradient flow we just stated works for any "geometric" energy, not necessarily defined on closed curves. In fact, this is the usual way for defining gradient flows of geometric functionals.

We are interested here in the gradient flow defined by the $(L^p, L^{p'})$-duality $\langle\langle \cdot, \cdot\rangle\rangle = \langle \cdot, \cdot\rangle_{L^{p'}, L^p}$. We shall see in Section 2.3 how to explicitly calculate and define such gradient flows when (M^m, g) is an arbitrary Riemannian manifold. It turns out that the flow of $\mathcal{E}_p$ with respect to the $(L^p, L^{p'})$-duality is defined by the evolution equation

$$\begin{cases} \partial_t\gamma = -\left(\nabla^2|k|^{p-2}k + \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k, \tau)\tau\right) & \text{on } [0, T) \times \mathbb{S}^1, \\ \gamma(0, \cdot) = \gamma_0(\cdot) & \text{on } \mathbb{S}^1, \end{cases} \tag{2.1}$$

where $\gamma_0 : \mathbb{S}^1 \to M$ is a given smooth curve, k is the curvature vector of $\gamma : [0,T) \times \mathbb{S}^1 \to M$, R is the Riemann tensor on (M, g), and ∇ is the normal connection along γ, that is

$$\nabla \phi \coloneqq D_\tau \phi - g(D_\tau \phi, \tau)\tau,$$

for any vector field $\phi : [0,T) \times \mathbb{S}^1 \to TM$ such that $\phi(t,x) \in T_{\gamma(t,x)}M$. Observe that in case $p = 2$ and $M = \mathbb{R}^m$ is the Euclidean space, then the flow reduces to the classical evolution

$$\begin{cases} \partial_t \gamma = -\left(\nabla^2 k + \frac{1}{2}|k|^2 k - k\right) & \text{on } [0,T) \times \mathbb{S}^1, \\ \gamma(0,\cdot) = \gamma_0(\cdot) & \text{on } \mathbb{S}^1, \end{cases}$$

and ∇ is just the composition of the normal projection along γ with the arclength derivative:

$$\nabla \phi = \partial_s \phi - \langle \partial_s \phi, \tau \rangle \tau,$$

for ϕ as above. Without loss of generality, by Nash Theorem [Nas56], we will always assume that (M, g) is smoothly isometrically embedded in the Euclidean space $\mathbb{R}^n$, for some n sufficiently large. In this way it is meaningful to say that a sequence of curves $\{\gamma_l\}_{l \in \mathbb{N}}$ converges in C^k to a curve γ.

For a given gradient flow, a number of questions can be investigated, starting from the existence and uniqueness of a solution once an initial datum is given. In the case of geometric evolution equations, as the energy functional and the velocity of the flow are independent of the parametrization of the curves, uniqueness is always understood up to reparametrization (see Remark 2.3.30 for additional comments). Short time existence and uniqueness results have been studied in the literature, mainly in the case $p = 2$, starting from [Pol96] (see also [HP99]). However, these evolution equations can be seen as parabolic evolution equations in the unknown given by the parametrization of the curve, and thus we will refer to general results like the one in [MM12] for short time existence and uniqueness results in the context of smooth curves.

Our study concerns the long time behavior of the solution of the gradient flow. One can hope that the solution $\gamma(t,\cdot)$ admits a limit, for example in some C^k-topology, as $t \to T^-$, where $T \in (0, +\infty]$ is the maximal time of existence of the solution. In such a case the limit should be a curve γ_∞ which is a critical point for the energy $\mathcal{E}_p$, as the flow "stops" at time T. The description of the long time behavior is quite often not an easy task, especially in case of evolution equations where high order space-derivatives appear. Nevertheless, it is often possible to prove very strong estimates on the solution that are uniform in time, and this is done by means of parabolic techniques. Let us say that $M = \mathbb{R}^m$ and $p = 2$, then it is known that these bounds lead to the conclusion that the flow sub-converges, that is, $T = +\infty$ and there are a sequence of times $t_n \nearrow +\infty$ and a sequence of points $p_n \in \mathbb{R}^m$ such that the sequence $\gamma(t_n, \cdot) - p_n$ converges to a critical point curve γ_∞ in C^k for any k, up to reparametrization. This has been studied in [Pol96] and then a complete proof is given in [DKS02]; in [DS17] and [Dal+18] the same conclusion is proved for the flow taking place in the hyperbolic plane $\mathbb{H}^2$ or in the unit 2-sphere $\mathbb{S}^2$ respectively.

However, the sole sub-convergence cannot tell anything about the full limit as $t \to +\infty$, and actually it does not prevent from the possibility that two different sequences $\gamma(t_n, \cdot), \gamma(\tau_n, \cdot)$ with $t_n, \tau_n \to +\infty$ converge to different critical points of $\mathcal{E}_p$, always up to reparametrization and, more importantly, isometry of the ambient. The sub-convergence does not imply that the flow remains in a compact region for any time either. In fact, in the evolution equations we will deal with, high order space derivatives appears, and this implies that no maximum principles hold, and then it is not possible to conclude that the flow always stays in a bounded region of the space by means of comparison arguments.

We formalize and apply a method firstly appeared in [CFS09] for promoting the sub-convergence of a flow to the existence of the full limit as $t \to T^-$. In [CFS09] the authors apply these techniques to the Willmore flow of closed surfaces, that is the L^2-gradient flow of the Willmore energy of surfaces. The key ingredient to run the argument is a Łojasiewicz–Simon gradient inequality for the energy functional under consideration. Such an inequality estimates the difference in energy between a chosen critical point and points sufficiently close to it in terms of some norm of the first variation functional of the energy (Corollary 2.2.7). As the norm of the first variation functional coincides with the norm of the velocity of the gradient flow, by its very definition, this furnishes an additional inequality that eventually can imply the full convergence of the flow. This method has been successfully applied in [DPS16] for proving the full convergence of the elastic flow of open curves subject to clamped boundary conditions; we will borrow useful notations from [DPS16]. We remark that the functional analytic tool used here and in those works, namely the Łojasiewicz–Simon-type inequality, is ultimately based on the important results contained in [Chi03]. Anticipated in [Ło84], the idea of using these inequalities for proving convergence of solutions to parabolic equations was firstly developed in the seminal paper of Simon [Sim83a], that contributed to add his name to the inequality.

Let us conclude this introduction by mentioning some related results in this area. Very recently many contributions have been given to the theory of gradient flows of networks, both in the context of elastic flows and of the curve shortening flow. Roughly speaking, a network is given by a suitable union of open immersed curves joined at their endpoints, possibly prescribing the angles that such curves must define at their junctions. Results about short and long time behavior of these flows are contained in [DLP19; Dal14; GMP19; GMP20; MNP17; MNP19]. It is an open problem to try to apply the method presented here to unsolved problems in the context of these flows, as well as for high order flows of higher dimensional manifolds like the ones in [Man02]. We finally mention that different ideas appeared in the literature for proving the full convergence of a flow; we recall for example [NO17] and [MS20b], that are based either on a priori hypotheses on the critical points of the energy functional or on a known classification of such critical points.

We believe that our methods could be applied for proving convergence of high order flows out of their sub-convergence from a unified point of view.

2.2 Elastic flow in the Euclidean space: outline of the proof and functional analysis

This section is devoted to the presentation of the general method for improving the sub-convergence of a flow to its full convergence. We consider here regular curves $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ and the elastic energy with exponent $p = 2$, that is

$$\mathcal{E}_2(\gamma) = \int_{\mathbb{S}^1} 1 + \frac{1}{2}|k|^2 \, ds,$$

where $\gamma \in W^{2,2}$ is a regular curve. If $\varphi : \mathbb{S}^1 \to \mathbb{R}^n$ is a differentiable vector field we define

$$\nabla\varphi := \partial_s\varphi - \langle \partial_s\varphi, \tau\rangle\tau,$$

that is the normal projection of the arclength derivative of φ. Moreover, we will denote with the symbol $\gamma^\top$ (resp. $\gamma^\perp$) the projection onto the tangent space (resp. normal space) of γ, i.e., if $\varphi : \mathbb{S}^1 \to \mathbb{R}^n$ is any field, then $\gamma^\top\varphi(x) := \langle\varphi(x), \tau(x)\rangle\tau(x)$ (resp. $\gamma^\perp\varphi(x) := \varphi(x) - \gamma^\top\varphi(x)$).

In the rest of the section we just want to collect the most crucial steps of the method, focusing on the proof of an abstract Łojasiewicz–Simon inequality (Section 2.2.2). A posteriori, this part will be a particular case of the theory developed in Section 2.3. As the study of these gradient flows into Riemannian manifolds (Section 2.3) is more involved, for the convenience of the reader we preferred to first present the significant steps of the proof here in the case of curves in $\mathbb{R}^n$ and exponent $p = 2$. In the case of the gradient flow of $\mathcal{E}_2$ in $\mathbb{R}^n$ the strategy can actually be simplified and we refer to [MP20] for such a case.

2.2.1 First and second variations

The strategy starts from a careful study of the properties of the first and second variations of $\mathcal{E}_2$. To this aim we need to define precisely the Banach spaces of vector fields $\varphi : \mathbb{S}^1 \to \mathbb{R}^n$ along a curve γ defining variations of the given curve.

Definition 2.2.1. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a regular curve of class H^4. For $k \in \mathbb{N}$ we define

$$H(\gamma)^{k,\perp} := \Big\{\varphi \in W^{k,2}(\mathbb{S}^1, \mathbb{R}^n) \ : \ \langle \tau(x), \varphi(x)\rangle = 0 \ \text{a.e. } x\Big\},$$

where we understand that $W^{0,2}(\mathbb{S}^1, \mathbb{R}^n) = L^2(\mathbb{S}^1, \mathbb{R}^n)$. Also we denote $H(\gamma)^{0,\perp}$ by $L^2(\gamma)^\perp$.

Remark 2.2.2. If $k \geq 2$ and $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is a regular curve of class H^4, there exists $\rho > 0$ such that $\gamma + \varphi$ is still a regular curve for any $\varphi \in B_\rho(0) \subset H^k(\mathbb{S}^1, \mathbb{R}^n)$ and for any $\varphi \in B_\rho(0) \subset H(\gamma)^{k,\perp}$. In the following we will always assume that $\rho = \rho(\gamma)$ is such that variations $\gamma + \varphi$ are regular curves for any φ as before.

Adopting the notation of [DPS16], it is worth to introduce the following functionals.

Definition 2.2.3. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a regular curve of class H^4. For suitable $\rho > 0$ we define

$$\begin{aligned} E &: B_\rho(0) \subset H(\gamma)^{4,\perp} \to \mathbb{R} & E(\varphi) &:= \mathcal{E}_2(\gamma + \varphi), \\ \mathbf{E} &: B_\rho(0) \subset H^4(\mathbb{S}^1, \mathbb{R}^n) \to \mathbb{R} & \mathbf{E}(\varphi) &:= \mathcal{E}_2(\gamma + \varphi). \end{aligned}$$

Using E and $\mathbf{E}$ we can classically see first and second variations of $\mathcal{E}_2$ as elements of dual spaces; the reason for distinguishing between normal or arbitrary fields along a curve will also be clear soon. More precisely, if γ is fixed, we have

$$\delta\mathbf{E} : B_\rho(0) \subset H^4(\mathbb{S}^1, \mathbb{R}^n) \to (H^4(\mathbb{S}^1, \mathbb{R}^n))^\star \qquad \delta\mathbf{E}(\varphi)[\psi] = \frac{d}{d\varepsilon}\bigg|_0 \mathcal{E}_2(\gamma + \varphi + \varepsilon\psi),$$

and similarly

$$\delta E : B_\rho(0) \subset H(\gamma)^{4,\perp} \to (H(\gamma)^{4,\perp})^\star \qquad \delta E(\varphi)[\psi] = \frac{d}{d\varepsilon}\bigg|_0 \mathcal{E}_2(\gamma + \varphi + \varepsilon\psi).$$

We refer to Proposition 2.3.12 and Corollary 2.3.14 for the general computation of the first variation functionals (see also [MP20]). In the case we are considering, for $\varphi, \psi \in B_\rho(0) \subset H^4(\mathbb{S}^1, \mathbb{R}^n)$ one obtains

$$\begin{aligned} \delta\mathbf{E}(\varphi)[\psi] &= \int_{\mathbb{S}^1} \left\langle \nabla^2_{\gamma+\varphi} k_{\gamma+\varphi} + \frac{1}{2}|k_{\gamma+\varphi}|^2 k_{\gamma+\varphi} - k_{\gamma+\varphi}, \psi \right\rangle ds_{\gamma+\varphi} \\ &=: \left\langle \nabla_{L^2(ds_{\gamma+\varphi})}\mathbf{E}(\varphi), \psi \right\rangle_{L^2(ds_{\gamma+\varphi})}, \end{aligned}$$

and the very same formula holds for E and $\varphi, \psi \in H(\gamma)^{4,\perp}$. In particular we write

$$\nabla_{L^2(dx)}\mathbf{E}(\varphi) = |\gamma' + \varphi'|\nabla_{L^2(ds_{\gamma+\varphi})}\mathbf{E}(\varphi) = |\gamma' + \varphi'| \left(\nabla^2_{\gamma+\varphi} k_{\gamma+\varphi} + \frac{1}{2}|k_{\gamma+\varphi}|^2 k_{\gamma+\varphi} - k_{\gamma+\varphi} \right),$$

and

$$\nabla_{L^2(dx)}E(\varphi) = \gamma^\perp \nabla_{L^2(dx)}\mathbf{E}(\varphi).$$

Setting $\varphi = 0$ we see that $\nabla_{L^2(ds_\gamma)}\mathbf{E}(0)$ is normal along γ and then

$$\delta\mathbf{E}(0)[\psi] = \delta E(0)[\gamma^\perp\psi] = \delta\mathbf{E}(0)[\gamma^\perp\psi],$$

that is, the variation of $\mathcal{E}_2$ at γ only depends on normal vector fields along γ. This is ultimately due to the geometric nature of the functional $\mathcal{E}_2$ and underlines the fact that $\mathcal{E}_2$ presents a degeneracy with respect to variations defined by tangential fields along γ. This is the true reason why one introduces the distinction between normal fields along γ and general fields.

As we will be interested in invertibility properties of the variations of $\mathcal{E}_2$, we will only need to evaluate the second variation of $\mathcal{E}_2$ along normal fields, ruling out the tangential degeneracy of the functional. Therefore we define the operator $\mathcal{L} = \delta^2 E(0) : H(\gamma)^{4,\perp} \to (H(\gamma)^{4,\perp})^\star$ by

$$\mathcal{L}(\varphi)[\psi] := \frac{d}{d\varepsilon}\bigg|_0 \frac{d}{d\eta}\bigg|_0 \mathcal{E}_2\left(\gamma + \eta\varphi + \varepsilon\psi\right) \qquad \forall\, \varphi, \psi \in H(\gamma)^{4,\perp}.$$

Observe that $\mathcal{L}$ is symmetric, that is $\mathcal{L}(\varphi)[\psi] = \mathcal{L}(\psi)[\varphi]$ for any $\varphi, \psi \in H(\gamma)^{4,\perp}$.

Remark 2.2.4 (Fredholm operators)**.** We recall here some basic definitions and facts about Fredholm operators, for which we refer to [HÖ7, Section 19.1]. A continuous linear operator $T : V_1 \to V_2$ between Banach spaces is said to be *Fredholm* if its kernel $\ker T$ has finite dimension and its image $\operatorname{Im} T$ has finite codimension, i.e., the quotient $V_2/\operatorname{Im} T$ is finite dimensional. In such a case one has that $\operatorname{Im} T$ is closed and there exists a finite dimensional subspace $\operatorname{coker} T$ of V_2 such that

$$V_2 = \operatorname{Im} T \oplus \operatorname{coker} T, \tag{2.2}$$

and the dimension of $\operatorname{coker} T$ is clearly equal to the codimension of $\operatorname{Im} T$ (see [HÖ7, Lemma 19.1.1]). With a little abuse of notation, we will denote by $\operatorname{coker} T$ any finite dimensional subspace satisfying (2.2).

Now, if T is Fredholm, we define its *index* to be the integer number

$$\operatorname{Ind} T := \dim\ker T - \dim\operatorname{coker} T,$$

where $\dim(\cdot)$ denotes the dimension of a finite dimensional vector space.

We finally recall that if T is Fredholm and $K : V_1 \to V_2$ is a compact operator, then $T + K$ is Fredholm and $\operatorname{Ind}(T + K) = \operatorname{Ind} T$ (see [HÖ7, Corollary 19.1.8]).

Now the first key observation is the fact that for a fixed regular curve γ of class H^4 and suitable φ, the operators $\delta E(\varphi)$ and $\delta\mathbf{E}(\varphi)$ actually belong to $(L^2(\gamma)^\perp)^\star$ and $(L^2(\mathbb{S}^1, \mathbb{R}^n))^\star$ respectively, as they are represented by the L^2 fields $\nabla_{L^2(dx)}E(\varphi)$ and $\nabla_{L^2(dx)}\mathbf{E}(\varphi)$ respectively. Moreover, the same holds for the second variation functional $\mathcal{L}$, and more precisely we can state the following result.

Lemma 2.2.5. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth regular curve and $\varphi \in H(\gamma)^{4,\perp}$. The operator $\mathcal{L}(\varphi)$ is an element of $(L^2(\gamma)^\perp)^\star$ represented by the pairing*

$$\mathcal{L}(\varphi)[\psi] = \left\langle |\gamma'| \left(\nabla^4\varphi + \Omega(\varphi)\right), \psi\right\rangle_{L^2(dx)} \qquad \forall\, \psi \in H(\gamma)^{4,\perp},$$

where $\Omega : H(\gamma)^{4,\perp} \to L^2(\gamma)^\perp$ is a compact operator.

Moreover the operator $\nabla^4 : H(\gamma)^{4,\perp} \to L^2(\gamma)^\perp$ is Fredholm of index zero, and then so is the operator

$$\mathcal{L} : H(\gamma)^{4,\perp} \to (L^2(\gamma)^\perp)^\star.$$

Proof. The calculation of $\mathcal{L}$ for curves in $\mathbb{R}^n$ can be easily carried out explicitly, and we refer to Proposition 2.3.16 for the computation in the general case of curves in manifolds (see also [MP20] for the case of $\mathbb{R}^n$). The complete thesis will follow from Lemma 2.3.23. □

The second classical ingredient needed for obtaining the Łojasiewicz–Simon gradient inequality is the analiticity of the energy functional and of its first variation. In our case, we have that for a fixed smooth regular curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ and suitable $\rho > 0$ the maps

$$E : B_\rho(0) \to \mathbb{R}, \qquad \delta E : B_\rho(0) \to (L^2(\gamma)^\perp)^\star,$$

are analytic. We refer to [DPS16, Lemma 3.4] for a detailed proof of this fact.

2.2.2 An abstract Łojasiewicz–Simon gradient inequality

In this section we prove a general statement collecting some conditions under which a Łojasiewicz–Simon inequality holds for a given energy functional. This result does not depend on whether we are considering curves in $\mathbb{R}^n$ or in manifolds, actually it is stated at a purely functional analytic level for an abstract energy functional, thus it can be possibly applied to different evolution equations.

We need to recall the functional analytic setting of [Chi03]. We assume that V is a Banach space, $U \subset V$ is open, and $E : U \to \mathbb{R}$ is a map of class C^2. We denote by $\mathscr{M} : U \to V^\star$ the Fréchet first derivative and by $\mathscr{L} : U \to L(V; V^\star)$ the Fréchet second derivative. We also assume that $0 \in U$. Let us denote

$$\mathcal{L} \coloneqq \mathscr{L}(0) \in L(V; V^\star), \qquad V_0 \coloneqq \ker \mathcal{L} \subset V.$$

We recall that a closed subspace $S \subset V$ is said to be *complemented* if there exists a continuous projection $P : V \to V$ such that $\operatorname{Im} P = S$ (see [Bre11, p. 38]). A continuous projection is a linear continuous map $P : V \to V$ such that $P \circ P = P$. In such a case, we denote by $P^\star : V^\star \to V^\star$ the adjoint projection. Recall that any finite dimensional subspace of a Banach space is complemented (see [Bre11, p. 38]).

Proposition 2.2.6 ([Chi03, Corollary 3.11])**.** *In the above notation, assume that E is analytic and 0 is a critical point of E, i.e., $\mathscr{M}(0) = 0$. Assume that V_0 is finite dimensional, and therefore complemented with a projection map P with $\operatorname{Im} P = V_0$. Moreover there exists a Banach space $W \hookrightarrow V^\star$ such that:*

1. *$\mathscr{M} : U \to W$ is W-valued and analytic;*

2. *$P^\star(W) \subset W$;*

3. $\mathcal{L}(V) = \ker P^\star \cap W$.

Then there exist $C, \rho > 0$ *and* $\theta \in (0, \frac{1}{2}]$ *such that*

$$|E(\psi) - E(\varphi)|^{1-\theta} \le C\|\mathscr{M}(\psi)\|_W,$$

for any $\psi \in B_\rho(\varphi)$.

Proposition 2.2.6 is exactly [Chi03, Corollary 3.11] with $X = V$ and $Y = W$ therein. Indeed one can check that the hypotheses of [Chi03, Corollary 3.11], that include Hypotheses 3.2 and 3.4 in [Chi03], reduce to the assumptions considered here in Proposition 2.2.6.

Applying Proposition 2.2.6 we can prove the following Łojasiewicz–Simon gradient inequality.

Corollary 2.2.7 (Abstract Łojasiewicz–Simon gradient inequality)**.** *Let* $E : B_{\rho_0}(0) \subset V \to \mathbb{R}$ *be an analytic map, where* V *is a Banach space and* 0 *is a critical point of* E*. Suppose that* $W \equiv Z^\star \hookrightarrow V^\star$ *is a Banach space with* $V \hookrightarrow Z$*, and that* $\mathscr{M} : B_\rho(0) \to W$ *is* W*-valued and analytic. Suppose also that* $\mathcal{L} := \mathscr{L}(0) \in L(V, W)$ *and* $\mathcal{L} : V \to W$ *is Fredholm of index zero.*

Then the hypotheses of Proposition 2.2.6 are satisfied. In particular there exist C*,* $\rho > 0$ *and* $\theta \in (0, \frac{1}{2}]$ *such that*

$$|E(\psi) - E(0)|^{1-\theta} \le C\|\mathscr{M}(\psi)\|_W,$$

for any $\psi \in B_\rho(0)$.

Proof. By hypothesis $V_0 := \ker \mathcal{L}$ is finite dimensional, and thus it is closed and complemented with a projection $P : V \to V$ such that $\operatorname{Im} P = V_0$. Moreover Item 1 of Proposition 2.2.6 is satisfied by assumption.

We can write that $V = V_0 \oplus V_1$ where $V_1 = \ker P$. If $P^\star : V^\star \to V^\star$ is the adjoint projection, we have that also $V^\star = V_0^\star \oplus V_1^\star$ and

$$V_0^\star = \operatorname{Im} P^\star, \qquad V_1^\star = \ker P^\star.$$

Let us introduce the canonical isometric injection $J_0 : Z \to Z^{\star\star}$. Let us denote $J : V \to (Z)^{\star\star}$ the restriction of J_0 to V. Hence

$$J : V \to J(V) \subset Z^{\star\star}.$$

We claim that $\mathcal{L} : V \to W$ satisfies that if

$$\mathcal{L}^\star : W^\star \to V^\star$$

is the adjoint of $\mathcal{L}$, then

$$\mathcal{L}^\star \circ J = \mathcal{L}. \tag{2.3}$$

Indeed, using that $\mathcal{L}$ is symmetric because it is a second Fréchet derivative, for any $\varphi, \psi \in V$ and $F := J(\psi) \in J(V) \subset Z^{\star\star}$ we find

$$(\mathcal{L}^\star \circ J)(\psi)[\varphi] = \mathcal{L}^\star(F)[\varphi] = F(\mathcal{L}\varphi) = (J(\psi))(\mathcal{L}\varphi) = (\mathcal{L}\varphi)[\psi] = \mathcal{L}(\psi)[\varphi].$$

As a general consequence of the fact that $\mathcal{L}$ is Fredholm of index zero, we have that

$$\dim\ker\mathcal{L} = \dim\ker\mathcal{L}^\star.$$

Indeed, index zero means that $\dim\ker\mathcal{L} = \dim(\operatorname{coker}\mathcal{L})$, where we split W as

$$W = \operatorname{Im}\mathcal{L} \oplus \operatorname{coker}\mathcal{L},$$

and $\operatorname{coker} \mathcal{L}$ is finite dimensional. Therefore $W^\star = (\operatorname{Im} \mathcal{L})^\star \oplus (\operatorname{coker} \mathcal{L})^\star$. And since $\ker \mathcal{L}^\star = (\operatorname{Im} \mathcal{L})^\perp = (\operatorname{coker} \mathcal{L})^\star$ (by [Bre11, Corollary 2.18]), we conclude that $\dim \ker \mathcal{L}^\star = \dim(\operatorname{coker} \mathcal{L})^\star = \dim(\operatorname{coker} \mathcal{L}) = \dim \ker \mathcal{L}$.

We claim that

$$J(\operatorname{Im} P) = \ker \mathcal{L}^\star \cap J(V). \tag{2.4}$$

Indeed by (2.3) we see that

$$\ker \mathcal{L} = \ker(\mathcal{L}^\star \circ J) = J^{-1}(\ker \mathcal{L}^\star).$$

Applying J on both sides we get $J(\operatorname{Im} P) = \ker \mathcal{L}^\star \cap J(V)$, that is (2.4). Since $\operatorname{Im} P = \ker \mathcal{L}$ and J is injective, we have $\dim \ker \mathcal{L} = \dim(J(\operatorname{Im} P)) = \dim(\ker \mathcal{L}^\star \cap J(V))$. Since $\dim \ker \mathcal{L} = \dim \ker \mathcal{L}^\star$, it follows that $\ker \mathcal{L}^\star \cap J(V) = \ker \mathcal{L}^\star$, and then

$$J(\operatorname{Im} P) = \ker \mathcal{L}^\star.$$

Therefore, recalling that $V^{\star\star} \hookrightarrow W^\star$ and that $W \hookrightarrow V^\star$, we get

$$\begin{aligned}
(\ker \mathcal{L}^\star)^\perp &= \{w \in W \ : \ \langle f, w\rangle_{W^\star, W} = 0 \ \ \forall f \in J(\operatorname{Im} P)\} \\
&= \{w \in W \ : \ \langle J(v), w\rangle_{W^\star, W} = 0 \ \ \forall v \in \operatorname{Im} P\} \\
&= \{w \in W \ : \ \langle w, v\rangle_{V^\star, V} = 0 \ \ \forall v \in \operatorname{Im} P\} \\
&= (\operatorname{Im} P)^\perp \cap W.
\end{aligned}$$

Finally, since $\operatorname{Im} \mathcal{L}$ is closed, using [Bre11, Corollary 2.18], this implies

$$\begin{aligned}
\operatorname{Im} \mathcal{L} = (\ker \mathcal{L}^\star)^\perp &= (\operatorname{Im} P)^\perp \cap W = \{f \in V^\star \ : \ \langle f, Pv\rangle_{V^\star, V} = 0 \ \ \forall v \in V\} \cap W \\
&= \ker P^\star \cap W,
\end{aligned}$$

and then Item 3 of Proposition 2.2.6 is verified.

We are just left with proving Item 2 of Proposition 2.2.6, that is, $P^\star(Z^\star) \subset Z^\star$. Observe that if we check that $P^\star(Z^\star \cap V_0^\star) \subset Z^\star \cap V_0^\star$, then we are done, indeed we would get

$$P^\star(Z^\star) = P^\star(Z^\star \cap V_0^\star \oplus Z^\star \cap V_1^\star) = P^\star(Z^\star \cap V_0^\star) \subset Z^\star \cap V_0^\star \subset Z^\star.$$

Now if $f_0 \in Z^\star \cap V_0^\star$, writing any $\varphi \in V$ as $\varphi = \varphi_0 \oplus \varphi_1 \in V_0 \oplus V_1$, we get

$$P^\star(f_0)[\varphi] = f_0(P\varphi) = f_0(\varphi_0) = f_0(\varphi_0) + f_0(\varphi_1) = f_0(\varphi),$$

indeed $f_0(\varphi_1) = (P^\star f_0)(\varphi_1) = f_0(P\varphi_1) = f_0(0) = 0$. Hence we proved that $P^\star f_0 = f_0$ for any $f_0 \in Z^\star \cap V_0^\star$, and thus got that $P^\star(Z^\star \cap V_0^\star) \subset Z^\star \cap V_0^\star$. $\square$

Let us mention that a result equivalent to Corollary 2.2.7 has been proved independently in the recent [Rup20].

2.2.3 Convergence of the elastic flow in the Euclidean space

If $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is a smooth critical point of $\mathcal{E}_2$, the analysis on the first and second variations, together with Lemma 2.2.5, implies that we can apply Corollary 2.2.7 with the spaces $V = H(\gamma)^{4,\perp}$ and $Z = L^2(\gamma)^\perp$ on the energy functional $E : B_{\rho_0}(0) \subset V \to \mathbb{R}$ defined by $E(\varphi) = \mathcal{E}_2(\gamma + \varphi)$. This gives that for some $\rho > 0$ it holds the Łojasiewicz–Simon inequality

$$|\mathcal{E}_2(\gamma + \varphi) - \mathcal{E}_2(\gamma)|^{1-\theta} \le C\|\delta E(\varphi)\|_{(L^2(\gamma)^\perp)^\star} \le C\|\nabla_{L^2(dx)} E(\varphi)\|_{L^2(dx)},$$

for any $\varphi \in B_\rho(0) \subset H(\gamma)^{4,\perp}$. Now, using the geometric nature of $\mathcal{E}_2$, the above inequality can be easily generalized to fields in $H^4(\mathbb{S}^1, \mathbb{R}^n)$ with suitably small norm. More precisely, one has that for some $\sigma > 0$ for any $\psi \in B_\sigma(0) \subset H^4(\mathbb{S}^1, \mathbb{R}^n)$ there is $\varphi \in B_\rho(0) \subset H(\gamma)^{4,\perp}$ such that the curves $\gamma + \psi$ and $\gamma + \varphi$ coincide up to reparametrization (see Lemma 2.3.27). As $|\nabla_{L^2(dx)} E(\varphi)| \le |\nabla_{L^2(dx)} \mathbf{E}(\varphi)|$ for any $\varphi \in B_\rho(0) \subset H(\gamma)^{4,\perp}$ and both $\mathcal{E}_2$ and $\nabla_{L^2(ds_{\gamma+\psi})} \mathbf{E}(\psi)$ are invariant under reparametrization, we eventually find that

$$|\mathcal{E}_2(\gamma + \psi) - \mathcal{E}_2(\gamma)|^{1-\theta} \le C \|\nabla_{L^2(dx)} \mathbf{E}(\psi)\|_{L^2(dx)}, \tag{2.5}$$

for any $\psi \in B_\sigma(0) \subset H^4(\mathbb{S}^1, \mathbb{R}^n)$ (see Corollary 2.3.28).

Following the ideas of [Sim83a], [CFS09], and [DPS16], we can now see how to use (2.5) in order to derive the convergence of the gradient flow of $\mathcal{E}_2$. Let us recall that by gradient flow of $\mathcal{E}_2$ we mean here the evolution equation

$$\begin{cases} \partial_t \gamma = -\nabla^2 k - \frac{1}{2}|k|^2 k + k, \\ \gamma(0, \cdot) = \gamma_0(\cdot), \end{cases} \tag{2.6}$$

for a given smooth curve $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^n$, where one looks for a smooth solution $\gamma : [0, T) \times \mathbb{S}^1 \to \mathbb{R}^n$. In this context, short time existence and sub-convergence of the flow as $t \to +\infty$ have been proved, and more precisely we can state the following.

Theorem 2.2.8 (Existence and sub-convergence, [Pol96], [DKS02, Theorem 3.2])**.** *For a given smooth curve $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^n$, a global solution $\gamma : [0, +\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$ to the flow defined in (2.6) exists and it is unique. Moreover there exist a sequence of times $t_j \to +\infty$ and a sequence of points $p_j \in \mathbb{R}^n$ such that the immersions*

$$\gamma(t_j, \cdot) - p_j,$$

converge in C^m to a critical point γ_∞ of $\mathcal{E}_2$, up to reparametrization, for any $m \in \mathbb{N}$.

We can now illustrate the argument that leads to the convergence as $t \to +\infty$ of the solution of this gradient flow. Here we sketch the proof we will use for the flow of curves in manifolds in Section 2.3; however, as already mentioned, in this case where the ambient is the Euclidean space, the proof can be simplified and we refer the reader to [MP20] for such a proof.

Let γ_0 be fixed, and let $\gamma, \gamma_\infty, t_j, p_j$ be given by Theorem 2.2.8. Without loss of generality we assume that γ_∞ is parametrized with constant speed. Fix $m \ge 8$ and let $\varepsilon \in (0, 1)$ to be chosen. By Theorem 2.2.8 there exists p_{j_0} such that

$$\|\bar{\gamma}(t_{j_0}, \cdot) - p_{j_0} - \gamma_\infty(\cdot)\|_{C^m(\mathbb{S}^1)} \le \varepsilon,$$

where $\bar{\gamma}(t, \cdot)$ is the constant speed reparametrization of $\gamma(t, \cdot)$. We want to show that if ε is sufficiently small, then actually $\bar{\gamma}$ smoothly converges. We rename $\gamma_0(\cdot) = \bar{\gamma}(t_{j_0}, \cdot)$. By short time existence and uniqueness results (Theorem 2.3.29, [Pol96]) there exists a solution $\widetilde{\gamma} : [0, +\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$ of

$$\begin{cases} \partial_t \widetilde{\gamma} = -\nabla^2 k_{\widetilde{\gamma}} - \frac{1}{2}|k_{\widetilde{\gamma}}|^2 k_{\widetilde{\gamma}} + k_{\widetilde{\gamma}}, \\ \widetilde{\gamma}(0, \cdot) = \gamma_0(\cdot). \end{cases}$$

We denote by $\widehat{\gamma}$ the constant speed reparametrization of $\widetilde{\gamma}$. For ε sufficiently small we can write $\widehat{\gamma}$ as a variation of γ_∞. More precisely, there is some maximal $T' \in (0, +\infty]$ such that for any $t \in [0, T')$ there exists $\psi_t \in B_\sigma(0) \subset H^4(\mathbb{S}^1, \mathbb{R}^n)$ such that $\widehat{\gamma}(t, \cdot) = \gamma_\infty(\cdot) + \psi_t(\cdot)$, where σ is as in (2.5).

Suppose by contradiction that $T' < +\infty$. Suitable parabolic estimates give that for any $t < T'$, knowing that the flow $\widehat{\gamma}$ remains close in $W^{4,p}$ to the fixed γ_∞, the norm $\|k_{\widehat{\gamma}}\|_{W^{l,2}}$ is bounded by a term depending on $\|k_{\gamma_0}\|_{W^{l,2}}$, on l, and on $\|k_{\gamma_\infty}\|_{W^{2,2}}$ for any $l \in \mathbb{N}$. This fact is technical but rather classical in the theory of parabolic geometric equations and the details are carried out in Proposition 2.3.31. In particular, as γ_0 is close to γ_∞ in C^m, these parabolic estimates applied for $l = m-2$ together with Sobolev embeddings eventually imply that

$$\sup_{[0,T')} \|\widehat{\gamma}(t,\cdot) - \gamma_0(\cdot)\|_{C^{m-3}(\mathbb{S}^1)} \leq C(\gamma_\infty),$$

and the key fact here is that the constant on the right deos not depend on ε. By triangular inequality we then deduce

$$\sup_{[0,T')} \|\widehat{\gamma}(t,\cdot) - \gamma_\infty(\cdot)\|_{C^{m-3}(\mathbb{S}^1)} \leq C(\gamma_\infty). \tag{2.7}$$

Now we consider the evolution of

$$H(t) := (\mathcal{E}_2(\widehat{\gamma}(t,\cdot)) - \mathcal{E}_2(\gamma_\infty))^\theta,$$

where θ is the Łojasiewicz–Simon exponent of (2.5). Using (2.5) it is immediate to estimate that

$$-\frac{d}{dt}H(t) \geq C(\gamma_\infty)\|\partial_t^\perp \widehat{\gamma}\|_{L^2(dx)}, \tag{2.8}$$

where $\partial_t^\perp \widehat{\gamma}$ is just the projection of the velocity $\partial_t \widehat{\gamma}$ onto the normal space of $\widehat{\gamma}$. Using (2.8) and the fact that $\|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})} = \|\partial_t \widetilde{\gamma}\|_{L^2(ds_{\widetilde{\gamma}})}$, one can show that the parametrization of $\widetilde{\gamma}$ does not degenerate, that is, the speed $|\partial_x \widetilde{\gamma}(t,x)|$ is bounded away from zero uniformly in time, and it is actually close to the speed of $\widehat{\gamma}$.

Therefore one shows that

$$\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{L^2(dx)} \leq C(\gamma_\infty)\varepsilon^\theta,$$

for $t \in [0,T')$. Suitable interpolation inequalities (see Remark 2.3.32) together with (2.7) imply that

$$\begin{aligned}
\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{W^{4,2}} &\leq C\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{C^5}^\alpha \|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{L^2(dx)}^{1-\alpha} \\
&\leq C\|\gamma_0 - \gamma_\infty\|_{C^2(\mathbb{S}^1)}^{\theta(1-\alpha)} \leq C\varepsilon^{\theta(1-\alpha)}
\end{aligned}$$

for $t \in [0,T')$ and some $\alpha \in (0,1)$. Hence if ε is sufficiently small this implies that $\|\psi_t\|_{W^{4,2}} \leq \frac{1}{2}\sigma$ for any $t \in [0,T')$, contradicting the maximality of T'.

Hence we have that for any $t \in [0,+\infty)$ the flow $\widehat{\gamma}(t,\cdot)$ can be written as $\gamma_\infty + \psi_t$ for some uniformly bounded fields ψ_t; in particular the evolution $\widehat{\gamma}(t,x)$ stays in a compact set for any t. Once boundedness in space is achieved, the above estimates eventually imply that $\widehat{\gamma}$ smoothly converges to a translation of γ_∞, and then the same holds for the original flow γ.

As a result of this argument, or as a particular case of Theorem 2.3.33, we can state the following theorem.

Theorem 2.2.9 (Smooth convergence in $\mathbb{R}^n$, [MP20])**.** *For a given smooth curve $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^n$, a global solution $\gamma : [0,+\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$ to the flow defined in (2.6) exists, it is unique, and it converges as $t \to +\infty$ to a critical point γ_∞ of $\mathcal{E}_2$ in C^m for any $m \in \mathbb{N}$, up to reparametrization. In particular, the flow stays in a compact set of $\mathbb{R}^n$ for any time.*

The variational approach leading to the above theorem and the abstract tool in Corollary 2.2.7 suggest that we can try to extend the result to the gradient flow of elastic functionals of curves immersed into Riemannian manifolds. The rest of the chapter is, in fact, devoted to prove that, under suitable hypotheses, the sub-convergence of the gradient flow of the p-elastic energy can be improved to full convergence of the flow also on Riemannian manifolds. This will fill the gaps in the heuristic proof presented above in the case of the flow in the Euclidean space for $p = 2$.

2.3 Convergence of p-elastic flows into manifolds

In this section we employ the techniques and the strategy discussed in Section 2.2 for proving the sub-convergence to convergence improvement for the p-elastic flow of curves in complete Riemannian manifolds (M, g). Let us start with a few definitions.

In the following (M^m, g) will be a fixed complete Riemannian manifold of dimension $m \geq 2$. By Nash Theorem [Nas56] we can assume without loss of generality that $(M^m, g) \hookrightarrow \mathbb{R}^n$ isometrically and that a smooth curve into M is a smooth regular curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ with $\gamma(x) \in M$ for any x. The exponential map of M will be denoted by $\exp : TM \to M$.

Having identified M with a subset of $\mathbb{R}^n$, we will denote by $\langle \cdot, \cdot \rangle$ both the Euclidean product and the metric on M. If V is a vector field in $\mathbb{R}^n$ and $x \in M$, by $M^\top V(x)$ (resp. $M^\perp V(x)$) we denote tangent (resp. normal) projection of V on the tangent space of $T_x M$ (resp. the normal space of $T_x M^\perp$). We denote by ∂_v a directional derivative in $\mathbb{R}^n$, and by D the Levi-Civita connection on M, so that

$$(D_v X)(x) = M^\top (\partial_v X)(x),$$

for tangent fields v, X on M.

The symbol ∇ will denote the normal connection along a curve γ in M, that is

$$\begin{aligned}(\nabla_v \phi)(x) &= (M^\top - \gamma^\top)(\partial_v \phi)(x) = (M^\top - \gamma^\top M^\top)(\partial_v \phi)(x) \\ &= D_v \phi(x) - \langle D_v \phi(x), \tau(x) \rangle \tau(x),\end{aligned}$$

for any smooth field $\phi \in TM \cap (T\gamma)^\perp$. Unless otherwise stated we will always denote

$$\gamma^\perp := M^\top - \gamma^\top,$$

that is, $\gamma^\perp$ is the normal projection along γ as a submanifold of M. We will also write $\nabla := \nabla_\tau$, in analogy with the notation used for curves in $\mathbb{R}^n$.

Remark 2.3.1. If $\varphi, \psi \in C^1(\mathbb{S}^1, \mathbb{R}^n)$ are fields such that $\varphi, \psi \in TM \cap (T\gamma)^\perp$ for a given regular curve $\gamma \in W^{4,p}(\mathbb{S}^1, M)$, then

$$\int_{\mathbb{S}^1} \langle \nabla \varphi, \psi \rangle \, ds = \int \langle \gamma^\perp M^\top |\gamma'|^{-1} \partial_x \varphi, \psi \rangle |\gamma'| \, dx = - \int_{\mathbb{S}^1} \langle \varphi, \nabla \psi \rangle \, ds,$$

that is, integration by parts holds for normal fields with respect to the normal connection ∇ and the arclength measure ds.

Remark 2.3.2. The curvature k of γ into $M \subset \mathbb{R}^n$ is the geodesic curvature of the curve on M. In particular we can write that

$$k = D_\tau \tau = M^\top (\partial_s \tau) = M^\top (\partial_s^2 \gamma).$$

Let us also define the Sobolev spaces

$$W^{k,p}(\mathbb{S}^1, M) := \left\{ \gamma : \mathbb{S}^1 \to \mathbb{R}^n \mid \gamma \in W^{k,p}(\mathbb{S}^1, \mathbb{R}^n),\ \gamma(x) \in M \text{ a.e. } x \right\},$$

for $k \in \mathbb{N}$ with $k \geq 1$ and $p \in [1, +\infty)$. For $k \geq 2$ and $p > 1$ we denote by $W^{k,p}_{imm}(\mathbb{S}^1, M)$ the open subset of $W^{k,p}(\mathbb{S}^1, M)$ of immersions, that is the subset of functions γ such that $|\gamma'| \geq c(\gamma) > 0$. The symbols introduced above for the curvature or the tangent vector of smooth curves will be analogously used for sufficiently regular Sobolev curves. Spaces $L^p(\mathbb{S}^1, M)$ are defined analogously.

Now we need to define the Banach spaces of vector fields along curves that we will use to produce variations of a given curve.

Definition 2.3.3. If γ is a fixed immersion of class C^1, for $k \in \mathbb{N}$ we define

$$T(\gamma)^{k,p} := \left\{ \varphi \in W^{k,p}(\mathbb{S}^1, \mathbb{R}^n) \ :\ \varphi(p) \in T_{\gamma(p)} M \ \forall\, p \in \mathbb{S}^1 \right\},$$

$$T(\gamma)^{k,p,\perp} := \left\{ \varphi \in W^{k,p}(\mathbb{S}^1, \mathbb{R}^n) \ :\ \langle \varphi, \partial_x \gamma \rangle \equiv 0,\ \varphi(p) \in T_{\gamma(p)} M \ \forall\, p \in \mathbb{S}^1 \right\},$$

for any $k \geq 1$, and

$$T(\gamma)^p = T(\gamma)^{0,p} := \left\{ \varphi \in L^p(\mathbb{S}^1, \mathbb{R}^n) \ :\ \varphi(p) \in T_{\gamma(p)} M \text{ a.e. on } \mathbb{S}^1 \right\},$$

$$T(\gamma)^{p,\perp} = T(\gamma)^{0,p,\perp} := \left\{ \varphi \in L^p(\mathbb{S}^1, \mathbb{R}^n) \ :\ \langle \varphi, \partial_x \gamma \rangle = 0,\ \varphi(p) \in T_{\gamma(p)} M \text{ a.e. on } \mathbb{S}^1 \right\}.$$

When nothing is specified, L^p-spaces are equipped with the Lebesgue measure. If for a given curve γ we want to employ the induced length measure on $\mathbb{S}^1$ we will specify $L^p(ds_\gamma)$.

The following lemma shows that the spaces $T(\gamma)^{k,p}$ do not depend on the embedding of M into $\mathbb{R}^n$.

Lemma 2.3.4. *Let γ be a fixed immersion of class C^1. Let $k \in \mathbb{N}$ and $\varphi \in T(\gamma)^{k,p}$. If there exists $\phi \in T(\gamma)^p$ such that*

$$\int \langle D_s^k \varphi, D_s \psi \rangle \, ds = - \int \langle \phi, \psi \rangle \, ds,$$

for any $\psi : \mathbb{S}^1 \to \mathbb{R}^n$ of class C^∞ such that $\psi(x) \in T_{\gamma(x)} M$, then $\varphi \in T(\gamma)^{k+1,p}$.

Proof. Let us first prove by induction that for any $n \in \mathbb{N}$ with $n \geq 1$ if $\alpha \in T(\gamma)^{n,p}$ then

$$D_s^n \alpha = \partial_s^n \alpha + \Omega_n(\alpha, ..., \partial_s^{n-1} \alpha), \tag{2.9}$$

where Ω_n is smooth in its entries, it only depends on M, and $\Omega_n(\alpha, ..., \partial_s^{n-1}\alpha) \in W^{1,p}$. In fact for $n = 1$ we have

$$D_s \alpha = \partial_s \alpha + \langle \alpha, \partial_s N_j \rangle N_j,$$

where $\{N_j\}$ is a local orthonormal frame of $TM^\perp$, and summation over j is understood. Since

$$\partial_s M^\top - M^\top \partial_s = -\partial_s N_j \otimes N_j - N_j \otimes \partial_s N_j,$$

for $n \geq 1$ we get

$$\begin{aligned} D_s^{n+1} \alpha &= M^\top \partial_s \left(\partial_s^n \alpha + \Omega_n(\alpha, ..., \partial_s^{n-1} \alpha) \right) \\ &= \partial_s \left(\partial_s^n \alpha - \langle \partial_s^n \alpha, N_j \rangle N_j \right) + \partial_s (M^\top \Omega_n) + (\partial_s N_j \otimes N_j + N_j \otimes \partial_s N_j)(\partial_s^n \alpha + \Omega_n) \\ &= \partial_s^{n+1} \alpha + \partial_s (M^\perp \Omega_n) + \partial_s (M^\top \Omega_n) + (\partial_s N_j \otimes N_j + N_j \otimes \partial_s N_j)(\partial_s^n \alpha + \Omega_n) \\ &= \partial_s^{n+1} \alpha + \partial_s \Omega_n + (\partial_s N_j \otimes N_j + N_j \otimes \partial_s N_j)(\partial_s^n \alpha + \Omega_n), \end{aligned}$$

that proves (2.9).

For $\Psi : \mathbb{S}^1 \to \mathbb{R}^n$ of class C^∞ we write $\Psi = \psi + \Psi^\perp$ where $\psi \in TM$. We have

$$\langle D_s^k\varphi, \partial_s\Psi\rangle = \langle D_s^k\varphi, D_s\psi\rangle + \langle \Psi, N_j\rangle\langle D_s^k\varphi, \partial_s N_j\rangle = \langle D_s^k\varphi, D_s\psi\rangle - \langle \Psi, B(\tau, D_s^k\varphi)\rangle,$$

where $\{N_j\}$ is a local orthonormal frame of $TM^\perp$. Hence

$$\int \langle D_s^k\varphi, \partial_s\Psi\rangle\, ds = \int -\langle \phi, \psi\rangle - \langle \Psi, B(\tau, D_s^k\varphi)\rangle\, ds = -\int \langle \phi + B(\tau, D_s^k\varphi), \Psi\rangle\, ds,$$

which shows that $D_s^k\varphi \in T(\gamma)^{1,p}$. And therefore by (2.9) also $\partial_s^k\varphi \in W^{1,p}$. $\square$

Let us state here another simple lemma about the regularity of the objects we will deal with.

Lemma 2.3.5. *Let $g \in W^{k,p}((0,1), B_r(0))$ with $B_r(0) \subset \mathbb{R}^N$. Let $f : B_{2r}(0) \to \mathbb{R}$ be a bounded function of class C^k with bounded continuous derivatives up to order k. Then $f \circ g \in W^{k,p}(0,1)$ and the operator*

$$W^{k,p}((0,1), B_r(0)) \ni\; g \mapsto f \circ g \;\in W^{k,p}(0,1)$$

is of class C^k.

Proof. Since $W^{k,p}((0,1), B_r(0)) \subset C^{k-1,\alpha}((0,1), B_r(0))$ we see that $f \circ g \in W^{k-1,p}(0,1)$. Now for a function $\widetilde{g} \in C^\infty((0,1), B_{\frac{3}{2}r}(0))$ the chain rule gives

$$(f \circ \widetilde{g})^{(k)} = (\nabla^k f)(\widetilde{g})[\widetilde{g}', \widetilde{g}', ..., \widetilde{g}'] + P((\nabla^{k-1} f)(\widetilde{g}), ..., (\nabla^2 f)(\widetilde{g}), \widetilde{g}', ..., \widetilde{g}^{(k-1)}) + \langle (\nabla f)(\widetilde{g}), \widetilde{g}^{(k)}\rangle,$$

where P is some polynomial. Considering a sequence $g_n \in C_c^\infty((0,1), B_{\frac{3}{2}r}(0))$ converging in $W^{k,p}$ to g and thus also strongly in C^{k-1} we see that

$$(\nabla^k f)(g_n)[g_n', g_n', ..., g_n'] \to (\nabla^k f)(g)[g', g', ..., g']$$

uniformly, and

$$\langle (\nabla f)(g_n), g_n^{(k)}\rangle \to \langle (\nabla f)(g), g^{(k)}\rangle$$

in L^p, and therefore $f \circ g \in W^{k,p}(0,1)$.

If now $g_h \in W^{k,p}((0,1), B_r(0))$ is a sequence converging to g in $W^{k,p}$, and then in C^{k-1}, we have that $f \circ g_h \to f \circ g$ in C^{k-1} and $(f \circ g_h)^{(k)} \to (f \circ g)^{(k)}$ in L^p by the above formulas, and then $g \mapsto f \circ g$ is continuous between the corresponding Sobolev spaces. Since $f \in C^k$ with bounded derivatives, an analogous argument shows that $g \mapsto f \circ g$ is of class C^k. $\square$

Corollary 2.3.6. *Let $\gamma : \mathbb{S}^1 \to M^m$ be a fixed regular curve of class C^1 and let $F : TM \to N^n$ a smooth map between manifolds. If $\varphi \in T(\gamma)^{k,p}$ then $F \circ \varphi$ is of class $W^{k,p}$, in the sense that for any local chart (U, ζ) on N, the map $\zeta \circ F \circ \varphi$ is of class $W^{k,p}$. Moreover for any local chart (U, ζ) in N the operator*

$$T(\gamma)^{k,p} \ni\; \varphi \mapsto \zeta \circ F \circ \varphi \;\in W^{k,p}((0,1), \mathbb{R}^n)$$

is of class C^∞.

Proof. For any local chart (V, ξ) on TM, Lemma 2.3.5 implies that $\xi \circ \varphi \in W^{k,p}(\mathbb{S}^1, \mathbb{R}^{2m})$. Since $\zeta \circ F \circ \xi^{-1}$ is smooth and $\xi \circ \varphi$ is bounded, we get that $\zeta \circ F \circ \xi^{-1} \circ \xi \circ \varphi$ is of class $W^{k,p}$. Smoothness of the operator $\varphi \mapsto \zeta \circ F \circ \varphi$ follows by applying Lemma 2.3.5. $\square$

In Corollary 2.3.6 a map fitting the hypothesis is the exponential map $\exp : TM \to M$. This leads to the following definition.

Definition 2.3.7. Let $\gamma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$. A map $\Phi : (-\varepsilon_0, \varepsilon_0) \times \mathbb{S}^1 \to M$ is a *variation* of γ if

$$\Phi(0,\cdot) = \gamma(\cdot), \qquad \Phi(s,\cdot) \in W^{4,p}_{imm}(\mathbb{S}^1, M) \quad \forall\, s, \qquad \Phi(\cdot, x) \in W^{4,p}((-\varepsilon_0, \varepsilon_0), \mathbb{R}^n) \quad \forall\, x.$$

In such a case we write that $\Phi \in \mathrm{Var}(\gamma)$ with *variation field* $\varphi(x) := \partial_\varepsilon \Phi(0, x)$. If it also occurs that $\varphi \in T\gamma^\perp$, then we say that Φ is a normal variation and we write that $\Phi \in \mathrm{Var}^\perp(\gamma)$.

Using the exponential map of M, we will always use a typical construction of variations of a curve given a variation field. More precisely, suppose that $\varphi \in T(\gamma)^{4,p}$ for an immersed curve $\gamma \in W^{4,p}(\mathbb{S}^1, M)$. We then define the variation

$$\Phi = \Phi(\varepsilon, x) : (-\varepsilon_0, \varepsilon_0) \times \mathbb{S}^1 \to M \qquad\qquad \Phi(\varepsilon, x) = \exp_{\gamma(x)}(\varepsilon\varphi(x)),$$

where $\exp_p : T_pM \to M$ is the exponential map of M. Since $\mathbb{S}^1$ is compact, the definition of Φ is well posed for ε_0 small enough. It holds that

$$\Phi(0, x) = \gamma(x), \qquad \partial_\varepsilon \Phi(0, x) = \varphi(x).$$

We also set $\gamma_\varepsilon(\cdot) = \Phi(\varepsilon, \cdot)$. Finally for any $v \in T_pM$ we will denote by $\sigma_v : [0, l_v) \to M$ the geodesic in M such that $\sigma(0) = p$ and $\sigma'(0) = v$. In this way we can write that

$$\gamma_\varepsilon(x) = \Phi(\varepsilon, x) = \sigma_{\varepsilon\varphi(x)}(1) = \sigma_{\varphi(x)}(\varepsilon).$$

Corollary 2.3.6 implies the following lemma.

Lemma 2.3.8. *Fix an immersed curve $\gamma \in W^{4,p}(\mathbb{S}^1, M)$. Then there exist a radius $\rho(\gamma) > 0$ and $\varepsilon_0(\gamma) > 1$ such that $\Phi(\varepsilon, x) = \exp_{\gamma(x)}(\varepsilon\varphi(x))$ is a variation of γ in the sense of Definition 2.3.7 with variation field φ for any $|\varepsilon| < \varepsilon_0$ and any $\varphi \in T(\gamma)^{4,p}$ with $\|\varphi\|_{W^{4,p}} \le \rho$.*

Finally, as in Section 2.2 we introduce the following functionals.

Definition 2.3.9. Let $\gamma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$ be fixed and let ρ be given by Lemma 2.3.8. For $\varphi \in B_\rho(0) \subset T(\gamma)^{4,p}$ we define

$$E : B_\rho(0) \subset T(\gamma)^{4,p,\perp} \to \mathbb{R} \qquad\qquad E(\varphi) := \mathcal{E}_p(\Phi(1,\cdot)),$$

and for $\varphi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$ we define

$$\mathbf{E} : B_\rho(0) \subset T(\gamma)^{4,p} \to \mathbb{R} \qquad\qquad \mathbf{E}(\varphi) := \mathcal{E}_p(\Phi(1,\cdot)),$$

where Φ is the variation associated with the given field φ.

2.3.1 First and second variations

Let $\gamma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$ be fixed. We want to compute the variations of E and $\mathbf{E}$. For φ in the suitable domains of the two functionals we recall that the first variations are defined as

$$\delta\mathbf{E}(\varphi) \in (T(\gamma)^{4,p})^\star \qquad\qquad \delta\mathbf{E}(\varphi)[\psi] := \frac{d}{ds}\bigg|_0 \mathbf{E}(\varphi + s\psi),$$

$$\delta E(\varphi) \in (T(\gamma)^{4,p,\perp})^\star \qquad\qquad \delta E(\varphi)[\psi] := \frac{d}{ds}\bigg|_0 E(\varphi + s\psi).$$

Let us collect some computations first.

Lemma 2.3.10. *Let $\sigma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$ and $\Phi \in \mathrm{Var}(\sigma)$ with variation field φ. Denote $\sigma_\varepsilon(x) = \Phi(\varepsilon, x)$. Then*

$$\frac{d}{d\varepsilon}\bigg|_0 ds_{\sigma_\varepsilon} = \langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle ds_\sigma, \tag{2.10}$$

$$\frac{d}{d\varepsilon}\bigg|_0 \int_{\mathbb{S}^1} ds_{\sigma_\varepsilon} = -\int_{\mathbb{S}^1} \langle \varphi, k_\sigma\rangle \, ds_\sigma, \tag{2.11}$$

$$\frac{d}{d\varepsilon}\bigg|_0 \tau_{\sigma_\varepsilon} = \partial_{s_\sigma}\varphi - \langle \partial_{s_\sigma}\varphi, \tau_\sigma\rangle \tau_\sigma, \tag{2.12}$$

$$(M^\top - \sigma^\top)\left(\frac{d}{d\varepsilon}\bigg|_0 k_{\sigma_\varepsilon}\right) = (M^\top - \sigma^\top)\partial_s((M^\top - \sigma^\top)\partial_s\varphi) - \langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle k_\sigma + R(\varphi^\perp, \tau_\sigma)\tau_\sigma, \tag{2.13}$$

where $\varphi^\perp = \sigma^\perp\varphi$.
If also $\Phi \in \mathrm{Var}^\perp(\sigma)$, i.e. $\varphi \in T\sigma^\perp$, then

$$\frac{d}{d\varepsilon}\bigg|_0 ds_{\sigma_\varepsilon} = -\langle k_\sigma, \varphi\rangle ds_\sigma, \tag{2.14}$$

$$\frac{d}{d\varepsilon}\bigg|_0 \tau_{\sigma_\varepsilon} = \nabla\varphi, \tag{2.15}$$

$$(M^\top - \sigma^\top)\left(\frac{d}{d\varepsilon}\bigg|_0 k_{\sigma_\varepsilon}\right) = \nabla^2\varphi + \langle \varphi, k_\sigma\rangle k_\sigma + R(\varphi, \tau_\sigma)\tau_\sigma. \tag{2.16}$$

Proof. Equation (2.10) follows by a direct calculation. Then (2.14) follows by the fact that $\langle \tau_\sigma, \partial_x\varphi\rangle = -\langle \partial_x\tau_\sigma, \varphi\rangle$ for $\varphi \in T\sigma^\perp$, and $\langle \partial^2_{s_\sigma}\sigma, \varphi\rangle = \langle M^\top(\partial^2_{s_\sigma}\sigma), \varphi\rangle$. Moreover for general variation field φ we have that

$$\begin{aligned}\langle \partial_{s_\sigma}\varphi, \tau_\sigma\rangle &= \left\langle \partial_{s_\sigma}\left(\langle\varphi, \tau_\sigma\rangle\tau_\sigma + \sum_2^m \langle\varphi, e_i\rangle e_i\right), \tau_\sigma\right\rangle = \partial_s(\langle\varphi, \tau_\sigma\rangle) + \sum_2^m \langle\varphi, e_i\rangle\langle\partial_s(e_i\circ\sigma), \tau_\sigma\rangle \\ &= \partial_s(\langle\varphi, \tau_\sigma\rangle) + -\langle\varphi, k_\sigma\rangle,\end{aligned}$$

where $\{\tau, e_2, ..., e_m\}$ is a local orthonormal frame of TM. Equation (2.11) then follows by integration.

Equation (2.12) and Equation (2.15) also follows by direct calculations and the definition of the normal connection ∇.

Now for $j = m+1, ..., n$ let $N_j : U \to \mathbb{S}^{n-1}$ be unit vector fields locally defined on a neighborhood of $\sigma(x)$ in M such that $\{N_j : j = m+1, ..., n\}$ is a local orthonormal frame of $(TM)^\perp$. Writing

$$k_{\sigma_\varepsilon} = D_{\tau_{\sigma_\varepsilon}}\tau_{\sigma_\varepsilon} = \partial_{s_{\sigma_\varepsilon}}\tau_{\sigma_\varepsilon} - \sum_j \langle \partial_{s_{\sigma_\varepsilon}}\tau_{\sigma_\varepsilon}, N_j\circ\sigma_\varepsilon\rangle N_j\circ\sigma_\varepsilon,$$

we have that

$$\begin{aligned}\frac{d}{d\varepsilon}\bigg|_0 k_{\sigma_\varepsilon} =& -\langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle\partial_{s_\sigma}\tau_\sigma + \partial_{s_\sigma}\left(\partial_{s_\sigma}\varphi - \langle\partial_{s_\sigma}\varphi, \tau_\sigma\rangle\tau_\sigma\right) + \sum_j \langle\tau_\sigma, \partial_{s_\sigma}\varphi\rangle\langle\partial_{s_\sigma}\tau_\sigma, N_j\circ\sigma\rangle N_j\circ\sigma + \\ & - \langle\partial_{s_\sigma}\left(\partial_{s_\sigma}\varphi - \langle\partial_{s_\sigma}\varphi, \tau_\sigma\rangle\tau_\sigma\right), N_j\circ\sigma\rangle N_j\circ\sigma - \langle\partial_{s_\sigma}\tau_\sigma, \partial_\varepsilon|_0(N_j\circ\sigma_\varepsilon)\rangle N_j\circ\sigma + \\ & - \langle\partial_{s_\sigma}\tau_\sigma, N_j\circ\sigma\rangle\partial_\varepsilon|_0(N_j\circ\sigma_\varepsilon).\end{aligned}$$

Denoting by $S_{N_j}(v) := -M^\top(\partial_v N_j)$ the shape operator of M defined by N_j, we have that $M^\top[\partial_\varepsilon|_0(N_j \circ \sigma_\varepsilon)] = -S_{N_j}(\varphi)$, and also

$$\begin{aligned}
&(M^\top - \sigma^\top)\partial_s((M^\top - \sigma^\top)\partial_s\varphi) \\
&\qquad = (M^\top - \sigma^\top)(\partial_s^2\varphi) - \langle \partial_s\varphi, \tau_\sigma\rangle k_\sigma - \sum_j \langle \partial_s\varphi, N_j\rangle (M^\top - \sigma^\top)(\partial_s N_j) \\
&\qquad = (M^\top - \sigma^\top)(\partial_s^2\varphi) - \langle \partial_s\varphi, \tau_\sigma\rangle k_\sigma - \sum_j \langle S_{N_j}(\tau_\sigma), \varphi\rangle \left(-S_{N_j}(\tau_\sigma) + \langle N_j, \partial_s^2\sigma\rangle\tau_\sigma\right) \\
&\qquad = (M^\top - \sigma^\top)(\partial_s^2\varphi) - \langle \partial_s\varphi, \tau_\sigma\rangle k_\sigma - \sum_j \langle B(\tau_\sigma, \varphi), N_j\rangle \left(\langle B(\tau_\sigma,\tau_\sigma), N_j\rangle\tau_\sigma - S_{N_j}(\tau_\sigma)\right),
\end{aligned}$$

where B is the second fundamental form of M in $\mathbb{R}^n$. Observe that if $\varphi \in (T\sigma)^\perp$, then actually $(M^\top - \sigma^\top)\partial_s((M^\top - \sigma^\top)\partial_s\varphi) = \nabla^2\varphi$; for sake of readability, in this proof we will denote by $\nabla^2\varphi$ the vector $(M^\top - \sigma^\top)\partial_s((M^\top - \sigma^\top)\partial_s\varphi)$ for any $\varphi \in TM$, that is, not only for normal fields along σ.

We have

$$\begin{aligned}
M^\top\left(\left.\frac{d}{d\varepsilon}\right|_0 k_{\sigma_\varepsilon}\right) &= -\langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle k_\sigma + M^\top(\partial_{s_\sigma}^2\varphi) - \partial_{s_\sigma}\left(\langle \partial_{s_\sigma}\varphi, \tau_\sigma\rangle\right)\tau_\sigma - \langle \partial_{s_\sigma}\varphi, \tau_\sigma\rangle k_\sigma + \\
&\quad - \sum_j \langle \partial_{s_\sigma}\tau_\sigma, N_j \circ \sigma\rangle(-S_{N_j}(\varphi)) \\
&= -2\langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle k_\sigma + (M^\top - \sigma^\top)(\partial_{s_\sigma}^2\varphi) - \langle \partial_{s_\sigma}\varphi, \partial_{s_\sigma}\tau_\sigma\rangle\tau_\sigma + \\
&\quad + \sum_j \langle B(\tau_\sigma,\tau_\sigma), N_j\rangle S_{N_j}(\varphi) \\
&= \nabla^2\varphi - \langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle k_\sigma - \langle \partial_{s_\sigma}\varphi, \partial_s^2\sigma\rangle\tau_\sigma + \\
&\quad + \sum_j \langle B(\tau_\sigma, \varphi), N_j\rangle \left(\langle B(\tau_\sigma,\tau_\sigma), N_j\rangle\tau_\sigma - S_{N_j}(\tau_\sigma)\right) + \langle B(\tau_\sigma,\tau_\sigma), N_j\rangle S_{N_j}(\varphi),
\end{aligned}$$

and

$$\begin{aligned}
\sigma^\top\left(\left.\frac{d}{d\varepsilon}\right|_0 k_{\sigma_\varepsilon}\right) &= \langle \partial_{s_\sigma}^2\varphi, \tau_\sigma\rangle\tau_\sigma - \partial_{s_\sigma}(\langle \partial_{s_\sigma}\varphi, \tau_\sigma\rangle)\tau_\sigma - \sum_j \langle \partial_{s_\sigma}\tau_\sigma, N_j \circ \sigma\rangle\, \sigma^\top(\partial_\varepsilon|_0(N_j \circ \sigma_\varepsilon)) \\
&= -\langle \partial_{s_\sigma}\varphi, \partial_{s_\sigma}\tau_\sigma\rangle\tau_\sigma + \sum_j \langle B(\tau_\sigma,\tau_\sigma), N_j\rangle\langle B(\varphi,\tau_\sigma), N_j\rangle\tau_\sigma.
\end{aligned}$$

Hence

$$\begin{aligned}
(M^\top - \sigma^\top)\left(\left.\frac{d}{d\varepsilon}\right|_0 k_{\sigma_\varepsilon}\right) &= \nabla^2\varphi - \langle \tau_\sigma, \partial_{s_\sigma}\varphi\rangle k_\sigma + \\
&\quad + \sum_j \langle B(\tau_\sigma,\tau_\sigma), N_j\rangle S_{N_j}(\varphi) - \langle B(\tau_\sigma,\varphi), N_j\rangle S_{N_j}(\tau_\sigma).
\end{aligned} \tag{2.17}$$

Understanding summation over repeated indices and letting $\{e_1, e_2, ..., e_m\}$ be a local orthonormal frame of TM along σ with $e_1 = \tau$ we have that

$$\begin{aligned}
&\langle B(\tau_\sigma,\tau_\sigma),N_j\rangle S_{N_j}(\varphi) - \langle B(\tau_\sigma,\varphi),N_j\rangle S_{N_j}(\tau_\sigma)\\
&\quad = \langle B(\tau_\sigma,\tau_\sigma),N_j\rangle \langle S_{N_j}(\varphi),e_i\rangle e_i - \langle B(\tau_\sigma,\varphi),N_j\rangle \langle S_{N_j}(\tau_\sigma),e_i\rangle e_i\\
&\quad = \langle B(\tau_\sigma,\tau_\sigma),N_j\rangle \langle B(\varphi,e_i),N_j\rangle e_i - \langle B(\tau_\sigma,\varphi),N_j\rangle \langle B(\tau_\sigma,e_i),N_j\rangle e_i\\
&\quad = \left(\langle B(\tau_\sigma,\tau_\sigma),B(\varphi,e_i)\rangle - \langle B(\tau_\sigma,\varphi),B(\tau_\sigma,e_i)\rangle\right) e_i\\
&\quad = R(\tau_\sigma,e_i,\tau_\sigma,\varphi)e_i\\
&\quad = R(e_i,\tau_\sigma,\varphi,\tau_\sigma)e_i\\
&\quad = \langle R(\varphi,\tau_\sigma)\tau_\sigma,e_i\rangle e_i,
\end{aligned}$$

where we used Gauss equation (Theorem 1.1.6) and the symmetries of the Riemann tensor. Since $R(e_i,\tau_\sigma,\tau_\sigma,\tau_\sigma)=0$, the above calculation and (2.17) imply (2.13) and (2.16). □

Remark 2.3.11. We remark that if $k\in[2,+\infty)$ and $\gamma\in W^{4,p}_{imm}(\mathbb{S}^1;M)$, then

$$\partial_s(|k|^{p-2}k) = (p-2)|k|^{p-4}\langle k,\partial_s k\rangle k + |k|^{p-2}\partial_s k \quad \in C^0(\mathbb{S}^1,\mathbb{R}^n),$$

in the classical sense. Indeed, at any point x, if $k(x)\neq 0$ then $|k|^{p-2}$ is differentiable at x and the above formula holds, while if $k(x)=0$ then $\frac{1}{h}|k(x+h)|^{p-2}k(x+h)\to|k(x)|^{p-2}\partial_x k(x)=0$ as $h\to 0$.

