Partial regularity for the optimal p-compliance problem with length penalization

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Abstract

We establish a partial $C^{1,\alpha}$ regularity result for minimizers of the optimal p-compliance problem with length penalization in any spatial dimension $N \geq 2$, extending some of the results obtained in [15, 9]. The key feature is that the $C^{1,\alpha}$ regularity of minimizers for some free boundary type problem is investigated with a free boundary set of codimension $N-1$. We prove that every optimal set cannot contain closed loops, cannot contain quadruple points, and it is $C^{1,\alpha}$ regular at $\mathcal{H}^1$-a.e. point for every $p \in (N-1, +\infty)$.

1. Introduction

1.1. General overview

A spatial dimension $N \geq 2$ and an exponent $p \in (1, +\infty)$ are given. Let Ω be an open bounded set in $\mathbb{R}^N$ and let f belong to $L^{q_0}(\Omega)$ with

$$q_0 = (p^*)' \text{ if } 1 < p < N, \qquad q_0 > 1 \text{ if } p = N, \qquad q_0 = 1 \text{ if } p > N, \tag{1.1}$$

where $p^* = Np/(N-p)$ and $(1/p^*) + (1/(p^*)') = 1$. We define the energy functional $E_{f,\Omega}$ over $W_0^{1,p}(\Omega)$ as follows

$$E_{f,\Omega}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \ dx - \int_\Omega fu \ dx.$$

Thanks to the Sobolev embeddings (see [21, Theorem 7.10]), $E_{f,\Omega}$ is finite on $W_0^{1,p}(\Omega)$. It is well known that for any closed proper subset Σ of $\overline{\Omega}$ the functional $E_{f,\Omega}$ admits a unique minimizer $u_{f,\Omega,\Sigma}$ over $W_0^{1,p}(\Omega \backslash \Sigma)$. Also $u_{f,\Omega,\Sigma}$ is a unique solution to the Dirichlet problem

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega\backslash\Sigma \\ \quad u &= 0 \text{ on } \Sigma \cup \partial\Omega, \end{cases} \tag{1.2}$$

which means that $u_{f,\Omega,\Sigma} \in W_0^{1,p}(\Omega\backslash\Sigma)$ and

$$\int_\Omega \langle |\nabla u_{f,\Omega,\Sigma}|^{p-2}\nabla u_{f,\Omega,\Sigma}, \nabla\varphi\rangle \ dx = \int_\Omega f\varphi \ dx \tag{1.3}$$

for all $\varphi \in W_0^{1,p}(\Omega\backslash\Sigma)$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. However, if a closed set $\Sigma \subset \overline{\Omega}$ has zero p-capacity (for the definition of capacity, see Section 2), then $u_{f,\Omega,\Sigma} = u_{f,\Omega,\emptyset}$ (see Remark 2.15). The dependence of $u_{f,\Omega,\Sigma}$ on p is neglected in this paper and in the sequel, when it is appropriate, in order to lighten the notation, we shall simply write u_Σ instead of $u_{f,\Omega,\Sigma}$. For each closed proper subset Σ of $\overline{\Omega}$ we define the p-compliance functional at Σ by

$$C_{f,\Omega}(\Sigma) = -E_{f,\Omega}(u_\Sigma) = \frac{1}{p'}\int_\Omega |\nabla u_\Sigma|^p \ dx = \frac{1}{p'}\int_\Omega fu_\Sigma \ dx.$$

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In two dimensions, following [15], we can interpret Ω as a membrane which is attached along $\Sigma \cup \partial\Omega$ to some fixed base and subjected to a given force f. Then u_Σ is the displacement of the membrane. The rigidity of the membrane is measured through the p-compliance functional which is equal to the product of the coefficient $\frac{1}{p'}$ and the work $\int_\Omega f u_\Sigma \; dx$ performed by the force f.

We study the following shape optimization problem.

Problem 1.1. *Let $p \in (N-1, +\infty)$. Given $\lambda > 0$, find a set $\Sigma \subset \overline{\Omega}$ minimizing the functional $\mathcal{F}_{\lambda,f,\Omega}$ defined by*

$$\mathcal{F}_{\lambda,f,\Omega}(\Sigma') = C_{f,\Omega}(\Sigma') + \lambda \mathcal{H}^1(\Sigma')$$

among all sets Σ' in the class $\mathcal{K}(\Omega)$ of all closed connected proper subsets of $\overline{\Omega}$.

In Proposition 2.24 it will be proved that there is a solution to Problem 1.1. Also, according to [8], the connectedness of admissible sets in the statement of Problem 1.1 is necessary for the existence of a solution to this problem.

It is worth noting that any closed set $\Sigma' \subset \overline{\Omega}$ with $\mathcal{H}^1(\Sigma') < +\infty$ is removable for the Sobolev space $W_0^{1,p}(\Omega)$ if $p \in (1, N-1]$ (see Theorem 2.6 and Remark 2.15), namely, $W_0^{1,p}(\Omega \backslash \Sigma') = W_0^{1,p}(\Omega)$ and this implies that $C_{f,\Omega}(\Sigma') = C_{f,\Omega}(\emptyset)$. Thus, defining Problem 1.1 for some exponent $p \in (1, N-1]$, we would get only trivial solutions to this problem: every point x_0 in $\overline{\Omega}$ and the empty set. On the other hand, if $\Sigma' \subset \overline{\Omega}$ is a closed set such that $\Sigma' \cap \Omega$ is of Hausdorff dimension one and with finite $\mathcal{H}^1$-measure, then Σ' is not removable for $W_0^{1,p}(\Omega)$ if and only if $p \in (N-1, +\infty)$ (see Corollary 2.8 and Remark 2.15). Therefore, Problem 1.1 is interesting only in the case when $p \in (N-1, +\infty)$.

We assume that $f \neq 0$ in $L^{q_0}(\Omega)$, because otherwise the p-compliance functional $C_{f,\Omega}(\cdot)$ would be reduced to zero, and then each solution to Problem 1.1 would be either a point $x_0 \in \overline{\Omega}$ or the empty set.

One of the main questions about minimizers of Problem 1.1 is the question of whether a minimizer containing at least two points is a finite union of C^1 curves. In dimension 2 and for $p = 2$ in [15], the authors established that locally inside Ω a minimizer of Problem 1.1 containing at least two points is a finite union of $C^{1,\alpha}$ curves that can only meet at their ends, by sets of three and with 120° angles. Again, in dimension 2, this result was partially generalized in [9] for all exponents $p \in (1, +\infty)$, namely, it was proved that if Σ is a solution to Problem 1.1, then it cannot contain closed loops (i.e., homeomorphic images of the circle S^1), it is Ahlfors regular if it contains at least two points (up to the boundary for a Lipschitz domain Ω), $\Sigma \cap \Omega$ cannot contain quadruple points (i.e., there is no point $x \in \Sigma \cap \Omega$ such that for some fairly small radius $r > 0$ the set $\Sigma \cap \overline{B}_r(x)$ is a union of four distinct C^1 arcs, each of which meets at point x exactly one of the other three at an angle of 180 degrees, and each of the other two at an angle of 90 degrees), and $\Sigma \cap \Omega$ is $C^{1,\alpha}$ regular at $\mathcal{H}^1$-a.e. point for every $p \in (1, +\infty)$. The main tool that was used in [15] to establish the *ε-regularity theorem* (if a minimizer Σ of Problem 1.1 is sufficiently close, in a ball $\overline{B}_r(x_0)$ such that $B_r(x_0) \subset \Omega$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$, then there exists a constant $a \in (0,1)$ such that $\Sigma \cap \overline{B}_{ar}(x_0)$ is a $C^{1,\alpha}$ arc) is a so-called *monotonicity formula* that was inspired by Bonnet on the Mumford-Shah functional (see [5]). This monotonicity formula was also a key tool in the classification of blow-up limits in [15] (in the case when $N = p = 2$), because it implies that for any point $x_0 \in \Sigma$ there exists the limit

$$\lim_{r \to 0+} \frac{1}{r} \int_{B_r(x_0)} |\nabla u_\Sigma|^2 \; dx = e(x_0) \in [0, +\infty).$$

According to [15], all blow-up limits at any $x_0 \in \Sigma \cap \Omega$ are of the same type: either $e(x_0) > 0$ and all blow-up limits at x_0 must be a half-line, or $e(x_0) = 0$. In the latter case, either there is a blow-up at x_0 which is a line, and then all other blow-ups at x_0 must also be a line, or there is no line, and then all blow-ups at x_0 are propellers (i.e., a union of three half-lines emanating from x_0 and making 120° angles). More precisely, given any point $x_0 \in \Sigma \cap \Omega$ we only have one of the following three possibilities:

(i) x_0 belongs to the interior of a single smooth arc; in this case x_0 is called a *regular* (or *flat*) point.

(ii) x_0 is a common endpoint of three distinct arcs which form at x_0 three equal angles of $120°$; in this case x_0 is called a *triple point*.

(iii) x_0 is the endpoint of one and only one arc; in this case x_0 is called a *crack-tip*.

However, the approach in [15] does not work for the cases when $p \neq 2$. The main obstruction to a full generalization of the result established in [15] is the lack of a good monotonicity formula, when the Dirichlet energy is not quadratic ($p \neq 2$).

Notice that in two dimensions and for $p \neq 2$ some monotonicity formula can still be established for the p-energy. Indeed, assume for simplicity that $f \in L^\infty(\Omega)$, $N = 2$, $p \in (1, +\infty)$, Σ is a closed proper subset of $\overline{\Omega}$, $x_0 \in \overline{\Omega}$, $0 \leq r_0 < r_1 \leq 1$, $(\Sigma \cup \partial\Omega) \cap \partial B_r(x_0) \neq \emptyset$ for all $r \in (r_0, r_1)$ and $\gamma \in [\gamma_\Sigma(x_0, r_0, r_1), 2\pi] \backslash \{0\}$, where

$$\gamma_\Sigma(x_0, r_0, r_1) = \sup\left\{ \frac{\mathcal{H}^1(S)}{r} : r \in (r_0, r_1),\ \ S \text{ is a connected component of } \partial B_r(x_0) \backslash (\Sigma \cup \partial\Omega) \right\}.$$

Assume also that $2 > \lambda_p/\gamma$, where λ_p denotes the L^p version of the Poincaré-Wirtinger constant and is defined by

$$\lambda_p = \min\left\{ \frac{\|g'\|_{L^p(0,1)}}{\|g\|_{L^p(0,1)}} : g \in W_0^{1,p}(0,1) \backslash \{0\} \right\}.$$

The value of λ_p was computed explicitly, for example, in [16, Corollary 2.7] and [38, Inequality (7a)], where the following equality was established

$$\lambda_p = 2\left(\frac{1}{p'}\right)^{\frac{1}{p}} \left(\frac{1}{p}\right)^{\frac{1}{p'}} \Gamma\left(\frac{1}{p'}\right) \Gamma\left(\frac{1}{p}\right),$$

in which Γ is the usual Gamma function. Then we can prove that the function

$$r \in (r_0, r_1) \mapsto \frac{1}{r^\beta} \int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx + C r^{2-\beta}$$

is nondecreasing with $\beta = \lambda_p/\gamma$ and $C = C(p, \lambda_p, \|f\|_\infty, |\Omega|, \gamma) > 0$. However, if $B_{r_1}(x_0) \subset \Omega$ and $\Sigma \cap \overline{B}_{r_1}(x_0)$ is a diameter of $\overline{B}_{r_1}(x_0)$, then $\gamma_\Sigma(x_0, 0, r_1) = \pi$ and λ_p/γ is strictly less than one for all $\gamma \in [\gamma_\Sigma(x_0, 0, r_1), 2\pi]$ if $p \in (1, 2) \cup (2, +\infty)$. So the resulting power of r in this monotonicity formula is not large enough and this formula cannot be used to prove $C^{1,\alpha}$ estimates as in the case $N = p = 2$. On the other hand, we do not know if there is a similar monotonicity formula for the p-energy in dimension $N \geq 3$, but we guess that there is no. Thus, we do not have a tool that would allow us to establish a classification of blow-up limits in the case when $p \neq 2$. For this reason, as in [9], we prove a partial $C^{1,\alpha}$ regularity result for the solutions to Problem 1.1. Although we guess that any minimizer of Problem 1.1 with at least two points is a finite union of $C^{1,\alpha}$ curves.

Suppose that Σ is a solution to Problem 1.1. As will be explained a little later, in order to establish the desired partial regularity result for Σ (Theorem 1.3) and to prove that Σ cannot contain closed loops (Theorem 1.4), we first prove the following: under some conditions (depending on N and p, where $p > N - 1$) on the integrability of the source f, there exist constants $b \in (0, 1)$ and $C > 0$ such that if $\Sigma \cap \overline{B}_r(x_0)$ remains fairly flat for all $r \in [r_0, r_1]$, $B_{r_1}(x_0) \subset \Omega$, r_1 is sufficiently small, $r_0 > 0$ is small enough with respect to r_1, then

$$\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx \leq C \left(\frac{r}{r_1}\right)^{1+b} \int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \ dx + C r^{1+b} \text{ for all } r \in [r_0, r_1] \qquad (\mathcal{E})$$

(see Lemma 3.6). Notice that one of the key differences between the approaches used in [15] and in [9] is the method of proving ($\mathcal{E}$). In [15], the idea was to use the aforementioned monotonicity formula, but in [9], in view of the lack of a good monotonicity formula when the Dirichlet energy is not quadratic, another method was used to prove ($\mathcal{E}$), which in many places is similar to the one we use in the present paper. However, in the proof of $(\mathcal{E})$ in [9], the crucial factor was that in dimension 2 an affine line (1-dimensional plane) is a set of codimension 1. More precisely, in [9] a reflection method was used to estimate a weak solution to the p-Laplace equation in $B_1(0)\backslash(\{0\}\times(-1,1))$, which vanishes p-q.e. on $\{0\}\times(-1,1)$ (see [9, Lemma 4.4]). This method is no more valid for a weak solution to the p-Laplace equation in $B_1(0)\backslash(\{0\}^{N-1}\times(-1,1))$, which vanishes p-q.e. on $\{0\}^{N-1}\times(-1,1)$, if $N\geq 3$. In the present paper, in any spatial dimension $N\geq 2$, we first use a certain barrier function, that we constructed in Lemma A.1, but which is in some sense weaker and slightly simpler than those that were constructed in [27, 28], in order to estimate a nonnegative p-harmonic function in $B_1\backslash(\{0\}^{N-1}\times(-1,1))$, continuous in B_1 and vanishing on $\{0\}^{N-1}\times(-1,1)$. Then we deduce the same kind of estimate for merely a weak solution to the p-Laplace equation in $B_1\backslash(\{0\}^{N-1}\times(-1,1))$ vanishing p-q.e. on $\{0\}^{N-1}\times(-1,1)$.

It is worth noting that our proofs of the partial regularity and the absence of quadruple points differ from those used in [9]. Many proofs in [9] are based on the fact that (only) in dimension 2 the "free boundary" Σ is of codimension 1, thus many standard arguments and competitors are available. Let us emphasize the most important places where the proofs used in [9] do not extend in a trivial manner to higher dimensions. Firstly, in the proof of the Ahlfors regularity in the "internal case" in [9] (see [9, Theorem 3.3]), the set $(\Sigma\backslash B_r(x))\cup\partial B_r(x)$ was used as a competitor for a minimizer Σ of Problem 1.1, which contains at least two points. But in dimension $N\geq 3$ we cannot effectively use such a competitor, because $\partial B_r(x)$ has infinite $\mathcal{H}^1$-measure. Secondly, as mentioned earlier, in [9], a reflection method was used to estimate a p-harmonic function in $(B_1\backslash[a_1,a_2])\subset\mathbb{R}^2$ that vanishes on $[a_1,a_2]\cap B_1$, where $[a_1,a_2]$ is a diameter of $\overline{B}_1$, which is no more available if $N\geq 3$ for a p-harmonic function in $(B_1\backslash[a_1,a_2])\subset\mathbb{R}^N$ which vanishes on $[a_1,a_2]\cap B_1$, where $[a_1,a_2]$ is a diameter of $\overline{B}_1$. Thirdly, in the density estimate in [9], when the minimizer Σ is εr-close, in a ball $\overline{B}_r(x)$ and in the Hausdorff distance, to a diameter $[a,b]$ of $\overline{B}_r(x)$, the set $\Sigma'=(\Sigma\backslash B_r(x))\cup[a,b]\cup W$ was used as a competitor for Σ, where $W=\partial B_r(x)\cap\{y:\operatorname{dist}(y,[a,b])\leq\varepsilon r\}$ (see [9, Proposition 6.8]). However, in dimension $N\geq 3$ we cannot effectively use the above competitor Σ', because it has infinite $\mathcal{H}^1$-measure. Fourthly, in dimension $N\geq 3$ we cannot effectively use the same type of competitors as in the proof of [9, Proposition 7.3] (where the fact that the minimizer is a set of codimension 1 was used to construct such competitors) in order to establish the absence of quadruple points in Ω for solutions to Problem 1.1.

The optimal p-compliance problem can also be formulated under length constraints. Namely, consider the following problem.

Problem 1.2. *Let $p\in(N-1,+\infty)$. Given $L>0$, find a set $\Sigma\subset\overline{\Omega}$ minimizing the p-compliance functional $C_{f,\Omega}$ among all sets Σ' in the class $\mathcal{A}_L(\Omega)$ of all closed connected subsets of $\overline{\Omega}$ satisfying the constraint $0<\mathcal{H}^1(\Sigma')\leq L$.*

This problem was studied in [12, 30, 31], and in [12] it was proved that it admits a solution. However, the question of whether every its solution is a finite union of C^1 curves is still open even in dimension 2 and for the linear case $p=2$. Taking into account the peculiarity of Problem 1.2, it seems that the main difficulty in solving this question consists in the fact that for Problem 1.2 we have no some kind of "the local minimization" of the one-dimensional Hausdorff measure, in contrast to Problem 1.1. Nevertheless, by establishing the regularity result for the solutions to Problem 1.1, we automatically establish the same result for some solutions to Problem 1.2. Indeed, if $\Sigma\subset\overline{\Omega}$ is a solution to Problem 1.1 such that $\operatorname{diam}(\Sigma)>0$, then Σ solves Problem 1.2 provided that $L=\mathcal{H}^1(\Sigma)$. It is worth mentioning that, according to the Γ-convergence result established in [12], in some sense the limit of Problem 1.1 as $p\to+\infty$ corresponds to the minimization of the functional

$$\mathcal{K}(\Omega)\ni\Sigma\mapsto\int_\Omega \operatorname{dist}(x,\Sigma\cup\partial\Omega)f(x)\ dx+\lambda\mathcal{H}^1(\Sigma),$$

which, as well as in its constrained form, was widely studied in the literature (the reader may consult [11, 13, 32, 35, 37, 10, 39, 25, 36]). It is known that minimizers of this functional may not be C^1 regular (see [36]).

1.2. Main results

In this paper, we establish a partial regularity result and some topological properties for minimizers of Problem 1.1 in any spatial dimension $N \geq 2$ and for every $p \in (N-1, +\infty)$, thus generalizing some of the results obtained in [15, 9]. Several of our results will hold under some integrability condition on the source f. We define

$$q_1 = \frac{Np}{Np - N + 1} \quad \text{if} \quad 2 \leq p < +\infty, \qquad q_1 = \frac{2p}{3p-3} \quad \text{if} \quad 1 < p < 2. \tag{1.4}$$

It is worth noting that $q_1 \geq q_0$. The condition $f \in L^{q_1}(\Omega)$ for $p \in [2, +\infty)$ is natural, since q_1 in this case seems to be the right exponent which implies an estimate of the type $\int_{B_r(x_0)} |\nabla u|^p \ dx \leq Cr$ for the solution u to the Dirichlet problem

$$-\Delta_p v = f \ \text{ in } \ B_r(x_0), \ \ v \in W_0^{1,p}(B_r(x_0)),$$

the kind of estimate we are looking for to establish regularity properties on a minimizer Σ of Problem 1.1. The main regularity result established in this paper is the following.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $p \in (N-1, +\infty)$, $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Then there exists a constant $\alpha \in (0,1)$ such that the following holds. Let Σ be a solution to Problem 1.1. Then for $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$ one can find a radius $r_0 > 0$ depending on x such that $\Sigma \cap \overline{B}_{r_0}(x)$ is a $C^{1,\alpha}$ regular curve.*

It is one of the first times that the regularity of minimizers for some free boundary type problem is investigated with a free boundary set of codimension $N-1$. Notice that in Theorem 1.3, when we say that a solution Σ to Problem 1.1 is $C^{1,\alpha}$ regular at $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$, we mean that the set of points $\Sigma \cap \Omega$ around which Σ is not a $C^{1,\alpha}$ regular curve has zero $\mathcal{H}^1$-measure. Thus, Theorem 1.3 is interesting only in the case when $\mathrm{diam}(\Sigma) > 0$, which happens to be true at least for some small enough values of λ (see Proposition 2.25).

We have also proved that if Σ is a solution to Problem 1.1, then Σ cannot contain closed loops (i.e., homeomorphic images of the circle S^1).

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Let Σ be a solution to Problem 1.1. Then Σ cannot contain closed loops (i.e., homeomorphic images of the circle S^1).*

Furthermore, we have proved that if Σ is a solution to Problem 1.1, then $\Sigma \cap \Omega$ cannot contain quadruple points, namely, there is no point $x \in \Sigma \cap \Omega$ such that for some fairly small radius $r > 0$ the set $\Sigma \cap \overline{B}_r(x)$ is a union of four distinct C^1 arcs, each of which meets at point x exactly one of the other three at an angle of 180 degrees, and each of the other two at an angle of 90 degrees.

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$ defined in (1.4). If Σ is a solution to Problem 1.1, then $\Sigma \cap \Omega$ cannot contain quadruple points.*

After all, when we are talking about one dimensional sets, it is not so obvious how horrible they can be. Incidentally, in codimension 2 it seems like a good idea to understand whether the minimizer contains knots, since our story of loops suggests the question.

1.3. Discussion of the proofs

Let us now outline the proofs of our main results.

Decay of the p-energy under flatness control.
To prove the partial $C^{1,\alpha}$ regularity result and the absence of closed loops, we first establish a decay

behavior of the p-energy $r \mapsto \int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx$ under flatness control on Σ at $x_0 \in \Omega$, namely, we prove $(\mathcal{E})$ (see Lemma 3.6). For this we use the following strategy consisting of four steps.

Step 1. We prove that there exist $\alpha, \delta \in (0,1)$ and $C > 0$, depending only on N and p, such that for any weak solution u to the p-Laplace equation in $B_1(0)\backslash(\{0\}^{N-1} \times (-1,1))$ vanishing p-q.e. on $\{0\}^{N-1} \times (-1,1)$, the estimate

$$\int_{B_r(0)} |\nabla u|^p \ dx \leq C r^{1+\alpha} \int_{B_1(0)} |\nabla u|^p \ dx$$

holds for all $r \in (0, \delta]$ (see Lemma 3.1).

Step 2. Arguing by contradiction and compactness, we establish a similar estimate as in *Step 1* for a weak solution to the p-Laplace equation in $B_r(x_0)\backslash\Sigma$ that vanishes on $\Sigma \cap B_r(x_0)$ in the case when $\Sigma \cap \overline{B}_r(x_0)$ is fairly close in the Hausdorff distance to a diameter of $\overline{B}_r(x_0)$. Recall that the Hausdorff distance for any two nonempty sets A, $B \subset \mathbb{R}^N$ is defined by

$$d_H(A,B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x,B),\ \sup_{y \in B} \operatorname{dist}(y,A)\right\}.$$

For each nonempty set $A \subset \mathbb{R}^N$, we immediately agree to define $d_H(\emptyset, A) = d_H(A, \emptyset) = +\infty$ and $d_H(\emptyset,\emptyset) = 0$. Let α, δ, C be as in *Step 1.* We prove that for each $\varrho \in (0,\delta]$ there exists $\varepsilon_0 \in (0,\varrho)$ such that if u is a weak solution to the p-Laplace equation in $B_r(x_0)\backslash\,\Sigma$ vanishing p-q.e. on $\Sigma \cap B_r(x_0)$, where Σ is a closed set such that $(\Sigma \cap B_r(x_0)) \cup \partial B_r(x_0)$ is connected and

$$\frac{1}{r} d_H(\Sigma \cap \overline{B}_r(x_0), L \cap \overline{B}_r(x_0)) \leq \varepsilon_0$$

for some affine line $L \subset \mathbb{R}^N$ passing through x_0, then the following estimate holds

$$\int_{B_{\varrho r}(x_0)} |\nabla u|^p \ dx \leq (C\varrho)^{1+\alpha} \int_{B_r(x_0)} |\nabla u|^p \ dx$$

(see Lemma 3.2).

Step 3. Recall that we want to establish a decay estimate for the solution u_Σ to the Dirichlet problem $-\Delta_p u = f$ in $\Omega\backslash\Sigma$, $u \in W_0^{1,p}(\Omega\backslash\Sigma)$ in a ball $B_r(x_0) \subset \Omega$ whenever Σ is sufficiently close, in $\overline{B}_r(x_0)$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$. For that purpose, we first control the difference between u_Σ and its p-Dirichlet replacement in $B_r(x_0)\backslash\Sigma$, where by the p-Dirichlet replacement of u_Σ in $B_r(x_0)\backslash\Sigma$ we mean the solution $w \in W^{1,p}(B_r(x_0))$ to the Dirichlet problem $-\Delta_p u = 0$ in $B_r(x_0)\backslash\Sigma$, $u - u_\Sigma \in W_0^{1,p}(B_r(x_0)\backslash\Sigma)$. Then, for some sufficiently small $a = a(N,p) \in (0,1)$, using the estimate for the local energy $\int_{B_{ar}(x_0)} |\nabla w|^p \ dx$ coming from *Step 2* and also the estimate for the difference between u_Σ and w in $B_r(x_0)\backslash\Sigma$, we arrive at the following decay estimate for u_Σ:

$$\frac{1}{ar}\int_{B_{ar}(x_0)} |\nabla u_\Sigma|^p \ dx \leq \frac{1}{2}\left(\frac{1}{r}\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx\right) + C r^{\gamma(N,p,q)},$$

where $\gamma(N,p,q) \in (0,1)$ provided that $q > q_1$, $f \in L^q(\Omega)$ and q_1 is defined in (1.4) (see Lemma 3.3 and Lemma 3.5).

Step 4. Finally, by iterating the result of *Step 3* in a sequence of balls $\{B_{a^l r_1}(x_0)\}_l$, we obtain the desired decay behavior of the p-energy $r \mapsto \int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx$ under flatness control on Σ at $x_0 \in \Omega$. Namely, there exist $b \in (0,1)$ and $C > 0$ such that if $\Sigma \cap \overline{B}_r(x_0)$ remains fairly flat for all r in $[r_0, r_1]$, $B_{r_1}(x_0) \subset \Omega$ and r_1 is sufficiently small, $r_0 > 0$ is small enough with respect to r_1, then the following estimate holds

$$\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx \leq C\left(\frac{r}{r_1}\right)^{1+b} \int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \ dx + C r^{1+b} \ \text{ for all } \ r \in [r_0, r_1]$$

(see Lemma 3.6). Thus, if $x_0 \in \Sigma \cap \Omega$ and $\Sigma \cap \overline{B}_r(x_0)$ remains fairly flat for all sufficiently small $r > 0$, then the energy $r \mapsto \frac{1}{r}\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx$ converges to zero no slower than Cr^b for some $b \in (0,1)$ and

$C > 0$. This will be used to prove the desired $C^{1,\alpha}$ result, and the same kind of estimate will also be used to prove the absence of closed loops.