Proposition 2.3.12 (First variation)**.** *Let $p\in[2,+\infty)$. Let $\gamma\in W^{4,p}(\mathbb{S}^1,M)$ be a regular curve. For any $\psi\in B_\rho(0)\subset T(\gamma)^{4,p,\perp}$ it holds that*

$$\delta E(0)[\psi] = \int -\langle\nabla(|k|^{p-2}k),\nabla\psi\rangle + \left\langle \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k,\tau)\tau,\psi\right\rangle ds.$$

For any $\psi\in B_\rho(0)\subset T(\gamma)^{4,p}$ it holds that

$$\delta\mathbf{E}(0)[\psi] = \delta\mathbf{E}(0)[\psi^\perp] = \delta E(0)[\psi^\perp],$$

where $\psi^\perp := (\mathrm{id}-\gamma^\top)\psi$.

Proof. Let us consider $\psi\in B_\rho(0)\subset T(\gamma)^{4,p}$. Using Lemma 2.3.10 with γ,ψ in place of σ,φ, if $\gamma_\varepsilon(\cdot)=\Phi(\varepsilon,\cdot)$ is the variation of γ, computations show that

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_0 \mathcal{E}_p(\Phi(\varepsilon,\cdot))\\
&\quad = \int -\langle k,\psi\rangle + \frac{1}{p}|k|^p\langle\tau,\partial_s\psi\rangle\,ds + \int |k|^{p-2}\langle k,\partial_\varepsilon|_0 k_{\gamma_\varepsilon}\rangle\,ds\\
&\quad = \int -\langle k,\psi\rangle + \frac{1}{p}|k|^p\langle\tau,\partial_s\psi\rangle - |k|^p\langle\tau,\partial_s\psi\rangle + |k|^{p-2}R(k,\tau,\psi^\perp,\tau) + \langle|k|^{p-2}k,\partial_s(M^\top-\gamma^\top)\partial_s\psi\rangle\,ds\\
&\quad = \int -\langle k,\psi\rangle - \frac{1}{p'}|k|^p\langle\tau,\partial_s\psi\rangle + |k|^{p-2}R(\psi^\perp,\tau,k,\tau) - \langle\nabla(|k|^{p-2}k),\partial_s\psi\rangle\,ds.
\end{aligned}$$

Moreover $\partial_s(\psi^\perp)=\partial_s(\psi-\langle\psi,\tau\rangle\tau)=\partial_s\psi-\partial_s(\langle\psi,\tau\rangle)\tau-\langle\psi,\tau\rangle\partial_s\tau$ and then

$$\begin{aligned}
&\int -\langle\nabla(|k|^{p-2}k),\partial_s\psi\rangle\,ds\\
&\qquad = \int -\langle\nabla(|k|^{p-2}k),\partial_s(\psi^\perp)\rangle - \langle\nabla(|k|^{p-2}k),\langle\psi,\tau\rangle k\rangle\,ds\\
&\qquad = \int -\langle\nabla(|k|^{p-2}k),\nabla(\psi^\perp)\rangle\,ds + \int |k|^p\langle\partial_s\psi,\tau\rangle + |k|^p\langle\psi,k\rangle + \langle\psi,\tau\rangle|k|^{p-2}\langle k,\nabla k\rangle\,ds.
\end{aligned}$$

Using that $-\frac{1}{p'}|k|^p\langle\tau,\partial_s\psi\rangle+|k|^p\langle\partial_s\psi,\tau\rangle+\langle\psi,\tau\rangle|k|^{p-2}\langle k,\nabla k\rangle=\partial_s\left(\frac{1}{p}|k|^p\langle\tau,\psi\rangle\right)-\frac{1}{p}|k|^p\langle k,\psi\rangle$, we conclude that

$$\frac{d}{d\varepsilon}\bigg|_0\mathcal{E}_p(\Phi(\varepsilon,\cdot))=\int-\langle\nabla(|k|^{p-2}k),\nabla(\psi^\perp)\rangle+\frac{1}{p'}|k|^p\langle k,\psi^\perp\rangle-\langle k,\psi^\perp\rangle+|k|^{p-2}R(\psi^\perp,\tau,k,\tau)\,ds. \tag{2.18}$$

□

Remark 2.3.13 (Definition of the p-elastic flow)**.** We observe that from Proposition 2.3.12 the definition of the p-elastic flow on manifolds given in (2.1) follows. Indeed, in the notation of Proposition 2.3.12, if γ is smooth, formally integrating by parts in the formula for $\delta E(0)[\psi]$, we get

$$\delta\mathbf{E}(0)[\psi]=\left\langle\nabla^2(|k|^{p-2}k)+\frac{1}{p'}|k|^pk-k+R(|k|^{p-2}k,\tau)\tau,\psi\right\rangle_{L^{p'}(ds_\gamma),L^p(ds_\gamma)}.$$

Corollary 2.3.14. *Let $p\in[2,+\infty)$. Let $\gamma:\mathbb{S}^1\to M$ be a fixed smooth regular curve. For any $\varphi,\psi\in B_\rho(0)\subset T(\gamma)^{4,p}$ it holds that*

$$\begin{aligned}\delta\mathbf{E}(\varphi)[\psi]=&-\left\langle\nabla_{\gamma_\varphi}|k_{\gamma_\varphi}|^{p-2}k_{\gamma_\varphi},\nabla_{\gamma_\varphi}((\gamma_\varphi^\perp)\mathcal{T}(\psi))\right\rangle_{L^{p'}(ds_{\gamma_\varphi}),L^p(ds_{\gamma_\varphi})}+\\&+\left\langle\mathcal{T}^\star\left(\frac{1}{p'}|k_{\gamma_\varphi}|^pk_{\gamma_\varphi}-k_{\gamma_\varphi}+R(|k_{\gamma_\varphi}|^{p-2}k_{\gamma_\varphi},\tau_{\gamma_\varphi})\tau_{\gamma_\varphi}\right),\psi\right\rangle_{L^{p'}(ds_{\gamma_\varphi}),L^p(ds_{\gamma_\varphi})}\end{aligned}$$

where $\gamma_\varphi(\cdot)=\Phi(1,\cdot)$, Φ is the variation of γ given by φ, and $\mathcal{T}:T_{\gamma(x)}M\to T_{\Phi(1,x)}M$ is the function $\mathcal{T}(\psi)=d[\exp_{\gamma(x)}]_\varphi(\psi)$ and $\mathcal{T}^\star$ is its adjoint.

Proof. Let us denote by $\Phi_V^\alpha(t,x)=\sigma_V(t)$ the variation of a curve α with respect to a field $V(x)$ along α. We need to consider the curve $\Phi^\gamma_{\varphi+\varepsilon\psi}(1,\cdot)=\exp_{\gamma(x)}(\varphi+\varepsilon\psi)$. We have that

$$\frac{d}{d\varepsilon}\bigg|_0\Phi^\gamma_{\varphi+\varepsilon\psi}(1,\cdot)=d[\exp_{\gamma(x)}]_\varphi(\psi).$$

For any x denote by $\mathcal{T}:T_{\gamma(x)}M\to T_{\Phi^\gamma_\varphi(1,x)}M$ the function

$$\mathcal{T}(\psi)=d[\exp_{\gamma(x)}]_\varphi(\psi).$$

By chain rule we have that

$$\frac{d}{d\varepsilon}\bigg|_0\mathcal{E}_p(\Phi^\gamma_{\varphi+\varepsilon\psi}(1,\cdot))=\frac{d}{d\varepsilon}\bigg|_0\mathcal{E}_p\left(\Phi^{\Phi^\gamma_\varphi(1,\cdot)}_{\mathcal{T}(\psi)}(\varepsilon,\cdot)\right).$$

Roughly speaking, differentiation of the variation of γ with respect to the field $\varphi+\varepsilon\psi$ is equivalent to differentiation at the varied curve $\gamma_\varphi:=\Phi^\gamma_\varphi(1,\cdot)$ with respect to the field $\mathcal{T}(\psi)$.

Therefore if we consider $\varphi,\psi\in B_\rho(0)\subset T(\gamma)^{4,p}$, Equation (2.18) implies that

$$\begin{aligned}\frac{d}{d\varepsilon}\bigg|_0\mathcal{E}_p(\Phi^\gamma_{\varphi+\varepsilon\psi}(1,\cdot))=\int&-\langle\nabla_{\gamma_\varphi}|k_{\gamma_\varphi}|^{p-2}k_{\gamma_\varphi},\nabla_{\gamma_\varphi}(\gamma_\varphi^\perp\mathcal{T}(\psi))\rangle+\\&+\frac{1}{p'}|k_{\gamma_\varphi}|^p\langle k_{\gamma_\varphi},\gamma_\varphi^\perp\mathcal{T}(\psi)\rangle+\\&-\langle k_{\gamma_\varphi},\gamma_\varphi^\perp\mathcal{T}(\psi)\rangle+|k_{\gamma_\varphi}|^{p-2}R(\gamma_\varphi^\perp\mathcal{T}(\psi),\tau_{\gamma_\varphi},k_{\gamma_\varphi},\tau_{\gamma_\varphi})\,ds_{\gamma_\varphi}.\end{aligned}$$

□

Now we want to calculate the second variation of $\mathcal{E}_p$. More precisely, as in the case of Section 2.2, for a smooth immersion $\gamma : \mathbb{S}^1 \to M$ we consider normal fields $\varphi, \psi \in T(\gamma)^{4,p,\perp}$ along γ and we compute

$$\mathcal{L} := \delta^2 E(0) : T(\gamma)^{4,p,\perp} \to (T(\gamma)^{4,p,\perp})^\star,$$

that is

$$\mathcal{L}(\varphi)[\psi] = \frac{d}{d\varepsilon}\bigg|_0 \frac{d}{d\eta}\bigg|_0 E(\varepsilon\varphi + \eta\psi) = \frac{d}{d\varepsilon}\bigg|_0 \frac{d}{d\eta}\bigg|_0 \mathcal{E}_p(\Phi^\gamma_{\varepsilon\varphi+\eta\psi}(1,\cdot)),$$

where $\Phi^\gamma_{\varepsilon\varphi+\eta\psi}(1,\cdot)$ is the variation of γ via the field $\varepsilon\varphi + \eta\psi$. Observe that $\mathcal{L}(\varphi)[\psi] = \mathcal{L}(\psi)[\varphi]$.

We need a technical tool first.

Lemma 2.3.15. *Let $\gamma : \mathbb{S}^1 \to M$ be a fixed smooth immersion and let ρ be given by Lemma 2.3.8. Let $\gamma_\varepsilon(\cdot) = \Phi(\varepsilon, \cdot)$ with Φ the variation of γ with variation field $\varphi \in T(\gamma)^{4,p,\perp}$.*

If $V(\varepsilon) \in T(\gamma_\varepsilon)^{1,p}$ is a field along γ_ε differentiable with respect to ε with $\partial_\varepsilon V(\varepsilon) \in T(\gamma_\varepsilon)^{1,p}$, we have that

$$\frac{d}{d\varepsilon}\bigg|_0 \partial_{s_{\gamma_\varepsilon}} V(\varepsilon) = \langle k_\gamma, \varphi\rangle \partial_{s_\gamma} V(0) + \partial_{s_\gamma} \frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon). \tag{2.19}$$

If also $V(\varepsilon) \in TM \cap (T\gamma_\varepsilon)^\perp$ for any ε, then

$$\begin{aligned}(M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 (\nabla_{\gamma_\varepsilon} V(\varepsilon)) = {}& (M^\top - \gamma^\top)\partial_{s_\gamma}((M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon)) + \\ & - \langle V(0), \nabla_\gamma\varphi\rangle k_\gamma + \langle k_\gamma, \varphi\rangle \nabla_\gamma V(0) + \langle V(0), k_\gamma\rangle \nabla_\gamma \varphi + \\ & + (\mathrm{id} - \gamma^\top) R(\varphi, \tau_\gamma) V(0).\end{aligned} \tag{2.20}$$

Proof. Equation (2.19) follows by a direct calculation. In order to derive (2.20) let $\{N_j\}$ be a local orthonormal frame of $(TM)^\perp$. Understanding summation over j it holds that

$$\begin{aligned}(M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 (\nabla_{\gamma_\varepsilon} V(\varepsilon)) &= \langle k_\gamma, \varphi\rangle \nabla_\gamma V(0) + (M^\top - \gamma^\top)\partial_{s_\gamma}\frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon) - \langle \partial_{s_\gamma} V(0), \tau_\gamma\rangle \nabla_\gamma\varphi + \\ &\quad - \langle \partial_{s_\gamma} V(0), N_j\rangle (M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 N_j \circ \gamma_\varepsilon \\ &= (M^\top - \gamma^\top)\partial_{s_\gamma}\frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon) + \langle k_\gamma, \varphi\rangle \nabla_\gamma V(0) + \langle V(0), k_\gamma\rangle \nabla_\gamma\varphi + \\ &\quad + \langle V(0), S_{N_j}(\tau_\gamma)\rangle (S_{N_j}(\varphi) - \langle S_{N_j}(\varphi), \tau_\gamma\rangle \tau_\gamma).\end{aligned} \tag{2.21}$$

Moreover for any field $W \in T(\gamma)^{1,p}$ we have that

$$(M^\top - \gamma^\top)\partial_{s_\gamma}((M^\top - \gamma^\top)W) = (M^\top - \gamma^\top)\partial_{s_\gamma} W - \langle W, \tau_\gamma\rangle k_\gamma + \langle W, N_j\rangle (S_{N_j}(\tau_\gamma) - \langle S_{N_j}(\tau_\gamma), \tau_\gamma\rangle \tau_\gamma).$$

Using $W = \frac{d}{d\varepsilon}\big|_0 V(\varepsilon)$ we deduce

$$\begin{aligned}(M^\top - \gamma^\top)\partial_{s_\gamma}\frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon) &= (M^\top - \gamma^\top)\partial_{s_\gamma}((M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon)) + \left\langle \frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon), \tau_\gamma \right\rangle k_\gamma + \\ &\quad - \left\langle \frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon), N_j \right\rangle (S_{N_j}(\tau_\gamma) - \langle S_{N_j}(\tau_\gamma), \tau_\gamma\rangle \tau_\gamma) \\ &= (M^\top - \gamma^\top)\partial_{s_\gamma}((M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 V(\varepsilon)) - \langle V(0), \nabla_\gamma\varphi\rangle k_\gamma + \\ &\quad - \langle V(0), S_{N_j}(\varphi)\rangle (S_{N_j}(\tau_\gamma) - \langle S_{N_j}(\tau_\gamma), \tau_\gamma\rangle \tau_\gamma).\end{aligned}$$

Moreover

$$
\begin{aligned}
&\langle V(0), S_{N_j}(\tau_\gamma)\rangle (S_{N_j}(\varphi) - \langle S_{N_j}(\varphi), \tau_\gamma\rangle \tau_\gamma) - \langle V(0), S_{N_j}(\varphi)\rangle (S_{N_j}(\tau_\gamma) - \langle S_{N_j}(\tau_\gamma), \tau_\gamma\rangle \tau_\gamma) \\
&\qquad = \langle B(V(0), \tau_\gamma), N_j\rangle S_{N_j}(\varphi) - \langle B(V(0), \varphi), N_j\rangle S_{N_j}(\tau_\gamma) + \\
&\qquad\quad + (\langle B(V(0), \varphi), N_j\rangle \langle B(\tau_\gamma, \tau_\gamma), N_j\rangle - \langle B(V(0), \tau_\gamma), N_j\rangle \langle B(\varphi, \tau_\gamma), N_j\rangle)\, \tau_\gamma \\
&\qquad = (\langle B(V(0), \tau_\gamma), B(\varphi, e_i)\rangle - \langle B(V(0), \varphi), B(\tau_\gamma, e_i)\rangle)\, e_i + \\
&\qquad\quad + (\langle B(V(0), \varphi), B(\tau_\gamma, \tau_\gamma)\rangle - \langle B(V(0), \tau_\gamma), B(\varphi, \tau_\gamma)\rangle)\, \tau_\gamma \\
&\qquad = R(V(0), e_i, \tau_\gamma, \varphi) e_i + R(V(0), \tau_\gamma, \varphi, \tau_\gamma)\tau_\gamma \\
&\qquad = -R(\tau_\gamma, \varphi)V(0) + \langle R(\tau_\gamma, \varphi)V(0), \tau_\gamma\rangle \tau_\gamma \\
&\qquad = (\mathrm{id} - \gamma^\top) R(\varphi, \tau_\gamma) V(0),
\end{aligned}
$$

where $\{e_1, ..., e_m\} = \{\tau_\gamma, e_2, ..., e_m\}$ is a local orthonormal frame of TM and we used Gauss equation (Theorem 1.1.6), and summation over repeated indices was understood. Inserting the previous identities in (2.21) yields (2.20). $\square$

In the following proposition we calculate the second variation $\mathcal{L}(\varphi)[\psi]$ of $\mathcal{E}_p$ with respect to normal variation fields φ, ψ along the given curve γ. In the statement we isolate an integral depending on second order derivatives is ψ, a second integral depending at most on first order derivatives in ψ, and a third integral in which the first variation of the energy appears. The complete calculation is explicit in Equation (2.31), and we shall also use such complete expression (2.31) later on.

Proposition 2.3.16 (Second variation)**.** *Let $p \in [2, +\infty)$. Let $\gamma : \mathbb{S}^1 \to M$ be a fixed smooth immersion and let ρ be given by Lemma 2.3.8. For any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$ it holds that*

$$
\begin{aligned}
\mathcal{L}(\varphi)[\psi] = \int &\left\langle |k|^{p-2}\nabla^2\varphi, \nabla^2\psi \right\rangle + (p-2)|k|^{p-4}\langle k, \nabla^2\varphi\rangle\langle k, \nabla^2\psi\rangle + \\
&+ \left\langle |k|^{p-2}R(\varphi, \tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + |k|^{p-2}\langle k, \varphi\rangle k, \nabla^2\psi \right\rangle ds + \\
+ \int A(\varphi, \psi)\, ds - &\int \left(\left\langle |k|^{p-2}k, \nabla^2\psi \right\rangle + \left\langle \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k, \tau)\tau, \psi \right\rangle \right) \langle k, \varphi\rangle\, ds,
\end{aligned}
$$

where $A(\cdot,\cdot)$ is bilinear with $A(\varphi,\psi)$ depending at most on first order derivatives in ψ, and, more precisely, the precise expression for $\mathcal{L}(\varphi)[\psi]$ is given by (2.31).

Proof. Denoting by $\gamma_\varepsilon(\cdot) = \Phi(\varepsilon, \cdot)$, by Φ the variation of γ with variation field φ, and by $\mathcal{T} : T_{\gamma(x)}M \to T_{\gamma_\varepsilon(x)}M$ the map $\mathcal{T}\psi = d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\psi)$, we have that

$$
\begin{aligned}
\mathcal{L}(\varphi)[\psi] = \frac{d}{d\varepsilon}\bigg|_0 \int &\left\langle |k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}, \nabla^2_{\gamma_\varepsilon}(\gamma_\varepsilon^\perp(\mathcal{T}\psi)) \right\rangle + \\
&+ \left\langle \frac{1}{p'}|k_{\gamma_\varepsilon}|^p k_{\gamma_\varepsilon} - k_{\gamma_\varepsilon} + R(|k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}, \tau_{\gamma_\varepsilon})\tau_{\gamma_\varepsilon}, \mathcal{T}\psi \right\rangle ds_{\gamma_\varepsilon}.
\end{aligned}
$$

Now we calculate term by term the above identity. Using (2.20) twice we have that

$$
\begin{aligned}
(M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 \left(\nabla^2_{\gamma_\varepsilon}(\gamma_\varepsilon^\perp(\mathcal{T}\psi))\right) &= \nabla\left((M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 \nabla_{\gamma_\varepsilon}(\gamma_\varepsilon^\perp(\mathcal{T}\psi))\right)+\\
&\quad-\langle\nabla\psi,\nabla\varphi\rangle k+\langle k,\varphi\rangle\nabla^2\psi+\langle\nabla\psi,k\rangle\nabla\varphi+\gamma^\perp R(\varphi,\tau)\nabla\psi\\
&=\nabla\left(\nabla(M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0(\gamma_\varepsilon^\perp(\mathcal{T}\psi))\right)+\\
&\quad+\nabla\left[-\langle\psi,\nabla\varphi\rangle k+\langle k,\varphi\rangle\nabla\psi+\langle\psi,k\rangle\nabla\varphi+\gamma^\perp R(\varphi,\tau)\psi\right]+\\
&\quad-\langle\nabla\psi,\nabla\varphi\rangle k+\langle k,\varphi\rangle\nabla^2\psi+\langle\nabla\psi,k\rangle\nabla\varphi+\gamma^\perp R(\varphi,\tau)\nabla\psi.
\end{aligned}
$$

We can compute

$$
\begin{aligned}
(M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0(\gamma_\varepsilon^\perp(\mathcal{T}\psi)) &= (M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0\left(d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\psi)-\langle d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\psi),\tau_{\gamma_\varepsilon}\rangle\tau_{\gamma_\varepsilon}\right)\\
&=(M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\psi),
\end{aligned}
\tag{2.22}
$$

where we used that $\langle d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\psi),\tau_{\gamma_\varepsilon}\rangle\big|_0=\langle d[\exp_{\gamma(x)}]_0(\psi),\tau_\gamma\rangle=\langle\psi,\tau\rangle=0$. Now the field

$$
\varepsilon\mapsto J(\varepsilon)=d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\varepsilon\psi)
$$

is a Jacobi field along the geodesic σ_φ such that $\sigma_\varphi(0)=\gamma(x)$, $\sigma'_\varphi(0)=\varphi$, $J(0)=0$, and $J'(0)=\psi$ (see [Car92, Chapter 5, Corollary 2.5]). Then

$$
\frac{d}{d\varepsilon}\bigg|_0 d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\psi)=\frac{d}{d\varepsilon}\bigg|_0\left(\frac{1}{\varepsilon}d[\exp_{\gamma(x)}]_{\varepsilon\varphi}(\varepsilon\psi)\right)=\frac{d}{d\varepsilon}\bigg|_0\left(\frac{1}{\varepsilon}J(\varepsilon)\right). \tag{2.23}
$$

We claim that $\frac{d}{d\varepsilon}\Big|_0\left(\frac{1}{\varepsilon}J(\varepsilon)\right)=0$ for any x. In fact let $\{E_i(\cdot)\}$ be an orthonormal parallel frame along σ_φ, and write $J(\varepsilon)=J^i(\varepsilon)E_i(\varepsilon)$. The Jacobi equation (see [Car92, Chapter 5, Definition 2.1]) for J then reads

$$
(J^i)''(\varepsilon)E_i(\varepsilon)+R(J^i(\varepsilon)E_i(\varepsilon),\sigma'_\varphi(\varepsilon))\sigma'_\varphi(\varepsilon)=0.
$$

Therefore since $J^i(\cdot)$ is of class C^2 with $J^i(0)=0$ we conclude that

$$
\frac{d}{d\varepsilon}\bigg|_0\left(\frac{1}{\varepsilon}J(\varepsilon)\right)=\frac{d}{d\varepsilon}\bigg|_0\left(\frac{J^i(\varepsilon)}{\varepsilon}\right)E_i(0)=\frac{1}{2}(J^i)''(0)E_i(0)=-R(J(0),\varphi(x))\varphi(x)=0, \tag{2.24}
$$

and thus

$$
(M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0(\gamma_\varepsilon^\perp(\mathcal{T}\psi))=0. \tag{2.25}
$$

Eventually we deduce that

$$
\begin{aligned}
&(M^\top-\gamma^\top)\frac{d}{d\varepsilon}\bigg|_0\left(\nabla^2_{\gamma_\varepsilon}(\gamma_\varepsilon^\perp(\mathcal{T}\psi))\right)\\
&\qquad=\nabla\left[-\langle\psi,\nabla\varphi\rangle k+\langle k,\varphi\rangle\nabla\psi+\langle\psi,k\rangle\nabla\varphi+\gamma^\perp R(\varphi,\tau)\psi\right]+\\
&\qquad\quad-\langle\nabla\psi,\nabla\varphi\rangle k+\langle k,\varphi\rangle\nabla^2\psi+\langle\nabla\psi,k\rangle\nabla\varphi+\gamma^\perp R(\varphi,\tau)\nabla\psi.
\end{aligned}
\tag{2.26}
$$

On the other hand we have that

$$
\begin{aligned}
(M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 &\left(|k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}\right) \\
&= (M^\top - \gamma^\top)\Big((p-2)|k|^{p-4}\langle k, \nabla^2\varphi + \langle\varphi,k\rangle k + \\
&\quad + R(\varphi,\tau)\tau\rangle k + |k|^{p-2}(\nabla^2\varphi + \langle\varphi,k\rangle k + R(\varphi,\tau)\tau)\Big) \\
&= |k|^{p-2}\nabla^2\varphi + |k|^{p-2}\langle\varphi,k\rangle k + \\
&\quad + |k|^{p-2}\gamma^\perp R(\varphi,\tau)\tau + (p-2)\Big(|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + |k|^{p-2}\langle\varphi,k\rangle k + \\
&\quad + |k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k\Big) \\
&= |k|^{p-2}\nabla^2\varphi + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + \\
&\quad + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + \\
&\quad + (p-1)|k|^{p-2}\langle\varphi,k\rangle k,
\end{aligned}
\tag{2.27}
$$

where we used that $R(\varphi,\tau)\tau \in T\gamma^\perp$. Similarly

$$
\begin{aligned}
(M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0 (|k_{\gamma_\varepsilon}|^p k_{\gamma_\varepsilon}) &= p|k|^{p-2}\langle\nabla^2\varphi, k\rangle k + p|k|^p\langle\varphi,k\rangle k + \\
&\quad + p|k|^{p-2}\langle R(\varphi,\tau)\tau, k\rangle k + \\
&\quad + |k|^p\left(\nabla^2\varphi + \langle\varphi,k\rangle k + \gamma^\perp R(\varphi,\tau)\tau\right) \\
&= |k|^p\nabla^2\varphi + p|k|^{p-2}\langle\nabla^2\varphi,k\rangle k + \\
&\quad + p|k|^{p-2}\langle R(\varphi,\tau)\tau,k\rangle k + |k|^p R(\varphi,\tau)\tau + \\
&\quad + (p+1)|k|^p\langle\varphi,k\rangle k.
\end{aligned}
\tag{2.28}
$$

Also, we already know that

$$
(M^\top - \gamma^\top)\frac{d}{d\varepsilon}\bigg|_0(-k_{\gamma_\varepsilon}) = -\left(\nabla^2\varphi + \langle\varphi,k\rangle k + R(\varphi,\tau)\tau\right). \tag{2.29}
$$

Finally, since

$$
\langle R(|k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}, \tau_{\gamma_\varepsilon})\tau_{\gamma_\varepsilon}, \gamma_\varepsilon^\perp\mathcal{T}\psi\rangle = \langle R(\gamma_\varepsilon^\perp\mathcal{T}\psi, \tau_{\gamma_\varepsilon})\tau_{\gamma_\varepsilon}, |k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}\rangle,
$$

denoting $\mathcal{R}$ the tensor $\mathcal{R}(X,Y,Z) = R(X,Y)Z$ we have that

$$
\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_0 &\langle R(|k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}, \tau_{\gamma_\varepsilon})\tau_{\gamma_\varepsilon}, \gamma_\varepsilon^\perp\mathcal{T}\psi\rangle \\
&= \langle(D_\varphi\mathcal{R})(\psi,\tau,\tau) + \mathcal{R}(\partial_\varepsilon|_0(\gamma_\varepsilon^\perp\mathcal{T}\psi),\tau,\tau) + \\
&\quad + \mathcal{R}(\psi,\nabla\varphi,\tau) + \mathcal{R}(\psi,\tau,\nabla\varphi), |k|^{p-2}k\rangle + \\
&\quad + \langle R(\psi,\tau)\tau, (M^\top - \gamma^\top)\partial_\varepsilon|_0(|k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon})\rangle.
\end{aligned}
$$

Using (2.22), (2.23), and (2.24) we see that

$$
\frac{d}{d\varepsilon}\bigg|_0 \gamma_\varepsilon^\perp\mathcal{T}\psi = -\langle\psi,\nabla\varphi\rangle\tau.
$$

Therefore

$$
\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_0 \langle R(|k_{\gamma_\varepsilon}|^{p-2}k_{\gamma_\varepsilon}, \tau_{\gamma_\varepsilon})\tau_{\gamma_\varepsilon}, \gamma_\varepsilon^\perp \mathcal{T}\psi\rangle = {} & \langle (D_\varphi \mathcal{R})(\psi,\tau,\tau) + R(\psi,\nabla\varphi)\tau + R(\psi,\tau)\nabla\varphi, |k|^{p-2}k\rangle + \\
& + \big\langle R(\psi,\tau)\tau\,,\, |k|^{p-2}\nabla^2\varphi + (p-2)|k|^{p-4}\langle k, \nabla^2\varphi\rangle k + \\
& + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + \\
& + (p-1)|k|^{p-2}\langle \varphi, k\rangle k\big\rangle.
\end{aligned}
\tag{2.30}
$$

Putting together (2.26), (2.27), (2.28), (2.29), and (2.30) we conclude that

$$
\mathcal{L}(\varphi)[\psi] = I_1 + I_2 + I_3 + I_4 + I_5,
$$

where

$$
\begin{aligned}
I_1 = \int \bigg\langle & |k|^{p-2}k, \nabla\Big[-\langle\psi,\nabla\varphi\rangle k + \langle k,\varphi\rangle\nabla\psi + \langle\psi,k\rangle\nabla\varphi + \gamma^\perp R(\varphi,\tau)\psi\Big] + \\
& - \langle\nabla\psi,\nabla\varphi\rangle k + \langle k,\varphi\rangle\nabla^2\psi + \\
& + \langle\nabla\psi,k\rangle\nabla\varphi + \gamma^\perp R(\varphi,\tau)\nabla\psi\bigg\rangle\, ds_\gamma,
\end{aligned}
$$

$$
\begin{aligned}
I_2 = \int \bigg\langle & |k|^{p-2}\nabla^2\varphi + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + \\
& + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + \\
& + (p-1)|k|^{p-2}\langle\varphi,k\rangle k, \nabla^2\psi\bigg\rangle\, ds_\gamma,
\end{aligned}
$$

$$
\begin{aligned}
I_3 = \int \bigg\langle & |k|^p\nabla^2\varphi + p|k|^{p-2}\langle\nabla^2\varphi, k\rangle k + \\
& + p|k|^{p-2}\langle R(\varphi,\tau)\tau, k\rangle k + |k|^p R(\varphi,\tau)\tau + \\
& + (p+1)|k|^p\langle\varphi,k\rangle k - \big(\nabla^2\varphi + \langle\varphi,k\rangle k + R(\varphi,\tau)\tau\big)\,,\psi\bigg\rangle\, ds_\gamma,
\end{aligned}
$$

$$
\begin{aligned}
I_4 = \int & \big\langle (D_\varphi\mathcal{R})(\psi,\tau,\tau) + R(\psi,\nabla\varphi)\tau + R(\psi,\tau)\nabla\varphi, |k|^{p-2}k\big\rangle + \\
& + \bigg\langle R(\psi,\tau)\tau, |k|^{p-2}\nabla^2\varphi + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + \\
& + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + \\
& + (p-1)|k|^{p-2}\langle\varphi,k\rangle k\bigg\rangle\, ds_\gamma,
\end{aligned}
$$

$$
I_5 = -\int \bigg(\big\langle |k|^{p-2}k, \nabla^2\psi\big\rangle + \bigg\langle \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k,\tau)\tau, \psi\bigg\rangle\bigg)\, \langle k,\varphi\rangle\, ds_\gamma.
$$

Integrating by parts and rearranging the terms we end up with

$$
\begin{aligned}
\mathcal{L}(\varphi)[\psi] = & \int \langle k,\varphi\rangle\langle |k|^{p-2}k,\nabla^2\psi\rangle + \Big\langle |k|^{p-2}\nabla^2\varphi + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + \\
& \quad + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k, \nabla^2\psi \Big\rangle\, ds_\gamma + \\
& + \int \big\langle -\nabla|k|^{p-2}k, \langle k,\varphi\rangle\nabla\psi\big\rangle - (p-1)\big\langle \nabla(\langle\varphi,k\rangle|k|^{p-2}k),\nabla\psi\big\rangle\, ds_\gamma + \\
& + \int \big\langle -\nabla|k|^{p-2}k, R(\varphi,\tau)\psi\big\rangle + \langle |k|^{p-2}k, R(\varphi,\tau)\nabla\psi\rangle + \\
& \quad + \big\langle (D_\varphi\mathcal{R})(\psi,\tau,\tau) + R(\psi,\nabla\varphi)\tau + R(\psi,\tau)\nabla\varphi, |k|^{p-2}k\big\rangle + \Big\langle R(\psi,\tau)\tau, |k|^{p-2}\nabla^2\varphi + \\
& \quad + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + \\
& \quad + (p-1)|k|^{p-2}\langle\varphi,k\rangle k \Big\rangle\, ds_\gamma + \\
& + \int \Big\langle -\nabla|k|^{p-2}k, -\langle\psi,\nabla\varphi\rangle k + \langle\psi,k\rangle\nabla\varphi\Big\rangle + \Big\langle \nabla(|k|^p\nabla\varphi) + \\
& \quad - \nabla(\langle |k|^{p-2}k,\nabla\varphi\rangle k) + |k|^p\nabla^2\varphi + p|k|^{p-2}\langle\nabla^2\varphi,k\rangle k + \\
& \quad + p|k|^{p-2}\langle R(\varphi,\tau)\tau,k\rangle k + |k|^p R(\varphi,\tau)\tau + (p+1)|k|^p\langle\varphi,k\rangle k + \\
& \quad - \big(\nabla^2\varphi + \langle\varphi,k\rangle k + R(\varphi,\tau)\tau\big), \psi\Big\rangle\, ds_\gamma + \\
& - \int \Big(\big\langle |k|^{p-2}k,\nabla^2\psi\big\rangle + \Big\langle \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k,\tau)\tau, \psi\Big\rangle\Big)\langle k,\varphi\rangle\, ds_\gamma,
\end{aligned}
\tag{2.31}
$$

that is

$$
\begin{aligned}
\mathcal{L}(\varphi)[\psi] = & \int \langle k,\varphi\rangle\langle |k|^{p-2}k,\nabla^2\psi\rangle + \Big\langle |k|^{p-2}\nabla^2\varphi + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + \\
& \quad + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k, \nabla^2\psi \Big\rangle\, ds_\gamma + \\
& + \int \big\langle -\nabla|k|^{p-2}k, \langle k,\varphi\rangle\nabla\psi\big\rangle - (p-1)\big\langle \nabla(\langle\varphi,k\rangle|k|^{p-2}k),\nabla\psi\big\rangle\, ds_\gamma + \\
& + \int \big\langle -\nabla|k|^{p-2}k, R(\varphi,\tau)\psi\big\rangle + \langle |k|^{p-2}k, R(\varphi,\tau)\nabla\psi\rangle + \\
& \quad + \big\langle (D_\varphi\mathcal{R})(\psi,\tau,\tau) + R(\psi,\nabla\varphi)\tau + R(\psi,\tau)\nabla\varphi, |k|^{p-2}k\big\rangle + \Big\langle R(\psi,\tau)\tau, |k|^{p-2}\nabla^2\varphi + \\
& \quad + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle k + |k|^{p-2}R(\varphi,\tau)\tau + (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + \\
& \quad + (p-1)|k|^{p-2}\langle\varphi,k\rangle k \Big\rangle\, ds_\gamma + \\
& + \int \langle\Omega(\varphi),\psi\rangle\, ds_\gamma + \\
& - \int \Big(\big\langle |k|^{p-2}k,\nabla^2\psi\big\rangle + \Big\langle \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k,\tau)\tau, \psi\Big\rangle\Big)\langle k,\varphi\rangle\, ds_\gamma,
\end{aligned}
\tag{2.32}
$$

where $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p,\perp}$ is a compact operator, and the thesis follows. □

We remark that we kept track of every term in (2.31) and in (2.32) as we will need to recall the precise expression of the second variation a couple of times in the sequel.

2.3.2 Critical points

As we are interested in the properties of the variations evaluated at a critical point of $\mathcal{E}_p$, we now consider the variations at such a curve. Let $p \in [2, +\infty)$, $\gamma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$ be fixed and let ρ be given by Lemma 2.3.8. Recall that by Proposition 2.3.12 the curve γ is a critical point if and only if for any $\psi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$ it holds that

$$\int -\langle \nabla(|k|^{p-2}k), \nabla\psi\rangle + \left\langle \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k, \tau)\tau, \psi \right\rangle ds = 0.$$

Lemma 2.3.17. *Let $p \in [2, +\infty)$. Let $\gamma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$ be a critical point. Then $|k|^{p-2}k \in C^{3,\alpha}(\mathbb{S}^1, M)$ for some $\alpha \in (0, 1)$, and either γ is a smooth geodesic and $k \equiv 0$ or the set $\{x \,:\, k(x) = 0\}$ is finite.*

Proof. Let $V = |k|^{p-2}k$. By Remark 2.3.11 we know that $V \in T(\gamma)^{1,\infty,\perp}$. Moreover $\langle D_s V, \tau\rangle = -\langle V, k\rangle$ and $\nabla V = D_s V + \langle V, k\rangle\tau$, and then V solves

$$\int \langle D_s V, D_s\psi\rangle\, ds = \int \left\langle \frac{1}{p'}|k|^p k - k + |k|^p k + R(V, \tau)\tau, \psi \right\rangle ds,$$

for any $\psi \in T(\gamma)^{1,p,\perp}$ for ρ small enough. In particular the weak derivative $D_s(D_s V)$ exists in L^∞, and $V \in T(\gamma)^{2,\infty,\perp} \subset C^{1,\alpha}$ for $\alpha \in (0, 1)$ that may change from line to line. By assumption also $k \in C^{1,\alpha}$ and thus $D^2_s V \in C^{1,\alpha}$, that implies $V \in C^{3,\alpha}$.

Now if at some x_0 it holds that $V(x_0) = 0$ and $D_s(V)(x_0) = 0$, since V now solves

$$D^2_s V = -\frac{1}{p'}|k|^p k + k - |k|^p k - R(V, \tau)\tau$$

pointwise in the classical sense, by existence and uniqueness we would get that $V(x) = 0$ and thus $k(x) = 0$ in a neighborhood of x_0. Iterating the argument this would imply that $k \equiv 0$. It follows that if $k \not\equiv 0$ then the set $\{x \,:\, k(x) = 0\}$ has to be finite, for otherwise by compactness and since $V \in C^{3,\alpha}$ this would imply the existence of a point x_0 with $V(x_0) = 0$ and $D_s(V)(x_0) = 0$. □

Proposition 2.3.18 (Variations at critical points)**.** *Let $p \in [2, +\infty)$. Let $\gamma \in W^{4,p}_{imm}(\mathbb{S}^1, M)$ be a critical point and let ρ be given by Lemma 2.3.8. Then*

$$\delta E(0)[\psi] = \left\langle \nabla^2(|k|^{p-2}k) + \frac{1}{p'}|k|^p k - k + R(|k|^{p-2}k, \tau)\tau, \psi \right\rangle_{L^{p'}(ds), L^p(ds)},$$

for any $\psi \in B_\rho(0) \subset T(\gamma)^{4,p}$, and

$$\begin{aligned}\mathcal{L}(\varphi)[\psi] = \int & |k|^{p-2}\langle \nabla^2\varphi, \nabla^2\psi\rangle + (p-2)|k|^{p-4}\langle k, \nabla^2\varphi\rangle\langle k, \nabla^2\psi\rangle + \\ & + \left\langle (p-2)|k|^{p-4}\langle k, R(\varphi, \tau)\tau\rangle k + |k|^{p-2}R(\varphi, \tau)\tau, \nabla^2\psi \right\rangle ds + \int \langle \Omega(\varphi), \psi\rangle\, ds,\end{aligned} \tag{2.33}$$

for any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$, where $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p',\perp}$ is compact.

Proof. We need to prove (2.33). By Lemma 2.3.17 and by (2.32), integration by parts yields

$$\begin{aligned}
\mathcal{L}(\varphi)[\psi] = \int & |k|^{p-2}\langle\nabla^2\varphi,\nabla^2\psi\rangle + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle\langle k,\nabla^2\psi\rangle + \\
& + \left\langle (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + |k|^{p-2}R(\varphi,\tau)\tau, \nabla^2\psi \right\rangle \, ds + \\
+ \int & \langle (D_\varphi\mathcal{R})(\psi,\tau,\tau), |k|^{p-2}k\rangle + \Big\langle R(\psi,\tau)\tau, |k|^{p-2}\nabla^2\varphi + \\
& + (p-2)|k_\gamma|^{p-4}\langle k_\gamma,\nabla^2\varphi\rangle k_\gamma + |k_\gamma|^{p-2}R(\varphi,\tau_\gamma)\tau_\gamma + (p-2)|k_\gamma|^{p-4}\langle k_\gamma, R(\varphi,\tau_\gamma)\tau_\gamma\rangle k_\gamma \Big\rangle \, ds + \\
+ \int & \langle \Omega(\varphi),\psi\rangle \, ds,
\end{aligned}$$

where by Lemma 2.3.17 we could use that

$$R(\nabla|k|^{p-2}k,\psi,\varphi,\tau) = -R(\psi,\nabla|k|^{p-2}k,\varphi,\tau),$$

$$\int -R(\nabla\psi,|k|^{p-2}k,\varphi,\tau) = \int \langle\psi,\nabla(R(\varphi,\tau)|k|^{p-2}k)\rangle,$$

$$\langle R(\psi,\tau)\tau,\langle\varphi,k\rangle|k|^{p-2}k\rangle = \langle\psi, R(\langle\varphi,k\rangle|k|^{p-2}k,\tau)\tau\rangle.$$

Moreover $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p',\perp}$ is compact. For a given local reference frame $\{\partial_j\}$ in M we can also write

$$(D_\varphi\mathcal{R})(\psi,\tau,\tau) = \varphi^m\psi^i\tau^j\tau^k\left(\partial_m R^l_{ijk} + \Gamma^\alpha_{mi}R^l_{\alpha jk} + \Gamma^\beta_{mi}R^l_{i\beta k} + \Gamma^\gamma_{mi}R^l_{ij\gamma}\right)\partial_l,$$

$$R(\psi,\tau)\tau = \psi^i\tau^j\tau^k R^l_{ijk},$$

where $\varphi = \varphi^m\partial_m$, $\psi = \psi^i\partial_i$, $\tau = \tau^a\partial_a$, $R(\partial_i,\partial_j)\partial_k = R^l_{ijk}\partial_l$, and $\{\Gamma^k_{ij}\}$ are the Christoffel symbols of M. This means that

$$\begin{aligned}
\mathcal{L}(\varphi)[\psi] = \int & |k|^{p-2}\langle\nabla^2\varphi,\nabla^2\psi\rangle + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle\langle k,\nabla^2\psi\rangle + \\
& + \left\langle (p-2)|k|^{p-4}\langle k, R(\varphi,\tau)\tau\rangle k + |k|^{p-2}R(\varphi,\tau)\tau, \nabla^2\psi \right\rangle \, ds + \int \langle\Omega(\varphi),\psi\rangle\, ds,
\end{aligned}$$

where $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p',\perp}$ is compact. ◻

We conclude with the following two observations about the regularity of critical points.

Remark 2.3.19. Let $p = 2$. Then critical points are smooth up to reparametrization with respect to constant speed. Indeed Lemma 2.3.17 implies that a constant speed critical point γ verifies that $k \in C^{3,\alpha}$, but then a bootstrap argument on the equation

$$D_s^2 k = -\frac{1}{2}|k|^2 k + k - |k|^2 - R(k,\tau)\tau,$$

gives that $k \in C^\infty$ and thus $\gamma \in C^\infty$.

Remark 2.3.20. Let $p > 2$. If γ is a constant speed critical point of $\mathcal{E}_p$ for some $p > 2$ and if k never vanishes, then γ is smooth. Indeed Lemma 2.3.17 implies that a constant speed critical point γ verifies that $|k|^{p-2}k \in C^{3,\alpha}$, and thus the equation

$$D_s^2(|k|^{p-2}k) = -\frac{1}{p'}|k|^p k + k - |k|^p - R(|k|^{p-2}k,\tau)\tau,$$

is classically satisfied. By a bootstrap argument we get that $|k|^{p-2}k$ is smooth, and then $|k|^p \in C^1$. Hence bootstrap on the equality

$$k = D_s^2(|k|^{p-2}k) + \frac{1}{p'}|k|^p k + |k|^p + R(|k|^{p-2}k, \tau)\tau,$$

implies that k is smooth, and then so is γ.

2.3.3 Analysis of the second variation and Łojasiewicz–Simon inequality

In the following we study the properties of the variations of $\mathcal{E}_p$, leading to the application of Corollary 2.2.7 and then to the proof of the convergence of the gradient flows. We shall distinguish between the cases $p = 2$ and $p > 2$, indeed, as also Proposition 2.3.18 suggests, we will see that the properties of the second variation at a curve γ depend on the zeros of the curvature of γ if $p > 2$, while for $p = 2$ the scenario is more regular.

For the convenience of the reader, let us start by recollecting the formulas for first and second variations at critical points under the form we will use them. We will always assume without loss of generality that critical points are parametrized with constant speed.

Proposition 2.3.21. *Let $p = 2$. Let $\gamma : \mathbb{S}^1 \to M$ be a fixed smooth immersion and let ρ be given by Lemma 2.3.8.*

For any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,2,\perp}$ it holds that

$$\delta E(\varphi)[\psi] = \left\langle \gamma^\perp \mathcal{T}^\star \left(\nabla^2_{\gamma_\varphi} k_{\gamma_\varphi} + \frac{1}{2}|k_{\gamma_\varphi}|^2 k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right), \psi \right\rangle_{L^2(ds_{\gamma_\varphi}), L^2(ds_{\gamma_\varphi})} \tag{2.34}$$

where $\gamma_\varphi(\cdot) = \Phi(1, \cdot)$, Φ is the variation of γ given by φ, and $\mathcal{T} : T_{\gamma(x)}M \to T_{\Phi(1,x)}M$ is the function $\mathcal{T}(\psi) = d[\exp_{\gamma(x)}]_\varphi(\psi)$ and $\mathcal{T}^\star$ is its adjoint.

If γ is a critical point, then

$$\mathcal{L}(\varphi)[\psi] = \int \left\langle \nabla^2 \varphi, \nabla^2 \psi \right\rangle \, ds + \int \langle \Omega(\varphi), \psi \rangle \, ds, \tag{2.35}$$

for any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,2,\perp}$, where $\Omega : T(\gamma)^{4,2,\perp} \to T(\gamma)^{2,\perp}$ is compact.

Proof. Equation (2.34) and Equation (2.35) immediately follow from Corollary 2.3.14 and (2.33). $\square$

In case $p = 2$, for a given immersion γ, Proposition 2.3.21 implies that the operator $\delta E(\varphi) \in (T(\gamma)^{4,2,\perp})^\star$ is represented by the function

$$\nabla_{T(\gamma)^2, T(\gamma)^2} E(\varphi) = |\partial_x \gamma_\varphi| \gamma^\perp \mathcal{T}^\star \left(\nabla^2_{\gamma_\varphi} k_{\gamma_\varphi} + \frac{1}{2}|k_{\gamma_\varphi}|^2 k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right) \qquad \in T(\gamma)^{2,\perp},$$

in the notation of Proposition 2.3.21. In this way we can say that $\delta E : T(\gamma)^{4,2,\perp} \to (T(\gamma)^{2,\perp})^\star$ via the paring

$$\delta E(\varphi)[\psi] = \left\langle \nabla_{T(\gamma)^2, T(\gamma)^2} E(\varphi), \psi \right\rangle_{L^2(dx), L^2(dx)}.$$

Similarly we have that $\mathcal{L} : T(\gamma)^{4,2,\perp} \to (T(\gamma)^{2,\perp})^\star$ with

$$\mathcal{L}(\varphi)[\psi] = \int \langle \nabla^4 \varphi, \psi \rangle + \langle \Omega(\varphi), \psi \rangle \, ds,$$

in the notation of Proposition 2.3.21.

Now we analogously consider $p > 2$.

Proposition 2.3.22. *Let $p > 2$. Let $\gamma : \mathbb{S}^1 \to M$ be a fixed smooth immersion and ρ be given by Lemma 2.3.8.*

For any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$ it holds that

$$\delta E(\varphi)[\psi] = -\left\langle \nabla_{\gamma_\varphi} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \nabla_{\gamma_\varphi} T \gamma_\varphi^\perp \mathcal{T} \psi \right\rangle_{L^{p'}(ds_{\gamma_\varphi}), L^p(ds_{\gamma_\varphi})} + \\ + \left\langle \gamma^\perp \mathcal{T}^\star \left(\frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi}) \tau_{\gamma_\varphi} \right), \psi \right\rangle_{L^{p'}(ds_{\gamma_\varphi}), L^p(ds_{\gamma_\varphi})}$$

where $\gamma_\varphi(\cdot) = \Phi(1, \cdot)$, Φ is the variation of γ given by φ, and $\mathcal{T} : T_{\gamma(x)} M \to T_{\Phi(1,x)} M$ is the function $\mathcal{T}(\psi) = d[\exp_{\gamma(x)}]_\varphi(\psi)$ and $\mathcal{T}^\star$ is its adjoint.

If γ is a critical point such that $|k_\gamma(x)| \neq 0$ for any x, then

$$\delta E(\varphi)[\psi] = \left\langle \gamma^\perp \mathcal{T}^\star \left(\nabla^2_{\gamma_\varphi} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi} + \frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi}) \tau_{\gamma_\varphi} \right), \psi \right\rangle_{L^{p'}(ds_{\gamma_\varphi}), L^p(ds_{\gamma_\varphi})} \tag{2.36}$$

and

$$\mathcal{L}(\varphi)[\psi] = \int |k|^{p-2} \left\langle \nabla^2 \varphi, \nabla^2 \psi \right\rangle + (p-2) |k|^{p-4} \langle k, \nabla^2 \varphi \rangle \langle k, \nabla^2 \psi \rangle \, ds + \int \langle \Omega(\varphi), \psi \rangle \, ds,$$

for any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$, where $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p',\perp}$ is compact.

If γ is a geodesic then

$$\mathcal{L}(\varphi)[\psi] = \int \langle \nabla \varphi, \nabla \psi \rangle - R(\psi, \tau, \varphi, \tau) \, ds,$$

for any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$.

Proof. The statements immediately follow from Corollary 2.3.14, (2.31), and (2.33), together with Remark 2.3.20. □

It is clear from Proposition 2.3.22 that whenever k vanishes, the leading terms in the bilinear form defining $\mathcal{L}$ disappear, and we cannot expect strong Fredholmness properties on $\mathcal{L}$.

However, if $p > 2$, for a given smooth critical point γ with $|k(x)| \neq 0$ for any x, Proposition 2.3.22 implies that the operator $\delta E(\varphi) \in (T(\gamma)^{4,p,\perp})^\star$ is represented by the function

$$\nabla_{T(\gamma)^{p'}, T(\gamma)^p} E(\varphi) \ \in T(\gamma)^{p',\perp},$$

$$\nabla_{T(\gamma)^{p'}, T(\gamma)^p} E(\varphi) = |\partial_x \gamma_\varphi| \gamma^\perp \mathcal{T}^\star \left(\nabla^2_{\gamma_\varphi} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi} + \frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi}) \tau_{\gamma_\varphi} \right),$$

in the notation of Proposition 2.3.22. In this way we can say that $\delta E : T(\gamma)^{4,p,\perp} \to (T(\gamma)^{p,\perp})^\star$ via the paring

$$\delta E(\varphi)[\psi] = \left\langle \nabla_{T(\gamma)^{p'}, T(\gamma)^p} E(\varphi), \psi \right\rangle_{L^{p'}(dx), L^p(dx)}.$$

Similarly we have that $\mathcal{L} : T(\gamma)^{4,p,\perp} \to (T(\gamma)^{p,\perp})^\star$ with

$$\mathcal{L}(\varphi)[\psi] = \int \langle \nabla^2 \left(|k|^{p-2} \nabla^2 \varphi \right), \psi \rangle + (p-2) \left\langle \nabla^2 \left(|k|^{p-4} \langle k, \nabla^2 \varphi \rangle k \right), \psi \right\rangle \, ds + \int \langle \Omega(\varphi), \psi \rangle \, ds,$$

in the notation of Proposition 2.3.22.

With the above results we can now derive the desired Fredholmenss properties on the second variation functionals. Once again, we shall divide the cases $p = 2$ and $p > 2$, as also the technical part of the two proofs is different.

Lemma 2.3.23. *Let $p = 2$. Let $\gamma : \mathbb{S}^1 \to M$ be a smooth critical point and let $\rho > 0$ be given by Lemma 2.3.8. Then the operator $\mathcal{L} : T(\gamma)^{4,2,\perp} \to (T(\gamma)^{2,\perp})^\star$ represented by the function*

$$\mathcal{L}(\varphi) = \nabla^4\varphi + \Omega(\varphi) \qquad \in T(\gamma)^{2,\perp},$$

where $\Omega : T(\gamma)^{4,2,\perp} \to T(\gamma)^{2,\perp}$ is compact, is Fredholm of index zero.

Proof. Since $\Omega : T(\gamma)^{4,2,\perp} \to T(\gamma)^{2,\perp}$ is compact, it is equivalent to prove that

$$\mathrm{id} + \nabla^4 : T(\gamma)^{4,2,\perp}(ds) \to T(\gamma)^{2,\perp}(ds),$$

is Fredholm of index zero (see Remark 2.2.4). Indeed we claim that it is actually invertible. It is clearly injective, indeed if $\varphi + \nabla^4\varphi = 0$, then multiplying by φ and integrating one has

$$\int |\nabla^2\varphi|^2 + |\varphi|^2\, ds = 0,$$

and then $\varphi = 0$. So we need to prove the surjectivity.

Let $a : T(\gamma)^{2,2,\perp} \times T(\gamma)^{2,2,\perp} \to \mathbb{R}$ the continuous bilinear form defined by

$$a(\varphi, \psi) = \int_{\mathbb{S}^1} \langle \nabla^2\varphi, \nabla^2\psi\rangle + \langle \varphi, \psi\rangle\, ds,$$

For $\varphi \in T(\gamma)^{2,2,\perp}$ it holds that

$$\nabla^2\varphi = D_s^2\varphi - \langle D_s^2\varphi, \tau\rangle\tau - \langle D_s\varphi, \tau\rangle k = D_s^2\varphi + (2\langle D_s\varphi, k\rangle + \langle \varphi, D_s k\rangle)\,\tau - \langle D_s\varphi, \tau\rangle k, \quad (2.37)$$

and

$$D_s^2\varphi = \partial_s^2\varphi + \left(2\langle \partial_s\varphi, \partial_s N_j\rangle + \langle \varphi, \partial_s^2 N_j\rangle\right) N_j + \langle \varphi, S_{N_j}(\tau)\rangle S_{N_j}(\tau), \quad (2.38)$$

where $\{N_j\}$ is a local orthonormal frame of $TM^\perp$, and we understood sum over j. Therefore for $\varphi \in T(\gamma)^{2,2,\perp}$ we have that

$$\begin{aligned}\int \langle D_s^2\varphi, D_s^2\varphi\rangle\, ds &= \int |\nabla^2\varphi|^2 + |\langle D_s\varphi, \tau\rangle k|^2 + 2\langle D_s\varphi, \tau\rangle\langle \nabla^2\varphi, k\rangle + |2\langle D_s\varphi, k\rangle + \langle \varphi, D_s k\rangle|^2\, ds\\ &\le \int \frac{3}{2}|\nabla^2\varphi|^2 + C(\gamma)\left(|\varphi|^2 + |D_s\varphi|^2\right)\, ds\\ &\le \int \frac{3}{2}|\nabla^2\varphi|^2 + C(\gamma)\left(|\varphi|^2 + |\partial_s\varphi|^2\right)\, ds\end{aligned}$$

and using also $\int |\partial_s\varphi|^2\, ds = -\int \langle \varphi, \partial_s^2\varphi\rangle\, ds \le \frac{1}{2}\int \frac{1}{\eta}|\varphi|^2 + \eta|\partial_s^2\varphi|^2\, ds$ for any $\eta > 0$ we conclude that

$$\begin{aligned}\int |\partial_s^2\varphi|^2\, ds &\le C(M, \gamma)\int |\varphi|^2 + |\partial_s\varphi|^2 + |D_s^2\varphi|^2\, ds\\ &\le C(M, \gamma, \varepsilon)\int |\varphi|^2 + |\nabla^2\varphi|^2\, ds + \varepsilon\int |\partial_s^2\varphi|^2\, ds.\end{aligned}$$

Hence we see that

$$\int |\varphi|^2 + |\partial_s\varphi|^2 + |\partial_s^2\varphi|^2\, ds \le C(\gamma)a(\varphi, \varphi),$$

that is, a is coercive on $T(\gamma)^{2,2,\perp}$.

Now, if $X \in T(\gamma)^{2,\perp}$ is fixed, we look at the energy functional $F : T(\gamma)^{2,2,\perp} \to \mathbb{R}$ given by

$$F(\varphi) := \int \frac{1}{2}|\nabla^2\varphi|^2 + \frac{1}{2}|\varphi|^2 - \langle X, \varphi\rangle \, ds.$$

Since $-\langle X, \varphi\rangle \geq -\frac{1}{4}|\varphi|^2 - |X|^2$, the coercivity of a implies that F has a minimizer $\varphi \in T(\gamma)^{2,2,\perp}$. Such minimizer φ satisfies the integral Euler–Lagrange equation

$$a(\varphi, \psi) = \int \langle \nabla^2\varphi, \nabla^2\psi\rangle + \langle \varphi, \psi\rangle \, ds = \int \langle X, \psi\rangle,$$

for any $\psi \in T(\gamma)^{2,2,\perp}$. If we show that $\varphi \in T(\gamma)^{4,2,\perp}$, we will have proved that for any $X \in T(\gamma)^{2,\perp}$ there exists $\varphi \in T(\gamma)^{4,2,\perp}$ such that $\nabla^4\varphi + \varphi = X$, and this will prove the required surjectivity. We are going to prove that $\varphi \in T(\gamma)^{3,2,\perp}$ first, and then $\varphi \in T(\gamma)^{4,2,\perp}$.

Let $\Psi \in C^\infty(\mathbb{S}^1; \mathbb{R}^n)$ be any field. By (2.37) and (2.38) we can write that

$$\langle \partial_s^2\varphi, \partial_s\Psi\rangle = \langle \nabla^2\varphi, \partial_s\Psi\rangle - \langle A(\varphi), \partial_s\Psi\rangle,$$

where $A(\varphi)$ is linear in φ and contains at most first order derivatives of φ. Writing $\psi := (M^\top - \gamma^\top)\Psi$ we have that

$$\langle \partial_s^2\varphi, \partial_s\Psi\rangle = \langle \nabla^2\varphi, \nabla\psi\rangle + \langle \Psi, N_j\rangle\langle \nabla^2\varphi, \partial_s N_j\rangle + \langle \Psi, \tau\rangle\langle \nabla^2\varphi, k\rangle - \langle A(\varphi), \partial_s\Psi\rangle,$$

understanding summation over j, for a local orthonormal frame $\{N_j\}$ of $TM^\perp$. Let $\eta \in C_c^\infty(\mathbb{S}^1 \setminus \{0\})$ such that $\eta \geq 0$ and $\int_{\mathbb{S}^1} \eta \, ds = 1$. Define

$$\Phi(x) = (M^\top - \gamma^\top) \int_0^x \psi(t) - \eta(t)\psi_0 \, ds(t),$$

where $\psi_0 := \int_{\mathbb{S}^1} \psi \, ds$. By construction we see that $\Phi \in C^\infty(\mathbb{S}^1; \mathbb{R}^n)$ and $\Phi \in TM \cap T\gamma^\perp$.

Since for any differentiable $\zeta : \mathbb{S}^1 \to \mathbb{R}^n$ we have that

$$\begin{aligned} \partial_s((M^\top - \gamma^\top)\zeta) &= (M^\top - \gamma^\top)\partial_s\zeta - (\partial_s N_j \otimes N_j + N_j \otimes \partial_s N_j)\zeta - (k \otimes \tau + \tau \otimes k)\zeta \\ &=: (M^\top - \gamma^\top)\partial_s\zeta + B(\zeta), \end{aligned} \tag{2.39}$$

we get that

$$\nabla\Phi = \psi - \eta(M^\top - \gamma^\top)\psi_0 + B\left(\int_0^x \psi - \eta\psi_0 \, ds\right),$$

and thus

$$\nabla^2\Phi = \nabla\psi - \partial_s\eta(M^\top - \gamma^\top)\psi_0 + (M^\top - \gamma^\top)\partial_s\left(B\left(\int_0^x \psi - \eta\psi_0 \, ds\right)\right).$$

Hence finally

$$\begin{aligned} \left|\int \langle \partial_s^2\varphi, \partial_s\Psi\rangle\right| &\leq \left|\int \langle \nabla^2\varphi, \nabla\psi\rangle\right| + \left|\int \langle \partial_s(A(\varphi)), \Psi\rangle\right| + C(M, \gamma, \|\varphi\|_{W^{2,2}})\|\Psi\|_{L^2} \\ &\leq \left|\int \langle \nabla^2\varphi, \nabla^2\Phi\rangle\right| + C(M, \gamma, \|\varphi\|_{W^{2,2}}, \eta)\|\Psi\|_{L^2} \\ &= \left|\int \langle X, \Phi\rangle - \langle \varphi, \Phi\rangle\right| + C(M, \gamma, \|\varphi\|_{W^{2,2}}, \eta)\|\Psi\|_{L^2} \\ &\leq C(M, \gamma, \|\varphi\|_{W^{2,2}}, \eta, X)\|\Psi\|_{L^2}, \end{aligned}$$

that implies that $\varphi \in T(\gamma)^{3,2,\perp}$. Once again let $\Psi \in C^\infty(\mathbb{S}^1;\mathbb{R}^n)$ be any field. Using (2.39) twice and writing $\psi = (M^\top - \gamma^\top)\Psi$ as before, we have

$$\begin{aligned}
(M^\top - \gamma^\top)\partial_s^2\Psi &= \partial_s((M^\top - \gamma^\top)\partial_s\Psi) + B(\partial_s\Psi) \\
&= \partial_s(\nabla\psi + (M^\top - \gamma^\top)\left[\partial_s(\langle\Psi, N_j\rangle N_j) + \partial_s(\langle\Psi,\tau\rangle\tau)\right]) + B(\partial_s\Psi) \\
&= \partial_s\nabla\psi + \partial_s\left(\langle\Psi, N_j\rangle\partial_s N_j + \langle\Psi,\tau\rangle k\right) + B(\partial_s\Psi).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\int\langle\partial_s^3\varphi,\partial_s\Psi\rangle\right| &= \left|\int\langle\partial_s^2\varphi,\partial_s^2\Psi\rangle\right| \\
&\le \left|\int\langle\nabla^2\varphi,\partial_s^2\Psi\rangle\right| + \left|\int\langle\partial_s^2(A(\varphi)),\Psi\rangle\right| \\
&\le \left|\int\langle\nabla^2\varphi,\nabla^2\psi\rangle\right| + \left|\int\langle\partial_s\nabla^2\varphi, \langle\Psi,N_j\rangle\partial_s N_j + \langle\Psi,\tau\rangle k\rangle\right| + \left|\int\langle\nabla^2\varphi, B(\partial_s\Psi)\rangle\right| + \\
&\quad + \left|\int\langle\partial_s^2(A(\varphi)),\Psi\rangle\right| \\
&\le \left|\int\langle X,\psi\rangle - \langle\varphi,\psi\rangle\right| + C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2} + \left|\int\langle\nabla^2\varphi, B(\partial_s\Psi)\rangle\right| \\
&\le C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2},
\end{aligned}$$

where the last inequality follows integrating by parts using the definition of B, and this implies that $\varphi \in T(\gamma)^{4,2,\perp}$. ☐

Lemma 2.3.24. *Let $p > 2$. Let $\gamma : \mathbb{S}^1 \to M$ be a smooth critical point with $|k(x)| \neq 0$ for any x. There exists $\rho > 0$ such that the operator $\mathcal{L} : T(\gamma)^{4,p,\perp} \to (T(\gamma)^{p,\perp})^\star$ represented by the function*

$$\mathcal{L}(\varphi) = \nabla^2\left(|k|^{p-2}\nabla^2\varphi\right) + (p-2)\nabla^2\left(|k|^{p-4}\langle k,\nabla^2\varphi\rangle k\right) + \Omega(\varphi) \qquad \in T(\gamma)^{p',\perp},$$

where $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p',\perp}$ is compact, is Fredholm of index zero.

Proof. Since $\Omega : T(\gamma)^{4,p,\perp} \to T(\gamma)^{p',\perp}$ is compact, it is equivalent to prove that the map

$$T(\gamma)^{4,p,\perp}(ds) \ni \quad \varphi \;\mapsto\; \mathscr{T}(\varphi) := \nabla^2\left(|k|^{p-2}\nabla^2\varphi\right) + (p-2)\nabla^2\left(|k|^{p-4}\langle k,\nabla^2\varphi\rangle k\right) + \varphi \quad \in T(\gamma)^{p',\perp}(ds),$$

is Fredholm of index zero (see Remark 2.2.4). Indeed we claim that it is actually invertible. The operator $\mathscr{T}$ is clearly injective, indeed if $\mathscr{T}(\varphi) = 0$, then integration by parts on $\int\langle\varphi,\mathscr{T}(\varphi)\rangle\,ds$ yields

$$0 = \int |k|^{p-2}|\nabla^2\varphi|^2 + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle^2 + |\varphi|^2\,ds,$$

and then $\varphi = 0$. Hence we are left to prove the surjectivity.

This time we consider $a : T(\gamma)^{2,p,\perp} \times T(\gamma)^{2,p,\perp} \to \mathbb{R}$ to be the continuous bilinear form

$$a(\varphi,\psi) := \int_{\mathbb{S}^1} |k|^{p-2}\langle\nabla^2\varphi,\nabla^2\psi\rangle + (p-2)|k|^{p-4}\langle k,\nabla^2\varphi\rangle\langle k,\nabla^2\psi\rangle + \langle\varphi,\psi\rangle\,ds.$$

By hypothesis there are constants c_1, c_2 depending on γ such that $c_1 \le |k(x)| \le c_2$ for any x.

From the proof of Lemma 2.3.23 we know that

$$\int |\varphi|^2 + |\partial_s\varphi|^2 + |\partial_s^2\varphi|^2\,ds \le C(M,\gamma)\int |\nabla^2\varphi|^2 + |\varphi|^2\,ds,$$

for any $\varphi \in T(\gamma)^{2,p,\perp}$. Therefore

$$\int |\varphi|^2 + |\partial_s\varphi|^2 + |\partial_s^2\varphi|^2 \, ds \le C(M,\gamma) a(\varphi,\varphi),$$

for any $\varphi \in T(\gamma)^{2,p,\perp}$. It follows that if $X \in T(\gamma)^{p',\perp}$ is fixed, the convex functional $F : T(\gamma)^{2,p,\perp} \to \mathbb{R}$ defined by

$$F(\varphi) = \int \frac{1}{2}|k|^{p-2}|\nabla^2\varphi|^2 + \frac{p-2}{2}|k|^{p-4}\langle k, \nabla^2\varphi\rangle^2 + \frac{1}{2}|\varphi|^2 - \langle X, \varphi\rangle \, ds$$

has a unique minimizer $\varphi \in T(\gamma)^{2,p,\perp}$. Such minimizer φ satisfies

$$a(\varphi,\psi) = \int \langle X, \Phi\rangle \, ds,$$

for any $\Phi \in T(\gamma)^{2,p,\perp}$. If we show that $\varphi \in T(\gamma)^{4,p,\perp}$, we will have proved that for any $X \in T(\gamma)^{p',\perp}$ there exists $\varphi \in T(\gamma)^{4,p,\perp}$ such that $\mathscr{T}(\varphi) = X$, and this will prove the required surjectivity of $\mathscr{T}$. We are going to show that $\varphi \in T(\gamma)^{3,p,\perp}$ first, and then $\varphi \in T(\gamma)^{4,p,\perp}$.

Let $\Psi \in C^\infty(\mathbb{S}^1;\mathbb{R}^n)$ be any field and let $\psi := (M^\top - \gamma^\top)\Psi$. Let $\nu \in C^\infty(\mathbb{S}^1;\mathbb{R}^n)$ be the normal field along γ defined by

$$\nu = \frac{|k|^{2-p}}{p-1}\left\langle \psi, \frac{k}{|k|}\right\rangle \frac{k}{|k|} + |k|^{2-p}\sum_{i=1}^{m-2}\langle \psi, e_i\rangle e_i,$$

where $\{e_1, ..., e_{m-2}, \frac{k}{|k|}\}$ is a fixed orthonormal frame of the normal bundle $T\gamma^\perp$ along γ in M. We remark that $\nu \in TM \cap T\gamma^\perp$. Letting also $\eta \in C^\infty_c(\mathbb{S}^1 \setminus \{0\})$ such that $\eta \ge 0$ and $\int_{\mathbb{S}^1}\eta \, ds = 1$, we define

$$\nu_0 = \int_{\mathbb{S}^1}\nu \, ds, \qquad \Phi(x) = (M^\top - \gamma^\top)\int_0^x \nu(y) - \eta(y)\nu_0 \, ds(y).$$

In this way, in the notation of (2.39) we have

$$\begin{aligned}
\nabla\Phi &= (M^\top - \gamma^\top)\left[(M^\top - \gamma^\top)(\nu - \eta\nu_0) + B\left(\int_0^x \nu(x) - \eta(x)\nu_0\right)\right] \\
&= \nu - \eta(M^\top - \gamma^\top)\nu_0 - \left\langle N_j, \int_0^x \nu - \eta\nu_0\right\rangle (M^\top - \gamma^\top)(\partial_s N_j) - \left\langle \tau, \int_0^x \nu - \eta\nu_0\right\rangle k.
\end{aligned}$$

By construction

$$\psi = |k|^{p-2}\nu + (p-2)|k|^{p-2}\left\langle \nu, \frac{k}{|k|}\right\rangle \frac{k}{|k|},$$

and then

$$\begin{aligned}
&|k|^{p-2}\nabla\Phi + (p-2)|k|^{p-2}\left\langle \nabla\Phi, \frac{k}{|k|}\right\rangle \frac{k}{|k|} = \\
&\quad = \psi + |k|^{p-2}\left[-\eta(M^\top - \gamma^\top)\nu_0 - \left\langle N_j, \int_0^x \nu - \eta\nu_0\right\rangle (M^\top - \gamma^\top)(\partial_s N_j) - \left\langle \tau, \int_0^x \nu - \eta\nu_0\right\rangle k\right] + \\
&\qquad + (p-2)|k|^{p-4}\Big\langle -\eta(M^\top - \gamma^\top)\nu_0 - \left\langle N_j, \int_0^x \nu - \eta\nu_0\right\rangle (M^\top - \gamma^\top)(\partial_s N_j) + \\
&\qquad - \left\langle \tau, \int_0^x \nu - \eta\nu_0\right\rangle k, k\Big\rangle k.
\end{aligned}$$

Therefore, as in the proof of Lemma 2.3.23, we estimate

$$
\begin{aligned}
\left|\int \langle \partial_s^2\varphi, \partial_s\Psi\rangle\right| &\le \left|\int \langle \nabla^2\varphi, \nabla\psi\rangle\right| + C(M,\gamma,\|\varphi\|_{W^{2,2}},\eta)\|\Psi\|_{L^2} \\
&\le \left|\int |k|^{p-2}\langle \nabla^2\varphi, \nabla^2\Phi\rangle + (p-2)|k|^{p-4}\langle k, \nabla^2\varphi\rangle\langle k, \nabla^2\Phi\rangle\right| + C(M,\gamma,\|\varphi\|_{W^{2,2}},\eta)\|\Psi\|_{L^2} \\
&= \left|\int \langle X, \Phi\rangle - \langle \varphi, \Phi\rangle\right| + C(M,\gamma,\|\varphi\|_{W^{2,2}},\eta)\|\Psi\|_{L^2} \\
&\le C(M,\gamma,\|\varphi\|_{W^{2,2}},\eta)\|\Psi\|_{L^2},
\end{aligned}
$$

that implies $\varphi \in T(\gamma)^{3,p,\perp}$.

Now the definition of ν implies that

$$
\langle \nabla\nu, e_i\rangle = |k|^{2-p}\langle \nabla\psi, e_i\rangle + A_i(\psi), \qquad \left\langle \nabla\nu, \frac{k}{|k|}\right\rangle = \frac{|k|^{2-p}}{p-1}\left\langle \nabla\psi, \frac{k}{|k|}\right\rangle + A_k(\psi),
$$

where $A_i(\psi), A_k(\psi)$ depend on γ and M, and they depend linearly on ψ and they are independent of the derivatives of ψ. Therefore we have

$$
\nabla\psi = |k|^{p-2}\nabla\nu + (p-2)|k|^{p-4}\langle k, \nabla\nu\rangle k + A_0(\psi),
$$

with A_0 having the same properties of A_i, A_k. Finally we can estimate

$$
\begin{aligned}
\left|\int \langle \partial_s^3\varphi, \partial_s\Psi\rangle\right| &= \left|\int \langle \partial_s^2\varphi, \partial_s^2\Psi\rangle\right| \\
&\le \left|\int \langle \nabla^2\varphi, \nabla^2\psi\rangle\right| + C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2} \\
&\le \left|\int \left\langle \nabla^2\varphi, \nabla\left(|k|^{p-2}\nabla\nu + (p-2)|k|^{p-4}\langle k, \nabla\nu\rangle k\right)\right\rangle\right| + \left|\int \langle \nabla^2\varphi, \nabla(A_0(\psi))\rangle\right| + \\
&\quad + C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2} \\
&\le \left|\int |k|^{p-2}\langle \nabla^2\varphi, \nabla^2\nu\rangle + (p-2)|k|^{p-4}\langle k, \nabla^2\varphi\rangle\langle k, \nabla^2\nu\rangle\right| + \left|\int \langle \nabla^3\varphi, A_0(\psi)\rangle\right| + \\
&\quad + C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2} \\
&\le \left|\int \langle X, \nu\rangle - \langle \varphi, \nu\rangle\right| + C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2} \\
&\le C(M,\gamma,\|\varphi\|_{L^2})\|\nu\|_{L^2} + C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2} \\
&\le C(M,\gamma,\|\varphi\|_{W^{3,2}})\|\Psi\|_{L^2},
\end{aligned}
$$

and we have proved that $\varphi \in T(\gamma)^{4,p,\perp}$. □

A last fact needed for applying Corollary 2.2.7 is the analyticity of the operators, as stated in the next lemma, for which we mainly refer to [DPS16]. Here the analyticity of the ambient comes into play.

Lemma 2.3.25. *Let $\gamma : \mathbb{S}^1 \to M$ be a smooth regular curve and let $\rho > 0$ be given by Lemma 2.3.8. Let $p \ge 2$. Suppose that (M,g) is an analytic complete Riemannian manifold endowed with an analytic metric tensor g.*

1. *If $p = 2$, then the maps*

$$E : B_\rho(0) \to \mathbb{R}, \qquad \delta E : B_\rho(0) \to (T(\gamma)^{2,\perp})^\star,$$

are analytic.

2. *If $p > 2$ and $|k(x)| \neq 0$ for any x, then the maps*

$$E : B_\rho(0) \to \mathbb{R}, \qquad \delta E : B_\rho(0) \to (T(\gamma)^{p,\perp})^\star,$$

are analytic, up to decrease ρ.

Proof. We adopt the notation used in (2.34) and (2.36). For a fixed $\varphi(x) \in T_{\gamma(x)}M$ we have that $\gamma_\varphi(x) = \exp_{\gamma(x)}(\varphi(x)) = \sigma_{\varphi(x)}(1)$, where $\sigma_{\varphi(x)}$ is the geodesic starting at $\gamma(x)$ with initial velocity $\sigma'_{\varphi(x)}(0) = \varphi(x)$. As the manifold and the metric are assumed to be analytic, so are the connection D and the Christoffel symbols Γ^k_{ij} on M. It follows that, as $\sigma_{\varphi(x)}$ solves a semi-linear ordinary differential equation with analytic coefficients, it depends analytically on the initial data. In particular the exponential map is analytic and the dependence of $\gamma_\varphi(x)$ on $\varphi(x) \in T_{\gamma(x)}M$ is analytic. Also, since the exponential map on M is analytic, so is its differential, and it follows that $d[\exp_{\gamma(x)}]$ is analytic as a map defined on $T(\gamma)^{4,p,\perp}$.

Now that we know that the map $\varphi \mapsto \gamma_\varphi$ is analytic, following the exhaustive proof of [DPS16, Lemma 3.4], one can check that the formulas for E and δE in (2.34) and (2.36) are sums of compositions of analytic functions of the parametrization γ_φ (in case $p > 2$ we use that $|k_{\gamma_\varphi}|$ never vanishes for ρ small enough). □

We now have all the ingredients for applying Corollary 2.2.7, thus getting the Łojasiewicz–Simon gradient inequality for the functional E.

Corollary 2.3.26. *Suppose that (M, g) is an analytic complete Riemannian manifold endowed with an analytic metric tensor g. Let $p \geq 2$. Let $\gamma : \mathbb{S}^1 \to M$ be a smooth critical point of $\mathcal{E}_p$. In case $p > 2$ assume that $|k(x)| \neq 0$ for any x. There exist $C, \rho > 0$ and $\theta \in (0, \frac{1}{2}]$ such that*

$$|E(\varphi) - E(0)|^{1-\theta} \leq C \|\nabla_{T(\gamma)^{p'}, T(\gamma)^p} E(\varphi)\|_{L^{p'}(ds_{\gamma_\varphi})}, \tag{2.40}$$

for any $\varphi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$, where $\gamma_\varphi(\cdot) = \Phi(1, \cdot)$ and Φ is the variation of γ given by φ.

Proof. Collecting the results of Proposition 2.3.21, Proposition 2.3.22, Lemma 2.3.23, Lemma 2.3.24, and Lemma 2.3.25, the statement follows from the direct application of Corollary 2.2.7 taking $V = T(\gamma)^{4,p,\perp}$, $Z = T(\gamma)^{p,\perp}$, and $\rho_0 > 0$ depending on γ given by Lemma 2.3.8. □

As outlined in the strategy of Section 2.2, we can exploit the geometric invariance of the energy for extending the inequality (2.40) to the functional $\mathbf{E}$ and generic variations in $T(\gamma)^{4,p}$. We need the following reparametrization result first.

Lemma 2.3.27. *Let $\gamma : \mathbb{S}^1 \to M$ be a smooth immersion and $p \geq 2$. Let $\rho_0(\gamma) > 0$ be given by Lemma 2.3.8. Then for any $\rho \in (0, \rho_0)$ there is $\sigma > 0$ such that for any $\psi \in B_\sigma(0) \subset T(\gamma)^{4,p}$ there exists a diffeomorphism $L : \mathbb{S}^1 \to \mathbb{S}^1$ of class $W^{4,p}$ and $\varphi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$ such that*

$$\gamma_\psi \circ L = \gamma_\varphi,$$

where $\gamma_\varphi(\cdot) = \Phi(1, \cdot)$, $\gamma_\psi(\cdot) = \Psi(1, \cdot)$, and Φ, Ψ are the variation of γ given by φ, ψ respectively.

Proof. By compactness there exists $\tau > 0$ such that $\gamma|_{(x-\tau,x+\tau)} : (x-\tau, x+\tau) \to M$ is an embedding for any $x \in \mathbb{S}^1$. Fix $\rho \in (0, \rho_0)$. If we choose $\sigma' > 0$ is sufficiently small, depending only on γ, and any $\psi \in B_{\sigma'}(0) \subset T(\gamma)^{4,p}$, we have that

$$\gamma_\psi \left(x - \frac{\tau}{2}, x + \frac{\tau}{2} \right) \subset U_x,$$

where $U_x \subset M$ is an open neighborhood of $\gamma\left(\left[x-\frac{\tau}{2}, x+\frac{\tau}{2}\right]\right)$ parametrized by the exponential map restricted to the normal bundle of γ. More precisely, there exists an open connected set

$$\Omega_x \subset \bigcup_{y \in (x-\tau, x+\tau)} T_{\gamma(y)}\gamma^\perp,$$

containing the origin of $T_{\gamma(y)}\gamma^\perp$ for any $y \in (x-\tau, x+\tau)$ such that any $q \in U_x$ can be uniquely written as $q = \exp^\perp(v_q)$ for some $v_q \in \Omega_x$, where $\exp^\perp$ is the restriction of the exponential map to the normal bundle of γ.

Hence for any $y \in \left(x-\frac{\tau}{2}, x+\frac{\tau}{2}\right)$ there exists a unique $G(y) \in (x-\tau, x+\tau)$ and a unique $\varphi \in T_{\gamma(G(y))}\gamma^\perp$ such that

$$\gamma_\psi(y) = \exp^\perp(\varphi(G(y))).$$

By defining $L = G^{-1} : G\left(x-\frac{\tau}{2}, x+\frac{\tau}{2}\right) \to \left(x-\frac{\tau}{2}, x+\frac{\tau}{2}\right)$ we see that

$$\gamma_\psi \circ L\,(y) = \exp(\varphi(y)),$$

for any $y \in G\left(x-\frac{\tau}{2}, x+\frac{\tau}{2}\right)$. Moreover since for $y \in \left(x-\frac{\tau}{2}, x+\frac{\tau}{2}\right)$ we can write explicitly

$$G(y) = \left(\gamma|_{(x-\tau,x+\tau)}\right)^{-1} \circ \pi \circ (\exp^\perp)^{-1} \circ \gamma_\psi\,(y),$$

where π is the projection of the normal bundle, we see that G is of class $W^{4,p}$, and then so is L. Also

$$\varphi(y) = (\exp^\perp)^{-1} \circ \gamma_\psi \circ L\,(y),$$

and then φ is of class $W^{4,p}$. By arbitrariness of x, one can then define a normal field φ along γ and a diffeomorphism L of $\mathbb{S}^1$ satisfying $\gamma_\psi \circ L = \gamma_\varphi$.

Finally, it follows from the construction that if $\psi_n \in T(\gamma)^{4,p}$ converges to 0, then the corresponding L_n converges to the identity in $W^{4,p}$, and then also $\varphi_n \to 0$ in $W^{4,p}$. This proves that for the chosen ρ, taking a suitable $0 < \sigma \le \sigma'$, for any $\psi \in B_\sigma(0) \subset T(\gamma)^{4,p}$ the resulting $\varphi \in T(\gamma)^{4,p,\perp}$ has norm less than the desired ρ. □

For the convenience of the reader, let us recall here that for a fixed smooth curve $\gamma : \mathbb{S}^1 \to M$ and $\rho > 0$ sufficiently small we have that

$$\delta \mathbf{E}(\varphi)[\psi] = \left\langle \mathcal{T}^\star \left(\nabla^2_{s_{\gamma_\varphi}} k_{\gamma_\varphi} + \frac{1}{2}|k_{\gamma_\varphi}|^2 k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right), \psi \right\rangle_{L^2(ds_{\gamma_\varphi}), L^2(ds_{\gamma_\varphi})}$$

for any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,2}$, where $\gamma_\varphi(\cdot) = \Phi(1, \cdot)$, Φ is the variation of γ given by φ, and $\mathcal{T} : T_{\gamma(x)}M \to T_{\Phi(1,x)}M$ is the function $\mathcal{T}(\psi) = d[\exp_{\gamma(x)}]_\varphi(\psi)$ and $\mathcal{T}^\star$ is its adjoint. We therefore write

$$\nabla_{T(\gamma)^2, T(\gamma)^2} \mathbf{E}(\varphi) = |\partial_x \gamma_\varphi| \mathcal{T}^\star \left(\nabla^2_{s_{\gamma_\varphi}} k_{\gamma_\varphi} + \frac{1}{2}|k_{\gamma_\varphi}|^2 k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right).$$

Analogously if $\gamma : \mathbb{S}^1 \to M$ is a smooth critical point of $\mathcal{E}_p$ for some $p > 2$ and $|k(x)| \neq 0$ for any x, for $\rho > 0$ sufficiently small we have that

$$\delta\mathbf{E}(\varphi)[\psi] = \left\langle \mathcal{T}^\star \left(\nabla^2_{s_{\gamma_\varphi}} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi} + \frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right), \psi \right\rangle_{L^{p'}(ds_{\gamma_\varphi}), L^p(ds_{\gamma_\varphi})}$$

for any $\varphi, \psi \in B_\rho(0) \subset T(\gamma)^{4,p}$, where γ_φ and $\mathcal{T}$ are as above. Therefore

$$\nabla_{T(\gamma)^{p'}, T(\gamma)^p} \mathbf{E}(\varphi) = |\partial_x \gamma_\varphi| \mathcal{T}^\star \left(\nabla^2_{s_{\gamma_\varphi}} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi} + \frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right).$$

As anticipated, using Lemma 2.3.27 we can now improve (2.40) to fields in $T(\gamma)^{4,p}$.

Corollary 2.3.28 (Łojasiewicz–Simon gradient inequality for elastic energies)**.** *Suppose that (M, g) is an analytic complete Riemannian manifold endowed with an analytic metric tensor g. Let $p \geq 2$. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth critical point of. In case $p > 2$ assume that $|k(x)| \neq 0$ for any x. There exist $C, \sigma > 0$ and $\theta \in (0, \frac{1}{2}]$ such that*

$$|\mathbf{E}(\psi) - \mathbf{E}(0)|^{1-\theta} \leq C \|\nabla_{T(\gamma)^{p'}, T(\gamma)^p} \mathbf{E}(\psi)\|_{L^{p'}(ds_{\gamma_\psi})}, \tag{2.41}$$

for any $\psi \in B_\sigma(0) \subset T(\gamma)^{4,p}$.

Proof. Let ρ be as in Corollary 2.3.26. Without loss of generality $\rho < \rho_0(\gamma)$, where $\rho_0(\gamma)$ is given by Lemma 2.3.8, and then let $\sigma > 0$ be the corresponding radius given by Lemma 2.3.27. Let $\psi \in B_\sigma(0) \subset T(\gamma)^{4,p}$ and let $L : \mathbb{S}^1 \to \mathbb{S}^1$ and $\varphi \in B_\rho(0) \subset T(\gamma)^{4,p,\perp}$ such that $\gamma_\psi \circ L = \gamma_\varphi$ in the notation of Lemma 2.3.27. Then

$$\mathbf{E}(\psi) = \mathcal{E}_p(\gamma_\psi) = \mathcal{E}_p((\gamma_\psi) \circ L) = \mathcal{E}_p(\gamma_\varphi) = E(\varphi).$$

Moreover by compactness and continuity of $w \mapsto d[\exp_{\gamma(x)}]_w$ we see that there exists $R = R(\gamma) > 0$ such that for any $x \in \mathbb{S}^1$ and any $w \in T_{\gamma(x)}M$ with $|w| \leq R$ it holds that $d[\exp_{\gamma(x)}]_w$ is invertible with $\|d[\exp_{\gamma(x)}]_w\| \leq C(\gamma)$, and then the same holds for $(d[\exp_{\gamma(x)}]_w)^\star$ and their inverse. Up to take smaller ρ and σ, we can assume that $|\varphi| \leq R$ and $|\psi| \leq R$.

Understanding that if $p = 2$ then $|v|^{p-2} \equiv 1$ for any vector v, we deduce that

$$\begin{aligned}
&\|\nabla_{T(\gamma)^{p'}, T(\gamma)^p} E(\varphi)\|^{p'}_{L^{p'}(ds_{\gamma_\varphi})} \\
&\leq \int_{\mathbb{S}^1} \left| (d[\exp_{\gamma(x)}]_\varphi)^\star \left(\nabla^2_{\gamma_\varphi} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi} + \frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right) \right|^{p'} ds_{\gamma_\varphi} \\
&\leq C \int_{\mathbb{S}^1} \left| \nabla^2_{\gamma_\varphi} |k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi} + \frac{1}{p'} |k_{\gamma_\varphi}|^p k_{\gamma_\varphi} - k_{\gamma_\varphi} + R(|k_{\gamma_\varphi}|^{p-2} k_{\gamma_\varphi}, \tau_{\gamma_\varphi})\tau_{\gamma_\varphi} \right|^{p'} ds_{\gamma_\varphi} \\
&= C \int_{\mathbb{S}^1} \left| \nabla^2_{\gamma_\psi \circ L} |k_{\gamma_\psi \circ L}|^{p-2} k_{\gamma_\psi \circ L} + \frac{1}{p'} |k_{\gamma_\psi \circ L}|^p k_{\gamma_\psi \circ L} - k_{\gamma_\psi \circ L} + R(|k_{\gamma_\psi \circ L}|^{p-2} k_{\gamma_\psi \circ L}, \tau_{\gamma_\psi \circ L})\tau_{\gamma_\psi \circ L} \right|^{p'} ds_{\gamma_\psi \circ L} \\
&= C \int_{\mathbb{S}^1} \left| \nabla^2_{\gamma_\psi} |k_{\gamma_\psi}|^{p-2} k_{\gamma_\psi} + \frac{1}{p'} |k_{\gamma_\psi}|^p k_{\gamma_\psi} - k_{\gamma_\psi} + R(|k_{\gamma_\psi}|^{p-2} k_{\gamma_\psi}, \tau_{\gamma_\psi})\tau_{\gamma_\psi} \right|^{p'} ds_{\gamma_\psi} \\
&= C \int_{\mathbb{S}^1} \Big| \left[(d[\exp_{\gamma(x)}]_\psi)^\star \right]^{-1} \cdot \\
&\qquad \cdot (d[\exp_{\gamma(x)}]_\psi)^\star \left(\nabla^2_{\gamma_\psi} |k_{\gamma_\psi}|^{p-2} k_{\gamma_\psi} + \frac{1}{p'} |k_{\gamma_\psi}|^p k_{\gamma_\psi} - k_{\gamma_\psi} + R(|k_{\gamma_\psi}|^{p-2} k_{\gamma_\psi}, \tau_{\gamma_\psi})\tau_{\gamma_\psi} \right) \Big|^{p'} ds_{\gamma_\psi} \\
&\leq C \int_{\mathbb{S}^1} \left| (d[\exp_{\gamma(x)}]_\psi)^\star \left(\nabla^2_{\gamma_\psi} |k_{\gamma_\psi}|^{p-2} k_{\gamma_\psi} + \frac{1}{p'} |k_{\gamma_\psi}|^p k_{\gamma_\psi} - k_{\gamma_\psi} + R(|k_{\gamma_\psi}|^{p-2} k_{\gamma_\psi}, \tau_{\gamma_\psi})\tau_{\gamma_\psi} \right) \right|^{p'} ds_{\gamma_\psi} \\
&\leq C \|\nabla_{T(\gamma)^{p'}, T(\gamma)^p} \mathbf{E}(\psi)\|^{p'}_{L^{p'}(ds_{\gamma_\psi})},
\end{aligned}$$

and therefore (2.41) readily follows from (2.40). $\square$

2.3.4 Convergence of elastic flows into manifolds

In this part we apply Corollary 2.3.28 and the strategy presented in Section 2.2 to prove the full convergence of the gradient flow of $\mathcal{E}_p$ out of its sub-convergence.

Let us start by recalling the definitions of the gradient flows we are considering. For a given smooth curve $\gamma_0 : \mathbb{S}^1 \to M$, we say that $\gamma : [0,T) \times \mathbb{S}^1 \to M$ is the solution of the gradient flow of $\mathcal{E}_p$ with datum γ_0 if it classically satisfies the equation

$$\begin{cases} \partial_t \gamma = -\left(\nabla^2 |k|^{p-2} k + \frac{1}{p'} |k|^p k - k + R(|k|^{p-2}k, \tau)\tau \right) & \text{on } [0,T) \times \mathbb{S}^1, \\ \gamma(0,\cdot) = \gamma_0(\cdot) & \text{on } \mathbb{S}^1, \end{cases} \tag{2.42}$$

where we understand that $|k|^{p-2} \equiv 1$ in case $p = 2$.

In order to prove the convergence of the flow we need a local existence and uniqueness result. This is contained in the next theorem, whose proof is based on rather classical arguments about parabolic equations, and then we will just comment on that.

Theorem 2.3.29 (Local existence and uniqueness)**.** *Let $p \geq 2$ and let $\gamma_0 : \mathbb{S}^1 \to M$ be a smooth curve. If $p > 2$ assume also that $|k_{\gamma_0}(x)| \neq 0$ for any x. There exists $T > 0$ and a unique $\gamma : [0,T) \times \mathbb{S}^1 \to M$ such that $\gamma(t,x)$ is a smooth solution of (2.42).*

The outline of the proof of Theorem 2.3.29 goes as follows. We can fix finitely many local charts $\{(U_i, \Phi_i)\}_{i=1}^N$ on M such that $\gamma_0(\mathbb{S}^1) \subset \cup_i U_i$. We can choose such charts so that for some $r_i > 0, x_i \in \mathbb{S}^1$ we have $U_i = B_{r_i}(\gamma_0(x_i))$, $\gamma_0(\frac{1}{2}(x_i + x_{i+1})) \in U_i \cap U_{i+1}$, and $B_{\frac{r_i}{2}}(x_i) \cap U_j = \emptyset$ for $i \neq j$, where we understand that $N + 1 = 1$. Fix also points $a_i, b_i \in \mathbb{S}^1$ such that

$$a_i < \tfrac{1}{2}(x_i + x_{i-1}) < b_{i-1} < x_i < a_{i+1},$$

so that $\gamma_0(a_i), \gamma_0(b_{i-1}) \in U_{i-1} \cap U_i$. Next we consider the curves $\gamma_{0,i} := \Phi_i \circ \gamma_0|_{a_i,b_i} : (a_i, b_i) \to \mathbb{R}^m$.

Consider first $p = 2$. In such local coordinates one checks that (2.42) in terms of $\gamma_i = \Phi_i \circ \gamma$ becomes

$$\begin{cases} \partial_t \gamma_i^l = -\frac{1}{|\partial_x \gamma_i|^4} \partial_x^4 \gamma_i^l + X^l(\gamma_i, \partial_x \gamma_i, \partial_x^2 \gamma_i, \partial_x^3 \gamma_i) & l = 1, ..., m, \\ \gamma_i(0,\cdot) = \gamma_{0,i}, \end{cases} \tag{2.43}$$

where $X^l : U_1 \times U_2 \times U_3 \times U_4 \to \mathbb{R}^m$ is smooth and $U_j \subset \mathbb{R}^m$ is a suitable open bounded set for any $j = 1, ..., 4$. It is possible to prove local existence and uniqueness with continuity with respect to the datum for (2.43), thus getting a solution $\gamma_i : [0, T_i) \times (a_i, b_i) \to \mathbb{R}^m$; indeed (2.43) is a parabolic quasi-linear system and one can replicate the very flexible strategy of [MM12]. Now if T_i is sufficiently small, one has that $\Phi_i^{-1} \circ \gamma_i$ makes sense and solves (2.42) up to reparametrization on $[0, T_i) \times (a_i, b_i)$. Also, a solution of the flow in (2.42) is independent of the parametrization of the curve at time t, and therefore $\Phi_i^{-1} \circ \gamma_i$ and $\Phi_{i-1}^{-1} \circ \gamma_{i-1}$ coincide on (a_i, b_{i-1}) up to a reparametrization on the interval (x_{i-1}, x_i). Hence we can glue together the solutions obtaining a flow solving (2.42) as stated in Theorem 2.3.29.

If instead $p > 2$, assuming $k \neq 0$, rewriting $k = |\partial_x \gamma|^{-2} \gamma^\perp(\partial_x^2 \gamma)$, where $\gamma^\perp = M^\top - \gamma^\top$, the evolution equation becomes

$$\partial_t \gamma = -\frac{1}{|\partial_x \gamma|^{2p}} \partial_x^2 \left(|\gamma^\perp \partial_x^2 \gamma|^{p-2} \partial_x^2 \gamma \right) + X(\gamma, \partial_x \gamma, \partial_x^2 \gamma, \partial_x^3 \gamma),$$

where $X : U_1 \times U_2 \times U_3 \times U_4 \to \mathbb{R}^n$ always denotes a smooth function and $U_j \subset \mathbb{R}^m$ is a suitable open bounded set for any $j = 1, ..., 4$. Exploiting the fact that by hypothesis the curvature of the datum does not vanish, the strong parabolicity of the system is preserved for short times and one proves Theorem 2.3.29 by means of the same techniques.

Remark 2.3.30 (Uniqueness and reparametrizations)**.** Let us remark here a well known fact about the uniqueness up to reparametrizations in the theory of evolution equation of geometric nature. Let us say that $\gamma : [0, T) \times \mathbb{S}^1 \to M$ solves

$$\begin{cases} \partial_t \gamma(t, x) = V_\gamma(t, x), \\ \gamma(0, \cdot) = \gamma_0(\cdot), \end{cases} \tag{2.44}$$

everything is smooth, and $V_\gamma \in T\gamma^\perp$ for any time is a normal velocity field computed in terms of the curve γ at any time, and γ is the unique solution of (2.44). Suppose that the velocity is geometric in the sense that if $\chi : [0, T) \times \mathbb{S}^1 \to \mathbb{S}^1$ satisfies that $\chi(t, \cdot)$ is a diffeomorphism for any t and $\sigma(t, x) := \gamma(t, \chi(t, x))$, then $V_\sigma(t, x) = V_\gamma(t, \chi(t, x))$. Observe that this is exactly the case of the family of flows we are considering. In such a case, it is immediate to check that σ solves

$$\begin{cases} \partial_t \sigma(t, x) = V_\sigma(t, x) + W(t, x)\tau_\sigma(t, x), \\ \sigma(0, \cdot) = \gamma_0(\chi(0, \cdot)), \end{cases}$$

and W can be computed explicitly in terms of χ and γ. In complete analogy, if $\beta : [0, T) \times \mathbb{S}^1 \to M$ is given and solves

$$\begin{cases} \partial_t \beta(t, x) = V_\beta(t, x) + w(t, x)\tau_\beta(t, x), \\ \beta(0, \cdot) = \gamma_0(\chi_0(\cdot)), \end{cases}$$

where $\chi_0 : \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism, then letting $\psi : [0, T) \times \mathbb{S}^1 \to \mathbb{S}^1$ be the smooth solution of

$$\begin{cases} \partial_t \psi(t, x) = -|(\partial_x \beta)(t, \psi(t, x))|^{-1} w(t, \psi(t, x)), \\ \psi(0, \cdot) = \chi_0^{-1}(\cdot), \end{cases}$$

it immediately follows that $\widehat{\gamma}(t, x) := \beta(t, \psi(t, x))$ solves (2.44), and then $\widehat{\gamma} = \gamma$ by uniqueness. We shall use this sort of geometric uniqueness up to reparametrizations several times.

The next proposition contains a result about parabolic estimates we will need for the proof of the main theorem.

Proposition 2.3.31 (Parabolic estimates)**.** *Let $\gamma : [0, \tau) \times \mathbb{S}^1 \to M \subset \mathbb{R}^n$ be a smooth solution of (2.42) and let $\widehat{\gamma}$ be its constant speed reparametrization. Let $\Gamma : \mathbb{S}^1 \to M \subset \mathbb{R}^n$ be a fixed smooth curve parametrized with constant speed. If $p > 2$ assume that $|k_\Gamma(x)| \neq 0$ for any x. Then there is $\bar{\sigma} = \bar{\sigma}(k_\Gamma) > 0$ depending only on k_Γ such that if*

$$\|\widehat{\gamma}(t, \cdot) - \Gamma\|_{W^{4,p}} \le \bar{\sigma},$$

for any $t \in [0, \tau)$, then

$$\|k_{\widehat{\gamma}}(t, \cdot)\|_{W^{m,2}} \le C(m, \|k_\Gamma\|_{W^{2,2}}, \bar{\sigma}, \mathcal{U}, \Lambda)(1 + \|k_{\widehat{\gamma}}(0, \cdot)\|_{W^{m,2}}),$$

for any $t \in [0, \tau)$ and any $m \in \mathbb{N}$, where $\widehat{\gamma}$ is the constant speed reparametrization of γ. Also, $\mathcal{U} = \mathcal{U}(\bar{\sigma})$ is a bounded neighborhood of Γ in M such that the flow γ is contained in $\mathcal{U}$ for any $t \in [0, \tau)$, and $\Lambda := \sup_{x \in \mathcal{U}} |B_x|$ is the maximal norm assumed by the second fundamental form of $M \hookrightarrow \mathbb{R}^n$ on $\mathcal{U}$.

In Proposition 2.3.31, writing that a constant depends on an open set $\mathcal{U} \subset M$ is a shortcut for saying that such constant depends on the metric g of M on $\mathcal{U}$, and thus possibly on all the intrinsic geometric quantities depending on g on $\mathcal{U}$.

The proof of Proposition 2.3.31 is a bit technical but based on classical arguments in the theory of geometric parabolic equations, and it is postponed to Section 2.3.5.

Remark 2.3.32 (Interpolation inequalities)**.** We recall separately some interpolation inequalities we shall employ. For $\alpha \in (0,1)$ and $k \in \mathbb{N}$ with $k \geq 1$ such that $s = \alpha k$ is not an integer, [Lun18, Example 5.15] and [Lun18, Corollary 1.7] imply that

$$\|f\|_{W^{s,p}(\mathbb{R})} \leq c(\alpha, p, k)\|f\|^{1-\alpha}_{L^p(\mathbb{R})}\|f\|^{\alpha}_{W^{k,p}(\mathbb{R})},$$

for any $f \in W^{k,p}(\mathbb{R})$. By taking $\alpha \in (0,1)$ such that $\alpha k > k-1$ and it is not an integer, we have the inequality

$$\|f\|_{W^{k-1,p}(\mathbb{R})} \leq c(\alpha, p, k)\|f\|^{1-\alpha}_{L^p(\mathbb{R})}\|f\|^{\alpha}_{W^{k,p}(\mathbb{R})}.$$

Hence for a function $F \in W^{k,p}(\mathbb{S}^1;\mathbb{R}^n)$, using a continuous extension operator $T : W^{k,p}(\mathbb{S}^1;\mathbb{R}^n) \to W^{k,p}(\mathbb{R};\mathbb{R}^n)$ we deduce the inequality

$$\|F\|_{W^{k-1,p}(\mathbb{S}^1;\mathbb{R}^n)} \leq c(\alpha, p, k)\|F\|^{1-\alpha}_{L^p(\mathbb{S}^1;\mathbb{R}^n)}\|F\|^{\alpha}_{W^{k,p}(\mathbb{S}^1;\mathbb{R}^n)}. \tag{2.45}$$

Similarly, setting $k = 1$ and thus $\alpha = s$ it holds that

$$\|F\|_{W^{s,p}(\mathbb{S}^1;\mathbb{R}^n)} \leq c(s,p)\|F\|^{1-s}_{L^p(\mathbb{S}^1;\mathbb{R}^n)}\|F\|^{s}_{W^{1,p}(\mathbb{S}^1;\mathbb{R}^n)}, \tag{2.46}$$

for any $F \in W^{1,p}(\mathbb{S}^1;\mathbb{R}^n)$.

The same references also imply the following interpolation inequality. Let $k \in \mathbb{N}$ with $k \geq 2$ and $0 < \delta < \beta < 1$; then there exists $\theta' \in (0,1)$ such that

$$\|f\|_{C^{k,\delta}(\mathbb{R})} \leq c(k, \beta, \varepsilon, \theta')\|f\|^{1-\theta'}_{C^{k-2}(\mathbb{R})}\|f\|^{\theta'}_{C^{k,\beta}(\mathbb{R})},$$

for any $f \in C^{k,\beta}(\mathbb{R})$. More precisely, we can choose $\theta' = \frac{2+\delta}{2+\beta}$, so that $k + \delta = (1-\theta')(k-2) + \theta'(k+\beta)$. By suitable extension of a function $F \in C^{k,\beta}(\mathbb{S}^1;\mathbb{R}^n)$, we have the inequality

$$\|F\|_{C^{k,\delta}(\mathbb{S}^1;\mathbb{R}^n)} \leq c(k, \beta, \varepsilon, \theta')\|F\|^{1-\theta'}_{C^{k-2}(\mathbb{S}^1;\mathbb{R}^n)}\|F\|^{\theta'}_{C^{k,\beta}(\mathbb{S}^1;\mathbb{R}^n)}. \tag{2.47}$$

Let us adopt the following notation. If $\gamma(t,\cdot)$ is some curve, we will denote by

$$T(t,x) = \frac{2\pi}{L(\gamma(t,\cdot))}\int_0^x |\partial_x \gamma(t,y)|\,dy,$$

so that the reparametrization

$$\gamma(t, T^{-1}(t,\cdot))$$

is a constant speed curve.

We are finally ready to prove the following theorem, which promotes sub-convergence to full convergence of the flow.

Theorem 2.3.33 (Smooth convergence)**.** *Suppose that (M,g) is an analytic complete Riemannian manifold endowed with an analytic metric tensor g. Let $p \geq 2$ and suppose that $\gamma : [0,+\infty) \times \mathbb{S}^1 \to M$ is a smooth solution of (2.42). Suppose that there exist a sequence of* *isometries $I_n : M \to M$, a sequence of times $t_n \nearrow +\infty$, and a smooth critical point $\gamma_\infty : \mathbb{S}^1 \to M$ of $\mathcal{E}_p$ such that*

$$I_n \circ \gamma(t_n, T^{-1}(t_n, \cdot)) - \gamma_\infty(\cdot) \xrightarrow[n\to\infty]{} 0 \qquad \text{in } C^m(\mathbb{S}^1),$$

for any $m \in \mathbb{N}$. If $p > 2$ assume also that $|k_{\gamma_\infty}(x)| \neq 0$ for any x.

Then the flow $\gamma(t, \cdot)$ converges in $C^m(\mathbb{S}^1)$ to a critical point as $t \to +\infty$, for any m and up to reparametrization.

Proof. In this proof constants may change from line to line and their dependence on universal parameters will be omitted. Let $m \geq 8$ be fixed. Let $\varepsilon \in (0, 1)$ that will be fixed later on. By hypothesis there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|I_l \circ \gamma(t_l, T^{-1}(t_l, \cdot)) - \gamma_\infty(\cdot)\|_{C^m(\mathbb{S}^1)} \leq \varepsilon \qquad \forall\, l \geq n_\varepsilon.$$

Let us rename $\gamma_\varepsilon(\cdot) := I_l \circ \gamma(t_l, T^{-1}(t_l, \cdot))$ for a chosen $l \geq n_\varepsilon$. If $p > 2$, for ε small we can assume that $|k_{\gamma_\varepsilon}(x)| \neq 0$ for any x. By Theorem 2.3.29 there exists a solution $\widetilde{\gamma} : [0, T) \times \mathbb{S}^1 \to M$ of

$$\begin{cases} \partial_t \widetilde{\gamma} = -\left(\nabla_s^2 |k_{\widetilde{\gamma}}|^{p-2} k_{\widetilde{\gamma}} + \frac{1}{p'} |k_{\widetilde{\gamma}}|^p k_{\widetilde{\gamma}} - k_{\widetilde{\gamma}} + R(|k_{\widetilde{\gamma}}|^{p-2} k_{\widetilde{\gamma}}, \tau_{\widetilde{\gamma}})\tau_{\widetilde{\gamma}}\right) & \text{on } [0, T) \times \mathbb{S}^1, \\ \widetilde{\gamma}(0, \cdot) = \gamma_\varepsilon(\cdot), & \end{cases}$$

for some $T > 0$. Since I_{n_ε} is an isometry, we have that $\widetilde{\gamma}(t, x) = I_{n_\varepsilon} \circ \gamma(t_{n_\varepsilon} + t, x)$ up to reparametrization. Hence, recalling Remark 2.3.30, as γ exists for any time by hypothesis, we get that $T = +\infty$.

We denote by $\widehat{\gamma}$ the constant speed reparametrization of $\widetilde{\gamma}$. For any ε sufficiently small, we can write $\widehat{\gamma}$ as a variation of γ_∞, at least for small times. More precisely, there is some $T' \in (0, +\infty]$ such that for any $t \in [0, T')$ there exists $\psi_t \in T(\gamma_\infty)^{4,p}$ such that $\widehat{\gamma}(t, \cdot) = \exp \psi_t(x)$ and $\|\psi_t\|_{W^{4,p}} < \sigma$, with σ is as in Corollary 2.3.28 applied to γ_∞. We assume that T' is the maximal time such that $\widehat{\gamma}$ can be written in such a way with fields ψ_t with $\|\psi_t\|_{W^{4,p}} < \sigma$.

Suppose by contradiction that $T' < +\infty$. Up to choosing a smaller $\sigma = \sigma(\gamma_\infty)$, we can apply Proposition 2.3.31 with $\Gamma = \gamma_\infty$ and $\tau = T'$ on the flow $\widetilde{\gamma}$. In the notation on Proposition 2.3.31, we obtain

$$\begin{aligned}
\sup_{[0,T')} \|\widehat{\gamma}(t, \cdot) - \gamma_\varepsilon\|_{C^{m-3}(\mathbb{S}^1)} &\leq \sup_{[0,T')} c\|\widehat{\gamma}(t, \cdot) - \gamma_\varepsilon(\cdot)\|_{W^{m-2,2}(\mathbb{S}^1)} \\
&\leq \sup_{[0,T')} c\|\widehat{\gamma}(t, \cdot) - \gamma_\varepsilon\|_{C^1(\mathbb{S}^1)} + C(\sigma(\gamma_\infty))\|k_{\widehat{\gamma}}(t, \cdot) - k_{\gamma_\varepsilon}\|_{W^{m-4,2}(\mathbb{S}^1)} \\
&\leq \sup_{[0,T')} c\|\widehat{\gamma}(t, \cdot) - \gamma_\infty\|_{C^1(\mathbb{S}^1)} + c\|\gamma_\varepsilon - \gamma_\infty\|_{C^1(\mathbb{S}^1)} + \\
&\qquad\qquad + C(\sigma(\gamma_\infty))(\|k_{\widehat{\gamma}}(t, \cdot)\|_{W^{m-4,2}(\mathbb{S}^1)} + \|k_{\gamma_\varepsilon}\|_{W^{m-4,2}(\mathbb{S}^1)}) \\
&\leq C(\sigma(\gamma_\infty)) + c\varepsilon + C(m, \|k_{\gamma_\infty}\|_{W^{2,2}}, \sigma(\gamma_\infty), \mathcal{U}, \Lambda)(1 + \|k_{\gamma_\varepsilon}\|_{W^{m-2,2}}) \\
&\leq C(m, \gamma_\infty, \mathcal{U}, \Lambda)(1 + \|k_{\gamma_\varepsilon}\|_{W^{m-2,2}}) \\
&\leq C(m, \gamma_\infty, \mathcal{U}, \Lambda)(1 + \varepsilon + \|k_{\gamma_\infty}\|_{C^{m-2}(\mathbb{S}^1)}) \\
&\leq C(m, \gamma_\infty, \mathcal{U}, \Lambda).
\end{aligned}$$

Also, $\mathcal{U}$ and Λ only depend on γ_∞ and $\sigma = \sigma(\gamma_\infty)$, then the above estimate becomes

$$\sup_{[0,T')} \|\widehat{\gamma}(t, \cdot) - \gamma_\varepsilon(\cdot)\|_{C^{m-3}(\mathbb{S}^1)} \leq C(m, \gamma_\infty),$$

and we observe that the constant on the right hand side is independent of ε. By triangular inequality we also have

$$\sup_{[0,T')} \|\widehat{\gamma}(t, \cdot) - \gamma_\infty(\cdot)\|_{C^{m-3}(\mathbb{S}^1)} \leq C(m, \gamma_\infty). \tag{2.48}$$

Now define

$$H(t) = (\mathcal{E}_p(\widehat{\gamma}(t,\cdot)) - \mathcal{E}_p(\gamma_\infty))^\theta,$$

where θ is the Łojasiewicz–Simon exponent of Corollary 2.3.28. Observe that since $\mathcal{E}_p(\gamma_0) \geq \mathcal{E}_p(\gamma_\infty)$, by uniqueness of the flow we also have that $\mathcal{E}_p(\widehat{\gamma}(t,\cdot)) \geq \mathcal{E}_p(\gamma_\infty)$ for any t, and we can also assume that $0 < \mathcal{E}_p(\widehat{\gamma}(t,\cdot)) - \mathcal{E}_p(\gamma_\infty) < 1$ without loss of generality. In particular H is well defined and positive.

By Remark 2.3.30 we have that $\partial_t^\perp \widehat{\gamma} = -|\partial_x \widehat{\gamma}|^{-1} \nabla_{T(\widehat{\gamma})^{p'}, T(\widehat{\gamma})^p} E(0)$. Using Corollary 2.3.28 get that

$$\begin{aligned}
-\frac{d}{dt} H(t) &= -\theta H^{\frac{\theta-1}{\theta}}(t) \langle \nabla_{T(\widehat{\gamma})^{p'}, T(\widehat{\gamma})^p} E(0), \partial_t^\perp \widehat{\gamma} \rangle_{L^{p'}(dx), L^p(dx)} \\
&= \theta H^{\frac{\theta-1}{\theta}}(t) \|\partial_t^\perp \widehat{\gamma}\|^2_{L^2(ds_{\widehat{\gamma}})} \\
&\geq C(L(\widehat{\gamma})) \theta H^{\frac{\theta-1}{\theta}}(t) \|\partial_t^\perp \widehat{\gamma}\|_{L^{p'}(ds_{\widehat{\gamma}})} \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})} \\
&\geq C(L(\widehat{\gamma}), \gamma_\infty) \theta H^{\frac{\theta-1}{\theta}}(t) \|\nabla_{T(\gamma_\infty)^{p'}, T(\gamma_\infty)^p} \mathbf{E}(\psi_t)\|_{L^{p'}(ds_{\widehat{\gamma}})} \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})} \\
&\geq C(L(\widehat{\gamma}), \gamma_\infty) \theta H^{\frac{\theta-1}{\theta}}(t) |\mathbf{E}(\psi_t) - \mathbf{E}(0)|^{1-\theta} \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})} \\
&= C(L(\widehat{\gamma}), \gamma_\infty) \theta H^{\frac{\theta-1}{\theta}}(t) |\mathcal{E}_p(\widehat{\gamma}(t,\cdot)) - \mathcal{E}_p(\gamma_\infty)|^{1-\theta} \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})} \\
&= C(L(\widehat{\gamma}), \gamma_\infty) \theta \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})},
\end{aligned} \tag{2.49}$$

for $t \in [0, T')$.

Now let us write more explicitly $\widehat{\gamma}$ as $\widehat{\gamma}(t,x) = \widetilde{\gamma}(t, \chi(t,x))$, where $\chi(t,\cdot)$ is the inverse of

$$x \mapsto \frac{2\pi}{L(\widetilde{\gamma}(t,\cdot))} \int_0^x ds_{\widetilde{\gamma}},$$

and $|\partial_x \widetilde{\gamma}|(0,x) = |\partial_x \widehat{\gamma}|(0,x) = \frac{L(\gamma_\varepsilon)}{2\pi}$, as the initial datum γ_ε is parametrized with constant speed.

Using (2.49) we have that

$$\begin{aligned}
\left| \int_0^x ds_{\widetilde{\gamma}} - \frac{L(\gamma_\varepsilon)}{2\pi} x \right| &= \left| \int_0^t \partial_t \int_0^x ds_{\widetilde{\gamma}}\, dt \right| \leq \int_0^t \left| \int_0^x \langle k_{\widetilde{\gamma}}, \partial_t \widetilde{\gamma} \rangle \, ds_{\widetilde{\gamma}} \right| dt \leq C(\gamma_\infty) \int_0^t \|\partial_t \widetilde{\gamma}\|_{L^2(ds_{\widetilde{\gamma}})}\, dt \\
&= C(\gamma_\infty) \int_0^t \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})}\, dt \leq C(\gamma_\infty) H(0) \leq C(\gamma_\infty) \|\gamma_\varepsilon - \gamma_\infty\|^\theta_{C^2} \leq C(\gamma_\infty) \varepsilon^\theta,
\end{aligned}$$

for any $t < T'$ and any $x \in \mathbb{S}^1$. It follows that $\left| \frac{2\pi}{L(\widetilde{\gamma})} \int_0^x ds_{\widetilde{\gamma}} - x \right| \leq C(\gamma_\infty) \varepsilon^\theta$ as well. If ε is sufficiently small, we deduce that

$$|\chi(t,x) - x| \leq C(\gamma_\infty) \varepsilon^\theta, \tag{2.50}$$

for any $t < T'$ and any $x \in \mathbb{S}^1$.

Therefore, writing $\widetilde{\gamma}(t,y) = \widehat{\gamma}(t, \varphi(t,y))$ and letting $\widetilde{\gamma}_\infty := \gamma_\infty(t, \varphi(t,y))$, we have

$$\begin{aligned}
\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|^2_{L^2(dx)} \leq C(\gamma_\infty) \|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|^2_{L^2(ds_{\widehat{\gamma}})} &= \int |\widetilde{\gamma}(t,y) - \gamma_\infty(\varphi(t,y))|^2 \, ds_{\widetilde{\gamma}} \\
&\leq C(\gamma_\infty) \int |\widetilde{\gamma}(t,y) - \widetilde{\gamma}(0,y)|^2 \, ds_{\widetilde{\gamma}} + |\widetilde{\gamma}(0,y) - \widetilde{\gamma}_\infty(t,y)|^2 \, ds_{\widetilde{\gamma}}.
\end{aligned}$$

We estimate the two terms above as

$$\left(\int |\widetilde{\gamma}(t,y) - \widetilde{\gamma}(0,y)|^2 \, ds_{\widetilde{\gamma}} \right)^{\frac{1}{2}} \leq \int_0^t \|\partial_t \widetilde{\gamma}\|_{L^2(ds_{\widetilde{\gamma}})}\, dt = \int_0^t \|\partial_t^\perp \widehat{\gamma}\|_{L^2(ds_{\widehat{\gamma}})}\, dt \leq C(\gamma_\infty) H(0) \leq C(\gamma_\infty) \varepsilon^\theta,$$

and

$$\begin{aligned}
\int |\widetilde{\gamma}(0,y) - \widetilde{\gamma}_\infty(t,y)|^2\, ds_{\widetilde{\gamma}} = \int |\gamma_\varepsilon(y) - \widetilde{\gamma}_\infty(t,y)|^2\, ds_{\widetilde{\gamma}} &= \int |\gamma_\varepsilon(\chi(t,x)) - \gamma_\infty(x)|^2\, ds_{\widehat{\gamma}} \\
&\le 2\int |\gamma_\varepsilon(\chi(t,x)) - \gamma_\varepsilon(x)|^2 + |\gamma_\varepsilon(x) - \gamma_\infty(x)|^2\, ds_{\widehat{\gamma}} \\
&\le C(\gamma_\infty)\left((\mathrm{Lip}(\gamma_\varepsilon))^2 \int_{\mathbb{S}^1} |\chi(t,x) - x|^2\, dx + \|\gamma_\varepsilon - \gamma_\infty\|_\infty^2 \right) \\
&\le C(\gamma_\infty)\varepsilon^{2\theta},
\end{aligned}$$

where we used (2.50). It follows that

$$\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{L^2(dx)} \le C(\gamma_\infty)\varepsilon^\theta, \tag{2.51}$$

for any $t < T'$. By (2.45) and (2.46), using that for $s \in (\frac{1}{2}, 1)$ we have the continuous embeddings $W^{s,2} \hookrightarrow C^{0,\alpha'} \hookrightarrow L^p$, for some $\alpha \in (0,1)$ we can write

$$\begin{aligned}
\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{W^{4,p}} &\le C\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{C^5}^\alpha \|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{L^p(dx)}^{1-\alpha} \\
&\le C\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{C^5}^\alpha \|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{W^{s,2}(dx)}^{1-\alpha} \\
&\le C\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{C^5}^\alpha \left(C\|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{L^2(dx)}^{1-s} \|\widehat{\gamma}(t,\cdot) - \gamma_\infty\|_{W^{1,2}(dx)}^{s} \right)^{1-\alpha} \\
&\le C(\gamma_\infty)\varepsilon^{\theta(1-s)(1-\alpha)}
\end{aligned}$$

for $t \in [0,T')$, where in the last inequality we used (2.48) and (2.51). This means that

$$\|\exp(\psi_t) - \gamma_\infty\|_{W^{4,p}} \le C(\gamma_\infty)\varepsilon^{\theta(1-s)(1-\alpha)},$$

for $t \in [0,T')$, and if ε is sufficiently small this implies that $\|\psi_t\|_{W^{4,p}} \le \frac{1}{2}\sigma$ for any $t \in [0,T')$, contradicting the maximality of T'.

Hence we proved that for some now fixed $\varepsilon_0 > 0$, for any $l \ge n_{\varepsilon_0}$ the constant speed parametrized flow $\widehat{\gamma}_l$ starting at $\gamma_l := I_l \circ \gamma(t_l, T^{-1}(t_l,\cdot)))$ exists for any time and it can be written as a variation of γ_∞ with uniformly bounded fields. In particular the evolution $\widehat{\gamma}_l$ stays in the compact set $\mathcal{U}$ for any t for any $l \ge n_{\varepsilon_0}$. Similarly, it follows that the original flow γ definitely remains in $I_{n_{\varepsilon_0}}^{-1}(\mathcal{U}) =: \mathcal{V}$.

For $l \ge n_{\varepsilon_0}$ consider the sequence of isometries $I_l : \mathcal{V} \to \mathcal{U}$. By Ascoli-Arzelà Theorem, up to subsequence we have that I_l uniformly converges to a map $I : \mathcal{V} \to \mathcal{U}$, which is still an isometry, and thus it is smooth by Myers–Steenrod Theorem (see [KN96, p. 169]). Moreover, observe that by compactness there exists a constant $\Delta > 0$ such that

$$|x-y| \le d_{(M,g)}(x,y) \le \Delta|x-y|,$$

for any couple $x,y \in \mathcal{U}$ or $x,y \in \mathcal{V}$, where $d_{(M,g)}$ is the geodesic distance on M.

Let us denote $\widehat{\gamma}_0 := \widehat{\gamma}_{n_{\varepsilon_0}}$. By uniqueness and Remark 2.3.30 we have that

$$\widehat{\gamma}_0 = I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \widehat{\gamma}_l,$$

up to a translation in time depending on l, for any $l > n_{\varepsilon_0}$.

We know that for any $\varepsilon \in (0,\varepsilon_0)$ and any $l \ge n_\varepsilon$ the flow $\widehat{\gamma}_l$ verifies that

$$\|\widehat{\gamma}_l(t,\cdot) - \gamma_\infty\|_{L^2(dx)} \le C(\gamma_\infty)\varepsilon^\theta,$$

for any $t > 0$. Indeed, this follows from (2.51) applied on the flow $\widehat{\gamma}_l$. Therefore we have

$$\|\widehat{\gamma}_0(t+T_l,\cdot) - I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \gamma_\infty\|_{L^2(dx)} = \|I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \widehat{\gamma}_l(t,\cdot) - I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \gamma_\infty\|_{L^2(dx)} \le \bar{C}\varepsilon^\theta,$$

for any ε, $l \ge n_\varepsilon$, and any $t > 0$, where $\bar{C} = \bar{C}(\gamma_\infty, \Delta)$ and T_l depends on l. Also, we have that

$$I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \gamma_\infty \xrightarrow[l]{} I_{n_{\varepsilon_0}} \circ I^{-1} \circ \gamma_\infty,$$

uniformly on $\mathbb{S}^1$, and then in $L^2(dx)$. Hence, finally, for any given $\bar{\varepsilon}$ we can set $\varepsilon = (\bar{C}\bar{\varepsilon})^{\frac{1}{\theta}}$ and take $l \ge n_\varepsilon$ such that $\|I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \gamma_\infty - I_{n_{\varepsilon_0}} \circ I^{-1} \circ \gamma_\infty\|_{L^2(dx)} \le \bar{\varepsilon}$, and we obtain

$$\begin{aligned}
&\|\widehat{\gamma}_0(t,\cdot) - I_{n_{\varepsilon_0}} \circ I^{-1} \circ \gamma_\infty\|_{L^2(dx)} \\
&\quad \le \|\widehat{\gamma}_0(t,\cdot) - I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \gamma_\infty\|_{L^2(dx)} + \|I_{n_{\varepsilon_0}} \circ I_l^{-1} \circ \gamma_\infty - I_{n_{\varepsilon_0}} \circ I^{-1} \circ \gamma_\infty\|_{L^2(dx)} \\
&\quad \le 2\bar{\varepsilon},
\end{aligned}$$

for any $t > T_l$. This implies that

$$\exists \lim_{t\to+\infty} \widehat{\gamma}_0(t,\cdot) = I_{n_{\varepsilon_0}} \circ I^{-1} \circ \gamma_\infty =: \bar{\gamma} \qquad \text{in } L^2(dx).$$

Now we can use (2.45) to get that

$$\|\widehat{\gamma}_0(t,\cdot) - \bar{\gamma}\|_{W^{m-4,2}} \le C\|\widehat{\gamma}_0(t,\cdot) - \bar{\gamma}\|^{\alpha}_{C^{m-3}}\|\widehat{\gamma}_0(t,\cdot) - \bar{\gamma}\|^{1-\alpha}_{L^2(dx)},$$

and using (2.48) we see that $\widehat{\gamma}_0 \to \bar{\gamma}$ in $W^{m-4,2}$. Taking higher m and interpolating using (2.47), one gets that $\widehat{\gamma}_0$ converge smoothly. Finally, since the original flow γ is a reparametrization of $I^{-1}_{n_{\varepsilon_0}} \circ \widehat{\gamma}_0$, it smoothly converges as desired, up to reparametrization. $\square$

Let us conclude by stating some consequences of Theorem 2.3.33. As we already mentioned, sub-convergence of the flow for $p = 2$ has been proved in the literature in some ambient spaces; thus we can apply Theorem 2.3.33 to get the following consequence.

Corollary 2.3.34. *Let $p = 2$ and suppose that $\gamma : [0,+\infty) \times \mathbb{S}^1 \to M$ is a smooth solution of (2.42). Assume that M is either the Euclidean space $\mathbb{R}^m$, the hyperbolic plane $\mathbb{H}^2$, or the unit 2-sphere $\mathbb{S}^2$.*

Then γ smoothly converges as $t \to +\infty$ to a critical point γ_∞ of $\mathcal{E}_2$ up to reparametrization. In particular, the flow stays in a compact set of M for any time.

The proof of Corollary 2.3.34 follows from Theorem 2.3.33 by the fact that sub-convergence of the flow has been proved in [DKS02] if $M = \mathbb{R}^m$, in [DS17] if $M = \mathbb{H}^2$, and in [Dal+18] if M is a 2-sphere.

Theorem 2.3.33 is clearly applicable in the special case where the isometries I_n appearing in the statement are the identity on M for any n. This is precisely the case in which one already knows that the flow remains in a compact subset of M. Such a hypothesis is automatically satisfied if the ambient manifold M is compact. Therefore we can state the following.

Corollary 2.3.35. *Suppose that (M, g) is an analytic compact Riemannian manifold endowed with an analytic metric tensor g. Let $p = 2$ and suppose that $\gamma : [0,+\infty) \times \mathbb{S}^1 \to M$ is a smooth solution of (2.42). Suppose that $\|\gamma(t,\cdot)\|_{C^m(\mathbb{S}^1)} \le C(m) < +\infty$ for any $t \ge 0$.*

Then the flow $\gamma(t,\cdot)$ converges in $C^m(\mathbb{S}^1)$ to a critical point as $t \to +\infty$, for any m and up to reparametrization.

Corollary 2.3.35 follows from the fact that, since M is compact, uniform bounds in C^m for any m guarantee the existence of a sequence of times $t_n \nearrow +\infty$ and of a critical point γ_∞ such that $\gamma(t_n, \cdot) \to \gamma_\infty$ in $C^m(\mathbb{S}^1)$ for any m as $n \to +\infty$ up to reparametrization. Hence we can apply Theorem 2.3.33 and Corollary 2.3.35 follows.

Let us remark that under the assumptions that M is an analytic compact manifold with an analytic metric g and $p = 2$, the uniform bounds in C^m in the hypotheses of Corollary 2.3.35 are likely to be true in general. Indeed, one should be able to derive the usual parabolic estimates in the same fashion of [DKS02], thus getting the desired uniform bounds.

Let us conclude with a few comments.

Remark 2.3.36. Theorem 2.3.33 remains true if one considers the analogously defined flow of the energy $\int \lambda + \frac{1}{p}|k|^p$ for any $\lambda > 0$. We believe that the statement of Corollary 2.3.34 continues to hold true if M is any hyperbolic space $\mathbb{H}^m$ or a sphere $\mathbb{S}^m$ with $m \geq 2$, but the sub-convergence of the flow has not been proved explicitly in the literature in these ambients, up to the knowledge of the author. More generally, it is likely that Corollary 2.3.34 remains true whenever (M, g) is a homogeneous manifold, that is a Riemannian manifold such that the group of isometries acts transitively on M; indeed, in such a case, one should be able to prove sub-convergence of the flow for $p = 2$ exactly as in [DKS02].

We remark that the hypothesis of (M^m, g) being of bounded geometry is not sufficient to imply that the solution of the flow converges. Indeed, in Section 2.3.6 we construct a simple example of a solution to the flow of the elastic energy with $p = 2$ in a surface in $\mathbb{R}^3$ that does not converge.

Remark 2.3.37. We remark that, even if Corollary 2.3.34 implies that the solution of the elastic flow for $p = 2$ in $\mathbb{R}^m$ stays in a compact region, this result does not tell anything about the size of the compact set containing the flow. We believe it is a nice open question to quantify, if possible, the size of such compact set depending on the given initial datum γ_0. We also mention that a related problem which is still open, up to the author's knowledge, is to prove or disprove the Huisken's conjecture stating that for the flow of $\mathcal{E}_2$ in $\mathbb{R}^2$, if the datum γ_0 does not intersect a closed halfplane, then the solution $\gamma(t, \cdot)$ is never completely contained in such halfplane.

2.3.5 Proof of Proposition 2.3.31

Throughout this section we assume the hypotheses of Proposition 2.3.31. More precisely we consider $\gamma : [0, \tau) \times \mathbb{S}^1 \to M$ to be a smooth solution of (2.42), and $\Gamma : \mathbb{S}^1 \to M$ is a fixed smooth curve parametrized with constant speed. Let $\widehat{\gamma}$ be the constant speed reparametrization of γ. We assume that $\bar{\sigma} > 0$ is such that

$$\|\widehat{\gamma}(t, \cdot) - \Gamma\|_{W^{4,p}} \leq \bar{\sigma},$$

for any $t \in [0, \tau)$. From now on we denote by $k = k(t, x)$ and $\widehat{k} = \widehat{k}(t, x)$ the curvature of $\gamma(t, \cdot)$ and $\widehat{\gamma}(t, \cdot)$ at the point x respectively. We are going to prove that if $\bar{\sigma}$ is small enough, we can bound the Sobolev norm $\|\widehat{k}\|_{W^{m,2}}$ uniformly in time in terms of the initial datum and a suitable constant. We will deal here only with the case $p > 2$. The very same argument can be replicated for the case $p = 2$, and the calculations become even simpler. Therefore we also assume that $|k_\Gamma(x)| \neq 0$ for any x.

We assume that $\bar{\sigma} > 0$ is small enough so that $\|\widehat{\gamma}(t, \cdot) - \Gamma\|_{C^3}$ is so small that

$$|k(t, x)| \geq c > 0,$$

for any $t \in [0,\tau)$ and x, where $c = c(\bar\sigma, k_\Gamma)$. Observe that therefore the choice of $\bar\sigma$ only depends on the curvature k_Γ of Γ. Moreover, by assumptions there exists a bounded neighborhood $\mathcal{U} = \mathcal{U}(\bar\sigma)$ of Γ in M such that the flow γ is contained in $\mathcal{U}$ for every time. We then let $\Lambda := \sup_{x\in\mathcal{U}} |B_x|$, where $|B|$ is the norm of the second fundamental form of $M \hookrightarrow \mathbb{R}^n$. In the forthcoming constants denoted by the capital letter C we will usually omit the dependence on $\bar\sigma$, $\mathscr{E}(\Gamma)$, $\mathcal{U}$, Λ, and a chosen index $m \in \mathbb{N}$.

Let us introduce the so-called scale invariant norms on k. We let

$$\|k\|_{n,q} := \sum_{j=0}^{n} \|\nabla^j k\|_q,$$

where

$$\|\nabla^j k\|_q := L(\gamma(t,\cdot))^{j+1-\frac{1}{q}} \left(\int |\nabla^j k|^q \, ds \right)^{\frac{1}{q}}.$$

As the integrand is a geometric quantity integrated with respect to arclegth, one can verify that

$$\|k\|_{n,q} = \|\widehat{k}\|_{n,q}.$$

First we need to recall a few fact about these norms and some interpolation inequalities.