Partial regularity.

We shall now try to explain how we use the decay of the p-energy under flatness control to prove the partial $C^{1,\alpha}$ regularity of the minimizers inside Ω. The first step in the proof is to show that every minimizer Σ of Problem 1.1 with $\mathrm{diam}(\Sigma) > 0$ is an *almost minimizer* for the length at any point in $\Sigma\cap\Omega$ around which Σ is flat enough and remains fairly flat on a large scale. More precisely, we need to prove that there exists $\beta \in (0, 1)$ such that for any competitor Σ' being τr-close, in a ball $\overline{B}_r(x_0) \subset \Omega$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$ for some small $\tau \in (0, 1)$ and satisfying $\Sigma'\Delta\Sigma \subset \overline{B}_r(x_0)$, it holds

$$\mathcal{H}^1(\Sigma \cap \overline{B}_r(x_0)) \leq \mathcal{H}^1(\Sigma' \cap \overline{B}_r(x_0)) + Cr^{1+\beta}$$

whenever $x_0 \in \Sigma\cap\Omega$, Σ is flat enough in $\overline{B}_r(x_0)$ and remains fairly flat on a large scale. In our framework, the term $Cr^{1+\beta}$ may only come from the p-compliance part of the functional $\mathcal{F}_{\lambda,f,\Omega}$. Thus, we need to prove that

$$C_{f,\Omega}(\Sigma') - C_{f,\Omega}(\Sigma) \leq Cr^{1+\beta}$$

whenever $x_0 \in \Sigma \cap \Omega$, Σ is flat enough in $\overline{B}_r(x_0)$ and remains fairly flat on a large scale, $\Sigma' \in \mathcal{K}(\Omega)$ is τr-close, in $\overline{B}_r(x_0) \subset \Omega$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$ for some small $\tau \in (0, 1)$ and $\Sigma'\Delta\Sigma \subset \overline{B}_r(x_0)$. Hereinafter in this section, C denotes a positive constant that can only depend on N, p, q_0, q, $\|f\|_q$, $|\Omega|$ (q_0 is defined in (1.1), $q \geq q_0$, $f \in L^q(\Omega)$) and can be different from line to line. Notice that one of the difficulties in obtaining the above estimate is a nonlocal behavior of the p-compliance functional. Namely, changing Σ locally in Ω, we change u_Σ in the whole Ω. This can be overcome, using a cut-off argument. Actually, we have shown that if Σ' is a competitor for Σ and $\Sigma'\Delta\Sigma \subset \overline{B}_r(x_0)$, then

$$C_{f,\Omega}(\Sigma') - C_{f,\Omega}(\Sigma) \leq C \int_{B_{2r}(x_0)} |\nabla u_{\Sigma'}|^p \ dx + Cr^{N+p'-\frac{Np'}{q}}$$

(see Corollary 2.21). However, the right-hand side in the above estimate depends on the competitor Σ', which pushes us to introduce the quantity

$$w^\tau_\Sigma(x_0, r) = \sup_{\substack{\Sigma' \in \mathcal{K}(\Omega),\, \Sigma'\Delta\Sigma\subset\overline{B}_r(x_0),\\ \mathcal{H}^1(\Sigma')\leq 100\mathcal{H}^1(\Sigma),\, \beta_{\Sigma'}(x_0,r)\leq\tau}} \frac{1}{r}\int_{B_r(x_0)} |\nabla u_{\Sigma'}|^p \ dx,$$

where $\beta_{\Sigma'}(x_0, r)$ is the flatness defined by

$$\beta_{\Sigma'}(x_0, r) = \inf_{L\ni x_0} \frac{1}{r} d_H(\Sigma' \cap \overline{B}_r(x_0), L \cap \overline{B}_r(x_0)),$$

where the infimum is taken over the set of all affine lines (1-dimensional planes) L in $\mathbb{R}^N$ passing through x_0. The quantity $w^\tau_\Sigma(x_0, r)$ is a variant of the one introduced in [15] and already used in [9]. Also notice that the assumption $\mathcal{H}^1(\Sigma') \leq 100\mathcal{H}^1(\Sigma)$ in the definition of $w^\tau_\Sigma(x_0, r)$ is rather optional, however, it guarantees that if Σ' is a maximizer in this definition, then it is arcwise connected. Thus, if $\Sigma' \in \mathcal{K}(\Omega)$, $\Sigma'\Delta\Sigma \subset \overline{B}_r(x_0)$, $\mathcal{H}^1(\Sigma') \leq 100\mathcal{H}^1(\Sigma)$ and $\beta_{\Sigma'}(x_0, 2r) \leq \tau$, we arrive at the estimate

$$C_{f,\Omega}(\Sigma') - C_{f,\Omega}(\Sigma) \leq Crw^\tau_\Sigma(x_0, 2r) + Cr^{N+p'-\frac{Np'}{q}}.$$

Next, applying the decay estimate established in *Step 4* above to the function $u_{\widetilde{\Sigma}}$, where $\widetilde{\Sigma}$ is a maximizer in the definition of $w^\tau_\Sigma(x_0, 2r)$, we obtain the following control

$$w^\tau_\Sigma(x_0, 2r) \leq C\Big(\frac{r}{r_1}\Big)^b w^\tau_\Sigma(x_0, r_1) + Cr^b,$$

provided that $\beta_\Sigma(x_0,\varrho)$ remains fairly small for all $\varrho\in[2r,r_1]$, $r_1>0$ is small enough, $B_{r_1}(x_0)\subset\ \Omega$ and $r>0$ is sufficiently small with respect to r_1 (see Proposition 5.6). Using also that $b<N-1+p'-Np'/q$, altogether we get

$$\mathcal{H}^1(\Sigma\cap\overline{B}_r(x_0))\leq\mathcal{H}^1(\Sigma'\cap\overline{B}_r(x_0))+Cr\Big(\frac{r}{r_1}\Big)^b w^\tau_\Sigma(x_0,r_1)+Cr^{1+b}$$

whenever Σ is a minimizer of Problem 1.1, $\beta_\Sigma(x_0,\varrho)$ remains fairly small for all $\varrho\in[r,r_1]$, $r_1>0$ is small enough, $B_{r_1}(x_0)\subset\Omega$, $r>0$ is sufficiently small with respect to r_1, $\Sigma'\in\mathcal{K}(\Omega)$ is τr-close, in $\overline{B}_r(x_0)$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$, $\Sigma'\Delta\Sigma\subset\overline{B}_r(x_0)$ and $\mathcal{H}^1(\Sigma')\leq 100\mathcal{H}^1(\Sigma)$.

The next step is to find a nice competitor Σ' for a minimizer Σ. More precisely, assume that $x_0\in\Sigma$, $\overline{B}_r(x_0)\subset\Omega$, r is sufficiently small, $\beta_\Sigma(x_0,r)$ is small enough and remains fairly small on a large scale. The task is to find a competitor Σ' such that $\Sigma'\Delta\Sigma\subset\overline{B}_r(x_0)$, Σ' is τr-close, in $\overline{B}_r(x_0)$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$ for some small $\tau\in(0,1)$ and, in addition, the length (i.e., $\mathcal{H}^1$-measure) of $\Sigma'\cap\overline{B}_r(x_0)$ is fairly close to the length of a diameter of $\overline{B}_r(x_0)$. Recall that in two dimensions we can take

$$\Sigma'=(\Sigma\backslash B_r(x_0))\cup(\partial B_r(x_0)\cap\{x:\operatorname{dist}(x,L)\leq\beta_\Sigma(x_0,r)r\})\cup(L\cap B_r(x_0))$$

provided $\beta_\Sigma(x_0,r)=d_H(\Sigma\cap\overline{B}_r(x_0),L\cap\overline{B}_r(x_0))/r$. But in dimension $N\geq3$ such a competitor is not effective, since it has Hausdorff dimension $N-1\geq2$. Notice that the main difficulty arising in the construction of a nice competitor in dimension $N\geq3$ is that we do not know whether the quantity $\mathcal{H}^0(\Sigma\cap\partial B_\varrho(x_0))$ is uniformly bounded from above for $x_0\in\Sigma$ and $\varrho>0$. However, according to the coarea inequality (see, for instance, [33, Theorem 2.1]), we know that for all $\varrho>0$,

$$\mathcal{H}^1(\Sigma\cap B_\varrho(x_0))\geq\int_0^\varrho\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))\ dt.$$

If, moreover, $\varrho<\operatorname{diam}(\Sigma)/2$, then $\Sigma\cap\partial B_t(x_0)\neq\emptyset$ for all $t\in(0,\varrho]$, since $x_0\in\Sigma$ and Σ is arcwise connected (see Remark 2.17). Thus, assuming that $\varrho<\operatorname{diam}(\Sigma)/2$ and $\kappa\in(0,1/4]$, for any $s\in[\kappa\varrho,2\kappa\varrho]$ we deduce the following

$$\mathcal{H}^1(\Sigma\cap B_\varrho(x_0))\geq\int_0^\varrho\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))\ dt>\int_s^{(1+\kappa)s}\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))\ dt.$$

The latter inequality implies that there exists $t\in[s,(1+\kappa)s]$ for which

$$\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))\leq\frac{1}{\kappa^2}\theta_\Sigma(x_0,\varrho),\ \text{ where }\ \theta_\Sigma(x_0,\varrho)=\frac{1}{\varrho}\mathcal{H}^1(\Sigma\cap B_\varrho(x_0)).$$

So if $\kappa\in(0,1/4]$, $x_0\in\Sigma$, $r>0$ is sufficiently small and $B_r(x_0)\subset\Omega$, then for all $s\in[\kappa r,2\kappa r]$ we can construct the following competitor

$$\Sigma'=(\Sigma\backslash B_t(x_0))\cup\left(\bigcup_{i=1}^{\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))}[z_i,z_i']\right)\cup(L\cap\overline{B}_t(x_0)),$$

where $t\in[s,(1+\kappa)s]$ is such that $\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))\leq\theta_\Sigma(x_0,r)/\kappa^2$, L is an affine line realizing the infimum in the definition of $\beta_\Sigma(x_0,t)$, $z_i\in\Sigma\cap\partial B_t(x_0)$ and z_i' denotes the projection of z_i to $L\cap\overline{B}_t(x_0)$. The flatness $\beta_{\Sigma'}(x_0,t)$ is less than or equal to $\beta_\Sigma(x_0,t)$ by construction. Assuming in addition that $\beta_\Sigma(x_0,r)$ is fairly small and $\theta_\Sigma(x_0,r)$ is also small enough, for the competitor Σ' constructed above it holds: $\beta_{\Sigma'}(x_0,t)$ is sufficiently small, since $\beta_{\Sigma'}(x_0,t)\leq\beta_\Sigma(x_0,t)$ and $\beta_\Sigma(x_0,\varrho)$ remains small for all ϱ in $(0,r)$ which are not too far from r (see (5.1)); the length of $\Sigma'\cap\overline{B}_t(x_0)$ is fairly close to the length of a diameter of $\overline{B}_t(x_0)$; the following estimate holds

$$\mathcal{H}^1(\Sigma\cap B_s(x_0))\leq\mathcal{H}^1(\Sigma\cap B_t(x_0))\leq\mathcal{H}^1(\Sigma'\cap B_t(x_0))+Ct\Big(\frac{t}{r}\Big)^b w^\tau_\Sigma(x_0,r)+Ct^{1+b}.$$

This allows us to prove the following fact: there exist ε, $\kappa \in (0, 1/100)$ such that if Σ is a minimizer of Problem 1.1, $x_0 \in \Sigma$, $r > 0$ is sufficiently small, $B_r(x_0) \subset \Omega$ and the following condition holds

$$\beta_\Sigma(x_0, r) + w^\tau_\Sigma(x_0, r) \leq \varepsilon, \;\; \theta_\Sigma(x_0, r) \leq 10\overline{\mu} \tag{$\mathcal{C}$}$$

with $\overline{\mu}$ being a unique positive solution to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$ (we shall explain a bit later why we take this particular bound), then there exists $s \in [\kappa r, 2\kappa r]$ for which $\mathcal{H}^0(\Sigma \cap \partial B_s(x_0)) = 2$ (see Proposition 5.8 (i)), the two points $\{\xi_1, \xi_2\}$ of $\Sigma \cap \partial B_s(x_0)$ belong to two different connected components of

$$\partial B_s(x_0) \cap \{x : \operatorname{dist}(x, L) \leq \beta_\Sigma(x_0, s)s\}$$

(see Proposition 5.8 (ii-1)), where L is an affine line realizing the infimum in the definition of $\beta_\Sigma(x_0, s)$, $\Sigma \cap \overline{B}_s(x_0)$ is arcwise connected (see Proposition 5.8 (ii-2)). Moreover, $(\Sigma \backslash B_s(x_0)) \cup [\xi_1, \xi_2]$ is a nice competitor for Σ (see Proposition 5.8 (ii-3)). Using this result together with the decay behavior of the local energy w^τ_Σ, we prove that there exists a constant $a \in (0, 1/100)$ such that if $x_0 \in \Sigma$, $r > 0$ is small enough, $B_r(x_0) \subset \Omega$ and the condition ($\mathcal{C}$) holds with some sufficiently small $\varepsilon > 0$, then

$$\beta_\Sigma(x_0, ar) \leq C(w^\tau_\Sigma(x_0, r))^{\frac{1}{2}} + Cr^{\frac{b}{2}} \;\text{ and }\; w^\tau_\Sigma(x_0, ar) \leq \frac{1}{2} w^\tau_\Sigma(x_0, r) + C(ar)^b$$

(see Proposition 5.12 (i), (ii)). Next, we need to control the density θ_Σ from above on a smaller scale by its value on a larger scale. Notice that in this paper we do not prove the Ahlfors regularity for a minimizer of Problem 1.1 in the spatial dimension $N \geq 3$ (for a proof in dimension 2, see [9, Theorem 3.3]), namely, that there exist constants $0 < c_1 < c_2$ and a radius $r_0 > 0$ such that if Σ is a minimizer of Problem 1.1 with at least two points, then for all $x \in \Sigma$ and $r \in (0, r_0)$, it holds

$$c_1 \leq \theta_\Sigma(x, r) \leq c_2.$$

In dimension $N \geq 3$ this problem seems very difficult and interesting. However, adapting some of the approaches of Paolini and Stepanov in [32], we prove the following fact: for each $a \in (0, 1/20]$ there exists $\varepsilon \in (0, 1/100)$ such that if $x_0 \in \Sigma$, $B_r(x_0) \subset \Omega$, $r > 0$ is sufficiently small and $\beta_\Sigma(x_0, r) + w^\tau_\Sigma(x_0, r) \leq \varepsilon$, then

$$\theta_\Sigma(x_0, ar) \leq 5 + \theta_\Sigma(x_0, r)^{1-\frac{1}{N}}$$

(see Proposition 5.11). Notice that if $\theta_\Sigma(x_0, r) \leq 10\overline{\mu}$, then, using the above estimate, we get

$$\theta_\Sigma(x_0, ar) \leq 5 + (10\overline{\mu})^{1-\frac{1}{N}} \leq 10(5 + \overline{\mu}^{1-\frac{1}{N}}) = 10\overline{\mu}.$$

The factor 10 in the estimate $\theta_\Sigma(x_0, r) \leq 10\overline{\mu}$ is rather important, it appears in the proof of Corollary 5.14.

Altogether we prove that there exist constants $a, \varepsilon \in (0, 1/100)$, $b \in (0, 1)$ such that if $x_0 \in \Sigma$, $r > 0$ is sufficiently small, $B_r(x_0) \subset \Omega$ and the condition ($\mathcal{C}$) holds with ε, then for all $n \in \mathbb{N}$,

$$\beta_\Sigma(x_0, a^{n+1}r) \leq C(w^\tau_\Sigma(x_0, a^n r))^{\frac{1}{2}} + C(a^n r)^{\frac{b}{2}} \;\text{ and }\; w^\tau_\Sigma(x_0, a^{n+1}r) \leq \frac{1}{2} w^\tau_\Sigma(x_0, a^n r) + C(a^{n+1}r)^b$$

(see Proposition 5.12). This, in particular, implies that the estimate $\beta_\Sigma(x_0, \varrho) \leq \widetilde{C}\varrho^\alpha$ holds for some $\alpha \in (0, 1)$, $\widetilde{C} = \widetilde{C}(N, p, q_0, q, \|f\|_q, |\Omega|, r) > 0$ and for all sufficiently small $\varrho > 0$ with respect to r (see Proposition 5.13).

Finally, we arrive to the so-called "ε-regularity" theorem, which says the following: there exist constants τ, a, ε, α, $\overline{r}_0 \in (0, 1)$ such that whenever $x \in \Sigma$, $0 < r < \overline{r}_0$, $B_r(x) \subset \Omega$,

$$\beta_\Sigma(x, r) + w^\tau_\Sigma(x, r) \leq \varepsilon \;\text{ and }\; \theta_\Sigma(x, r) \leq \overline{\mu},$$

then for some $\widetilde{C} = \widetilde{C}(N, p, q_0, q, \|f\|_q, |\Omega|, r) > 0$, $\beta_\Sigma(y, \varrho) \leq \widetilde{C}\varrho^\alpha$ for any point $y \in \Sigma \cap B_{ar}(x)$ and any radius $\varrho \in (0, ar)$ (see Corollary 5.14). In particular, there exists $t \in (0, 1)$ such that $\Sigma \cap \overline{B}_t(x)$ is a $C^{1,\alpha}$

regular curve. On the other hand, notice that, since closed connected sets with finite $\mathcal{H}^1$-measure are $\mathcal{H}^1$-rectifiable (see, for instance, [18, Proposition 30.1]), $\beta_\Sigma(x,r) \to 0$ as $r \to 0+$ at $\mathcal{H}^1$-a.e. $x \in \Sigma$ and hence at $\mathcal{H}^1$-a.e. $x \in \Sigma \cap \Omega$, $w_\Sigma^\tau(x,r) \to 0$ as $r \to 0+$. Moreover, in view of Besicovitch-Marstrand-Mattila Theorem (see [2, Theorem 2.63]),

$$\theta_\Sigma(x,r) \to 2 \text{ as } r \to 0+ \text{ at } \mathcal{H}^1\text{-a.e. } x \in \Sigma.$$

At the end, observing that for each $N \geq 2$, the unique positive solution $\overline{\mu}$ to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$ is strictly greater than 5, we bootstrap all the estimates and prove that every minimizer Σ of Problem 1.1 is $C^{1,\alpha}$ regular at $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$.

Absence of loops.

If a minimizer Σ of Problem 1.1 contained a homeomorphic image Γ of the circle S^1, then there would exist a point $x_0 \in \Gamma \cap \Omega$ such that there would be a sequence of relatively open sets $D_n \subset \Sigma$ satisfying: $x_0 \in D_n$ for all sufficiently large n; $\Sigma \setminus D_n$ are connected for all n; $\operatorname{diam}(D_n) \searrow 0$ as $n \to +\infty$; D_n are connected for all n; there exists the affine line T_{x_0} such that $x_0 \in T_{x_0}$ and

$$\frac{1}{r} d_H(\Sigma \cap \overline{B}_r(x_0), T_{x_0} \cap \overline{B}_r(x_0)) \to 0 \text{ as } r \to 0+$$

(see Lemma 4.1). So we could "cut out" D_n, for which $\mathcal{H}^1(D_n) \geq \operatorname{diam}(D_n)$, and estimate the resulting variation of the p-compliance in terms of $(\operatorname{diam}(D_n))^{1+b}$ for all sufficiently large n, where $b \in (0,1)$ is some fixed constant. Namely, we would obtain that for all sufficiently large n,

$$C_{f,\Omega}(\Sigma \setminus D_n) - C_{f,\Omega}(\Sigma) \leq C(\operatorname{diam}(D_n))^{1+b} \text{ and } \operatorname{diam}(D_n) \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \setminus D_n),$$

where C is a positive constant independent of n, which would lead to a contradiction with the optimality of Σ.

Absence of quadruple points.

The idea behind the proof is as follows. Assuming that a minimizer Σ of Problem 1.1 contains a quadruple point $x_0 \in \Sigma \cap \Omega$, one can change Σ in all sufficiently small neighborhoods of x_0 in order to obtain sequences $(r_n)_{n\in\mathbb{N}}$, $(\Sigma_n)_{n\in\mathbb{N}}$ such that: $r_n > 0$, $r_n \to 0$ as $n \to +\infty$; Σ_n are competitors for Σ;

$$C_{f,\Omega}(\Sigma_n) - C_{f,\Omega}(\Sigma) \leq C r_n^{1+b} \text{ and } \widetilde{C} r_n \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma_n) \tag{$\mathcal{Q}$}$$

for all sufficiently large n, where $\widetilde{C}, C > 0$ and $b \in (0,1)$ are constants independent of n. More precisely, if a minimizer Σ of Problem 1.1 contained a quadruple point $x_0 \in \Sigma \cap \Omega$, then there would be a radius $\varrho \in (0, \operatorname{diam}(\Sigma)/2)$ such that $\Sigma \cap \overline{B}_\varrho(x_0)$ would consist of exactly four distinct C^1 arcs, each of which would meet at point x_0 exactly one of the other three at an angle of 180 degrees, and each of the other two at an angle of 90 degrees. So there would be a cross K passing through x_0 (K consists of two mutually perpendicular affine lines passing through x_0) such that

$$\frac{1}{r} d_H(\Sigma \cap \overline{B}_r(x_0), K \cap \overline{B}_r(x_0)) \to 0 \text{ as } r \to 0+$$

(it is worth noting that the estimate (3.28) still holds if the affine line in Lemma 3.6 is replaced by a suitable cross, see Lemma 6.2). Furthermore, $\Sigma \cap \overline{B}_\varrho(x_0)$ would be Ahlfors regular, which would imply that for some positive constant $C_0 > 0$, without loss of generality,

$$\mathcal{H}^1(\Sigma \cap B_r(x_0)) \leq C_0 r \text{ for all } r \in (0, \varrho].$$

On the other hand, using the coarea inequality (see [33, Theorem 2.1]), for each $r \in (0, \varrho/2]$ we would obtain that

$$\mathcal{H}^1(\Sigma \cap B_{2r}(x_0)) \geq \int_0^{2r} \mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) \, dt > \int_r^{2r} \mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) \, dt.$$

Then there would exist $t \in [r, 2r]$ such that

$$\mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) \leq \frac{1}{r}\mathcal{H}^1(\Sigma \cap B_{2r}(x_0)) \leq 2C_0.$$

So we could construct a nice competitor Σ_t for Σ. Namely, let $D_t = K \cap \partial B_t(x_0)$ and let $S_4(D_t) \subset \overline{B}_t(x_0)$ be a closed set of minimum $\mathcal{H}^1$-measure in the ball $\overline{B}_t(x_0)$ which connects the all four points of D_t (as in [11], we shall call it a *Steiner connection* of the points of D_t; for more details on Steiner connections, see, for instance [22, 34, 20]). Without loss of generality, we could also assume that Σ is fairly close, in $\overline{B}_\varrho(x_0)$ and in the Hausdorff distance to K. For each point $z_i \in \Sigma \cap \partial B_t(x_0)$, let γ_i denote the geodesic in $\partial B_t(x_0)$ connecting z_i with the point of D_t closest to z_i. Let G denote the union of all arcs γ_i, and let Σ_t be defined by

$$\Sigma_t = (\Sigma \backslash B_t(x_0)) \cup G \cup S_4(D_t).$$

Then we would obtain that there exists a positive constant $\widetilde{C} > 0$ independent of t such that

$$\mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma_t) \geq \widetilde{C}t,$$

where the facts that $\mathcal{H}^1(\Sigma \cap \overline{B}_t(x_0)) \geq 4t$, $\mathcal{H}^1(G) \leq \delta t$ for some fairly small $\delta \in (0, 1)$ and that $\mathcal{H}^1(S_4(D_t)) = \sqrt{2}(\sqrt{3}+1)t \approx 3.86t$ were used. Altogether we would obtain that there exist sequences $(r_n)_{n\in\mathbb{N}}$ and $(\Sigma_n)_{n\in\mathbb{N}}$ satisfying $(\mathcal{Q})$. Next, letting n tend to $+\infty$, we would obtain a contradiction with the optimality of Σ.

1.4. Structure of the paper

In Section 2, we recall the basic definitions and notions used in the paper and prove some preparatory results that will be used in Sections 3-6. In Section 3, we establish the decay behavior of the p-energy under flatness control (we prove $(\mathcal{E})$), which will be used in Section 4 to prove Theorem 1.4 and in Section 5 to prove Theorem 1.3. In Section 6, we prove Proposition 1.5. Finally, in Appendix A, we prove several auxiliary results for the reader's convenience.

2. Preliminaries

2.1. Conventions and Notation

Conventions: in this paper, we say that a value is positive if it is strictly greater than zero, and a value is nonnegative if it is greater than or equal to zero. Euclidean spaces are endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. By N we denote an integer greater than or equal to 2. Throughout this paper, Ω will denote an open bounded set in $\mathbb{R}^N$.

Notation: we denote by $B_r(x)$, $\overline{B}_r(x)$, and $\partial B_r(x)$, respectively, the open ball, the closed ball, and the N-sphere with center x and radius r. If the center is at the origin, we write B_r, $\overline{B}_r$ and ∂B_r the corresponding balls and the N-sphere. For each set $A \subset \mathbb{R}^N$, the set A^c will denote its complement in $\mathbb{R}^N$, that is, $A^c = \mathbb{R}^N \backslash A$. We denote by $\operatorname{dist}(x, A)$, $\operatorname{diam}(A)$, $A\Delta B$, and $|A|$, respectively, the Euclidean distance from $x \in \mathbb{R}^N$ to $A \subset \mathbb{R}^N$, the diameter of A, the symmetric difference of A and $B \subset \mathbb{R}^N$, and the N-dimensional Lebesgue measure of A. We shall sometimes write points of $\mathbb{R}^N$ as $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. If $U \subset \mathbb{R}^N$ is Lebesgue measurable, then for $p \in [1, +\infty) \cup \{+\infty\}$, $L^p(U)$ will denote the space consisting of all real measurable functions on U that are p^{th}-power integrable on U if $p \in [1, +\infty)$ and are essentially bounded if $p = +\infty$; $L^p(U; \mathbb{R}^N)$ is the respective space of functions with values in $\mathbb{R}^N$. By $L^1_{loc}(U)$ we denote the space of functions u such that $u \in L^1(V)$ for all $V \subset\subset U$. The norm on $L^p(U)$ ($L^p(U; \mathbb{R}^N)$) is denoted by $\|\cdot\|_{L^p(U)}$ ($\|\cdot\|_{L^p(U;\mathbb{R}^N)}$) or $\|\cdot\|_p$ when it is appropriate. We use the standard notation for Sobolev spaces. For an open set $U \subset \mathbb{R}^N$, denote by $W_0^{1,p}(U)$ the closure of $C_0^\infty(U)$ in the Sobolev space $W^{1,p}(U)$, where $C_0^\infty(U)$ is the space of functions in $C^\infty(U)$ with compact support in U. Recall that $W^{1,p}_{loc}(U)$ is the space of functions u such that $u \in W^{1,p}(V)$ for all $V \subset\subset U$. We shall denote by $\mathcal{H}^d(A)$ the d-dimensional Hausdorff measure of A.