Lemma 2.3.38. *Let $\gamma(t,x)$ be as above.*

1. *For any $n \in \mathbb{N}$ and $q \in [1,\infty)$, it holds that*

$$\frac{1}{C}\|k\|_{n,q} \le \|\widehat{k}\|_{W^{n,q}} \le C\|k\|_{n,q}. \tag{2.52}$$

2. *For any $n \in \mathbb{N}$ it holds that*

$$\|k\|^2_{n,2} \le c(n)(\|\nabla^n k\|_2^2 + \|k\|_2^2). \tag{2.53}$$

3. *For any $n \in \mathbb{N}$, $q \ge 2$, and $0 \le i < n$, it holds that*

$$\|\nabla^i k\|_q \le C(q,n)\|k\|_2^{1-\alpha}\|k\|^{\alpha}_{n,2}, \tag{2.54}$$

   with $\alpha = \frac{1}{n}\left(i + \frac{1}{2} - \frac{1}{q}\right)$.

4. *Suppose $\nu \ge 2$ and $1 \le i_1 \le ... \le i_\nu \le n-1$ are integers. Let $\mu = \sum_{j=1}^{\nu} i_j$. If $\mu + \frac{\nu}{2} < 2n+1$, then for any $\varepsilon > 0$ it holds that*

$$\int \Pi_{j=1}^{\nu} |\nabla^{i_j} k| \le \varepsilon \int |\nabla^n k|^2 \, ds + C(\varepsilon). \tag{2.55}$$

Proof. We prove the items separately.

1. If ϕ is a smooth normal field along γ on M, it holds that

$$\nabla^n \phi = \partial_s^n \phi + \sum_{j=0}^{n-1} L_j^n(\partial_s^j \phi), \tag{2.56}$$

for any n, where L^n_j is a smooth tensor defined on $\mathcal{U}$, and $\|L^n_j\| \le C(n, \bar\sigma, k_\Gamma, \mathcal{U}, \Lambda)$. Indeed the equality immediately follows for $n = 1$, and then

$$\nabla^{n+1}\phi = \partial_s\nabla^n\phi - \langle\partial_s\nabla^n\phi, \tau\rangle\tau - \langle\partial_s\nabla^n\phi, N_j\rangle N_j = \partial_s\nabla^n\phi + \langle\nabla^n\phi, \partial_s\tau\rangle\tau - \langle\nabla^n\phi, \partial_s N_j\rangle N_j,$$

where $\{N_j\}$ is a locally defined orthonormal frame of M, and then (2.56) follows by induction on n.

Now we consider (2.56) with $\phi = k$. Observe that for any $n \in \mathbb{N}$ we have

$$\int |\partial^n_{s_\gamma} k|^q \, ds_\gamma = \int |\partial^n_{s_{\widehat\gamma}} \widehat{k}|^q \, ds_{\widehat\gamma}.$$

Since $0 < L(\Gamma) - \eta \le L(\gamma) \le L(\Gamma) + \eta$ for $\bar\sigma$ small, we get that (2.52) follows from (2.56) by induction on n using that $\widehat\gamma$ is parametrized with constant speed.

2. By scaling invariance, we can assume $L(\gamma(t, \cdot)) = 1$. For $n = 2$ the inequality follows from $\|\nabla k\|_2^2 = -\int\langle k, \nabla^2 k\rangle \, ds$; for higher n the inequality follows by induction.

3. By scaling invariance, we can assume that $L(\gamma(t, \cdot)) = 1$ and also that $\gamma(t, \cdot)$ is arclength parametrized. Then since $|\partial_s|\phi|| \le |\nabla\phi|$ for any normal field ϕ along γ on M, the standard proof of [Aub98, Theorem 3.70] applies.

4. Equation (2.55) follows by the very same arguments leading to [DKS02, (2.16)], observing that the proof only relies on Hölder inequality, on (2.53), and on (2.54). The constant $C(\varepsilon)$ appearing on the right hand side of (2.55) depends on ε and we omitted dependence on $\int |k|^2 \, ds$ as it is estimated in terms of $\mathscr{E}(\Gamma)$ and $\bar\sigma$.

□

By Item 1 in Lemma 2.3.38, our aim is then to bound the norms $\|\nabla^m k\|_2$ uniformly in time for any $m \in \mathbb{N}$, as this will imply Proposition 2.3.31.

Let us denote by V the velocity of the flow γ, that is

$$V := -\nabla^2(|k|^{p-2}k) - \frac{1}{p'}|k|^p k + k - R(|k|^{p-2}k, \tau)\tau.$$

We denote $\partial_t^\perp := \gamma^\perp\partial_t$, where recall that $\gamma^\perp = M^\top - \gamma^\top$. First of all, we need to derive the evolution equations for the derivatives of the curvature.

Lemma 2.3.39. *Let $\gamma(t, x)$ be as above.*

1. *It holds that*
$$\partial_t^\perp k = \nabla^2 V + \langle V, k\rangle k + R(V, \tau)\tau. \tag{2.57}$$

2. *For any smooth normal field ϕ along γ on M it holds that*
$$\partial_t^\perp\nabla\phi - \nabla\partial_t^\perp\phi = \langle\phi, k\rangle\nabla V + \langle k, V\rangle\nabla\phi - \langle\phi, \nabla V\rangle k + R(V, \tau)\phi. \tag{2.58}$$

3. *For any $m \in \mathbb{N}$, denoting by $\phi_m = \nabla^m k$, we have that*
$$\begin{aligned}\partial_t^\perp\phi_m + \nabla^{m+4}(|k|^{p-2}k) = \nabla^{m+2}\left(-\frac{1}{p'}|k|^p k + k - R(|k|^{p-2}k, \tau)\tau\right) + \\ + \nabla^m\left(\langle V, k\rangle k + R(V, \tau)\tau\right) + \sum_{j=0}^{m-1}\nabla^{m-1-j}(X(\phi_j)),\end{aligned} \tag{2.59}$$

where

$$X(\phi) = \langle \phi, k\rangle \nabla V + \langle k, V\rangle \nabla\phi - \langle \phi, \nabla V\rangle k + R(V,\tau)\phi,$$

for any smooth normal field ϕ along γ on M.

Proof. We prove the items separately.

1. Equation (2.57) follows from (2.16).

2. For ϕ, ψ normal along γ on M we compute

$$\langle \partial_t^\perp \nabla\phi, \psi\rangle = \langle \partial_t(\partial_s\phi + \langle\phi, k\rangle\tau + \langle \phi, \partial_s N_j\rangle N_j), \psi\rangle,$$

where $\{N_j\}$ is a locally defined orthonormal frame of M. Using that $\partial_t\partial_s = \partial_s\partial_t + \langle k, V\rangle\partial_s$, writing

$$\partial_t\phi = \partial_t^\perp\phi - \langle \phi, \partial_t^\perp\tau\rangle\tau - \langle\phi, \partial_t^\perp N_j\rangle N_j,$$

using that $\partial_t^\perp\tau = \nabla V$ by (2.15) and using Gauss equation, one obtains (2.58).

3. We know that (2.59) holds for $m=0$ by (2.57). The cases $m \geq 1$ then easily follow by induction on m using (2.58).

□

We introduce the following notation. If $\phi_1, ..., \phi_r$ are vector fields along γ, we denote by

$$\phi_1 * ... * \phi_r,$$

a generic contraction of the given fields by some tensor whose norm is locally bounded on M. The outcome of the contraction may be a vector or a scalar function. In particular we have that $|\phi_1 * ... * \phi_r| \leq C|\phi_1|...|\phi_r|$ on $\mathcal{U}$ for some constant C clearly depending also on the specific tensor.

We need a few last inequalities and then we will be able to prove the desired bounds using the evolution equations (2.59).

Lemma 2.3.40. *Let $\gamma(t,x)$ be as above.*

1. *Let $n \in \mathbb{N}$ with $n \geq 1$ and $j \in \{1, ..., n\}$. For any $\varepsilon > 0$ it holds that*

$$\int |\nabla^{n-j}k|^2\, ds \leq \varepsilon \|\nabla^n k\|_2^2 + C(\varepsilon).$$

2. *Let $n \in \mathbb{N}$ with $n \geq 1$ and $j \in \{1, ..., n\}$. For any $\varepsilon > 0$ it holds that*

$$\int |\nabla^{n-j}k||\nabla^n k|\, ds \leq \varepsilon \|\nabla^n k\|_2^2 + C(\varepsilon).$$

3. *For any $q \in \mathbb{R}$ and $L \in \mathbb{N}$ with $L \geq 1$ it holds that*

$$\partial_s^L |k|^q = C_{q,L}|k|^{q-2L}\langle k, \nabla k\rangle^L + \sum_{\substack{l_1+...+l_\lambda = L \\ 1\leq l_1 \leq ... \leq l_\lambda < L}} C_{\bar{l}}|k|^{q_{\bar{l}}} k * ... * k * \nabla^{l_1}k * ... * \nabla^{l_\lambda}k +$$
$$+ q|k|^{q-2}\langle k, \nabla^L k\rangle,$$

where $C_{q,L} \in \mathbb{R}$ and we denoted by $\bar{l}$ the array $(l_1, ..., l_\lambda)$, and $C_{\bar{l}}$, $q_{\bar{l}}$ are some numbers depending on $\bar{l}$.

Proof. We prove the items separately.

1. Applying (2.54) and then (2.53) we find

$$\int |\nabla^{n-j}k|^2\,ds \le C\|\nabla^{n-j}k\|_2^2 \le C\|k\|_2^{2(1-\alpha)}\|k\|_{n,2}^{2\alpha} \le C\|k\|_2^{2(1-\alpha)}(\|\nabla^n k\|_2^2+\|k\|_2^2)^{\alpha}$$
$$\le C\|\nabla^n k\|_2^{2\alpha}+C \le \varepsilon\|\nabla^n k\|_2^2 + C(\varepsilon),$$

where we used that $\alpha = \frac{n-j}{n} = 1-\frac{j}{n} < 1$, and then $2\alpha < 2$ and we used Young inequality.

2. Arguing as in Item 1 we obtain

$$\int |\nabla^{n-j}k||\nabla^n k|\,ds \le C\|\nabla^{n-j}k\|_2\|\nabla^n k\|_2 \le C\|k\|_2^{1-\alpha}\|k\|_{n,2}^{\alpha}\|\nabla^n k\|_2$$
$$\le C(\|\nabla^n k\|_2^2+\|k\|_2^2)^{\frac{\alpha}{2}}\|\nabla^n k\|_2 \le C\|\nabla^n k\|_2 + C\|\nabla^n k\|_2^{1+\alpha},$$

where $\alpha = 1-\frac{j}{n}$, then $1+\alpha<2$, and thus the conclusion follows again by Young inequality.

3. For $L=1$ we have $\partial_s|k|^q = q|k|^{q-2}\langle k,\nabla k\rangle$, and then for $L\ge 2$ the claim follows by induction.

□

We are ready for estimating the norms $\|\nabla^m k\|_2$ uniformly in time. Recall that $\|\nabla k\|_{L^\infty} \le C = C(\bar\sigma, \|k_\Gamma\|_{W^{2,2}})$ and also $\|\nabla^2 k\|_p \le C = C(\bar\sigma, \|\nabla^2 k_\Gamma\|_{L^2})$ uniformly in time by assumptions. From now on let

$$m \ge 3,$$

and assume by induction that

$$\|\widehat{k}\|_{W^{m-1,2}}(t) \le C(1+\|\widehat{k}\|_{W^{m-1,2}}(0)),$$

for any $t\in[0,\tau)$, where always $C = C(m-1,\bar\sigma,\mathcal{U},\|k_\Gamma\|_{W^{2,2}},\Lambda)$. In particular $|\nabla^r k|$ is uniformly bounded in L^∞ for any $r=0,...,m-2$, and $\nabla^{m-1}k\in L^2(ds_\gamma)$.

Multiplying (2.59) by $\nabla^m k$ and integrating we get the evolutions

$$\begin{aligned}\partial_t\left(\frac12\int|\nabla^m k|^2\,ds\right) &+ \int\langle\nabla^{m+2}k,\nabla^{m+2}(|k|^{p-2}k)\rangle\,ds \\ &= \int\left\langle \nabla^m k,\nabla^{m+2}\left(-\frac{1}{p'}|k|^p k + k - R(|k|^{p-2}k,\tau)\tau\right)\right\rangle ds - \frac12\int|\nabla^m k|^2\langle V,k\rangle\,ds+ \\ &+\int\left\langle\nabla^m k,\nabla^m\left(\langle V,k\rangle k + R(V,\tau)\tau\right)+\sum_{j=0}^{m-1}\nabla^{m-1-j}(X(\nabla^j k))\right\rangle ds,\end{aligned} \tag{2.60}$$

in the notation of (2.59). We now estimate each term in (2.60). We are going to see that, once the second summand on the left hand side is correctly estimated, by the same arguments also the remaining terms will be controlled in the right way.

We have

$$\begin{aligned}\int\langle\nabla^{m+2}k,\nabla^{m+2}(|k|^{p-2}k)\rangle &= \sum_{j=0}^{m+2}\binom{m+2}{j}\int\langle\nabla^{m+2}k,\partial_s^j(|k|^{p-2})\nabla^{m+2-j}k\rangle\\ &= \int|k|^{p-2}|\nabla^{m+2}k|^2+\sum_{j=1}^{m+2}\binom{m+2}{j}\int\langle\nabla^{m+2}k,\partial_s^j(|k|^{p-2})\nabla^{m+2-j}k\rangle\\ &\geq C\int|\nabla^{m+2}k|^2+\sum_{j=1}^{m+2}\binom{m+2}{j}\int\langle\nabla^{m+2}k,\partial_s^j(|k|^{p-2})\nabla^{m+2-j}k\rangle.\end{aligned}\tag{2.61}$$

Let $j\in\{1,...,m+2\}$. By Lemma 2.3.40 we find

$$\begin{aligned}\int\langle\nabla^{m+2}k,\partial_s^j(|k|^{p-2})\nabla^{m+2-j}k\rangle &= C_{p-2,j}\int|k|^{p-2-2j}\langle k,\nabla k\rangle^j\langle\nabla^{m+2}k,\nabla^{m+2-j}k\rangle+\\ &+\sum_{\substack{l_1+...+l_\lambda=j\\ 1\leq l_1\leq...\leq l_\lambda<j}}C_{\bar{l}}\int|k|^{q_{\bar{l}}}k*...*k\nabla^{l_1}k*...*\nabla^{l_\lambda}k\langle\nabla^{m+2}k,\nabla^{m+2-j}k\rangle+\\ &+(p-2)\int|k|^{p-4}\langle k,\nabla^j k\rangle\langle\nabla^{m+2}k,\nabla^{m+2-j}k\rangle.\end{aligned}\tag{2.62}$$

We study separately the case $j=m+2$. So let $j=m+2$. In this case

$$\begin{aligned}\int\langle\nabla^{m+2}k,\partial_s^{m+2}(|k|^{p-2})k\rangle &= C_{p-2,m+2}\int|k|^{p-2-2(m+2)}\langle k,\nabla k\rangle^{m+2}\langle\nabla^{m+2}k,k\rangle+\\ &+\sum_{\substack{l_1+...+l_\lambda=m+2\\ 1\leq l_1\leq...\leq l_\lambda<m+2}}C_{\bar{l}}\int|k|^{q_{\bar{l}}}k*...*k\nabla^{l_1}k*...*\nabla^{l_\lambda}k\langle\nabla^{m+2}k,k\rangle+\\ &+(p-2)\int|k|^{p-4}\langle\nabla^{m+2}k,k\rangle^2\\ &\geq C_{p-2,m+2}\int|k|^{p-2-2(m+2)}\langle k,\nabla k\rangle^{m+2}\langle\nabla^{m+2}k,k\rangle+\\ &+C_{(1,m+1)}\int|k|^{q_{(1,m+1)}}k*...*k\nabla k*\nabla^{m+1}k\langle\nabla^{m+2}k,k\rangle+\\ &+\sum_{\substack{l_1+...+l_\lambda=m+2\\ 1\leq l_1\leq...\leq l_\lambda<m+1}}C_{\bar{l}}\int|k|^{q_{\bar{l}}}k*...*k\nabla^{l_1}k*...*\nabla^{l_\lambda}k\langle\nabla^{m+2}k,k\rangle.\end{aligned}$$

Clearly

$$\left|C_{p-2,m+2}\int|k|^{p-2-2(m+2)}\langle k,\nabla k\rangle^{m+2}\langle\nabla^{m+2}k,k\rangle\right|\leq\varepsilon\int|\nabla^{m+2}k|^2+C(\varepsilon),$$

$$\left|C_{(1,m+1)}\int|k|^{q_{(1,m+1)}}k*...*k\nabla k*\nabla^{m+1}k\langle\nabla^{m+2}k,k\rangle\right|\leq\varepsilon\int|\nabla^{m+2}k|^2+C(\varepsilon).$$

We want to prove the very same estimate for the generic term

$$C_{\bar{l}}\int|k|^{q_{\bar{l}}}k*...*k\nabla^{l_1}k*...*\nabla^{l_\lambda}k*\nabla^{m+2}k,\tag{2.63}$$

with $1\leq l_1\leq...\leq l_\lambda<m+1$ and $l_1+...+l_\lambda=m+2$. We divide two cases.

- Suppose that writing $l_1 \le ... \le l_{i-1} < m-1 \le l_i \le ... \le l_\lambda \le m$ it occurs that $\sum_{r=i}^{\lambda} l_r = m+2$, or suppose that there is some s such that $l_s = m-2$.

  In the first situation, since $\sharp\{l_r \ge m-1\} \le 1 + \frac{3}{m-1}$, if $m > 4$ then $\sharp\{l_r \ge m-1\} \le 1$, and it is impossible to satisfy that $\sum_{r=i}^{\lambda} l_r = m+2$. If instead $m = 3$, then it must be that $l_\lambda = 3, l_{\lambda-1} = 2$, while if $m = 4$, then it must occur that $l_\lambda = l_{\lambda-1} = 3$. We can handle these two cases individually:

$$\begin{aligned}\int |\nabla^2 k||\nabla^3 k||\nabla^5 k| &\le C\|\nabla^2 k\|_p \|\nabla^3 k\|_q \|\nabla^5 k\|_2 \le C\|\nabla^5 k\|_2 (\|\nabla^5 k\|_2^\alpha + C)\\ &\le \varepsilon \int |\nabla^5 k|^2 + C(\varepsilon),\end{aligned} \tag{2.64}$$

  where $\frac{1}{2} + \frac{1}{p} + \frac{1}{q} = 1$, and then $q > 2$, and $\alpha = \frac{1}{5}(3 + \frac{1}{2} - \frac{1}{q}) = \frac{3}{5} + \frac{1}{5p}$ by Lemma 2.3.38, so that $1 + \alpha < 2$;

$$\begin{aligned}\int |\nabla^3 k|^2 |\nabla^6 k| &\le C(\|\nabla^3 k\|_4^4)^{\frac{1}{2}} \|\nabla^6 k\|_2 \le \left(\|\nabla^3 k\|_2^{2(1-s)} \|\nabla^3 k\|_q^{sq}\right)^{\frac{1}{2}} \|\nabla^6 k\|_2\\ &\le \|\nabla^3 k\|_2^{(1-s)} (C + \|\nabla^6 k\|_2^\alpha)^{\frac{sq}{2}} \|\nabla^6 k\|_2 \le C\|\nabla^6 k\|_2 + \|\nabla^6 k\|_2^{1+\frac{\alpha s q}{2}},\end{aligned}$$

  where we used that $\|\nabla^3 k\|_2$ is uniformly bounded by induction, $4 = 2(1-s) + qs$, and $\alpha = \frac{1}{6}(3 + \frac{1}{2} - \frac{1}{q})$ by Lemma 2.3.38. Since $sq = 2 + 2s$, for s small we get $\frac{\alpha s q}{2} = \alpha(1+s) < (\frac{1}{2} + \frac{1}{12})(1+s) < 1$, and we conclude the desired estimate by Young inequality.

  In the other situation, when there is some $l_s = m-2$, we see that if $\sharp\{l_r \ge m-1\} < 2$ then $\{l_r \ge m-1\}$ is either empty or only contains l_λ, and thus by induction the integral is estimated by

$$C\int |\nabla^{l_\lambda} k||\nabla^{m+2} k|, \qquad \text{or} \qquad C\int |\nabla^{m+2} k|,$$

  and we can apply Lemma 2.3.40. If instead $\sharp\{l_r \ge m-1\} \ge 2$, then, from $\sum_1^\lambda l_r = m+2$, we are in the case $m = 3$, and one estimates a term like

$$C\int |\nabla^3 k||\nabla^5 k|,$$

  by Lemma 2.3.40, or like

$$C\int |\nabla^2 k|^2 |\nabla^5 k|,$$

  in the same way we treated (2.64).

- In this second case we have that writing $l_1 \le ... \le l_{i-1} < m-1 \le l_i \le ... \le l_\lambda \le m$ it occurs that $\sum_{r=i}^{\lambda} l_r < m+2$, and $l_r \ne m-2$ for any r.

  In this case we first integrate by parts in (2.63) once, and thus we get something of the form

$$\begin{aligned}\left| C_{\bar{l}} \int |k|^{q_{\bar{l}}} k * ... * k \nabla^{l_1} k * ... * \nabla^{l_\lambda} k * \nabla^{m+2} k \right| &\le C \int \Pi_{r=1}^{\lambda} |\nabla^{j_r} k|\ |\nabla^{m+1} k|\\ &\le C \int \Pi_{r=i}^{\lambda} |\nabla^{j_r} k|\ |\nabla^{m+1} k|,\end{aligned}$$

  where we wrote $1 \le j_1 \le ... j_{i-1} < m-1 \le j_i \le ... \le j_\lambda \le m+1$, where we used that since $l_r \ne m-2$ for any r, then $\sharp\{l_r \ge m-1\} = \sharp\{j_r \ge m-1\}$. It follows that $\sum_{r=i}^{\lambda} j_r \le m+2$, and then $\sharp\{j_r \geq m-1\} \leq \frac{m+2}{m-1}$. Therefore the desired estimate follows by applying (2.55) with

$$\mu = m+1+\sum_{r=i}^{\lambda} j_r \leq 2m+3, \qquad \nu = 1+\sharp\{j_r \geq m-1\} \leq \frac{2m+1}{m-1}, \qquad n = m+2,$$

observing that $\mu + \frac{\nu}{2} \leq 2m+3+\frac{1}{2}\frac{2m+1}{m-1} < 2(m+2)+1$ for any $m \geq 3$.

Consider now the remaining cases of $j \in \{1, ..., m+1\}$ in (2.62). For $j = 1$, by Lemma 2.3.40, we get

$$\left| \int \langle \nabla^{m+2} k, \partial_s (|k|^{p-2}) \nabla^{m+1} k \rangle \right| \leq C \int |\nabla^{m+2} k| |\nabla^{m+1} k| \leq \varepsilon \int |\nabla^{m+2} k|^2 + C(\varepsilon).$$

For $j \in \{2, ..., m+1\}$ we see that (2.62) can be rewritten

$$\begin{aligned} \int \langle \nabla^{m+2} k, \partial_s^j (|k|^{p-2}) \nabla^{m+2-j} k \rangle = C_{p-2,j} \int |k|^{p-2-2j} \langle k, \nabla k \rangle^j \langle \nabla^{m+2} k, \nabla^{m+2-j} k \rangle + \\ + \sum_{\substack{l_1+...+l_\lambda = m+2 \\ 1 \leq l_1 \leq ... \leq l_\lambda < m+1}} C_{\bar{l}} \int |k|^{q_{\bar{l}}} k * ... * k \nabla^{l_1} k * ... * \nabla^{l_\lambda} k * \nabla^{m+2} k + \\ + (p-2) \int |k|^{p-4} \langle k, \nabla^j k \rangle \langle \nabla^{m+2} k, \nabla^{m+2-j} k \rangle, \end{aligned}$$

and we have that $\left| C_{p-2,j} \int |k|^{p-2-2j} \langle k, \nabla k \rangle^j \langle \nabla^{m+2} k, \nabla^{m+2-j} k \rangle \right| \leq \varepsilon \int |\nabla^{m+2} k|^2 + C(\varepsilon)$ by Lemma 2.3.40, the terms in the sum have exactly the form of the generic term (2.63) we studied above, and thus they can be estimated as desired, and finally the last summand

$$(p-2) \int |k|^{p-4} \langle k, \nabla^j k \rangle \langle \nabla^{m+2} k, \nabla^{m+2-j} k \rangle,$$

is estimated by Lemma 2.3.40 if $j = m+1$, otherwise it is again of the form (2.63).

Putting together the estimates we found, coming back to (2.61), we get that the estimate becomes

$$C(\varepsilon) + \varepsilon \int |\nabla^{m+2} k|^2 + \int \langle \nabla^{m+2} k, \nabla^{m+2} (|k|^{p-2} k) \rangle \geq C \int |\nabla^{m+2} k|^2,$$

and choosing $\varepsilon > 0$ sufficiently small this becomes

$$C_1 \int |\nabla^{m+2} k|^2 \leq \int \langle \nabla^{m+2} k, \nabla^{m+2} (|k|^{p-2} k) \rangle + C_2.$$

Finally it is easy to check that the terms on the right hand side of (2.60) can be estimated by analogous quantities. More precisely, after integration by parts, we need to estimate

$$\int \left\langle \nabla^{m+2} k, \nabla^m \left(-\frac{1}{p'} |k|^p k + k - R(|k|^{p-2} k, \tau) \tau \right) \right\rangle,$$

and

$$\int \left\langle \nabla^m k, \nabla^m \left(\langle V, k \rangle k + R(V, \tau) \tau \right) + \sum_{j=0}^{m-1} \nabla^{m-1-j} (X(\nabla^j k)) \right\rangle.$$

Noticing that in a term of the form $\nabla^{m-1-j}(X(\nabla^j k))$ the sum of the orders of the derivatives of k is $m+2$, one can check that the above two integrals can be estimated by exactly the same arguments employed before. Moreover, also the remained integral on the right hand side of (2.60), namely $\int |\nabla^m k|^2 \langle V, k\rangle \, ds$, can be easily estimated using Lemma 2.3.38.

In the end (2.60) becomes

$$\begin{aligned}\partial_t\left(\frac{1}{2}\int |\nabla^m k|^2\, ds\right) &+ C_1 \int |\nabla^{m+2} k|^2 \\ &\leq \partial_t\left(\frac{1}{2}\int |\nabla^m k|^2\, ds\right) + \int \langle \nabla^{m+2} k, \nabla^{m+2}(|k|^{p-2}k)\rangle + C_2 \\ &\leq \varepsilon \int |\nabla^{m+2} k|^2 + C(\varepsilon) + C_2,\end{aligned}$$

and taking $\varepsilon > 0$ sufficiently small we get the estimate

$$\partial_t\left(\frac{1}{2}\int |\nabla^m k|^2\, ds\right) + C_3 \int |\nabla^{m+2} k|^2 \leq C_4.$$

By (2.53) one finally obtains

$$\partial_t\left(\int |\nabla^m k|^2\, ds\right) + C_5 \int |\nabla^m k|^2 \leq C_6,$$

and by comparison we get the desired bound

$$\|\nabla^m k\|_2^2(t) \leq \|\nabla^m k\|_2^2(0) + C,$$

for any $t \in [0, \tau)$.

2.3.6 An example of a non-converging flow

In this section we construct an example of an analytic 2-dimensional submanifold M of $\mathbb{R}^3$ and of a solution to the gradient flow of the elastic energy with exponent $p = 2$ such that it does not converge. Indeed, the resulting flow will leave any compact set of M for times sufficiently large.

In $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$, consider the curve

$$\sigma : (0, +\infty) \to \mathbb{R}^3, \qquad \sigma(t) = (f(t), 0, t),$$

that parametrizes the graph of a function $f\colon (0, +\infty) \to \mathbb{R}$ on the plane $\{y = 0\}$, and assume that f is analytic and $f(t) > 0$ for any $t > 0$. Consider the surface of revolution about the z-axis of the curve σ, that is, the analytic surface parametrized by the immersion

$$F(\theta, t) = (f(t)\cos\theta, f(t)\sin\theta, t)\,,$$

for $t > 0$ and $\theta \in [0, 2\pi]$. We denote by M such a complete analytic submanifold of $\mathbb{R}^3$. Consider $t_0 > 0$ and the closed curve γ_0 given by the intersection $M \cap \{z = t_0\}$. Letting

$$f(t) = 1 + \frac{1}{t},$$

we want to show that if t_0 is sufficiently big, the resulting solution γ of the gradient flow of the elastic energy with exponent $p = 2$ starting from γ_0 does not converge, and, in fact, it escapes from any compact set of M for times sufficiently large.

As the manifold M lacks of a "good" family of isometries that could satisfy the hypotheses of Theorem 2.3.33, it may be not too surprising that the flow does not converge. However, we notice that (M,g) is a manifold of bounded geometry, that is, the Riemann curvature tensor, that we will compute later, is uniformly pointwise bounded together with all its derivatives and the injectivity radius of (M,g) is strictly positive. Therefore, this example shows that not even such hypotheses are sufficient for the convergence of the flow.

Since M is a surface of revolution, its Gaussian curvature K can be computed in terms of f (see [AT12, Example 4.5.22]) and it equals

$$K(\theta,t)=\frac{-f''(t)}{f(t)(1+(f'(t))^2)^2}.$$

Hence, as M is a 2-dimensional, by (1.1) the Riemann tensor is

$$R(X,Y)Z=K(\langle Y,Z\rangle X-\langle X,Z\rangle Y),$$

for any tangent vectors X,Y,Z. Moreover, a unit normal field along M is given by

$$\nu(\theta,t)=\frac{1}{\sqrt{1+(f'(t))^2}}\left(-\cos\theta,-\sin\theta,f'(t)\right).$$

By the rotational symmetry of M and the choice of γ_0, the flow γ is of the form $\gamma(t,\mathbb{S}^1)=M\cap\{z=G(t)\}$ for some function G. Hence the desired conclusion follows once we prove the following two facts.

1. There exists $\tau_0>0$ such that for any $\tau\geq\tau_0$, the curve $\alpha(\theta)=(f(\tau)\cos\theta,f(\tau)\sin\theta,\tau)$ is not a critical point of the energy.
2. There exists $\tau_0>0$ such that for any $\tau\geq\tau_0$, letting $\alpha(\theta)=(f(\tau)\cos\theta,f(\tau)\sin\theta,\tau)$, we have that $-\nabla_{T(\alpha)^2,T(\alpha)^2}E(0)$ is a positive multiple of the curvature k_α of α, and the third component of k_α is positive.

Both the above items will follow from the direct calculation of $\nabla_{T(\alpha)^2,T(\alpha)^2}E(0)$, that is, from the computation of the first variation at the curve α.

We have that $|\alpha'|=f(\tau)$, and then

$$\begin{aligned}k_\alpha=M^\top(\partial^2_{s_\alpha}\alpha)&=-\frac{1}{f(\tau)}\left[(\cos\theta,\sin\theta,0)-\langle(\cos\theta,\sin\theta,0),\nu\rangle\,\nu\right]\\&=-\frac{1}{f(\tau)}\left[\frac{(f'(\tau))^2}{1+(f'(\tau))^2}(\cos\theta,\sin\theta,0)+\left(0,0,\frac{f'(\tau)}{1+(f'(\tau))^2}\right)\right],\end{aligned}$$

and we see that the third component of k_α is always strictly positive, indeed $f'(\tau)=-\frac{1}{\tau^2}$. Denoting as usual $\alpha^\perp:=M^\top-\alpha^\top$, we compute

$$\nabla k_\alpha=\alpha^\perp(\partial_{s_\alpha}k_\alpha)=\frac{1}{f(\tau)}\alpha^\perp(\partial_\theta k_\alpha)=\frac{1}{f(\tau)}\frac{(f'(\tau))^2}{1+(f'(\tau))^2}\alpha^\perp\left(-\sin\theta,\cos\theta,0\right)=0,$$

and then also $\nabla^2k_\alpha=0$. Therefore

$$\begin{aligned}\frac{1}{|\alpha'|}\nabla_{T(\alpha)^2,T(\alpha)^2}E(0)&=\nabla^2k_\alpha+\frac12|k_\alpha|^2k_\alpha-k_\alpha+R(k_\alpha,\tau_\alpha)\tau_\alpha=\frac12|k_\alpha|^2k_\alpha-k_\alpha+Kk_\alpha\\&=\left(\frac12|k_\alpha|^2+K-1\right)k_\alpha=\left(\frac12\frac{(f'(\tau))^2}{f^2(\tau)(1+(f'(\tau))^2)}-\frac{f''(\tau)}{f(\tau)(1+(f'(\tau))^2)^2}-1\right)k_\alpha\\&=\left(\frac12(f'(\tau))^2(1+(f'(\tau))^2)-f(\tau)f''(\tau)-f^2(\tau)(1+(f'(\tau))^2)^2\right)\frac{k_\alpha}{f^2(\tau)(1+(f'(\tau))^2)^2}\\&=\big(-1+O(\tau^{-1})\big)\frac{k_\alpha}{f^2(\tau)(1+(f'(\tau))^2)^2}.\end{aligned}$$

From the above computation we see that for τ_0 sufficiently large and $\tau \geq \tau_0$, we have that $\nabla_{T(\alpha)^2,T(\alpha)^2}E(0) \neq 0$, and then α is not a critical point, and $-\nabla_{T(\alpha)^2,T(\alpha)^2}E(0)$ is a positive multiple of k_α.

We deduce that if $t_0 \geq \tau_0$, then the flow γ does not remain in a bounded subset of M, and, in fact, it sweeps the set $M \cap \{z \geq t_0\}$.

Chapter 3

A varifold perspective on the elastic energies of planar sets

Contents

We address the problem of finding a weak definition of the p-elastic energy $\mathcal{E}_p$ on 2-dimensional subsets of the plane. As we shall discuss, we want to construct such a weak definition by relaxation with respect to the convergence in L^1 of characteristic functions of sets. In this way, we define a naturally lower semicontinuous functional, and, by the suitable choice of sets with bounded p-elastic energy, we end up with a relaxed functional that allows to consider sets whose boundary is the image of non-embedded curves. In the first part of the chapter we characterize the relaxed functional and in the second one we discuss several properties and applications. The results of this chapter are contained in [Poz20a].

3.1 A notion of relaxation for the p-elastic energy of planar sets

We are interested in studying the elastic properties of the boundaries of measurable sets in $\mathbb{R}^2$, that is, we want to discuss whether a weak definition of p-elastic energy like $\mathcal{E}_p$ can be defined on subsets of $\mathbb{R}^2$. As we shall see, several weak definitions for the elastic energy of non-smooth sets have been considered, leading to a distinction between sets with finite or infinite elastic energy. Such definitions usually make use of a parametrized curve related to the boundary of the given set.

Our first purpose is then to give a new definition of the sets which are considered to be enough regular for having finite p-elastic energy. We want such a definition to be intrinsically dependent on the given set, using immersions of curves only as a tool for the calculation of the energy. In order to get a variationally meaningful functional, we will then construct it by relaxation.

As already said, the very first step in the construction of a relaxed functional is the choice of the *regular* objects, that is, the family of sets that we assume they have finite energy in some suitable classical sense. One could classically say that a set E is regular if its boundary is a finite disjoint union of images of embedded smooth closed curves of class C^2. In this way the p-elastic energy of E would be just the classical p-elastic energy of such a parametrization of ∂E. This definition is, indeed, the one considered in the most important classical papers about this problem ([BDMP93], [BM04] and [BM07]). However, with this definition it turns out that sets like the one in Fig. 3.1 not only have infinite energy, but they also have infinite relaxed energy (as always, defined with respect to the L^1-convergence of sets). More generally, this happens for sets whose boundary is the image of an immersed curve with transversal self-intersections.

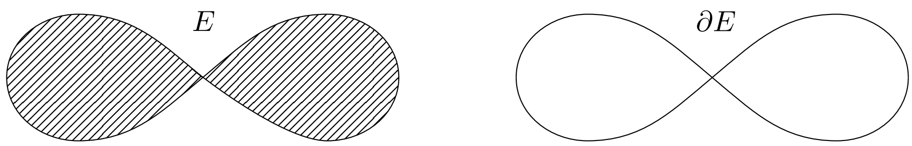

Figure 3.1: A set of finite perimeter E with boundary ∂E that can be parametrized by a smooth non-injective immersion.

However the p-elastic energy $\mathcal{E}_p$ is perfectly well-defined on immersed curves, even if those are not embedded. Also, for many applications one would like to consider sets like the one in Fig. 3.1 as regular sets, or at least as sets with finite relaxed energy. Actually, a suitable parametrization of ∂E in Figure 3.1 is even a critical point of $\mathcal{E}_2$ (see for example [DP17] for the so-called "Figure Eight" critical point).

An alternative definition of regular elastic set, i.e., a definition of set with finite elastic energy, comes intrinsically from the geometric properties of the boundary of sets of finite perimeter studied in the context of varifolds. In fact, by De Giorgi Theorem 1.4.6, if E is a set of finite perimeter in $\mathbb{R}^2$ then the reduced boundary $\mathcal{F}E$ is 1-rectifiable, and therefore the integer rectifiable varifold $V_E \coloneqq \mathbf{v}(\mathcal{F}E, 1)$ is well-defined. As we defined the p-elastic energy $\mathcal{E}_p$ on varifolds, the energy of V_E, for a given finite perimeter set E, associates an elastic energy to the set E in an intrinsic way.

We define the class of *elastic varifolds* as the integer rectifiable varifolds $V = \mathbf{v}(\Gamma, \theta_V)$ such that there exists a finite family of Lipschitz maps $\gamma_i : \mathbb{S}^1 \to \mathbb{R}^2$ such that

$$V = \sum_{i=1}^{N} (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)), \tag{3.1}$$

where each $(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$ is the image varifold of $\mathbb{S}^1$ induced by γ_i. We shall see that a representation like (3.1) is not ambiguous and that the curves appearing in the formula can be used to compute the p-elastic energy of V, in case they are sufficiently regular.

In this way, the class of regular sets that we choose is the class of finite perimeter sets E such that

$$\mathbf{v}(\mathcal{F}E, 1) = \sum_{i=1}^{N} (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)),$$

for some C^2-immersions $\gamma_i : \mathbb{S}^1 \to \mathbb{R}^2$, that is, the varifold V_E associated to E is elastic and admits a representation with C^2-curves. In such a way the set in Fig. 3.1 belongs to the initial class of regular sets having finite elastic energy.

We have to mention that a significant attempt in order to give a good definition of the elastic energy on sets that are not natural limits of smooth sets with bounded energy is contained in [BP95] (see also the references therein). Here the authors consider an interesting generalization of the elastic energy functional whose relaxation is able to take into account the energy of angles and cusps.

Let us mention that a good notion of p-elastic energy on sets $E \subset \mathbb{R}^2$ which is lower semicontinuous in the L^1-topology is also useful as a starting point for the study of a weak gradient flow of $\mathcal{E}_p$. The characterization of the relaxed energy allows us to define the gradient flow on a huge family of sets, and therefore to try to obtain a generalized flow, for example using a minimizing movements technique in the spirit of [LS95] and [ATW93]. In a generalized flow one certainly wants to consider sets like the one in Fig. 3.1, hence a definition in which its energy is finite is required. Moreover, high order evolution equations, like the ones appearing as gradient flows of $\mathcal{E}_p$, lack of a maximum principle, and then self-intersections of the evolving curve can naturally occur. This is a second reason why it is relevant to give a good definition taking into account sets like in Figure 3.1. We mention that [OPG19] contains a recent formulation of a generalized gradient flow of the p-elastic energy by means of minimizing movements.

3.1.1 Setting and notation

In what follows, if γ is any parametrization of a curve, we denote by (γ) its image. The letter E will usually denote a measurable set in $\mathbb{R}^2$.

Let us recall here some basic properties of sets of finite perimeter, together with the choice of a convention and of the notation. The following observations actually work for sets of finite perimeter in any dimension. Recall from Section 1.4 that, for a given set E, we denote by E^t the set of points with density t, for $t \in [0,1]$, and by $\mathcal{F}E$ and $\partial^* E$ the reduced and essential boundary of E respectively, if E has finite perimeter.

By Lebesgue Differentiation Theorem, characteristic functions defining finite perimeter sets are defined up to Lebesgue-negligible sets. In this chapter it will be convenient to choose, for a finite perimeter set identified by a characteristic function $\chi_E \in BV(\mathbb{R}^2)$, the representative E that is the set given by the points having density equal to 1, i.e.

$$E = E^1 := \left\{ x \in \mathbb{R}^2 \;\middle|\; \lim_{\rho \searrow 0} \frac{\int_{B_\rho(x)} \chi_E}{|B_\rho(x)|} = 1 \right\}. \tag{3.2}$$

Assuming (3.2), we have that

$$\partial E \equiv \partial E^1 = \partial^m E := \left\{ x \in \mathbb{R}^2 \mid \int_{B_\rho(x)} \chi_E > 0, \int_{B_\rho(x)} 1 - \chi_E > 0 \;\forall \rho > 0 \right\}. \tag{3.3}$$

Indeed if $x \in \partial^m E$, as E^1 is a representative of E, there are sequences $x_n^1 \in E^1, x_n^2 \in \mathbb{R}^2 \setminus E^1$ converging to x, and thus $x \in \partial E^1$. Conversely if $x \in \partial E^1$, then $\int_{B_\rho(x)} \chi_E > 0$ for any $\rho > 0$, for otherwise any point in E^1 sufficiently close to x would have density equal to zero. Arguing similarly on the complement of E^1, (3.3) follows.

It also follows that the reduced boundary is dense in the boundary of E, that is

$$\overline{\mathcal{F}E} = \partial^m E = \partial E. \tag{3.4}$$

Indeed $\mathcal{F}E \subset \partial E$ and if by contradiction there is $x \in \partial^m E \setminus \overline{\mathcal{F}E}$, then for some $\rho_0 > 0$ we have $B_{\rho_0}(x) \cap \overline{\mathcal{F}E} = \emptyset$ and $0 < |E \cap B_{\rho_0}(x)| < \pi\rho_0^2$. Hence by relative isoperimetric inequality [AFP00, Equation (3.43)] in the ball $B_{\rho_0}(x)$ we get that $P(E, B_{\rho_0}(x)) > 0$, but since $B_{\rho_0}(x) \cap \overline{\mathcal{F}E} = \emptyset$ using De Giorgi Theorem 1.4.6 we also have $P(E, B_{\rho_0}(x)) = \mathcal{H}^1(\mathcal{F}E \cap B_{\rho_0}(x)) = 0$, which gives a contradiction.

Observe that it also follows that

$$\mathrm{diam}(\mathcal{F}E) = \mathrm{diam}(\partial E).$$

In the rest of this chapter we will always assume that a finite perimeter set E is of the form (3.2).

3.1.2 Elastic varifolds

Here we prove some important remarks about varifolds defined through immersions of curves, that we shall call *elastic varifolds*. The next definition comes from [BM04].

Definition 3.1.1. Given a family of regular C^1 curves $\alpha_i : (-a_i, a_i) \to \mathbb{R}^2$ for $i = 1, ..., N$ and a point $p \in \mathbb{R}^2$ such that $\alpha_i(t_i) = p$ for some times t_i and the curves $\{\alpha_i\}$ are tangent at p, let $v \in \mathbb{S}^1$ such that $\alpha_i'(t_i)$ and v are parallel for any i. We say that $R_v(p)$ is a *nice rectangle at p for the curves $\{\alpha_i\}$ with side parameters $a, b > 0$* if

$$R_v(p) = \{z \in \mathbb{R}^2 \ : \ |\langle z-p, v\rangle| < a, \, |\langle z-p, v^\perp\rangle| < b\},$$

and

$$R_v(p) \cap \left(\bigcup_{i=1}^N (\alpha_i) \right) = \bigcup_{i=1}^M \mathrm{graph}(f_i),$$

for distinct C^1 functions $f_i : [-a, a] \to (-b, b)$, where $\mathrm{graph}(f_i)$ denotes the graph of f_i constructed on the lower side of the rectangle.

We also give the following definition.

Definition 3.1.2. Let $V = \mathbf{v}(\cup_{i\in I}(\gamma_i), \theta_V)$ be a varifold defined by the $W^{2,p}$ immersions $\gamma_i : \mathbb{S}^1 \to \mathbb{R}^2$, and assume that $\mathcal{E}_p(V) < +\infty$, $\theta_V \le C < +\infty$.

For any $p \in \cup_{i\in I}(\gamma_i)$ and any $v \in \mathbb{S}^1$ denote by $g_1, ..., g_r : [-\varepsilon, \varepsilon] \hookrightarrow \mathbb{R}^2$ arclength parametrized injective arcs such that: $g_i(0) = p$, $\dot{g}_i(0) = v$, $g_i([-\varepsilon, 0]) \neq g_j([-\varepsilon, 0])$ or $g_i([0, \varepsilon]) \neq g_j([0, \varepsilon])$ for $i \neq j$, and $\cup_{i=1}^r (g_i) \cap \overline{B}_\rho(p) = \cup_{i\in I}(\gamma_i) \cap \overline{B_\rho(p)}$. Observe that for any such p, v and ρ small enough, the arcs g_i are well defined.

We say that V verifies the *flux property* if: $\forall\, p \in \cup_{i\in I}(\gamma_i)$, $\forall\, v \in \mathbb{S}^1$, and ρ small enough there exists a nice rectangle $R_v(p) \subset B_\rho(p)$ for the family of arcs $\{g_i\}$ such that it holds that

$$\forall |c| < a : \qquad \sum_{z \in \cup_{i=1}^r (g_i) \cap \{y \,|\, \langle y-p, v\rangle = c\}} \theta_V(z) = M,$$

for a constant $M \in \mathbb{N}$ with $M \le \theta_V(p)$.

Roughly speaking, Definition 3.1.2 requires that the “incoming” total amount of multiplicity at p in direction v equals the “outcoming” total amount of multiplicity at p in direction v.

Observe that if $V = \sum_{i\in I} (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$ with $\gamma_i \in W^{2,p}$ immersions and $\mathcal{E}_p(V) < +\infty$, $\theta_V \le C < +\infty$, then V verifies the flux property.

Remark 3.1.3. Let E be a set of finite perimeter in $\mathbb{R}^2$, let $\Gamma = \cup_{i=1}^N(\gamma_i)$ with $\gamma_i \in C^1(\mathbb{S}^1;\mathbb{R}^2)$ and regular for any i. Assume that $V_E \coloneqq \mathbf{v}(\mathcal{F}E,1) = \sum_{i=1}^N(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1))$. Then $\mathcal{H}^1(\partial E\setminus\mathcal{F}E) = 0$, and we can equivalently write $V_E = \mathbf{v}(\partial E,1)$.

Indeed by assumption $\mathcal{H}^1$-almost every point $p\in\Gamma$ is contained in $\mathcal{F}E$, $\mathrm{spt}V_E = \Gamma$, and $\Gamma = \mathrm{spt}V_E = \mathrm{spt}(\mathcal{H}^1 \llcorner \mathcal{F}E) = \partial E$. Therefore $0 = \mathcal{H}^1(\Gamma\setminus\mathcal{F}E) = \mathcal{H}^1(\partial E\setminus\mathcal{F}E)$.

Lemma 3.1.4. *Assume $p>1$. If an integer rectifiable varifold $V = \mathbf{v}(\Gamma,\theta_V)$ is such that $V = \sum_{i=1}^N(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1))$ for some regular curves $\gamma_i \in W^{2,p}(\mathbb{S}^1;\mathbb{R}^2)$ and $\mathcal{E}_p(V)<+\infty$, then V has generalized curvature*

$$k_V(p) = \frac{1}{\theta_V(p)}\sum_{i=1}^N\sum_{t\in\gamma_i^{-1}(p)} k_{\gamma_i}(t) \qquad \text{at } \mathcal{H}^1\text{-ae } p\in\Gamma, \tag{3.5}$$

the generalized boundary $\sigma_V = 0$, and

$$\mathcal{E}_p(V) = \sum_{i=1}^N \mathcal{E}_p(\gamma_i). \tag{3.6}$$

In particular, since k_V is uniquely defined, the value $\mathcal{E}_p(V)$ is independent of the choice of the family of curves $\{\gamma_i\}$ defining V.

Proof. Suppose first that $N=1$, and then denote $\gamma_1=\gamma$. Up to rescaling, assume without loss of generality that γ is parametrized by arclength. By assumption $\gamma\in C^{1,\alpha}$ for $\alpha\le\frac{1}{p'}$, and clearly $\Gamma=(\gamma)$ and $\mu_V = \theta_V\mathcal{H}^1\llcorner(\gamma) = \gamma_\sharp(\mathcal{H}^1\llcorner\mathbb{S}^1)$ (by the arclength parametrization assumption). If $X\in C^1_c(\mathbb{R}^2,\mathbb{R}^2)$ is a vector field, using the area formula (Theorem 1.2.8) and the fact that $\theta_V\ge 1$ $\mathcal{H}^1$-ae on Γ, we have

$$\begin{aligned}
\int \mathrm{div}_{T_p\Gamma}X\,d\mu_V(p) &= \int \mathrm{div}_{T_p(\gamma)}X\,d\gamma_\sharp(\mathcal{H}^1\llcorner\mathbb{S}^1)(p) = \int_{\mathbb{S}^1}\langle\gamma'(t),(\nabla X)_{\gamma(t)}(\gamma'(t))\rangle\,dt\\
&= \int_{\mathbb{S}^1}\langle\gamma',(X\circ\gamma)'\rangle\,dt = -\int_{\mathbb{S}^1}\langle\gamma''(t),X(\gamma(t))\rangle\,dt\\
&= -\int\int_{\gamma^{-1}(p)}\langle\gamma''(t),X(\gamma(t))\rangle\,d\mathcal{H}^0\,d\mathcal{H}^1\llcorner\Gamma(p)\\
&= -\int\left\langle X(p),\int_{\gamma^{-1}(p)}k_\gamma(t)\,d\mathcal{H}^0\right\rangle\frac{\theta_V(p)}{\theta_V(p)}\,d\mathcal{H}^1\llcorner\Gamma(p)\\
&= -\int\left\langle X(p),\frac{1}{\theta_V(p)}\sum_{t\in\gamma^{-1}(p)}k_\gamma(t)\right\rangle d\mu_V(p).
\end{aligned}$$

If now $N>1$, by linearity of the first variation we get

$$\begin{aligned}
\int \mathrm{div}_{T_p\Gamma}X\,d\mu_V(p) &= -\sum_{i=1}^N\int\left\langle X(p),\sum_{t\in\gamma_i^{-1}(p)}k_{\gamma_i}(t)\right\rangle d\mathcal{H}^1\llcorner(\gamma_i)(p)\\
&= -\int\left\langle X(p),\frac{1}{\theta_V(p)}\sum_{i=1}^N\sum_{t\in\gamma_i^{-1}(p)}k_{\gamma_i}(t)\right\rangle\theta_V(p)\,d\mathcal{H}^1\llcorner\left(\cup_{i=1}^n(\gamma_i)\right)\\
&= -\int\left\langle X(p),\frac{1}{\theta_V(p)}\sum_{i=1}^N\sum_{t\in\gamma_i^{-1}(p)}k_{\gamma_i}(t)\right\rangle d\mu_V.
\end{aligned}$$

Now we want to prove (3.6). Let us consider the set $W = \{p \in \Gamma \,|\, \theta_V(p) > 1\}$. Up to redefining some γ_i on circles of different radii, we can suppose from now on that γ_i is parametrized by arclength for any i. We can decompose $W = T \cup X \cup Y \cup Z$ with

$$\begin{aligned}
T &= \big\{p \in W \,|\, \exists i,j,t,\tau: \ \gamma_i(t) = \gamma_j(\tau) = p, \, \gamma_i'(t) \neq \alpha\gamma_j'(\tau) \ \forall \alpha \in \mathbb{R}\big\}, \\
X &= \big\{p \in W \setminus T \,|\, \exists i,t: \ \gamma_i(t) = p, \, t \text{ is not a Lebesgue point of } \gamma_i''\big\}, \\
Y &= \big\{p \in W \setminus (T \cup X) \,|\, \forall i,j,t,\tau: \ \gamma_i(t) = \gamma_j(\tau) = p \ \Rightarrow \ \gamma_i''(t) = \gamma_j''(\tau)\big\}, \\
Z &= \big\{p \in W \setminus (T \cup X) \,|\, \exists i,j,t,\tau: \ \gamma_i(t) = \gamma_j(\tau) = p, \, \gamma_i''(t) \neq \gamma_j''(\tau)\big\}.
\end{aligned}$$

We are going to prove that T, Z are at most countable, so that, since $\mathcal{H}^1(X) = 0$, we will get that $\mathcal{H}^1(W) = \mathcal{H}^1(Y)$. And then, using (3.5), we will get (3.6).

Let $p \in \Gamma$ and $C \in \mathbb{N}$ such that $\theta_V \leq C$, which exists by Theorem 1.3.1. Let $v_1(p), ..., v_k(p) \in \mathbb{S}^1$ with $k = k(p) \leq C$ such that if $\gamma_i(t) = p$ then $\gamma_i'(t)$ is proportional to some v_j. For any $i = 1, ..., k$ let $R_{v_i}(p)$ be a nice rectangle at p for the curves $\{\alpha_j\}_{j \in J(i)}$ which are suitable restrictions of the curves $\{\gamma_i\}$. Then let $f_1^i, ..., f_l^i$ with $l = l(i)$ be C^1 functions $f_s^i : [-a_i, a_i] \to (-b_i, b_i)$ given by the definition of nice rectangle.

Let $q \in \cup_{s=1}^l \text{graph}(f_s^i)$, and assume $q \in T$. If a_i is chosen sufficiently small, the fact that q belongs to T means that the transversal intersection happens between some of the curves $\{\alpha_j\}_{j \in J(i)}$. This means that there is some $\delta_q > 0, x_q \in (-a_i, a_i), r, s \in \{1, ..., l\}$ such that

$$f_r^i(x_q) = f_s^i(x_q), \quad (x_q, f_r^i(x_q)) = q, \quad \text{graph}\big(f_r^i|_{(x_q - \delta_q, x_q + \delta_q)}\big) \cap \text{graph}\big(f_s^i|_{(x_q - \delta_q, x_q + \delta_q)}\big) = \{q\}.$$

Letting $A_i = \{x \in (-a_i, a_i) \,|\, f_r^i \neq f_s^i\}$, which is open, we see that x_q belongs to the boundary of some connected component of A_i. This implies that $T \cap \big(\cup_{s=1}^l \text{graph}(f_s^i)\big)$ is countable, and this is true for any $i = 1, ..., k(p)$.

For any $p \in \Gamma$ take a ball $B_{r(p)}(p) \subset \cap_{i=1}^{k(p)} R_{v_i(p)}(p)$ for suitable rectangles $R_{v_i(p)}(p)$ as above. Then $T \cap B_{r(p)}(p)$ is countable. Since Γ is compact, taking a finite cover of such balls $B_{r(p_1)}(p_1), ..., B_{r(p_L)}(p_L)$, we conclude that T is countable.

Consider now $q \in \cup_{s=1}^l \text{graph}(f_s^i)$, and assume $q \in Z$. If a_i is chosen sufficiently small, the fact that q belongs to Z means that the tangential intersection occurs between some of the curves $\{\alpha_j\}_{j \in J(i)}$. Hence at some $x_q \in (-a_i, a_i)$ for some $r, s \in \{1, ..., l\}$ we find that x_q is a Lebesgue point for $(f_r^i)''$ and $(f_s^i)''$, and

$$f_r^i(x_q) = f_s^i(x_q), \quad (x_q, f_r^i(x_q)) = q, \quad (f_r^i)''(x_q) \neq (f_s^i)''(x_q).$$

This implies that there exists $\varepsilon > 0$ such that for any $0 < |t| < \varepsilon$ we have $(f_r^i)'(x_q + t) \neq (f_s^i)'(x_q + t)$. By continuity of the first derivative we have that, for example, $(f_r^i)'(x_q + t) > (f_s^i)'(x_q + t)$ for any $0 < |t| < \varepsilon$, and therefore $f_r^i(x_q + t) > f_s^i(x_q + t)$ for any $0 < |t| < \varepsilon$. So we find that x_q belongs to the boundary of a connected component of an A_i defined as above as in the case of the set T. Arguing as before we eventually get that Z is countable. $\square$

Lemma 3.1.5. *Let $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$, where $\gamma_1, ..., \gamma_N : \mathbb{S}^1 \to \mathbb{R}^2$ are Lipschitz curves. Assume that $\mathcal{H}^1(\{x \,|\, \theta_V(x) > 1\}) = 0$, and define*

$$E = \left\{ p \in \mathbb{R}^2 \setminus \Gamma \,:\, \left| \sum_{i=1}^N \text{Ind}_{\gamma_i}(p) \right| \text{ is odd} \right\}. \tag{3.7}$$

Then E is a set of finite perimeter and $V = V_E$. Moreover, the set E is uniquely determined by V, i.e. if $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) = \sum_{i=1}^M (\sigma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$ then the corresponding set defined using (3.7) with the family $\{\gamma_i\}$ is the same set defined using (3.7) with the family $\{\sigma_i\}$.

Proof. The set E is open and bounded and $\partial E = \Gamma$, hence E is a set of finite perimeter.

Let us first check that if $V = \sum_{i=1}^{N} (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) = \sum_{i=1}^{M} (\sigma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$, then the definition of E by (3.7) is independent of the choice of the family of curves. The fact that a point $p \in \mathbb{R}^2 \setminus \Gamma$ belongs to E depends on the class

$$\left(\sum_{i=1}^{N} \mathrm{Ind}_{\gamma_i}(p)\right) \mod 2, \qquad \text{or} \qquad \left(\sum_{i=1}^{M} \mathrm{Ind}_{\sigma_i}(p)\right) \mod 2.$$

Without loss of generality we can take $p = 0$. Since the curves $\{\gamma_i\}, \{\sigma_i\}$ define the same varifold, for $\mathcal{H}^1$-ae point $q \in \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ we have that

$$\sum_{i=1}^{N} \sharp\left(\frac{\gamma_i}{|\gamma_i|}\right)^{-1}(q) = \sum_{i=1}^{M} \sharp\left(\frac{\sigma_i}{|\sigma_i|}\right)^{-1}(q). \tag{3.8}$$

In the following we denote by $\deg(f, y)$ the degree of a map f at y and by $\deg_2(f, y)$ the degree mod 2 of f at y (we refer to [Mil65]). Since the curves are Lipschitz almost every point $q \in \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ is a regular value for $\frac{\gamma_i}{|\gamma_i|}, \frac{\sigma_i}{|\sigma_i|}$ and we can perform the calculation

$$\begin{aligned}\left(\sum_{i=1}^{N} \mathrm{Ind}_{\gamma_i}(p)\right) \mod 2 &= \left(\sum_{i=1}^{N} \deg\left(\frac{\gamma_i}{|\gamma_i|}, q\right)\right) \mod 2 = \left(\sum_{i=1}^{N} \deg_2\left(\frac{\gamma_i}{|\gamma_i|}, q\right)\right) \mod 2 \\ &= \left(\sum_{i=1}^{N} \sharp\left(\frac{\gamma_i}{|\gamma_i|}\right)^{-1}(q) \mod 2\right) \mod 2,\end{aligned}$$

that together with the same expression using the curves σ_i, implies that E is uniquely defined by (3.8).

Now we prove that $V = V_E$. Let

$$X = \{p \in \Gamma \,|\, \theta_V(p) = 1,\ \gamma_i(t) = p \Rightarrow \gamma_i \text{ is differentiable at } t\}.$$

We want to prove that

$$\mathcal{H}^1\big(\mathcal{F}E \Delta X\big) = 0, \tag{3.9}$$

which implies that $V = V_E$.

If $\gamma_i(t) = p \in X$, then there is $\varepsilon > 0$ such that $\gamma_i\big((t-\varepsilon, t+\varepsilon)\big) \subset \{\theta_V = 1\} \subset \Gamma = \partial E$. By Rademacher we therefore have that $\mathcal{H}^1(X \cap \gamma_i\big((t-\varepsilon, t+\varepsilon)\big) \setminus \mathcal{F}E) = 0$. Hence $\mathcal{H}^1(X \setminus \mathcal{F}E) = 0$.

Now let $p \in \mathcal{F}E$, we want to prove that $\mathcal{H}^1(\mathcal{F}E \setminus X) = 0$, and this will complete the claim (3.9). If $\theta_V(p) = 1$ only a curve passes (once) trough p, say $\gamma_1(t_1) = p$, and since $p \in \mathcal{F}E$ such curve has to be differentiable at t_1. Conversely if $p = \gamma_i(t_i)$ for some $\{i, t_i\}$'s, assuming that each γ_i is differentiable at t_i, we want to prove that $\theta_V(p) = 1$. Suppose by contradiction that $\theta_V(p) > 1$, then there are $\alpha, \beta : (-\varepsilon, \varepsilon) \to \Gamma$ Lipschitz different arcs such that $\alpha(0) = \beta(0) = p$ and α, β are differentiable at time 0; moreover the hypothesis $\mathcal{H}^1(\{x \,|\, \theta_V(x) > 1\}) = 0$ implies that $\mathcal{H}^1\big((\alpha) \cap (\beta)\big) = 0$. Therefore $\mathcal{H}^1$-ae point $p \in (\alpha) \cup (\beta)$ is contained in X, and thus $\mathcal{H}^1$-ae point $p \in (\alpha) \cup (\beta)$ is contained in $\mathcal{F}E$, since we already know that $\mathcal{H}^1(X \setminus \mathcal{F}E) = 0$. So for any $\varepsilon > 0$ there is $r > 0$ such that

$$\mathcal{H}^1\big([(\alpha) \cup (\beta)] \cap B_r(p)\big) \geq (2 - \varepsilon)2r,$$

and thus

$$\mathcal{H}^1(\mathcal{F}E \cap B_r(p)) \geq \mathcal{H}^1\big([(\alpha) \cup (\beta)] \cap B_r(p)\big) \geq (2 - \varepsilon)2r,$$

which is a contradiction with the fact that any point in $\mathcal{F}E$ has 1-dimensional density equal to 1, that is, $\lim_{r\searrow 0}\frac{\mathcal{H}^1(\mathcal{F}E\cap B_r(p))}{2r}=1$, which follows from Theorem 1.4.6.

So we have proved that a point $p\in\mathcal{F}E$ verifies that: if $\theta_V(p)=1$ then $p\in X$, and if any curve passing through p at some time is differentiable at that time then $p\in X$. In any case we conclude that $\mathcal{H}^1$-almost every point in $\mathcal{F}E$ belongs to X, and then $\mathcal{H}^1(\mathcal{F}E\setminus X)=0$. $\square$

3.1.3 Relaxation

In this section we finally give a definition of $\mathcal{E}_p$ on measurable subsets of $\mathbb{R}^2$ and we characterize its L^1-relaxation.

From now on and for the rest of section let $p>1$ be fixed. For any measurable set $E\subset\mathbb{R}^2$ we define the energy

$$\mathcal{E}_p(E)=\begin{cases}\mathcal{E}_p(V_E) & \text{if } \ V_E=\sum_{i\in I}(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1)), \quad \gamma_i:\mathbb{S}^1\to\mathbb{R}^2 \ C^2\text{-immersion}, \quad \sharp I<+\infty,\\ +\infty & \text{otherwise.}\end{cases}$$

We write $\mathcal{E}_p(E)$ understanding that $\mathcal{E}_p$ is defined on the set of equivalence classes of characteristic functions endowed with L^1 norm. We want to calculate the relaxed functional $\overline{\mathcal{E}_p}$ with respect to the L^1-sense of convergence of characteristic functions, that is

$$\overline{\mathcal{E}_p}(E):=\inf\left\{\liminf_n \mathcal{E}_p(E_n)\ |\ E_n\to E \text{ in } L^1(\mathbb{R}^2)\right\}.$$

By Remark 3.1.3 and Lemma 3.1.4, if $\mathcal{E}_p(E)<\infty$, we have that

$$\mathcal{H}^1(\partial E\setminus\mathcal{F}E)=0, \qquad \mathcal{E}_p(V_E)=\sum_{i\in I}\mathcal{E}_p(\gamma_i),$$

if $V_E=\sum_{i\in I}(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1))$. Also, up to replacing E with its complement, we can suppose that E is bounded.

If $E\subset\mathbb{R}^2$ is measurable, we define

$$\mathcal{A}(E)=\left\{V=\mathbf{v}(\Gamma,\theta_V)=\sum_{i\in I}(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1))\ \middle|\ \begin{aligned}&\gamma_i:\mathbb{S}^1\to\mathbb{R}^2\ W^{2,p}\text{-immersion},\ \sharp I<+\infty,\\ &\sum_{i\in I}\mathcal{E}_p(\gamma_i)<+\infty,\\ &\partial E\subset\Gamma,\ V_E\le V,\\ &\mathcal{F}E\subset\{x\in\mathbb{R}^2\,|\,\theta_V(x)\text{ is odd}\},\\ &\mathcal{H}^1(\{x\,|\,\theta_V(x)\text{ is odd}\}\setminus\mathcal{F}E)=0\end{aligned}\right\},$$

Remark 3.1.6. Observe that if $V\in\mathcal{A}(E)$, then $\mathcal{E}_p(V)<+\infty$, and then $\theta_V(x)=\lim_{\rho\searrow 0}\frac{\mu_V(B_\rho(x))}{2\rho}$ exists and it is uniformly bounded on Γ by Theorem 1.3.1. Moreover, the condition $\partial E\subset\Gamma$ and the bound on the energy of the curves imply that $\mathcal{H}^1(\partial E)<\infty$, and then E is a set of finite perimeter.

The main result of this part is the following characterization of $\overline{\mathcal{E}_p}$. We recall that we say that a set E is essentially unbounded if $|E\setminus B_r(0)|>0$ for any $r>0$, while we say that it is essentially bounded in the opposite case.

Theorem 3.1.7 (Relaxation)**.** *For any measurable set $E \subset \mathbb{R}^2$ we have that the following holds.*

1. *If $\mathcal{A}(E) \neq \emptyset$ and E is essentially bounded, then the minimum*
$$\min\left\{\mathcal{E}_p(V) \mid V \in \mathcal{A}(E)\right\}$$
exists.

2. *It holds that*
$$\overline{\mathcal{E}_p}(E) = \begin{cases} +\infty & \text{if } \mathcal{A}(E) = \emptyset \ \text{ or } \ E \text{ is ess. unbounded,} \\ \min\left\{\mathcal{E}_p(V) \mid V \in \mathcal{A}(E)\right\} & \text{otherwise.} \end{cases} \tag{3.10}$$

The proof of Theorem 3.1.7 will be completed at the end of the section.

Remark 3.1.8. Choosing for a measurable set E the L^1 representative defined in (3.2), then the set E is essentially unbounded if and only if it is unbounded. Hence, in the statement of Theorem 3.1.7 one can actually write unbounded in place of essentially unbounded.

Remark 3.1.9. The characterization of $\overline{\mathcal{E}_p}$ given by Theorem 3.1.7 immediately implies the following "stability property". It holds that
$$\mathcal{E}_p(E) < +\infty \qquad \Rightarrow \qquad \overline{\mathcal{E}_p}(E) = \mathcal{E}_p(E) < +\infty,$$
that is, the relaxed energy of a regular set equals to the initial energy defined on it.

Indeed if $\mathcal{E}_p(E) < +\infty$, then $V_E \in \mathcal{A}(E)$. Consider any $W = \mathbf{v}(\Gamma, \theta_W) \in \mathcal{A}(E) \setminus \{V_E\}$, then by definition we have that $V_E \leq V$ in the sense of measures and $\mathcal{F}E \subset \{x \,|\, \theta_W(x) \text{ is odd}\}$, and this implies that $\mathcal{H}^1(\mathcal{F}E \setminus \Gamma) = 0$. Therefore $\mu_W(\mathbb{R}^2) \geq \mathcal{H}^1(\mathcal{F}E) = \mu_{V_E}(\mathbb{R}^2)$, and then also $\mathcal{E}_p(W) \geq \mathcal{E}_p(V_E)$ by locality of the generalized curvature of 1-dimensional varifolds (see [LM09, Theorem 2.1]).