2.2. Definitions and some preparatory results

We begin by defining weak solutions to the p-Poisson equation,

$$-\Delta_p u := -div(|\nabla u|^{p-2}\nabla u) = f.$$

Definition 2.1. Let $U \subset \mathbb{R}^N$ be open and bounded, $p \in (1, +\infty)$ and $f \in L^1_{loc}(U)$. We say that u is a weak solution to the p-Poisson equation $-\Delta_p v = f$ in U provided $u \in W^{1,p}_{loc}(U)$ and

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \ dx = \int_U f\varphi \ dx, \tag{2.1}$$

whenever $\varphi \in C_0^\infty(U)$. If u is a weak solution to the p-Poisson equation $-\Delta_p v = f$ in U and $f = 0$, then we say that u is a weak solution to the p-Laplace equation in U.

Definition 2.2. Let $U \subset \mathbb{R}^N$ be open and bounded, $p \in (1, +\infty)$. We say that u is a *weak subsolution (supersolution)* to the p-Laplace equation in U provided $u \in W^{1,p}_{loc}(U)$ and

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \ dx \le (\ge)\, 0, \tag{2.2}$$

whenever $\varphi \in C_0^\infty(U)$ is nonnegative. If u is an upper (lower) semicontinuous weak subsolution (supersolution) to the p-Laplace equation in U, then we say that u is *p-subharmonic (p-superharmonic)* in U. If u is a continuous weak solution to the p-Laplace equation in U, then we say that u is p-harmonic in U.

Remark 2.3. We read $|0|^{p-2}0$ as 0 also when $1 < p < 2$. If $u \in W^{1,p}(U)$ is a weak solution to the p-Poisson equation $-\Delta_p v = f$ in U, where $f \in L^{q_0}(U)$ with q_0 defined in (1.1), then (2.1) holds for all $\varphi \in W_0^{1,p}(U)$.

The following basic result for weak solutions holds (see [26, Theorem 2.7]).

Theorem 2.4. *Let U be a bounded open set in $\mathbb{R}^N$ and let $u \in W^{1,p}(U)$. The following two assertions are equivalent.*

(i) u is minimizing:

$$\int_U |\nabla u|^p \ dx \le \int_U |\nabla v|^p \ dx, \quad \textit{when } \ v - u \in W_0^{1,p}(U).$$

(ii) the first variation vanishes:

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\zeta\rangle \ dx = 0, \quad \textit{when } \ \zeta \in W_0^{1,p}(U).$$

Now we introduce the notion of the Bessel capacity (see e.g. [1], [41]).

Definition 2.5. *For $p \in (1, +\infty)$, the Bessel $(1, p)$-capacity of a set $E \subset \mathbb{R}^N$ is defined as*

$$\mathrm{Cap}_p(E) = \inf\big\{\|f\|^p_{L^p(\mathbb{R}^N)} : \ g * f \ge 1 \ \textit{ on } \ E, \ \ f \in L^p(\mathbb{R}^N), \ \ f \ge 0\big\},$$

where the Bessel kernel g is defined as that function whose Fourier transform is

$$\hat{g}(\xi) = (2\pi)^{-\frac{N}{2}}(1 + |\xi|^2)^{-\frac{1}{2}}.$$

We say that a property holds p-quasi everywhere (abbreviated as p-q.e.) if it holds except on a set A where $\mathrm{Cap}_p(A) = 0$. It is worth mentioning that by [1, Corollary 2.6.8], for every $p \in (1, +\infty)$, the notion of the Bessel capacity Cap_p is equivalent to the following

$$\widetilde{\mathrm{Cap}_p}(E) = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p \ dx + \int_{\mathbb{R}^N} |u|^p \ dx : u \geq 1 \text{ on some neighborhood of } E \right\}$$

in the sense that there exists $C = C(N, p) > 0$ such that for any set $E \subset \mathbb{R}^N$,

$$\frac{1}{C} \widetilde{\mathrm{Cap}_p}(E) \leq \mathrm{Cap}_p(E) \leq C \widetilde{\mathrm{Cap}_p}(E).$$

The notion of capacity is crucial in the investigation of the pointwise behavior of Sobolev functions.

For convenience, we recall the next theorems and propositions.

Theorem 2.6. *Let $E \subset \mathbb{R}^N$ and $p \in (1, N]$. Then $\mathrm{Cap}_p(E) = 0$ if $\mathcal{H}^{N-p}(E) < +\infty$. Conversely, if $\mathrm{Cap}_p(E) = 0$, then $\mathcal{H}^{N-p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$.*

Proof. For a proof of the fact that $\mathrm{Cap}_p(E) = 0$ if $\mathcal{H}^{N-p}(E) < +\infty$, we refer to [1, Theorem 5.1.9]. The fact that if $\mathrm{Cap}_p(E) = 0$, then $\mathcal{H}^{N-p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$ follows from [1, Theorem 5.1.13]. ☐

Remark 2.7. Let $p \in (N, +\infty)$. Then there exists $C = C(N, p) > 0$ such that for any nonempty set $E \subset \mathbb{R}^N$, $\mathrm{Cap}_p(E) \geq C$. We can take $C = \mathrm{Cap}_p(\{0\})$, which is positive by [1, Proposition 2.6.1 (a)], and use the fact that the Bessel $(1, p)$-capacity is invariant under translations and is nondecreasing with respect to set inclusion.

Recall that for all $E \subset \mathbb{R}^N$ the number

$$\dim_{\mathrm{H}}(E) = \sup\{s \in [0, +\infty) : \mathcal{H}^s(E) = +\infty\} = \inf\{t \in [0, +\infty) : \mathcal{H}^t(E) = 0\}$$

is called the Hausdorff dimension of E.

Corollary 2.8. *Let $p \in (1, +\infty)$, $E \subset \mathbb{R}^N$, $\dim_{\mathrm{H}}(E) = 1$ and $\mathcal{H}^1(E) < +\infty$. Then $\mathrm{Cap}_p(E) > 0$ if and only if $p \in (N-1, +\infty)$.*

Proof of Corollary 2.8. If $p > N$, then by Remark 2.7, $\mathrm{Cap}_p(E) > 0$. Assume by contradiction that $\mathrm{Cap}_p(E) = 0$ for some $p \in (N-1, N]$. Taking $\varepsilon = (p - N + 1)/2$ so that $0 < N - p + \varepsilon < 1$, by Theorem 2.6 we get, $\mathcal{H}^{N-p+\varepsilon}(E) = 0$, but this leads to a contradiction with the fact that $\dim_{\mathrm{H}}(E) = 1$. On the other hand, if $p \in (1, N-1]$, then $\mathcal{H}^{N-p}(E) < +\infty$ and by Theorem 2.6, $\mathrm{Cap}_p(E) = 0$. This completes the proof of Corollary 2.8. ☐

Proposition 2.9. *Let $\Sigma \subset \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, $0 \leq r_0 < r_1$ and $p \in (1, N]$. Assume that*

$$\Sigma \cap \partial B_r(x_0) \neq \emptyset \ \textit{ for all } \ r \in (r_0, r_1).$$

Then there exists a constant $C > 0$, possibly depending only on N and p, such that

$$\mathrm{Cap}_p(\{0\}^{N-1} \times [0, r_1 - r_0]) \leq C \mathrm{Cap}_p(\Sigma \cap B_{r_1}(x_0)).$$

Proof. The proof is straightforward if $p \in (1, N-1]$, since in this case $\mathrm{Cap}_p(\{0\}^{N-1} \times [0, r_1 - r_0]) = 0$ according to Corollary 2.8. Assume that $p \in (N-1, N]$. Let $A(x_0, r_0) = \overline{B}_{r_0}(x_0)$ if $r_0 > 0$ and $A(x_0, r_0) = \{x_0\}$ if $r_0 = 0$. For each $x \in \Sigma \cap (B_{r_1}(x_0) \backslash A(x_0, r_0))$, we define $\Phi(x) = (\{0\}^{N-1}, |x - x_0|)$. Since Φ is 1-Lipschitz, by [1, Theorem 5.2.1], there exists $C = C(N, p) > 0$ such that

$$\mathrm{Cap}_p(\{0\}^{N-1} \times (r_0, r_1)) = \mathrm{Cap}_p(\Phi(\Sigma \cap (B_{r_1}(x_0) \backslash A(x_0, r_0)))) \leq C \mathrm{Cap}_p(\Sigma \cap (B_{r_1}(x_0) \backslash A(x_0, r_0))).$$

Notice that $\mathrm{Cap}_p(\{0\}^{N-1} \times [r_0, r_1]) \leq \mathrm{Cap}_p(\{0\}^{N-1} \times (r_0, r_1))$, since $\mathrm{Cap}_p(\cdot)$ is a subadditive set function (see, for instance, [1, Proposition 2.3.6]) and $\mathrm{Cap}_p(\{0\}^{N-1} \times \{r_i\}) = 0$ for $i = 0, 1$ by Theorem 2.6. So

$\mathrm{Cap}_p(\{0\}^{N-1} \times [r_0, r_1]) \leq C\mathrm{Cap}_p(\Sigma \cap (B_{r_1}(x_0) \backslash A(x_0, r_0)))$ for some $C = C(N,p) > 0$. Then, using the fact that the Bessel capacity is nondecreasing with respect to set inclusion and, if necessary, the fact that it is invariant under translations, we recover the desired estimate. This completes the proof of Proposition 2.9. $\square$

Corollary 2.10. *Let $\Sigma \subset \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, $0 \leq r_0 < r_1$ and $p \in (1, N]$. Assume that $\Sigma \cap \overline{B}_{r_0}(x_0) \neq \emptyset$ if $r_0 > 0$ and $x_0 \in \Sigma$ if $r_0 = 0$. Assume also that $(\Sigma \cap B_{r_1}(x_0)) \cup \partial B_{r_1}(x_0)$ is connected. Then there exists a constant $C > 0$, possibly depending only on N and p, such that*

$$\mathrm{Cap}_p(\{0\}^{N-1} \times [0, r_1 - r_0]) \leq C\mathrm{Cap}_p(\Sigma \cap B_{r_1}(x_0)).$$

Proof of Corollary 2.10. It follows from the conditions of Corollary 2.10 that $\Sigma \cap \partial B_r(x_0) \neq \emptyset$ for all $r \in (r_0, r_1)$. Then it only remains to use Proposition 2.9. This completes the proof of Corollary 2.10. $\square$

Proposition 2.11. *Let $r \in (0, 1]$ and $A_r = \{0\}^{N-1} \times [0, r]$. The following assertions hold.*

(i) If $p \in (N-1, N)$, then there exists a constant $C = C(N,p) > 0$ such that

$$r^{N-p} \leq C\mathrm{Cap}_p(A_r).$$

(ii) If $p = N$, then there exists a constant $C = C(N) > 0$ such that

$$\left(\log\left(\frac{C}{r}\right)\right)^{1-p} \leq C\mathrm{Cap}_p(A_r).$$

Proof. Since $\mathrm{diam}(A_r) \leq 1$, (i) and (ii) follows from [1, Corollary 5.1.14]. $\square$

Corollary 2.12. Let $p \in (N-1, N]$ and $\Sigma = (\{0\}^{N-1} \times (-1, 1)) \cup \partial B_1$. Then there exist $r_0, C_0 > 0$ such that

$$\frac{\mathrm{Cap}_p(\Sigma \cap B_r(x_0))}{\mathrm{Cap}_p(B_r(x_0))} \geq C_0 \tag{2.3}$$

whenever $0 < r < r_0$ and $x_0 \in \Sigma$.

Proof of Corollary 2.12. Since Σ is arcwise connected and $\mathrm{diam}(\Sigma) = 2$, setting $r_0 = 1$, we observe that $\Sigma \cap \partial B_r(x_0) \neq \emptyset$ whenever $0 < r < r_0$ and $x_0 \in \Sigma$. Then Proposition 2.9 says that for some $C = C(N,p) > 0$,

$$\mathrm{Cap}_p(\{0\}^{N-1} \times [0, r]) \leq C\mathrm{Cap}_p(\Sigma \cap B_r(x_0))$$

whenever $0 < r < r_0$ and $x_0 \in \Sigma$. However, this, together with Proposition 2.11, [1, Proposition 5.1.2], [1, Proposition 5.1.3] and [1, Proposition 5.1.4], proves that there exists a constant $C_0 > 0$ such that the desired estimate (2.3) holds for C_0 whenever $0 < r < r_0$ and $x_0 \in \Sigma$. This completes the proof of Corollary 2.12. $\square$

Definition 2.13. Let the function u be defined p-q.e. on $\mathbb{R}^N$ or on some open subset. Then u is said to be p-quasi continuous if for every $\varepsilon > 0$ there is an open set A with $\mathrm{Cap}_p(A) < \varepsilon$ such that the restriction of u to the complement of A is continuous in the induced topology.

Theorem 2.14. *Let $Y \subset \mathbb{R}^N$ be an open set and $p \in (1, +\infty)$. Then for each $u \in W^{1,p}(Y)$ there exists a p-quasi continuous function $\widetilde{u} \in W^{1,p}(Y)$, which is uniquely defined up to a set of Cap_p-capacity zero and $u = \widetilde{u}$ a.e. in Y.*

Proof. We refer the reader, for instance, to the proof of [9, Theorem 2.8], which actually applies for the general spatial dimension $N \geq 2$. $\square$

Remark 2.15. A Sobolev function $u \in W^{1,p}(\mathbb{R}^N)$ belongs to $W_0^{1,p}(Y)$ if and only if its p-quasi continuous representative $\widetilde{u}$ vanishes p-q.e. on Y^c (see [3, Theorem 4] and [23, Lemma 4]). Thus, if Y' is an open subset of Y and $u \in W_0^{1,p}(Y)$ such that $\widetilde{u} = 0$ p-q.e. on $Y \backslash Y'$, then the restriction of u to Y' belongs to $W_0^{1,p}(Y')$ and conversely, if we extend a function $u \in W_0^{1,p}(Y')$ by zero in $Y \backslash Y'$, then $u \in W_0^{1,p}(Y)$. It is worth mentioning that if $\Sigma \subset \overline{Y}$ and $\mathrm{Cap}_p(\Sigma) = 0$, then $W_0^{1,p}(Y) = W_0^{1,p}(Y \backslash \Sigma)$. Indeed, $u \in W_0^{1,p}(Y)$ if and only if $u \in W^{1,p}(\mathbb{R}^N)$ and $\widetilde{u} = 0$ p-q.e. on Y^c that is equivalent to say $u \in W^{1,p}(\mathbb{R}^N)$ and $\widetilde{u} = 0$ p-q.e. on $Y^c \cup \Sigma$ (since $\mathrm{Cap}_p(\Sigma) = 0$ and $\mathrm{Cap}_p(\cdot)$ is a subadditive set function, see [1, Proposition 2.3.6]) or $u \in W_0^{1,p}(Y \backslash \Sigma)$. In the sequel we shall always identify $u \in W^{1,p}(Y)$ with its p-quasi continuous representative $\widetilde{u}$.

Proposition 2.16. *Let $D \subset \mathbb{R}^N$ be a bounded extension domain (see [41, Remark 2.5.2]) and let $u \in W^{1,p}(D)$. Consider $E = \overline{D} \cap \{x : u(x) = 0\}$. If $\mathrm{Cap}_p(E) > 0$, then there exists a constant $C = C(N, p, D) > 0$ such that*

$$\int_D |u|^p \, dx \leq C(\mathrm{Cap}_p(E))^{-1} \int_D |\nabla u|^p \, dx.$$

Proof. For a proof, see, for instance, [41, Corollary 4.5.3, p. 195]. □

It is also worth recalling the following fact, which will be used several times in this paper.

Remark 2.17. Every closed and connected set $\Sigma \subset \mathbb{R}^N$ satisfying $\mathcal{H}^1(\Sigma) < +\infty$ is arcwise connected (see, for instance, [18, Corollary 30.2, p. 186]).

2.3. Poincaré inequality

Proposition 2.18. *Let $\Sigma \subset \mathbb{R}^N$, $\xi \in \mathbb{R}^N$ and $r > 0$ be such that $\Sigma \cap \partial B_s(\xi) \neq \emptyset$ for every $s \in [r, 2r]$. Let $p \in (N-1, +\infty)$ and $u \in W^{1,p}(B_{2r}(\xi))$ satisfying $u = 0$ p-q.e. on $\Sigma \cap B_{2r}(\xi)$. Then there exists a constant $C = C(N, p) > 0$ such that*

$$\int_{B_{2r}(\xi)} |u|^p \, dx \leq C r^p \int_{B_{2r}(\xi)} |\nabla u|^p \, dx.$$

Proof. We refer the reader to the proof of [9, Corollary 2.12], which also applies for the present geometric assumptions. □

2.4. Estimate for $E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'})$

We begin by proving the following "localization lemma".

Lemma 2.19. *Let $p \in (1, +\infty)$ and $f \in L^{q_0}(\Omega)$ with q_0 defined in (1.1). Let Σ and Σ' be closed proper subsets of $\overline{\Omega}$ and $x_0 \in \mathbb{R}^N$. Assume that $0 < r_0 < r_1$ and $\Sigma' \Delta \Sigma \subset \overline{B}_{r_0}(x_0)$. Then there exists $C = C(p) > 0$ such that for any $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ satisfying $\varphi = 1$ over $B_{r_1}^c(x_0)$, $\varphi = 0$ over $B_{r_0}(x_0)$ and $\|\varphi\|_\infty \leq 1$ on $\mathbb{R}^N$, one has*

$$E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}) \leq C \int_{B_{r_1}(x_0)} |\nabla u_{\Sigma'}|^p \, dx + C \int_{B_{r_1}(x_0)} |u_{\Sigma'}|^p |\nabla \varphi|^p \, dx + \int_{B_{r_1}(x_0)} f u_{\Sigma'} (1 - \varphi) \, dx.$$

Proof. We refer the reader to the proof of [9, Lemma 4.1], that actually applies for the general spatial dimension $N \geq 2$. □

Lemma 2.20. *Let $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q \geq q_0$, where q_0 is defined in (1.1). Assume that Σ is a closed arcwise connected proper subset of $\overline{\Omega}$ such that for some $x_0 \in \mathbb{R}^N$ and $0 < 2r_0 \leq r_1 \leq 1$ it holds*

$$\Sigma \cap B_{r_0}(x_0) \neq \emptyset, \quad \Sigma \backslash B_{r_1}(x_0) \neq \emptyset. \tag{2.4}$$

Then for any $r \in [r_0, r_1/2]$, for any $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ such that $\|\varphi\|_\infty \leq 1$, $\varphi = 1$ over $B_{2r}^c(x_0)$, $\varphi = 0$ over $B_r(x_0)$ and $\|\nabla \varphi\|_\infty \leq 1/r$, the following assertions hold.

(i) There exists $C = C(N,p) > 0$ such that

$$\int_{B_{2r}(x_0)} |u_\Sigma|^p |\nabla\varphi|^p \ dx \le C \int_{B_{2r}(x_0)} |\nabla u_\Sigma|^p \ dx. \tag{2.5}$$

(ii) There exists $C = C(N,p,q_0,q,\|f\|_q) > 0$ such that

$$\int_{B_{2r}(x_0)} f u_\Sigma (1-\varphi) \ dx \le C \int_{B_{2r}(x_0)} |\nabla u_\Sigma|^p \ dx + C r^{N+p'-\frac{Np'}{q}}. \tag{2.6}$$

Proof. Thanks to (2.4), $\Sigma \cap \partial B_s(x_0) \neq \emptyset$ for all $s \in [r, 2r]$. Then, since $u_\Sigma = 0$ p-q.e. on Σ and $u_\Sigma \in W^{1,p}(B_{2r}(x_0))$, we can use Proposition 2.18, which says that there exists $C = C(N,p) > 0$ such that

$$\int_{B_{2r}(x_0)} |u_\Sigma|^p \ dx \le C r^p \int_{B_{2r}(x_0)} |\nabla u_\Sigma|^p \ dx. \tag{2.7}$$

Therefore,

$$\int_{B_{2r}(x_0)} |u_\Sigma|^p |\nabla\varphi|^p \ dx \le \frac{1}{r^p} \int_{B_{2r}(x_0)} |u_\Sigma|^p \ dx \le C \int_{B_{2r}(x_0)} |\nabla u_\Sigma|^p \ dx.$$

This proves (2.5).

Let us now prove (2.6). First, notice that thanks to (2.7) and the fact that $2r \le 1$, there exists $C_0 = C_0(N,p) > 0$ such that

$$\|u_\Sigma\|_{W^{1,p}(B_{2r}(x_0))} \le C_0 \|\nabla u_\Sigma\|_{L^p(B_{2r}(x_0))}. \tag{2.8}$$

Next, using the Sobolev embeddings (see [21, Theorem 7.26]) together with (2.8) and the fact that $u_\Sigma = 0$ p-q.e. on Σ, we deduce that there exists $\widetilde{C} = \widetilde{C}(N,p,q_0) > 0$ such that

$$\|u_\Sigma\|_{L^{q_0'}(B_{2r}(x_0))} \le \widetilde{C} r^\beta \|\nabla u_\Sigma\|_{L^p(B_{2r}(x_0))}, \tag{2.9}$$

where

$$\beta = 0 \ \text{ if } \ N-1 < p < N, \quad \beta = \frac{N}{q_0'} \ \text{ if } \ p = N, \quad \beta = 1 - \frac{N}{p} \ \text{ if } \ N < p < +\infty.$$

Thus, using the fact that $|f u_\Sigma(1-\varphi)| \le |f u_\Sigma|$, Hölder's inequality, the estimate (2.9) and Young's inequality, we get

$$\begin{aligned}
\int_{B_{2r}(x_0)} f u_\Sigma (1-\varphi) \ dx \le \|f\|_{L^{q_0}(B_{2r}(x_0))} \|u_\Sigma\|_{L^{q_0'}(B_{2r}(x_0))} &\le |B_{2r}(x_0)|^{\frac{1}{q_0}-\frac{1}{q}} \|f\|_{L^q(\Omega)} \|u_\Sigma\|_{L^{q_0'}(B_{2r}(x_0))} \\
&\le C r^{N(\frac{1}{q_0}-\frac{1}{q})+\beta} \|\nabla u_\Sigma\|_{L^p(B_{2r}(x_0))} \\
&= C r^{\frac{N}{p'}+1-\frac{N}{q}} \|\nabla u_\Sigma\|_{L^p(B_{2r}(x_0))} \\
&\le C r^{N+p'-\frac{Np'}{q}} + C \|\nabla u_\Sigma\|^p_{L^p(B_{2r}(x_0))},
\end{aligned}$$

where $C = C(N,p,q_0,q,\|f\|_q) > 0$. This concludes the proof of Lemma 2.20. □

The following corollary follows directly from Lemma 2.19 and Lemma 2.20, thus, we omit the proof.

Corollary 2.21. *Let $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q \ge q_0$, where q_0 is defined in (1.1). Let Σ and Σ' be closed arcwise connected proper subsets of $\overline{\Omega}$ and let $x_0 \in \mathbb{R}^N$. Suppose that $0 < 2r_0 \le r_1 \le 1$, $\Sigma' \,\Delta\, \Sigma \subset \overline{B}_{r_0}(x_0)$ and*

$$\Sigma' \cap B_{r_0}(x_0) \neq \emptyset, \ \ \Sigma' \backslash B_{r_1}(x_0) \neq \emptyset.$$

Then for every $r \in [r_0, r_1/2]$,

$$E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}) \le C \int_{B_{2r}(x_0)} |\nabla u_{\Sigma'}|^p \ dx + C r^{N+p'-\frac{Np'}{q}}, \tag{2.10}$$

where $C = C(N,p,q_0,q,\|f\|_q) > 0$.

2.5. Uniform boundedness of $u_{f,\Omega,\Sigma}$ with respect to Σ

In this subsection, we prove a uniform bound, with respect to Σ, for the unique solution $u_{f,\Omega,\Sigma}$ to the Dirichlet problem (1.2). It is worth mentioning that the estimate (2.13) will never be used in the sequel, however, we find it interesting enough to keep it in the present paper. Also notice that we can extend $u_{f,\Omega,\Sigma}$ by zero outside $\Omega\backslash\Sigma$ to an element of $W^{1,p}(\mathbb{R}^N)$, we shall use the same notation for this extension as for $u_{f,\Omega,\Sigma}$.

Proposition 2.22. *Let Σ be a closed proper subset of $\overline{\Omega}$, $p \in (1,+\infty)$ and $f \in L^{q_0}(\Omega)$ with q_0 defined in (1.1). Then there exists a constant $C > 0$, possibly depending only on N, p and q_0, such that*

$$\int_\Omega |\nabla u_{f,\Omega,\Sigma}|^p \ dx \leq C|\Omega|^\alpha \|f\|^\beta_{L^{q_0}(\Omega)}, \tag{2.11}$$

where

$$(\alpha,\beta) = (0,p') \ \textit{if } 1 < p < N, \ \ (\alpha,\beta) = \left(\frac{N'}{q_0'}, N'\right) \ \textit{if } p = N, \ \ (\alpha,\beta) = \left(\frac{p-N}{N(p-1)}, p'\right) \ \textit{if } p > N. \tag{2.12}$$

Moreover, if $f \in L^q(\Omega)$ with $q > \frac{N}{p}$ if $p \in (1,N]$ and $q = 1$ if $p > N$, then there exists a constant $C = C(N,p,q,\|f\|_q, |\,\Omega\,|) > 0$ such that

$$\|u_{f,\Omega,\Sigma}\|_{L^\infty(\mathbb{R}^N)} \leq C. \tag{2.13}$$

Proof. To establish the estimate (2.13), we use [9, Lemma A.2] with $U = \Omega\backslash\Sigma$, which provides a constant $C = C(N,p,q,\|f\|_q,|U|) > 0$ such that $\|u_{f,\Omega,\Sigma}\|_{L^\infty(\mathbb{R}^N)} \leq C$, but observing that C is increasing with respect to $|U|$, we recover (2.13). Now let $f \in L^{q_0}(\Omega)$. Using $u_{f,\Omega,\Sigma}$ as a test function in (1.3), we get

$$\int_\Omega |\nabla u_{f,\Omega,\Sigma}|^p \ dx = \int_\Omega f u_{f,\Omega,\Sigma} \ dx \leq \|f\|_{L^{q_0}(\Omega)} \|u_{f,\Omega,\Sigma}\|_{L^{q_0'}(\Omega)}, \tag{2.14}$$

where the above estimate comes by using Hölder's inequality. Next, recalling that by the Sobolev inequalities (see [21, Theorem 7.10]) there is $C = C(N,p) > 0$ such that $\|u_{f,\Omega,\Sigma}\|_{L^{q_0'}(\Omega)} \leq C|\Omega|^\gamma \|\nabla u_{f,\Omega,\Sigma}\|_{L^p(\Omega)}$, where

$$\gamma = 0 \quad \text{if} \quad 1 < p < N, \qquad \gamma = \frac{1}{N} - \frac{1}{p} \quad \text{if} \quad p > N,$$

and using (2.14), we recover (2.11) in the case when $p \neq N$. If $p = N$ and $1 < q_0 \leq N$, then for $\varepsilon \in (0, N-1]$ such that $\frac{1}{q_0'} = \frac{1}{N-\varepsilon} - \frac{1}{N}$, we get

$$\begin{aligned}
\|u_{f,\Omega,\Sigma}\|_{L^{q_0'}(\Omega)} &\leq C\|\nabla u_{f,\Omega,\Sigma}\|_{L^{N-\varepsilon}(\Omega)} \ \text{(by the Sobolev inequality)} \\
&\leq C|\Omega|^{\frac{1}{q_0'}} \|\nabla u_{f,\Omega,\Sigma}\|_{L^N(\Omega)} \ \text{(by Hölder's inequality).}
\end{aligned}$$

The latter estimate together with (2.14) yields (2.11) in the case when $p = N$ and $1 < q_0 \leq N$. Assume now that $p = N$ and $q_0 > N$. Then $q_0' < N' \leq N$. Using Hölder's inequality and the fact that

$$\|u_{f,\Omega,\Sigma}\|_{L^{N'}(\Omega)} \leq C|\Omega|^{\frac{1}{N'}} \|\nabla u_{f,\Omega,\Sigma}\|_{L^N(\Omega)},$$

which was proved above, we obtain that

$$\|u_{f,\Omega,\Sigma}\|_{L^{q_0'}(\Omega)} \leq |\Omega|^{\frac{1}{q_0'} - \frac{1}{N'}} \|u_{f,\Omega,\Sigma}\|_{L^{N'}(\Omega)} \leq C|\Omega|^{\frac{1}{q_0'}} \|\nabla u_{f,\Omega,\Sigma}\|_{L^N(\Omega)}.$$

This, together with (2.14), yields (2.11) in the case when $p = N$ and $q_0 > N$, and completes the proof of Proposition 2.22. $\square$

2.6. Existence

Theorem 2.23. *Let $p \in (N-1,+\infty)$, $f \in L^{q_0}(\Omega)$ with q_0 defined in (1.1). Let $(\Sigma_n)_n \subset \mathcal{K}(\Omega)$ be a sequence converging to $\Sigma \in \mathcal{K}(\Omega)$ in the Hausdorff distance. Then $u_{\Sigma_n} \underset{n\to+\infty}{\longrightarrow} u_\Sigma$ strongly in $W^{1,p}(\Omega)$.*

Proof. For a proof, see [42] for the case $N = p = 2$ and [7] for the general case. □

Proposition 2.24. *Problem 1.1 admits a minimizer.*

Proof. Let $(\Sigma_n)_n \subset \mathcal{K}(\Omega)$ be a minimizing sequence for Problem 1.1. We can assume that $\Sigma_n \neq \emptyset$ and $C_{f,\Omega}(\Sigma_n)+\lambda\mathcal{H}^1(\Sigma_n) \leq C_{f,\Omega}(\emptyset)$ at least for a subsequence still denoted by n, because otherwise the empty set would be a minimizer. Then, by Blaschke's theorem (see [2, Theorem 6.1]), there exists $\Sigma \in \mathcal{K}(\Omega)$ such that, up to a subsequence still denoted by the same index, Σ_n converges to Σ in the Hausdorff distance as $n \to +\infty$. Furthermore, by Theorem 2.23, u_{Σ_n} converges to u_Σ strongly in $W_0^{1,p}(\Omega)$ and hence $C_{f,\Omega}(\Sigma_n) \to C_{f,\Omega}(\Sigma)$ as $n \to +\infty$. Then, using Gołąb's theorem (see, for instance, [33, Theorem 3.3]), we deduce that Σ is a minimizer of Problem 1.1. □

The next proposition says that, at least for some range of values of $\lambda > 0$, solutions to Problem 1.1 are nontrivial.

Proposition 2.25. *Let $p \in (N-1,+\infty)$, $f \in L^{q_0}(\Omega)$, $f \neq 0$ and q_0 is defined in (1.1). Then there exists a number $\lambda_0 = \lambda_0(N,p,f,\Omega) > 0$ such that if Problem 1.1 is defined for $\lambda \in (0,\lambda_0]$, then every solution to this problem has positive $\mathcal{H}^1$-measure. Moreover, if $p > N$ and Problem 1.1 is defined for an arbitrary $\lambda > 0$, then the empty set will not be a solution to this problem.*

Proof. For a proof in the case when $N = 2$, we refer the reader to [9, Proposition 2.17], the proof for the general case is similar. □

3. Decay of the p-energy under flatness control

In this section, we prove that if $p \in (N-1,+\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4), then there exist $\varepsilon_0, b, \overline{r} \in (0,1)$ and $C = C(N,p,q_0,q,\|f\|_q,|\Omega|) > 0$, where q_0 is defined in (1.1), such that the following holds. Assume that $\Sigma \subset \overline{\Omega}$ is a closed arcwise connected set, $0 < 2r_0 \leq r_1 \leq \overline{r}$, $B_{r_1}(x_0) \subset \Omega$ and that for each $r \in [r_0,r_1]$ there exists an affine line $L = L(r)$ passing through x_0 such that $d_H(\Sigma\cap\overline{B}_r(x_0), L\cap\overline{B}_r(x_0)) \leq \varepsilon_0 r$. Assume also that $\Sigma \setminus B_{r_1}(x_0) \neq \emptyset$. Then for all $r \in [r_0,r_1]$,

$$\int_{B_r(x_0)} |\nabla u_\Sigma|^p \, dx \leq C\Big(\frac{r}{r_1}\Big)^{1+b} \int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \, dx + Cr^{1+b}.$$

We begin by establishing a control for the functional $r \mapsto \int_{B_r} |\nabla u|^p \, dx$, where u is a weak solution to the p-Laplace equation in $B_1\backslash(\{0\}^{N-1}\times(-1,1))$ vanishing p-q.e. on $\{0\}^{N-1}\times(-1,1)$. In [28] it was shown that if u is a positive p-harmonic function in $B_1\backslash(\{0\}^{N-1}\times(-1,1))$, continuous in B_1 with $u = 0$ on $\{0\}^{N-1}\times(-1,1)$, then there exists $\delta = \delta(N,p) \in (0,1)$ such that $u \in C^{0,\beta}(B_\delta)$, where $\beta = (p-N+1)/(p-1)$ and $C^{0,\beta}(U)$ denotes the space of Hölder continuous functions in the open set U. Furthermore, β is the optimal Hölder exponent for u. In fact, comparing the function u with the p-superharmonic and p-subharmonic functions constructed in [28, Lemma 3.4], [27, Lemma 3.7], it was shown that there exists $C = C(N,p) > 0$ and $\delta = \delta(N,p) \in (0,1)$ such that

$$C^{-1}\operatorname{dist}(x,\{0\}^{N-1}\times(-1,1))^\beta \leq \frac{u(x)}{u(A_{1/2})} \leq C\operatorname{dist}(x,\{0\}^{N-1}\times(-1,1))^\beta \tag{3.1}$$

whenever $x \in B_\delta$, where $A_{1/2}$ is a point in $\{|x'| = 1/2\}\cap\partial B_{1/2}$. The upper bound in (3.1) implies that $u \in C^{0,\beta}(B_\delta)$ (see [28, Corollary 3.7]), and the lower bound proves that β is optimal.

However, for the purposes of this paper, the optimal regularity for a p-harmonic function vanishing on a 1-dimensional plane is not necessary. It so happened that for every $p \in (N-1,+\infty)$ we also constructed a nice barrier function in order to estimate a nonnegative p-subharmonic function vanishing on a 1-dimensional plane. More precisely, for any fixed $\gamma \in (0,\beta)$ and some $\delta = \delta(N,p,\gamma) \in (0,1)$, we constructed a p-superharmonic function in Lemma A.1, such that comparing this function with a nonnegative p-subharmonic function u in $B_1 \backslash (\{0\}^{N-1} \times (-1,1))$, continuous in B_1 and with $u = 0$ on $\{0\}^{N-1} \times (-1,1)$, we obtain the following control

$$u(x) \leq Cu(A_{1/2}) \operatorname{dist}(x, \{0\}^{N-1} \times (-1,1))^{\gamma},$$

where $x \in B_\delta$ and $C = C(N,p,\gamma) > 0$. If γ is close enough to β, using the above control, we deduce the estimate (3.2) which is sufficient to obtain the desired decay behavior of the p-energy under flatness control. Finally, since our barrier function is slightly simpler than those in [28, Lemma 3.4], [27, Lemma 3.7] and in order to make the presentation self-contained, we shall use it in the proof of Lemma 3.1.

Lemma 3.1. *Let $p \in (N-1,+\infty)$. There exist $\alpha, \delta \in (0,1)$ and $C > 0$, depending only on N and p, such that if $u \in W^{1,p}(B_1)$ is a weak solution to the p-Laplace equation in $B_1 \backslash (\{0\}^{N-1} \times (-1,1))$ satisfying $u = 0$ p-q.e. on $\{0\}^{N-1} \times (-1,1)$, then*

$$\int_{B_r} |\nabla u|^p \, dx \leq Cr^{1+\alpha} \int_{B_1} |\nabla u|^p \, dx \ \text{for all } r \in (0,\delta]. \tag{3.2}$$

Proof. To lighten the notation, we denote $\{0\}^{N-1} \times (-1,1)$ by S.
Step 1. We prove the estimate (3.2) in the case when u is continuous and nonnegative in B_1 with $u = 0$ on S. Let $\gamma = \frac{1}{2}\left(\frac{p-N+1}{p} + \frac{p-N+1}{p-1}\right)$. By Lemma A.1, there exists $\delta_0 = \delta_0(N,p) \in (0,1)$ such that $\hat{u}(x) = |x'|^\gamma + x_N^2$ is p-superharmonic in $\{0 < |x'| < \delta_0\} \cap \{|x_N| < 1\}$. On the other hand, according to Lemma A.3, there exists $\varepsilon = \varepsilon(N,p) \in (0,1)$ and $C = C(N,p) > 0$ such that $u \leq Cu(A_\varepsilon)$ in $\overline{B}_\varepsilon$, where A_ε denotes a point with $\operatorname{dist}(A_\varepsilon, \{0\}^{N-1} \times \mathbb{R}) = \varepsilon$ and $A_\varepsilon \in \partial B_\varepsilon$. Without loss of generality, we can assume that $\delta_0 \leq \varepsilon$. Hereinafter in this proof, C denotes a positive constant that can only depend on N, p and can be different from line to line. Using Harnack's inequality (see, for instance, [24, Theorem 6.2]), we deduce that $u(A_\varepsilon) \leq Cu(A_{1/2})$ and hence $u \leq Cu(A_{1/2})$ in $\overline{B}_{\delta_0}$ for a point $A_{1/2} \in \{|x'| = 1/2\} \cap \partial B_{1/2}$. Next, since

$$\hat{u}(x) = \left(\frac{\delta_0}{\sqrt{2}}\right)^\gamma + x_N^2 \geq \left(\frac{\delta_0}{\sqrt{2}}\right)^\gamma \ \text{if } |x'| = \frac{\delta_0}{\sqrt{2}} \ \text{and } \hat{u}(x) = |x'|^\gamma + \frac{\delta_0^2}{2} \geq \frac{\delta_0^2}{2} \ \text{if } |x_N| = \frac{\delta_0}{\sqrt{2}},$$

the estimate $u \leq Cu(A_{1/2})\hat{u}$ holds on $\partial(\{|x'| < \delta_0/\sqrt{2}\} \cap \{|x_N| < \delta_0/\sqrt{2}\})$; see Figure 3.1. Notice also that $u(x) \leq Cu(A_{1/2})\hat{u}(x)$ if $x \in S$. Thus, using the comparison principle (see e.g. [24, Theorem 7.6]), we obtain

$$u \leq Cu(A_{1/2})\hat{u} \ \text{in } \{|x'| < \delta_0/\sqrt{2}\} \cap \{|x_N| < \delta_0/\sqrt{2}\}. \tag{3.3}$$

Now we set $\delta := \delta_0/10$. According to Lemma A.4, u is a p-subharmonic function in B_1. Then, using Caccioppoli's inequality (see e.g. [26, Lemma 2.9] or [24, Lemma 3.27]), which is applicable to nonnegative p-subharmonic functions, and also using (3.3), for all $r \in (0,\delta]$, we deduce that

$$\begin{aligned}
\int_{B_r} |\nabla u|^p \, dx &\leq p^p r^{-p} \int_{B_{2r}} u^p \, dx \\
&\leq Cu^p(A_{1/2}) r^{-p} \int_{B_{2r}} \hat{u}^p \, dx \\
&\leq Cu^p(A_{1/2}) r^{-p} \int_{B_{2r}} (r^\gamma + r^2)^p \, dx \\
&\leq Cu^p(A_{1/2}) r^{\gamma p + N - p} \\
&= Cu^p(A_{1/2}) r^{1+\alpha},
\end{aligned} \tag{3.4}$$

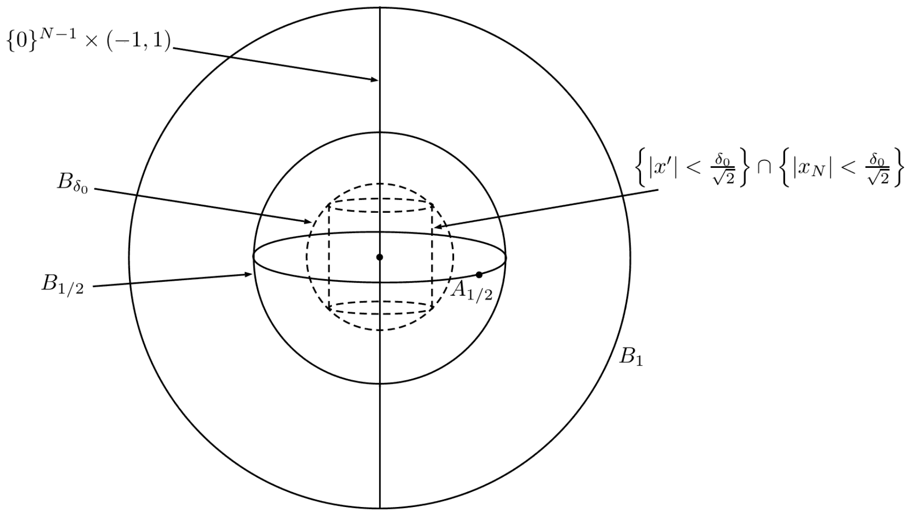

Figure 3.1: In the proof of Lemma 3.1 we estimate on $\partial\Big(\Big\{|x'|<\frac{\delta_0}{\sqrt{2}}\Big\}\cap\Big\{|x_N|<\frac{\delta_0}{\sqrt{2}}\Big\}\Big)$ a nonnegative p-harmonic function u in $B_1\backslash(\{0\}^{N-1}\times(-1,1))$, continuous in B_1 with $u=0$ on $\{0\}^{N-1}\times(-1,1)$.

where $\alpha=\gamma p-p+N-1$ is positive, since $\gamma>(p-N+1)/p$. On the other hand, by Harnack's inequality, $u(A_{1/2})\le Cu(x)$ for all $x\in B_{1/4}(A_{1/2})$ and then

$$\begin{aligned} u^p(A_{1/2}) &= \frac{1}{|B_{1/4}|}\int_{B_{1/4}(A_{1/2})} u^p(A_{1/2})\ dx\\ &\le C\int_{B_{1/4}(A_{1/2})} u^p\ dx\\ &\le C\int_{B_1} u^p\ dx\\ &\le C\int_{B_1}|\nabla u|^p\ dx, \end{aligned}\tag{3.5}$$

where we have used Proposition 2.18. Gathering together (3.4) and (3.5), we deduce (3.2).

Step 2. We prove (3.2) in the case when $u\in W^{1,p}(B_1)$ and $u=0$ p-q.e. on S. Let us fix a sequence $(\varphi_n)_{n\in\mathbb{N}}\subset C^\infty(\overline{B}_1)$ such that for each $n\in\mathbb{N}$, $\varphi_n=0$ on S and, furthermore, $\varphi_n\to u$ in $W^{1,p}(B_1)$. Let us briefly explain why such a sequence exists. For an arbitrary open set U with $B_1\subset\subset U$, according to [21, Theorem 7.25], there exists $\widetilde{u}\in W_0^{1,p}(U)$ such that $\widetilde{u}=u$ a.e. in B_1. By [1, Theorem 6.1.4], $\widetilde{u}=u$ p-q.e. in B_1 and hence $\widetilde{u}=0$ p-q.e. on S. Then $\widetilde{u}\in W_0^{1,p}(U\backslash S)$ (see Remark 2.15). So there exists a sequence $(\varphi_n)_n\subset C_0^\infty(U\backslash S)$ such that $\varphi_n\to\widetilde{u}$ in $W^{1,p}(U)$. It remains to note that $\varphi_n\to u$ in $W^{1,p}(B_1)$. Next, for each $n\in\mathbb{N}$, let u_n be the unique solution to the p-Laplace equation in $B_1\backslash S$ such that $u_n-\varphi_n\in W_0^{1,p}(B_1\backslash S)$. Notice that, by [26, Theorem 2.19], u_n is continuous in $B_1\backslash S$. On the other hand, since φ_n is continuous in $\overline{B}_1$, according to [24, Theorem 6.27], we can show that u_n is continuous in $\overline{B}_1$ if we can prove that there exist constants $C_0>0$ and $r_0>0$ such that

$$\frac{\mathrm{Cap}_p((S\cup\partial B_1)\cap B_r(x))}{\mathrm{Cap}_p(B_r(x))}\ge C_0\tag{3.6}$$

whenever $0<r<r_0$ and $x\in S\cup\partial B_1$. However, the estimate (3.6) in the case when $p\in(N-1,N]$ follows from Corollary 2.12; in the case when $p>N$, using Remark 2.7 and the fact that the Bessel capacity is invariant under translations and is nondecreasing with respect to set inclusion, it is easy to see that the estimate (3.6) holds for $C_0=\mathrm{Cap}_p(\{0\})/\mathrm{Cap}_p(B_1)$ whenever $x\in S\cup\partial B_1$ and $0<r<1$. Thus, for each $n\in\mathbb{N}$, u_n is continuous in $\overline{B}_1$. Then, by Lemma A.4, $u_n^+=\max\{u_n,0\}$ and $u_n^-=-\min\{u_n,0\}$ are nonnegative p-subharmonic functions in B_1 such that $u_n^+=u_n^-=0$ on S.

Now let v_n be the unique solution to the p-Laplace equation in $B_1\backslash S$ such that $v_n - u_n^+ \in W_0^{1,p}(B_1\backslash S)$. As before, by [26, Theorem 2.19] and [24, Theorem 6.27], v_n is continuous in $\overline{B}_1$ and also $v_n = u_n^+$ on $S \cup \partial B_1$. Then, by the comparison principle, $u_n^+ \leq v_n$ in $\overline{B}_1$. Let $\delta = \delta(N,p)$, $\alpha = \alpha(N,p) \in (0,1)$ be the constants from *Step 1.* Next, applying Caccioppoli's inequality to u_n^+, using the fact that $u_n^+ \leq v_n$ in $\overline{B}_1$ and applying the result of *Step 1* to v_n, for all $r \in (0,\delta]$ we deduce that

$$\begin{aligned}\int_{B_r} |\nabla u_n^+|^p \ dx &\leq p^p r^{-p} \int_{B_{2r}} u_n^{+p} \ dx \\ &\leq p^p r^{-p} \int_{B_{2r}} v_n^p \ dx \\ &\leq C r^{1+\alpha} \int_{B_1} |\nabla v_n|^p \ dx \\ &\leq C r^{1+\alpha} \int_{B_1} |\nabla u_n^+|^p \ dx, \end{aligned} \tag{3.7}$$

where the last estimate comes from the fact that v_n minimizes the functional $v \mapsto \int_{B_1} |\nabla v|^p \ dx$ among all $v \in W^{1,p}(B_1)$ such that $v - u_n^+ \in W_0^{1,p}(B_1\backslash S)$ (see Theorem 2.4) and u_n^+ is a competitor. Arguing by the same way as for u_n^+, we deduce that for all $r \in (0,\delta]$,

$$\int_{B_r} |\nabla u_n^-|^p \ dx \leq C r^{1+\alpha} \int_{B_1} |\nabla u_n^-|^p \ dx. \tag{3.8}$$

Next, since $\varphi_n \to u$ in $W^{1,p}(B_1)$ and u solves the Dirichlet problem $-\Delta_p v = 0$ in $B_1\backslash S$ with its own trace on $S \cup \partial B_1$, by [7, Theorem 3.5], $u_n \to u$ in $W^{1,p}(B_1)$ and hence $u_n^+ \to u^+$, $u_n^- \to u^-$ in $W^{1,p}(B_1)$. This, together with (3.7) and (3.8), implies that for all $r \in (0,\delta]$,

$$\begin{aligned}\int_{B_r} |\nabla u|^p \ dx = \int_{B_r} |\nabla u^+ - \nabla u^-|^p \ dx &\leq 2^{p-1} \int_{B_r} |\nabla u^+|^p \ dx + 2^{p-1} \int_{B_r} |\nabla u^-|^p \ dx \\ &\leq C r^{1+\alpha} \int_{B_1} |\nabla u^+|^p \ dx + C r^{1+\alpha} \int_{B_1} |\nabla u^-|^p \ dx \\ &\leq C r^{1+\alpha} \int_{B_1} |\nabla u|^p \ dx. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Now we establish an estimate for a weak solution to the p-Laplace equation in $B_r(x_0)\backslash\Sigma$ that vanishes on $\Sigma \cap B_r(x_0)$ in the case when Σ is close enough, in $\overline{B}_r(x_0)$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$.

Lemma 3.2. *Let $p \in (N-1, +\infty)$ and let $\alpha, \delta \in (0,1)$, $C > 1$ be as in Lemma 3.1. Then for each $\varrho \in (0,\delta]$ there exists $\varepsilon_0 \in (0,\varrho)$ such that the following holds. Let $\Sigma \subset \mathbb{R}^N$ be a closed set such that $(\Sigma \cap B_r(x_0)) \cup \partial B_r(x_0)$ is connected and assume that for some affine line L passing through x_0, $d_H(\Sigma \cap \overline{B}_r(x_0), L \cap \overline{B}_r(x_0)) \leq \varepsilon_0 r$. Then for any weak solution $u \in W^{1,p}(B_r(x_0))$ to the p-Laplace equation in $B_r(x_0)\backslash\Sigma$ vanishing p-q.e. on $\Sigma \cap B_r(x_0)$, the following estimate holds*

$$\int_{B_{\varrho r}(x_0)} |\nabla u|^p \ dx \leq (C\varrho)^{1+\alpha} \int_{B_r(x_0)} |\nabla u|^p \ dx.$$

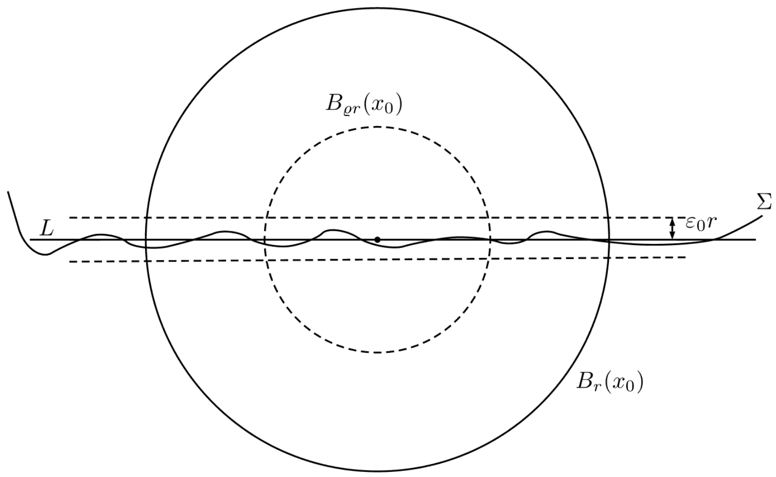

Figure 3.2: The geometry in Lemma 3.2.

Proof. Since the p-Laplacian is invariant under scalings, rotations and translations, we can assume that $B_r(x_0) = B_1$ and $L \cap \overline{B}_r(x_0) = \{0\}^{N-1} \times [-1, 1]$. To simplify the notation, we denote $\{0\}^{N-1} \times [-1, 1]$ by S. By contradiction, suppose that for some $\varrho \in (0, \delta]$ there exist sequences $(\varepsilon_n)_n$, $(\Sigma_n)_n$ and $(u_n)_n$ such that for each $n \in \mathbb{N}$: $\varepsilon_n \in (0, \varrho)$, $\varepsilon_n \downarrow 0$ as $n \to +\infty$; Σ_n is closed, $(\Sigma_n \cap B_1) \cup \partial B_1$ is connected, $d_H(\Sigma_n \cap \overline{B}_1, S) \le \varepsilon_n$ implying that

$$d_H(\Sigma_n \cap \overline{B}_1, S) \to 0 \ \text{ as } \ n \to +\infty; \tag{3.9}$$

u_n is a weak solution to the p-Laplace equation in $B_1 \setminus \Sigma_n$, $u_n = 0$ p-q.e. on $\Sigma_n \cap B_1$ and

$$\int_{B_\varrho} |\nabla u_n|^p \ dx > (C\varrho)^{1+\alpha} \int_{B_1} |\nabla u_n|^p \ dx. \tag{3.10}$$

Next, for each $n \in \mathbb{N}$, we define $v_n \in W^{1,p}(B_1)$ as

$$v_n(\cdot) = \frac{u_n(\cdot)}{\left(\int_{B_1} |\nabla u_n|^p \ dx\right)^{\frac{1}{p}}}. \tag{3.11}$$

Notice that $v_n = 0$ p-q.e. on $\Sigma_n \cap B_1$ and

$$\int_{B_1} |\nabla v_n|^p \ dx = 1. \tag{3.12}$$

On the other hand, for each $n \in \mathbb{N}$, $\Sigma_n \cap B_\delta \neq \emptyset$. This, together with the fact that $(\Sigma_n \cap B_1) \cup \partial B_1$ is connected, according to Corollary 2.10 and Proposition 2.11 in the case when $p \in (N-1, N]$, and according to Remark 2.7 in the case when $p \in (N, +\infty)$, implies that there exists a constant $\widetilde{C} > 0$ (independent of n) such that for each $n \in \mathbb{N}$,

$$\operatorname{Cap}_p(\Sigma_n \cap B_1) \ge \widetilde{C}.$$

Using the above estimate together with Proposition 2.16 and with (3.12), we conclude that the sequence $(v_n)_n$ is bounded in $W^{1,p}(B_1)$. Hence, up to a subsequence still denoted by the same index, we have

$$v_n \rightharpoonup v \ \text{ weakly in } \ W^{1,p}(B_1) \tag{3.13}$$

$$v_n \to v \ \text{ strongly in } \ L^p(B_1), \tag{3.14}$$

for some $v \in W^{1,p}(B_1)$. Let us now show that $v = 0$ p-q.e. on $S \cap B_1$. For each $t \in (0, 1)$, we fix a function $\psi \in C_0^1(B_1)$ such that $\psi = 1$ on $\overline{B}_t$ and $0 \le \psi \le 1$. Since $(\Sigma_n \cap B_1) \cup \partial B_1$ is connected for each $n \in \mathbb{N}$ and $d_H(\Sigma_n \cap \overline{B}_1, S) \to 0$ as $n \to +\infty$, it follows (see Section 6 in [7]) that the sequence of Sobolev spaces $W_0^{1,p}(B_1 \setminus \Sigma_n)$ converges in the sense of Mosco to $W_0^{1,p}(B_1 \setminus S)$. Notice that for each $n \in \mathbb{N}$, $v_n\psi \in W_0^{1,p}(B_1 \setminus \Sigma_n)$ and by (3.13), $v_n\psi \rightharpoonup v\psi$ weakly in $W^{1,p}(\mathbb{R}^N)$. Then, using the definition of limit in the sense of Mosco, we deduce that $v\psi \in W_0^{1,p}(B_1 \setminus S)$. This implies that $v = 0$ p-q.e. on $\{0\}^{N-1} \times [-t, t]$ (see Remark 2.15). As $t \in (0, 1)$ was arbitrarily chosen, $v = 0$ p-q.e. on $S \cap B_1$.