We conclude this part by showing some properties of varifolds $V \in \mathcal{A}(E)$.

Lemma 3.1.10. *Let $E \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Let $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$ with $\gamma_1, ..., \gamma_N : \mathbb{S}^1 \to \mathbb{R}^2$ Lipschitz curves. Suppose that $\mathcal{F}E \subset \Gamma$ and*
$$\mathcal{H}^1(\mathcal{F}E \,\Delta\, \{x \mid \theta_V(x) \text{ is odd}\}) = 0.$$
Then
$$E = \left\{ p \in \mathbb{R}^2 \setminus \Gamma \,:\, \left| \sum_{i=1}^N \mathrm{Ind}_{\gamma_i}(p) \right| \text{ is odd} \right\}.$$

Proof. Fix $p \in \mathbb{R}^2 \setminus \Gamma$. In the following we suppose without loss of generality that $p = 0$. By hypotheses and by the calculations in the proof of Lemma 3.1.5, there exists a vector $v \in \mathbb{R}^2 \setminus \{0\}$ such that the ray $L = \{tv \,|\, t \in [0, \infty)\}$ verifies the following properties.

i) L intersects Γ at points y such that for any $i = 1, ..., N$ if $\gamma_i(t) = y$ then γ_i is differentiable at t.

ii) L intersects $\mathcal{F}E$ a finite number $M \in \mathbb{N}$ of times at points z in $\mathcal{F}E \cap \{x \mid \theta_V(x) \text{ is odd}\}$ where $\nu_E(z), v$ are independent.

iii) L intersects $\Gamma \setminus \mathcal{F}E$ a finite number of times at points w in $\{x \mid \theta_V(x) \text{ is even}\}$ where $\gamma_i'(t), v$ are independent whenever $\gamma_i(t) = w$.

iv) It holds that

$$\left(\sum_{i=1}^{N}\mathrm{Ind}_{\gamma_i}(p)\right) \mod 2 = \left(\sum_{i=1}^{N}\sum_{y\in L\cap(\gamma_i)}\sharp\left(\frac{\gamma_i}{|\gamma_i|}\right)^{-1}\left(\frac{y}{|y|}\right) \mod 2\right) \mod 2$$
$$= \left(\sum_{i=1}^{N}\sum_{y\in L\cap(\gamma_i)\cap\mathcal{F}E}\sharp\left(\frac{\gamma_i}{|\gamma_i|}\right)^{-1}\left(\frac{y}{|y|}\right) \mod 2\right) \mod 2,$$

where in iv) the second equality follows from ii) and iii). By hypothesis we have that $\partial E\subset\Gamma$, and then we can assume that $E=E^1$ is open. Now if $p\in E$, since E is also bounded, the number M has to be odd, and then $\left(\sum_{i=1}^{N}\mathrm{Ind}_{\gamma_i}(p)\right) \mod 2=1$. Conversely if p is in the interior of E^c, then M is even, and then $\left(\sum_{i=1}^{N}\mathrm{Ind}_{\gamma_i}(p)\right) \mod 2=0$. □

Remark 3.1.11. We observe that Lemma 3.1.10 applies to couples E,V with $V\in\mathcal{A}(E)$.

Lemma 3.1.12. *Let $V=\mathbf{v}(\Gamma,\theta_V)\in\mathcal{A}(E)$ for some measurable set E. Letting $\Sigma:=\overline{\Gamma\setminus\partial E}$, it holds that if $\Sigma\neq\emptyset$ then for any $x\in\Sigma\cap\partial E$ at least one of the following holds.*

1. *$\exists y\in\Sigma\cap\partial E$, $\exists f:[0,T]\to\mathbb{R}^2$ $C^1\cap W^{2,p}$, $T>0$, f regular curve from x to y with $(f)\subset\Gamma$,*
2. *x is not isolated in $\Sigma\cap\partial E$.*

The alternative above is not exclusive.

Proof. Write $V=\sum_{i=1}^{N}(\sigma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1))$. Assume $\Sigma\neq\emptyset$, that is equivalent to $\Gamma\setminus\partial E=:S\neq\emptyset$. Suppose $x\in\Sigma\cap\partial E$ is isolated in $\Sigma\cap\partial E$, then we want to prove that condition 1 holds true. There exists $r_0>0$ such that $B_r(x)\cap\Sigma\cap\partial E=\{x\}$ for any $r\le r_0$. Up to reparametrization we can say that $\sigma_1|_{(-\varepsilon,\varepsilon)}:(-\varepsilon,\varepsilon)\to B_{r_0}(x)$ passes through x at time 0. Up to reparametrize $\sigma_1(t)$ into $\sigma_1(-t)$, we can say that there exists a time $T>0$ such that $\sigma_1|_{(0,T)}\subset S$ and $y:=\sigma_1(T)\in\partial S=\Sigma\cap\partial E$, looking at S as topological subspace of Σ. Indeed, otherwise x would not be isolated in $\partial S=\Sigma\cap\partial E$. Defining $f(t)=\sigma_1(t)$ for $t\in[0,T]$ gives alternative 1. □

Necessary conditions.

Here we prove that a set $E\subset\mathbb{R}^2$ with $\overline{\mathcal{E}_p}(E)<+\infty$ has the necessary properties that inspire formula (3.10).

Let E_n be any sequence of sets such that $\mathcal{E}_p(E_n)\le C$ and $\chi_{E_n}\to\chi_E$ in $L^1(\mathbb{R}^2)$. Let us adopt the notation $V_{E_n}=\sum_{i\in I_n}(\gamma_{i,n})_\sharp(\mathbf{v}(\mathbb{S}^1,1))=\mathbf{v}(\Gamma_n,\theta_{V_{E_n}})$, so that $\mathcal{E}_p(E_n)=\sum_{i\in I_n}\mathcal{H}^1(\gamma_{i,n})+\frac{1}{p}\int|k_{\gamma_{i,n}}|^p\,ds$. Using also (1.13) we have that $0<c\le\mathcal{H}^1(\gamma_{i,n})\le C<\infty$ for any i,n. Also $\mathcal{E}_p(\gamma_{i,n})\ge c>0$ for any i,n, thus $\sharp I_n<+\infty$ for large n and then we can suppose that $I_n=I$ for any n. Also we can choose E_n bounded and by L^1 convergence we have that $|E|<+\infty$.

Moreover, we observe that in order to calculate the relaxation of $\mathcal{E}_p$ we can suppose that the sequence E_n is actually uniformly bounded, hence getting that E is bounded. Indeed if (up to subsequence) we have that, say, $\gamma_{1,n}\cap B_n(0)^c\neq\emptyset$, then by boundedness of the length we have $\gamma_{1,n}\subset(B_{n-c}(0))^c$ for any n for some c. Let Λ_n be the connected component of $\cup_{i\in I}(\gamma_i)$ containing (γ_1). The component Λ_n is equal to some union $\cup_{j\in J_n}(\gamma_{j,n})$. Up to relabeling we can suppose that $J_n=J$ for any n. Since the length of each curve is uniformly bounded, there exist open sets U_n such that $\Lambda_n \subset U_n$, $U_n \cap \big(\cup_{i\in I\setminus J} (\gamma_{i,n})\big) = \emptyset$, and $U_n \cap B_{R_n}(0) = \emptyset$ for some sequence $R_n \to \infty$. Therefore the set $E'_n := E_n \setminus U_n$ still converges to E in $L^1(\mathbb{R}^2)$, and $\mathcal{E}_p(E'_n) < \mathcal{E}_p(E_n)$.

Lemma 3.1.13. *Suppose $E \subset \mathbb{R}^2$ verifies that $\overline{\mathcal{E}_p}(E) < +\infty$. Let $E_n \subset \mathbb{R}^2$ be a sequence of uniformly bounded sets such that $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^2)$ with $\mathcal{E}_p(E_n) \le C$. Suppose that for any n the set $\{p \,|\, \theta_{V_{E_n}}(p) > 1\}$ is finite, then any subsequence of V_{E_n} converging in the sense of varifolds converge to an element of $\mathcal{A}(E)$.*

Proof. We adopt the notation $V_{E_n} = \sum_{i\in I_n} (\gamma_{i,n})_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) = \mathbf{v}(\Gamma_n, \theta_{V_{E_n}})$ used above. The arclength parametrizations $\sigma_{i,n}$ corresponding to $\gamma_{i,n}$ are uniformly bounded in $W^{2,p}$ for any $i \in I_n = I$ and for any n. Therefore, since the sequence is uniformly bounded in $\mathbb{R}^2$, up to subsequence $\sigma_{i,n} \to \sigma_i$ strongly in $C^{1,\alpha}$ for some $\alpha \le \frac{1}{p'}$ and weakly in $W^{2,p}(\mathbb{R}^2)$ for any $i \in I$. Each σ_i is then a closed curve parametrized by arclength, and we denote by γ_i the parametrization on $\mathbb{S}^1$ with constant velocity. Hence the varifolds V_{E_n} converge to some integer rectifiable varifold $V = \mathbf{v}(\Gamma, \theta_V)$ in the sense of varifolds, and $V = \sum_{i\in I} (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$. The multiplicity function θ_V is upper semicontinuous and pointwise bounded by Proposition 1.2.13 and Theorem 1.3.1 . Also, the sets E_n converge to E weakly* in $BV(\mathbb{R}^2)$, that is $\chi_{E_n} \to \chi_E$ and $D\chi_{E_n} \overset{\star}{\rightharpoonup} D\chi_E$, thus E is a set of finite perimeter. Observe that $|D\chi_{E_n}| = \mu_{V_{E_n}} \overset{\star}{\rightharpoonup} \mu_V$.

From now on we denote $\Gamma = \cup_{i\in I}(\sigma_i)$, $\Sigma = \overline{\Gamma \setminus \partial E}$, $S = \Gamma \setminus \partial E$.

Let $x \in \partial E$, so that for any $\rho > 0$ we have

$$\lim_n \int_{B_\rho(x)} \chi_{E_n} > 0, \qquad \lim_n \int_{B_\rho(x)} \chi_{E_n^c} > 0.$$

Then for $\rho > 0$ there is $n(\rho)$ such that there exist $\xi_n \in E_n \cap B_\rho(x)$, $\eta_n \in E_n^c \cap B_\rho(x)$ for any $n \ge n(\rho)$ and thus there exists $w_n \in \partial E_n \cap B_\rho(x)$ for any $n \ge n(\rho)$. Taking some sequence $\rho_k \searrow 0$, we find a sequence w_n converging to x. Therefore, also by density (3.4), we have proved that $\mathcal{F}E \subset \partial E \subset \big\{ y \,|\, y = \lim_n y_n,\ y_n \in \mathcal{F}E_n \big\} = \Gamma$. In particular ∂E is 1-rectifiable.

Now we prove that $\mathcal{F}E \subset \{x \mid \theta_V(x) \text{ is odd}\}$. So let $p \in \mathcal{F}E$, and let $\{\gamma^j_k \,|\, j = 1, ..., N, i = 1, ..., n_j\}$ be distinct curves which are suitable disjoint restrictions of the γ_i's such that $(\gamma^j_k) \subset (\gamma_j)$ for any k (up to relabeling the γ_i's) and

$$\Gamma \cap B_{r_0}(p) = \bigcup_{j,k} (\gamma^j_k).$$

Without loss of generality we write $\gamma^j_k(t^j_k) = p$. We want to prove that $\sum_{j=1}^N n_j = \theta_V(p)$ is odd. Since $p \in \mathcal{F}E$ there is $q \in E \cap B_{r_0}(p)$ such that the segment

$$s(t) = q + \frac{p-q}{|p-q|} t \qquad t \in [0, 2|p-q|]$$

is such that

$$0 < \left| \left\langle \frac{p-q}{|p-q|}, (\gamma^j_k)'(t^j_k) \right\rangle \right| < 1, \tag{3.11}$$

and $s|_{[0,|p-q|)} \subset E$, $s|_{[|p-q|,2|p-q|]} \subset E^c$. Also denote $b := s(2|p-q|)$. Moreover, we could choose q so that $B_{r_q}(q) \subset E_n$ and $B_{r_b}(b) \subset E_n^c$ for n sufficiently big for some $r_q, r_b > 0$. Also since $\gamma_{i,n} \to \gamma_i$ strongly in $C^{1,\alpha}$, by (3.11) we get that s intersects transversely $\gamma_{i,n}$ for any i for n big enough, and the number of such intersections is $\theta_V(p)$.

We know that for any $\varepsilon > 0$ there is $a_\varepsilon \in E_n^{c*}$, where $(\cdot)^*$ will always denote the unbounded connected component of $(\cdot)$, such that

$$\left| \frac{p-q}{|p-q|} - \frac{a_\varepsilon - b}{|a_\varepsilon - b|} \right| < \varepsilon,$$

$$\sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(b) \mod 2 = \sum_{i\in I} \sharp \left(\frac{\gamma_{i,n}}{|\gamma_{i,n}|} \right)^{-1} \left(\frac{a_\varepsilon - b}{|a_\varepsilon - b|} \right) \mod 2.$$

Hence up to a small smooth deformation which is different from the identity only on $\big\{x + t\frac{p-q}{|p-q|} \mid x \in B_{r_b}(b), t \in \mathbb{R}_{\geq 0}\big\} \setminus B_{r_b}(b)$ we can suppose that

$$\left\{ b + \mathbb{R}_{\geq 0}\left(\frac{p-q}{|p-q|} \right) \right\} \cap \left(\bigcup_{i\in I} (\gamma_{i,n}) \right) \subset \mathcal{F}E_n,$$

and for $M > 0$ sufficiently big it holds that

$$a_0 \coloneqq b + M\frac{p-q}{|p-q|} \in E_n^{c*},$$

and also

$$\sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(b) \mod 2 = \sum_{i\in I} \sharp \left(\frac{\gamma_{i,n}}{|\gamma_{i,n}|} \right)^{-1} \left(\frac{a_0 - b}{|a_0 - b|} \right) \mod 2. \tag{3.12}$$

Taking into account Lemma 3.1.5, by construction we have that the quantity in (3.12) equals $0 \mod 2$. Moreover we have that

$$1 \mod 2 = \sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(q) \mod 2 = \left(\theta_V(p) + \sum_{i\in I} \mathrm{Ind}_{\gamma_{i,n}}(b) \right) \mod 2,$$

and then $\theta_V(p)$ is odd.

It remains to prove that $\mathcal{H}^1(\{x \mid \theta_V(x) \text{ odd}\} \setminus \mathcal{F}E) = 0$. Here we are going to use the facts recalled in Section 1.5 and the correspondence between sets of finite perimeter and currents (see, in particular, Remark 1.5.3).

The use of currents here is just a choice of a language, as everything can be rephrased in terms of sets of finite perimeters. However, we believe this approach has some advantages on the presentation of the proof.

We have that $[|E_n|] \to [|E|]$ in the sense of currents, and thus $\partial[|E_n|] \to \partial[|E|]$. Taking on $\mathbb{S}^1$ the standard positive orientation with respect to the disk it encloses, we can write $\partial[|E_n|] = \sum_{i=0}^{\infty} (\alpha_{i,n})_\sharp([|\mathbb{S}^1|])$ for countably many Lipschitz parametrizations $\alpha_{i,n}$ ordered so that $L(\alpha_{i+1,n}) \leq L(\alpha_{i,n})$ for any i, n. Such immersions positively orient the boundary ∂E_n^i of E_n^i, where E_n^i is one of the open connected components of E_n, which are at most countable. The length of each $\alpha_{i,n}$ is uniformly bounded, then we can assume that the parametrizations $\alpha_{i,n}$ are L-Lipschitz with constant L independent of i, n. Since the parametrizations $\sigma_{i,n}$ converge strongly in C^1, the immersions $\alpha_{i,n}$ uniformly converge to L-Lipschitz curves $\alpha_i : \mathbb{S}^1 \to \mathbb{R}^2$ as $n \to \infty$. We can also suppose that each α_i is parametrized by constant speed, almost everywhere where α_i is differentiable. In the sense of currents we have that

$$\sum_{i=0}^{\infty} (\alpha_{i,n})_\sharp([|\mathbb{S}^1|]) = \partial[|E_n|] \to \partial[|E|] = \boldsymbol{\tau}\big(\mathcal{F}E, 1, \xi_E\big),$$

where ξ_E is the standard orientation of Remark 1.5.3. Let us define

$$T \coloneqq \sum_{i=0}^{\infty} (\alpha_i)_\sharp([|\mathbb{S}^1|]).$$

Since each $(\alpha_{i,n})$ is contained in some $(\sigma_{i_0,n})$ we have that $d_{\mathcal{H}}(\alpha_{i,n},\alpha_i) \le N \max_{i=1,\dots,N} \|\sigma_{i,n} - \sigma_i\|_\infty \le \varepsilon$ for any $n \ge n_\varepsilon$. Writing the Hausdorff distance as $d_{\mathcal{H}}(A,B) = \inf\big\{\varepsilon > 0 \mid A \subset \mathcal{N}_\varepsilon(B),\, B \subset \mathcal{N}_\varepsilon(A)\big\}$ where $\mathcal{N}_\varepsilon(X) = \{x \mid d(x,X) < \varepsilon\}$, we have that

$$\forall \varepsilon > 0\, \exists n_\varepsilon : \qquad d_{\mathcal{H}}\big(\cup_i (\alpha_{i,n}), \cup_i(\alpha_i)\big) < \varepsilon \qquad n \ge n_\varepsilon. \tag{3.13}$$

Thus $\cup_i(\alpha_{i,n})$ converges in Hausdorff distance to the set $\overline{\cup_i(\alpha_i)}$. Moreover, writing $\cup_i(\alpha_{i,n}) = \sqcup_1^{k_n} C_n^j$ as a disjoint union of finitely many compact connected components, by a diagonal argument, applying Gołab Theorem [Fal86, Theorem 3.18] on each component, we can assume $k_n = k$ for any n and that C_n^j converges in Hausdorff distance to a compact connected set C^j for any $j = 1,\dots,n$. Therefore $\overline{\cup_i(\alpha_i)} = \cup_1^k C^j = \Gamma$, and then $\mathcal{H}^1(\overline{\cup_i(\alpha_i)}) = \mathcal{H}^1(\Gamma)$ is finite and $\overline{\cup_i(\alpha_i)}$ is closed and 1-rectifiable.

Let $x \in \mathbb{R}^2 \setminus \Gamma$. By (3.13) we have that there is $\rho > 0$ such that $B_\rho(x) \cap \big(\overline{\cup_i(\alpha_i)} \cup \cup_i(\alpha_{i,n})\big) = \emptyset$ for any n large. Then there exists n_x such that for any i the index $\mathrm{Ind}_{\alpha_{i,n}}(x)$ is the same for any $n \ge n_x$.

Indeed suppose by contradiction for any n there is $i_n, N_1, N_2 \ge n$ with $1 = \mathrm{Ind}_{\alpha_{i_n,N_1}}(x) \ne \mathrm{Ind}_{\alpha_{i_n,N_2}}(x) = 0$ and $i_n \to \infty$ without loss of generality. Then $L(\alpha_{i_n,N_1}) \ge C(\rho)$ for a constant $C(\rho) > 0$ depending only on ρ by isoperimetric inequality. Since $L(\alpha_{i+1,n}) \le L(\alpha_{i,n})$ for any i,n and $i_n \to \infty$, this implies $P(E_n)$ is arbitrarily big that for n large enough.

Now let $x \in \mathbb{R}^2 \setminus \Gamma$ such that there exists $\lim_n \chi_{E_n}(x)$. Since $\chi_{E_n}(x) = \sum_i \mathrm{Ind}_{\alpha_{i,n}}(x)$ for n big such that $B_\rho(x) \cap \big(\overline{\cup_i(\alpha_i)} \cup \cup_i(\alpha_{i,n})\big) = \emptyset$ for some $\rho > 0$, from the above discussion we have that

$$\begin{aligned} \lim_n \sum_i \mathrm{Ind}_{\alpha_{i,n}}(x) = 1 \quad &\Leftrightarrow \quad \forall n \ge n_0\, \exists i : \quad \mathrm{Ind}_{\alpha_{i,n}}(x) = 1 \\ &\Leftrightarrow \quad \exists i(x)\, \forall n \ge n_0 \quad \mathrm{Ind}_{\alpha_{i(x),n}}(x) = 1 \\ &\Leftrightarrow \quad \exists i(x) : \quad \mathrm{Ind}_{\alpha_{i(x)}}(x) = 1. \end{aligned}$$

Hence

$$x \in E \quad \Leftrightarrow \quad \lim_n \sum_i \mathrm{Ind}_{\alpha_{i,n}}(x) = 1 \quad \Leftrightarrow \quad \exists i(x) : \quad \mathrm{Ind}_{\alpha_{i(x)}}(x) = 1 \quad \Leftrightarrow \quad \sum_i \mathrm{Ind}_{\alpha_i}(x) = 1.$$

In particular

$$E = \left\{ x \in \mathbb{R}^2 \setminus \Gamma \mid \sum_{i=0}^{\infty} \mathrm{Ind}_{\alpha_i}(x) = 1 \right\} = \left\{ x \in \mathbb{R}^2 \setminus \Gamma \ : \ \left| \sum_{i \in I} \mathrm{Ind}_{\sigma_i}(x) \right| \text{ is odd} \right\}, \tag{3.14}$$

where the second equality follows by the uniform convergence of the finitely many curves $\sigma_{i,n}$. Also for any $i \ne j$ it holds that $|\{x \in \mathbb{R}^2 \setminus (\alpha_i) \mid \mathrm{Ind}_{\alpha_i}(x) = 1\} \cap \{x \in \mathbb{R}^2 \setminus (\alpha_j) \mid \mathrm{Ind}_{\alpha_j}(x) = 1\}| = 0$, because the equality holds for any n for $\alpha_{i,n}, \alpha_{j,n}$. Hence, by convergence in the sense of currents, we have

$$\begin{aligned} \sum_i \int_{(\alpha_{i,n})} \langle \omega, \tau_{i,n} \rangle = \int_{E_n} d\omega \ \underset{n}{\to} \ &\int_{\left\{ \sum_{i=0}^{\infty} \mathrm{Ind}_{\alpha_i}(x) = 1 \right\}} d\omega \\ &= \sum_i \int_{\left\{ \mathrm{Ind}_{\alpha_i}(x) = 1 \right\}} d\omega = \sum_i \int_{(\alpha_i)} \langle \omega, \tau_i \rangle, \end{aligned}$$

for any 1-form ω on $\mathbb{R}^2$. This means that

$$\sum_{i=0}^{\infty}(\alpha_{i,n})_\sharp([|\mathbb{S}^1|]) = \partial[|E_n|] \to T = \sum_{i=0}^{\infty}(\alpha_i)_\sharp([|\mathbb{S}^1|]) = \partial[|E|] = \boldsymbol{\tau}\big(\mathcal{F}E, 1, \xi_E\big),$$

in the sense of currents. In particular we can write the multiplicity function of the current $\partial[|E|]$ as

$$m(x) = \sum_{i=0}^{\infty}\sum_{y\in\alpha_i^{-1}(x)} S(y), \tag{3.15}$$

for $\mathcal{H}^1$-ae $x \in \mathbb{R}^2$, where $S(y) = +1$ if $d(\alpha_i)_y$ preserves the orientation and $S(y) = -1$ in the opposite case. Note that since θ_V is bounded, $\Gamma = \overline{\cup_i(\alpha_i)}$, and $\theta_V(p) \geq \sum_i \sharp\alpha_i^{-1}(p)$, then the series in (3.15) is actually a finite sum. Also observe that since E is a set of finite perimeter, by the identity $\partial[|E|] = \boldsymbol{\tau}\left(\mathcal{F}E, 1, \xi_E\right)$, the multiplicity function m is equal to 1 $\mathcal{H}^1$-ae on $\mathcal{F}E$, $\mathcal{H}^1(\{x \,|\, |m(x)| \geq 1\} \setminus \mathcal{F}E) = 0$, and $m = 0$ $\mathcal{H}^1$-ae on $\mathbb{R}^2 \setminus \mathcal{F}E$. Also, $m(x) = 0$ at $\mathcal{H}^1$-ae $x \in \Gamma \setminus \cup_i(\alpha_i)$.

Now since $\Gamma = \overline{\cup_i(\alpha_i)}$ and $\mathcal{H}^1(\alpha_i(\{t : \nexists\alpha_i'(t)\})) = 0$, then

$$\mathcal{H}^1\big(\{p \in \Gamma \mid \exists\, t, i \,:\, \alpha_i(t) = p,\, \nexists\alpha_i'(t)\}\big) = 0. \tag{3.16}$$

So let $p \in \cup_i(\alpha_i)$ be such that if $\alpha_i(t) = p$ then $\exists\alpha_i'(t)$. We want to check that $\theta_V(p)$ and $\sum_i \sharp\alpha_i^{-1}(p)$ have the same parity. In fact if without loss of generality $\theta_V(p) > \sum_i \sharp\alpha_i^{-1}(p)$, taking into account (3.14), following a segment s intersecting $\cup_i(\alpha_i)$ only at p and transversely (as in the first part of the proof) we have that:

i) s passes from E to E^c if and only if $\theta_V(p)$ is odd, or equivalently if and only if $\sum_i \sharp\alpha_i^{-1}(p)$ is odd;

ii) s passes from E to E if and only if $\theta_V(p)$ is even, or equivalently if and only if $\sum_i \sharp\alpha_i^{-1}(p)$ is even.

Hence by (3.15) we conclude that $\theta(p)$ is odd if and only if alternative i) above holds, if and only if the summands in (3.15) are odd, if and only if $m(p)$ is odd. By (3.16) this holds for $\mathcal{H}^1$-ae point in $\cup_i(\alpha_i)$. Therefore $\mathcal{H}^1(\{x \,|\, m(x) \text{ is odd}\}\Delta\{x \,|\, \theta_V(x) \text{ is odd}\}) = 0$. So finally since $\mathcal{H}^1(\{x \,|\, m(x) = 1\} \setminus \mathcal{F}E) = 0$, then

$$\begin{aligned} 0 &= \mathcal{H}^1(\{x \,|\, m(x) \text{ odd}\} \setminus \{x \,|\, m(x) = 1\}) = \mathcal{H}^1(\{x \,|\, \theta_V(x) \text{ odd}\} \setminus \{x \,|\, m(x) = 1\}) \\ &= \mathcal{H}^1(\{x \,|\, \theta_V(x) \text{ is odd}\} \setminus \mathcal{F}E), \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.1.7.

First we want to prove the following approximation result.

Proposition 3.1.14. *Let $E \subset \mathbb{R}^2$ be measurable and bounded with $\mathcal{A}(E) \neq \emptyset$. Then for any $V \in \mathcal{A}(E)$ there exists a sequence E_n of uniformly bounded sets such that*

$$\mathcal{E}_p(E_n) < +\infty, \quad \chi_{E_n} \to \chi_E \quad \text{in } L^1(\mathbb{R}^2), \quad V_{E_n} \to V \text{ as varifolds}, \quad \lim_n \mathcal{E}_p(E_n) = \mathcal{E}_p(V).$$

Moreover for any n we have that $V_{E_n} = \sum_{i=1}^N(\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) = \mathbf{v}(\Gamma_n, \theta_{V_{E_n}})$ and $\{p \,|\, \theta_{V_{E_n}}(p) > 1\}$ is finite.

Proof. Let $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1)) \in \mathcal{A}(E)$ with $\gamma_i \in W^{2,p}$ and regular. For any i let $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of analytic regular immersions such that $\gamma_{i,n} \to \gamma_i$ in $W^{2,p}$ as $n \to \infty$. Hence the set

$$\{x \in \mathbb{R}^2 \,|\, \exists i,j,t \neq \tau : \; \gamma_i(t) = \gamma_j(\tau)\} \tag{3.17}$$

is finite. Let $V_n = \sum_{i=1}^N (\gamma_{i,n})_\sharp(\mathbf{v}(\mathbb{S}^1,1))$. By (3.17) we can define E_n as in Lemma 3.1.5, so that $V_n = V_{E_n}$. Moreover we have that

$$\mathcal{E}_p(E_n) < +\infty, \qquad \lim_{n\to\infty} \mathcal{E}_p(V_{E_n}) = \lim_{n\to\infty} \mathcal{E}_p(E_n) = \mathcal{E}_p(V), \qquad V_{E_n} \to V.$$

By uniform convergence of $\gamma_{i,n}$ we get that for any $\varepsilon > 0$ there is n_ε such that

$$\bigcup_{i=1}^N (\gamma_{i,n}) \subset I_{\frac{\varepsilon}{2}}\left(\bigcup_{i=1}^N (\gamma_i)\right) \qquad \forall n \geq n_\varepsilon,$$

where $I_{\frac{\varepsilon}{2}}$ denotes the $\frac{\varepsilon}{2}$ open tubolar neighborhood. Hence up to passing to a subsequence by Riesz-Fréchet-Kolmogorov Theorem [Bre11, Theorem 4.26] we have that χ_{E_n} converges strongly in $L^2(\mathbb{R}^2)$, and then in $L^1(\mathbb{R}^2)$ and pointwise almost everywhere to the characteristic function of a closed set F. Using the definition of E_n and Lemma 3.1.10 together with Remark 3.1.11 we have that $F = E$, and the proof is completed. □

Corollary 3.1.15. *Let $E \subset \mathbb{R}^2$ be measurable and bounded with $\mathcal{A}(E) \neq \emptyset$. Then*

$$\exists \min \big\{\mathcal{E}_p(V) \mid V \in \mathcal{A}(E)\big\}.$$

Proof. Let V_k be a minimizing sequence in $\mathcal{A}(E)$. Up to subsequence we can assume that $V_k \to V$ in the sense of varifolds and the supports sptV_k are uniformly bounded. By Proposition 3.1.14 using a diagonal argument we find a sequence of uniformly bounded sets E_k such that

$$\begin{aligned} &\chi_{E_k} \to \chi_E \quad \text{in } L^1(\mathbb{R}^2), && \mathcal{F}_p(E_k) \leq C < +\infty, \\ &V_{E_k} \to V \quad \text{as varifolds}, && \lim_k \mathcal{F}_p(E_k) = \lim_k \mathcal{F}_p(V_k) = \inf_{\mathcal{A}(E)} \mathcal{F}_p \geq \mathcal{F}_p(V), \end{aligned}$$

and $\{p \,|\, \theta_{V_{E_k}}(p) > 1\}$ is finite. Hence E_k is a possible approximating sequence of E by regular sets, i.e. a competitor in the calculation of the relaxation $\overline{\mathcal{E}_p}(E)$. Then by Lemma 3.1.13 we get that $V \in \mathcal{A}(E)$, and therefore V minimizes $\mathcal{E}_p$ on $\mathcal{A}(E)$. □

Now Proposition 3.1.14 together with Corollary 3.1.15 readily imply Theorem 3.1.7.

Comments on the case $p = 1$.

The characterization of the relaxed energy given by Theorem 3.1.7 fails in the $p = 1$ case. This is ultimately due to the fact that, if $I \subset \mathbb{R}$ is a bounded interval, functions $u \in W^{2,1}(I)$ do not satisfy good sequential compactness properties, that is, strong compactness in C^1. Indeed even if $u \in W^{2,1}(I)$ implies that $u' \in W^{1,1}(I) = AC(\bar{I})$ and hence $u \in C^1$, the immersion $W^{2,1}(I) \hookrightarrow C^1(\bar{I})$ is not continuous. Since $W^{2,1}(I) \hookrightarrow W^{1,p}(I)$ for any $p \in [1,\infty)$, we have that $W^{2,1}(I)$ compactly embeds only in $C^{0,\alpha}(\bar{I})$ for any $\alpha \in (0,1)$. This implies that the convergence of the curves defining the boundary of sets E_n with $\mathcal{E}_1(E_n) \leq C$ is much weaker than in the $p > 1$ case.

From the geometric point of view, in the characterization of the corresponding relaxed functional with $p = 1$, a main difference is the following. As we will show in Section 3.2.3, the $\overline{\mathcal{E}_p}$

energy of polygons, i.e. sets whose boundary is the image of an injective piecewise C^2 closed curve, is infinite if $p > 1$. Instead if E is a regular polygon, it can happen that $\overline{\mathcal{E}_1}(E) < +\infty$. For instance, consider a square Q in the plane: in small neighborhoods of the four vertices the boundary ∂Q can be approximated by a piece of circumference of radius converging to 0 with finite bounded energy. This is ultimately due to the invariance of the energy $\mathcal{E}_1$ under rescaling, a property that is absent if $p > 1$. We notice that this implies that limit varifolds possibly do not verify the flux property (see also of the arguments in the proof of Proposition 3.2.8).

We believe that the presence of such "vertices", like the ones of a polygon, in the boundary of the limit set is the main difference with the $p > 1$ case, and that sets E with $\overline{\mathcal{E}_1}(E) \le C$ have at most countably many vertices, each of them giving an additional positive contribution to the energy proportional to the angle determined at the vertex.

3.2 Qualitative properties and applications

In this section we discuss applications, examples, and qualitative properties related to the notion of relaxation characterized in Theorem 3.1.7. We first start with an example that compares the relaxed energy $\overline{\mathcal{E}_p}$ with the classical notion of relaxation described at the beginning of the chapter studied in [BDMP93; BM04; BM07]. Then we discuss a simple application to the so-called inpainting problem. Finally we collect additional examples and qualitative properties of sets with finite relaxed energy.

3.2.1 Comparison with the classical relaxation

For simplicity we limit ourselves to the case $p = 2$. Let us define the classical notion of relaxation considered in [BDMP93; BM04; BM07]. Let $E \subset \mathbb{R}^2$ be measurable and define the energy

$$G(E) = \begin{cases} \int_{\partial E} 1 + \frac{1}{2}|k_{\partial E}|^2 \, d\mathcal{H}^1 & E \text{ is of class } C^2, \\ +\infty & \text{otherwise,} \end{cases}$$

where $k_{\partial E}$ is the curvature of the boundary of the C^2-smooth set E. Then the functional $\overline{G}$ is the L^1-relaxation of G. We clearly have that

$$G(E) < +\infty \quad \Rightarrow \quad \mathcal{E}_2(E) = G(E),$$

and

$$\overline{\mathcal{E}_2}(E) \le \overline{G}(E) \qquad \forall E.$$

The precise characterization of $\overline{G}$ is discussed in [BM04] and [BM07]. In this part we observe that

$$\exists E : \qquad \overline{\mathcal{E}_2}(E) < \overline{G}(E) < +\infty.$$

Indeed an example is the set E_0 in Fig. 3.2 described in [BM07, Example 4.4]. Let γ_1, γ_2 be as in Fig. 3.2. In [BM07, Example 4.4] it is proved that

$$\overline{G}(E_0) > \mathcal{E}_2(\gamma_1) + \mathcal{E}_2(\gamma_2).$$

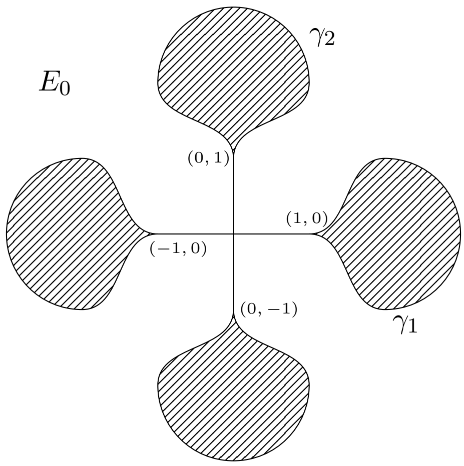

Figure 3.2: Picture of the set E_0 in [BM07, Example 4.4]. The curve γ_1 parametrizes the left and the right components, while γ_2 parametrizes the upper and lower components. The varifold $(\gamma_1)_\sharp(\mathbf{v}(\mathbb{S}^1,1)) + (\gamma_1)_\sharp(\mathbf{v}(\mathbb{S}^1,1))$ belongs to $\mathcal{A}(E_0)$, and it has multiplicity equal to 1 on ∂E_0 and equal to 2 on the cross in the middle of the picture.

Here we want to prove that

$$\overline{\mathcal{E}_2}(E_0) = \mathcal{E}_2(\gamma_1) + \mathcal{E}_2(\gamma_2). \tag{3.18}$$

Observe that inside $B_1(0)$ the curves γ_1, γ_2 have zero curvature and their total length equals 8. Since $\overline{\mathcal{E}_2}(E_0) < +\infty$ there exists a varifold $V = \sum_{i=1}^N (\sigma_i)_\sharp(\mathbf{v}(\mathbb{S}^1,1)) \in \mathcal{A}(E_0)$. Up to renaming and reparametrization assume $\sigma_1(0) = (1,0)$, $\sigma_1'(0) = -(1,0)$, and $\sigma_1|_{(-T,0)}$ joins $(1,0)$ and $(1,0)$ having support contained in $\mathcal{F}E_0 \setminus \overline{B_1(0)}$. Since σ_1 is C^1 and closed, by the above discussion there exists a first time $\tau > 0$ such that $\sigma_1(\tau) \in \{(1,0),(0,1),(-1,0),(0,-1)\}$. We distinguish two cases.

i) If $\sigma_1(\tau) \in \{(0,1),(0,-1)\}$, arguing like in the proof of inequality (1.13) one has

$$\frac{\pi}{2} \le \left[L(\sigma_1|_{(0,\tau)})\right]^{\frac{1}{2}} \left[\int_0^\tau |k_{\sigma_1}|^2\,ds\right]^{\frac{1}{2}} \le \frac{1}{2}\left(L(\sigma_1|_{(0,\tau)}) + \int_0^\tau |k_{\sigma_1}|^2\,ds\right),$$

and then $\mathcal{E}_2(\sigma_1|_{(0,\tau)}) \ge \frac{\pi}{2} + \frac{1}{2}L(\sigma_1|_{(0,\tau)}) \ge \frac{\pi}{2} + \frac{\sqrt{2}}{2} > 2$.

ii) If $\sigma_1(\tau) = (1,0)$ by an analogous argument one gets $\mathcal{E}_2(\gamma_1|_{(0,\tau)}) \ge 2\pi > 2$.

Therefore, comparing with γ_1, we get that it is convenient for σ_1 to pass first through the point $(-1,0)$ among the points $\{(1,0),(0,1),(-1,0),(0,-1)\}$. Hence, comparing with γ_1, γ_2 and using the characterization in Theorem 3.1.7, we see that (3.18) immediately follows.

3.2.2 Inpainting

Here we describe a simple but significant application of the relaxed functional $\overline{\mathcal{E}_p}$ given by Theorem 3.1.7. Such an application arises from the inpainting problem that, roughly speaking, consists in the reconstruction of a part of an image, knowing how the remaining part of the picture looks like. This problem as stated is quite involved [Ber+11]. Assuming the only two colors of the image are black and white, as already pointed out for example in [AM03], one can think that the black shape contained in lost part of the image is consistent with the shape minimizing a suitable functional depending on length and curvature of its boundary. In such a setting, the known part of the image plays the role of the boundary conditions. On different scales one can ask for the optimal unknown shape to minimize a weighted functional, where one can give more importance to the length or to the curvature term. Now we formalize the problem and we give a variational result.

Fix $p \in (1, \infty)$. In $\mathbb{R}^2$ consider the set H defined as follows. Let Q_1, Q_2 be the squares $Q_1 = \{(x, y) : 0 \le x \le 10, 0 \le y \le 10\}, Q_2 = \{(x, y) : -10 \le x \le 0, -10 \le y \le 0\}$, modify the squares in small neighborhoods of the vertices into convex sets $\widetilde{Q}_1, \widetilde{Q}_2$ with smooth boundary. Let

$$H := \big(\widetilde{Q}_1 \cup \widetilde{Q}_2\big) \setminus B_1(0).$$

For $\lambda \in (0, \infty)$ let $\mathcal{F}_{\lambda,p}$ be the functional

$$\mathcal{F}_{\lambda,p}(E) = \begin{cases} \lambda \mu_{V_E}(\mathbb{R}^2) + \int |k_{V_E}|^p \, d\mu_{V_E} & \text{if} \quad \begin{array}{l} V_E = \sum_{i \in I} (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)), \\ \gamma_i : \mathbb{S}^1 \to \mathbb{R}^2 \ C^2\text{-immersion}, \ \sharp I < +\infty, \end{array} \\ +\infty & \text{otherwise.} \end{cases}$$

Observe that the functional $\mathcal{F}_{\lambda,p}$ is just a modification of $\mathcal{E}_p$, where the constant weight in front of length and curvature are changed. As bounds on $\mathcal{F}_{\lambda,p}$ are equivalent to bounds on $\mathcal{E}_p$, the characterization for $\overline{\mathcal{F}_{\lambda,p}}$ is completely analogous to the one of $\overline{\mathcal{E}_p}$, and one can state a result like Theorem 3.1.7 for the relaxation $\overline{\mathcal{F}_{\lambda,p}}$.

We want to solve the minimization problem

$$\mathfrak{P} = \min \big\{ \overline{\mathcal{F}_{\lambda,p}}(E) \ \big| \ E \subset \mathbb{R}^2 \text{ measurable s.t. } E \setminus B_1(0) = H \big\}, \tag{3.19}$$

under the hypothesis of λ suitably small. The heuristic idea is that a good candidate minimizer is given by the set

$$E_0 = \big[(Q_1 \cup Q_2) \cap \overline{B_1(0)}\big] \cup H,$$

which has finite $\mathcal{E}_p$ energy. For a qualitative picture see Fig. 3.3.

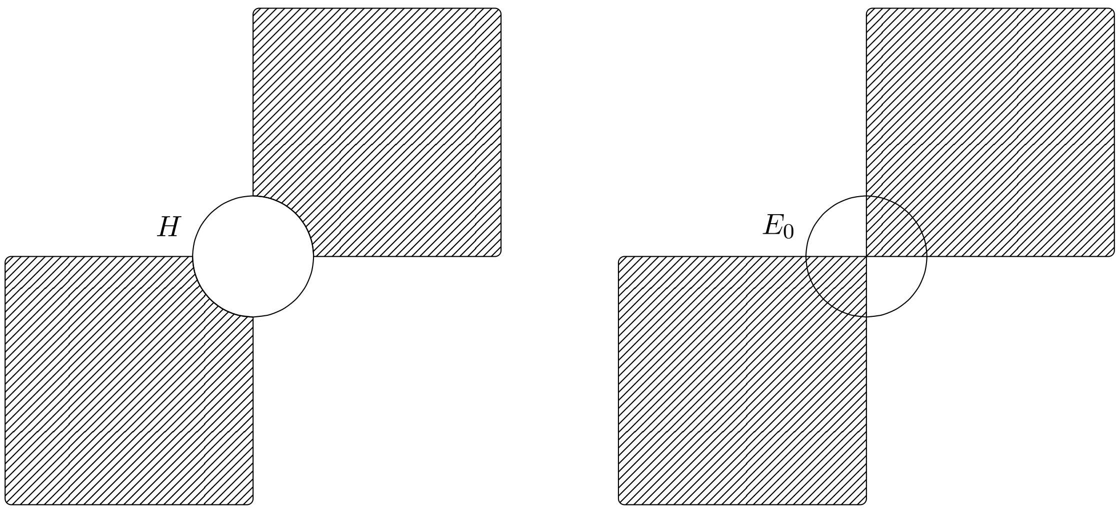

Figure 3.3: Qualitative pictures of the datum H and the minimizer E_0.

Remark 3.2.1. Observe that if $\overline{G}$ is the relaxed functional defined in Section 3.2.1, then $\overline{G}(E_0) = +\infty$, and thus E_0 will never be detected by a minimization problem (3.19) analogously defined with the functional $\overline{G}$.

We have the following result.

Proposition 3.2.2. *There exists $\lambda_0 \in \left(0, \frac{\pi}{2}\right)$ such that for any $\lambda \in (0, \lambda_0)$ the set E_0 is the unique minimizer of problem $\mathfrak{P}$.*

Proof. Let us first observe that varifolds associated to sets with energy sufficiently close to the infimum of the problem have mass uniformly bounded independently of λ. More precisely, suppose that E is a competitor such that $\overline{\mathcal{F}_{\lambda,p}}(E) \le \inf \mathfrak{P} + 1$, and let $\overline{\mathcal{F}_{\lambda,p}}(E) = \mathcal{F}_{\lambda,p}(V)$ for some $V \in \mathcal{A}(E)$. Then

$$\int |k_V|^p \, d\mu_V \le 1 + \overline{\mathcal{F}_{\lambda,p}}(E_0) \le 1 + \mathcal{F}_{\frac{\pi}{2},p}(E_0) =: C_1.$$

Applying Theorem 1.3.1 with $\sigma = 1$ and $\rho \to +\infty$ on the monotone function $M_{V,0}(\cdot)$ we get

$$\begin{aligned}
\mu_V(B_1(0)) &\le -\int_{\mathbb{R}^2 \setminus B_1} \left\langle k_V, \frac{x}{|x|} \right\rangle d\mu_V(x) - \int_{B_1} \langle k_V, x \rangle \, d\mu_V(x) \\
&\le C(H) + \int_{B_1} |k_V| \, d\mu_V \\
&\le C(H) + C_1^{\frac{1}{p}} \mu_V(B_1(0))^{\frac{1}{p'}}.
\end{aligned}$$

Hence $\mu_V(\mathbb{R}^2) \le C(H) + \mu_V(B_1(0)) \le \overline{C} = \overline{C}(H, p, E_0)$ and $\overline{C}$ is independent of λ.

Now let E_n be a minimizing sequence of problem $\mathfrak{P}$. By Theorem 3.1.7 and Lemma 3.1.4 we can write $\overline{\mathcal{F}_{\lambda,p}}(E_n) = \sum_{i \in I_n} \mathcal{F}_{\lambda,p}(\gamma_{i,n})$ for some curves $\gamma_{i,n}$. Up to subsequence $I_n = I$ and the curves converge strongly in C^1 and weakly in $W^{2,p}$ to curves γ_i. In particular $E_n \to E$ in the L^1-sense, and

$$\overline{\mathcal{F}_{\lambda,p}}(E) \le \inf \mathfrak{P} \le \overline{\mathcal{F}_{\lambda,p}}(E_0) = \mathcal{F}_{\lambda,p}(E_0). \tag{3.20}$$

by lower semicontinuity. Moreover, by C^1 strong convergence we have that

$$\forall i \, \forall p \in \big((\gamma_i) \cap \partial B_1(0)\big) \setminus \{(1,0), (0,1), (-1,0), (0,-1)\} \quad \Rightarrow \quad (\gamma_i) \text{ is tangent to } \partial B_1(0) \text{ at } p.$$

Observe that the energy of E_0 "inside" $B_1(0)$, that is the $\mathcal{F}_{\lambda,p}$-energy of the curves $\partial E_0 \cap B_1(0)$, is equal to 4λ. We now argue as in Section 3.2.1.

Since $\overline{\mathcal{F}_{\lambda,p}}(E) < +\infty$ there exists a varifold $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) \in \mathcal{A}(E)$. Up to renaming and reparametrization assume $\gamma_1(0) = (1,0)$, $\gamma_1'(0) = -(1,0)$, and $\gamma_1|_{(-T,0)}$ joins $(0,1)$ and $(1,0)$ having support contained in $\mathcal{F}H \setminus \overline{B_1(0)}$. Since γ_1 is C^1 and closed, by the above discussion there exists a first time $\tau > 0$ such that γ_1 intersects transversely $\partial B_1(0)$. Also such transversal intersection can take place only at one of the points in $\{(1,0), (0,1), (-1,0), (0,-1)\}$. We distinguish two cases.

i) If $\gamma_1(\tau) \in \{(0,1), (0,-1)\}$, recalling that there is $\overline{C} > 0$ independent of λ such that $L(\gamma_i) \le \overline{C}$ for any i, then arguing like in (1.13) we get

$$\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)}) \ge \lambda\sqrt{2} + \frac{\pi}{2} \frac{1}{L(\gamma_1|_{(0,\tau)})^{\frac{p}{p'}}} \ge \lambda\sqrt{2} + \frac{\pi}{2} \frac{1}{\overline{C}} > 2\lambda,$$

where the last inequality holds choosing λ_0 small enough.

ii) If $\gamma_1(\tau) = (1, 0)$, then by the same argument leading to (1.13) one has

$$\pi \leq \frac{L(\gamma_1|_{(0,\tau)})}{p'} + \frac{1}{p}\int_0^\tau |k_{\gamma_1}|^p \, ds. \tag{3.21}$$

If $\lambda p' \geq 1$, then $\pi \leq \mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})$. If instead $\lambda p' < 1$, then also $\frac{\lambda p'}{p} < 1$, and multiplying (3.21) by $\lambda p'$ one has $\lambda p' \pi \leq \mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)})$. So we can write that $\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)}) \geq \min\{1, \lambda p'\}\pi$. Choosing $\lambda_0 < \frac{\pi}{2}$ then $\pi > 2\lambda$, and since $p' > 1 > \frac{2}{\pi}$ then $\lambda p' \pi > 2\lambda$.

Hence in any case we have

$$\mathcal{F}_{\lambda,p}(\gamma_1|_{(0,\tau)}) > 2\lambda.$$

By inequality (3.20) we conclude that $\gamma_1(\tau) = (-1, 0)$ and, by characterization of $\overline{\mathcal{F}_{\lambda,p}}$ as in Theorem 3.1.7, we have that $\partial E_0 \subset \cup_{i=1}^N (\gamma_i)$. It follows that $E = E_0$, and thus $\mathfrak{P}$ has a unique minimizer, that is E_0. □

3.2.3 Examples and qualitative properties

In this section we fix $p \in (1, \infty)$ and we collect some observations about the qualitative properties of sets E having $\overline{\mathcal{E}_p}(E) < +\infty$.

First we want to prove a result that is strongly related to [BDMP93, Theorem 6.5], that states the relation between the elastic energy of a set and the number of cusps on its boundary. To this aim we need some definitions.

Definition 3.2.3. Let $E \subset \mathbb{R}^2$ be closed. A point $p \in \partial E$ is a *(simple) cusp* if there is $r > 0$ such that up to rotation and translation the set $B_r(p) \cap \partial E$ is the union of the graphs of two functions $f_1, f_2 : [0, a] \to \mathbb{R}$ of class $W^{2,p}$ with $f_i(0) = f_i'(0) = 0$, $f_1(x) \leq f_2(x)$, and $f_1(x) = f_2(x)$ if and only if $x = 0$.

Also, we need the following definitions in the context of planar graphs.

Definition 3.2.4. Let $G \subset \mathbb{R}^2$ be a planar finite graph, i.e. a set given by the union of finitely many embeddings of $[0, 1]$ of class $C^1 \cap W^{2,p}$, called *edges* of G, possibly meeting only at the endpoints, called *vertices* of G. The symbols E_G, V_G respectively denote the set of edges of G and the set of vertices of G. Together with the topology of a graph G, it is assigned a *multiplicity function* $m : E_G \to \mathbb{N}$.

For any vertex $v \in V_G$ there is $r_v > 0$ such that for $0 < r < r_v$ the set $H := G \cap B_r(v)$ is a finite connected graph whose edges only meet at v and with multiplicity inherited from G. In this notation, the *local density* of G at v is the number $\rho_G(v) := \sum_{e \in E_H} m(e)$.

Now assume also that for any $v \in V_G$ and $0 < r < r_v$, if f_i are regular parametrizations of the edges e_i of the graph $H = G \cap B_r(v)$ with $f_i(1) = v$, then for any i there is j such that the arclength derivatives $\dot{f}_i, \dot{f}_j$ satisfy $\dot{f}_i(1) = -\dot{f}_j(1)$. Under this assumption, we denote by $w_1(v), ..., w_{N_v}(v)$ unit norm vectors identifying the possible tangent directions given by $\{\dot{f}_i(1)\}_i$. Hence, being $w_i(v)^\perp$ the counterclockwise rotation of $w_i(v)$ of an angle equal to $\pi/2$, we define

$$I^+(w_i(v)) := \left\{ e_i \in E_H \mid \dot{f}_i(1) = \pm w_i(v), \ [\dot{f}_i(1), w_i(v)] \text{ is a negative basis of } \mathbb{R}^2 \right\},$$

$$I^-(w_i(v)) := \left\{ e_i \in E_H \mid \dot{f}_i(1) = \pm w_i(v), \ [\dot{f}_i(1), w_i(v)] \text{ is a positive basis of } \mathbb{R}^2 \right\},$$

and

$$\rho_G^+(v, w_i(v)) := \sum_{e_i \in I^+(w_i(v))} m(e_i),$$

$$\rho_G^-(v, w_i(v)) := \sum_{e_i \in I^-(w_i(v))} m(e_i).$$

The graph G is said to be *regular* if for any $v \in V_G$ and for any $w_i(v)$ it holds that $\rho_G^+(v, w_i(v)) = \rho_G^-(v, w_i(v))$.

Remark 3.2.5. Let $V = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1))$ be a varifold in $\mathcal{A}(E)$ for some set E. Suppose that $\Gamma = \cup(\gamma_i)$ defines a finite planar graph G_Γ. To any edge e of G_Γ we assign the multiplicity function $m_\Gamma(e) = \theta_V(p)$ for any $p \in e$ that is not a vertex. Such a definition of multiplicity is well-posed since any edge of G_Γ is assumed to be an embedded curve that never touches other edges, hence θ_V is constant along an edge of G_Γ. We observe that by the flux property (Definition 3.1.2), the graph G_Γ with the multiplicity m_Γ is regular.

We are ready to prove the following result about the energy of sets which are smooth out of finitely many cusp points. The strategy follows ideas from [BDMP93], but it is different in the technical parts.

Theorem 3.2.6. *Let $E \subset \mathbb{R}^2$ be a closed set whose boundary is $W^{2,p}$ smooth at every point but at finitely many ones which are simple cusps $q_1, ..., q_k$. Then*

$$\overline{\mathcal{E}_p}(E) < +\infty \qquad \Leftrightarrow \qquad k \text{ is even.}$$

Proof. If k is even, [BDMP93, Theorem 6.4] implies that the relaxed energy $\overline{G}(E)$ studied in [BDMP93] is finite. Since $\overline{\mathcal{E}_p} \leq \overline{G}$, we have one implication.

Now suppose that $\overline{\mathcal{E}_p}(E)$ is finite, i.e. $\mathcal{A}(E) \neq \emptyset$ by Theorem 3.1.7. Let $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) \in \mathcal{A}(E)$. We are going to construct a set $\widetilde{E}$ satisfying the hypotheses of the theorem and having the same unknown number of cusps of E, together with a varifold $\widetilde{V} = \mathbf{v}(\widetilde{\Gamma}, \widetilde{\theta}_{\widetilde{V}}) \in \mathcal{A}(\widetilde{E})$ with the additional property that $\widetilde{\Gamma}$ defines a finite graph $G_{\widetilde{\Gamma}}$ with multiplicity as given in Remark 3.2.5. Once the support of a varifold in $\mathcal{A}(\widetilde{E})$ is a finite graph, we can prove that the number of cusps is even.

Step 1. Here we construct $\widetilde{E}$ and $\widetilde{\Gamma}$ as claimed. Let $\mathcal{C}(\Gamma)$ be the set of points $p \in \Gamma$ such that in any neighborhood of p it is impossible to write Γ as a single graph, i.e. p is a crossing or a branching point of two pieces of some curves γ_i, γ_j. Denote by K the set of accumulation points of $\mathcal{C}(\Gamma)$. Observe that K is compact.

Now fix $\varepsilon \in (0, 1)$ and let $q \in K$. Let $v_1(q), ..., v_{N_q}(q)$ be unit vectors identifying the tangent directions at q of the curves passing through q. For $j = 1, ..., N_q$ let $\sigma_1^j, ..., \sigma_{M_{q,j}}^j$ be suitable restrictions of the curves $\{\gamma_i\}$ on disjoint intervals $I_i^j = \text{domain}(\sigma_i^j)$ such that each σ_i^j passes through q with tangent parallel to $v_j(q)$. Also, for $i = 1, ..., N_q$ let $R_i(q)$ be open rectangles with two sides parallel to $v_i(q)$. Up to restriction we assume that each σ_i^j is contained in $\overline{R_j(q)}$ with endpoints on the boundary of the rectangle. We can assume that the following properties are satisfied.

i) Each rectangle contains at most one cusp and cusps do not lie on the boundary of any rectangle. Also if $q \in \Gamma \setminus \partial E$, then $\overline{R_i(q)} \cap \partial E = \emptyset$.

ii) The set $R_i(q) \cap \partial E$ is homeomorphic to a closed segment such that: if no cusps lie in $R_i(q)$ then $R_i(q) \cap \partial E$ is the graph of a $W^{2,p}$ function, if a cusp lies in $R_i(q)$ then $R_i(q) \cap \partial E$ is the union of the graphs of two $C^1 \cap W^{2,p}$ functions as in the definition of simple cusp.

iii) Each σ_i^j can be parametrized as graph inside $R_j(q)$, and $|\dot{\sigma}_i^j(\cdot)+v_j(q)|\le\varepsilon$ or $|\dot{\sigma}_i^j(\cdot)-v_j(q)|\le\varepsilon$.

iv) The curve σ_i^j intersects $\partial R_j(q)$ only on the sides perpendicular to $v_j(q)$ and transversely, and σ_i^j intersects σ_k^l only in the open set $R_j(q)\cup R_l(q)\setminus(\partial R_j(q)\cup\partial R_k(q))$.

v) If $a\in\partial I_i^j, b\in\partial I_k^j$ and $\sigma_i^j(a)=\sigma_i^k(b)$, then $\dot{\sigma}_i^j(a)=\pm\dot{\sigma}_k^j(b)$.

vi) For $a\in\partial I_i^j$: if $\sigma_i^j(a)\in\mathcal{F}E$, then $\theta_V(\sigma_i^j(a))=\sharp\{k\mid\sigma_k^j$ passes through $\sigma_i^j(a)\}$ is odd; if $\sigma_i^j(a)\in\Gamma\setminus\partial E$, then $\theta_V(\sigma_i^j(a))=\sharp\{k\mid\sigma_k^j$ passes through $\sigma_i^j(a)\}$ is even.

Item v) follows by the fact that transverse crossings of two curves are at most countable (as proved in Lemma 3.1.4), while Item vi) follows from the fact that $V\in\mathcal{A}(E)$ and thus θ_V is odd (resp. even) at $\mathcal{H}^1$-ae point of $\mathcal{F}E$ (resp. $\Gamma\setminus\partial E$).

Since the set K is compact, we can extract a finite covering of rectangles corresponding to points $q_1,...,q_L$. By Theorem 1.3.1 the numbers N_{q_i} of the rectangles of q_i are uniformly bounded in terms of the energy, which is finite. Hence we can add to the cover the possibly remaining rectangles corresponding to each cusp q_i, yielding a covering that is still finite. For any $j=1,...,L$ and $i=1,...,N_{q_j}$ we are going to modify the curves σ_i^j in a finite number of steps. We start from the family $\{\sigma_i^1\}_{i=1}^{M_{q_1,1}}$ corresponding to $R_1(q_1)$, then one modifies the curves corresponding to $R_2(q_1)$ and so on up to $R_{N_{q_1}}(q_1)$, then one changes the curves of the families corresponding to q_2 and so on up to q_L. Since the procedure is the same at any step, let us describe only the case of the family $\{\sigma_i^1\}_{i=1}^{M_{q_1,1}}$ corresponding to $R_1(q_1)$. In the end we will end up with the desired $\widetilde{E},\widetilde{\Gamma}$.

We modify a σ_i^1 as follows, depending on the cases $q_1\in\Gamma\setminus\partial E$, or $q_1\in\mathcal{F}E$, or q_1 is a cusp.

1. Assume that $q_1\in\Gamma\setminus\partial E$. Fix σ_i^1 and split it into the two pieces divided by q_1. Let us say that one such piece of σ_i^1 is parametrized as graph by $f:[0,\alpha]\to\mathbb{R}$ with $f(0)=f'(0)=0$ corresponding to q_1. Let u_i^1 be the solution of

$$\begin{cases} u(x)=\lambda x^3+\mu x^2+\nu x+\omega,\\ u(0)=u'(0)=0,\\ u(\alpha)=f(\alpha),\ \ u'(\alpha)=f'(\alpha),\end{cases}\tag{3.22}$$

for the suitable constants λ,μ,ν,ω. Doing the same with the other piece of σ_i^1, we substitute each σ_i^1 with the graphs of the obtained functions u_i^1 (such modification is then a change in one of the original curves γ_i's). Observe that by properties v), vi) one obtains a new varifold still in the class $\mathcal{A}(E)$, indeed graphs of finitely many polynomials meet in at most finitely many points.

2. Suppose now that $q\in\mathcal{F}E$. By construction, say $R_1(q_1)$ contains some curves with endpoints on $\mathcal{F}E\cap\partial R_1(q_1)$. In this case we modify the curves exactly as before following the system (3.22); moreover we declare that the boundary $\mathcal{F}E$ is modified inside $R_1(q_1)$ following the new modified curves having endpoints on $\mathcal{F}E\cap\partial R_1(q_1)$. This leads to a new set, which we already denote by $\widetilde{E}$, satisfying the hypotheses of the theorem and having the same number of cusps of E, together with a new varifold, already denoted by $\widetilde{V}$, in the class $\mathcal{A}(\widetilde{E})$ (as before this is due to the properties v) and vi), together with the fact that the new curves are graphs of polynomials).

3. Finally suppose q_1 is a cusp of ∂E. In this case we modify the curves (and the set E) exactly in the same way of the case 2). This preserves the cusp in the new set $\widetilde{E}$.

After performing these modifications in any $R_i(q_j)$ we end up with a varifold $\widetilde{V}$ given by curves $\widetilde{\gamma}_i$ such that the set $\mathcal{C}(\widetilde{\Gamma})$ of the points $p \in \widetilde{\Gamma}$ such that in any neighborhood of p it is impossible to write $\widetilde{\Gamma}$ as a single graph is finite. Indeed the points of this type belonging to the union of the closure of the rectangles $R_i(q_j)$ are finite. So, if by contradiction there are points of $\mathcal{C}(\widetilde{\Gamma})$ accumulating to some limit point q, this would be outside the union of the rectangles $R_i(q_j)$, and q would be a limit of a sequence in $\mathcal{C}(\Gamma)$. Hence q would be in K, and thus in the interior of some rectangle $R_i(q_j)$, that is a contradiction.

Step 2. Now we show that if a set E is as in the hypotheses of the theorem and if $V = \mathbf{v}(\Gamma, \theta_V) \in \mathcal{A}(E)$ is such that Γ defines a finite graph, then the number of cusps of E is even. Together with Step 1, this implies the thesis. Here we essentially generalize the strategy of [BDMP93]. Let G_Γ be the finite regular graph given by Γ with multiplicity m_Γ as assigned in Remark 3.2.5. Let us construct a new graph G with multiplicity m as follows. For $e \in E_{G_\Gamma}$, define the multiplicity

$$m(e) := \begin{cases} \frac{m_\Gamma(e)}{2} & \text{if } m_\Gamma(e) \text{ even,} \\ \frac{m_\Gamma(e)-1}{2} & \text{if } m_\Gamma(e) \text{ odd,} \end{cases}$$

with the convention that if $m(e) = 0$, then the edge e does not appear in G. Now let $y \in V_G$. We want to evaluate the parity of $\rho_G(y)$ dividing some cases.

i) Suppose $y \notin \partial E$. Then any edge e of G_Γ with endpoint at y has $\rho^+_{G_\Gamma}(y, w_i(y)) = \rho^-_{G_\Gamma}(y, w_i(y))$ even for any $w_i(y)$. Hence by definition we have that $\rho_G(y)$ is even.

ii) Suppose $y \in \mathcal{F}E$. Then exactly two edges e_1, e_2 of G_Γ having an endpoint at y have odd multiplicity: $m_\Gamma(e_i) = 2k_i + 1$ for $i = 1, 2$. Up to relabeling suppose that $e_1 \in I^+(w_1(y))$ and $e_2 \in I^-(w_1(y))$. Every other edge of G_Γ having an endpoint at y has even multiplicity. Since G_Γ is regular we have that

$$2k_1 + 1 + 2a_1^+ = \rho^+_{G_\Gamma}(y, w_1(y)) = \rho^-_{G_\Gamma}(y, w_1(y)) = 2k_2 + 1 + 2a_1^-,$$

and similarly

$$2a_i^+ = \rho^+_{G_\Gamma}(y, w_i(y)) = \rho^-_{G_\Gamma}(y, w_i(y)) = 2a_i^-,$$

for any possible $i \geq 2$. Then

$$\rho_G(y) = k_1 + a_1^+ + k_2 + a_1^- + \sum_{i\geq 2} a_i^+ + a_i^- = 2\left(k_1 + a_1^+ + \sum_{i\geq 2} a_i^+\right)$$

is even.

iii) Finally suppose that y is a cusp of ∂E. Then exactly two edges e_1, e_2 of G_Γ having an endpoint at y have odd multiplicity: $m_\Gamma(e_i) = 2k_i + 1$ for $i = 1, 2$. Up to relabeling suppose that $e_1, e_2 \in I^+(w_1(y))$. Every other edge of G_Γ having an endpoint at y has even multiplicity. Since G_Γ is regular we have that

$$2k_1 + 1 + 2k_2 + 1 + 2a_1^+ = \rho^+_{G_\Gamma}(y, w_1(y)) = \rho^-_{G_\Gamma}(y, w_1(y)) = 2a_1^-,$$

and similarly

$$2a_i^+ = \rho^+_{G_\Gamma}(y, w_i(y)) = \rho^-_{G_\Gamma}(y, w_i(y)) = 2a_i^-,$$

for any possible $i \geq 2$. Then

$$\rho_G(y) = k_1 + k_2 + a_1^+ + a_1^- + \sum_{i\geq 2} a_i^+ + a_i^- = 2(k_1 + k_2 + a_1^+) + 1 + 2\sum_{i\geq 2} a_i^+,$$

that is odd.

It follows that the cusps of ∂E coincides with the vertices y of G having odd local density $\rho_G(y)$. By Theorem 1.2.1 in [Ore62], the vertices of a finite graph with odd local density are even. Hence the cusps are even and the proof is completed. □

Now we turn our attention to another class of sets. Let us give the following definition.

Definition 3.2.7. A closed measurable set $E \subset \mathbb{R}^2$ is a *p-polygon* if $\partial E = (\gamma)$ for a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ such that:

1. γ is injective;
2. there exist finitely many times $t_1 < t_2 < ... < t_K$ such that $\gamma|_{(t_i,t_{i+1})} \in W^{2,p}$ for $i = 1, ..., K$ (with $t_{K+1} = t_1$), and $\gamma'(t_i^-), \gamma'(t_i^+)$ are linearly independent for $i = 1, ..., K$.

We can prove the following result about the energy of polygons.