We claim that v is a weak solution to the p-Laplace equation in $B_1 \setminus S$, that is,

$$\int_{B_1} \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \, dx = 0 \text{ for all } \varphi \in C_0^\infty(B_1 \setminus S). \tag{3.15}$$

In order to get the equality above, it suffices to show that $|\nabla v_n|^{p-2}\nabla v_n \rightharpoonup |\nabla v|^{p-2}\nabla v$ weakly in $L^{p'}(B_1; \mathbb{R}^N)$. In fact, if $\varphi \in C_0^\infty(B_1 \setminus S)$, then $\{\varphi \neq 0\} \subset\subset B_1 \setminus S$ and thanks to (3.9), for all n large enough, $\{\varphi \neq 0\} \subset\subset B_1 \setminus \Sigma_n$, so we can write the following

$$\int_{B_1} \langle |\nabla v_n|^{p-2} \nabla v_n, \nabla \varphi \rangle \, dx = 0.$$

Next, letting n tend to $+\infty$ in the above equality and using that $|\nabla v_n|^{p-2}\nabla v_n \rightharpoonup |\nabla v|^{p-2}\nabla v$ weakly in $L^{p'}(B_1; \mathbb{R}^N)$, we would obtain (3.15). We first prove that, at least for a subsequence, $\nabla v_n \to \nabla v$ a.e. in B_1. For each integer $m \geq 10$, we define $\Omega_m := \{x \in B_1 : \mathrm{dist}(x, S) > 1/m\}$. Notice that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega_m)$ and for all n large enough (with respect to m), v_n is a weak solution to the p-Laplace equation in Ω_m. Then, according to [4, Theorem 2.1], there exists a subsequence $(v_{n(m,k)})_{k \in \mathbb{N}}$ such that $\nabla v_{n(m,k)} \to \nabla v$ a.e. in Ω_m. For each m as above, let $(v_{n(m+1,k)})_{k \in \mathbb{N}}$ be a subsequence of $(v_{n(m,k)})_{k \in \mathbb{N}}$ satisfying $\nabla v_{n(m+1,k)} \to \nabla v$ a.e. in Ω_{m+1}. Thus, for the diagonal subsequence $(v_{n(m,m)})_{m \in \mathbb{N}}$, $\nabla v_{n(m,m)} \to \nabla v$ a.e. in B_1. So, at least for a subsequence, $\nabla v_n \to \nabla v$ a.e. in B_1. On the other hand, since $(v_n)_n$ is bounded in $W^{1,p}(B_1)$, there exists $w \in L^{p'}(B_1; \mathbb{R}^N)$ such that, up to a subsequence still denoted by the same index, $|\nabla v_n|^{p-2}\nabla v_n \rightharpoonup w$ weakly in $L^{p'}(B_1; \mathbb{R}^N)$. Then, using the fact that, up to a subsequence, $|\nabla v_n|^{p-2}\nabla v_n \to |\nabla v|^{p-2}\nabla v$ a.e. in B_1 (we read $|0|^{p-2}0$ as 0 also when $1 < p < 2$) and using Mazur's lemma (see [40, Theorem 2, p.120]), we deduce that $w = |\nabla v|^{p-2}\nabla v$. We can now conclude that $|\nabla v_n|^{p-2}\nabla v_n \rightharpoonup |\nabla v|^{p-2}\nabla v$ weakly in $L^{p'}(B_1; \mathbb{R}^N)$. This proves the claim.

We now want to prove the strong convergence of ∇v_n to ∇v in $L^p(B_\delta; \mathbb{R}^N)$. Since $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^p(B_\delta; \mathbb{R}^N)$, we only need to prove that $\|\nabla v_n\|_{L^p(B_\delta; \mathbb{R}^N)}$ tends to $\|\nabla v\|_{L^p(B_\delta; \mathbb{R}^N)}$. By the weak convergence, we already have that

$$\int_{B_\delta} |\nabla v|^p \, dx \leq \liminf_{n \to +\infty} \int_{B_\delta} |\nabla v_n|^p \, dx.$$

Thus, it remains to prove the reverse inequality with a limsup. For this, for an arbitrary $\varepsilon \in (\delta, 1)$, we fix $\chi_\varepsilon \in C_0^\infty(B_1)$ such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon = 1$ on $\overline{B}_\delta$, $\chi_\varepsilon = 0$ on B_ε^c and $\|\nabla \chi_\varepsilon\|_\infty \leq 2/(\varepsilon - \delta)$. Notice that $v_n\chi_\varepsilon \in W_0^{1,p}(B_1 \setminus \Sigma_n)$. Then, since $v_n \in W^{1,p}(B_1)$ is a weak solution to the p-Laplace equation in $B_1 \setminus \Sigma_n$ and $\chi_\varepsilon = 0$ on B_ε^c,

$$\int_{B_\varepsilon} \chi_\varepsilon |\nabla v_n|^p \, dx = -\int_{B_\varepsilon} v_n \langle |\nabla v_n|^{p-2} \nabla v_n, \nabla \chi_\varepsilon \rangle \, dx.$$

On the other hand, from the fact that $\|\nabla \chi_\varepsilon\|_\infty \leq 2/(\varepsilon - \delta)$, (3.12), (3.14) and since $|\nabla v_n|^{p-2}\nabla v_n$ weakly converges to $|\nabla v|^{p-2}\nabla v$ in $L^{p'}(B_\varepsilon; \mathbb{R}^N)$, it follows that

$$\lim_{n \to +\infty} -\int_{B_\varepsilon} v_n \langle |\nabla v_n|^{p-2} \nabla v_n, \nabla \chi_\varepsilon \rangle \, dx = -\int_{B_\varepsilon} v \langle |\nabla v|^{p-2} \nabla v, \nabla \chi_\varepsilon \rangle \, dx = \int_{B_\varepsilon} \chi_\varepsilon |\nabla v|^p \, dx,$$

where to get the latter equality we have used that $v \in W^{1,p}(B_1)$ is a weak solution to the p-Laplace equation in $B_1 \setminus S$, $v\chi_\varepsilon \in W_0^{1,p}(B_1 \setminus S)$ and $\chi_\varepsilon = 0$ on B_ε^c. So we obtain that

$$\limsup_{n \to +\infty} \int_{B_\delta} |\nabla v_n|^p \, dx \leq \limsup_{n \to +\infty} \int_{B_\varepsilon} \chi_\varepsilon |\nabla v_n|^p \, dx = \int_{B_\varepsilon} \chi_\varepsilon |\nabla v|^p \, dx.$$

Next, letting ε tend to $\delta+$ and using Lebesgue's dominated convergence theorem, we get

$$\limsup_{n\to+\infty} \int_{B_\delta} |\nabla v_n|^p \ dx \leq \int_{B_\delta} |\nabla v|^p \ dx.$$

Thus, we have proved the strong convergence of ∇v_n to ∇v in $L^p(B_\delta;\mathbb{R}^N)$. Using (3.10), (3.11) and passing to the limit, we therefore arrive at

$$\int_{B_\varrho} |\nabla v|^p \ dx \geq (C\varrho)^{1+\alpha}. \tag{3.16}$$

However, Lemma 3.1, together with (3.12) and (3.13), says the following

$$\int_{B_\varrho} |\nabla v|^p \ dx \leq C\varrho^{1+\alpha},$$

which leads to a contradiction with (3.16), since $\alpha, \varrho > 0$ and $C > 1$. This completes the proof of Lemma 3.2.

□

Now we want to establish an estimate for a weak solution to the p-Poisson equation in $B_r(x_0)\backslash\Sigma$ that vanishes on $\Sigma \cap B_r(x_0)$ in the case when Σ is sufficiently close, in $\overline{B}_r(x_0)$ and in the Hausdorff distance, to a diameter of $\overline{B}_r(x_0)$. For that purpose, in the following lemma we control the difference between a weak solution to the p-Poisson equation and its p-Dirichlet replacement in a ball with a crack.

Lemma 3.3. *Let $p \in (N-1, +\infty)$ and $f \in L^q(B_{r_1}(x_0))$ with $q > q_0$, where q_0 is defined in (1.1). Let Σ be a closed arcwise connected set in $\mathbb{R}^N$ and $0 < 2r_0 \leq r_1 \leq 1$ satisfy*

$$\Sigma \cap B_{r_0}(x_0) \neq \emptyset, \ \ \Sigma \backslash B_{r_1}(x_0) \neq \emptyset \ \text{ and } \ B_{r_1}(x_0)\backslash\Sigma \neq \emptyset. \tag{3.17}$$

Let $u \in W^{1,p}(B_{r_1}(x_0))$ satisfying $u = 0$ p-q.e. on $\Sigma \cap B_{r_1}(x_0)$ be a weak solution to the p-Poisson equation $-\Delta_p v = f$ in $B_{r_1}(x_0)\backslash\Sigma$. Let $w \in W^{1,p}(B_{r_1}(x_0))$ be the unique solution to the p-Laplace equation in $B_{r_1}(x_0)\backslash\Sigma$ such that $w - u \in W_0^{1,p}(B_{r_1}(x_0)\backslash\Sigma)$. If $2 \leq p < +\infty$, then

$$\int_{B_{r_1}(x_0)} |\nabla u - \nabla w|^p \ dx \leq Cr_1^{N+p'-\frac{Np'}{q}}, \tag{3.18}$$

where $C = C(N, p, q_0, q, \|f\|_q) > 0$.
If $1 < p < 2$, then

$$\int_{B_{r_1}(x_0)} |\nabla u - \nabla w|^p \ dx \leq C(I(u))^p(r_1^{p-1})^{2+p'-\frac{2p'}{q}}, \tag{3.19}$$

where $C = C(p, q_0, q, \|f\|_q) > 0$ and $I(u) = 2^{\frac{2}{p}}\left(\int_{B_{r_1}(x_0)} |\nabla u|^p \ dx\right)^{\frac{2-p}{p}}$.

Remark 3.4. Observe that for any $N \geq 2$ and any $p \in (N-1, +\infty)$, $N + p' - Np'/q$ is positive if $q > q_0$, where q_0 is defined in (1.1).

Proof. We provide a proof of the estimate (3.18), for a proof of the estimate (3.19) see [9, Lemma 4.9]. For convenience, we define $z = u - w$. Thanks to (3.17) and the fact that $z = 0$ p-q.e. on $\Sigma \cap B_{r_1}(x_0)$, by Proposition 2.18, there exists $C_0 = C_0(N, p) > 0$ such that

$$\|z\|_{L^p(B_{r_1}(x_0))} \leq C_0 r_1 \|\nabla z\|_{L^p(B_{r_1}(x_0))}.$$

Since $r_1 \leq 1$, the above estimate leads to the following

$$\|z\|_{W^{1,p}(B_{r_1}(x_0))} \leq C\|\nabla z\|_{L^p(B_{r_1}(x_0))}, \tag{3.20}$$

where $C = C(N,p) > 0$. Then, using the Sobolev embeddings (see [21, Theorem 7.26]) together with (3.20) and in the case when $N < p < +\infty$ using also that $z(\xi) = 0$ for some $\xi \in \Sigma \cap B_{r_1}(x_0)$, yielding that $|z(x)| = |z(x) - z(\xi)| \le C'(2r_1)^{1-\frac{N}{p}} \|z\|_{W^{1,p}(B_{r_1}(x_0))}$ for some $C' = C'(N,p) > 0$, we deduce the following

$$\|z\|_{L^{q_0'}(B_{r_1}(x_0))} \le \widetilde{C} r_1^{\alpha} \|\nabla z\|_{L^p(B_{r_1}(x_0))}, \tag{3.21}$$

where $\widetilde{C} = \widetilde{C}(N,p,q_0) > 0$ and

$$\alpha = 0 \ \text{ if } \ 2 \le N-1 < p < N, \quad \alpha = \frac{N}{q_0'} \ \text{ if } \ p = N, \quad \alpha = 1 - \frac{N}{p} \ \text{ if } \ N < p < +\infty.$$

Next, according to [17, Lemma 2.2], there exists $c_0 = c_0(p) > 0$ such that,

$$\int_{B_{r_1}(x_0)} |\nabla z|^p \ dx \le c_0 \int_{B_{r_1}(x_0)} \langle |\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w, \nabla z\rangle \ dx,$$

and, since $z \in W_0^{1,p}(B_{r_1}(x_0)\backslash\Sigma)$, we get

$$\int_{B_{r_1}(x_0)} |\nabla z|^p \ dx \le c_0 \int_{B_{r_1}(x_0)} \langle |\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w, \nabla z\rangle \ dx = c_0 \int_{B_{r_1}(x_0)} fz \ dx.$$

Applying Hölder's inequality to the right-hand side of the above formula and using (3.21), we obtain

$$\begin{aligned}\int_{B_{r_1}(x_0)} |\nabla z|^p \ dx \le c_0 \|f\|_{L^{q_0}(B_{r_1}(x_0))} \|z\|_{L^{q_0'}(B_{r_1}(x_0))} &\le c_0 |B_{r_1}(x_0)|^{\frac{1}{q_0}-\frac{1}{q}} \|f\|_{L^q(B_{r_1}(x_0))} \|z\|_{L^{q_0'}(B_{r_1}(x_0))} \\ &\le C r_1^{N(\frac{1}{q_0}-\frac{1}{q})+\alpha} \left(\int_{B_{r_1}(x_0)} |\nabla z|^p \ dx\right)^{\frac{1}{p}}\end{aligned}$$

for some $C = C(N,p,q_0,q,\|f\|_q) > 0$. Therefore,

$$\int_{B_{r_1}(x_0)} |\nabla z|^p \ dx \le C^{p'} r_1^{Np'(\frac{1}{q_0}-\frac{1}{q})+p'\alpha} = C^{p'} r_1^{N+p'-\frac{Np'}{q}}.$$

This completes the proof of Lemma 3.3. $\square$

Using together Lemma 3.2 and Lemma 3.3, we obtain the following estimate for the solution u_Σ to the Dirichlet problem $-\Delta_p u = f$ in $\Omega\backslash\Sigma$, $u \in W_0^{1,p}(\Omega\backslash\Sigma)$. Notice that in the following statement the definition of $\gamma(p,q)$ also depends on N, but we decided not to mention it explicitly to simplify the notation.

Lemma 3.5. *Let $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q > q_0$, where q_0 is defined in (1.1). Then there exist $a \in (0,1/2)$, $\varepsilon_0 \in (0,a)$ and $C = C(N,p,q_0,q,\|f\|_q,|\Omega|) > 0$ such that the following holds. Assume that $\Sigma \subset \overline{\Omega}$ is a closed arcwise connected set, $0 < 2r_0 \le r_1 \le 1$, $B_{r_1}(x_0) \subset \Omega$,*

$$\Sigma \cap B_{r_0}(x_0) \neq \emptyset \ \text{ and } \ \Sigma \backslash B_{r_1}(x_0) \neq \emptyset.$$

In addition, suppose that there exists an affine line $L \subset \mathbb{R}^N$ passing through x_0 such that

$$d_H(\Sigma \cap \overline{B}_{r_1}(x_0), L \cap \overline{B}_{r_1}(x_0)) \le \varepsilon_0 r_1. \tag{3.22}$$

Then

$$\frac{1}{ar_1} \int_{B_{ar_1}(x_0)} |\nabla u_\Sigma|^p \ dx \le \frac{1}{2}\left(\frac{1}{r_1}\int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \ dx\right) + C r_1^{\gamma(p,q)}, \tag{3.23}$$

where

$$\gamma(p,q) = N - 1 + p' - \frac{Np'}{q} \ \text{ if } \ 2 \le p < +\infty, \qquad \gamma(p,q) = 3p - 3 - \frac{2p}{q} \ \text{ if } \ 1 < p < 2. \tag{3.24}$$

Proof. Let $w \in W^{1,p}(B_{r_1}(x_0))$ be the unique solution to the p-Laplace equation in $B_{r_1}(x_0)\backslash\Sigma$ such that $w - u_\Sigma \in W_0^{1,p}(B_{r_1}(x_0)\backslash\Sigma)$. Let $I(\cdot)$ be as in Lemma 3.3. Using (2.11) and Hölder's inequality, it is easy to see that

$$I(u_\Sigma) \leq C_1 \tag{3.25}$$

for some $C_1 = C_1(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$. Then, applying Lemma 3.3 and using (3.25), we get

$$\int_{B_{r_1}(x_0)} |\nabla u_\Sigma - \nabla w|^p \ dx \leq C r_1^{1+\gamma(p,q)}, \tag{3.26}$$

where $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$ and $\gamma(p,q)$ is defined in (3.24). Now let $\alpha, \delta \in (0,1)$ and $\widetilde{C} > 1$, depending only on N and p, be as in Lemma 3.1, where $\widetilde{C}$ is such that the estimate (3.2) holds with C replaced by $\widetilde{C}$. Define $a = \min\{\delta, (2^{-p}\widetilde{C}^{-1-\alpha})^{\frac{1}{\alpha}}\}$. For each $N \geq 2$ and $p \in (N-1, +\infty)$, the constant a is fixed. Applying Lemma 3.2 with $r = r_1$ and $\varrho = a$, we obtain some $\varepsilon_0 \in (0, a)$ such that under the condition (3.22),

$$\frac{1}{a}\int_{B_{ar_1}(x_0)} |\nabla w|^p \ dx \leq \widetilde{C}^{1+\alpha} a^\alpha \int_{B_{r_1}(x_0)} |\nabla w|^p \ dx \leq 2^{-p} \int_{B_{r_1}(x_0)} |\nabla w|^p \ dx. \tag{3.27}$$

Hereinafter in this proof, C denotes a positive constant that can only depend on N, p, q_0, q, $\|f\|_q$, $|\Omega|$ and can be different from line to line. Since for any nonnegative numbers c and d, $(c+d)^p \leq 2^{p-1}(c^p + d^p)$, we have

$$\begin{aligned}
\frac{1}{a}\int_{B_{ar_1}(x_0)} |\nabla u_\Sigma|^p \ dx \quad &\leq \quad \frac{2^{p-1}}{a}\int_{B_{ar_1}(x_0)} |\nabla w|^p \ dx + \frac{2^{p-1}}{a}\int_{B_{ar_1}(x_0)} |\nabla u_\Sigma - \nabla w|^p \ dx \\
&\leq \quad \frac{1}{2}\int_{B_{r_1}(x_0)} |\nabla w|^p \ dx + \frac{2^{p-1}}{a}\int_{B_{r_1}(x_0)} |\nabla u_\Sigma - \nabla w|^p \ dx \\
&\leq \quad \frac{1}{2}\int_{B_{r_1}(x_0)} |\nabla w|^p \ dx + C r_1^{1+\gamma(p,q)} \\
&\leq \quad \frac{1}{2}\int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \ dx + C r_1^{1+\gamma(p,q)},
\end{aligned}$$

where we have used (3.27), (3.26), and to obtain the last estimate, Theorem 2.4. The proof of Lemma 3.5 follows by dividing the resulting inequality by r_1. □

Finally, by iterating Lemma 3.5 in a sequence of balls $\{B_{a^l r_1}(x_0)\}_l$, we obtain the desired decay behavior of the p-energy $r \mapsto \int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx$ under flatness control on Σ at $x_0 \in \Omega$.

Lemma 3.6. *Let $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Then there exist $\varepsilon_0, b, \overline{r} \in (0,1)$ and $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$, where q_0 is defined in (1.1), such that the following holds. Assume that $\Sigma \subset \overline{\Omega}$ is a closed arcwise connected set, $0 < 2r_0 \leq r_1 \leq \overline{r}$, $B_{r_1}(x_0) \subset \Omega$ and that for each $r \in [r_0, r_1]$ there exists an affine line $L = L(r)$ passing through x_0 such that $d_H(\Sigma \cap \overline{B}_r(x_0), L \cap \overline{B}_r(x_0)) \leq \varepsilon_0 r$. Assume also that $\Sigma \setminus B_{r_1}(x_0) \neq \emptyset$. Then for all $r \in [r_0, r_1]$,*

$$\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx \leq C\Big(\frac{r}{r_1}\Big)^{1+b} \int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \ dx + C r^{1+b}. \tag{3.28}$$

Proof. Let $a \in (0, 1/2)$, $\varepsilon_0 \in (0, a)$ and $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$ be the constants given by Lemma 3.5. The definition of q_1 and the assumption $q > q_1$ have been made in order to guarantee that $\gamma(p,q) > 0$, where $\gamma(p,q)$ is defined in (3.24). Let us now define

$$b = \min\left\{\frac{\gamma(p,q)}{2}, \frac{\ln(3/4)}{\ln(a)}\right\}, \quad \bar{r} = \left(\frac{1}{4}\right)^{\frac{1}{b}}.$$

Notice that for all $t \in (0, \overline{r}]$,

$$\frac{1}{2}t^b + t^{\gamma(p,q)} \leq (at)^b. \tag{3.29}$$

Indeed, since $0 < 2b \leq \gamma(p,q)$, $b \leq \ln(3/4)/\ln(a)$ and $a, \overline{r} \in (0,1)$, $t^{\gamma(p,q)} \leq t^{2b} \leq \overline{r}^b t^b$ and $3/4 \leq a^b$, so

$$\frac{1}{2}t^b + t^{\gamma(p,q)} \leq \frac{1}{2}t^b + \overline{r}^b t^b \leq \frac{3}{4}t^b \leq (at)^b.$$

It is worth noting that $\Sigma \cap B_{r_0}(x_0) \neq \emptyset$, which comes from the assumption

$$d_H(\Sigma \cap \overline{B}_{r_0}(x_0), L(r_0) \cap \overline{B}_{r_0}(x_0)) \leq \varepsilon_0 r_0.$$

Under the assumptions of Lemma 3.6, we can apply Lemma 3.5 in all the balls $B_{a^l r_1}(x_0)$, $l \in \{0, ..., k\}$, where $k \in \mathbb{N}$ is such that $a^{k+1}r_1 < r_0 \leq a^k r_1$. Next, we define $\Psi(r) = \frac{1}{r}\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx$, $r \in (0, r_1]$ and prove by induction that for each $l \in \{0, ..., k\}$,

$$\Psi(a^l r_1) \quad \leq \quad \frac{1}{2^l}\Psi(r_1) + C(a^l r_1)^b. \tag{3.30}$$

It is clear that (3.30) holds for $l = 0$. Assume that (3.30) holds for some $l \in \{0, ..., k-1\}$. Then, applying Lemma 3.5 and using the induction hypothesis, we get

$$\Psi(a^{l+1}r_1) \leq \frac{1}{2}\Psi(a^l r_1) + C(a^l r_1)^{\gamma(p,q)} \leq \frac{1}{2}\left(\frac{1}{2^l}\Psi(r_1) + C(a^l r_1)^b\right) + C(a^l r_1)^{\gamma(p,q)}.$$

Thanks to (3.29), we finally conclude that

$$\Psi\big(a^{l+1}r_1\big) \leq \frac{1}{2^{l+1}}\Psi(r_1) + C\big(a^{l+1}r_1\big)^b.$$

Thus (3.30) is proved. Now let $r \in [r_0, r_1]$ and $l \in \{0, ..., k\}$ be such that $a^{l+1}r_1 < r \leq a^l r_1$. Then

$$\begin{aligned}\Psi(r) \leq \frac{1}{a}\Psi(a^l r_1) \leq \frac{1}{a}\frac{1}{2^l}\Psi(r_1) + \frac{C}{a}(a^l r_1)^b &\leq \frac{2}{a}(a^{l+1})^b \Psi(r_1) + C'(a^{l+1}r_1)^b \\ &\leq C''\left(\frac{r}{r_1}\right)^b \Psi(r_1) + C'' r^b,\end{aligned}$$

where $C'' = C''(a, N, p, q_0, q, \|f\|_q, |\Omega|) > 0$. Since a is fixed for each $N \geq 2$ and $p \in (N-1, +\infty)$, we can assume that C'' depends only on $N, p, q_0, q, \|f\|_q$ and $|\Omega|$. This completes the proof of Lemma 3.6. $\square$

4. Absence of loops

In this section, we prove Theorem 1.4. The next lemma will be used in the proof of Theorem 1.4.