Proposition 3.2.8. *Let E be a p-polygon, then $\overline{\mathcal{E}_p}(E) = +\infty$.*

Proof. Let γ be as in the definition of p-polygon. Without loss of generality we can assume that $0 = \gamma(0)$ is such that $\gamma'(0^-)$ and $\gamma'(0^+)$ are linearly independent. Suppose by contradiction that there is a varifold $V = \mathbf{v}(\Gamma, \theta_V) = \sum_{i=1}^N (\gamma_i)_\sharp(\mathbf{v}(\mathbb{S}^1, 1)) \in \mathcal{A}(E)$. Let $v = \gamma'(0^-)$, then since V verifies the flux property we find a nice rectangle $R_v(p)$ at p with side parameters a, b for the curves $\{g_j\}_{j=1}^r$ given by the definition of flux property. We can suppose that $g_1|_{[-\varepsilon,0)} \subset \mathcal{F}E$, $g_1|_{(0,\varepsilon]} \subset \Gamma \setminus \partial E$, and that $(g_i) \cap \partial E = \{0\}$ for $i = 2, ..., r$. Hence

$$g_1|_{[-\varepsilon,0)} \subset \{\theta_V \text{ odd}\},$$

$$\mathcal{H}^1\left(\left(\bigcup_{i=2}^r (g_i) \cup g_1\big((0,\varepsilon]\big)\right) \setminus \{\theta_V \text{ even}\}\right) = 0.$$

Then there exists $c_1 \in (-a, 0)$ such that

$$\sum_{z \in \cup_{j=1}^r (g_i) \cap \{y \,|\, \langle y-p, v\rangle = c_1\}} \theta_V(z) = M_1$$

with M_1 odd, and there exists $c_2 \in (0, a)$ such that

$$\sum_{z \in \cup_{j=1}^r (g_i) \cap \{y \,|\, \langle y-p, v\rangle = c_2\}} \theta_V(z) = M_2$$

with M_2 even. But by the flux property M_1 and M_2 should be equal, thus we have a contradiction. □

Remark 3.2.9. It follows from the proof of Proposition 3.2.8 that, roughly speaking, $\overline{\mathcal{E}_p}(E) = +\infty$ whenever the boundary ∂E "has an angle" (in the same sense of the definition of polygon).

With the strategy employed in the proof of Proposition 3.2.8 we can construct an example of a set $E \subset \mathbb{R}^2$ such that E is a set of finite perimeter such that the associated varifold V_E verifies that

$$\sigma_{V_E} = 0, \qquad k_{V_E} \in L^2(\mu_{V_E}), \qquad \overline{\mathcal{E}_p}(E) = +\infty.$$

Such set is discussed in the next example.

Example 3.2.10. Consider a positive angle $\theta > 0$ which will be taken very small and the vectors in the plane identified by the complex numbers

$$e^{-i\theta}, \qquad e^{-i2\theta}, \qquad e^{i(-\pi+\theta)}, \qquad e^{i(-\pi+2\theta)}. \tag{3.23}$$

The sum of such vectors gives the point $(0, -2(\sin(\theta) + \sin(2\theta)))$. Now let $\varphi > 0$ be another positive angle and consider the vectors

$$e^{i\varphi}, \qquad e^{i(\pi-\varphi)}, \tag{3.24}$$

so that the sum of these last vectors gives the point $(0, 2\sin(\varphi))$. Then for $\theta \to 0$, since $\sin(\theta) + \sin(2\theta) = 3\theta + o(\theta^2)$ there exists $\varphi = 3\theta + o(\theta^2)$ such that the sum of the vectors in (3.23) and (3.24) is zero.

Given these vectors we can define a set E as in Fig. 3.4 whose boundary is the image of three smooth closed immersions σ_i of the interval $[0, 1]$ having $\sigma_i(0) = \sigma_i(1) = 0$ with derivative $\sigma_i'(0), \sigma_i'(1)$ proportional to the vectors in (3.23), (3.24). In such a way the varifold V_E clearly verifies that $\sigma_{V_E} = 0$ and $k_{V_E} \in L^2(\mu_{V_E})$. However arguing as in the proof of Proposition 3.2.8 and assuming $\overline{\mathcal{E}_p}(E) < +\infty$, one immediately gets a contradiction. Hence $\overline{\mathcal{E}_p}(E) = +\infty$.

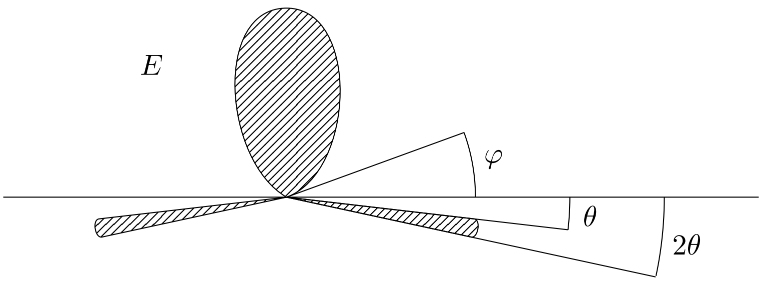

Figure 3.4: Picture describing the set E of Example 3.2.10. The set is symmetric with respect to the reflection about the vertical axis.

We can also construct a simple example showing that there are sets E with $\overline{\mathcal{E}_p}(E) < \infty$, but such that $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) > 0$ and ∂E is the support of a C^∞ immersion σ.

Example 3.2.11. Let us construct a set E such that $\partial E = (\gamma)$ for a C^∞ immersion $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) > 0$, and $\overline{\mathcal{E}_p}(E) < +\infty$.

Let $\{q_n\}_{n\geq1} = \mathbb{Q} \cap [0, 1]$ be an enumeration of the rationals in $[0, 1]$, and define $K = [0, 1] \setminus \cup_{n\geq1}(q_n - 2^{-n-2}, q_n - 2^{-n-2})$. The set K is compact and $\mathcal{L}^1(K) \geq 1 - \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2}$. Consider a C^∞ nonincreasing function $\varphi : [0, \infty) \to [0, 1]$ such that $\varphi(0) = 1$, $\varphi(t) = 0$ for $t \geq 1$ and let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi\left(\frac{(x - q_n)^2}{(2^{n+2})^2}\right) \qquad \forall x \in [0, 1].$$

By construction we have that $K = f^{-1}(0)$. Moreover $f \in C^\infty([0, 1])$, indeed $\varphi \leq 1$ and $|\varphi^{(k)}| \leq c_k$ for any $k \geq 1$ for some $c_k > 0$, so that both the series defining f and the one of the derivatives totally converge. Then we can define a C^∞ parametrization $\sigma : [0, 4] \to \mathbb{R}^2$ such that $\sigma(t) = (t, f(t))$ for $t \in [0, 1]$, $\sigma(t) = (3 - t, -f(t))$ for $t \in [2, 3]$, while $\sigma|_{[1,2]}$ and $\sigma|_{[3,4]}$ parametrize two drops with vertices respectively at $(1, 0)$ and $(0, 0)$. Therefore σ parametrizes the boundary of a bounded set E which is the planar surface enclosed by the two drops and lying between the graphs of f and $-f$.

By construction $\partial E = (\sigma)$ and $\mathcal{F}E = (\sigma) \setminus K$, hence $\mathcal{H}^1(\partial E \setminus \mathcal{F}E) \geq \frac{1}{2}$. However approximating f with $f_n(x) = f(x) + \frac{1}{n}\psi(x)$, where $\psi \in C^\infty([0,1],[0,1])$ is such that $\psi(0) = \psi(1) = 0$, $\psi|_{(0,1)} > 0$, and defining σ_n in analogy with σ, we conclude that $\overline{\mathcal{E}_p}(E) < +\infty$.

We conclude the chapter with some additional examples.

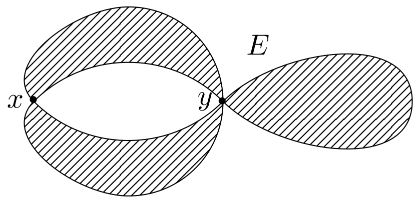

Figure 3.5: An example of a set of finite perimeter E such that $\mathcal{F}_p(E) = \mathcal{F}_p(V) < +\infty$ for any $p \in [1,\infty)$, where $V \in \mathcal{A}(E)$ is the varifold induced by a smooth immersion γ parametrizing ∂E. Here $\partial E = \mathcal{F}E \sqcup \{x, y\}$ and the strict inclusions $\mathcal{F}E \subsetneq \{x \mid \theta_V(x) \text{ is odd}\} = \mathcal{F}E \sqcup \{y\} \subsetneq \partial E$ occur.

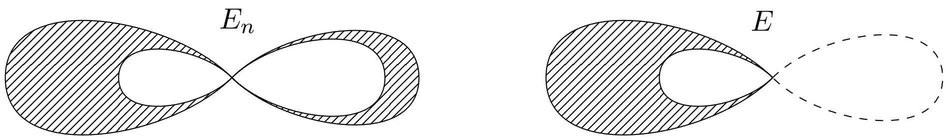

Figure 3.6: An example of a set E with finite relaxed energy such that $\partial E \setminus \mathcal{F}E$ is a singleton. A sequence of sets E_n converging to E with uniformly bounded energy is, for example, given by like in the one on the left in the picture. The dashed line represents the corresponding ghost line given by the collapsing of the right part of the sets E_n.

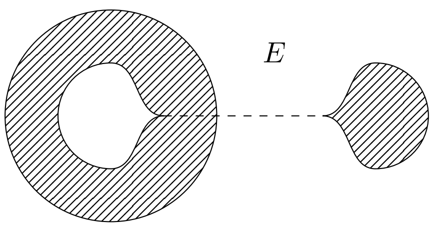

Figure 3.7: An example of a set E with finite relaxed energy such that, by Lemma 3.1.12, the multiplicity θ_V is not locally constant on connected components of $\mathcal{F}E$.

Chapter 4

Willmore energy of surfaces with boundary in weak and strong realms

Contents

This chapter is devoted to the study of some minimization problems on the Willmore energy of surfaces with boundary. We first give an overview of some of the main results present in the literature in the context of the existence theory for minimization problems on the Willmore energy. Then we consider the minimization of the Willmore energy among surfaces having a fixed planar boundary and we prove two non-existence results. We also discuss some Li–Yau-type inequalities in the context of surfaces with boundary. In the second part of the chapter we consider the minimization problem of the Willmore energy among varifolds having a fixed boundary and connected support, proving some properties and existence theorems. Some of the results in this chapter are contained in [Poz20c] and [NP20].

Let us recall some definitions first. If $\varphi : M^2 \to \mathbb{R}^n$ is a smooth immersion, its Willmore energy $\mathcal{W}(\varphi)$ is defined by

$$\mathcal{W}(\varphi) := \int_M |H|^2 \, d\mu_\varphi, \tag{4.1}$$

where H is the mean curvature vector and μ_φ is the volume measure induced by φ. If $\partial M \neq \emptyset$, we will denote by ds_φ the length measure induced by φ on ∂M. We also define the functional

$$D(\varphi) := \int_M |B|^2 \, d\mu_\varphi,$$

where $|B|$ is the norm of the second fundamental form B of the immersion φ.

Let us introduce a further notation. If $\varphi : M^2 \to \mathbb{R}^n$ is a smooth immersion of a surface with boundary ∂M and co_φ is its conormal, denoting by $\gamma := \varphi|_{\partial M}$ the parametrization of the boundary of the surface, we denote by

$$k_\varphi(x) := \langle k_\gamma, -co_\varphi \rangle,$$

the *scalar geodesic curvature* of φ, where $k_\gamma(x)$ is the curvature of γ. Observe that the scalar geodesic curvature is the quantity integrated over the boundary in the Gauss–Bonnet Theorem 1.1.10. Moreover, in such a setting, if $k_\varphi \in L^1(ds_\varphi)$, we define

$$G(\varphi) := \int_{\partial M} k_\varphi \, ds_\varphi.$$

4.1 A brief history of minimization problems on the Willmore energy

In this section we collect some very important results in the existence theory of minimization problems on the Willmore energy. We follow a chronological order, trying to explain the reasons why the variational study of this functional gained importance.

The work of Thomas Willmore [Wil65] in 1965 gave rise to a considerable interest in the variational study of the geometric functional $\mathcal{W}$ defined in (4.1), that today is named after him. We have to mention that a similar energy already appeared in [Tho24], where Thomsen introduced the functional $\int_M |B^\circ|^2 \, d\mu_g$, where $B^\circ = B - 2Hg$ is the tracefree second fundamental form. In [Tho24] the author studied properties of *conformal invariance* of integral functionals like $\mathcal{W}$, on which we will come back later; some authors actually assert that such properties where already known to Schadow in 1922. Observe that by Gauss–Bonnet Theorem 1.1.10, on compact surfaces without boundary the two integral quantities, namely $\int |B^\circ|^2$ and $\int |H|^2$, differ by an additive topological constant depending only on the Euler characteristic of M (see Remark 1.1.11). This means that these integral functionals are equivalent in the context of many *topologically constrained minimization problems*, i.e., problems in which the topology is fixed. In fact, in [Wil65] the author firstly considered topological constraints on the competitors of minimization problems on $\mathcal{W}$. It is probably this combination of topological, geometric, and variational properties that increased the interest towards the Willmore functional.

Let us begin with the first result in the variational study of $\mathcal{W}$, that is the identification of global minimizers of this energy among all possible closed immersed surfaces.

Theorem 4.1.1 (Global minimizers, [Wil65], [Che71a], [Che71b])**.** *If $\varphi : M^2 \to \mathbb{R}^n$ is a smooth immersion of a compact surface without boundary, then*

$$\mathcal{W}(\varphi) \geq 4\pi, \tag{4.2}$$

with equality if and only if φ is a round sphere, that is, M^2 is topologically a sphere and φ embeds M as a round 2-sphere of some radius contained in an affine 3-dimensional subspace of $\mathbb{R}^n$.

The proof of Theorem 4.1.1 in the case $n = 3$ goes back to Willmore and it is contained in [Wil65], while Chen proved the result in higher codimension in [Che71a] and [Che71b].

We remark that we already proved the inequality (4.2) at the level of varifolds in Corollary 1.3.8 as a consequence of the monotonicity formula. We will prove that the rigidity part of Theorem 4.1.1 in codimension 1 also holds in the setting of varifolds (see Proposition 4.3.16).

Let us also remark that inequality (4.2) and the rigidity part of Theorem 4.1.1 remain true for generalized Willmore functionals of higher dimensional manifolds. More precisely, if $\varphi : M^n \to \mathbb{R}^{n+m}$ is a smooth immersion of a closed manifold, then

$$\int_M |H|^n \, d\mu_g \geq |\mathbb{S}^n|,$$

where $|\mathbb{S}^n|$ is the volume of the standard Euclidean n-dimensional sphere. The proof of this fact is contained in [Che71a] and [Che71b] as well. We also mention that these kind of Willmore-type inequalities have been proved also in the case of embedded submanifolds in Riemannian manifolds with non-negative Ricci curvature by means of arguments based on potential theory. For a complete treatise of these results one can see the recent Ph.D thesis of Fogagnolo [Fog20] (see also [AFM20] and [AM20]).

From Theorem 4.1.1 it is clear that asking for uniqueness of minimizers of $\mathcal{W}$ in such unconstrained problems is meaningless, given that every round sphere has energy equal to 4π. This degeneracy is ultimately due to the following fundamental invariance property of the Willmore energy, that is, the fact that the functional is invariant under conformal transformation of the ambient.

Definition 4.1.2. A *conformal map* is a smooth diffeomorphism $F : (X, g_X) \to (Y, h_Y)$ between two Riemannian manifolds (X, g_X) and (Y, g_Y) such that $F^* g_Y = e^{2\lambda} g_X$ for some $\lambda \in C^\infty(X)$, that is, the pull back metric $F^* g_Y$ is a multiple of the metric g_X on X.

It follows form the definition that the differential of a conformal map preserves angles between tangent vectors. We will be mainly interested in conformal transformations of open sets in the Euclidean space. The following classical theorem, due to Liouville, completely classify conformal maps in Euclidean spaces $\mathbb{R}^n$ for $n \geq 3$. Before giving the statement, let us point out that translations in $\mathbb{R}^n$ are denoted by

$$x \mapsto T_q(x) := x + q,$$

for some $q \in \mathbb{R}^n$, a dialation in $\mathbb{R}^n$ is a map

$$x \mapsto D_\alpha(x) = \alpha x,$$

for some $\alpha > 0$, while the *spherical inversion*, or *spherical reflection*, at the unit sphere is the map $I_{1,0} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by

$$I_{1,0}(x) := \frac{x}{|x|^2}.$$

Composition of $I_{1,0}$ with a translation and a dialation gives arbitrary spherical inversions. More precisely, we define

$$I_{r,c} : \mathbb{R}^n \setminus \{c\} \to \mathbb{R}^n \setminus \{0\} \qquad\qquad I_{r,c}(x) := r^2 \frac{x - c}{|x - c|^2},$$

for any $c \in \mathbb{R}^n$ and $r > 0$.

Theorem 4.1.3 (Liouville, [Spi99, p. 209])**.** *Let $F : U \to V$ be a smooth diffeomorphism between two connected open subsets of the Euclidean space $\mathbb{R}^n$, and let $n \geq 3$. Then F is conformal if and only if F is a composition of isometries, dialations, and spherical inversions.*

We now state the result about the conformal invariance of $\mathcal{W}$ in the setting of smooth surfaces with boundary.

Theorem 4.1.4 (Conformal invariance in $\mathbb{R}^n$, [Che74], [Whi73], [Wei78])**.** *Let $\varphi : M^2 \to \mathbb{R}^n$ be a smooth immersion of a compact surface with boundary ∂M. Let $\Sigma = \varphi(M) \subset U$ with $U \subset \mathbb{R}^n$ open. Suppose that $F : U \to V \subset \mathbb{R}^n$ is a conformal diffeomorphism. Let $\psi = F \circ \varphi$, and denote by H_φ, H_ψ, μ_φ, μ_ψ, ds_φ, ds_ψ the mean curvature vectors and the induced 2-dimensional and 1-dimensional measures induced on M and ∂M by φ, ψ respectively. Then*

$$\int_M |H_\varphi|^2 \, d\mu_\varphi + \int_{\partial M} k_\varphi \, ds_\varphi = \int_M |H_\psi|^2 \, d\mu_\psi + \int_{\partial M} k_\psi \, ds_\psi,$$

where k_φ and k_ψ denote the scalar geodesic curvature of the immersions $\varphi|_{\partial M} : \partial M \to \mathbb{R}^n$ and $\psi|_{\partial M} : \partial M \to \mathbb{R}^n$.

From the previous theorem we see that uniqueness of minimizers of $\mathcal{W}$ among a class of closed surfaces can be only understood up to conformal transformation of the ambient.

Remark 4.1.5 (Conformal invariance on manifolds)**.** The conformal invariance of $\mathcal{W}$ is actually even more general, in the sense that it holds for immersions $\varphi : M^2 \to (\overline{M}^n, \overline{g})$ of surfaces in any Riemannian manifold. More precisely, given φ, it holds that the quantity

$$\int_M |H_\varphi|^2_{\overline{g}} + \overline{K}(\varphi_* T_x M) \, d\mu_\varphi + \int_{\partial M} k_\varphi \, ds_\varphi,$$

where $\overline{K}(\varphi_* T_x M)$ is the sectional curvature of $\overline{M}$ calculated on the plane $\varphi_* T_x M$, is invariant under conformal changes of the metric $\overline{g}$ of the ambient manifold. The proof of this fact is due to Weiner [Wei78]. Since the Euclidean space has constant sectional curvature equal to zero, this general result implies Theorem 4.1.4.

Coming back to variational problems regarding the Willmore energy, we see that Theorem 4.1.1 solves the minimization problem of $\mathcal{W}$ in Euclidean spaces among closed surfaces. The next step is then to study minimization problems of $\mathcal{W}$ among families of surfaces satisfying some constraints. This plan already started in [Wil65], when Willmore stated his celebrated conjecture. A nice discussion about the formulation of the conjecture is contained in Willmore's book [Wil93].

Conjecture 4.1.6 (Willmore conjecture, [Wil65])**.** *The infimum of the Willmore energy among smooth immersions of the 2-dimensional torus into $\mathbb{R}^3$ is $2\pi^2$. This infimum is only achieved by the immersion*

$$[0, 2\pi] \times [0, 2\pi] \ni (u, v) \mapsto \big((\sqrt{2} + \cos u) \cos v, (\sqrt{2} + \cos u) \sin v, \sin u \big) \in \mathbb{R}^3,$$

called Willmore torus*, up to conformal transformations of the ambient.*

The statement of the Willmore conjecture motivated the fact that the first constraints imposed on the minimization of the Willmore energy have been of topological type, that is, constraints on the genus of competitors. We then introduce the following class of problems.

Problem 4.1.7. Fix an integer $\mathfrak{g} \geq 1$ and let $M_\mathfrak{g}$ be the 2-dimensional closed surface of genus $\mathfrak{g}$. Fix also an integer $n \geq 3$. Prove existence of minimizers for the minimization problem

$$\min \left\{ \mathcal{W}(\varphi) \mid \varphi : M_\mathfrak{g} \to \mathbb{R}^n \text{ smooth immersion} \right\}.$$

We also define

$$\beta_\mathfrak{g} := \inf \left\{ \mathcal{W}(\varphi) \mid \varphi : M_\mathfrak{g} \to \mathbb{R}^n \text{ smooth immersion} \right\}. \tag{4.3}$$

Problem 4.1.7 has been completely solved putting together the remarkable results of Simon [Sim93], in which the author studies and solves the case of $\mathfrak{g} = 1$, and the important contribution by Bauer and Kuwert [BK03], who built on this result to prove existence of minimizers for any higher genus. Putting together these works, as well as the ones in some related papers, we can state the following answer to Problem 4.1.7.

Theorem 4.1.8 (Topologically constrained problems, [Law70],[Sim93],[BK03],[KLS10])**.** *For any integers $\mathfrak{g} \geq 1, n \geq 3$, the minimization Problem 4.1.7 admits a minimizer $F : M_\mathfrak{g} \to \mathbb{R}^n$. Any such minimizer is a smooth embedding. Moreover, it holds that*

$$4\pi \leq \beta_\mathfrak{g} < 8\pi, \qquad\qquad \exists \lim_{\mathfrak{g}\to\infty} \beta_\mathfrak{g} = 8\pi.$$

In Theorem 4.1.8, the proof that $\lim_\mathfrak{g} \beta_\mathfrak{g} = 8\pi$ is contained in [KLS10], while the estimate $4\pi \leq \beta_\mathfrak{g} < 8\pi$ follows from Theorem 4.1.1 and the results in [Law70]. More precisely, in [Law70] the author constructs embedded minimal surfaces in the sphere $\mathbb{S}^3$ of any given genus having area strictly less than 8π. By Remark 4.1.5 this implies the existence of closed surfaces of any given genus in $\mathbb{R}^3$ having Willmore energy strictly less than 8π.

Indeed the stereographic projection $\pi : (\mathbb{S}^3 \setminus \{p_N\}, g_{\mathbb{S}^3}) \to (\mathbb{R}^3, g_{\mathbb{R}^3})$ is conformal, where p_N is a chosen "north pole" on $\mathbb{S}^3$, and then, by Remark 4.1.5, if $\psi : M_\mathfrak{g} \to \mathbb{S}^3$ is an embedded minimal surface in $\mathbb{S}^3$, then

$$\mathcal{W}(\pi \circ \varphi) = \int_{M_\mathfrak{g}} d\mu_\psi = Area(\psi),$$

where we used that the sectional curvature in the sphere is constantly equal to 1. Therefore, testing the identity on Lawson's minimal surfaces of [Law70] gives the desired estimate.

However, Theorem 4.1.8 and the works related to its proof do not contain any information about the explicit values of the numbers $\beta_\mathfrak{g}$, neither about characterization or qualitative properties of the minimizers. In particular there has been no answer to the Willmore conjecture until the outstanding work of Fernando Codá Marques and André Neves [MN14], whose results imply the Willmore conjecture.

Theorem 4.1.9 (Marques–Neves, [MN14])**.** *Let $\psi : M^2_\mathfrak{g} \to \mathbb{S}^3$ be an embedding of a surface of genus $\mathfrak{g} \geq 1$ into the sphere $\mathbb{S}^3$. Then*

$$\int_{M_\mathfrak{g}} 1 + |H_\psi|^2_{g_{\mathbb{S}^3}} \, d\mu_\psi \geq 2\pi^2, \tag{4.4}$$

where $H_\psi \in T\mathbb{S}^3 \cap \psi_*(TM)^\perp$ is the mean curvature vector of the embedding ψ inside the sphere $\mathbb{S}^3$. Moreover, equality holds if and only if $\psi(M)$ is the Clifford torus $\mathbb{S}^1_{\frac{1}{\sqrt{2}}} \times \mathbb{S}^1_{\frac{1}{\sqrt{2}}}$, *up to conformal transformation of $\mathbb{S}^3$, where $\mathbb{S}^1_{\frac{1}{\sqrt{2}}}$ is the circle of radius $\frac{1}{\sqrt{2}}$ and center the origin in $\mathbb{R}^2$.*

Using again the fact that the stereographic projection π is conformal, if $\varphi : M_{\mathfrak{g}} \to \mathbb{R}^3$ is a smooth immersion, then the map $\psi = \pi^{-1} \circ \varphi : M_{\mathfrak{g}} \to \mathbb{S}^3$ satisfies that

$$\mathcal{W}(\varphi) = \int_{M_{\mathfrak{g}}} |H_\psi|^2_{g_{\mathbb{S}^3}} + \overline{K}(\psi_* T_x M_{\mathfrak{g}})\, d\mu_\psi = \int_{M_{\mathfrak{g}}} 1 + |H_\psi|^2_{g_{\mathbb{S}^3}}\, d\mu_\psi,$$

which is precisely the quantity appearing in (4.4). Since it turns out that the Willmore torus is the stereographic projection of the Clifford torus, Theorem 4.1.9 proves the Willmore conjecture.

Corollary 4.1.10 (Marques–Neves, [MN14])**.** *The Willmore conjecture 4.1.6 holds true.*

Let us now briefly discuss here another consequence of the conformal properties of the Willmore functional, which will be strongly related to the results we shall present.

Proposition 4.1.11. *Let $\varphi : M^2 \to \mathbb{R}^n$ be a smooth immersion of a closed surface. Assume $0 \in \Sigma = \varphi(M)$. Denote by $I : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ the spherical inversion $I(x) = \frac{x}{|x|}$ and let $\psi = I \circ \varphi|_{M \setminus \varphi^{-1}(0)}$. Then*

$$\mathcal{W}(\psi) = \mathcal{W}(\varphi) - 4\pi\, \sharp\varphi^{-1}(0),$$

where $\sharp(\cdot)$ denotes the cardinality of $(\cdot)$.

A short proof of this result can be found in [BK03], but the estimate was already present in the literature as a consequence of the mentioned results of Chen [Che71a; Che71b; Che74] and Weiner [Wei78]. As a corollary we get the already mentioned Li–Yau inequality [LY82], which is also true as suitably stated in the context of varifolds (Corollary 1.3.8).

Corollary 4.1.12 (Li–Yau inequality)**.** *Let $\varphi : M^2 \to \mathbb{R}^n$ be a smooth immersion of a closed surface. Then for any $p \in \varphi(M)$ it holds that*

$$\mathcal{W}(\varphi) \geq 4\pi\, \sharp\varphi^{-1}(p).$$

In particular, if φ is not an embedding, then $\mathcal{W}(\varphi) \geq 8\pi$.

In Section 4.2.2 and Section 4.A we will come back on Li–Yau-type inequalities like the one in Corollary 4.1.12. We will prove a completely analogous version of Corollary 4.1.12 for surfaces whose boundary is a circle (see Theorem 4.2.6) and we will also discuss the case of surfaces with an arbitrary planar boundary (see Theorem 4.A.4).

We observe that using Proposition 4.1.11 and Corollary 4.1.12, one can clearly prove that the infimum of the Willmore energy among closed surfaces is 4π and that the unique minimizers are round spheres, recovering Theorem 4.1.1. We remark the similarities of such a strategy in the study of different conformally invariant energies, like the energies of knots introduced by O'Hara [O'H91; O'H92; O'H03]. In fact, we find analogous arguments and results in [FHW94] in the variational study of such knot energies.

We will similarly exploit the conformal properties of $\mathcal{W}$ to solve some minimization problems regarding surfaces with boundary, except we will prove non-existence of minimizers instead (see Theorem 4.2.1 and Corollary 4.2.7).

Let us also mention that, in the last years, outstanding contributions to the variational theory of the Willmore energy have been given by Rivière. In [Riv07] and [Riv08] the author identified a strong relation between conformally invariant geometric energies and suitable conservation laws. This theory has deep consequences on the theory of *Willmore surfaces*, that is, critical points of the Willmore energy. From the point of view of direct methods in Calculus of Variations, strongly related results are [Riv13] and [Riv14]. In these works the author formulates and employs a *parametrization approach* in the study of existence and regularity of minimizers for minimization problems on the Willmore functional. For instance, the existence of minimizers stated in Theorem 4.1.8 is recovered as a corollary of the theory developed.

This approach has to be compared to the more classical one introduced by Simon in [Sim93]. We can refer today to this method as the *ambient approach*, as the direct methods employed to get existence and regularity of minimizers are achieved by means of the techniques of Geometric Measure Theory. Here the use of varifolds is fundamental.

Both the approaches had many subsequent applications, on which we will come back also later, especially about the measure theoretic methods of Simon.

From the point of view of the existence theory in the context of closed surfaces, we can say that many questions found satisfactory answers. This is not the case of existence theory in the setting of surfaces with boundary. We mention here one of the first fundamental results achieved by means of Simon's approach. The following theorem is a general existence results of critical points for the Willmore energy under the constraint of *clamped boundary conditions*. By clamped boundary conditions we mean that both the boundary and the conormal of the considered immersions are fixed.

Theorem 4.1.13 (Willmore surfaces with clamped boundary conditions, [Sch10])**.** *Fix $n \geq 3$. Let $\Gamma \subset \mathbb{R}^n$ be a smooth embedded closed 1-dimensional manifold. Let $N : \Gamma \to \mathbb{R}^n$ be a smooth unit vector field along Γ which is normal along Γ. Then there exists a compact oriented surface with boundary $\Sigma_0 \subset \mathbb{R}^n$ and a map $\varphi : \Sigma_0 \to \mathbb{R}^n$ such that:*

1. *$\partial\Sigma_0 = \Gamma$,*
2. *$\varphi|_{\partial\Sigma_0} : \partial\Sigma_0 \to \Gamma$ is a continuous embedding of Γ,*
3. *$co_\varphi = N$,*
4. *φ is a C^∞-Willmore surface on $\Sigma_0 \setminus \{p_1, ..., p_k\}$, where $p_1, ..., p_k$ are finitely many points, called branching points, and φ is continuous at those points.*

Let us stress that, despite Theorem 4.1.13 is a statement of existence of Willmore surfaces, the result is obtained via an adaptation of Simon's ambient approach, that is, by means of a direct method that studies varifold limits of minimizing sequences of suitable minimization problems.

We conclude this part by stating a problem, which is the direct analogue with boundary of Problem 4.1.7. We believe that a full understanding of such problem would be a great achievement in the context of the existence theory we described.

Problem 4.1.14. Fix $\Gamma_1, ..., \Gamma_k \subset \mathbb{R}^n$ finitely many smooth embedded curves, and $n \geq 3$. Fix an integer $\mathfrak{g} \geq 0$ and let $\Sigma_\mathfrak{g}$ be the 2-dimensional closed surface of genus $\mathfrak{g}$ with k disks removed. Characterize existence of minimizers and infimum for the minimization problem

$$\min\left\{\mathcal{W}(\varphi) \mid \varphi : \Sigma_\mathfrak{g} \to \mathbb{R}^n \text{ smooth immersion, } \varphi|_{\partial\Sigma_\mathfrak{g}} : \partial\Sigma_\mathfrak{g} \to \sqcup_i\Gamma_i \text{ smooth embedding}\right\}.$$

We will refer to Problem 4.1.14 as the general *Plateau-Douglas problem for the Willmore energy* (see [Poz20c]). Indeed, in the case of the area functional, the analogous problem goes under this name in the literature (see [DHT10]). In the following, also some results in the direction of this problem are presented. However a satisfactory answer to Problem 4.1.14 is still quite far today.

4.2 On the Plateau–Douglas problem for the Willmore energy

In this section we collect some results from [Poz20c] in the direction of Problem 4.1.14 in the 3-dimensional Euclidean space $\mathbb{R}^3$. We will focus on a non-existence result for the minimization problem of the Willmore energy among surfaces having a prescribed circle as boundary (see Theorem 4.2.1 and Corollary 4.2.7). We also prove a result in the spirit of the Li–Yau-type inequality in the context of surfaces with planar boundary (Theorem 4.2.6).

Let us first mention here some related results in the literature. Minimization problems of the Willmore energy for surfaces with boundary and the study of corresponding critical points are already present in the literature under two main formulations, depending on the chosen boundary conditions.

As we already said, proceeding by analogy with the Plateau–Douglas problem for the area functional, we are interested in Problem 4.1.14 in studying the minimization of the Willmore energy under the sole constraint of having a fixed boundary. We believe this is also the most reasonable assumption in case the existence of minimizers is the main object of study. The direct calculation of the first variation of the Willmore energy shows that surfaces which are critical points under the sole constraint that fixes the boundary get an additional boundary condition, that is just what happens in the case of Neumann boundary conditions. Such a condition is the so-called *Navier (natural) condition*, and it consists in the prescription $H = 0$ on the boundary. Critical points satisfying Navier conditions have been studied mainly under the assumption that the surfaces have rotational symmetry. Under this symmetry assumptions, recent results are contained in [BDF10; BDF13; DDG08; DG09; DDW13; Dal+11; Eic16; EG19].

The second formulation under which critical points of the Willmore energy with fixed boundary have been studied is in presence of *clamped boundary conditions*. In this case also the conormal at the boundary is prescribed. Two of the most important contribution in these area are the already mentioned [Sch10] and [Eic19], where the authors construct branched immersions which are critical points out of finitely many branching points by means of a refinement of Simon's ambient approach developed in [Sim93].

We remark that also Rivière's parametrization approach has been employed in the study of minimization problems of surfaces with boundary for example in [DLPR20] and [MS20a].

Finally, let us mention that a result about symmetry breaking of Willmore surfaces with Navier boundary conditions is contained in [Man18]. We also mention that an interesting problem about Willmore surfaces in a free boundary setting is considered in [AK16]. Other remarkable related results are the rigidity theorems for Willmore surfaces proved in [Pal00] and [Dal12]. A study of the Willmore energy under both Navier and clamped conditions on surfaces that are assumed to be graphs is contained in [DGR17].

In this chapter we define an *asymptotically flat surface of genus $\mathfrak{g}$ without boundary with K ends* to be a complete orientable immersed 2-dimensional manifold $\varphi : M \to \mathbb{R}^3$ such that the following properties hold.

1. $M \simeq M_{\mathfrak{g}} \setminus \sqcup_{i=1}^{K} \overline{D_i}$, i.e. M is diffeomorphic to a genus $\mathfrak{g}$ surface with finitely many disjoint closed topological disks removed;

2. For any $i = 1, ..., K$ there is U_i open boundary chart at D_i such that U_i is diffeomorphic to an annulus with $\partial U_i = \partial D_i \sqcup \gamma_i$ for a curve $\gamma_i \simeq \mathbb{S}^1$, and there is an affine plane Π_i such that for any $\varepsilon > 0$ there is $R > 0$ such that $\varphi(U_i) \setminus B_R(0)$ is the graph over $\Pi_i \setminus K_R$ of a function f_R with $\|f_R\|_{C^1} \leq \varepsilon$ where $K_R \subset \Pi_i$ is compact.

3. $D(\varphi) < +\infty$.

If φ defines an asymptotically flat surface of genus $\mathfrak{g}$ without boundary with K ends as above, we call *end* of the surface one of the sets $\varphi(U_i)$.

In the following we will also consider *asymptotically flat surfaces* Σ *of genus* $\mathfrak{g}$ *with* K *ends with boundary* Γ, meaning that Γ is a smooth complete embedding of $\mathbb{R}$ and $\Sigma \subset \mathbb{R}^3$ is a subset such that the following properties hold.

1. $\Sigma = \varphi(\psi(M) \setminus L)$ where ψ is an embedding defining an asymptotically flat surface of genus $\mathfrak{g}$ without boundary with K end, $\varphi : \psi(M) \to \mathbb{R}^3$ is a complete immersion, L is diffeomorphic to an open half-plane and it is contained in one end, say E_1, of $\psi(M)$. Moreover $\varphi|_{\partial L} : \partial L \to \Gamma$ is an embedding.

2. For any $i = 2, ..., K$ for any end E_i of ψ there is an affine plane Π_i such that for any $\varepsilon > 0$ there is $R > 0$ such that $\varphi(E_i) \setminus B_R(0)$ is the graph over $\Pi_i \setminus K_R$ of a function f_R with $\|f_R\|_{C^1} \le \varepsilon$ where $K_R \subset \Pi_i$ is compact.

3. There is an affine plane Π_1 such that for any $\varepsilon > 0$ there is $R > 0$ such that $\varphi(E_1 \setminus (B_R(0) \cup L))$ is the graph over $\Pi_1 \setminus H$ of a function f_R with $\|f_R\|_{C^1} \le \varepsilon$ where $H \subset \Pi_1$ is smooth and diffeomorphic to a halfplane.

4. $D(\varphi) < +\infty$.

Recalling Theorem 4.1.8 and the definition of $\beta_{\mathfrak{g}}$ in (4.3), we also define

$$e_{\mathfrak{g}} := \beta_{\mathfrak{g}} - 4\pi.$$

The number $e_{\mathfrak{g}}$ plays an important role in the study of the Willmore energy of asymptotically flat surfaces. Indeed it holds that

$$e_{\mathfrak{g}} = \inf \left\{ \mathcal{W}(\varphi) \;\middle|\; \varphi \quad \begin{array}{l} \text{asymptotically flat surface of genus } \mathfrak{g} \\ \text{without boundary with one end} \end{array} \right\}.$$

Indeed one can verify that if φ is an immersion of $M_{\mathfrak{g}}$ and $0 \in \varphi(M_{\mathfrak{g}})$, then $I \circ \varphi|_{M_{\mathfrak{g}} \setminus \varphi^{-1}(0)}$ defines an asymptotically flat surface of genus $\mathfrak{g}$ without boundary, where $I(x) = \frac{x}{|x|^2}$. Conversely, if φ defines an asymptotically flat surface of genus $\mathfrak{g}$ without boundary with one end, then $I \circ \varphi$ extends to a $C^{1,1}$ immersion of $M_{\mathfrak{g}}$; in particular $I \circ \varphi$ extends to an element of the class $\mathcal{E}_{M_{\mathfrak{g}}}$ defined in [Riv14, p. 46]. Therefore by Proposition 4.1.11 (see [BK03, Theorem 2.2]) together with [Riv14, Theorem 1.7], we have that the infimum of the Willmore energy among asymptotically flat surfaces of genus $\mathfrak{g}$ without boundary with one end is equal to $e_{\mathfrak{g}}$, and such infimum is achieved only by immersions of the form $I \circ \varphi|_{M_{\mathfrak{g}} \setminus \varphi^{-1}(0)}$ for $\varphi : M_{\mathfrak{g}} \to \mathbb{R}^3$ embedding such that $0 \in \varphi(M_{\mathfrak{g}})$ and $\mathcal{W}(\varphi) = \beta_{\mathfrak{g}}$.

It follows from the above characterization of $e_{\mathfrak{g}}$ if $\Gamma \subset \mathbb{R}^2$ is an embedded closed planar curve and $\mathfrak{g} \in \mathbb{N}$, then

$$\inf \left\{ \mathcal{W}(\varphi) \;\middle|\; \begin{array}{ll} \varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3 & \text{smooth immersion of a surface of genus } \mathfrak{g} \\ \varphi|_{\partial \Sigma_{\mathfrak{g}}} : \partial \Sigma_{\mathfrak{g}} \to \Gamma \text{ smooth embedding} & \end{array} \right\} \le e_{\mathfrak{g}}. \tag{4.5}$$

Indeed we can consider an embedded asymptotically flat surface Σ without boundary with one end and genus $\mathfrak{g}$ such that $\mathcal{W}(\Sigma) = e_{\mathfrak{g}}$, that is, Σ is minimizing among asymptotically flat surfaces of its own genus. Without loss of generality we can assume that the set $\{x^2 + y^2 \le 1, z = 0\}$ is strictly contained in the open planar region Ω enclosed by Γ. Chosen $\varepsilon > 0$, up to a translation, a rotation, and a rescaling of Σ by a small factor one can construct a competitor Σ' for the infimum in (4.5) such that $\Sigma' \cap \{x^2+y^2 \le \frac{1}{2}\} = \Sigma \cap \{x^2+y^2 \le \frac{1}{2}\}$, $\Sigma' \cap \{\frac{1}{2} \le x^2+y^2 \le 1\}$ is the graph of a smooth function over the annulus $\{\frac{1}{2} \le x^2+y^2 \le 1\}$ with $\mathcal{W}(\Sigma' \cap \{\frac{1}{2} \le x^2+y^2 \le 1\}) \le \varepsilon$, and $\Sigma' \cap \{x^2+y^2 \ge 1\} = \Omega \cap \{x^2+y^2 \ge 1\}$. Therefore $\mathcal{W}(\Sigma') \le e_{\mathfrak{g}} + \varepsilon$. For the explicit construction of Σ' we refer to the proof of [Poz20c, Theorem 1.4], in which a similar construction is used several times.

4.2.1 Circle boundary datum

We consider here Problem 4.1.14 for surfaces in $\mathbb{R}^3$ having one boundary curve, which is assumed to be a unit circle. In the following we shall denote by $\mathbb{S}^1$ the unit circle understanding it is a subset of $\mathbb{R}^3$ as

$$\mathbb{S}^1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1, z = 0\}.$$

and we denote by D be the bounded planar disk enclosed by $\mathbb{S}^1$. In this setting, the minimization problem becomes

$$\min\{\mathcal{W}(\varphi) \mid \varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3 \text{ smooth immersion, } \varphi|_{\partial\Sigma_{\mathfrak{g}}} \to \mathbb{S}^1 \text{ smooth embedding}\}, \tag{4.6}$$

and $\Sigma_{\mathfrak{g}}$ is diffeomorphic to the closed orientable surface of genus $\mathfrak{g}$ with one smooth disk removed. Also, for a fixed $\mathfrak{g}$, we denote by

$$\mathcal{F} \coloneqq \{\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3 \text{ smooth immersion, } \varphi|_{\partial\Sigma_{\mathfrak{g}}} \to \mathbb{S}^1 \text{ smooth embedding}\},$$

the family of competitors.

If $\varphi \in \mathcal{F}$, since the curvature vector of the boundary curve is just $k_{\mathbb{S}^1}(p) = -p$ for any $p \in \mathbb{S}^1$, denoting by co_φ the unit outward conormal of φ, we see that the scalar geodesic curvature of the boundary is given by $k_\varphi(x) = \langle k_{\mathbb{S}^1}(\varphi(x)), -co_\varphi(x)\rangle = \langle \varphi(x), co_\varphi(x)\rangle$ for any $x \in \partial\Sigma_{\mathfrak{g}}$. Identifying $\partial\Sigma_{\mathfrak{g}} \equiv \mathbb{S}^1$, we can assume that φ is the identity on $\mathbb{S}^1$, and we write

$$\forall \varphi \in \mathcal{F}: \qquad k_\varphi(p) = \langle p, co_\varphi(p)\rangle \quad \forall\, p \in \mathbb{S}^1, \qquad G(\varphi) = \int_{\mathbb{S}^1} \langle p, co_\varphi(p)\rangle \, d\mathcal{H}^1(p).$$

Observe that if $\varphi \in \mathcal{F}$, then $G(\varphi) \le 2\pi$ with equality if and only if $co_\varphi(p) = p$ for any $p \in \mathbb{S}^1$.

The main result of this part is the following non-existence theorem.

Theorem 4.2.1 (Non-existence of minimizers)**.** *For any genus $\mathfrak{g} \ge 1$, problem (4.6) has no minimizers and the infimum equals $e_{\mathfrak{g}}$.*

The proof of Theorem 4.2.1 is based on the following tool.

Lemma 4.2.2. *Let $\varphi \in \mathcal{F}$ and denote $\Sigma = \varphi(\Sigma_{\mathfrak{g}})$. Then for any $\varepsilon > 0$ there is $F : U \to \mathbb{R}^3$ such that*

1. *$U \subset \mathbb{R}^3$ is open and $\Sigma \subset U$;*
2. *$F : U \to F(U)$ is a conformal diffeomorphism;*
3. *$\|co_{F\circ\varphi}(p) - p\|_{L^2(\mathbb{S}^1)} < \varepsilon$;*
4. *$2\pi - G(F \circ \varphi) < \varepsilon$.*

Proof. If $G(\varphi) = 2\pi$, then $co_\varphi(p) = p$ and the identity $F = \mathrm{id}|_{\mathbb{R}^3}$ works. So we can assume that $G(\varphi) < 2\pi$. Denote by T_q and D_α the translation and the dialation with respect to a vector $q \in \mathbb{R}^3$ or a factor $\alpha > 0$ respectively. Consider the point $(-1,0,0) =: v \in \mathbb{S}^1$ and the spherical inversion $I_{1,v}$. Note that $I_{1,v}$ maps $\mathbb{S}^1 \setminus \{v\}$ onto the line $r_{-v/2}$ passing through the point $-v/2$, lying in the plane of $\mathbb{S}^1$ and parallel to $T_v\mathbb{S}^1$. Let $\mathbf{R}$ be a rotation in $\mathbb{R}^3$ about the axis $\{x = z = 0\}$, we claim that the desired map F is

$$F(p) = \begin{cases} I_{1,v}^{-1} \circ T_{-v/2} \circ \mathbf{R} \circ D_\alpha \circ T_{v/2} \circ I_{1,v}(p) = v + \dfrac{\alpha \mathbf{R}\left[\frac{p-v}{|p-v|^2} + \frac{v}{2}\right] - \frac{v}{2}}{\left|\alpha \mathbf{R}\left[\frac{p-v}{|p-v|^2} + \frac{v}{2}\right] - \frac{v}{2}\right|^2} & p \in U \setminus \{v\}, \\ v & p = v. \end{cases}$$

for suitable choice of $\alpha \in (0,1)$ and rotation $\mathbf{R}$, and F is defined on

$$U = \mathbb{R}^3 \setminus \left\{ I_{1,v}^{-1}\left(-\frac{1}{2}\left(\frac{1}{\alpha}\mathbf{R}^{-1}[v] + v\right)\right)\right\}.$$

The surface $I_{1,v}(\Sigma)$ is an asymptotically flat manifold with K ends, where $K \geq 1$ is the multiplicity of v in Σ. For $\beta, \gamma, \delta \in (0,1)$ arbitrarily small there exist $\alpha = \alpha(\beta,\gamma) \in (0,1)$ sufficiently small and suitable $\mathbf{R} = \mathbf{R}(\delta)$ so that

$$d_{C^1}\left(\mathbf{R} \circ D_\alpha \circ T_{v/2} \circ I_{1,v}(\Sigma) \setminus B_\gamma(0), \bigcup_{i=1}^{K} \Pi_i \setminus B_\gamma(0)\right) < \beta, \tag{4.7}$$

for a half plane Π_1 and planes $\Pi_2, ..., \Pi_K$ passing through the origin with $\partial\Pi_1 = \{x = y = 0\}$, and

$$\begin{aligned} &\langle co_{\Pi_1}, (-1,0,0)\rangle > 1 - \delta, \\ &0 \notin T_{-v/2} \circ \mathbf{R} \circ D_\alpha \circ T_{v/2} \circ I_{1,v}(\Sigma), \end{aligned} \tag{4.8}$$

where co_{Π_1} is the conormal vector of Π_1. Equation (4.7) is a shortcut for saying that the functions given by the definition of asymptotically flat surface whose graphs parametrize $\mathbf{R} \circ D_\alpha \circ T_{v/2} \circ I_{1,v}(\Sigma) \setminus B_\gamma(0)$ have C^1-norm smaller than β.

Note that condition $0 \notin T_{-v/2} \circ \mathbf{R} \circ D_\alpha \circ T_{v/2} \circ I_{1,v}(\Sigma)$ in (4.8) is equivalent to $I_{1,v}(p) \neq -\frac{1}{2}\big(\frac{1}{\alpha}\mathbf{R}[v] + v\big)$ for any $p \in \Sigma$; and, since $\frac{1}{\alpha}\mathbf{R}[v] + v \neq 0$ for any $\alpha < 1$, such condition is equivalent to $I_{1,v}^{-1}\big(- \frac{1}{2}\big(\frac{1}{\alpha}\mathbf{R}[v] + v\big)\big) \notin \Sigma$, which justifies the definition of U. So by (4.8) the function F is well defined on U and $\Sigma \subset U$.

Now for any $p \in U \setminus \{v\}$ we have

$$F(p) = v + \frac{\alpha\mathbf{R}[p-v] + |p-v|^2 w}{\left|\alpha\mathbf{R}\left[\frac{p-v}{|p-v|}\right] + |p-v| w\right|^2}, \qquad w := \frac{\alpha}{2}\mathbf{R}[v] - \frac{v}{2},$$

which is readily checked to be of class $C^1(U)$ and conformal; in fact, at $p = v$, the differential fo F is $dF_v = \frac{1}{\alpha}\mathbf{R}$. By the regularity result in [LS14, Theorem 3.1] we conclude that F is actually smooth on U.

The inverse map $I_{1,v}^{-1}$ has differential

$$d(I_{1,v}^{-1})_q = \frac{1}{|q|^2}\left(\mathrm{id} - \frac{2}{|q|^2} q \otimes q\right).$$

Hence taking $q = (1/2, t, 0) \in r_{-v/2}$, $e_3 = (0,0,1)$ and $X \in (T_q(r_{-v/2}))^\perp$ we have $d(I_{1,v}^{-1})_q(e_3) = \frac{1}{|q|^2} e_3$ and thus

$$\begin{aligned} \left(\frac{d(I_{1,v}^{-1})_q(X)}{|d(I_{1,v}^{-1})_q(X)|} \right)_3 &:= \frac{\langle d(I_{1,v}^{-1})_q(X), e_3 \rangle}{|d(I_{1,v}^{-1})_q(X)|} = \frac{\langle d(I_{1,v}^{-1})_q(X), d(I_{1,v}^{-1})_q(e_3) \rangle}{|d(I_{1,v}^{-1})_q(X)||d(I_{1,v}^{-1})_q(e_3)|} = \frac{\langle X, e_3 \rangle}{|X|} \\ &=: \left(\frac{X}{|X|} \right)_3, \end{aligned} \tag{4.9}$$

that is, the third component of a vector is preserved. Hence putting together (4.7), (4.8), (4.9), and choosing α, β, γ sufficiently small, the thesis follows. ☐

Corollary 4.2.3. *If $\varphi \in \mathcal{F}$ is such that $G(\varphi) < 2\pi$, then there is $\varphi' \in \mathcal{F}$ such that $\mathcal{W}(\varphi') < \mathcal{W}(\varphi)$. In particular a minimizer φ of problem (4.6) must satisfy $co_\varphi(p) = p$ on $\mathbb{S}^1$.*

Proof. Apply Lemma 4.2.2 recalling that the quantity $\mathcal{W} + G$ is conformally invariant by Theorem 4.1.4, so that if G increases then $\mathcal{W}$ must decrease. ☐

We are now ready for proving Theorem 4.2.1.

Proof of Theorem 4.2.1. Assume by contradiction that a minimizer φ exists, and let $\Sigma = \varphi(\Sigma_\mathfrak{g})$. Then by Corollary 4.2.3 the conormal $co_\varphi(p)$ of Σ is identically equal to the position vector p on $\mathbb{S}^1$. Let $\Sigma^{ext} := \Sigma \cup \{z = 0, x^2 + y^2 \geq 1\}$, which defines a $C^{1,1}$-surface. Now two possibilities can occur.

Suppose first there exists $\bar{p} \in int(D)$ such that $\bar{p} \notin \Sigma$, where D is the closed disk enclosed by $\mathbb{S}^1$. Then $I_{1,\bar{p}}(\Sigma^{ext}) \cup \{0\} =: \Sigma'$ is a well defined surface of class $C^{1,1}$ without boundary with genus $\mathfrak{g}$. Also $\Sigma' \supset D$, then Σ' cannot be a minimizer for the Willmore energy among closed surfaces of genus $\mathfrak{g}$, otherwise Σ' would be analytic ([Riv08, Theorem I.3]) and then equal to the plane containing D. Hence $\mathcal{W}(\Sigma') > \beta_\mathfrak{g}$, and then $\mathcal{W}(\Sigma) = \mathcal{W}(\Sigma^{ext}) > \beta_\mathfrak{g} - 4\pi = e_\mathfrak{g}$ (by Proposition 4.1.11). Since by (4.5) the infimum of our problem is $\leq e_\mathfrak{g}$, this implies that Σ is not a minimizer.

Suppose now the other case: $D \subset \Sigma$. In this case the whole plane containing D is contained in Σ^{ext}, which has genus $\mathfrak{g} \geq 1$, then there exists a point $q \in \Sigma^{ext}$ with multiplicity ≥ 2. Now let $x \in \mathbb{R}^3 \setminus \Sigma^{ext}$, then $\Sigma' := I_{1,x}(\Sigma^{ext}) \cup \{0\}$ is a $C^{1,1}$ closed surface of genus $\mathfrak{g}$ with a point of multiplicity ≥ 2, then $\mathcal{W}(\Sigma') \geq 8\pi$ and $\mathcal{W}(\Sigma) = \mathcal{W}(\Sigma^{ext}) \geq 4\pi$ (by Proposition 4.1.11). Since by (4.5) the infimum of our problem is $\leq \beta_\mathfrak{g} - 4\pi < 4\pi$, this implies that Σ could not be a minimizer.

Finally, for any $\varepsilon > 0$ we know from Lemma 4.2.2 and the proof of Corollary 4.2.3 that

$$\inf_{\varphi \in \mathcal{F}} \mathcal{W} = \inf_{\substack{\varphi \in \mathcal{F} \\ G(\varphi) \geq 2\pi - \varepsilon}} \mathcal{W},$$

then, since $\inf_\mathcal{F} \mathcal{W} \leq \beta_\mathfrak{g} - 4\pi$, by the above argument we conclude that

$$\inf_{\varphi \in \mathcal{F}} \mathcal{W} = \inf_{\substack{\varphi \in \mathcal{F} \\ G(\varphi) = 2\pi}} \mathcal{W} = \beta_\mathfrak{g} - 4\pi = e_\mathfrak{g}.$$

☐

4.2.2 A Li–Yau-type inequality for surfaces with circular boundary

Throughout this section we denote by $\Sigma_{\mathfrak{g}}$ a surface with boundary which is diffeomorphic to the orientable surface $M_{\mathfrak{g}}$ of genus $\mathfrak{g}$ having a removed disk. In particular $\partial\Sigma_{\mathfrak{g}}$ is diffeomorphic to $\mathbb{S}^1$. Moreover, we still assume that $\mathbb{S}^1 = \{x^2+y^2=1, z=0\}$ is a subset of $\mathbb{R}^3$.

If $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ is an immersion and we denote by $\Sigma = \varphi(\Sigma_{\mathfrak{g}})$ its image, we will use the same symbol Σ to denote the image varifold $\operatorname{Im}\varphi$. Moreover, we will use the shortcuts

$$|\Sigma \cap U| := \mu_\Sigma(U), \qquad \int_{\Sigma\cap U} f := \int_U f\, d\mu_\Sigma, \qquad \int_\Sigma g := \int_{\Sigma_{\mathfrak{g}}} g\, d\mu_\varphi$$

for any open set $U \subset \mathbb{R}^3$ and continuous functions $f : U \to \mathbb{R}$ and $g : \Sigma_{\mathfrak{g}} \to \mathbb{R}$ whenever there is no risk of confusion.

In this section we are interested in proving a result related to the Li–Yau inequality (see Corollary 1.3.8 and Corollary 4.1.12). More precisely, given an immersion $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$, we want to prove that the presence of self-intersections implies a lower bound estimate on the Willmore energy $\mathcal{W}(\varphi)$.

We recall here a few concepts we will need. Let $\gamma : (0,1) \to \{z=0\} \subset \mathbb{R}^3$ be a planar regular curve of class C^2. The *normal vector* ν_γ along γ is the counterclockwise rotation of an angle $\frac{\pi}{2}$ of the tangent vector τ_γ, where we understand that $\{z=0\}$ is oriented with the oriented basis $[(1,0,0),(0,1,0)]$. In this way we can write the curvature vector of γ as $k_\gamma = \widetilde{k}_\gamma \nu_\gamma$, and the scalar function $\widetilde{k}_\gamma$ is the *oriented curvature* of γ. More generally, if ν is a unit normal vector along γ, the scalar function $\langle k_\gamma, \nu\rangle$ is the *oriented curvature of* γ *with respect to* ν.

We remark that if $\gamma : (0,1) \to \{z=0\}$, $\widetilde{k}_\gamma$ is the oriented curvature with respect to the normal vector ν_γ, and $0 < s < t < 1$, then

$$\int_s^t \widetilde{k}_\gamma\, ds = \theta(t) - \theta(s), \tag{4.10}$$

where $\theta : (0,1) \to \mathbb{R}$ is any C^1 function such that $\tau_\gamma(x) = (\cos\theta(x), \sin\theta(x))$. Indeed, assuming without loss of generality that γ is parametrized by arclength, writing the tangent vector of γ as $\tau_\gamma(x) = (\cos\theta(x), \sin\theta(x))$ we have

$$\widetilde{k}_\gamma = \langle k_\gamma, \nu_\gamma\rangle = \langle \theta'(-\sin\theta, \cos\theta), (-\sin\theta, \cos\theta)\rangle = \theta',$$

and then $\int_s^t \theta'(x)\, dx = \theta(t) - \theta(s)$.

It follows that if $\gamma : \mathbb{S}^1 \to \{z=0\}$ is an embedded closed planar curve of class C^2 that positively orients the boundary of the planar region it encloses, and $\widetilde{k}_\gamma$ is the oriented curvature, then

$$\int_{\mathbb{S}^1} \widetilde{k}_\gamma\, ds = 2\pi. \tag{4.11}$$

First we need to prove a sort of Gauss–Bonnet Theorem for surfaces with planar boundary.

Lemma 4.2.4. *The following identities hold true.*

1. *Let $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ be a smooth immersion such that $\varphi : \partial\Sigma_{\mathfrak{g}} \to \Gamma$ is the embedding of a closed planar curve $\Gamma \subset \{z=0\}$. Assume that there is $p_0 \in \partial\Sigma_{\mathfrak{g}}$ such that $\varphi^{-1}(\varphi(p_0)) = \{p_0\}$. Let*

$$\psi := I_{1,\varphi(p_0)} \circ \varphi|_{\Sigma_{\mathfrak{g}}\setminus\{p_0\}}.$$

Then

$$\mathcal{W}(\varphi) + G(\varphi) = \mathcal{W}(\psi) + G(\psi) + 2\pi.$$

2. *Let $p_0 \in \partial\Sigma_{\mathfrak{g}}$ and $\psi : \Sigma_{\mathfrak{g}} \setminus \{p_0\} \to \mathbb{R}^3$ be a smooth immersion such that $\psi : \partial\Sigma_{\mathfrak{g}} \to \Gamma$ is the embedding of a planar curve $\widetilde{\Gamma} \subset \{z = 0\}$ such that $\Gamma := I_{1,0}(\widetilde{\Gamma}) \cup \{0\}$ is a closed smooth curve and*

$$\varphi(p) := \begin{cases} I_{1,0} \circ \psi & \text{if } p \in \Sigma_{\mathfrak{g}} \setminus \{p_0\}, \\ 0 & \text{if } p = p_0. \end{cases}$$

is a proper smooth immersion of $\Sigma_{\mathfrak{g}}$. Then

$$\mathcal{W}(\varphi) + G(\varphi) = \mathcal{W}(\psi) + G(\psi) + 2\pi.$$

Proof. The two statements are equivalent, so here we prove the first one. Without loss of generality we can assume that $\varphi(p_0) = 0$ and we denote $I \equiv I_{1,0}$ the spherical inversion at the standard unit sphere. Moreover denote $\Sigma = \varphi(\Sigma_{\mathfrak{g}})$. Since the origin has a unique preimage in $\Sigma_{\mathfrak{g}}$, there is $r > 0$ so small that $D_r := \Sigma \cap B_r$ is homeomorphic to an embedded disk. Let

$$\varphi_r := \varphi|_{\Sigma_{\mathfrak{g}} \setminus \varphi^{-1}(D_r)}, \qquad \Sigma_r := \Sigma_{\mathfrak{g}} \setminus \varphi^{-1}(D_r)$$

that is the immersion of a smooth surface with piecewise smooth boundary. It is proved in [Che74] that the function $(|H|^2 - K)\sqrt{\det g}$, where g is the metric and H, K are the mean and Gaussian curvature on a given immersed surface, is pointwise conformally invariant. Hence, letting $\psi_r := I \circ \varphi_r$, we have

$$\int_{\Sigma_r} |H_{\varphi_r}|^2 - K_{\varphi_r} = \int_{\Sigma_r} |H_{\psi_r}|^2 - K_{\psi_r}, \tag{4.12}$$

for any $r > 0$ sufficiently small. Moreover, letting $r \to 0$, we have $\int_{\Sigma_r} |H_{\varphi_r}|^2 \to \mathcal{W}(\varphi)$ and $\int_{\Sigma_r} |H_{\psi_r}|^2 \to \mathcal{W}(\psi)$. So we want to study the limit of the two integrals of the Gaussian curvature as $r \to 0$.

Letting $\{z = 0\}$ be oriented by the oriented basis $[(1,0,0),(0,1,0)]$, we can assume that Γ is positively parametrized by $\varphi|_{\partial\Sigma_{\mathfrak{g}}}$ with respect to the planar region it encloses. Moreover, let $\Sigma_{\mathfrak{g}}$ be oriented accordingly to its boundary. Assume the same positive orientation for Σ_r and φ_r. We know from Gauss–Bonnet Theorem 1.1.10 that the sum

$$F(f) := \int_S K_S + G(f) + \alpha(f),$$

is a topological invariant, where $f : S \to \mathbb{R}^3$ is some immersed surface and $\alpha(f)$ denotes the sum of oriented angles in Theorem 1.1.10. In particular $F(\varphi_r) = F(\psi_r)$ for any r small.

The curve $\partial B_r(0) \cap \Sigma$ is closer and closer in C^2-norm to a half circle of radius r contained in $d\varphi_{p_0}(T_{p_0}\Sigma_{\mathfrak{g}})$ as $r \to 0$. Hence, letting $\gamma_r := \varphi|_{\varphi_r^{-1}(\partial B_r(0)\cap\Sigma)}$ the parametrization of $\varphi_r^{-1}(\partial B_r(0) \cap \Sigma)$, we have

$$\lim_{r\to 0} \int_{\varphi^{-1}(\partial B_r(0)\cap\Sigma)} \langle k_{\gamma_r}, -co_{\varphi_r} \rangle = -\pi,$$

and then $\lim_{r\to 0} G(\varphi_r) = -\pi + G(\varphi)$. Moreover, the choice of the orientation implies that $\lim_{r\to 0} \alpha(\varphi_r) = \pi$. Therefore $\lim_{r\to 0} G(\varphi_r) + \alpha(\varphi_r) = G(\varphi)$.

Since the inversion I maps points closer to the origin to points farther from it, we have that if $I \circ \gamma_r$ parametrizes the curve $\psi_r^{-1}(I(\partial B_r(0) \cap \Sigma))$, then

$$\lim_{r\to 0} \int_{\varphi^{-1}(\partial B_r(0)\cap\Sigma)} \langle k_{I\circ\gamma_r}, -co_{\psi_r} \rangle = \pi.$$

Moreover, since I is conformal, the sum of the oriented angles is preserved and $\lim_{r\to 0}\alpha(\psi_r)=\pi$. Therefore $\lim_{r\to 0} G(\psi_r)+\alpha(\psi_r)=G(\psi)+2\pi$.

Hence using that $F(\varphi_r)=F(\psi_r)$, passing to the limit $r\to 0$ in (4.12), that is equivalent to

$$\int_{\Sigma_r}|H_{\varphi_r}|^2-F(\varphi_r)+G(\varphi_r)+\alpha(\varphi_r)=\int_{\Sigma_r}|H_{\psi_r}|^2-F(\psi_r)+G(\psi_r)+\alpha(\psi_r),$$

yields the thesis. □

Since we set $\mathbb{S}^1=\{x^2+y^2=1, z=0\}\subset\mathbb{R}^3$, as we observed in Section 4.2.1 whenever an immersion $\varphi:\Sigma_{\mathfrak{g}}\to\mathbb{R}^3$ is such that $\varphi:\partial\Sigma_{\mathfrak{g}}\to\mathbb{S}^1$ is an embedding of the circle, we can identify $\partial\Sigma_{\mathfrak{g}}\equiv\mathbb{S}^1$ and assume that φ is the identity on $\mathbb{S}^1$. In such a setting we denote by $co_\varphi(p)$ the conormal of φ at $p\in\mathbb{S}^1$, and the unit outward conormal at $p\in\mathbb{S}^1$ of the planar disk enclosed by $\mathbb{S}^1$ is just $p=\varphi(p)$.

Lemma 4.2.5. *For any $\omega<4\pi$ exists $\varepsilon>0$ such that if $\varphi:\Sigma_{\mathfrak{g}}\to\mathbb{R}^3$ is an immersion such that $\varphi:\partial\Sigma_{\mathfrak{g}}\to\mathbb{S}^1$ is an embedding, with outward conormal field co_φ, and such that*

$$\exists p_0\in\mathbb{R}^3:\quad \sharp\varphi^{-1}(p_0)\geq 2,$$

$$\|co_\varphi(x)-\nu_{\mathbb{S}^1}(\varphi(x))\|_{L^2(\partial\Sigma_{\mathfrak{g}})}\leq\varepsilon,$$

where $\nu_{\mathbb{S}^1}$ is the unit outward conormal of the planar disk enclosed by $\mathbb{S}^1$, then $\mathcal{W}(\varphi)\geq\omega$.

Proof. As discussed above, we can identify without loss of generality $\partial\Sigma_{\mathfrak{g}}$ with $\mathbb{S}^1$ and set $\varphi(p)=p$ for any $p\in\mathbb{S}^1$. The conormal co_φ is then defined on $\mathbb{S}^1$ and $\nu_{\mathbb{S}^1}(p)=p=\varphi(p)$ for any $p\in\mathbb{S}^1$.

By smooth approximation of the surface we can prove the statement for $p_0\notin\mathbb{S}^1$. Let us assume by contradiction that

$$\begin{aligned}\exists\,\Sigma_n:=\varphi_n(\Sigma_{\mathfrak{g}}):\quad &\|co_n(p)-p\|_{L^2(\mathbb{S}^1)}\leq\frac{1}{n},\\ &\exists p_n\in\Sigma_n\setminus\mathbb{S}^1:\quad \sharp\varphi_n^{-1}(p_n)\geq 2,\\ &\mathcal{W}(\varphi_n)\leq\omega<4\pi,\end{aligned}\tag{4.13}$$

where co_n is the conormal of φ_n. Up to a small modification of the sequence which preserves (4.13), we can assume that for any n there is $q\in\mathbb{S}^1$ such that $\sharp\varphi_n^{-1}(q)=1$. Then we consider the sequence $\Sigma'_n:=I_{1,q}(\Sigma_n\setminus\{q\})$, that, up to isometry of $\mathbb{R}^3$, is an asymptotically flat manifold with one end such that

$$\begin{aligned}&\Gamma'_n:=\partial\Sigma'_n=\{(X_n,y,0)|y\in\mathbb{R}\}\qquad X_n\in\mathbb{R}_{>0},\\ &\theta'_n(0)\geq 2,\\ &0\notin\Gamma'_n,\\ &\mathcal{W}(\Sigma'_n)=\mathcal{W}(\varphi_n)+G(\varphi_n)-2\pi,\end{aligned}$$

where we write $\mathcal{W}(\Sigma'_n)$ denoting by Σ'_n also the image varifold induced by the immersion defining Σ'_n, $\theta'_n(p)$ is the multiplicity of Σ'_n, and the last equality follows from Lemma 4.2.4.

Consider now a blow up sequence $\Sigma''_n:=\frac{\Sigma_n}{r_n}$ for $r_n\searrow 0$ so small that, up to subsequence, Σ''_n converges in the sense of varifolds to the integer rectifiable varifold $\mu=\mathbf{v}\big(\bigcup_{i=1}^M\Pi_i,\theta\big)$, where each Π_i is a plane passing through the origin, and $M\geq 2$ or $M=1,\theta\geq 2$. Now we exploit the monotonicity formula of Theorem 1.3.7. Calculating $T_{\Sigma''_n,0}(\rho)$ we get

$$\exists\lim_{\rho\to\infty}T_{\Sigma''_n,0}(\rho)=\frac{1}{2}\int_{\Gamma''_n}\frac{\langle p,co''_n(p)\rangle}{|p|^2}\,d\mathcal{H}^1(p),$$

for any n, where co''_n is the conormal of Σ''_n on its boundary $\Gamma''_n := \partial\Sigma''_n = \{(R_n, y, 0) | y \in \mathbb{R}\}$, where $R_n := \frac{X_n}{r_n} \to +\infty$ choosing r_n sufficiently small. Indeed

$$\begin{aligned}\left|\frac{1}{2\rho^2}\int_{\Gamma''_n\cap B_\rho(0)}\langle p, co''_n(p)\rangle\, d\mathcal{H}^1(p)\right| &\le \left|\frac{1}{2\rho^2}\int_{\Gamma''_n\cap B_\rho(0)}\langle (R_n, y, 0), ((co''_n(p))_1, 0, (co''_n(p))_3)\rangle\, d\mathcal{H}^1(p)\right| \\ &\le \frac{R_n}{2\rho^2}\mathcal{H}^1(\Gamma''_n\cap B_\rho(0)) \le \frac{R_n}{\rho} \xrightarrow[\rho\to\infty]{} 0.\end{aligned}$$

Denoting by H''_n the mean curvature of Σ''_n, for any $0 < \sigma < \rho$ we have

$$\begin{aligned}\int_{\Sigma''_n\cap B_\rho(0)}\frac{\langle H''_n, p\rangle}{\rho^2} &= \int_{\Sigma''_n\cap B_\sigma(0)}\frac{\langle H''_n, p\rangle}{\rho^2} + \int_{\Sigma''_n\cap B_\rho(0)\setminus B_\sigma(0)}\frac{\langle H''_n, p\rangle}{\rho^2} \\ &\le \frac{1}{\rho}\int_{\Sigma''_n\cap B_\sigma(0)}|H''_n| + \frac{|\Sigma''_n\cap B_\rho(0)|^{1/2}}{\rho}\mathcal{W}(\Sigma''_n\setminus B_\sigma(0)),\end{aligned}$$

hence letting first $\rho\to\infty$ and then $\sigma\to\infty$, since $\lim_{\rho\to\infty}\frac{|\Sigma''_n\cap B_\rho(0)|^{1/2}}{\rho} = (\pi/2)^{1/2}$, we have

$$\lim_{\rho\to\infty}\int_{\Sigma''_n\cap B_\rho(0)}\frac{\langle H''_n, p\rangle}{\rho^2} = 0.$$

Hence by Theorem 1.3.7 we have

$$\begin{aligned}\exists \lim_{\rho\to\infty} A_{\Sigma''_n,0}(\rho) &= \lim_{\rho\to\infty}\frac{|\Sigma''_n\cap B_\rho(0)|}{\rho^2} + \frac{\mathcal{W}(\Sigma''_n)}{4} + \frac{1}{2}\int_{\Gamma''_n}\frac{\langle p, co''_n(p)\rangle}{|p|^2}\, d\mathcal{H}^1(p) \\ &= \frac{\pi}{2} + \frac{\mathcal{W}(\Sigma''_n)}{4} + \frac{1}{2}\int_{\Gamma''_n}\frac{\langle p, co''_n(p)\rangle}{|p|^2}\, d\mathcal{H}^1(p) \\ &= \frac{\pi}{2} + \frac{\mathcal{W}(\Sigma''_n)}{4} + \frac{1}{2}\int_{\mathbb{R}}\frac{R_n(co''_n(p))_1}{R_n^2 + y^2}\, dy \\ &\le \frac{\pi}{2} + \frac{\mathcal{W}(\Sigma''_n)}{4} + \frac{1}{2}\int_{\mathbb{R}}\frac{1}{1+u^2}\, du = \pi + \frac{\mathcal{W}(\Sigma''_n)}{4}.\end{aligned}$$

By the convergence $\Sigma''_n \to \mu$ in the sense of varifolds, the monotone quantity $A_{\Sigma''_n,0}(\rho)$ is lower semicontinuous for almost all $\rho > 0$, i.e. $\liminf_n A_{\Sigma''_n,0}(\rho) \ge A_{\mu,0}(\rho)$ for almost every $\rho > 0$. Indeed the first and second summands in the definition of $A_{\Sigma''_n,0}(\rho)$, that is the mass and the Willmore energy in the ball $B_\rho(0)$, are lower semicontinuous, while by continuity of the first variation under varifold convergence and the fact that $R_n \to +\infty$, the summand $R_{\Sigma''_n,0}(\rho)$ is continuous in the limit $n\to\infty$. Therefore

$$\begin{aligned}A_{\mu,0}(\rho) &\le \liminf_{n\to\infty} A_{\Sigma''_n,0}(\rho) \le \liminf_{n\to\infty}\lim_{\rho\to\infty} A_{\Sigma''_n,0}(\rho) \\ &\le \liminf_{n\to\infty} \pi + \frac{\mathcal{W}(\Sigma''_n)}{4} \\ &= \pi + \liminf_{n\to\infty}\frac{\mathcal{W}(\Sigma'_n)}{4} \\ &= \pi + \liminf_{n\to\infty}\frac{\mathcal{W}(\Sigma_n)}{4} < 2\pi,\end{aligned} \tag{4.14}$$

for almost every $\rho > 0$, where we used that $G(\varphi_n) \to 2\pi$ by the absurd hypothesis. Suitably passing to the limit $\rho \to +\infty$ in (4.14) we find

$$2\pi \le \lim_{\rho\to\infty}\frac{\mu(B_\rho(0))}{\rho^2} = \lim_{\rho\to\infty} A_\mu(\rho) < 2\pi,$$

which gives the desired contradiction. □

Using the above result we can deduce the following Li–Yau-type inequality, that states that the Willmore energy of a non-embedded surface whose boundary is a circle is greater or equal than 4π.

Theorem 4.2.6 (Li–Yau-type inequality with circle boundary)**.** *If $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ is an immersion such that $\varphi : \partial\Sigma_{\mathfrak{g}} \to \mathbb{S}^1$ is an embedding and there is $p_0 \in \mathbb{R}^3$ such that $\sharp\varphi^{-1}(p_0) \geq 2$, then $\mathcal{W}(\varphi) \geq 4\pi$.*

Proof. By approximation taking a small perturbation of the surface we can prove the statement for $p_0 \notin \mathbb{S}^1$. For any $\varepsilon \in (0,1)$ by Lemma 4.2.2 we get the existence of a surface $\varphi' : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ such that $\varphi' : \partial\Sigma_{\mathfrak{g}} \to \mathbb{S}^1$ is an embedding and

i) $\varphi' = F \circ \varphi$ for some F conformal;

ii) $G(\varphi') > G(\varphi)$;

iii) there exists $p' \notin \mathbb{S}^1$ with $\sharp(\varphi')^{-1}(p') \geq 2$;

iv) $\|(co_{\varphi'})(p) - \nu_{\mathbb{S}^1}(\varphi'(p))\|_{L^2(\mathbb{S}^1)} \leq \varepsilon$;

where $\nu_{\mathbb{S}^1}(q) = q$ is the outward unit conormal of the disk enclosed by $\mathbb{S}^1$. Taking ε as given by Lemma 4.2.5 for some $\omega < 4\pi$ we get

$$\mathcal{W}(\varphi) \geq \mathcal{W}(\varphi') \geq \omega,$$

and letting $\omega \to 4\pi$ we get the thesis. $\square$

Putting together Theorem 4.2.1 with Theorem 4.2.6 we get another non-existence result.

Corollary 4.2.7. *For any genus $\mathfrak{g} \geq 1$, the minimization problem*

$$\min\{\mathcal{W}(\varphi) \mid \varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3 \text{ smooth embedding},\ \varphi : \partial\Sigma_{\mathfrak{g}} \to \mathbb{S}^1 \text{ embedding}\}$$

has no minimizers and the infimum equals $e_{\mathfrak{g}} = \beta_{\mathfrak{g}} - 4\pi$.

4.3 Minimization of the Willmore energy of connected surfaces with boundary

In the second part of this chapter we want to address another family of minimization problems on the Willmore energy of surfaces with boundary. We want to consider the problem of finding an optimal elastic surface spanning a given boundary, under the constraint that such a surface is *connected*. Hence, as a first step, we need to give a sense to what we mean by "optimal elastic" surface. By such a terminology we mean that such a surface should minimize a given Willmore-type energy, subjected to suitable boundary conditions. Let us then introduce the minimization problems we will study.

If $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ is a finite disjoint union of smooth closed embedded curves in $\mathbb{R}^3$, a classical formulation of the Plateau's problem with datum γ may be to solve the minimization problem

$$\min\left\{\mu_\varphi(\Sigma) \mid \varphi : \Sigma \to \mathbb{R}^3 \text{ immersion},\ \varphi|_{\partial\Sigma} : \partial\Sigma \to \gamma \text{ embedding}\right\}, \tag{4.15}$$

where Σ is some 2-dimensional manifold, that is, one wants to look for the surface of least area having the given boundary. From a physical point of view, we know that solutions to the Plateau's problem are good models of soap elastic films having the given boundary (see [Mor09]). Moreover, critical points of the Plateau's problem are *minimal surfaces*, that is, they are characterized by having zero mean curvature; this is true also in the non-smooth context of varifolds in the appropriate sense, and it is an immediate consequence of Proposition 1.1.8 and Remark 1.1.9. In particular, minimal surfaces or varifolds with vanishing mean curvature have zero Willmore energy.

It is well-known that (4.15) can be solved under suitable assumptions on the boundary curves, or it can be solved in an appropriate generalized sense in the setting of currents (see [Mor09] and [Sim83b, Chapter 7]). However, as we are going to discuss, the Plateau's problem, and more generally the minimization of the area functional, may be incompatible with some constraints, such as the connectedness constraint we are interested in.

Therefore if we want to model an optimal elastic connected surface with boundary we should solve a minimization problem on a different energy, that possibly recovers the Plateau's problem. As the Willmore energy is sometimes associated to the "total bending" energy of a surface (see [Nit93; EFH17]), we will study suitable minimization problems on the Willmore energy of varifolds satisfying suitable boundary conditions. We will study both conditions of clamped or natural type on the generalized boundary, adding the constraint that the support of the varifold must connect some assigned curves $(\gamma^1), ..., (\gamma^\alpha) \subset \mathbb{R}^3$.

The minimization problems we will study have the form

$$\mathcal{P} := \min \{\mathcal{W}(V) \quad | \quad V = \mathbf{v}(M, \theta_V) : \quad \sigma_V = \sigma_0, \quad \mathrm{spt}V \cup \gamma \ \text{ compact, connected}\},$$

for some assigned vector valued Radon measure σ_0, or

$$\mathcal{Q} := \min \{\mathcal{W}(V) \quad | \quad V = \mathbf{v}(M, \theta_V) : \quad |\sigma_V| \le \mu, \quad \mathrm{spt}V \cup \gamma \ \text{ compact, connected}\},$$

for some assigned positive Radon measure μ with $\mathrm{spt}\mu = \gamma$.

Let us introduce a remarkable particular case that motivates our study. Let $\mathscr{C} = [0,1]^2/_\sim$ be a cylinder. Let $R \ge 1$ and $h > 0$. We define

$$\Gamma_{R,h} := \left\{x^2 + y^2 = 1, z = h\right\} \cup \left\{x^2 + y^2 = R^2, z = -h\right\}, \qquad R \ge 1, \quad h > 0,$$

that is a disjoint union of two parallel circles of possibly different radii. We consider the class of immersions

$$\mathscr{F}_{R,h} := \left\{\varphi : \mathscr{C} \to \mathbb{R}^3 \,|\, \varphi \text{ smooth immersion}, \ \varphi|_{\partial\mathscr{C}} : \partial\mathscr{C} \to \Gamma_{R,h} \text{ smooth embedding}\right\}.$$

By [Sch83, Corollary 3], if a minimal surface has $\Gamma_{R,h}$ as boundary, then it necessarily is a catenoid or a pair of planar disks. Moreover one can show there exists a threshold value $h_0 > 0$ such that $\Gamma_{R,h}$ is the boundary of a catenoid if and only if $h \le h_0$. For example, in the case of $R = 1$ one has $h_0 = \left(\min_{t>0} \frac{\cosh(t)}{t}\right)^{-1}$. In particular for any $h > h_0$ there are no minimal surfaces (and thus no solutions of the Plateau's problem) connecting the two components of $\Gamma_{R,h}$, even if $h = h_0 + \varepsilon > h_0$ is very close to h_0. This rigidity in the behavior of minimal surfaces suggests that in some cases an energy different from the area functional may be a good model for connected soap films. Since surfaces with zero Willmore energy recover critical points of the Plateau's problem, we expect the minimization of $\mathcal{W}$ to be a good process for describing optimal elastic surfaces under constraints, like connectedness ones, that do not match with the area functional.

In the following we prove two main existence theorems for the above problems $\mathcal{P}$ and $\mathcal{Q}$ (Theorem 4.3.8 and Theorem 4.3.9). This is done by establishing a relation between the varifold convergence of a sequence of varifolds with bounded Willmore energy and the convergence in Hausdorff distance of their supports (Theorem 4.3.4). In the final part of the section we analyze more deeply the motivating example of surfaces having two coaxial circles as boundary, i.e., immersions in $\mathscr{F}_{R,h}$.

4.3.1 Hausdorff distance and Willmore energy

The convergence of sets with respect to the Hausdorff distance will play an important role in our study. For every sets $X, Y \subset \mathbb{R}^3$ we recall that the Hausdorff distance $d_{\mathcal{H}}$ between X and Y is defined as

$$\begin{aligned} d_{\mathcal{H}}(X,Y) &\coloneqq \inf \{\varepsilon > 0 \,|\, X \subset \mathcal{N}_\varepsilon(Y),\, Y \subset \mathcal{N}_\varepsilon(X)\} \\ &= \max \left\{ \sup_{x\in X} \inf_{y \in Y} |x-y|,\, \sup_{y\in Y} \inf_{x\in X} |x-y| \right\}, \end{aligned}$$

where $\mathcal{N}_\varepsilon(X)$ is the tubular neighborhood of a set $X \subset \mathbb{R}^3$ given by

$$\mathcal{N}_\varepsilon(X) \coloneqq \left\{ y \in \mathbb{R}^3 \;|\; \inf_{x\in X} d(x,y) < \varepsilon \right\}.$$

Observe that, as defined here the Hausdorff distance is clearly not a distance on the power set of $\mathbb{R}^3$. However, we say that a sequence of sets X_n *converges in* $d_{\mathcal{H}}$ to a set X if $\lim_n d_{\mathcal{H}}(X_n, X) = 0$. In such a case we write that $X_n \to X$ in $d_{\mathcal{H}}$.

Now we prove some useful properties about the Hausdorff distance.

Lemma 4.3.1. *Suppose that $X_n \to X$ in $d_{\mathcal{H}}$. Then*

1. *$X_n \to \overline{X}$ in $d_{\mathcal{H}}$;*
2. *if X_n is connected for any sufficiently large n and X is bounded, then $\overline{X}$ is connected as well.*

Proof. The proof of 1 follows by noticing that if $X \subset \mathcal{N}_{\frac{\varepsilon}{2}}(X_n)$, then $\overline{X} \subset \mathcal{N}_\varepsilon(X_n)$. Now we prove 2. By 1 we can assume without loss of generality that X is closed, and thus compact. Suppose by contradiction that there exist two closed sets $A, B \subset X$ such that $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$, and $A \cup B = X$. Since X is compact, A and B are compact as well, and thus $d(A,B) \coloneqq \inf_{x\in A, y\in B} |x-y| = \varepsilon > 0$. By assumption, for any $n \geq n(\frac{\varepsilon}{4})$ we have that $X_n \subset \mathcal{N}_{\frac{\varepsilon}{4}}(X) = \mathcal{N}_{\frac{\varepsilon}{4}}(A) \cup \mathcal{N}_{\frac{\varepsilon}{4}}(B)$ and $\mathcal{N}_{\frac{\varepsilon}{4}}(A) \cap \mathcal{N}_{\frac{\varepsilon}{4}}(B) = \emptyset$. The sets $\mathcal{N}_{\frac{\varepsilon}{4}}(A) \cap X_n$ and $\mathcal{N}_{\frac{\varepsilon}{4}}(B) \cap X_n$ are disjoint and definitively non-empty, and open in X_n. This implies that X_n is not connected for n large enough, that gives a contradiction. $\square$

Lemma 4.3.2. *Suppose X_n is a sequence of uniformly bounded closed sets in $\mathbb{R}^3$ and let $X \subset \mathbb{R}^3$ be closed. Then $X_n \to X$ in $d_{\mathcal{H}}$ if and only if the following two properties hold:*

a) for any subsequence of points $y_{n_k} \in X_{n_k}$ such that $y_{n_k} \underset{k}{\to} y$, we have that $y \in X$;

b) for any $x \in X$ there exists a sequence $y_n \in X_n$ converging to x.

Proof. Suppose first that $d_{\mathcal{H}}(X_n, X) \to 0$. If there exists a converging subsequence $y_{n_k} \in X_{n_k}$ with limit $y \notin X$, then $d(y_{n_k}, X) \geq \varepsilon_0 > 0$, and thus $X_{n_k} \not\subset \mathcal{N}_{\frac{\varepsilon_0}{2}}(X)$ for k large, that is impossible; so we have proved a). Now let $x \in X$ be fixed. Consider a strictly decreasing sequence $\varepsilon_m \searrow 0$. For any $\varepsilon_m > 0$ let n_{ε_m} be such that $X \subset \mathcal{N}_{\varepsilon_m}(X_n)$ for any $n \geq n_{\varepsilon_m}$. This means that $B_{\varepsilon_m}(x) \cap X_n \neq \emptyset$ for any $n \geq n_{\varepsilon_m}$ and any $m \in \mathbb{N}$. We can define a sequence

$$n \mapsto x_n \in X_n \cap B_{\varepsilon_{m_n}}(x),$$

where

$$m_n = \sup \{m \in \mathbb{N} \,|\, X_n \cap B_{\varepsilon_m}(x) \neq \emptyset\},$$

understanding that $x_n = x$ if $m_n = \infty$, indeed since X_n is closed we have that $x \in X_n$ if $m_n = \infty$. The sequence ε_{m_n} converges to 0 as $n \to \infty$, otherwise there exists $\eta > 0$ such that $X_n \cap B_\eta(x) = \emptyset$ for any n large, but this contradicts the convergence in $d_{\mathcal{H}}$. Hence $x_n \to x$ and we have proved b).

Suppose now that a) and b) hold. If there is $\varepsilon_0 > 0$ such that $X_n \not\subset \mathcal{N}_{\varepsilon_0}(X)$ for n large, then a subsequence x_{n_k} converges to a point y such that $d(y, X) \geq \varepsilon_0 > 0$, that is impossible. If there is $\varepsilon_0 > 0$ such that $X \not\subset \mathcal{N}_{\varepsilon_0}(X_n)$ for n large, then there is a sequence $z_n \in X$ such that $d(z_n, X_n) \geq \varepsilon_0 > 0$. By b) we have that X is bounded, then a subsequence z_{n_k} converges to $z \in X$, and $d(z, X_{n_k}) \geq \frac{\varepsilon_0}{2}$ definitely in k. But then z is not the limit of any sequence $x_{n_k} \in X_{n_k}$. However z is the limit of a sequence $\bar{x}_n \in X_n$ by b), and thus it is the limit of the subsequence $\bar{x}_{n_k}$, and this gives a contradiction. □

Corollary 4.3.3. *Let X_n be a sequence of uniformly bounded closed sets. Suppose that $X_n \to X$ in $d_{\mathcal{H}}$ and $X_n \to Y$ in $d_{\mathcal{H}}$. If both X and Y are closed, then $X = Y$.*

Proof. Both X and Y are bounded. We can apply Lemma 4.3.2, that immediately implies that $X \subset Y$ and $Y \subset X$ using the characterization of convergence in $d_{\mathcal{H}}$ given by items a) and b). □

The above standard properties allow us to relate the convergence in the sense of varifolds to the convergence of their supports in Hausdorff distance, under the assumption of bounded Willmore energy.

We recall that by Theorem 1.3.7 we can assume that if $V = \mathbf{v}(M, \theta_V)$ is a varifold in $\mathbb{R}^3$ with bounded Willmore energy, compact support, and generalized boundary σ_V such that $\mathcal{H}^2(\mathrm{spt}\sigma_V) = 0$, then we can set

$$M = \left\{ x \in \mathbb{R}^3 \setminus \mathrm{spt}\sigma_V \;|\; \theta_V(x) \geq \frac{1}{2} \right\} \cup \mathrm{spt}\sigma_V,$$

which is compact, and θ_V is pointwise defined at any $x \in \mathbb{R}^3 \setminus \mathrm{spt}\sigma_V$ by the limit $\theta_V(x) = \lim_{r \searrow 0} \frac{\mu_V(B_r(x))}{\pi r^2}$. In the following, whenever a varifold $V = \mathbf{v}(M, \theta_V)$ fits these assumptions we will assume that M and θ_V are as above.

Theorem 4.3.4 (Varifold convergence and Hausdorff distance)**.** *Let $V_n = \mathbf{v}(M_n, \theta_{V_n}) \neq 0$ be a sequence of varifolds with uniformly bounded Willmore energy converging to $V = \mathbf{v}(M, \theta_V) \neq 0$. Suppose that the M_n's are connected and uniformly bounded.*
Suppose that $\mathrm{spt}\sigma_{V_n} = (\gamma_n^1) \cup ... \cup (\gamma_n^\alpha)$ where the γ_n^i's are disjoint smooth embedded closed curves, $\bar{\gamma}^1, ..., \bar{\gamma}^\beta$ with $\beta \leq \alpha$ are disjoint smooth embedded closed curves, and assume that $(\gamma_n^i) \to (\bar{\gamma}^i)$ in $d_{\mathcal{H}}$ for $i = 1, ..., \beta$ and that $\mathcal{H}^1(\gamma_n^i) \to 0$ for $i = \beta + 1, ..., \alpha$.

Then $M_n \to M \cup (\bar{\gamma}^1) \cup ... \cup (\bar{\gamma}^\beta)$ in Hausdorff distance $d_{\mathcal{H}}$ (up to subsequence) and $M \cup (\bar{\gamma}^1) \cup ... \cup (\bar{\gamma}^\beta)$ is connected. Moreover $(\gamma_n^i) \to \{p_i\}$ in $d_{\mathcal{H}}$ for any $i = \beta + 1, ..., \alpha$ for some points $p_i \in \mathbb{R}^3$, $p_i \in M$ for any $i = \beta + 1, ..., \alpha$, and $\mathrm{spt}\sigma_V \subset (\bar{\gamma}^1) \cup ... \cup (\bar{\gamma}^\beta) \cup \{p_{\beta+1}, ..., p_\alpha\}$.

Proof. Let us first observe that by the uniform boundedness of M_n, we get that (γ_n^i) converges to some compact set X^i in $d_{\mathcal{H}}$ up to subsequence for any $i = \beta+1, ..., \alpha$. Each X_i is connected by Lemma 4.3.1, then by Gołab Theorem (see [Fal86, Theorem 3.18]) we know that $\mathcal{H}^1(X^i) \le \liminf_n \mathcal{H}^1(\gamma_n^i) = 0$, hence $X^i = \{p_i\}$ for any $i = \beta+1, ..., \alpha$ for some points $p_{\beta+1}, ..., p_\alpha$. Denote $X = \{p_{\beta+1}, ..., p_\alpha\}$.

By assumption we know that $\mu_{V_n} \stackrel{\star}{\rightharpoonup} \mu_V$ as measures on $\mathbb{R}^3$, also M_n and M can be taken to be closed. Moreover $\mathrm{spt}\sigma_V \subset X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$. Indeed V_n are definitely varifolds without generalized boundary on any open set of the form $\mathcal{N}_\varepsilon(X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta))$ and they converge as varifolds to V on such an open set with equibounded Willmore energy.

We want to prove that the sets M_n and $M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$ satisfy points a) and b) of Lemma 4.3.2 and that $X \subset M$.

Let $x \in M \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta) \cup X$. If $x \in (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta) \cup X$, then by assumption and Lemma 4.3.2 there is a sequence of points in $\mathrm{spt}\sigma_{V_n}$ converging to x. So let $x \in M \setminus ((\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta) \cup X)$. We know that there exists the limit $\lim_{\rho \searrow 0} \frac{\mu_V(B_\rho(x))}{\pi\rho^2} \ge 1$, hence we can write that for any $\rho \in (0, \rho_0)$ with $\rho_0 < d(x, \mathrm{spt}\sigma_V)$ we have that $\mu_V(B_\rho(x)) \ge \frac{\pi}{2}\rho^2$. There exists a sequence $\rho_m \searrow 0$ such that $\lim_n \mu_{V_n}(B_{\rho_m}(x)) = \mu_V(B_{\rho_m}(x))$ for any m. Hence $M_n \cap B_{\rho_m}(x) \neq \emptyset$ for any m definitely in n. Arguing as in Lemma 4.3.2 we find a sequence $x_n \in M_n$ converging to x, and thus the property b) of Lemma 4.3.2 is achieved.

For any $\varepsilon > 0$ let $A_\varepsilon := \mathcal{N}_\varepsilon(X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta))$. Let us show that for any $\varepsilon > 0$ it occurs that $M_n \setminus A_\varepsilon$ converges to $\big(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)\big) \setminus A_\varepsilon = M \setminus A_\varepsilon$ in $d_{\mathcal{H}}$, i.e., we want to check property a) of Lemma 4.3.2 for such sets.

Once this convergence is established, we get that $M_n \to M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$ in $d_{\mathcal{H}}$ and we can show that the whole thesis follows. Indeed we have that for any $\varepsilon > 0$ for any $\eta > 0$ it holds that

$$M_n \setminus A_\varepsilon \subset \mathcal{N}_\eta\left(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta) \setminus A_\varepsilon\right),$$

$$\left(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)\right) \setminus A_\varepsilon \subset \mathcal{N}_\eta(M_n \setminus A_\varepsilon),$$

for any $n \ge n_{\varepsilon,\eta}$. In particular

$$M_n = M_n \setminus A_\varepsilon \cup A_\varepsilon \subset \mathcal{N}_\eta(M \setminus A_\varepsilon) \cup A_\varepsilon \subset \mathcal{N}_{\eta+2\varepsilon}(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)),$$

and

$$\begin{aligned} M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta) &= \left(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)\right) \setminus A_\varepsilon \cup A_\varepsilon \\ &\subset \mathcal{N}_\eta(M_n \setminus A_\varepsilon) \cup A_\varepsilon \subset \mathcal{N}_{\eta+2\varepsilon}(M_n), \end{aligned}$$

for any $n \ge n_{\varepsilon,\eta}$. Setting $\varepsilon = \eta$ we see that for any $\eta > 0$ it holds that

$$M_n \subset \mathcal{N}_{3\eta}\left(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)\right), \qquad \left(M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)\right) \subset \mathcal{N}_{3\eta}(M_n),$$

for any $n \ge n_{2\eta,\eta}$. Hence $M_n \to M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$ in $d_{\mathcal{H}}$. Therefore $M \cup X \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$ is closed and connected. Moreover we get that $X \subset M$, indeed for any $p_i \in X$ for any $K \in \mathbb{N}_{\ge 1}$ by connectedness of M_n we find some subsequence $y_{n_k} \in M_n \cap \partial B_{\frac{1}{K}}(p_i)$ converging to a point $y_K \in M \cap \partial B_{\frac{1}{K}}(p_i)$. Since M is closed, passing to the limit $K \to \infty$ we see that $p_i \in M$. In particular $M_n \to M \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$ in $d_{\mathcal{H}}$, by Lemma 4.3.1 we get that $M \cup (\bar\gamma^1) \cup ... \cup (\bar\gamma^\beta)$ is connected, and the proof is completed.

So we are left to prove that $M_n \setminus A_\varepsilon$ converges to $\big(M \cup X \cup (\bar{\gamma}^1) \cup ... \cup (\bar{\gamma}^\beta)\big) \setminus A_\varepsilon = M \setminus A_\varepsilon$ in $d_{\mathcal{H}}$ for any fixed $\varepsilon > 0$. Consider any converging sequence $y_{n_k} \in M_{n_k} \setminus A_\varepsilon$. For simplicity, let us denote y_n such sequence. Suppose by contradiction that $y_n \to y$ but $y \notin M \cup A_\varepsilon$. Since M is closed, there exist $\zeta > 0$ such that $B_\zeta(y) \cap M = \emptyset$ for n large. Since M_n is connected and $M \neq \emptyset$ we can write that $\partial B_\zeta(y) \cap M_n \neq \emptyset$ for any $\sigma \in (\frac{\zeta}{4}, \frac{\zeta}{2})$ for n large enough. Since $y_n \notin A_\varepsilon$, up to choosing a smaller ζ we can assume that $B_\zeta(y)$ does not intersect $\mathrm{spt}\sigma_{V_n}$ for n large. Fix $N \in \mathbb{N}$ with $N \geq 2$ and consider points

$$z_{n,k} \in \partial B_{(1+\frac{k}{N})\frac{\zeta}{4}}(y) \cap M_n \neq \emptyset,$$

for any $k = 1, ..., N-1$. The open balls

$$\Big\{ B_{\frac{1}{2N}\frac{\zeta}{4}}(z_{n,k}) \Big\}_{k=1}^{N-1}$$

are pairwise disjoint. Passing to the limit $\sigma \searrow 0$, setting $\rho = \frac{\zeta}{8N}$, and using Young's inequality in Equation (1.15) evaluated on the varifold V_n at the point $p_0 = z_{n,k}$ we get that

$$\begin{aligned} \pi &\leq \frac{\mu_{V_n}\Big(B_{\frac{\zeta}{8N}}(z_{n,k})\Big)}{\left(\frac{\zeta}{8N}\right)^2} + \frac{1}{4}\int_{B_{\frac{\zeta}{8N}}(z_{n,k})} |H_{V_n}|^2\, d\mu_{V_n} + \frac{1}{\left(\frac{\zeta}{8N}\right)^2}\int_{B_{\frac{\zeta}{8N}}(z_{n,k})} \langle H_{V_n}, p - z_{n,k}\rangle\, d\mu_{V_n}(p) \\ &\leq \frac{3}{2}\frac{\mu_{V_n}\Big(B_{\frac{\zeta}{8N}}(z_{n,k})\Big)}{\left(\frac{\zeta}{8N}\right)^2} + \frac{3}{4}\int_{B_{\frac{\zeta}{8N}}(z_{n,k})} |H_{V_n}|^2\, d\mu_{V_n}, \end{aligned} \tag{4.16}$$

for any n large and any $k = 1, ..., N-1$. Since

$$\limsup_n \mu_{V_n}\Big(B_{\frac{\zeta}{8N}}(z_{n,k})\Big) \leq \limsup_n \mu_{V_n}\Big(\overline{B_{\frac{\zeta}{2}}(y)}\Big) \leq \mu_V\Big(B_{\frac{3}{4}\zeta}(y)\Big) = 0,$$

summing over $k = 1, ..., N-1$ in (4.16) and passing to the limit $n \to \infty$ we get that

$$\pi(N-1) \leq \limsup_n \frac{3}{4}\sum_{k=1}^{N-1}\int_{B_{\frac{\zeta}{8N}}(z_{n_k})} |H_{V_n}|^2\, d\mu_{V_n} \leq \frac{3}{4}\limsup_n \mathcal{W}(V_n).$$

Since N can be chosen arbitrarily big from the beginning, we get a contradiction with the uniform bound on the Willmore energy of the V_n's. □

Remark 4.3.5. Arguing as in the second part of the proof of Theorem 4.3.4, we get the following useful statement.

Assuming $V_n = \mathbf{v}(M_n, \theta_{V_n}) \neq 0$ is a sequence of curvature varifolds with boundary with uniformly bounded Willmore energy converging to $V = \mathbf{v}(M, \theta_V) \neq 0$. Suppose that the M_n's are connected and closed and that M is closed. Suppose that $\mathrm{spt}\sigma_{V_n}$ is as in Theorem 4.3.4. If a subsequence $y_{n_k} \in M_{n_k}$ converges to y, then $y \in M \cup \bar{\gamma}^1 \cup ... \cup \bar{\gamma}^\beta$.

Observe that the supports M_n, M are not necessarily bounded here.

Remark 4.3.6. The connectedness assumption in Theorem 4.3.4 is essential. Consider indeed the following example: let $M_n = \partial B_1(0) \cup \partial B_{\frac{1}{n}(0)}$ and $\theta_{V_n}(p) = 1$ for any $p \in M_n$. Hence the varifolds $\mathbf{v}(M_n, \theta_{V_n})$ converge to $\mathbf{v}(\partial B_1(0), 1)$ as varifolds and they have uniformly bounded energy equal to 8π, but clearly M_n does not converge to $\partial B_1(0)$ in $d_{\mathcal{H}}$.

Remark 4.3.7. The statement of Theorem 4.3.4 also holds if we assume $\mathrm{spt}\sigma_{V_n} \subset (\gamma_n^1) \cup ... \cup (\gamma_n^\alpha)$ and $M_n \cup (\gamma_n^1) \cup ... \cup (\gamma_n^\alpha)$ connected. In this case, using the notation of the proof of Theorem 4.3.4, we have that $M_n \cup (\gamma_n^1) \cup ... \cup (\gamma_n^\alpha)$ converges to $M \cup X \cup (\bar{\gamma}^1) \cup ... \cup (\bar{\gamma}^\beta)$ in $d_{\mathcal{H}}$ and $M \cup X \cup (\bar{\gamma}^1) \cup ... \cup (\bar{\gamma}^\beta)$ is connected.

4.3.2 Existence of minimizers and asymptotics

Using the above results, we are ready for proving two existence theorems about boundary valued minimization problems on connected varifolds.

Theorem 4.3.8. *Let $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ be a disjoint union of smooth embedded closed curves with $\alpha \in \mathbb{N}_{\geq 2}$. Let*

$$\sigma_0 = m\,\nu_0\,\mathcal{H}^1 \llcorner \gamma$$

be a vector valued Radon measure, where $m : \gamma \to \mathbb{N}_{\geq 1}$ and $\nu_0 : \gamma \to (T\gamma)^\perp$ are $\mathcal{H}^1$-measurable functions with $m \in L^\infty(\mathcal{H}^1 \llcorner \gamma)$ and $|\nu_0| = 1$ $\mathcal{H}^1$-ae. Let $\mathcal{P}$ be the minimization problem

$$\mathcal{P} := \min \{\mathcal{W}(V) \quad | \quad V = \mathbf{v}(M, \theta_V) : \quad \sigma_V = \sigma_0, \quad \mathrm{spt}V \cup \gamma \quad \textit{compact, connected}\}.$$

If $\inf \mathcal{P} < 4\pi$, then $\mathcal{P}$ has minimizers.

Proof. Let $V_n = \mathbf{v}(M_n, \theta_{V_n})$ be a minimizing sequence for the problem $\mathcal{P}$. Denote $I = \inf \mathcal{P} < 4\pi$, and suppose without loss of generality that $\mathcal{W}(V_n) < 4\pi$ for any n. For any $p_0 \in M_n \setminus \gamma$ passing to the limits $\sigma \to 0$ and $\rho \to \infty$ in the monotonicity formula (1.15) we get

$$4\pi \leq \mathcal{W}(V_n) + 2\frac{|\sigma_0|(\gamma)}{d(p_0, \gamma)},$$

and then

$$\sup_{p_0 \in M_n \setminus \gamma} d(p_0, \gamma) \leq 2\frac{|\sigma_0|(\gamma)}{4\pi - \mathcal{W}(V_n)} \leq C(\sigma_0, I). \tag{4.17}$$

Hence the sequence M_n is uniformly bounded in $\mathbb{R}^3$. Integrating the tangential divergence of the field $X(p) = \chi(p)\,p$ where $\chi(p) = 1$ for any $p \in B_{R_0}(0) \supset M_n$ for any n we get that

$$\begin{aligned} 2\mu_{V_n}(\mathbb{R}^3) &= \int \mathrm{div}_{TM_n} X \, d\mu_{V_n} = -2\int \langle H_{V_n}, X\rangle \, d\mu_{V_n} + \int \langle X, \nu_0\rangle d|\sigma_0| \\ &\leq C(\sigma_0, I)\mu_{V_n}(\mathbb{R}^3)^{\frac{1}{2}} + C(\sigma_0, I), \end{aligned}$$

for any n, and then μ_{V_n} is uniformly bounded. By Theorem 1.2.14 we have that $V_n \to V = \mathbf{v}(M, \theta_V)$ in the sense of varifolds (up to subsequence), and M is compact.

By an argument analogous to the proof of Theorem 4.3.4 we now show that $V \neq 0$. Suppose by contradiction that $V = 0$. Since $\alpha \geq 2$ and the curves $\gamma^1, ..., \gamma^\alpha$ are disjoint and embedded, there exist an embedded torus $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3 \setminus \gamma$ dividing $\mathbb{R}^3$ into two connected components A_1, A_2 such that $A_1 \supset (\gamma^1)$ and $A_2 \supset (\gamma^2) \cup ... \cup (\gamma^\alpha)$. Since M_n is connected and uniformly bounded, there is a sequence of points $y_n \in M_n \cap \phi(\mathbb{S}^1 \times \mathbb{S}^1)$ with a converging subsequence $y_{n_k} \to y$. Observe that there is $\Delta > 0$ such that $d(y_n, \gamma) \geq \Delta$. Since $V = 0$ we have that $y \notin \mathrm{spt}V$. Let $N \geq 4$ be a natural number and consider the balls $\left\{B_{\frac{j}{N}\frac{\Delta}{2}}(y)\right\}_{j=1}^N$. Up to subsequence, for n sufficiently large there is $z_{n,j} \in \partial B_{\frac{j}{N}\frac{\Delta}{2}}(y) \cap M_n$. Also the balls

$$\left\{B_{\frac{\Delta}{4N}}(z_{n,j})\right\}_{j=1}^N$$

are pairwise disjoint. As in (4.16) we get that

$$\pi \leq \frac{3}{2} \frac{\mu_{V_n}\left(B_{\frac{\Delta}{4N}}(z_{n,j})\right)}{\left(\frac{\Delta}{4N}\right)^2} + \frac{3}{4}\int_{B_{\frac{\Delta}{4N}}(z_{n,j})} |H_{V_n}|^2\, d\mu_{V_n}$$

for any $j = 1, ..., N$. Since $\limsup_n \mu_{V_n}\left(B_{\frac{\Delta}{4N}}(z_{n,j})\right) \leq \mu_V(B_{\frac{3}{4}\Delta}(y)) = 0$, summing over $j = 1, ..., N$ and passing to the limit in n we get

$$4\pi \leq N\pi \leq \frac{3}{4}\lim_n \mathcal{W}(V_n) \leq 3\pi,$$

that gives a contradiction.

Hence Theorem 4.3.4 implies that $\mathrm{spt}V \cup \gamma = M \cup \gamma$ is connected. Since $\mathcal{W}(V) \leq I$ by Corollary 1.2.16, we are left to show that $\sigma_V = \sigma_0$. Since γ is smooth, for $q_0 \in \gamma$ we can write that

$$|(T\gamma)^\perp(p - q_0)| \leq C_\gamma |p - q_0|^2 \tag{4.18}$$

as $p \to q_0$ with $p \in \gamma$ for some constant C_γ depending on the curvature of γ. Let $0 < \sigma < s$ with $s = s(\gamma)$ such that (4.18) holds for $p \in \gamma \cap B_s(q)$ for any $q \in \gamma$. For any $q_0 \in \gamma$ the monotonicity formula (1.15) at q_0 on V_n gives

$$\begin{aligned}
\frac{\mu_{V_n}(B_\sigma(q_0))}{\sigma^2} \leq & -\frac{1}{\sigma^2}\int_{B_\sigma(q_0)} \langle H_{V_n}, p - q_0\rangle\, d\mu_{V_n}(p) + \\
& -\frac{1}{2}\int_{B_\sigma(q_0)} \left(\frac{1}{|p-q_0|^2} - \frac{1}{\sigma^2}\right)\langle p - q_0, \nu_0\rangle\, d|\sigma_0|(p) + \lim_{\rho\to\infty} A_{V_n,q_0}(\rho) \\
\leq & \, \mathcal{W}(V_n)^{\frac{1}{2}}\left(\frac{\mu_{V_n}(B_\sigma(q_0))}{\sigma^2}\right)^{\frac{1}{2}} + \frac{1}{2}\int_{B_\sigma(q_0)} \frac{C_\gamma|p-q_0|^2}{|p-q_0|^2} + \frac{1}{\sigma}\, d|\sigma_0|(p) + \pi + \\
& + \frac{1}{2}\int \frac{\langle p - q_0, \nu_0\rangle}{|p - p_0|^2}\, d|\sigma_0|(p) \\
\leq & \, \mathcal{W}(V_n)^{\frac{1}{2}}\left(\frac{\mu_{V_n}(B_\sigma(q_0))}{\sigma^2}\right)^{\frac{1}{2}} + C_\gamma|\sigma_0|(B_\sigma(q_0)) + \frac{1}{\sigma}|\sigma_0|(B_\sigma(q_0)) + \pi + \\
& + \frac{1}{2}\frac{1}{s}|\sigma_0|\left(\gamma \setminus B_\sigma(q)\right) \\
\leq & \, C(I)\left(\frac{\mu_{V_n}(B_\sigma(q_0))}{\sigma^2}\right)^{\frac{1}{2}} + C(\gamma, \sigma_0).
\end{aligned}$$

In particular

$$\mu_{V_n}(B_\sigma(q)) \leq C(I, \gamma, \sigma_0)\sigma^2$$

for any $q_0 \in \gamma$, any $\sigma \in (0, s)$, and any n. Consider now any $X \in C^0_c(B_r(q_0))$ for fixed $q_0 \in \gamma$ and $r \in (0, s)$. By (1.4) we have that

$$\lim_n -2\int \langle H_{V_n}, X\rangle\, d\mu_{V_n} + \int \langle X, \nu_0\rangle\, d|\sigma_0| = -2\int \langle H_V, X\rangle\, d\mu_V + \int \langle X, \nu_V\rangle\, d|\sigma_V|, \tag{4.19}$$

where we wrote $\sigma_V = \nu_V\, |\sigma_V|$. Now let $m \in \mathbb{N}$ be large and consider the cut off function

$$\Lambda_m(p) = \begin{cases} 1 - m d(p,\gamma) & d(p,\gamma) < \frac{1}{m}, \\ 0 & d(p,\gamma) \geq \frac{1}{m}. \end{cases} \tag{4.20}$$

Take now $X = \Lambda_m Y$ for arbitrary $Y \in C^0_c(B_r(q_0))$. We have that

$$\limsup_{m\to\infty} \lim_n \left| \int \langle H_{V_n}, X\rangle \, d\mu_{V_n} \right| = \limsup_{m\to\infty} \lim_n \left| \int_{B_r(q_0)\cap \mathcal{N}_{\frac{1}{m}}(\gamma)} \Lambda_m \langle H_{V_n}, Y\rangle \, d\mu_{V_n} \right|$$
$$\leq \|Y\|_\infty \limsup_m \lim_n \mathcal{W}(V_n)^{\frac{1}{2}} \mu_{V_n}\left(B_r(q_0)\cap \mathcal{N}_{\frac{1}{m}}(\gamma)\right)^{\frac{1}{2}}.$$

Moreover, there exists a constant $C(\gamma)$ such that $B_r(q_0)\cap \mathcal{N}_{\frac{1}{m}}(\gamma) \subset \cup_{i=1}^{C(\gamma)m} B_{\frac{2}{m}}(q_i)$ for some points $q_i \in \gamma$ and at most $C(\gamma)m$ balls $\left\{B_{\frac{2}{m}}(q_i)\right\}_i$. Hence for $\frac{2}{m} < s$ we can estimate

$$\mu_{V_n}\left(B_r(q_0)\cap \mathcal{N}_{\frac{1}{m}}(\gamma)\right) \leq \sum_{i=1}^{C(\gamma)m} \mu_{V_n}\left(B_{\frac{2}{m}}(q_i)\right) \leq C(\gamma) m C(I,\gamma,\sigma_0)\frac{4}{m^2}.$$

Therefore

$$\limsup_{m\to\infty} \lim_n \left| \int \langle H_{V_n}, X\rangle \, d\mu_{V_n} \right| \leq \|Y\|_\infty \limsup_m C(I,\gamma,\sigma_0)\frac{1}{\sqrt{m}} = 0. \tag{4.21}$$

Hence setting $X = \Lambda_m Y$ in (4.19) and letting $m\to\infty$ we obtain

$$\int \langle Y, \nu_0\rangle \, d|\sigma_0| = \int \langle Y, \nu_V\rangle \, d|\sigma_V|,$$

for any $Y \in C^0_c(B_r(q_0))$. Since $q_0 \in \gamma$ is arbitrary we conclude that $\sigma_V = \sigma_0$, and thus V is a minimizer. □

Theorem 4.3.9. *Let $\gamma = (\gamma^1)\cup ... \cup(\gamma^\alpha)$ be a disjoint union of smooth embedded closed curves with $\alpha \in \mathbb{N}_{\geq 2}$. Let $m : \gamma \to \mathbb{N}_{\geq 1}$ be $\mathcal{H}^1$-measurable with $m \in L^\infty(\mathcal{H}^1 \llcorner \gamma)$. Let $\mathcal{Q}$ be the minimization problem*

$$\mathcal{Q} := \min\left\{ \mathcal{W}(V) \quad \middle| \quad V = \mathbf{v}(M,\theta_V): \quad |\sigma_V| \leq m\mathcal{H}^1 \llcorner \gamma, \quad \mathrm{spt} V \cup \gamma \;\; \textit{compact, connected}\right\}.$$

If $\inf \mathcal{Q} < 4\pi$*, then* $\mathcal{Q}$ *has minimizers.*

Proof. We adopt the same notation used in the proof of Theorem 4.3.8. In this case the generalized boundaries of the minimizing sequence $V_n = \mathbf{v}(M_n, \theta_{V_n})$ are denoted by $\sigma_{V_n} = \nu_{V_n}|\sigma_{V_n}|$, and $|\sigma_{V_n}| \leq m\mathcal{H}^1 \llcorner \gamma$. The very same strategy used in Theorem 4.3.8 shows that V_n converges up to subsequence in the sense of varifolds to a limit $V = \mathbf{v}(M,\theta_V) \neq 0$ with $M\cup\gamma$ compact and connected by Theorem 4.3.4 and Remark 4.3.7, and $\mathcal{W}(V) \leq \inf \mathcal{Q}$. Hence, to see that V is a minimizer, we are left with showing that $|\sigma_V| \leq m\mathcal{H}^1 \llcorner \gamma$.

Denoting $\mu := m\mathcal{H}^1 \llcorner \gamma$, we find as in Theorem 4.3.8 that there exist constants $C = C(\inf \mathcal{Q}, \gamma, \mu)$ and $s = s(\gamma)$ such that

$$\mu_{V_n}(B_\sigma(q)) \leq C\sigma^2,$$

for any $q \in \gamma$, any $\sigma \in (0,s)$, and any n large. Now for any $X \in C^0_c(B_r(q_0))$ for fixed $q_0 \in \gamma$ and $r \in (0,s)$ the convergence of the first variation of varifolds (see (1.4)) reads

$$\lim_n -2\int \langle H_{V_n}, X\rangle \, d\mu_{V_n} + \int \langle X, \nu_{V_n}\rangle \, d|\sigma_{V_n}| = -2\int \langle H_V, X\rangle \, d\mu_V + \int \langle X, \nu_V\rangle \, d|\sigma_V|, \tag{4.22}$$

where we wrote $\sigma_V = \nu_V|\sigma_V|$. Now we set $X = \Lambda_m Y$ in (4.22) for $Y \in C^0_c(B_r(q_0))$ and Λ_m as in (4.20). Estimating as in (4.21) and taking the limit $m \to \infty$ we obtain

$$\lim_n \int \langle Y, \nu_{V_n}\rangle \, d|\sigma_{V_n}| = \int \langle Y, \nu_V\rangle \, d|\sigma_V|,$$

that is $\sigma_{V_n} \stackrel{\star}{\rightharpoonup} \sigma_V$, and thus $|\sigma_V|(A) \le \liminf_n |\sigma_{V_n}|(A) \le \mu(A)$ for any open set A. Hence $|\sigma_V| \le \mu$ and V is a minimizer of $\mathcal{Q}$. $\square$

Remark 4.3.10. Assuming in Theorem 4.3.8 and in Theorem 4.3.9 the connected components of the boundary datum are at least two (i.e. $\alpha \ge 2$) is technical, but it is also essential in order to obtain a non-trivial minimization problem. Moreover, in the case of the problem in Theorem 4.3.9, this is also crucial in order to obtain a problem that does not necessarily reduces to a Plateau's one. In fact if we consider a single closed embedded smooth (oriented) curve γ, then [Sim83b, Chapter 7, Lemma 2.1] guarantees the existence of an area minimizing integer rectifiable current $T = \tau(M, \theta, \xi)$ with compact support and with boundary $\partial T = \gamma$. Hence by [Sim83b, Chapter 7, Lemma 1.2] the integer rectifiable varifold $V = \mathbf{v}(M, \theta)$ is stationary, i.e., its mean curvature vanishes, and $\mathrm{spt}\sigma_V \subset \gamma$. Then we can consider $M = \mathrm{spt}T$, that is compact. Since $\partial T = \gamma$ and T is minimizing, the set $M \cup \gamma$ is connected and $\mathcal{W}(V)$ is trivially zero.

The existence theorems 4.3.8 and 4.3.9 can be applied in different perturbative regimes, as discussed in the following corollaries and remarks.

Corollary 4.3.11. *Let $\gamma = (\gamma^1) \cup \ldots \cup (\gamma^\alpha)$ be a disjoint union of smooth embedded closed sucrves with $\alpha \in \mathbb{N}_{\ge 2}$. Suppose that there exists a compact connected immersed surface $\Sigma \subset \mathbb{R}^3$ whose boundary is γ. Let $\varepsilon \in \mathbb{R}$ and $f_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth family of diffeomorphisms with $f_0 = id|_{\mathbb{R}^3}$. For any ε let*

$$\sigma_\varepsilon = co_{f_\varepsilon(\Sigma)} \mathcal{H}^1 \llcorner (f_\varepsilon(\gamma)),$$

where $co_{f_\varepsilon(\Sigma)}$ is the conormal of $f_\varepsilon(\Sigma)$. If $\mathcal{W}(\Sigma) < 4\pi$, there exists $\varepsilon_1 > 0$ such that if $\varepsilon_0 < \varepsilon_1$ the minimization problems

$$\mathcal{P}_\varepsilon := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ \sigma_V = \sigma_\varepsilon, \ \mathrm{spt}V \cup f_\varepsilon(\gamma) \ \ \textit{compact, connected} \right\},$$

$$\mathcal{Q}_\varepsilon := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ |\sigma_V| \le \mathcal{H}^1 \llcorner (f_\varepsilon(\gamma)), \ \mathrm{spt}V \cup f_\varepsilon(\gamma) \ \ \textit{compact, connected} \right\},$$

have minimizers for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Corollary 4.3.12. *Let $\gamma = (\gamma^1) \cup \ldots \cup (\gamma^\alpha)$ be a disjoint union of smooth closed curves with $\alpha \in \mathbb{N}_{\ge 2}$. Suppose that there exists a compact connected immersed minimal surface $\Sigma \subset \mathbb{R}^3$ whose boundary is γ. Let $\varepsilon \in \mathbb{R}$ and $f_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth family of diffeomorphisms with $f_0 = id|_{\mathbb{R}^3}$. For any ε let*

$$\sigma_\varepsilon = co_{f_\varepsilon(\Sigma)} \mathcal{H}^1 \llcorner (f_\varepsilon(\gamma)),$$

where $co_{f_\varepsilon(\Sigma)}$ is the conormal field of $f_\varepsilon(\Sigma)$. Then there exists $\varepsilon_1 > 0$ such that if $\varepsilon_0 < \varepsilon_1$ the minimization problems

$$\mathcal{P}_\varepsilon := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ \sigma_V = \sigma_\varepsilon, \ \mathrm{spt}V \cup f_\varepsilon(\gamma) \ \ \textit{compact, connected} \right\},$$

$$\mathcal{Q}_\varepsilon := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ |\sigma_V| \le \mathcal{H}^1 \llcorner (f_\varepsilon(\gamma)), \ \mathrm{spt}V \cup f_\varepsilon(\gamma) \ \ \textit{compact, connected} \right\},$$

have minimizers for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Remark 4.3.13. Many examples in which the existence theorems 4.3.8 and 4.3.9 and Corollary 4.3.11 apply are given by defining the following boundary data. We can consider any compact smooth surface $S \subset \mathbb{R}^3$ without boundary such that $\mathcal{W}(S) < 8\pi$. Then Corollary 1.3.8 implies that S is embedded. Considering any suitable plane π that intersects S in finitely many disjoint compact embedded curves $\gamma^1, ..., \gamma^\alpha$, we get that one halfspace determined by π contains a piece Σ of S with $\mathcal{W}(\Sigma) < 4\pi$ and $\partial\Sigma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$. Denoting co_Σ the conormal field of Σ we get that problems

$$\mathcal{P} := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ \sigma_V = co_\Sigma \mathcal{H}^1 \llcorner \partial\Sigma, \ \mathrm{spt}V \cup \partial\Sigma \ \text{ compact, connected} \right\},$$

$$\mathcal{Q} := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ |\sigma_V| \le \mathcal{H}^1 \llcorner \partial\Sigma, \ \mathrm{spt}V \cup \partial\Sigma \ \text{ compact, connected} \right\},$$

and suitably small perturbations $\mathcal{P}_\varepsilon$, $\mathcal{Q}_\varepsilon$ of them have minimizers.

Remark 4.3.14. Suppose that $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ is a disjoint union of smooth embedded closed curves and that γ is contained in some sphere $\mathbb{S}^2_R(c)$ of radius R and center c in $\mathbb{R}^3$. Up to translation let $c = 0$. If there is a point $N \in \mathbb{S}^2_R(0)$ such that for any i the image $\pi_N(\gamma^i)$ via the stereographic projection $\pi_N : \mathbb{S}^2_R(0) \setminus \{N\} \to \mathbb{R}^2$ is homotopic to a point in $\mathbb{R}^2 \setminus \cup_{i=1}^\alpha \pi_N(\gamma^i)$, then the problem

$$\mathcal{Q} := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ |\sigma_V| \le \mathcal{H}^1 \llcorner \gamma, \ \mathrm{spt}V \cup \gamma \ \text{ compact, connected} \right\},$$

has minimizers. Indeed under such assumption there exists a connected submanifold Σ of $\mathbb{S}^2_R(0)$ with $\partial\Sigma = \gamma$, thus $\mathcal{W}(\Sigma) < 4\pi$ and Theorem 4.3.9 applies.

Remark 4.3.15. Coming back to a motivating example discussed at in Section 4.3, for given $R \ge 1$ and $h > 0$ let us consider the curves

$$\Gamma_{R,h} = \{x^2 + y^2 = 1, z = h\} \cup \{x^2 + y^2 = R^2, z = -h\}.$$

Suppose that $h_0 > 0$ is the critical value for which a connected minimal surface Σ with $\partial\Sigma = \Gamma_{R,h}$ exists if and only if $h \le h_0$. Let Σ_0 be a minimal surface with $\partial\Sigma_0 = \Gamma_{R,h_0}$. Applying Corollary 4.3.12 we get that for $\varepsilon > 0$ sufficiently small the minimization problem

$$\mathcal{Q}_\varepsilon := \min \left\{ \mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : \ |\sigma_V| \le \mathcal{H}^1 \llcorner \Gamma_{R,h_0+\varepsilon}, \ \mathrm{spt}V \cup \Gamma_{R,h_0+\varepsilon} \ \text{ compact, connected} \right\}$$

has minimizers.

Let us anticipate that in the case of boundary data of the form $\Gamma_{R,h}$ we will see in Corollary 4.3.19 that actually existence of minimizers for the problem $\mathcal{Q}_\varepsilon$ is guaranteed for any $\varepsilon > 0$, indeed we will see that the hypotheses implying existence of minimizers actually hold for the boundary datum $\Gamma_{R,h}$ for any $h > 0$.

In the rest of the section we want to discuss what is possible to prove about minimizers of the problems in Theorem 4.3.8 and Theorem 4.3.9 in an asymptotic regime where the diameter of the support of the varifold is very large if compared with the length of the assigned boundary curves. As the Willmore energy is invariant under rescaling, this is equivalent to investigate what happens in case the length of the assigned boundary curves goes to zero.

First we need to prove an analogous of Theorem 4.1.1 in the setting of varifolds. More precisely, we are going to prove that, also among varifolds without boundary and compact support, the infimum of the Willmore energy is only achieved by spheres. This result is certainly expected by experts in the field, but, up our knowledge, it has not been proved yet without appealing to highly non-trivial and recent regularity theorems. Instead, our only tool will be the monotonicity formula of Theorem 1.3.7.

Proposition 4.3.16 (Uniqueness of global minimizers among varifolds)**.** *Let $V = \mathbf{v}(M, \theta_V)$ be an integer rectifiable varifold with $\sigma_V = 0$ and such that* $\mathrm{spt}V$ *is compact. If $\mathcal{W}(V) = 4\pi$, then $V = \mathbf{v}(\mathbb{S}^2_R(z), 1)$ for some 2-sphere $\mathbb{S}^2_R(z) \subset \mathbb{R}^3$.*

Proof. Passing to the limits $\sigma \to 0$ and $\rho \to \infty$ in the monotonicity formula (1.15) we get that

$$4\pi\theta_V(p_0) + 4\int_M \left|\frac{H}{2} + \frac{(p-p_0)^\perp}{|p-p_0|^2}\right|^2 d\mu_V = 4\pi,$$

for any $p_0 \in \mathbb{R}^3$, where we are assuming by Theorem 1.3.7 that θ_V coincides with the 2-dimensional density of μ_V and M coincides with the support of V. Hence $\theta_V(p_0) = 1$ for any $p_0 \in M$, and also

$$H(p) = -2\frac{(p-p_0)^\perp}{|p-p_0|^2}, \tag{4.23}$$

at $\mathcal{H}^2$-ae $p \in M$ and for every $p_0 \in M$.

Fix $\delta > 0$ small and two points $p_1, p_2 \in M$ with $p_2 \notin B_{2\delta}(p_1)$. For $\mathcal{H}^2$-ae $p \in M$ we can write

$$H(p) = \begin{cases} -2\frac{(p-p_1)^\perp}{|p-p_1|^2} & p \notin B_\delta(p_1), \\ -2\frac{(p-p_2)^\perp}{|p-p_2|^2} & p \notin B_\delta(p_2). \end{cases}$$

Since M is bounded, we get that $H \in L^\infty(\mu_V)$. Therefore, since $\theta_V = 1$ on M, by the Allard Regularity Theorem (see [Sim83b, Chapter 5, Theorem 5.2]) we get that M is a closed surface of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$. Since M is closed, it is also compact, and thus it is connected, for otherwise $\mathcal{W}(V) \geq 8\pi$.

Now let $p \in M$ be any fixed point such that (4.23) holds, and denote by ν_p a unit vector such that $\nu_p^\perp = T_pM$. Up to translation let $p = 0$. Consider the axis generated by ν_0 and any point $p_0 \in M \setminus \{0\}$. We can write $p_0 = q + w$ with $q = \alpha\nu_0$ and $\langle w, \nu_0\rangle = 0$. Writing analogously $(q + w') \in M \setminus \{0\}$ another point with the same component on the axis generated by ν_0, (4.23) implies that

$$-2\frac{-\langle q,\nu_0\rangle\nu_0}{|q|^2+|w|^2} = -2\frac{(0-q-w)^{\perp_0}}{|q+w|^2} = H(0) = -2\frac{(0-q-w')^{\perp_0}}{|q+w'|^2} = -2\frac{-\langle q,\nu_0\rangle\nu_0}{|q|^2+|w'|^2}.$$

Hence, whenever $q \neq 0$, we have that $|w| = |w'|$; that is points in M of the form $\alpha\nu_0 + w$ with $\alpha \neq 0$ and $w \in \nu_0^\perp$ lie on a circle. It follows that M is invariant under rotations about the axis $\{t\nu_0 \mid t \in \mathbb{R}\}$. This argument works at $\mathcal{H}^2$-almost any point of M. Therefore we have that for any $p \in M$, the set $M \setminus \nu_p^\perp$ is invariant under rotations about the axis $p + \{t\nu_p \mid t \in \mathbb{R}\}$.

Still assuming $0 \in M$, up to rotation suppose that $\nu_0 = (0,0,1)$. Let $a \in M$ be such that $\nu_a = (1,0,0)$. There exists a point $b \in M$ such that $b = t\nu_0 = (0,0,t)$ for some $t \in \mathbb{R} \setminus \{0\}$. We can write $0 = q + w$ and $b = q + w'$ for the same $q \in a + \{t\nu_a \mid t \in \mathbb{R}\}$ and some $w, w' \in \nu_a^\perp$. Since $|w| = |w'|$, it follows that $q \neq 0$, otherwise $b = 0$. Since $q \neq 0$, the rotation of the origin about the axis $a + \{t\nu_a \mid t \in \mathbb{R}\}$ implies that M contains a circle C of radius $r > 0$ passing through the origin, and the plane containing C is orthogonal to $\nu_0^\perp$. Since M is of class C^1, by rotational invariance about the axis $\{t\nu_0 \mid t \in \mathbb{R}\}$, the circle C has to be tangent at 0 to the subspace $\nu_0^\perp$. Thus by invariance with respect to the rotation about the axis $\{t\nu_0 \mid t \in \mathbb{R}\}$, we have that M contains the sphere S of positive radius given by the rotation of C about $\{t\nu_0 \mid t \in \mathbb{R}\}$. Since M is of class $C^{1,\alpha}$ and the Willmore energy of a sphere is 4π, it follows that M coincides with such a sphere S. $\square$

Now we can prove the above mentioned result on the asymptotic behavior of connected varifolds. Under suitable assumptions we can prove that if a sequence of varifolds is such that the "length" of their boundaries becomes negligible with respect to the diameter of their supports, then the sequence converges to a sphere, up to translation and rescaling.

Corollary 4.3.17. *Let $V_n = \mathbf{v}(M_n, \theta_{V_n})$ be a sequence of integer rectifiable curvature varifolds with boundary satisfying the hypotheses of the Compactness Theorem 1.2.20 for curvature varifolds with $k=2$. Suppose that M_n is compact and connected for any n. If*

$$\begin{aligned}
&\mathcal{W}(V_n) \le 4\pi + o(1) \qquad \text{as } n \to \infty,\\
&\operatorname{diam}(\operatorname{spt}V_n) \xrightarrow[n\to\infty]{} +\infty,\\
&\limsup_n \frac{|\sigma_{V_n}|(\mathbb{R}^3)}{\operatorname{diam}(\operatorname{spt}V_n)} = 0,
\end{aligned}$$

and $\operatorname{spt}\sigma_{V_n}$ is a disjoint union of uniformly finitely many closed embedded smooth curves with uniformly bounded length, then the sequence

$$\widetilde{V}_n := \mathbf{v}\left(\frac{M_n}{\operatorname{diam}(\operatorname{spt}V_n)}, \widetilde{\theta}_n\right)$$

where $\widetilde{\theta}_n(x) = \theta_{V_n}(\operatorname{diam}(\operatorname{spt}V_n)\,x)$, converges up to subsequence and translation to the varifold

$$V = \mathbf{v}(\mathbb{S}, 1),$$

where $\mathbb{S}$ is a sphere of diameter 1, in the sense of varifolds and in Hausdorff distance.

Proof. Up to translation let us assume that $0 \in \operatorname{spt}V_n$. Then $\operatorname{spt}\widetilde{V}_n$ is uniformly bounded with $\operatorname{diam}(\operatorname{spt}\widetilde{V}_n) = 1$. We have that

$$2\mu_{\widetilde{V}_n}(\mathbb{R}^3) = \int \operatorname{div}_{T\widetilde{V}_n} p \, d\mu_{\widetilde{V}_n}(p) \le C\mathcal{W}(\widetilde{V}_n)^{\frac{1}{2}} \left(\mu_{\widetilde{V}_n}(\mathbb{R}^3)\right)^{\frac{1}{2}} + C\frac{|\sigma_{V_n}|(\mathbb{R}^3)}{\operatorname{diam}(\operatorname{spt}V_n)},$$

and thus Theorem 1.2.20 implies that $\widetilde{V}_n$ converges to a limit varifold V (up to subsequence). Also $\sigma_{\widetilde{V}_n} \overset{\star}{\rightharpoonup} \sigma_V$, and thus $|\sigma_V|(\mathbb{R}^3) \le \liminf_n |\sigma_{\widetilde{V}_n}|(\mathbb{R}^3) \le \limsup_n \frac{|\sigma_{V_n}|(\mathbb{R}^3)}{\operatorname{diam}(\operatorname{spt}V_n)} = 0$; hence V has compact support and no generalized boundary.

Let us say that $\operatorname{spt}\sigma_{\widetilde{V}_n}$ is the disjoint union of the smooth closed curves $\gamma_n^1, ..., \gamma_n^\alpha$. By the uniform boundedness of $\operatorname{spt}\widetilde{V}_n$, we get that γ_n^i converges to some compact set X^i in $d_{\mathcal{H}}$ up to subsequence by Blaschke Theorem (see [Fal86, Theorem 3.16]). Each X_i is connected by Lemma 4.3.1, then by Gołab Theorem ([Fal86, Theorem 3.18]) we know that $\mathcal{H}^1(X^i) \le \liminf_n \mathcal{H}^1(\gamma_n^i) = 0$, hence $X^i = \{p_i\}$ for any i for some points $p_1, ..., p_\alpha$, and we can assume that $p_i \neq 0$ for any $i = 1, ..., \alpha$.