Lemma 4.1. *Let Σ be a closed connected set in $\mathbb{R}^N$ with $\mathcal{H}^1(\Sigma) < +\infty$. Then the following assertions hold.*

- *If Σ contains a simple closed curve Γ, then $\mathcal{H}^1$-a.e. point $x \in \Gamma$ is a "noncut" point, namely, there exists a sequence of relatively open sets $D_n \subset \Sigma$ satisfying*
  - *(i) $x \in D_n$ for all sufficiently large n;*
  - *(ii) $\Sigma \setminus D_n$ are connected for all n;*
  - *(iii) $\operatorname{diam} D_n \searrow 0$ as $n \to +\infty$;*
  - *(iv) D_n are connected for all n.*
- *"flatness" : for $\mathcal{H}^1$-a.e. point $x \in \Sigma$ there exists the "tangent" line T_x to Σ at x in the sense that $x \in T_x$ and*

$$\frac{1}{r}d_H(\Sigma \cap \overline{B}_r(x), T_x \cap \overline{B}_r(x)) \underset{r \to 0+}{\to 0}.$$

Proof. By [33, Lemma 5.6], $\mathcal{H}^1$-a.e. point $x \in \Gamma$ is a noncut point for Σ (i.e., a point such that $\Sigma \setminus \{x\}$ is connected). Then, by [32, Lemma 5.3], it follows that for each noncut point there are connected neighborhoods D_n that can be cut leaving the set connected and $\operatorname{diam}(D_n) \searrow 0$, so (i)-(iv) are satisfied for a suitable sequence D_n. Let us now prove the second assertion of Lemma 4.1. First, notice that, there is a Lipschitz surjective mapping $g : [0, L] \to \Sigma$, where $L = \mathcal{H}^1(\Sigma)$ (see, for instance, [18, Proposition 30.1]). Furthermore, in [29, Proposition 3.4], it was proved that $\mathcal{H}^1(\Sigma \setminus \Sigma_0) = 0$, where

$$\begin{aligned}\Sigma_0 = \{x \in \Sigma : t \in (0, L),\ \ g'(t) \text{ exists, } |g'(t)| = 1 \text{ whenever } g(t) = x,\ \ g^{-1}(x) \text{ is finite}\\ \text{and if } g(t) = g(s) = x, \text{ then } g'(t) = \pm g'(s)\},\end{aligned}$$

and that for all $x \in \Sigma_0$,

$$\frac{1}{r} \max_{y \in \Sigma \cap \overline{B}_r(x)} \operatorname{dist}(y, T_x \cap \overline{B}_r(x)) \underset{r \to 0+}{\to} 0, \tag{4.1}$$

where $T_x = x + Span(g'(t)),\ x = g(t)$. In order to prove that

$$\frac{1}{r} \max_{y \in T_x \cap \overline{B}_r(x)} \operatorname{dist}(y, \Sigma \cap \overline{B}_r(x)) \underset{r \to 0+}{\to} 0, \tag{4.2}$$

we shall follow the same approach as in [6, Proposition 2.2]. Observe that for each $x \in \Sigma_0$ there exists a mapping $h \mapsto \xi(h)$ such that $\xi(h) \to 0$ when $h \to 0$ and $g(t + h) = g(t) + hg'(t) + h\xi(h)$ when $|h| > 0$ is small enough, where $g(t) = x$. Next, let $\delta \in (0, 1)$ be given. We can choose a sufficiently small $r_0 > 0$ such that $|\xi(h)| < \delta/2$ for all $h \in (-r_0, r_0)\setminus\{0\}$. Then for each $r \in (0, r_0)$ and each $z \in T_x \cap \overline{B}_{(1-\delta/2)r}(x)$, there exists $\lambda \in [(\delta/2-1)r, (1-\delta/2)r]$ such that $z = g(t)+\lambda g'(t)$. So, defining $y = g(t+\lambda)$ and observing that $g(t + \lambda) = g(t) + \lambda g'(t) + \lambda\xi(\lambda)$, we deduce that $y \in \Sigma \cap B_r(x)$ and $|z - y| < \delta r/2$. This implies that $\max_{z \in T_x \cap \overline{B}_r(x)} \operatorname{dist}(z, \Sigma \cap \overline{B}_r(x)) < \delta r$ for all $r \in (0, r_0)$ and, therefore, proves (4.2). Observing that (4.1) and (4.2) together prove the second assertion of Lemma 4.1, we complete the proof. $\square$

Proof of Theorem 1.4. For the sake of contradiction, assume that for some $\lambda > 0$ a minimizer Σ of $\mathcal{F}_{\lambda,f,\Omega}$ over $\mathcal{K}(\Omega)$ contains a simple closed curve $\Gamma \subset \Sigma$. Notice that there is no a relatively open subset in Σ contained in both Γ and $\partial\Omega$, because otherwise, according to Lemma 4.1, there would be a relatively open subset $D \subset \Sigma$ such that $D \subset \partial\Omega$ and $\Sigma \setminus D$ would remain connected, but, observing that in this case $u_{\Sigma\setminus D} = u_\Sigma$ and $\mathcal{H}^1(D) > 0$, we would obtain a contradiction with the optimality of Σ. Thus, by Lemma 4.1, there is a point $x_0 \in \Gamma \cap \Omega$ which is a noncut point for Σ and such that Σ is flat at x_0. Therefore for x_0 there exist the sets $D_n \subset \Sigma$ and the tangent line T_{x_0} to Σ at x_0 as in Lemma 4.1. Let $\varepsilon_0, b, \overline{r}, C$ be the constants of Lemma 3.6 and let $B_{t_0}(x_0) \subset \Omega$ with $t_0 < \overline{r}$. We define $r_n := \operatorname{diam} D_n$ so that $D_n \subset \Sigma \cap \overline{B}_{r_n}(x_0)$. The flatness of Σ at x_0 implies that for any given $\varepsilon > 0$ there is $\delta \in (0, t_0]$ such that

$$d_H(\Sigma \cap \overline{B}_r(x_0), T_{x_0} \cap \overline{B}_r(x_0)) \leq \varepsilon r \text{ for all } r \in (0, \delta].$$

For each $n \in \mathbb{N}$, we define $\Sigma_n := \Sigma \setminus D_n$, which, by Lemma 4.1, remains closed and connected. We fix $\varepsilon = \varepsilon_0/2$ and $r \in (0, \delta]$. Next, we want to apply Lemma 3.6 to Σ_n, but we have to control the Hausdorff distance between $\Sigma_n \cap \overline{B}_r(x_0)$ and a diameter of $\overline{B}_r(x_0)$. We already know that Σ is εr-close, in $\overline{B}_r(x_0)$ and in the Hausdorff distance, to $T_{x_0} \cap \overline{B}_r(x_0)$ for all $r \in (0, \delta]$. Furthermore, if $r_n \leq \varepsilon_0 r/2$, then

$$\begin{aligned}d_H(\Sigma_n \cap \overline{B}_r(x_0), T_{x_0} \cap \overline{B}_r(x_0)) &\leq d_H(\Sigma_n \cap \overline{B}_r(x_0), \Sigma \cap \overline{B}_r(x_0)) + d_H(\Sigma \cap \overline{B}_r(x_0), T_{x_0} \cap \overline{B}_r(x_0))\\ &\leq r_n + \frac{\varepsilon_0 r}{2} \leq \frac{\varepsilon_0 r}{2} + \frac{\varepsilon_0 r}{2} = \varepsilon_0 r.\end{aligned}$$

Thus, if $2r_n/\varepsilon_0 < \delta/2$, we can apply Lemma 3.6 to Σ_n for the interval $[2r_n/\varepsilon_0, \delta]$, which says that

$$\int_{B_r(x_0)} |\nabla u_{\Sigma_n}|^p \ dx \leq C\Big(\frac{r}{\delta}\Big)^{1+b} \int_{B_\delta(x_0)} |\nabla u_{\Sigma_n}|^p \ dx + Cr^{1+b} \text{ for all } r \in \Big[\frac{2r_n}{\varepsilon_0}, \delta\Big],$$

where $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$ (q_0 is defined in (1.1)). Hereinafter in this proof, C denotes a positive constant that does not depend on r_n and can be different from line to line. Next, using the above estimate for $r = 2r_n/\varepsilon_0$ and using also (2.11), we get

$$\int_{B_{\frac{2r_n}{\varepsilon_0}}(x_0)} |\nabla u_{\Sigma_n}|^p \, dx \le C r_n^{1+b}$$

for each $n \in \mathbb{N}$ such that $2r_n/\varepsilon_0 < \delta/2$. Recall that the exponent b given by Lemma 3.6 is positive provided $q > q_1$, which is one of our assumptions. Now, since Σ is a minimizer of Problem 1.1 and Σ_n is a competitor for Σ, we get the following

$$\begin{aligned}
0 &\le \mathcal{F}_{\lambda,f,\Omega}(\Sigma_n) - \mathcal{F}_{\lambda,f,\Omega}(\Sigma) \le E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma_n}) - \lambda r_n \\
&\le C \int_{B_{2r_n}(x_0)} |\nabla u_{\Sigma_n}|^p \, dx + C r_n^{N+p'-\frac{Np'}{q}} - \lambda r_n \ \text{(by Corollary 2.21)} \\
&\le C \int_{B_{\frac{2r_n}{\varepsilon_0}}(x_0)} |\nabla u_{\Sigma_n}|^p \, dx + C r_n^{N+p'-\frac{Np'}{q}} - \lambda r_n \\
&\le C r_n^{1+b} + C r_n^{N+p'-\frac{Np'}{q}} - \lambda r_n.
\end{aligned}$$

Notice that $N + p' - Np'/q > 1$ if and only if $q > Np/(Np - N + 1)$, which is always true under the assumption $q > q_1$. Therefore, letting n tend to $+\infty$, we arrive to a contradiction. This completes the proof of Theorem 1.4. $\square$

5. Proof of partial regularity

In this section, we prove that every solution Σ to Problem 1.1 is locally $C^{1,\alpha}$ regular at $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$.

We recall that $\mathcal{K}(\Omega)$ is the class of all closed connected proper subsets of $\overline{\Omega}$. The factor λ in the statement of Problem 1.1 affects the shape of an optimal set minimizing the functional $\mathcal{F}_{\lambda,f,\Omega}$ over $\mathcal{K}(\Omega)$, and, according to Proposition 2.25, we know that there exists a number $\lambda_0 = \lambda_0(N, p, f, \Omega) > 0$ such that if $\lambda \in (0, \lambda_0]$, then each minimizer Σ of the functional $\mathcal{F}_{\lambda,f,\Omega}$ over $\mathcal{K}(\Omega)$ has positive $\mathcal{H}^1$-measure. Throughout this section, we assume that $\lambda = \lambda_0 = 1$ for simplicity. This is not restrictive regarding to the regularity theory.

As mentioned in Section 1.1, our approach differs from the one used in [9] to prove the partial regularity result in dimension 2. Since we deal with general dimensions, in Propositions 5.8, 5.12, 5.13 we assume, in addition, that the quantity θ_Σ (see Definition 5.3) at the corresponding scale is bounded from above by $10\overline{\mu}$, where μ is a unique positive solution to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$, compared to Propositions 6.8, 6.11, 6.12 in [9]. This condition allows us to construct a nice competitor for the minimizer Σ and derive the estimate (5.23), which we use to prove the assertion (i) of Proposition 5.8. All in all, we prove the estimate (5.16), which is crucial in the proof of Proposition 5.12. To prove the assertion (iii) of Proposition 5.12, we need to control the density θ_Σ on a smaller scale by its value on a larger scale. Adapting some of the approaches of Paolini and Stepanov in [32], we prove that for each $a \in (0, 1/20]$ there exists $\varepsilon \in (0, 1/100)$ such that if $x_0 \in \Sigma$, $B_r(x_0) \subset \Omega$, $r > 0$ is sufficiently small and $\beta_\Sigma(x_0, r) + w_\Sigma^\tau(x_0, r) \le \varepsilon$, then the estimate (5.31) holds. Altogether we prove Corollary 5.14 and, finally, we prove Theorem 1.3.

5.1. Control on defect of minimality

We begin with the definition of the flatness.

Definition 5.1. *For each closed set $\Sigma \subset \mathbb{R}^N$, each point $x \in \mathbb{R}^N$ and radius $r > 0$, we define the flatness of Σ in $\overline{B}_r(x)$ as follows*

$$\beta_\Sigma(x,r) = \inf_{L \ni x} \frac{1}{r} d_H(\Sigma \cap \overline{B}_r(x), L \cap \overline{B}_r(x)),$$

where the infimum is taken over the set of all affine lines (1-dimensional planes) L passing through x.

Notice that if $\beta_\Sigma(x,r) < +\infty$, then it is easy to prove that the infimum above is actually the minimum, and in this case $\beta_\Sigma(x,r) \in [0,\sqrt{2}]$ and $\beta_\Sigma(x,r) = \sqrt{2}$ if and only if $\Sigma \cap \overline{B}_r(x)$ is a point in $\partial B_r(x)$. Furthermore, it is worth noting that if $\kappa \in (0,1)$ and $\beta_\Sigma(x,\kappa r) < +\infty$, then the following inequality holds

$$\beta_\Sigma(x,\kappa r) \leq \frac{2}{\kappa}\beta_\Sigma(x,r) \tag{5.1}$$

(for a proof of the inequality (5.1), we refer the reader to the proof of [9, Proposition 6.1], which actually applies for the general spatial dimension $N \geq 2$).

Now we introduce the following notions of the local energy and the density, which will play a crucial role in the proof of partial regularity.

Definition 5.2. *Let $\Sigma \in \mathcal{K}(\Omega)$ and $\tau \in [0,\sqrt{2}]$. For each $x_0 \in \overline{\Omega}$ and $r > 0$, we define*

$$w_\Sigma^\tau(x_0,r) = \sup_{\substack{\Sigma' \in \mathcal{K}(\Omega),\, \Sigma' \Delta \Sigma \subset \overline{B}_r(x_0) \\ \mathcal{H}^1(\Sigma') \leq 100\mathcal{H}^1(\Sigma),\, \beta_{\Sigma'}(x_0,r) \leq \tau}} \frac{1}{r}\int_{B_r(x_0)} |\nabla u_{\Sigma'}|^p \, dx. \tag{5.2}$$

The condition $\mathcal{H}^1(\Sigma') \leq 100\mathcal{H}^1(\Sigma)$, together with the facts that $\mathcal{H}^1(\Sigma) < +\infty$, $\Sigma' \in \mathcal{K}(\Omega)$ in the definition of w_Σ^τ above, guarantees that Σ' is arcwise connected (see Remark 2.17).

Definition 5.3. *Let $\Sigma \subset \mathbb{R}^N$ be $\mathcal{H}^1$-measurable. For each $x_0 \in \Sigma$ and $r > 0$, we define*

$$\theta_\Sigma(x_0,r) = \frac{1}{r}\mathcal{H}^1(\Sigma \cap B_r(x_0)).$$

Remark 5.4. Assume that $\Sigma \in \mathcal{K}(\Omega)$, $\tau \in [0,\sqrt{2}]$, $x_0 \in \overline{\Omega}$ and $\beta_\Sigma(x_0,r) \leq \tau$. Then there exists a solution to problem (5.2). Indeed, Σ is a competitor in the definition of $w_\Sigma^\tau(x_0,r)$. Thus, according to Proposition 2.22, $w_\Sigma^\tau(x_0,r) \in [0,+\infty)$. We can then conclude using the direct method in the Calculus of Variations, standard compactness results and Gołąb's theorem (see, for instance, [33, Theorem 3.3]).

We shall use the following proposition in order to establish a decay behavior for $w_\Sigma^\tau(x_0,r)$ whenever Σ is flat enough in all balls $\overline{B}_r(x_0)$ with $r \in [r_0,r_1]$.

Proposition 5.5. *Let $\Sigma \subset \overline{\Omega}$ be closed and arcwise connected, $x \in \overline{\Omega}$, $\tau \in [0,1/10]$ and let $\beta_\Sigma(x,r_1) \leq \varepsilon$ for some $\varepsilon \in [0,\tau]$. In addition, assume that $0 < r_0 < r_1$, $\beta_\Sigma(x,r) \leq \tau$ for all $r \in [r_0,r_1]$ and $\Sigma \backslash \overline{B}_{r_1}(x) \neq \emptyset$. If $r \in [r_0,r_1]$, then for any closed arcwise connected set $\Sigma' \subset \overline{\Omega}$ such that $\Sigma' \,\Delta\, \Sigma \subset \overline{B}_r(x)$ and $\beta_{\Sigma'}(x,r) \leq \tau$ we have that*

(i)

$$\beta_{\Sigma'}(x,r_1) \leq \frac{5\tau r}{r_1} + \varepsilon, \tag{5.3}$$

(ii)

$$\beta_{\Sigma'}(x,s) \leq 6\tau \text{ for all } s \in [r,r_1]. \tag{5.4}$$

Proof. Every ball in this proof is centered at x. Let L_1, L and L' realize the infimum, respectively, in the definitions of $\beta_\Sigma(x,r_1)$, $\beta_\Sigma(x,r)$ and $\beta_{\Sigma'}(x,r)$. Notice that

$$d_H(\Sigma \cap \overline{B}_r, L \cap \overline{B}_r) \leq \tau r. \tag{5.5}$$

On the other hand,

$$\begin{aligned} d_H(\Sigma' \cap \overline{B}_{r_1}, L_1 \cap \overline{B}_{r_1}) &\le d_H(\Sigma' \cap \overline{B}_{r_1}, \Sigma \cap \overline{B}_{r_1}) + d_H(\Sigma \cap \overline{B}_{r_1}, L_1 \cap \overline{B}_{r_1}) \\ &\le d_H(\Sigma' \cap \overline{B}_r, \Sigma \cap \overline{B}_r) + \varepsilon r_1, \end{aligned} \tag{5.6}$$

where the latter inequality comes because $\Sigma' \Delta\, \Sigma \subset \overline{B}_r$ and $\beta_\Sigma(x, r_1) \le \varepsilon$. In addition,

$$\begin{aligned} d_H(\Sigma' \cap \overline{B}_r, \Sigma \cap \overline{B}_r) &\le d_H(\Sigma' \cap \overline{B}_r, L' \cap \overline{B}_r) + d_H(L \cap \overline{B}_r, L' \cap \overline{B}_r) + d_H(\Sigma \cap \overline{B}_r, L \cap \overline{B}_r) \\ &\le 2\tau r + d_H(L \cap \overline{B}_r, L' \cap \overline{B}_r), \end{aligned} \tag{5.7}$$

where we have used (5.5) and the assumption $\beta_{\Sigma'}(x, r) \le \tau$. Notice that, since $\Sigma \cap B_r \ne \emptyset$, $\Sigma \backslash \overline{B}_{r_1} \ne \emptyset$ and Σ is arcwise connected, there is a sequence $(x_n)_n \subset \Sigma \backslash \overline{B}_r$ converging to some point $y \in \partial B_r$. We conclude that $y \in \Sigma' \cap \Sigma \cap \partial B_r$ because $\Sigma' \Delta \Sigma \subset \overline{B}_r$ and Σ', Σ are closed. If $y \in L \cap L'$, then $L = L'$. Assume that $y \notin L$. Let Π be the 2-dimensional plane passing through L and y, and let $\xi \in L \cap \partial B_r$ be such that $|y - \xi| = \mathrm{dist}(y, L \cap \partial B_r)$. Denote by γ the geodesic in the circle $\Pi \cap \partial B_r$ connecting y with ξ. Then

$$\mathcal{H}^1(\gamma) \le \arcsin(\beta_\Sigma(x, r)) r \le \arcsin(\tau) r \le \frac{3}{2}\tau r,$$

where we have used the assumption $\beta_\Sigma(x, r) \le \tau$ and the fact that $\arcsin(t) \le 3t/2$ for all $t \in [0, 1/10]$. Notice that if $y \in L'$, then $d_H(L \cap \overline{B}_r, L' \cap \overline{B}_r) \le \mathcal{H}^1(\gamma)$, otherwise let $\xi' \in L' \cap \partial B_r$ be such that $|y - \xi'| = \mathrm{dist}(y, L' \cap \partial B_r)$ and let γ' be the geodesic in the circle $\Pi' \cap \partial B_r$ connecting y and ξ', where Π' is the 2-dimensional plane passing through L' and y. Then, using the assumption $\beta_{\Sigma'}(x, r) \le \tau$ and proceeding as before, we get

$$\mathcal{H}^1(\gamma') \le \frac{3}{2}\tau r.$$

Finally, we can conclude that

$$d_H(L \cap \overline{B}_r, L' \cap \overline{B}_r) \le \mathcal{H}^1(\gamma) + \mathcal{H}^1(\gamma') \le 3\tau r.$$

This, together with (5.7), gives the following

$$d_H(\Sigma' \cap \overline{B}_r, \Sigma \cap \overline{B}_r) \le 5\tau r. \tag{5.8}$$

Using (5.6) and (5.8), we get

$$d_H(\Sigma' \cap \overline{B}_{r_1}, L_1 \cap \overline{B}_{r_1}) \le 5\tau r + \varepsilon r_1.$$

Thus, we have proved (i). Now let $s \in [r, r_1]$ and let L_s be an affine line realizing the infimum in the definition of $\beta_\Sigma(x, s)$. As in the proof of (i), we get

$$\begin{aligned} d_H(\Sigma' \cap \overline{B}_s, L_s \cap \overline{B}_s) &\le d_H(\Sigma' \cap \overline{B}_s, \Sigma \cap \overline{B}_s) + d_H(\Sigma \cap \overline{B}_s, L_s \cap \overline{B}_s) \\ &\le d_H(\Sigma' \cap \overline{B}_r, \Sigma \cap \overline{B}_r) + d_H(\Sigma \cap \overline{B}_s, L_s \cap \overline{B}_s). \end{aligned}$$

This, together with (5.8) and the fact that $\beta_\Sigma(x, s) \le \tau$, implies

$$d_H(\Sigma' \cap \overline{B}_s, L_s \cap \overline{B}_s) \le 5\tau r + \tau s \le 6\tau s,$$

concluding the proof of Proposition 5.5. □

Hereinafter in this section, τ is a fixed constant such that $\tau \in (0, \varepsilon_0/6]$, where ε_0 is the constant of Lemma 3.6. Notice that ε_0 is fairly small.

Now we establish a decay behavior for $w_\Sigma^\tau(x, \cdot)$, provided that $\beta_\Sigma(x, \cdot)$ is small enough.

Proposition 5.6. *Let $p \in (N-1,+\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Let $\varepsilon_0, b, \overline{r} \in (0,1)$, $C > 0$ be the constants of Lemma 3.6. Assume that $\Sigma \in \mathcal{K}(\Omega)$, $\mathcal{H}^1(\Sigma) < +\infty$, $0 < r_0 \le r_1/10$ and $B_{r_1}(x_0) \subset \Omega$ with $r_1 \in (0, \min\{\overline{r}, \mathrm{diam}(\Sigma)/2\})$. Assume also that*

$$\beta_\Sigma(x_0, r) \le \tau/2$$

for all $r \in [r_0, r_1]$. Then, for all $r \in [r_0, r_1/10]$,

$$w^\tau_\Sigma(x_0, r) \le C\left(\frac{r}{r_1}\right)^b w^\tau_\Sigma(x_0, r_1) + Cr^b. \tag{5.9}$$

Proof. According to Remark 2.17, Σ is arcwise connected. From Remark 5.4 it follows that there is $\Sigma_r \subset \overline{\Omega}$ realizing the supremum in the definition of $w^\tau_\Sigma(x_0, r)$ which, by Remark 2.17, is arcwise connected. Furthermore, Proposition 5.5 says that

$$\beta_{\Sigma_r}(x_0, r_1) \le \tau \ \text{ and } \ \beta_{\Sigma_r}(x_0, s) \le 6\tau \le \varepsilon_0 \ \text{ for all } \ s \in [r, r_1].$$

Thus, we can apply Lemma 3.6 to u_{Σ_r}, which yields

$$w^\tau_\Sigma(x_0, r) = \frac{1}{r}\int_{B_r(x_0)} |\nabla u_{\Sigma_r}|^p \ dx \le C\Big(\frac{r}{r_1}\Big)^b \frac{1}{r_1}\int_{B_{r_1}(x_0)} |\nabla u_{\Sigma_r}|^p \ dx + Cr^b \le C\Big(\frac{r}{r_1}\Big)^b w^\tau_\Sigma(x_0, r_1) + Cr^b.$$

Notice that to obtain the last estimate we have used the definition of $w^\tau_\Sigma(x_0, r_1)$ and the fact that $\beta_{\Sigma_r}(x_0, r_1) \le \tau$. □

Now we are in position to control a defect of minimality via w^τ_Σ.

Proposition 5.7. *Let $p \in (N-1,+\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4), and let $\varepsilon_0, b, \overline{r} \in (0,1)$ be the constants of Lemma 3.6. Assume that $\Sigma \in \mathcal{K}(\Omega)$, $\mathcal{H}^1(\Sigma) < +\infty$, $0 < r_0 \le r_1/10$, $B_{r_1}(x_0) \subset \Omega$ with $r_1 \in (0, \min\{\overline{r}, \mathrm{diam}(\Sigma)/2\})$. Assume also that*

$$\beta_\Sigma(x_0, r) \le \tau/2$$

for all $r \in [r_0, r_1]$. Then there exists a constant $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$, where q_0 is defined in (1.1), such that if $r \in [r_0, r_1/10]$, then for any $\Sigma' \in \mathcal{K}(\Omega)$ satisfying $\Sigma' \Delta \Sigma \subset \overline{B}_r(x_0)$, $\mathcal{H}^1(\Sigma') \le 100\mathcal{H}^1(\Sigma)$ and $\beta_{\Sigma'}(x_0, r) \le \tau$,

$$E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}) \le Cr\Big(\frac{r}{r_1}\Big)^b w^\tau_\Sigma(x_0, r_1) + Cr^{1+b}. \tag{5.10}$$

Proof. According to Remark 2.17, Σ and Σ' are arcwise connected and by Corollary 2.21,

$$E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}) \le C\int_{B_{2r}(x_0)} |\nabla u_{\Sigma'}|^p \ dx + Cr^{N+p'-\frac{Np'}{q}}, \tag{5.11}$$

where $C = C(N, p, q_0, q, \|f\|_q) > 0$. On the other hand, by Proposition 5.5,

$$\beta_{\Sigma'}(x_0, r_1) \le \tau \ \text{ and } \ \beta_{\Sigma'}(x_0, s) \le \varepsilon_0 \ \text{ for all } \ s \in [r, r_1].$$

Thus, applying Lemma 3.6 to $u_{\Sigma'}$, we obtain that

$$\int_{B_{2r}(x_0)} |\nabla u_{\Sigma'}|^p \ dx \le C\Big(\frac{2r}{r_1}\Big)^{1+b}\int_{B_{r_1}(x_0)} |\nabla u_{\Sigma'}|^p \ dx + C(2r)^{1+b}, \tag{5.12}$$

where $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$. Hereinafter in this proof, C denotes a positive constant that can only depend on N, p, q_0, q, $\|f\|_q$, $|\Omega|$ and can be different from line to line. Using (5.11), (5.12) and the fact that $r^{N+p'-\frac{Np'}{q}} < r^{1+b}$ (because $r \in (0,1)$ and $0 < b < N-1+p'-Np'/q$), we deduce the following chain of estimates

$$
\begin{aligned}
E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}) &\le C\Big(\frac{r}{r_1}\Big)^{1+b} \int_{B_{r_1(x_0)}} |\nabla u_{\Sigma'}|^p \ dx + Cr^{1+b} \\
&\le Cr\Big(\frac{r}{r_1}\Big)^{b} \frac{1}{r_1} \int_{B_{r_1}(x_0)} |\nabla u_{\Sigma'}|^p \ dx + Cr^{1+b} \\
&\le Cr\Big(\frac{r}{r_1}\Big)^{b} w_\Sigma^\tau(x_0,r_1) + Cr^{1+b},
\end{aligned}
$$

where the last estimate is obtained using the definition of $w_\Sigma^\tau(x_0,r_1)$ and the fact that $\beta_{\Sigma'}(x_0,r_1) \le \tau$. This completes the proof of Proposition 5.7. □

5.2. Density control

The following proposition says that there exists a constant $\kappa \in (0,1/100)$ such that if Σ is a solution to Problem 1.1, $\beta_\Sigma(x_0,r)$, $w_\Sigma^\tau(x_0,r)$ are fairly small provided that $B_r(x_0) \subset \Omega$ with $x_0 \in \Sigma$, and if $\theta_\Sigma(x_0,r)$ is also small enough, then there exists $t \in [\kappa r, 2\kappa r]$ such that $\mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) = 2$. This allows to construct a nice competitor for Σ and derive the estimate (5.16) leading to the regularity.

Proposition 5.8. *Let $p \in (N-1,+\infty)$, $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Then there exist $\delta, \varepsilon, \kappa \in (0,1/100)$ and $C = C(N,p,q_0,q,\|f\|_q,|\Omega|) > 0$, where q_0 is defined in (1.1), such that the following holds. Assume that Σ is a solution to Problem 1.1, $x_0 \in \Sigma$, $0 < r < \min\{\delta, \mathrm{diam}(\Sigma)/2\}$, $B_r(x_0) \subset \Omega$ and*

$$\beta_\Sigma(x_0,r) + w_\Sigma^\tau(x_0,r) \le \varepsilon. \tag{5.13}$$

Assume also that

$$\theta_\Sigma(x_0,r) \le 10\overline{\mu}, \tag{5.14}$$

where $\overline{\mu}$ is a unique positive solution to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$. Then the following assertions hold.