Using ideas from the proof of Theorem 4.3.4, we can now show that $V \neq 0$. Indeed suppose by contradiction that $V = 0$. Fix $N \in \mathbb{N}$ with $N \ge 4$. By connectedness of M_n, since $\operatorname{diam}(\operatorname{spt}\widetilde{V}_n) \to 1$, and the boundary curves converge to a discrete sets, for $j = 1, ..., N$ there are points $z_{n,j} \in \partial B_{\frac{j}{2N}}(0) \cap \operatorname{spt}\widetilde{V}_n$ for n large. We can also choose N so that $d(z_{n,j}, \operatorname{spt}\sigma_{\widetilde{V}_n}) \ge \delta(N) > 0$ for n large. The open balls $\left\{B_{\frac{1}{4N}}(z_{n,j})\right\}_{j=1}^N$ are pairwise disjoint. Using Young inequality as in Theorem 4.3.4 in the monotonicity formula (1.15) applied on $\widetilde{V}_n$ at points $z_{n,j}$ with $\sigma \to 0$ and

$\rho = \frac{1}{4N}$ gives

$$\pi \le \frac{3}{2}\frac{\mu_{\widetilde{V}_n}(B_{\frac{1}{4N}}(z_{n,j}))}{\left(\frac{1}{4N}\right)^2} + \frac{3}{4}\int_{B_{\frac{1}{4N}}(z_{n,j})} |H_{\widetilde{V}_n}|^2 \, d\mu_{\widetilde{V}_n} + \\ + \frac{1}{2}\left| \int \left(\frac{1}{|p-z_{n,j}|^2} - \frac{1}{\left(\frac{1}{4N}\right)^2} \right) (p - z_{n,j}) \, d\sigma_{\widetilde{V}_n}(p) \right|, \tag{4.24}$$

for any n and $j = 1, ..., N$. Since $V = 0$ we have that

$$\limsup_n \mu_{\widetilde{V}_n}(B_{\frac{1}{4N}}(z_{n,j})) \le \limsup_n \mu_{\widetilde{V}_n}(\overline{B_2(0)}) = 0.$$

Also

$$\left| \int \left(\frac{1}{|p-z_{n,j}|^2} - \frac{1}{\left(\frac{1}{4N}\right)^2} \right) (p - z_{n,j}) \, d\sigma_{\widetilde{V}_n}(p) \right| \le C(\delta(N), N) |\sigma_{\widetilde{V}_n}|(\mathbb{R}^3) \xrightarrow[n\to\infty]{} 0.$$

Hence summing on $j = 1, ..., N$ in (4.24) and passing to the limit $n \to \infty$ we get

$$4\pi \le N\pi \le \frac{3}{4} \lim_n \mathcal{W}(\widetilde{V}_n) \le 3\pi,$$

that gives a contradiction. Therefore we can apply Theorem 4.3.4 to conclude that $\mathrm{spt}\widetilde{V}_n$ converges to M in $d_{\mathcal{H}}$. Finally, since V is a compact varifold without generalized boundary and

$$4\pi \le \mathcal{W}(V) \le \liminf_n \mathcal{W}(V_n) = 4\pi,$$

by Proposition 4.3.16 we conclude that V is a round sphere of multiplicity 1. By Lemma 4.3.2 the diameter of M is the limit $\lim_n \mathrm{diam}(\mathrm{spt}\widetilde{V}_n) = 1$. □

4.3.3 The case of two coaxial circles

In this section we want to discuss more in detail how the existence theorems 4.3.8 and 4.3.9 and the asymptotic behavior described in Corollary 4.3.17 relate with the remarkable case that motivates our study, namely the immersions in the class

$$\mathscr{F}_{R,h} := \left\{ \varphi : \mathscr{C} \to \mathbb{R}^3 \,|\, \varphi \text{ smooth immersion}, \ \varphi|_{\partial\mathscr{C}} : \partial\mathscr{C} \to \Gamma_{R,h} \text{ smooth embedding} \right\}.$$

where $\mathscr{C} = [0,1]^2/_\sim$ is a cylinder, $R \ge 1$, $h > 0$, and

$$\Gamma_{R,h} := \left\{ x^2 + y^2 = 1, z = h \right\} \cup \left\{ x^2 + y^2 = R^2, z = -h \right\}, \qquad R \ge 1, \quad h > 0,$$

that is a disjoint union of two parallel circles of possibly different radii.

First, the monotonicity formula provides the following estimates on immersions $\varphi \in \mathscr{F}_{R,h}$.

Lemma 4.3.18. *Fix $R \ge 1$ and $h > 0$. It holds that:*

1.

$$\inf \left\{ \mathcal{W}(\varphi) \mid \varphi \in \mathscr{F}_{R,h} \right\} \le 4\pi \frac{4h^2 + R^2 - 1}{\sqrt{(4h^2 + R^2 - 1)^2 + 16h^2}} < 4\pi,$$

2.

$$\lim_{h\to\infty} \inf \left\{ \mathcal{W}(\varphi) \mid \varphi \in \mathscr{F}_{R,h} \right\} = 4\pi.$$

Proof. We prove the items separately.

1. We can consider as competitor in $\mathscr{F}_{R,h}$ the truncated sphere

$$\Sigma = \mathbb{S}^2_{\sqrt{1+(z_0-h)^2}}(z_0) \cap \{|z| \le h\},$$

where $z_0 = \left(0, 0, \frac{1-R^2}{4h}\right)$ is the point on the z-axis located at the same distance from the two connected components of $\Gamma_{R,h}$. The surface Σ is contained in another truncated sphere Σ' having the same center and radius and symmetric with respect to the plane $\{z = \frac{1-R^2}{4h}\}$. The boundary of Σ' is the disjoint union of two circles of radius 1. Hence we simply have

$$\mathcal{W}(\Sigma) \le \mathcal{W}(\Sigma') = 4\pi \frac{4h^2 + R^2 - 1}{\sqrt{(4h^2 + R^2 - 1)^2 + 16h^2}}.$$

2. Let $\varphi \in \mathscr{F}_{R,h}$ and $\Sigma = \varphi(\mathscr{C})$. By connectedness there is a point $p \in \Sigma \setminus \partial\Sigma$ lying in the plane $z = 0$. Hence $d_{\mathcal{H}}(\Sigma, \partial\Sigma) \ge h$, and by (1.24) we have

$$4\pi \le \mathcal{W}(\Sigma) + 2\frac{2\pi(1+R)}{h}.$$

This holds for any $\varphi \in \mathscr{F}_{R,h}$. Then $4\pi \le \inf\left\{\mathcal{W}(\varphi) \mid \varphi \in \mathscr{F}_{R,h}\right\} + \frac{4\pi(1+R)}{h}$ and the thesis follows by Item 1 and by letting $h \to \infty$.

□

We already discussed in Remark 4.3.15 the existence of minimization problems arising by perturbations of minimal catenoids in some $\mathscr{F}_{R,h}$. By Lemma 4.3.18 we can complete the picture about existence of optimal connected elastic surfaces with boundary $\Gamma_{R,h}$ for any $R \ge 1$ and $h > 0$, as well as the asymptotic behavior of almost optimal surfaces having such boundaries.

Corollary 4.3.19. *Fix $R \ge 1$ and $h > 0$.*

1. *The minimization problem*

$$\mathcal{Q}_{R,h} := \min\Big\{\mathcal{W}(V) \mid V = \mathbf{v}(M, \theta_V) : |\sigma_V| \le \mathcal{H}^1 \llcorner \Gamma_{R,h}, \\ \operatorname{spt} V \cup \Gamma_{R,h} \ \ \textit{compact, connected}\Big\}$$

has minimizers.

2. *Let $h_k \to \infty$ be any sequence. Let $\Sigma_k = \varphi_k(\mathscr{C})$ for $\varphi_k \in \mathscr{F}_{R,h_k}$. Suppose that $\mathcal{W}(\varphi_k) \le 4\pi + o(1)$ as $k \to \infty$. Let $S_k = \frac{\Sigma_k}{\operatorname{diam}\Sigma_k}$.*

   Then (up to subsequence) S_k converges in Hausdorff distance to a sphere $\mathbb{S}$ of diameter 1, and the varifolds corresponding to S_k converge to $V = \mathbf{v}(\mathbb{S}, 1)$ in the sense of varifolds.

Proof. Item 1 follows from [Lemma 4.3.18, Item 1] by applying Corollary 4.3.11. So we need to prove Item 2. Identifying S_k with the varifold it defines, we estimate the total variation of the boundary measure by $|\partial S_k| \le \frac{\mathcal{H}^1(\Gamma_{R,h_k})}{\operatorname{diam}\Sigma_k}$. Moreover, by Remark 1.1.11, the L^2-norm of the second fundamental form of S_k is uniformly bounded. Hence Corollary 4.3.17 applies and the thesis follows. □

Using the notation of Item 2 in Corollary 4.3.19, we remark that even if we know that the rescalings S_k converge to a sphere in $d_{\mathcal{H}}$ and as varifolds, it remains open the question whether at a scale of order h_k the sequence Σ_k is actually "close to" a big sphere. More precisely it seems a delicate issue to understand if $\mathrm{diam}\Sigma_k \sim 2h_k$ as $k \to \infty$. We conclude with the following partial result in this direction: the monotonicity formula gives us some evidence in the case we assume that $\frac{\mathrm{diam}\Sigma_k}{h_k} \to \infty$.

Proposition 4.3.20. *Let $\Sigma_k = \varphi_k(\mathscr{C})$ for $\varphi_k \in \mathscr{F}_{R,h_k}$. Suppose that $\mathcal{W}(\varphi_k) \le 4\pi + o(1)$ as $k \to \infty$. Let $M_k = \frac{\Sigma_k}{h_k}$.*

Then M_k converges up to subsequence to a curvature varifold $Z = \mathbf{v}(M, \theta_Z)$ in the sense of varifolds. If also

$$\frac{\mathrm{diam}\Sigma_k}{h_k} \to \infty,$$

then M is a plane containing the axis $\{(0,0,t) \,|\, t \in \mathbb{R}\}$ and $\theta_Z \equiv 1$ on M.

Proof. We identify M_k with the varifold it defines. First we can establish the convergence up to subsequence in the sense of varifolds by using Theorem 1.2.20. Indeed we have that $\mathcal{H}^1(\partial M_k) \to 0$, $\int_{M_k} |B_{M_k}|^2$ is scaling invariant and thus finite by Remark 1.1.11. Moreover, since $d(0, \partial M_k) \ge 1$, by the monotonicity formula (1.15) we get that

$$\begin{aligned}
\frac{\mu_{M_k}(B_\sigma(0))}{\sigma^2} \le & -\frac{1}{\sigma^2}\int_{B_\sigma(0)} \langle H_{M_k}, p\rangle \, d\mu_{M_k}(p) - \frac{1}{2}\int_{B_\sigma(0)\cap \partial M_k} \left(\frac{1}{|p|^2} - \frac{1}{\sigma^2}\right) \langle p, co_{M_k}(p)\rangle \, d\mathcal{H}^1(p) \\
& + \lim_{\rho\to\infty} A_{M_k,0}(\rho) \\
\le & \; \pi + o(1) + \frac{1}{\sigma^2}\int_{B_\sigma(0)} |p||H_{M_k}| \, d\mu_{M_k}(p) + \frac{1}{2}\int_{\partial M_k \setminus B_\sigma(0)} \frac{d\mathcal{H}^1(p)}{|p|} \\
& + \frac{1}{2\sigma^2}\int_{\partial M_k \cap B_\sigma(0)} |p| \, d\mathcal{H}^1(p) \\
\le & \; \pi + o(1) + \frac{1}{\sigma}\mu_{M_k}(B_\sigma(0))^{\frac{1}{2}} \mathcal{W}(M_k)^{\frac{1}{2}} + \frac{1}{2}\mathcal{H}^1(\partial M_k) + \frac{1}{2\sigma}\mathcal{H}^1(\partial M_k),
\end{aligned}$$

and therefore $\mu_{M_k}(B_\sigma(0)) \le C(\sigma)$ for any $\sigma \ge 1$. Hence the hypotheses of Theorem 1.2.20 are satisfied and we denote $Z = \mathbf{v}(M, \theta_Z)$ the limit curvature varifold of M_k. Observe that $\sigma_Z = 0$ and $\mathcal{W}(Z) < +\infty$.

From now on assume that $\mathrm{diam}\Sigma_k / h_k \to \infty$. Arguing as in the proof of Corollary 4.3.17 we can prove that $Z \ne 0$. Indeed suppose by contradiction that $Z = 0$. Fix $N \in \mathbb{N}$ with $N \ge 4$. By connectedness of M_k, for $j = 1, ..., N$ there are points $z_{k,j} \in \partial B_{\frac{j}{N}}(0,0,1) \cap M_k$ and $z_{k,j} \notin \partial M_k$ for k large. The open balls $\left\{ B_{\frac{1}{2N}}(z_{k,j}) \right\}_{j=1}^N$ are pairwise disjoint. Hence the monotonicity formula (1.15) applied on M_k at points $z_{k,j}$ with $\sigma \to 0$ and $\rho = \frac{1}{2N}$ gives

$$\pi \le \frac{3}{2}\frac{\mu_{M_k}(B_{\frac{1}{2N}}(z_{k,j}))}{\left(\frac{1}{2N}\right)^2} + \frac{3}{4}\int_{B_{\frac{1}{2N}}(z_{k,j})} |H_{M_k}|^2 \, d\mu_{M_k}, \tag{4.25}$$

for any k and $j = 1, ..., N$. Since $Z = 0$ we have that

$$\limsup_k \mu_{M_k}(B_{\frac{1}{2N}}(z_{k,j})) \le \limsup_k \mu_{M_k}(B_2(0,0,1)) = 0.$$

Hence, summing on $j = 1, ..., N$ in (4.25) and passing to the limit $k \to \infty$ we get

$$4\pi \le N\pi \le \frac{3}{4}\lim_k \mathcal{W}(M_k) \le 3\pi,$$

that gives a contradiction.

Also the support of Z is unbounded. Indeed suppose by contradiction that $\mathrm{spt}Z \subset\subset B_R(0)$, and thus we can take $M = \mathrm{spt}Z$ to be compact by Theorem 1.3.7. Since M_k is connected, there exists $q'_k \in M_k \cap \partial B_{2R}(0)$ definitely in k for R sufficiently big. Up to subsequence $q'_k \to q'$. By Remark 4.3.5 we get that $q' \in \mathrm{spt}Z$, that contradicts the absurd hypothesis.

Since M is unbounded, by Corollary 1.3.9 we know that

$$\lim_{\rho\to\infty} \frac{\mu_Z(B_\rho(q))}{\rho^2} \geq \pi.$$

By construction

$$\lim_k \int_{B_\sigma(0)\cap\partial M_k} \left\langle \frac{p}{|p|^2}, co_{M_k} \right\rangle d\mathcal{H}^1(p) = 0,$$

hence passing to the limit $k \to \infty$ in the monotonicity formula (1.15) we get that

$$A_{Z,0}(\sigma) \leq \liminf_k A_{M_k,0}(\sigma),$$

for almost every $\sigma > 0$. By monotonicity

$$A_{Z,0}(\sigma) \leq \liminf_k \lim_{\sigma\to\infty} A_{M_k,0}(\sigma) \leq \liminf_k \frac{\mathcal{W}(M_k)}{4} + \mathcal{H}^1(\partial M_k) \leq \pi.$$

On the other hand, using (1.20), we have that

$$\lim_{r\to\infty} A_{Z,0}(r) = \frac{1}{4}\mathcal{W}(Z) + \lim_{r\to\infty} \frac{\mu_Z(B_r(q))}{r^2} \geq \frac{1}{4}\mathcal{W}(Z) + \pi.$$

Hence Z is stationary, $\lim_{\rho\to\infty} \frac{\mu_Z(B_\rho(q))}{\rho^2} = \pi$, and M can be taken to be closed by Theorem 1.3.7. If p_0 is any point in M, the monotonicity formula for Z centered at p_0 then reads

$$\frac{\mu_Z(B_\sigma(p_0))}{\sigma^2} + \int_{B_\rho(p_0)\setminus B_\sigma(p_0)} \frac{|(p-p_0)^\perp|^2}{|p-p_0|^4} = \frac{\mu_Z(B_\rho(q))}{\rho^2}. \tag{4.26}$$

In particular $\theta_Z(p_0) = 1$, and thus we can apply Allard Regularity Theorem (see [Sim83b, Chapter 7, Theorem 7.1]) at p_0. Thus we get that M is of class C^∞ around p_0 (and analogously everywhere), and thus there exists the limit

$$\lim_{\sigma\to 0} \int_{B_\rho(p_0)\setminus B_\sigma(p_0)} \frac{|(p-p_0)^\perp|^2}{|p-p_0|^4} = \int_{B_\rho(p_0)} \frac{|(p-p_0)^\perp|^2}{|p-p_0|^4}.$$

Passing to the limits $\rho \to \infty$ and $\sigma \searrow 0$ in (4.26), we get that

$$\lim_{\rho\to\infty} \int_{B_\rho(p_0)} \frac{|(p-p_0)^\perp|^2}{|p-p_0|^4} = 0.$$

Therefore $|(p-p_0)^\perp| = 0$ for any $p \in M$, where we recall that $(\cdot)^\perp$ is the orthogonal projection on $T_pM^\perp$. Since this is true for any $p_0 \in M$, we conclude that M is a plane. Finally Remark 4.3.5 implies that M contains the vertical axis $\{(0,0,t) \,|\, t \in \mathbb{R}\}$. □

4.3.4 Further results on the Helfrich energy

In this final section we mention two additional results following from the proofs of the existence theorems 4.3.8 and 4.3.9. We want to introduce a new Willmore-type energy, which is known as *Helfrich energy*, or *Canham–Helfrich energy*.

If $\varphi : M^2 \hookrightarrow \mathbb{R}^3$ is an isometric immersion of an oriented surface M (possibly with boundary), since $\mathbb{R}^3$ is oriented, there is a unique unit smooth normal vector field $\nu : M \to TM^\perp$ along φ such that $[d\varphi_x(e_1), d\varphi_x(e_2), \nu(x)]$ is an oriented basis of $\mathbb{R}^3$ for any $x \in M$ and any oriented basis $[e_1, e_2]$ of T_xM. We can then define $\overline{H} := \langle H, \nu \rangle$.

In the above setting, for fixed constants $\lambda > 0$ and $H_0 \in \mathbb{R}$, we define the *Helfrich energy* $\mathscr{H}_{\lambda,H_0}(\varphi)$ *of* φ by

$$\mathscr{H}_{\lambda,H_0}(\varphi) := \int_M \lambda + (\overline{H} - H_0)^2 \, d\mu_\varphi.$$

The above functional appeared in [Can70] and [Hel73] as a good model for the bending energy of the membranes of biological cells or lipids. In fact, we can look at the Willmore energy as a particular case of Helfrich energy.

From a variational point of view, it is clearly equivalent to consider $\lambda = 1$ and $H_0 = 0$, therefore obtaining the functional

$$\mathscr{H}(\varphi) := \int_M 1 + |H|^2 \, d\mu_\varphi,$$

for φ as before, but M not necessarily orientable. If V is an integer rectifiable varifold in $\mathbb{R}^3$ with mean curvature H, we then naturally define

$$\mathscr{H}(V) := \mu_V(\mathbb{R}^3) + \mathcal{W}(V).$$

Very recent contributions in the variational theory about the Helfrich energy are contained in [MS20a; Eic19; Eic20], in which existence theorems and lower semicontinuity properties are proved under different constraints on area, enclosed volume, topolgy, or boundary.

We shall now extend Theorem 4.3.8 and Theorem 4.3.9 to the Helfrich energy $\mathscr{H}$. Actually, the fact that $\mathscr{H}$ takes into account the mass $\mu_V(\mathbb{R}^3)$ is a simplification in variational purposes, as it immediately offers a uniform bound on the "area" of the varifolds of a minimizing sequence.

First, let us prove the following preliminary result.

Proposition 4.3.21. *Let $V = \mathbf{v}(M, \theta_V)$ be an integer rectifiable varifold in $\mathbb{R}^3$ with $\mathcal{W}(V) + |\sigma_V|(\mathbb{R}^3) \le C_0 < +\infty$ with $\operatorname{spt}|\sigma_V|$ compact and $\mathcal{H}^2(\operatorname{spt}\sigma_V) = 0$. Then*

1. *If $\mu_V(\mathbb{R}^3) \le C_1$, then $\operatorname{spt}V$ is compact and $\operatorname{diam}(K) \le C(C_0, C_1, \operatorname{diam}(\operatorname{spt}\sigma_V))$ for any connected component K of $\operatorname{spt}V$.*

2. *If $\operatorname{diam}(\operatorname{spt}V) \le C_3$, then $\mu_V(\mathbb{R}^3) \le C(C_0, C_3)$.*

Proof. We prove the items separately.

1. Since $\mu_V(\mathbb{R}^3)$ is finite, by Theorem 1.3.7 we can assume that $M = \operatorname{spt}V$ and that θ_V is pointwise well defined on $\mathbb{R}^3 \setminus \operatorname{spt}\sigma_V$ and coincides with the 2-dimensional density of μ_V. By Corollary 1.3.9 we have that M is bounded, and then compact. Let K be a connected component of M. Hence K is compact and there is an open set U containing K such that $U \cap M = K$. Thus $\mathbf{v}(K, \theta_V)$ is a varifold with mean curvature in U, and then in $\mathbb{R}^3$. Therefore we can assume without loss of generality that $M = K$ is connected.

Let us assume that $\sigma_V \neq 0$, the case $\sigma_V = 0$ being simpler. If there exist points $\xi_i \in M \setminus \mathrm{spt}\sigma_V$ such that the balls $\{B_1(\xi_i)\}$ are disjoint, passing to the limit $\sigma \to 0$ and taking $\rho = 1$ in (1.15) on the monotone function $A_{V,\xi_i}(\cdot)$ we find

$$\pi \leq \mu_V(B_1(\xi_i)) + \frac{1}{4}\int_{B_1(\xi_i)} |H|^2 \, d\mu_V + R_{V,\xi_i}(1). \tag{4.27}$$

Suppose now that there is a point $z \in M$ such that $d(z, \mathrm{spt}\sigma_V) \geq 2N \in \mathbb{N}$. Then, since M is connected, for $i = 1, ..., N-1$ there are points $\xi_i \in M \cap \partial B_{2i}(z)$ such that the balls $\{B_1(\xi_i)\}$ are disjoint and also $B_1(\xi_i) \cap \mathrm{spt}\sigma_V = \emptyset$. Therefore, for such points we have

$$|R_{V,\xi_i}(1)| \leq \int_{B_1(\xi_i)} |\langle H(x), x - \xi_i\rangle| \, d\mu_V(x) \leq \frac{1}{2}\left(\int_{B_1(\xi_i)} |H|^2 \, d\mu_V + \mu_V(B_1(\xi_i))\right).$$

Summing over i in (4.27) we get

$$(N-1)\pi \leq \frac{3}{2}\mu_V(\mathbb{R}^3) + \frac{3}{4}\mathcal{W}(V),$$

and then $N \leq C(C_0, C_1)$. It follows that $\mathrm{diam}(M) \leq C(C_0, C_1, \mathrm{diam}(\mathrm{spt}\sigma_V))$.

2. As $\mathrm{spt}V$ is bounded, then $\mu_V(\mathbb{R}^3) = \mu_V(B_{R_0}(0))$ for some $R_0 > 0$ is finite, and thus by Theorem 1.3.7 we can assume that $M = \mathrm{spt}V$ and that θ_V is pointwise well defined on $\mathbb{R}^3 \setminus \mathrm{spt}\sigma_V$ and coincides with the 2-dimensional density of μ_V. Up to translation, we can assume that $0 \in M$ and that $\frac{1}{2}R_0 \leq \mathrm{diam}(M) \leq 2R_0$.

   We now argue as in the proof of Theorem 4.3.8. Let $X(p) = \chi(p)\, p \in C^\infty_c(\mathbb{R}^3, \mathbb{R}^3)$ be a vector field, where $\chi \in C^\infty_c(\mathbb{R}^3)$ is a cut off function such that $\chi(p) = 1$ for $p \in B_{R_0}(0)$. Then

$$\begin{aligned} 2\mu_V(\mathbb{R}^3) &= \int \mathrm{div}_{TM} X \, d\mu_V = -2\int \langle H, X\rangle \, d\mu_V + \int X \, d\sigma_V \\ &\leq 2R_0 \left(\mathcal{W}(V)\right)^{\frac{1}{2}} \left(\mu_V(\mathbb{R}^3)\right)^{\frac{1}{2}} + R_0|\sigma_V|(\mathbb{R}^3), \end{aligned}$$

   and the thesis follows.

□

Proposition 4.3.21 implies that, once a sequence V_n of varifolds with fixed boundary has bounded Willmore energy, then a bound on $\mu_{V_n}(\mathbb{R}^3)$ is equivalent to a bound on the diameter of $\mathrm{spt}V_n$.

Hence we can prove the following existence theorem using Proposition 4.3.21 and the arguments already used in Theorem 4.3.8 and Theorem 4.3.9.

Theorem 4.3.22. *Let $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ be a disjoint union of smooth embedded closed curves with $\alpha \in \mathbb{N}_{\geq 2}$. Let*

$$\sigma_0 = m\,\nu_0\,\mathcal{H}^1 \llcorner \gamma$$

be a vector valued Radon measure, where $m : \gamma \to \mathbb{N}_{\geq 1}$ and $\nu_0 : \gamma \to (T\gamma)^\perp$ are $\mathcal{H}^1$-measurable functions with $m \in L^\infty(\mathcal{H}^1 \llcorner \gamma)$ and $|\nu_0| = 1$ $\mathcal{H}^1$-ae. Let $\mathcal{P}_{\mathscr{H}}$ be the minimization problem

$$\mathcal{P}_{\mathscr{H}} := \min\left\{\mathscr{H}(V) \quad | \quad V = \mathbf{v}(M, \theta_V): \quad \sigma_V = \sigma_0, \quad \mathrm{spt}V \cup \gamma \ \ \textit{compact, connected}\right\}.$$

If $\inf \mathcal{P}_{\mathscr{H}} < +\infty$, then $\mathcal{P}_{\mathscr{H}}$ has minimizers.

Proof. Since $\inf \mathcal{P}_{\mathscr{H}} < +\infty$, there exists a minimizing sequence V_n for $\mathcal{P}_{\mathscr{H}}$. Then $\mu_{V_n}(\mathbb{R}^3)$ is uniformly bounded, and then so is $\operatorname{diam}(\operatorname{spt} V_n)$ by Proposition 4.3.21. Hence V_n converges up to subsequence to a varifold V with compact support. The rest of the proof now follows by the very same arguments employed in the proof of Theorem 4.3.8. □

In the complete analogous way, we also obtain the following last existence result.

Theorem 4.3.23. *Let $\gamma = (\gamma^1) \cup ... \cup (\gamma^\alpha)$ be a disjoint union of smooth embedded closed curves with $\alpha \in \mathbb{N}_{\ge 2}$. Let $m : \gamma \to \mathbb{N}_{\ge 1}$ be $\mathcal{H}^1$-measurable with $m \in L^\infty(\mathcal{H}^1 \llcorner \gamma)$. Let $\mathcal{Q}_{\mathscr{H}}$ be the minimization problem*

$$\mathcal{Q}_{\mathscr{H}} := \min \left\{ \mathscr{H}(V) \quad | \quad V = \mathbf{v}(M, \theta_V) : \quad |\sigma_V| \le m\mathcal{H}^1 \llcorner \gamma, \quad \operatorname{spt} V \cup \gamma \ \ \textit{compact, connected} \right\}.$$

If $\inf \mathcal{Q}_{\mathscr{H}} < +\infty$, *then* $\mathcal{Q}_{\mathscr{H}}$ *has minimizers.*

Let us conclude the section with a final comment on the hypotheses of the existence theorems 4.3.8 and 4.3.9 on the Willmore energy, and the ones above on the Helfrich energy. In these theorems the key hypotheses are on the infimum of the considered problems.

Assuming $\inf < +\infty$ in the minimization problems on the Helfrich energy just means that there exists a competitor. Indeed in these problems one fixes a priori conditions on the generalized boundary, and it is not obvious to say whether a varifold fitting the desired boundary conditions exists, especially varifolds having an assigned generalized boundary σ_0. This leads to a first very important question in Geometric Measure Theory, which is widely open today up to the author's knowledge: given a varifold V and even assuming good structural hypotheses like the ones in Theorem 1.3.7, what can be said on the measure σ_V?

A further question arises analyzing the hypothesis $\inf < 4\pi$ in the minimization problems on the Willmore energy in Theorem 4.3.8 and Theorem 4.3.9. Not only this assumption is necessary for the existence of at least one competitor, as said above, but we also needed it in order to extract a *converging* minimizing sequence. Apparently, as we used the monotonicity formula in such an argument, the constant 4π estimating the infimum seemed to be optimal (precisely in inequality (4.17)). On the other hand, a sole bound on the Willmore energy on a sequence of varifolds with boundary clearly does not imply bounds on diameter or mass: consider for example a sequence of spheres with diverging radii with a disk removed, so that the Willmore energy is strictly less then 4π (remember also Proposition 4.3.21 on the equivalence between a bound on the mass and on the diameter). Hence, not only the bound of 4π on $\mathcal{W}$ was necessary in our argument, but also the fact that the sequence was actually *minimizing* was needed. A last question is then spontaneous: in problems like the ones in Theorem 4.3.8 or Theorem 4.3.8, just assuming that the infimum of the problem is $< +\infty$, do minimizing sequences converge to compactly supported varifolds?

Appendix

4.A More on Li–Yau-type inequalities for surfaces with boundary

In this appendix we derive another Li–Yau-type inequality for immersed surfaces whose boundary is an arbitrary smooth embedded closed planar curve.

Once again, we denote by $\Sigma_{\mathfrak{g}}$ a surface with boundary which is diffeomorphic to the orientable surface $M_{\mathfrak{g}}$ of genus $\mathfrak{g}$ having a removed disk. We will assume that $\mathbb{S}^1 = \{x^2 + y^2 = 1, z = 0\}$ is a subset of $\mathbb{R}^3$.

If $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ is an immersion and we denote by $\Sigma = \varphi(\Sigma_{\mathfrak{g}})$ its image, we will use the same symbol Σ to denote the image varifold $\operatorname{Im}\varphi$. We will also use the shortcuts

$$|\Sigma \cap U| \coloneqq \mu_\Sigma(U), \qquad \int_{\Sigma\cap U} f \coloneqq \int_U f \, d\mu_\Sigma, \qquad \int_\Sigma g \coloneqq \int_{\Sigma_{\mathfrak{g}}} g \, d\mu_\varphi$$

for any open set $U \subset \mathbb{R}^3$ and continuous functions $f : U \to \mathbb{R}$ and $g : \Sigma_{\mathfrak{g}} \to \mathbb{R}$ whenever there is no risk of confusion.

We discuss first a formula we will need about the relation between the curvature of a closed curve γ and the curvature of the curve given by the spherical inversion of γ having the center of inversion on γ.

Remark 4.A.1. Let $\gamma : \mathbb{S}^1 \to \{z = 0\} \subset \mathbb{R}^3$ be a closed embedded smooth planar curve. We identify $\{z = 0\}$ with $\mathbb{R}^2$, having the standard orientation given by the oriented basis $[(1,0,0),(0,1,0)]$. Let $\widetilde{k}$ be the oriented curvature of γ and assume that $0 \in \gamma(\mathbb{S}^1)$. The map $\gamma' \coloneqq I_{1,0} \circ \gamma|_{\mathbb{S}^1\setminus\gamma^{-1}(0)}$ parametrizes the spherical inversion of γ with respect to $I_{1,0}$. Let $\widetilde{k}'$ be the oriented curvature of γ' with respect to the normal vector along γ' given by the counterclockwise rotation of $\frac{\pi}{2}$ of the tangent vector $\tau_{\gamma'}$. Then

$$\widetilde{k}'(t) = -|\gamma(t)|^2 \widetilde{k}(t) + 2\det(\gamma(t), \tau_\gamma(t)) \qquad \forall\, t \in \mathbb{S}^1 \setminus \gamma^{-1}(0), \tag{4.28}$$

where $\det(\gamma, \tau_\gamma)$ is the determinant of the (2×2)-matrix whose columns are the first two components of the vectors $\gamma, \tau_\gamma \in \mathbb{R}^3$.

Indeed, without loss of generality we can assume that γ is parametrized by arclength, and we can use [AT12, Remark 1.3.16, Equation (1.11)] to get that

$$\widetilde{k}' = \frac{1}{|\partial_t\gamma'|^3} \det(\partial_t\gamma', \partial_t^2\gamma'),$$

where the determinant is understood as above. Using that $\gamma' = \frac{\gamma}{|\gamma|^2}$, the direct computation shows that

$$\partial_t\gamma' = \frac{1}{|\gamma|^2}\left(\tau_\gamma - \frac{2\langle\gamma,\tau_\gamma\rangle}{|\gamma|^2}\gamma\right),$$

$$\partial_t^2\gamma' = \left(\frac{8\langle\gamma,\tau_\gamma\rangle^2}{|\gamma|^6} - \frac{2\langle\gamma,k_\gamma\rangle}{|\gamma|^4} - \frac{2}{|\gamma|^4}\right)\gamma - \frac{4\langle\gamma,\tau_\gamma\rangle}{|\gamma|^4}\tau_\gamma + \frac{k_\gamma}{|\gamma|^2},$$

and then $|\partial_t\gamma'| = \frac{1}{|\gamma|^2}$. Then the direct computation using the above formula for $\widetilde{k}'$ readily yields (4.28).

Let us start with a simple observation that follows from the monotonicity formula. Roughly speaking, the next remark states that whenever an immersed surface has a point located very far from its boundary (compared with the length of such boundary), then the Willmore energy is bounded below by a constant close to 4π.

Remark 4.A.2. Let $\varphi:\Sigma_{\mathfrak{g}}\to\mathbb{R}^3$ be an immersion and let $\Gamma=\varphi(\partial\Sigma_{\mathfrak{g}})$. Suppose that there exists a point $p\in\Sigma\coloneqq\varphi(\Sigma_{\mathfrak{g}})$ such that $d(p,\Gamma)\ge R>0$. Then

$$\mathcal{W}(\varphi)\ge 4\pi\sharp\varphi^{-1}(p)-\frac{2}{R}L(\Gamma),$$

where $L(\Gamma)$ is the length of the immersed curve $\varphi:\partial\Sigma_{\mathfrak{g}}\to\mathbb{R}^3$.

The estimate follows from the monotonicity formula of Theorem 1.3.7. Indeed, denoting by Σ also the image varifold $\operatorname{Im}\varphi$, we can write

$$\pi\sharp\varphi^{-1}(p)=\lim_{r\to0}A_{\Sigma,p}(r)\le\lim_{r\to+\infty}A_{\Sigma,p}(r)=\frac{1}{4}\mathcal{W}(\varphi)+\frac{1}{2}\frac{1}{R}L(\Gamma),$$

where we also used (1.19) in the limit as $r\to0$.

Another preliminary estimate on the Willmore energy follows by exploiting its conformal properties and, more precisely, Lemma 4.2.4. In the next lemma we show the desired Li–Yau-type inequality for surfaces with arbitrary planar boundary under the additional hypothesis that a self-intersection occurs on the boundary.

Lemma 4.A.3. *Let Γ be an embedded planar closed smooth curve and let ν_Γ be the unit outward conormal of the planar region enclosed by Γ. Then for any any $\omega<4\pi$ there exists $\varepsilon>0$ such that if $\varphi:\Sigma_{\mathfrak{g}}\to\mathbb{R}^3$ is an immersion such that $\varphi:\partial\Sigma_{\mathfrak{g}}\to\Gamma$ is an embedding and*

$$\exists p_0\in\Gamma:\quad \sharp\varphi^{-1}(p_0)\ge2,$$
$$\|co_\varphi-\nu_\Gamma\circ\varphi\|_{L^2(\partial\Sigma_{\mathfrak{g}})}\le\varepsilon,$$

then $\mathcal{W}(\varphi)\ge\omega$.

Proof. Without loss of generality we can identify $\partial\Sigma_{\mathfrak{g}}$ with Γ, and then assume that both co_φ and ν_Γ are defined on Γ.

Let us suppose by contradiction that there exist $\omega<4\pi$ and a sequence $\varphi_n:\Sigma_{\mathfrak{g}}\to\mathbb{R}^3$ such that

$$\begin{aligned}\Sigma_n\coloneqq\varphi_n(\Sigma_{\mathfrak{g}}):\quad&\exists p_n\in\Gamma:\quad\sharp\varphi_n^{-1}(p_n)\ge2,\\&\|co_{\varphi_n}-\nu_\Gamma\|_{L^2(\Gamma)}<\frac{1}{n},\\&\mathcal{W}(\Sigma_n)<\omega.\end{aligned}$$

Up to isometry we can assume that $p_n=0$ for any n and that $\Gamma\subset\{z=0\}\subset\mathbb{R}^3$. Let $m_n=\sharp\varphi_n^{-1}(0)\ge2$. Recall that the first assertion in Lemma 4.2.4 concerns the Willmore energy of the spherical inversion of an immersion centered at a multiplicity one point on its boundary. Such result can be easily generalized to the case in which a self-intersection occurs at such boundary point; in fact this is just a possible proof of the classical Li–Yau inequality (see [BK03, Theorem 2.2]). More precisely, if $I(x) = \frac{x}{|x|^2}$ for $x \in \mathbb{R}^3 \setminus \{0\}$ and we define

$$\psi_n := I \circ \varphi_n|_{\Sigma_{\mathfrak{g}} \setminus \{\varphi_n^{-1}(0)\}},$$

it holds that

$$\mathcal{W}(\varphi_n) + G(\varphi_n) = \mathcal{W}(\psi_n) + G(\psi_n) + 2\pi + 4\pi(m_n - 1) \geq G(\psi_n) + 6\pi. \tag{4.29}$$

Let us denote by $\gamma : [0, L(\Gamma)] \to \{z = 0\}$ the arclength parametrization of Γ that positively orients the curve with respect to the planar region it encloses, with $\gamma(0) = 0$. Being $\nu_\Gamma = -\nu_\gamma$, as ν_γ is the counterclockwise rotation of τ_γ, by (4.33) we get

$$G(\varphi_n) = \int_\Gamma \langle \widetilde{k}_\gamma \nu_\gamma, -co_{\varphi_n} \rangle \xrightarrow[n\to\infty]{} \int_\Gamma \widetilde{k}_\gamma = 2\pi,$$

where in the last equality we used (4.11).

The function $(I \circ \gamma)(t)$ parametrizes the boundary of the immersion ψ_n (for $t \neq 0, L(\Gamma)$), which is a planar curve contained in $\{z = 0\}$. It is immediate to check that the sequence of conormals co_{ψ_n} converges to $\nu_{I(\Gamma)}$ in L^2 on compact sets, where $\nu_{I(\Gamma)}$ is the normal vector along $I(\Gamma)$ pointwise given by the counterclockwise rotation of $\frac{\pi}{2}$ of the tangent vector of the parametrization $I \circ \gamma$. Also, we can write the the oriented curvature of $I \circ \gamma$ with respect to $\nu_{I(\Gamma)}$ in terms of the parametrization γ as

$$\widetilde{k}_{I\circ\gamma}(t) = -|\gamma(t)|^2 \widetilde{k}_\gamma + 2\det(\gamma(t), \dot{\gamma}(t)) \qquad \forall t \in (0, L(\Gamma)),$$

by (4.28). Hence we have

$$|I(\gamma(t))|^2 \widetilde{k}_{I\circ\gamma}(t) = \frac{\widetilde{k}_{I\circ\gamma}(t)}{|\gamma(t)|^2} \leq C_0,$$

and we get

$$\int_{I(\Gamma)} |\widetilde{k}_{I\circ\gamma}| \, d\mathcal{H}^1 = \int_0^{L(\Gamma)} |\widetilde{k}_{I\circ\gamma}(t)| \left| \frac{d}{dt} I(\gamma) \right| dt = \int_0^{L(\Gamma)} |\widetilde{k}_{I\circ\gamma}(t)| \frac{1}{|\gamma(t)|^2} \, dt \leq C < +\infty. \tag{4.30}$$

So, by the local convergence in L^2 of co_{ψ_n} to $\nu_{I(\Gamma)}$ and taking into account (4.30), using (4.10) we get

$$|G(\psi_n)| \xrightarrow[n\to\infty]{} \left| \int_{I(\Gamma)} \widetilde{k}_{I\circ\gamma} \right| = 0 \tag{4.31}$$

Therefore passing to the limit $n \to +\infty$ in (4.29) we find

$$6\pi > \omega + 2\pi \geq 6\pi,$$

and this gives a contradiction. □

We can now prove a second Li–Yau-type estimate in case the boundary of an immersed surface is an arbitrary embedded planar closed smooth curve. In this case, we need to further assume that the conormal of the immersion is suitably close to the outward conormal of the planar set enclosed by the boundary curve, as we initially did in Lemma 4.2.5. We believe that this additional hypothesis can be removed, at least in case the boundary curve is convex, but this is still an open question, up to the author's knowledge.

Theorem 4.A.4 (Li–Yau-type inequality with planar boundary)**.** *Let Γ be an embedded planar closed smooth curve and let ν_Γ be the unit outward conormal of the planar region enclosed by Γ. Then for any $\omega < 4\pi$ there exists $\varepsilon > 0$ such that if $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ is an immersion such that $\varphi : \partial\Sigma_{\mathfrak{g}} \to \Gamma$ is an embedding and*

$$\exists p_0 \in \mathbb{R}^3 : \quad \sharp\varphi^{-1}(p_0) \geq 2,$$

$$\|co_\varphi - \nu_\Gamma \circ \varphi\|_{L^2(\partial\Sigma_{\mathfrak{g}})} \leq \varepsilon,$$

then $\mathcal{W}(\varphi) \geq \omega$.

Proof. By Theorem 4.2.6 we can assume that Γ is not a circle. In particular, we can assume that for any $q \in \Gamma$ the curve $I_{1,q}(\Gamma \setminus \{q\})$ is not a straight line, and then it is well defined a unique plane containing $I_{1,q}(\Gamma \setminus \{q\})$. Without loss of generality, we can identify $\partial\Sigma_{\mathfrak{g}}$ with Γ, and then assume that both co_φ and ν_Γ are defined on Γ.

We can assume without loss of generality that $\Gamma \subset \{z = 0\}$, and we assume that $\{z = 0\}$ is positively oriented by the oriented couple $[(1, 0, 0), (0, 1, 0)]$.

By Lemma 4.A.3 we can actually prove instead that for any $d > 0$ and any $\omega < 4\pi$ there exists $\varepsilon > 0$ such that if $\varphi : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ is an immersion such that $\varphi : \partial\Sigma_{\mathfrak{g}} \to \Gamma$ is an embedding and

$$\exists p_0 \in \mathbb{R}^3 : \quad \sharp\varphi^{-1}(p_0) \geq 2,$$

$$d(p_0, \Gamma) \geq d,$$

$$\|co_\varphi - \nu_\Gamma \circ \varphi\|_{L^2(\partial\Sigma_{\mathfrak{g}})} \leq \varepsilon,$$

then $\mathcal{W}(\varphi) \geq \omega$.

Let us assume by contradiction that there exist $d > 0$, $\omega < 4\pi$, and a sequence $\varphi_n : \Sigma_{\mathfrak{g}} \to \mathbb{R}^3$ such that

$$\begin{aligned} \Sigma_n := \varphi_n(\Sigma_{\mathfrak{g}}) : \quad & \exists p_n \in \Sigma_n : \quad \sharp\varphi_n^{-1}(p_n) \geq 2, \\ & d(p_n, \Gamma) \geq d, \\ & \|co_n - \nu_\Gamma\|_{L^2(\Gamma)} < \frac{1}{n}, \\ & \mathcal{W}(\Sigma_n) < \omega. \end{aligned}$$

Up to isometry and small smooth perturbation we can additionally assume that $0 \in \Gamma$ is a point of multiplicity 1 for any n, i.e. $\sharp\varphi_n^{-1}(0) = 1$, $|p_n| \geq \delta > 0$ for any n, and $\nu_\Gamma(0) = (0, -1, 0)$.

Moreover, by Remark 4.A.2, since Γ is fixed we can additionally assume that

$$d(p_n, \Gamma) \leq \overline{C} < +\infty, \tag{4.32}$$

for any n.

Part 1. In this part we construct a sequence of asymptotically flat surfaces with boundary with properties analogous to the ones of φ_n.

For $\eta_n \searrow 0$ we first consider smooth diffeomorphisms $F_n : \mathbb{R}^3 \to \mathbb{R}^3$ with $F_n(x, y, z) = F_n(x, y)$ and $\|F_n - \mathrm{id}|_{\mathbb{R}^3}\|_{C^2(\mathbb{R}^3)} \leq \eta_n$, $\mathcal{W}(F_n \circ \varphi_n) < \omega$, and $F_n(\Gamma \cap B_{\rho_n}(0))$ is an arc of a circle with curvature equal to the curvature of Γ at 0 for some $\rho_n \searrow 0$. Also, $F_n \circ \varphi_n$ still has a point $F_n(p_n)$ of multiplicity ≥ 2 with $d(F_n(p_n), F_n(\Gamma)) \geq d/2 > 0$, the origin 0 has multiplicity 1 for $F_n \circ \varphi_n$, and $|F_n(p_n)| \geq \delta/2 > 0$ for any n. It is immediate to check that the conormal $co_{F_n \circ \varphi_n}$ of $F_n \circ \varphi_n$ satisfies

$$\|co_{F_n \circ \varphi_n} - \nu_{F_n(\Gamma)}\|_{L^2(F_n(\Gamma))} \to 0, \tag{4.33}$$

as $n \to +\infty$.

Now let $\Sigma'_n := I_{1,0}(F_n(\Sigma_n) \setminus \{0\})$. By Lemma 4.2.4 we have

$$\mathcal{W}(\Sigma'_n) = \mathcal{W}(F_n \circ \varphi_n) + G(F_n \circ \varphi_n) - 2\pi - G(\Sigma'_n), \tag{4.34}$$

where we write Σ'_n in place of the immersion defining it in order to simplify the notation.

Let us denote by $\gamma_n : [0, L(F_n(\Gamma))] \to \{z = 0\}$ the arclength parametrization of $F_n(\Gamma)$, that positively orients the curve with respect to the planar region it encloses, with $\gamma_n(0) = 0$. Then the curvature vector of $F_n(\Gamma)$ is $\widetilde{k}_n \nu_n$, where $\widetilde{k}_n$ and ν_n are the oriented curvature and normal vector of γ_n respectively on the plane $\{z = 0\}$. Being $\nu_n = -\nu_{F_n(\Gamma)}$, by (4.33) we get

$$G(F_n \circ \varphi_n) = \int_{F_n(\Gamma)} \langle \widetilde{k}_n \nu_n, -co_{F_n \circ \varphi_n} \rangle \xrightarrow[n\to\infty]{} \int_\Gamma \widetilde{k}_\Gamma = 2\pi, \tag{4.35}$$

where in the last equality we used (4.11).

The function $I_{1,0}(\gamma_n)$ parametrizes the boundary Γ'_n of Σ'_n, which is a planar curve contained in $\{z = 0\}$ which coincides with a straight line outside a suitable compact set. By an argument analogous to the one leading to (4.33), we have that the conormal co'_n of Σ'_n converges to $\nu_{I_{1,0}(\Gamma)}$ in L^2 on compact sets, where $\nu_{I_{1,0}(\Gamma)}$ is the normal vector along $I_{1,0}(\Gamma)$ pointwise given by the limit of $\nu_{\Gamma'_n}$, that is the counterclockwise rotation of $\frac{\pi}{2}$ of the tangent vector of the parametrization $I_{1,0}(\gamma_n)$. Also, we can write the the oriented curvature of Γ'_n with respect to $\nu_{\Gamma'_n}$ in terms of the parametrization γ_n as

$$\widetilde{k}'_n(t) = -|\gamma_n(t)|^2 \widetilde{k}_n + 2\det(\gamma_n(t), \dot{\gamma}_n(t)) \qquad \forall t \in (0, L(F_n(\Gamma))),$$

by (4.28). Hence we have

$$|I_{1,0}(\gamma_n(t))|^2 \widetilde{k}'_n(t) = \frac{\widetilde{k}'_n(t)}{|\gamma_n(t)|^2} \le C_0, \tag{4.36}$$

and we get

$$\int_{\Gamma'_n} |\widetilde{k}'_n| \, d\mathcal{H}^1 = \int_0^{L(F_n(\Gamma))} |\widetilde{k}'_n(t)| \left| \frac{d}{dt} I_{1,0}(\gamma_n) \right| dt = \int_0^{L(F_n(\Gamma))} |\widetilde{k}'_n(t)| \frac{1}{|\gamma_n(t)|^2} \, dt \le C < +\infty, \tag{4.37}$$

for C independent of n, showing that $|\widetilde{k}'_n|$ has bounded integral uniformly in n. So, by the local convergence in L^2 of co'_n to $\nu_{I_{1,0}(\Gamma)}$ and taking into account (4.37), similarly as in (4.35), using (4.10) we get

$$|G(\Sigma'_n)| \xrightarrow[n\to\infty]{} \left| \int_{I_{1,0}(\Gamma)} \widetilde{k}' \right| = 0, \tag{4.38}$$

where $\widetilde{k}'$ is the oriented curvature of $I_{1,0}(\Gamma)$ with respect to $\nu_{I_{1,0}(\Gamma)}$. Hence by (4.34), (4.35) and (4.38) we conclude that

$$\liminf_{n\to\infty} \mathcal{W}(\Sigma'_n) = \liminf_{n\to\infty} \mathcal{W}(F_n \circ \varphi_n) \le \omega. \tag{4.39}$$

Part 2. In this second part we want to derive a contradiction using the sequence Σ'_n and the monotonicity formula of Theorem 1.3.7.

For any n apply the translations $p \mapsto p - I_{1,0}(F_n(p_n))$, which put at the origin the point with multiplicity ≥ 2 in Σ'_n. Since in the previous notation $|F_n(p_n)| \ge \delta/2 > 0$ and $d(F_n(p_n), F_n(\Gamma)) \ge$

$d/2 > 0$, after this isometry we have that the boundary of the translated Σ'_n, which we still denote Γ'_n, verifies

$$0 < C_1 \le d(\Gamma'_n, 0) \le C_2.$$

Also we can take a system of coordinates in which the asymptotic line of Γ'_n is parallel to the axis $\{(x,0,0)|x \in \mathbb{R}\}$. By (4.36), the fact that $|F_n(p_n)| \ge \delta/2$, and (4.32), the boundaries Γ'_n with curvature $k_{\Gamma'_n}$ still satisfy

$$|q|^2 k_{\Gamma'_n}(q) \le C'_0 \qquad \forall q \in \Gamma'_n,$$

for any n.

After such a translation we perform a blow up $\Sigma''_n = \frac{1}{r_n}\Sigma'_n$ with $r_n \searrow 0$ so small that

$$\Sigma''_n \xrightarrow[n\to\infty]{} \mu := \mathbf{v}\left(\bigcup_{i=1}^M \Pi_i, \theta\right)$$

in the sense of varifolds, where each Π_i is a plane passing through the origin and $M \ge 2$ or $M = 1, \theta \ge 2$.

Now we exploit the monotonicity formula in Theorem 1.3.7. Denoting by Σ''_n also the corresponding varifold, we can evaluate $T_{\Sigma''_n,0}(\rho)$ and we get

$$\forall n \,\exists \lim_{\rho\to\infty} T_{\Sigma''_n,0}(\rho) = \frac{1}{2}\int_{\Gamma''_n} \frac{\langle p, co''_n(p)\rangle}{|p|^2}\, d\mathcal{H}^1(p),$$

indeed for a suitable compact K_n we have $(co''_n)_1 = 0$ out of K_n, and thus

$$\begin{aligned}
\left|\frac{1}{2\rho^2}\int_{\Gamma''_n\cap B_\rho(0)} \langle p, co''_n(p)\rangle\, d\mathcal{H}^1(p)\right| &= \left|\frac{1}{2\rho^2}\left(\int_{\Gamma''_n\cap B_\rho(0)\cap K_n} x(co''_n)_1 + \int_{\Gamma''_n\cap B_\rho(0)} y(co''_n)_2 + z(co''_n)_3\right)\right| \\
&\le \frac{1}{2\rho^2}\left(\int_{\Gamma''_n\cap B_\rho(0)\cap K_n} |x| + \int_{\Gamma''_n\cap B_\rho(0)} |y| + |z|\right) \\
&\le \frac{1}{2\rho^2}\rho C(n) \xrightarrow[\rho\to\infty]{} 0,
\end{aligned}$$

where $(x,y,z) = p$ always denotes the coordinates of points in $\mathbb{R}^3$. Also, denoting by H''_n the mean curvature of Σ''_n, we have for any $0 < \sigma < \rho$ that

$$\begin{aligned}
\int_{\Sigma''_n\cap B_\rho(0)} \frac{\langle H''_n, p\rangle}{\rho^2} &= \int_{\Sigma''_n\cap B_\sigma(0)} \frac{\langle H''_n, p\rangle}{\rho^2} + \int_{\Sigma''_n\cap B_\rho(0)\setminus B_\sigma(0)} \frac{\langle H''_n, p\rangle}{\rho^2} \\
&\le \frac{1}{\rho}\int_{\Sigma''_n\cap B_\sigma(0)} |H''_n| + \frac{|\Sigma''_n\cap B_\rho(0)|^{1/2}}{\rho}\mathcal{W}(\Sigma''_n \setminus B_\sigma(0)),
\end{aligned}$$

hence letting first $\rho\to\infty$ and then $\sigma\to\infty$, since $\lim_{\rho\to\infty}\frac{|\Sigma''_n\cap B_\rho(0)|^{1/2}}{\rho} = (\pi/2)^{1/2}$, we have

$$\lim_{\rho\to\infty}\int_{\Sigma''_n\cap B_\rho(0)} \frac{\langle H''_n, p\rangle}{\rho^2} = 0.$$

So Theorem 1.3.7 implies

$$\begin{aligned}
\exists \lim_{\rho\to\infty} A_{\Sigma''_n,0}(\rho) &= \lim_{\rho\to\infty}\frac{|\Sigma''_n\cap B_\rho(0)|}{\rho^2} + \frac{\mathcal{W}(\Sigma''_n)}{4} + \frac{1}{2}\int_{\Gamma''_n}\frac{\langle p, co''_n(p)\rangle}{|p|^2}\, d\mathcal{H}^1(p) \\
&= \frac{\pi}{2} + \frac{\mathcal{W}(\Sigma'_n)}{4} + \frac{1}{2}\int_{\Gamma''_n}\frac{\langle p, co''_n(p)\rangle}{|p|^2}\, d\mathcal{H}^1(p).
\end{aligned} \tag{4.40}$$

Now we want to estimate the integral

$$\mathcal{I}_n := \int_{\Gamma''_n} \frac{\langle p, co''_n(p)\rangle}{|p|^2}.$$

Observe that $\mathcal{I}_n = \int_{\Gamma'_n} \frac{\langle p, co'_n(p)\rangle}{|p|^2}$, as it is scaling invariant. We distinguish two cases: the distance of the plane containing Γ'_n from the origin is bounded from below by a constant $\Delta > 0$, or the origin belongs to such a plane for any n. Indeed, if such distance goes to zero as $n \to \infty$, by smaller and smaller smooth modifications which do not modify the conormal neither (4.39) we would be back to the second case.

i) Let us start from the case in which the origin is uniformly bounded away from the plane containing Γ'_n. Recalling the above setting, after a rotation, we have for suitable compact sets K_n that

$$\begin{aligned} &\Gamma'_n \subset \{z = z_n\}, \qquad && \Gamma'_n \setminus K_n = \{(x, y_n, z_n) | x \in \mathbb{R}\} \setminus K_n, \\ &0 < \Delta \le z_n \le C_3, && |y_n| \le C_4. \end{aligned} \tag{4.41}$$

Let σ_n be a parametrization of $\frac{1}{2z_n}\Gamma'_n$ given by the suitable translation and rescaling of $I_{1,0} \circ \gamma_n$. Hence $I_n = \int_{\sigma_n} \frac{\langle p, co_{\sigma_n}(p)\rangle}{|p|^2}$, where co_{σ_n} is just the conormal of $\frac{1}{2z_n}\Sigma'_n$ defined along σ_n. Note that $(\sigma_n) \subset \{z = 1/2\}$ and $\left|\frac{y_n}{2z_n}\right| \le \frac{C_4}{2\Delta}$. We now perform the change of variable $q = I_{1,0}(p)$ and we use the area formula of Theorem 1.2.8. The tangential Jacobian of $I_{1,0}$ on (σ_n) is $(J^{\sigma_n} I_{1,0})(p) = \frac{1}{|p|^2}$, hence

$$\mathcal{I}_n = \int_{\alpha_n} \left\langle \frac{q}{|q|^2}, co_{\sigma_n}\left(\frac{q}{|q|^2}\right) \right\rangle d\mathcal{H}^1(q),$$

where $\alpha_n(t) = I_{1,0}(\sigma_n(t))$ for any $t \in (0, L(F_n(\Gamma)))$ and $\alpha_n(0) = 0$. Defining $V_n(q) := \left.\frac{dI_{1,0}(co_{\sigma_n})}{|dI_{1,0}(co_{\sigma_n})|}\right|_{\frac{q}{|q|^2}}$ for $q \in (\alpha_n)$, which is normal along α_n, we have $co_{\sigma_n}\left(\frac{q}{|q|^2}\right) = \left.\frac{dI_{1,0}(V_n)}{|dI_{1,0}(V_n)|}\right|_q$, and thus

$$\begin{aligned} \mathcal{I}_n &= \int_{\alpha_n} \left\langle \frac{q}{|q|^2}, \left.\frac{dI_{1,0}(V_n)}{|dI_{1,0}(V_n)|}\right|_q \right\rangle d\mathcal{H}^1(q) = \int_{\alpha_n} \left\langle \frac{q}{|q|^2}, V_n(q) - 2\frac{\langle q, V_n(q)\rangle}{|q|^2} q \right\rangle d\mathcal{H}^1(q) \\ &= -\int_{\alpha_n} \left\langle \frac{q}{|q|^2}, V_n(q) \right\rangle d\mathcal{H}^1(q). \end{aligned}$$

We have that the curve α_n is contained in the sphere $S := \{x^2 + y^2 + (z-1)^2 = 1\}$. Moreover α_n is given by the composition of γ_n with a spherical inversion $I_{1,0}$, a traslation, a dialation of a factor $\frac{1}{2z_n}$, and then a further spherical inversion $I_{1,0}$. Hence (4.41) implies that the curvature k_{α_n} of α_n is pointwise bounded uniformly in n. Hence α_n weakly converges in $W^{2,2}$ and strongly in C^1, up to subsequence, to a curve of class $W^{2,2}$. The length of α_n is uniformly bounded and by construction the field V_n is such that

$$\|V_n - \nu_{\mathcal{C}_n}\|_{L^2(\alpha_n)} \xrightarrow[n\to\infty]{} 0,$$

where $\nu_{\mathcal{C}_n} \in TS \cap (T\alpha_n)^\perp$ is the outward unit conormal of $\mathcal{C}_n \subset S$, which the connected component of $S \setminus (\alpha_n)$ positively oriented by the normal field $N(x, y, z) = (x, y, z-1) \in (TS)^\perp$ with respect to the parametrization α_n of its boundary. Then

$$\mathcal{I}_n = -\int_{\alpha_n} \left\langle \frac{q}{|q|^2}, \nu_{\mathcal{C}_n}(q) \right\rangle d\mathcal{H}^1(q) - \int_{\alpha_n} \left\langle (T\alpha_n)^\perp\left(\frac{q}{|q|^2}\right), V_n(q) - \nu_{\mathcal{C}_n}(q) \right\rangle d\mathcal{H}^1(q) =: \mathcal{J}_n + \mathcal{K}_n.$$

Since the tangent vectors τ_{α_n} are equi-Lispchitz, then $|(T\alpha_n)^\perp(\alpha_n(t))| \le C|t|^2$ for any n. Then $(T\alpha_n)^\perp\left(\frac{q}{|q|^2}\right)$ is bounded for $q \in (\alpha_n)$, and we have

$$|\mathcal{K}_n| \le \|V_n - \nu_{\mathcal{C}_n}\|_{L^2(\alpha_n)} \left\|(T\alpha_n)^\perp\left(\frac{q}{|q|^2}\right)\right\|_{L^2(\alpha_n)} \xrightarrow[n\to\infty]{} 0. \tag{4.42}$$

Moreover

$$\begin{aligned}
\mathcal{J}_n &= \lim_{k\to\infty} -\int_{\alpha_n \setminus B_{\frac{1}{k}}(0)} \left\langle \frac{q}{|q|^2}, \nu_{\mathcal{C}_n}(q) \right\rangle \\
&= \lim_{k\to\infty} -\int_{\partial\left(\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)\right)} \left\langle \frac{q}{|q|^2}, \nu_{\partial\left(\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)\right)}(q) \right\rangle + \int_{\mathcal{C}_n \cap \partial B_{\frac{1}{k}}(0)} \left\langle \frac{q}{|q|^2}, \nu_{\partial\left(\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)\right)}(q) \right\rangle \\
&= -\pi + \lim_{k\to\infty} -\int_{\partial\left(\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)\right)} \left\langle \frac{q}{|q|^2}, \nu_{\partial\left(\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)\right)}(q) \right\rangle \\
&= -\pi - \lim_{k\to\infty} \int_{\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)} \mathrm{div}_{\mathcal{C}_n}\left(\frac{q}{|q|^2}\right) + 2\left\langle H_{\mathcal{C}_n}, \frac{q}{|q|^2}\right\rangle,
\end{aligned}$$

where $\nu_{\partial\left(\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)\right)}$ is the unit outward conormal of $\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)$ and we used Proposition 1.1.8. The mean curvature vector of $\mathcal{C}_n$ is equal to $H_{\mathcal{C}_n}(q) = -N(q)$, while

$$\mathrm{div}_{\mathcal{C}_n}\left(\frac{q}{|q|^2}\right) = \mathrm{div}_{\mathbb{R}^3}\left(\frac{q}{|q|^2}\right) - \left\langle \left(\nabla\left(\frac{q}{|q|^2}\right)\right)(N), N \right\rangle = \frac{2}{|q|^4}\langle q, N\rangle^2.$$

Being $N(q) = q - (0,0,1)$ and $|q|^2 = 2z$ on S, we end up with

$$\begin{aligned}
\mathcal{J}_n &= -\pi - \lim_{k\to\infty} \int_{\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)} \frac{2z}{|q|^2}\left(\frac{z}{|q|^2} - 1\right) = -\pi - \lim_{k\to\infty} \int_{\mathcal{C}_n \setminus B_{\frac{1}{k}}(0)} -\frac{1}{2} \\
&= \frac{\mathcal{H}^2(\mathcal{C}_n)}{2} - \pi.
\end{aligned} \tag{4.43}$$

Since α_n converges to a $C^{1,1}$ curve of positive length, there is $\eta > 0$ such that $\mathcal{H}^2(\mathcal{C}_n) \le 4\pi - 2\eta$ for any large n. So, putting together (4.42) and (4.43) we find

$$\liminf_{n\to\infty} \mathcal{I}_n \le \pi - \eta,$$

and recalling (4.40) and (4.39) we get

$$\liminf_{n\to\infty} \lim_{\rho\to\infty} A_{\Sigma''_n,0}(\rho) \le \frac{\pi}{2} + \frac{\omega}{4} + \frac{1}{2}(\pi - \eta) = \pi + \frac{\omega}{4} - \frac{\eta}{2}. \tag{4.44}$$

ii) In the other case suppose that 0 lies in the plane of Γ'_n for any n. As in i), if σ_n is a suitable translation of $I_{1,0} \circ \gamma_n$, we look at the curve $\beta_n(t) = I_{1,0}(\sigma_n(t))$ for $t \in (0, L(F_n(\Gamma)))$ and $\beta_n(0) = 0$, which parametrizes the closure of $I_{1,0}(\Gamma'_n)$. Since $0 < C_1 \le d(\Gamma'_n, 0) \le C_2$ we have that the length of β_n and its curvature are uniformly bounded as in the previous case for the curve α_n. As before, the change of variable yields

$$\mathcal{I}_n = -\int_{\beta_n} \left\langle \frac{q}{|q|^2}, V_n(q) \right\rangle d\mathcal{H}^1(q),$$

with $V_n(q) = \frac{dI_{1,0}(co'_n)}{|dI_{1,0}(co'_n)|}\Big|_{\frac{q}{|q|^2}}$. Again we write

$$\mathcal{I}_n = -\int_{\beta_n}\left\langle \frac{q}{|q|^2}, \nu_{\mathcal{D}_n}\right\rangle - \int_{\beta_n}\left\langle \frac{q}{|q|^2}, V_n - \nu_{\mathcal{D}_n}\right\rangle,$$

where $\nu_{\mathcal{D}_n}$ is the unit outward conormal of the domain $\mathcal{D}_n$ on the plane with boundary β_n such that $\|V_n - \nu_{\mathcal{D}_n}\|_{L^2(\beta_n)} \to 0$. As before $\left|\int_{\beta_n}\langle \frac{q}{|q|^2}, V_n - \nu_{\mathcal{D}_n}\rangle\right| \underset{n}{\to} 0$. So we need to calculate

$$\int_{\beta_n}\left\langle \frac{q}{|q|^2}, \nu_{\mathcal{D}_n}\right\rangle = \lim_{k\to\infty}\int_{\beta_n}\langle \psi_k, \nu_{\mathcal{D}_n}\rangle = \lim_{k\to\infty}\int_{\mathcal{D}_n} \mathrm{div}_{\mathbb{R}^2}\psi_k = \pi,$$

where ψ_k is the truncated function

$$\psi_k(p) = k\chi_{B_{\frac{1}{k}}(0)}(p)\frac{p}{|p|} + \left(1 - \chi_{B_{\frac{1}{k}}(0)}(p)\right)\frac{p}{|p|^2},$$

which is such that $(T\beta_n)^\perp(\psi_k) \xrightarrow[k\to\infty]{} (T\beta_n)^\perp\left(\frac{p}{|p|^2}\right)$ in $L^1(\beta_n)$ and $\mathrm{div}_{\mathbb{R}^2}(\psi_k) = k\chi_{B_{\frac{1}{k}}(0)}(p)\frac{1}{|p|} \in L^1(\mathbb{R}^2)$.

Hence we see that in this case

$$\liminf_{n\to\infty}\mathcal{I}_n = \pi. \tag{4.45}$$

Hence recalling (4.40) and (4.39) we get from (4.45) that

$$\liminf_{n\to\infty}\lim_{\rho\to\infty} A_{\Sigma''_n,0}(\rho) \leq \frac{\pi}{2} + \frac{\omega}{4} + \frac{\pi}{2} = \pi + \frac{\omega}{4}. \tag{4.46}$$

Now we put cases i) and ii) together: by convergence in the sense of varifolds, and then of first variation of varifolds, the quantity $A_{\Sigma''_n,0}(\rho)$ is lower semicontinuous with respect to varifold convergence at almost every fixed $\rho > 0$. Hence by monotonicity, by (4.44) and (4.46) we have that in any case

$$A_{\mu,0}(\rho) \leq \liminf_{n\to\infty} A_{\Sigma''_n,0}(\rho) \leq \liminf_{n\to\infty}\lim_{\rho\to\infty} A_{\Sigma''_n,0}(\rho) \leq \pi + \frac{\omega}{4}. \tag{4.47}$$

for almost every $\rho > 0$. Suitably passing to the limit in (4.47), by definition of μ and $A_{\mu,0}$, we find

$$2\pi \leq \lim_{\rho\to\infty}\frac{\mu(B_\rho(0))}{\rho^2} = \lim_{\rho\to\infty} A_{\mu,0}(\rho) \leq \pi + \frac{\omega}{4} < 2\pi,$$

which gives the desired contradiction. □

The structure of the proof of Theorem 4.A.4 is actually completely analogous to the one of Lemma 4.2.5, except for the necessary technicalities. If compared with the inequality in Theorem 4.2.6, in Theorem 4.A.4 we still have the further assumption on the conormal, which is assumed to be suitably close to the normal vector on the boundary curve pointing outwards with respect to the bounded planar region enclosed. This is ultimately due to the fact that, if the boundary curve is a circle, as spherical inversions maps circles into straight lines and vice versa, it is possible to apply a conformal transformation of the ambient preserving the circular boundary (as exploited in Lemma 4.2.2). All this is clearly not possible in case the boundary curve is not a circle.

As already mentioned, at least in case the curve Γ in Theorem 4.A.4 is planar and convex, we expect that the hypothesis on the conormal can be removed, but the use of the conformal invariance of $\mathcal{W}$ and of the monotonicity formula (1.15) might not be sufficient anymore.

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