(i) There exists $t \in [\kappa r, 2\kappa r]$ such that

$$\mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) = 2. \tag{5.15}$$

(ii) Let $t \in [\kappa r, 2\kappa r]$ be such that $\mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) = 2$. Then

(ii-1) the two points of $\Sigma \cap \partial B_t(x_0)$ belong to two different connected components of

$$\partial B_t(x_0) \cap \{y : dist(y,L) \le \beta_\Sigma(x_0,t)t\},$$

where L is an affine line realizing the infimum in the definition of $\beta_\Sigma(x_0,t)$.

(ii-2) $\Sigma \cap \overline{B}_t(x_0)$ is arcwise connected.

(ii-3) If $\{z_1,z_2\} = \Sigma \cap \partial B_t(x_0)$, then

$$\mathcal{H}^1(\Sigma \cap B_t(x_0)) \le |z_2 - z_1| + Ct\Big(\frac{t}{r}\Big)^b w_\Sigma^\tau(x_0,r) + Ct^{1+b}, \tag{5.16}$$

where $b \in (0,1)$ is the constant given by Lemma 3.6.

Remark 5.9. If the situation of item (*ii*-1) occurs, we say that the two points lie "on different sides".

Proof. Let $\varepsilon_0, b, \overline{r} \in (0,1)$ be the constants of Lemma 3.6 and let $C = C(N,p,q_0,q,\|f\|_q,|\Omega|) > 0$ be the constant of Proposition 5.7. We define

$$\varepsilon = \frac{1}{\overline{\mu}C}\Big(\frac{\tau}{10}\Big)^{10}, \qquad k = \frac{\tau}{200}. \tag{5.17}$$

Fix $\delta \in (0, \overline{r})$ such that $\delta^b \le \varepsilon$ and hence

$$w_\Sigma^\tau(x_0, r) + \delta^b \le 2\varepsilon. \tag{5.18}$$

Step 1. Let us first prove (i). Thanks to (5.1) and (5.13), for all $s \in [\kappa r, r]$, it holds

$$\beta_\Sigma(x_0, s) \le \frac{2}{\kappa}\beta_\Sigma(x_0, r) \le \frac{2\varepsilon}{\kappa}. \tag{5.19}$$

On the other hand, for all $s \in [\kappa r, r]$,

$$\theta_\Sigma(x_0, s) \le \frac{r}{s}\theta_\Sigma(x_0, r) \le \frac{r}{\kappa r}\theta_\Sigma(x_0, r) \le \frac{10\overline{\mu}}{\kappa}, \tag{5.20}$$

where the last estimate is due to (5.14). Fix an arbitrary $s \in [\kappa r, 2\kappa r]$. By the coarea inequality (see, for instance, [33, Theorem 2.1]),

$$\mathcal{H}^1(\Sigma \cap B_{(1+\kappa)s}(x_0)) \ge \int_0^{(1+\kappa)s} \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0))\ d\varrho > \int_s^{(1+\kappa)s} \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0))\ d\varrho, \tag{5.21}$$

where the latter estimate comes from the fact that $\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \ge 1$ for all $\varrho \in (0, r]$, since $x_0 \in \Sigma$, Σ is arcwise connected and $r < \operatorname{diam}(\Sigma)/2$. Then there exists $\varrho \in [s, (1+\kappa)s]$ such that

$$\frac{1}{\kappa s}\mathcal{H}^1(\Sigma \cap B_{(1+\kappa)s}(x_0)) \ge \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)).$$

This, together with (5.20) and the fact that $s \in [\kappa r, 2\kappa r]$, implies that

$$\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \le \frac{1+\kappa}{\kappa}\theta_\Sigma(x_0, (1+\kappa)s) \le \frac{10(1+\kappa)\overline{\mu}}{\kappa^2}. \tag{5.22}$$

Let L realize the infimum in the definition of $\beta_\Sigma(x_0, \varrho)$ and let $\{\xi_1, \xi_2\} = \partial B_\varrho(x_0) \cap L$. For each $z_i \in \Sigma \cap \partial B_\varrho(x_0)$, let z_i' denote the projection of z_i to $[\xi_1, \xi_2]$. Define W and Σ' by

$$W := \bigcup_{i=1}^{\mathcal{H}^0(\Sigma\cap\partial B_\varrho(x_0))} [z_i, z_i'], \qquad \Sigma' := W \cup [\xi_1, \xi_2] \cup (\Sigma \backslash B_\varrho(x_0)).$$

Then $\Sigma' \in \mathcal{K}(\Omega)$, $\Sigma' \Delta \Sigma \subset \overline{B}_\varrho(x_0)$ and from (5.19) it follows that $\beta_{\Sigma'}(x_0, \varrho) \le 2\varepsilon/\kappa$. Furthermore, using (5.20) and the facts that Σ is arcwise connected and $r < \operatorname{diam}(\Sigma)/2$, it is easy to see that $\mathcal{H}^1(\Sigma') \le 100\mathcal{H}^1(\Sigma)$. Since Σ' is a competitor,

$$\mathcal{H}^1(\Sigma) \le \mathcal{H}^1(\Sigma') + E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}),$$

and then, using Proposition 5.7, we get

$$\begin{aligned}\mathcal{H}^1(\Sigma \cap B_s(x_0)) &\le \mathcal{H}^1(\Sigma \cap B_\varrho(x_0)) \le 2\varrho + \mathcal{H}^1(W) + C\varrho\Big(\frac{\varrho}{r}\Big)^b w_\Sigma^\tau(x_0, r) + C\varrho^{1+b} \\ &\le 2(1+\kappa)s + \frac{10(1+\kappa)^2\overline{\mu}}{\kappa^2}\beta_\Sigma(x_0, \varrho)s + C(1+\kappa)s\bigg(\frac{(1+\kappa)s}{r}\bigg)^b w_\Sigma^\tau(x_0, r) + C((1+\kappa)s)^{1+b},\end{aligned} \tag{5.23}$$

where we have used that $\mathcal{H}^1(W) \le (\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)))\beta_\Sigma(x_0, \varrho)\varrho$, (5.22) and the fact that $\varrho \le (1+\kappa)s$. Now we define the next three sets

$$E_1 := \{t \in (0, 2\kappa r] : \mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) = 1\}, \qquad E_2 := \{t \in (0, 2\kappa r] : \mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) = 2\},$$
$$E_3 := \{t \in (0, 2\kappa r] : \mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) \ge 3\}.$$

We claim that either $E_1 = \emptyset$ or $E_1 \subset (0, \kappa r/200)$. Assume by contradiction that there exists some $t \in [\kappa r/200, 2\kappa r]$ such that $\mathcal{H}^0(\Sigma \cap \partial B_t(x_0)) = 1$. Then the set

$$\Sigma'' = \Sigma \backslash B_t(x_0)$$

would be arcwise connected, $\Sigma''\Delta\,\Sigma\subset B_t(x_0)$, $\mathcal{H}^1(\Sigma'')<\mathcal{H}^1(\Sigma)$ and

$$\beta_{\Sigma''}(x_0,r)\le 2\kappa+\varepsilon<\tau. \tag{5.24}$$

Since Σ'' is a competitor, $\mathcal{H}^1(\Sigma)\le\mathcal{H}^1(\Sigma'')+E_{f,\Omega}(u_\Sigma)-E_{f,\Omega}(u_{\Sigma''})$. On the other hand, we observe that $t\le\mathcal{H}^1(\Sigma\cap B_t(x_0))$, because $t<\mathrm{diam}(\Sigma)/2$, $x_0\in\Sigma$ and Σ is arcwise connected. Thus

$$t\le\mathcal{H}^1(\Sigma\cap B_t(x_0))\le E_{f,\Omega}(u_\Sigma)-E_{f,\Omega}(u_{\Sigma''}). \tag{5.25}$$

Notice that, by assumption, the estimate (5.10) holds with C, but looking at the proof of Proposition 5.7, we observe that (2.10) in Corollary 2.21 also holds with C. Then, using (5.25), Corollary 2.21, the fact that $t^{N+p'-\frac{Np'}{q}}<t^{1+b}$ (because $t\in(0,1)$ and $0<b<N-1+p'-Np'/q$) and (5.24) together with the definition of $w^\tau_\Sigma(x_0,r)$, we obtain the following chain of estimates

$$\begin{aligned}
t\le\mathcal{H}^1(\Sigma\cap B_t(x_0))\le E_{f,\Omega}(u_\Sigma)-E_{f,\Omega}(u_{\Sigma''})&\le C\int_{B_{2t}(x_0)}|\nabla u_{\Sigma''}|^p\ dx+Ct^{N+p'-\frac{Np'}{q}}\\
&\le C\int_{B_r(x_0)}|\nabla u_{\Sigma''}|^p\ dx+Cr^{1+b}\\
&\le Crw^\tau_\Sigma(x_0,r)+Cr^{1+b},
\end{aligned}$$

leading to a contradiction with the fact that $\kappa r/200\le t$, since $Crw^\tau_\Sigma(x_0,r)+Cr^{1+b}\le 2Cr\varepsilon<\kappa r/200$ by (5.18) and (5.17). Thus, either $E_1=\emptyset$ or

$$E_1\subset(0,\kappa r/200). \tag{5.26}$$

Next, by the coarea inequality,

$$\mathcal{H}^1(\Sigma\cap B_{2\kappa r}(x_0))\ge\int_0^{2\kappa r}\mathcal{H}^0(\Sigma\cap\partial B_t(x_0))\ dt. \tag{5.27}$$

Also, applying (5.23) with $s=2\kappa r$ and using (5.17), (5.18) and the fact that $\beta_\Sigma(x_0,\varrho)\le 2\varepsilon/\kappa$, we get the following estimate

$$\mathcal{H}^1(\Sigma\cap B_{2\kappa r}(x_0))\le 4\kappa r+\frac{\kappa r}{200}. \tag{5.28}$$

Then, (5.26), (5.27) and (5.28) together imply

$$\begin{aligned}
4\kappa r+\frac{\kappa r}{200}&\ge\mathcal{H}^1(E_1)+2\mathcal{H}^1(E_2)+3\mathcal{H}^1(E_3)\\
&\ge\mathcal{H}^1(E_1)+2(2\kappa r-\mathcal{H}^1(E_1)-\mathcal{H}^1(E_3))+3\mathcal{H}^1(E_3)\\
&=4\kappa r-\mathcal{H}^1(E_1)+\mathcal{H}^1(E_3)\\
&>4\kappa r-\frac{\kappa r}{200}+\mathcal{H}^1(E_3)
\end{aligned}$$

and hence

$$\mathcal{H}^1(E_3)<\frac{\kappa r}{100}. \tag{5.29}$$

Notice that (5.26) and (5.29) yield the following estimate

$$\mathcal{H}^1(E_2\cap[\kappa r,2\kappa r])>\frac{\kappa r}{2}.$$

This completes the proof of (i).

Step 2. We prove (ii). Let $t\in E_2\cap[\kappa r,2\kappa r]$. Assume that $(ii$-1) does not hold for t. Let L be an affine line realizing the infimum in the definition of $\beta_\Sigma(x_0,t)$, $\{P_1,P_2\}=L\cap\partial B_t(x_0)$ and $\{z_1,z_2\}=\Sigma\cap\partial B_t(x_0)$. Assume that $\mathrm{dist}(z_i,\{P_1,P_2\})=\mathrm{dist}(z_i,P_2)$, $i=1,2$. Then we can take as a competitor the set

$$\Sigma'''=(\Sigma\setminus B_t(x_0))\cup\gamma_{z_1,P_2}\cup\gamma_{z_2,P_2},$$

where γ_{z_i,P_2} is the geodesic in $\partial B_t(x_0)$ connecting z_i and P_2 for $i=1,2$. So

$$\mathcal{H}^1(\Sigma \cap B_t(x_0)) \leq \mathcal{H}^1(\gamma_{z_1,P_2}) + \mathcal{H}^1(\gamma_{z_2,P_2}) + E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'''}).$$

Arguing as in the proof of the fact that $E_1 \subset (0, \kappa r/200)$ in *Step 1*, we obtain the estimate

$$E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'''}) < \frac{\kappa r}{200}.$$

In addition, thanks to (5.19) and to the fact that $\arcsin(s) \leq 2s$ for all $s \in [0, 1/10]$,

$$\mathcal{H}^1(\gamma_{z_1,P_2}) + \mathcal{H}^1(\gamma_{z_2,P_2}) \leq 2t \arcsin(\beta_\Sigma(x_0,t)) \leq \frac{8\varepsilon t}{\kappa}.$$

But then

$$\mathcal{H}^1(\Sigma \cap B_t(x_0)) < \frac{\kappa r}{100}$$

and this leads to a contradiction because $\mathcal{H}^1(\Sigma \cap B_t(x_0)) \geq t \geq \kappa r$. Therefore (*ii*-1) holds. Next, assume that $\Sigma \cap \overline{B}_t(x_0)$ is not arcwise connected. Then, from Lemma 5.10, it follows that $\Sigma \setminus B_t(x_0)$ is arcwise connected. Thus, taking the set $\Sigma \setminus B_t(x_0)$ as a competitor, by analogy with *Step 1*, we get

$$\mathcal{H}^1(\Sigma \cap B_t(x_0)) < \frac{\kappa r}{200},$$

which, as before, leads to a contradiction. Thus (*ii*-2) holds. Since $\Sigma \cap \partial B_t(x_0) = \{z_1, z_2\}$, where z_1, z_2 lie "on different sides", the set $(\Sigma \setminus B_t(x_0)) \cup [z_1, z_2]$ is a competitor for Σ, moreover, it fulfills the conditions of Proposition 5.7 and hence (5.16) holds. This proves (*ii*) and completes the proof of Proposition 5.8. ☐

Lemma 5.10. *Let $x_0 \in \mathbb{R}^N$, $r > 0$ and $\Sigma \subset \mathbb{R}^N$ be an arcwise connected set such that $\Sigma \cap \overline{B}_r(x_0)$ is not arcwise connected and $\mathcal{H}^0(\Sigma \cap \partial B_r(x_0)) = 2$. Then $\Sigma \setminus B_r(x_0)$ is arcwise connected.*

Proof. Let $\{z_1, z_2\} = \Sigma \cap \partial B_r(x_0)$. It suffices to prove that for every point $x \in \Sigma \setminus B_r(x_0)$ there exist two arcs $\gamma_1, \gamma_2 \subset \Sigma \setminus B_r(x_0)$ such that γ_i connects x with z_i for $i \in \{1,2\}$. Since Σ is arcwise connected and $\Sigma \cap \partial B_r(x_0) = \{z_1, z_2\}$, for every $x \in \Sigma \setminus B_r(x_0)$ there exists an arc $\gamma \subset \Sigma \setminus B_r(x_0)$ connecting x with z_1 or with z_2. Assume by contradiction that for some $x \in \Sigma \setminus B_r(x_0)$ there is no arc $\gamma \subset \Sigma \setminus B_r(x_0)$ connecting x with z_i, where $i \in \{1,2\}$. Let $\widetilde{z}_i = z_2$ if $i = 1$ and $\widetilde{z}_i = z_1$ if $i = 2$. Since Σ is arcwise connected, there is an arc $\gamma \subset \Sigma$ connecting x with z_i. We can conclude that $\gamma = \gamma_{ext} \cup \gamma_{int}$, where $\gamma_{ext} \subset \Sigma \setminus B_r(x_0)$ is an arc connecting x with $\widetilde{z}_i$ and $\gamma_{int} \subset \Sigma \cap \overline{B}_r(x_0)$ is an arc connecting $\widetilde{z}_i$ with z_i. On the other hand, if $y \in \Sigma \cap B_r(x_0)$, then there exists an arc in $\Sigma \cap \overline{B}_r(x_0)$ connecting y with $\widetilde{z}_i$ or with z_i. Therefore, $\Sigma \cap \overline{B}_r(x_0)$ is arcwise connected, which leads to a contradiction. This completes our proof of Lemma 5.10. ☐

Now our purpose is to control the density θ_Σ from above on a smaller scale by its value on a larger scale, provided that on a larger scale β_Σ and w_Σ^τ are small enough. Adapting some of the approaches of Paolini and Stepanov in [32], we prove the following proposition.

Proposition 5.11. *Let $p \in (N-1, +\infty)$, $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Then there exists $\delta \in (0,1)$ and for each $a \in (0, 1/20]$ there exists $\varepsilon \in (0,1)$ such that the following holds. Assume that Σ is a solution to Problem 1.1, $x_0 \in \Sigma$, $r \in (0, \min\{\delta, \mathrm{diam}(\Sigma)/2\})$, $B_r(x_0) \subset \Omega$ and*

$$\beta_\Sigma(x_0, r) + w_\Sigma^\tau(x_0, r) \leq \varepsilon. \tag{5.30}$$

Then the following estimate holds

$$\theta_\Sigma(x_0, ar) \leq 5 + \theta_\Sigma(x_0, r)^{1-\frac{1}{N}}. \tag{5.31}$$

Proof. Let $\varepsilon_0, b, \overline{r} \in (0,1)$ be the constants of Lemma 3.6 and let $C > 0$ be the constant of Proposition 5.7. Recall that $\tau \in (0, \varepsilon_0/6]$. We define $\delta, \varepsilon \in (0,1)$ as follows

$$\delta = \min\left\{\overline{r}, \left(\frac{1}{4C}\right)^{\frac{1}{b}}\right\}, \qquad \varepsilon = \frac{a^2 c_0 \tau}{10^7}, \tag{5.32}$$

where $c_0 > 0$ is a constant that will be fixed later for the proof to work. It is worth noting that, according to (5.1) and (5.30), for all $s \in [ar, r]$, it holds

$$\beta_\Sigma(x_0, s) \leq \frac{2}{a}\beta_\Sigma(x_0, r) \leq \frac{2\varepsilon}{a}. \tag{5.33}$$

Applying the coarea inequality (see, for instance, [33, Theorem 2.1]), we get

$$\mathcal{H}^1(\Sigma \cap B_r(x_0)) \geq \int_0^r \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0))\ d\varrho > \int_{ar}^{2ar} \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0))\ d\varrho,$$

where the latter inequality comes from the fact that for all $\varrho \in (0, r]$, $\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \geq 1$, since Σ is arcwise connected (see Remark 2.17), $x_0 \in \Sigma$ and $r < \operatorname{diam}(\Sigma)/2$. Then there exists $\varrho \in [ar, 2ar]$ such that

$$\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \leq \frac{1}{a}\theta_\Sigma(x_0, r). \tag{5.34}$$

Next, we construct the competitor Σ' for Σ such that $\Sigma' \Delta \Sigma \subset \overline{B}_\varrho(x_0)$, $\mathcal{H}^1(\Sigma') \leq 100\mathcal{H}^1(\Sigma)$ and $\beta_{\Sigma'}(x_0, \varrho) \leq \beta_\Sigma(x_0, \varrho)$. Let $L \subset \mathbb{R}^N$ be an affine line realizing the infimum in the definition of $\beta_\Sigma(x_0, \varrho)$. We denote by A_1 and A_2 the two points in $\partial B_\varrho(x_0) \cap L$ and denote by G_n the set of all points (x', x_N) in $[-1,1]^N$ such that $nx_i \in \mathbb{Z}$ for all $i = 1, ..., N$ except for at most one (i.e., G_n is a uniform 1-dimensional grid of step $1/n$ in $[-1,1]^N$). Notice that G_n is arcwise connected and

$$\mathcal{H}^1(G_n) \leq 2^N N(n+1)^{N-1},\ \ \operatorname{dist}(y, G_n) \leq \frac{\sqrt{N}}{2n} \tag{5.35}$$

for all $y \in [-1,1]^N$. Let $h : \mathbb{R}^N \to \mathbb{R}^N$ be the rotation around the origin such that $h(\mathbb{R}e_N) = L - x_0$, where $\{e_1, ..., e_N\}$ is the canonical basis for $\mathbb{R}^N$. Next, we define $Q_n^i := A_i + \beta_\Sigma(x_0, \varrho)\varrho h(G_n)$, $i = 1, 2$. In addition, we observe that

$$\Sigma \cap \partial B_\varrho(x_0) \subset \partial B_\varrho(x_0) \cap \{x \in \mathbb{R}^N : \operatorname{dist}(x, L) \leq \beta_\Sigma(x_0, \varrho)\varrho\} \subset \bigcup_{i=1}^2 \big(A_i + \beta_\Sigma(x_0, \varrho)\varrho h\big([-1,1]^N\big)\big).$$

For each point $z_j \in \Sigma \cap \partial B_\varrho(x_0)$, we denote by z_j^n an arbitrary projection of z_j to $Q_n^1 \cup Q_n^2$ and by $[z_j, z_j^n]$ the segment connecting these two points. Then the set

$$S_n = Q_n^1 \cup Q_n^2 \cup \left(\bigcup_{j=1}^{\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0))} [z_j, z_j^n]\right)$$

contains all the points of $\Sigma \cap \partial B_\varrho(x_0)$, $S_n \cup (L \cap B_\varrho(x_0))$ is arcwise connected, and, using (5.35), we have that

$$\mathcal{H}^1(S_n) \leq 2^{N+1}N(n+1)^{N-1}\beta_\Sigma(x_0, \varrho)\varrho + \frac{\sqrt{N}}{2n}\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0))\beta_\Sigma(x_0, \varrho)\varrho.$$

Let $\widetilde{S}_n$ be the projection of S_n to $\{x \in \mathbb{R}^N : \operatorname{dist}(x, L) \leq \beta_\Sigma(x_0, \varrho)\varrho\} \cap \overline{B}_\varrho(x_0)$. Since the projection onto a nonempty closed convex set is a 1-Lipschitz mapping, it follows that $\mathcal{H}^1(\widetilde{S}_n) \leq \mathcal{H}^1(S_n)$. Moreover, notice that $\widetilde{S}_n \cup (L \cap B_\varrho(x_0))$ is arcwise connected. Thus, defining

$$\Sigma' = (\Sigma \backslash B_\varrho(x_0)) \cup \widetilde{S}_n \cup (L \cap B_\varrho(x_0))$$

and choosing $n = \lfloor (\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)))^{\frac{1}{N}} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part, we observe that

$$\mathcal{H}^1(\widetilde{S}_n) \leq M_0(\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)))^{1-\frac{1}{N}}\beta_\Sigma(x_0, \varrho)\varrho, \tag{5.36}$$

where $M_0 = M_0(N) > 0$. Now we can set

$$c_0 = (M_0 C)^{-1}. \tag{5.37}$$

Thanks to (5.34) and (5.36), we obtain

$$\mathcal{H}^1(\widetilde{S}_n) < M_0 \left(\frac{1}{a} \theta_\Sigma(x_0, r) \right)^{1-\frac{1}{N}} \beta_\Sigma(x_0, \varrho)\varrho.$$

This, together with (5.33), (5.32), (5.37) and the fact that $2\varrho \le 4ar < \operatorname{diam}(\Sigma) \le \mathcal{H}^1(\Sigma)$, implies the following

$$\mathcal{H}^1(\Sigma') < 100\mathcal{H}^1(\Sigma).$$

Also notice that $\Sigma' \subset \overline{\Omega}$ is closed, arcwise connected, $\Sigma' \Delta \Sigma \subset \overline{B}_\varrho(x_0)$,

$$\beta_{\Sigma'}(x_0, \varrho) \le \beta_\Sigma(x_0, \varrho) \le \frac{2\varepsilon}{a} < \tau$$

(see (5.33), (5.32)). So we can apply Proposition 5.7 to Σ and Σ'. Thus, by the optimality of Σ and Proposition 5.7,

$$\mathcal{H}^1(\Sigma) \le E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma'}) + \mathcal{H}^1(\Sigma') \le C\varrho \Big(\frac{\varrho}{r}\Big)^b w_\Sigma^\tau(x_0, r) + C\varrho^{1+b} + \mathcal{H}^1(\Sigma').$$

Altogether we have

$$\mathcal{H}^1(\Sigma \cap B_{ar}(x_0)) \le \mathcal{H}^1(\Sigma \cap B_\varrho(x_0)) \le C\varrho \Big(\frac{\varrho}{r}\Big)^b w_\Sigma^\tau(x_0, r) + C\varrho^{1+b} + 2\varrho + M_0 \left(\frac{1}{a} \theta_\Sigma(x_0, r) \right)^{1-\frac{1}{N}} \beta_\Sigma(x_0, \varrho)\varrho.$$

Next, recalling that $\varrho \in [ar, 2ar]$, $r < \delta$, $(2a)^b < 1$ and (5.33), we obtain

$$\theta_\Sigma(x_0, ar) \le 2C\big(w_\Sigma^\tau(x_0, r) + \delta^b\big) + 4 + \frac{4\varepsilon M_0}{a} \left(\frac{1}{a} \theta_\Sigma(x_0, r) \right)^{1-\frac{1}{N}}.$$

However, this, together with (5.30), (5.32) and (5.37), yields the estimate

$$\theta_\Sigma(x_0, ar) \le 5 + \theta_\Sigma(x_0, r)^{1-\frac{1}{N}}$$

and completes the proof of Proposition 5.11. $\square$

5.3. Control of the flatness

The next proposition asserts that if $\beta_\Sigma(x, r)$ and $w_\Sigma^\tau(x, r)$ are pretty small and $\theta_\Sigma(x, r)$ is controlled from above by $10\overline{\mu}$, where $\overline{\mu}$ is a unique positive solution to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$, then β_Σ, w_Σ^τ stay small and θ_Σ remains controlled from above by $10\overline{\mu}$ on smaller scales, and, in addition, in some sense w_Σ^τ controls the square of β_Σ.

Proposition 5.12. *Let $p \in (N-1, +\infty)$, $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Then there exist constants $a, r_0 \in (0, 1/100)$, $b \in (0, 1)$, $0 < \delta_1 < \delta_2 < 1/100$ and $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$ with q_0 defined in (1.1) such that the following holds. Assume that Σ is a solution to Problem 1.1, $x \in \Sigma$, $0 < r < \min\{r_0, \operatorname{diam}(\Sigma)/2\}$, $B_r(x) \subset \Omega$,*

$$w_\Sigma^\tau(x, r) \le \delta_1, \ \ \beta_\Sigma(x, r) \le \delta_2 \ \text{ and } \ \theta_\Sigma(x, r) \le 10\overline{\mu}, \tag{5.38}$$

where $\overline{\mu} > 0$ is a unique positive solution to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$. Then

(i)

$$\beta_\Sigma(x, ar) \le C(w_\Sigma^\tau(x, r))^{\frac{1}{2}} + Cr^{\frac{b}{2}}; \tag{5.39}$$

(ii)

$$w^{\tau}_{\Sigma}(x,ar)\le \frac{1}{2}w^{\tau}_{\Sigma}(x,r)+C(ar)^{b};\tag{5.40}$$

(iii)

$$w^{\tau}_{\Sigma}(x,a^{n}r)\le \delta_1,\ \ \beta_{\Sigma}(x,a^{n}r)\le\delta_2,\ \ \theta_{\Sigma}(x,a^{n}r)\le 10\overline{\mu}\ \ \textit{for all}\ \ n\in\mathbb{N}.\tag{5.41}$$

Proof. Let C_0 be the constant such that the estimate (3.28) holds with C_0, and let C_1 be the constant such that the estimate (5.16) holds with C_1. Without loss of generality, we can assume that $C_0 < C_1$. Let $b \in (0,1)$ be the constant of Lemma 3.6, and let $a,\delta,\varepsilon,\kappa \in (0,1/100)$ be such that $\delta,\varepsilon,\kappa$ are the constants of Proposition 5.8 and, at the same time, a,δ,ε are the constants of Proposition 5.11 with

$$a=\min\left\{\kappa,\left(\frac{1}{2C_0}\right)^{\frac{1}{b}}\right\}.$$

Now we can set

$$\delta_2:=\frac{a\varepsilon}{2},\ \ \delta_1:=\left(\frac{a\delta_2}{50C_1}\right)^{2},\ \ C:=\frac{24C_1}{a}\tag{5.42}$$

and fix $r_0\in(0,\delta)$ such that

$$Cr_0^{\frac{b}{2}}\le\frac{\delta_1}{2}.\tag{5.43}$$

Step 1. Let us first prove (i). By Proposition 5.8, there exists $t\in[\kappa r,2\kappa r]$ such that $\Sigma\cap\partial B_t(x)=\{z_1,z_2\}$, z_1 and z_2 lie "on different sides" (see Remark 5.9). According to Proposition 5.8 (ii-3), we get

$$\mathcal{H}^1(\Sigma\cap B_t(x))\le |z_1-z_2|+C_1t\Big(\frac{t}{r}\Big)^{b}w^{\tau}_{\Sigma}(x,r)+C_1t^{1+b}:=|z_1-z_2|+M.$$

Recall that, by Proposition 5.8 (ii-2), $\Sigma\cap\overline{B}_t(x)$ is arcwise connected. Let $\Gamma\subset\Sigma\cap\overline{B}_t(x)$ be an arc connecting z_1 with z_2. Then, using Lemma A.5, we obtain

$$\max_{y\in\Gamma}\mathrm{dist}(y,[z_1,z_2])\le(2t(\mathcal{H}^1(\Gamma)-|z_1-z_2|))^{\frac{1}{2}}\le(4\kappa rM)^{\frac{1}{2}}.$$

Since $\Sigma\cap\overline{B}_t(x)$ is arcwise connected, $\Sigma\cap\partial B_t(x)=\{z_1,z_2\}$ and $\mathcal{H}^1(\Gamma)\ge|z_1-z_2|$,

$$\sup_{y\in(\Sigma\cap\overline{B}_t(x))\setminus(\Gamma\cap B_t(x))}\mathrm{dist}(y,\Gamma)\le\mathcal{H}^1(\Sigma\cap B_t(x)\setminus\Gamma)\le\mathcal{H}^1(\Sigma\cap B_t(x))-|z_1-z_2|\le M.$$

Thus

$$\max_{y\in\Sigma\cap\overline{B}_t(x)}\mathrm{dist}(y,[z_1,z_2])\le(4\kappa rM)^{\frac{1}{2}}+M$$

but this yields the following estimate

$$d_H(\Sigma\cap\overline{B}_t(x),[z_1,z_2])\le(4\kappa rM)^{\frac{1}{2}}+M,\tag{5.44}$$

because $\Sigma\cap\overline{B}_t(x)$ is arcwise connected and Σ escapes $\partial B_t(x)$ either through z_1 or through z_2. Without loss of generality, assume that $[z_1,z_2]$ is not a diameter of $\overline{B}_t(x)$, otherwise we can pass directly to the estimate (5.47). So let $\widetilde{L}$ be the line passing through x and collinear to $[z_1,z_2]$. Now observe that if Π is the 2-dimensional plane passing through $\widetilde{L}$ and $[z_1,z_2]$, then the intersection of Π with $\partial B_t(x)$ is the circle S on Π with center x and radius t. Then, denoting by ξ_1 and ξ_2 the two points in $\widetilde{L}\cap\partial B_t(x)$ in such a way that $\mathrm{dist}(\xi_i,\{z_1,z_2\})=\mathrm{dist}(\xi_i,z_i)$ for $i=1,2$, we get

$$d_H([z_1,z_2],\widetilde{L}\cap\overline{B}_t(x))\le\mathcal{H}^1(\gamma_{z_1,\xi_1})=\mathcal{H}^1(\gamma_{z_2,\xi_2}),\tag{5.45}$$

where γ_{z_i,ξ_i} is the geodesic in S connecting z_i with ξ_i. Since $\mathrm{dist}(x,[z_1,z_2])\le(4\kappa rM)^{\frac{1}{2}}+M$ (see (5.44)),

$$\mathcal{H}^1(\gamma_{z_1,\xi_1})\le\arcsin\bigg(\frac{(4\kappa rM)^{\frac{1}{2}}+M}{t}\bigg)t\le 2((4\kappa rM)^{\frac{1}{2}}+M),\tag{5.46}$$

where the latter estimate holds because $((4\kappa rM)^{\frac{1}{2}}+M)/t<1/10$ and $\arcsin(s)\le 2s$ for all $s\in[0,1/10]$. Using (5.44) together with (5.45) and (5.46), we obtain that

$$d_H(\Sigma\cap\overline{B}_t(x),\widetilde{L}\cap\overline{B}_t(x))\le 3((4\kappa rM)^{\frac{1}{2}}+M)$$

and hence $\beta_\Sigma(x,t)\le 3((4\kappa rM)^{\frac{1}{2}}+M)/t$. Next, since $t\in[\kappa r,2\kappa r]$ and $a\in(0,\kappa]$, if $ar=\lambda t$ for some $\lambda\in(0,1]$, then $2/\lambda\le 4\kappa/a$ and, thanks to (5.1),

$$\beta_\Sigma(x,ar)=\beta_\Sigma(x,\lambda t)\le\frac{4\kappa}{a}\beta_\Sigma(x,t)\le\frac{12}{ar}((4\kappa rM)^{\frac{1}{2}}+M). \tag{5.47}$$

On the other hand, since $\kappa, w_\Sigma^\tau(x,r), r\in(0,1/100)$ and $b\in(0,1)$, we can conclude the following

$$(4\kappa rM)^{\frac{1}{2}}\le\big(C_1r^2w_\Sigma^\tau(x,r)+C_1r^{2+b}\big)^{\frac{1}{2}}\le C_1r(w_\Sigma^\tau(x,r))^{\frac{1}{2}}+C_1r^{1+\frac{b}{2}} \tag{5.48}$$

and, moreover,

$$M=C_1t\Big(\frac{t}{r}\Big)^bw_\Sigma^\tau(x,r)+C_1t^{1+b}\le C_1r(w_\Sigma^\tau(x,r))^{\frac{1}{2}}+C_1r^{1+\frac{b}{2}}. \tag{5.49}$$

By (5.47)-(5.49),

$$\beta_\Sigma(x,ar)\le C(w_\Sigma^\tau(x,r))^{\frac{1}{2}}+Cr^{\frac{b}{2}},$$

with $C=24C_1/a$. Using (5.38), the above estimate, (5.42) and (5.43), we get

$$\beta_\Sigma(x,ar)\le C(\delta_1)^{\frac{1}{2}}+Cr_0^{\frac{b}{2}}<\delta_2.$$

Next, observe that $a<1/100$ and $\beta_\Sigma(x,s)$ is fairly small for all $s\in[ar,r]$, so we can apply Proposition 5.6 with $r_0=ar$ and $r_1=r$ to get the following

$$w_\Sigma^\tau(x,ar)\le C_0a^bw_\Sigma^\tau(x,r)+C_0(ar)^b\le\frac{1}{2}w_\Sigma^\tau(x,r)+C(ar)^b\le\frac{\delta_1}{2}+\frac{\delta_1}{2}=\delta_1,$$

where we have used that $a\le(1/2C_0)^{\frac{1}{b}}$, $C_0(ar)^b<C(ar)^b<Cr_0^{\frac{b}{2}}$, (5.38) and (5.43). We have proved the assertions (i), (ii) and that $w_\Sigma^\tau(x,ar)\le\delta_1$, $\beta_\Sigma(x,ar)\le\delta_2$.

Step 2. We prove (iii). Recall that $a,\delta,\varepsilon\in(0,1/100)$ are the constants of Proposition 5.11 and, by definition, $\delta_1<\delta_2=a\varepsilon/2$. Then, according to (5.38),

$$\beta_\Sigma(x,r)+w_\Sigma^\tau(x,r)\le\varepsilon.$$

Thus, applying Proposition 5.11 and using again (5.38), we get

$$\theta_\Sigma(x,ar)\le 5+\theta_\Sigma(x,r)^{1-\frac{1}{N}}\le 5+(10\overline{\mu})^{1-\frac{1}{N}}\le 10\big(5+\overline{\mu}^{1-\frac{1}{N}}\big)=10\overline{\mu}.$$

At this point, we have shown that (5.38) holds with r replaced by ar. So, repeating the arguments above, we observe that (5.38) holds with r replaced by a^2r. Therefore, iterating, we deduce (iii). This completes the proof of Proposition 5.12. □

Now we prove that there exist a critical threshold $\delta_0\in(0,1/100)$ and an exponent $\alpha\in(0,1)$ such that if $\beta_\Sigma(x,r)+w_\Sigma^\tau(x,r)$ falls below δ_0 and if $\theta_\Sigma(x,r)$ is small enough for $x\in\Sigma\cap\Omega$ and fairly small $r>0$, then $\beta_\Sigma(x,\varrho)\le C\varrho^\alpha$ for all sufficiently small $\varrho>0$, where $C>0$ is a constant independent of x but depending on r. This leads to the $C^{1,\alpha}$ regularity.

Proposition 5.13. *Let $p\in(N-1,+\infty)$, $f\in L^q(\Omega)$ with $q>q_1$, where q_1 is defined in (1.4). Let $a\in(0,1/100)$ be the constant of Proposition 5.12. Then there exist constants $\delta_0,\overline{r}_0\in(0,1/100)$ and $\alpha\in(0,1)$ such that the following holds. Assume that Σ is a solution to Problem 1.1. If $x\in\Sigma$ and $0<r<\min\{\overline{r}_0,\mathrm{diam}(\Sigma)/2\}$ satisfy $B_r(x)\subset\Omega$,*

$$\beta_\Sigma(x,r)+w_\Sigma^\tau(x,r)\le\delta_0\ \text{ and }\ \theta_\Sigma(x,r)\le 10\overline{\mu} \tag{5.50}$$

with $\overline{\mu}$ being a unique positive solution to the equation $\mu=5+\mu^{1-\frac{1}{N}}$, then

$$\beta_\Sigma(x,\varrho)\le C\varrho^\alpha\ \text{ for all }\ \varrho\in(0,ar) \tag{5.51}$$

and for some constant $C=C(N,p,q_0,q,\|f\|_q,|\Omega|,r)>0$, where q_0 is defined in (1.1).

Proof. Let $a, \delta_1, r_0 \in (0, 1/100)$, $b \in (0,1)$ and $C > 0$ be as in Proposition 5.12. We define

$$\delta_0 := \delta_1, \ \ \gamma := \min\left\{\frac{b}{2}, \frac{\ln(3/4)}{\ln(a)}\right\}, \ \ \overline{r}_0 := \min\left\{r_0, \left(\frac{1}{4}\right)^{\frac{1}{\gamma}}\right\}.$$

It is easy to check that for all $t \in (0, \overline{r}_0]$,

$$\frac{1}{2}t^\gamma + t^b \leq (at)^\gamma. \tag{5.52}$$

Indeed, since $0 < 2\gamma \leq b$, $\gamma \leq \ln(3/4)/\ln(a)$ and $a, \overline{r}_0 \in (0,1)$, $t^b \leq t^{2\gamma} \leq \overline{r}_0^\gamma t^\gamma$ and $3/4 \leq a^\gamma$, so

$$\frac{1}{2}t^\gamma + t^b \leq \frac{1}{2}t^\gamma + \overline{r}_0^\gamma t^\gamma \leq \frac{3}{4}t^\gamma \leq (at)^\gamma.$$

We prove by induction that for all $n \in \mathbb{N}$,

$$w_\Sigma^\tau(x, a^n r) \leq \frac{1}{2^n} w_\Sigma^\tau(x, r) + C(a^{n+1}r)^\gamma. \tag{5.53}$$

Clearly, (5.53) holds for $n = 0$. Suppose (5.53) holds for some $n \in \mathbb{N}$. Then, applying (5.40) with r replaced by $a^n r$ and using the induction hypothesis, we get

$$\begin{aligned} w_\Sigma^\tau(x, a^{n+1}r) &\leq \frac{1}{2} w_\Sigma^\tau(x, a^n r) + C(a^{n+1}r)^b \\ &\leq \frac{1}{2^{n+1}} w_\Sigma^\tau(x, r) + \frac{C}{2}(a^{n+1}r)^\gamma + C(a^{n+1}r)^b \\ &\leq \frac{1}{2^{n+1}} w_\Sigma^\tau(x, r) + C(a^{n+2}r)^\gamma, \end{aligned}$$

where the last estimate comes by using (5.52). This proves (5.53). Now let $\varrho \in (0, ar)$ and let $l \geq 1$ be the integer such that $a^{l+1}r < \varrho \leq a^l r$. Then, using if necessary (5.1), we see that $\beta_\Sigma(x, \varrho) \leq 2\beta_\Sigma(x, a^l r)/a$. Furthermore, Proposition 5.12 (i) says that

$$\beta_\Sigma(x, a^l r) \leq C(w_\Sigma^\tau(x, a^{l-1}r))^{\frac{1}{2}} + C(a^{l-1}r)^{\frac{b}{2}}.$$

On the other hand, using (5.53) and the fact that $w_\Sigma^\tau(x, r) < 1$, we get

$$\begin{aligned} w_\Sigma^\tau(x, a^{l-1}r) \leq \frac{1}{2^{l-1}} w_\Sigma^\tau(x, r) + C(a^l r)^\gamma \leq C'\left(\frac{3}{4}\right)^{l+1} + C'(a^{l+1}r)^\gamma &\leq C' a^{\gamma(l+1)} + C'\varrho^\gamma \\ &\leq C'\left(\frac{\varrho}{r}\right)^\gamma + C'\varrho^\gamma \end{aligned}$$

for some $C' = C'(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$. So we can control $\beta_\Sigma(x, \varrho)$ as follows

$$\begin{aligned} \beta_\Sigma(x, \varrho) \leq \frac{2}{a}\beta_\Sigma(x, a^l r) &\leq \frac{2C}{a}(w_\Sigma^\tau(x, a^{l-1}r))^{\frac{1}{2}} + \frac{2C}{a}(a^{l-1}r)^{\frac{b}{2}} \\ &\leq C''\left(\frac{\varrho}{r}\right)^{\frac{\gamma}{2}} + C''\varrho^{\frac{\gamma}{2}} + C''\varrho^{\frac{b}{2}} \\ &\leq \widetilde{C}\varrho^{\frac{\gamma}{2}} \ \ (\gamma \leq b/2), \end{aligned}$$

where $\widetilde{C} = \widetilde{C}(N, p, q_0, q, \|f\|_q, |\Omega|, r) > 0$. Setting $\alpha = \gamma/2$ and $C := \widetilde{C}$, we complete the proof of Proposition 5.13. ☐

Corollary 5.14. *Let Σ be a solution to Problem 1.1 and a, α, δ_0, $\overline{r}_0$, $\overline{\mu}$ be the constants as in the statement of Proposition 5.13. Assume that $x \in \Sigma$, $0 < r < \min\{\overline{r}_0, \operatorname{diam}(\Sigma)/2\}$, $B_r(x) \subset \Omega$,*

$$\beta_\Sigma(x, r) + w_\Sigma^\tau(x, r) \leq \varepsilon \ \text{ and } \ \theta_\Sigma(x, r) \leq \overline{\mu}$$

with $\varepsilon := \delta_0/200$. Then for any point $y \in \Sigma \cap B_{ar/10}(x)$ and radius $\varrho \in (0, ar/10)$ the following estimate holds

$$\beta_\Sigma(y, \varrho) \leq C\varrho^\alpha,$$

where $C = C(N, p, q_0, q, \|f\|_q, |\Omega|, r) > 0$. In particular, there exists $t_0 \in (0,1)$ such that $\Sigma \cap \overline{B}_{t_0}(x)$ is a $C^{1,\alpha}$ regular curve.

Proof of Corollary 5.14. Recall that $a \in (0, 1/100)$. Let $y \in \Sigma \cap B_{ar/10}(x)$ and L_x realize the infimum in the definition of $\beta_\Sigma(x, r)$. Notice that $d_H(\Sigma \cap \overline{B}_{r/10}(y), L_x \cap \overline{B}_{r/10}(y)) \leq 5\varepsilon r$. Let L be the affine line passing through y and collinear to L_x. It is easy to see that $d_H(L_x \cap \overline{B}_{r/10}(y), L \cap \overline{B}_{r/10}(y)) \leq 5\varepsilon r$ and hence

$$d_H\big(\Sigma \cap \overline{B}_{\frac{r}{10}}(y), L \cap \overline{B}_{\frac{r}{10}}(y)\big) \leq d_H\big(\Sigma \cap \overline{B}_{\frac{r}{10}}(y), L_x \cap \overline{B}_{\frac{r}{10}}(y)\big) + d_H\big(L_x \cap \overline{B}_{\frac{r}{10}}(y), L \cap \overline{B}_{\frac{r}{10}}(y)\big) \leq 10\varepsilon r.$$

Thus $\beta_\Sigma(y, r/10) \leq \delta_0/2$. Next, let Σ' realize the supremum in the definition of $w_\Sigma^\tau(y, r/10)$. Such Σ' exists due to the condition $\beta_\Sigma(y, r/10) \leq \delta_0/2 \leq \tau$ (see Remark 5.4). Then we have that

$$w_\Sigma^\tau\left(y, \frac{r}{10}\right) = \frac{10}{r} \int_{B_{\frac{r}{10}}(y)} |\nabla u_{\Sigma'}|^p \ dz \leq \frac{10}{r} \int_{B_r(x)} |\nabla u_{\Sigma'}|^p \ dz \leq 10 w_\Sigma^\tau(x, r) < \frac{\delta_0}{2},$$

where we have used the facts that $B_{r/10}(y) \subset B_{(1+a)r/10}(x)$, $\beta_\Sigma(y, r/10)$ and $\beta_\Sigma(x, r)$ are pretty small, namely, proceeding as in the proof of Proposition 5.5, we can show that $\beta_{\Sigma'}(x, r) \leq \tau$. Thus

$$\beta_\Sigma\left(y, \frac{r}{10}\right) + w_\Sigma^\tau\left(y, \frac{r}{10}\right) < \delta_0.$$

On the other hand, $\theta_\Sigma(y, r/10) \leq 10\theta_\Sigma(x, r) \leq 10\overline{\mu}$. Then, according to Proposition 5.13, $\beta_\Sigma(y, \varrho) \leq C\varrho^\alpha$ for all $\varrho \in (0, ar/10)$. Since the point y was arbitrarily chosen in $\Sigma \cap B_{ar/10}(x)$, there exists $t_0 \in (0, ar/10)$ such that $\Sigma \cap \overline{B}_{t_0}(x)$ is a $C^{1,\alpha}$ regular curve (see, for instance, [19, Proposition 9.1]). □

Proof of Theorem 1.3. Let $\varepsilon_0, b, \overline{r} \in (0, 1)$, $C > 0$ be the constants of Lemma 3.6. Since closed connected sets with finite $\mathcal{H}^1$-measure are $\mathcal{H}^1$-rectifiable (see [18, Proposition 30.1, p. 186]), then (see Lemma 4.1) for $\mathcal{H}^1$-a.e. point x in Σ there exists the affine line T_x passing through x such that

$$\frac{1}{r} d_H(\Sigma \cap \overline{B}_r(x), T_x \cap \overline{B}_r(x)) \underset{r \to 0+}{\to} 0. \tag{5.54}$$

On the other hand,

$$\theta_\Sigma(x, r) \underset{r \to 0+}{\to} 2 \tag{5.55}$$

for $\mathcal{H}^1$-a.e. $x \in \Sigma$, in view of Besicovitch-Marstrand-Mattila Theorem (see [2, Theorem 2.63]). Let $x \in \Sigma \cap \Omega$ be such a point that (5.54) and (5.55) hold with x. According to (5.54),

$$\beta_\Sigma(x, r) \underset{r \to 0+}{\to} 0\ . \tag{5.56}$$

We claim that $w_\Sigma^\tau(x, r) \to 0$ as $r \to 0+$. Indeed, by (5.56), for any $\varepsilon \in (0, \varepsilon_0)$ there is $t_\varepsilon \in (0, \overline{r})$ such that

$$\beta_\Sigma(x, r) \leq \varepsilon \ \text{ for all } \ r \in (0, t_\varepsilon]. \tag{5.57}$$

We assume that $B_{t_\varepsilon}(x) \subset \Omega$, $t_\varepsilon < \operatorname{diam}(\Sigma)/2$ and $\varepsilon < \tau/2$. Recall that $\tau \in (0, \varepsilon_0/6]$. Then, by Proposition 5.6, for all $r \in (0, t_\varepsilon/10]$,

$$w_\Sigma^\tau(x, r) \leq C\Big(\frac{r}{t_\varepsilon}\Big)^b w_\Sigma^\tau(x, t_\varepsilon) + Cr^b. \tag{5.58}$$

On the other hand, by Remark 5.4 and Proposition 2.22, $w_\Sigma^\tau(x, t_\varepsilon) < +\infty$. Thus, letting r tend to $0+$ in (5.58), we get

$$w_\Sigma^\tau(x, r) \underset{r \to 0+}{\to} 0\ . \tag{5.59}$$

By (5.56) and (5.59),

$$\beta_\Sigma(x, r) + w_\Sigma^\tau(x, r) \underset{r \to 0+}{\to} 0.$$

This, together with (5.55), Corollary 5.14 and the fact that for each integer $N \geq 2$, the unique positive solution $\overline{\mu}$ to the equation $\mu = 5 + \mu^{1-\frac{1}{N}}$ is strictly greater than 5, completes the proof of Theorem 1.3. □

6. Remark about singular points

In this section, we prove that if Σ is a solution to Problem 1.1, then $\Sigma \cap \Omega$ cannot contain quadruple points, namely, there is no point $x \in \Sigma \cap \Omega$ such that for some fairly small radius $r > 0$ the set $\Sigma \cap \overline{B}_r(x)$ is a union of four distinct C^1 arcs, each of which meets at point x exactly one of the other three at an angle of 180 degrees, and each of the other two at an angle of 90 degrees.

We shall say that a set $K \subset \mathbb{R}^N$ is a cross passing through a point $x \in \mathbb{R}^N$ if K consists of two mutually perpendicular affine lines passing through x. For convenience, let us denote the cross $(\mathbb{R} \times \{0\}^{N-1}) \cup (\{0\}^{N-1} \times \mathbb{R})$ passing through the origin by K_0.

Lemma 6.1. *Let $p \in (N-1, +\infty)$. There exist $\alpha, \delta \in (0,1)$ and $C > 0$, depending only on N and p, such that if $u \in W^{1,p}(B_1)$ is a weak solution to the p-Laplace equation in $B_1 \backslash E$, where*

$$E = ((-1,1) \times \{0\}^{N-1}) \cup (\{0\}^{N-1} \times (-1,1)),$$

satisfying $u = 0$ p-q.e. on E, then

$$\int_{B_r} |\nabla u|^p \ dx \leq C r^{1+\alpha} \int_{B_1} |\nabla u|^p \ dx \ \text{ for all } r \in (0, \delta]. \tag{6.1}$$

Proof. First, adapting the proof of Lemma A.3, one observes that there exist $\varepsilon = \varepsilon(N,p) \in (0,1)$ and $C = C(N,p) > 0$ such that for any nonnegative p-harmonic function v in $B_1 \backslash E$, continuous in B_1 and satisfying $v = 0$ on E, the following estimate holds

$$\max_{x \in \overline{B}_\varepsilon} v(x) \leq C v(A_\varepsilon), \tag{6.2}$$

where A_ε is a point in ∂B_ε such that $\mathrm{dist}(A_\varepsilon, E) = \varepsilon$. Next, assuming, as in *Step 1* in the proof of Lemma 3.1 that u is continuous and nonnegative in B_1, by virtue of (6.2) and the fact that we add the additional boundary condition (i.e., $u = 0$ on $(-1,1) \times \{0\}^{N-1}$) compared with the situation in Lemma 3.1, we observe that all the estimates established in the proof of Lemma 3.1 in *Step 1* for a nonnegative p-harmonic function in $B_1 \backslash (\{0\}^{N-1} \times (-1,1))$, that is continuous in B_1 and vanishes on $\{0\}^{N-1} \times (-1,1)$ are also valid for u. In the case when we only know that the weak solution $u \in W^{1,p}(B_1)$ vanishes p-q.e. on E, we can proceed in the same way as in *Step 2* in the proof of Lemma 3.1 changing $\{0\}^{N-1} \times (-1,1)$ by E. These observations complete our proof of Lemma 6.1. $\square$

The following lemma says that the estimate (3.28) still holds if the affine line in Lemma 3.6 is replaced by a suitable cross.

Lemma 6.2. *Let $p \in (N-1, +\infty)$, $f \in L^q(\Omega)$ with $q > q_1$, where q_1 is defined in (1.4). Then there exist $\varepsilon_0, \overline{r}, b \in (0,1)$, $C = C(N, p, q_0, q, \|f\|_q, |\Omega|) > 0$ such that the following holds. Let $\Sigma \subset \overline{\Omega}$ be a closed arcwise connected set. Assume that $0 < 2r_0 \leq r_1 \leq \overline{r}$, $B_{r_1}(x_0) \subset \Omega$ and that for all $r \in [r_0, r_1]$ there is a cross $K = K(r)$, passing through x_0, such that $d_H(\Sigma \cap \overline{B}_r(x_0), K \cap \overline{B}_r(x_0)) \leq \varepsilon_0 r$. Assume also that $\Sigma \setminus B_{r_1}(x_0) \neq \emptyset$. Then for every $r \in [r_0, r_1]$,*

$$\int_{B_r(x_0)} |\nabla u_\Sigma|^p \ dx \leq C \left(\frac{r}{r_1}\right)^{1+b} \int_{B_{r_1}(x_0)} |\nabla u_\Sigma|^p \ dx + C r^{1+b}.$$

Proof. The proof follows by reproducing the proofs of Lemma 3.2, Lemma 3.5 and Lemma 3.6 with a minor modification, namely, replacing the affine line by a suitable cross in the proofs of these lemmas, such a reproduction is possible thanks to Lemma 6.1. $\square$

We are now ready to prove Proposition 1.5.

Proof of Proposition 1.5. Assume by contradiction that for some $\lambda > 0$ a minimizer Σ of Problem 1.1 contains a quadruple point $x_0 \in \Sigma \cap \Omega$. Let $\varepsilon_0, b, \overline{r}, C$ be the constants of Lemma 6.2. Without loss of generality, we can assume that $0 < t_0 < \min\{\overline{r}, \operatorname{diam}(\Sigma)/2\}$, $B_{t_0}(x_0) \subset \Omega$, the set $\Sigma \cap \overline{B}_{t_0}(x_0)$ consists of exactly four distinct C^1 arcs, each of which meets at point x_0 exactly one of the other three at an angle of 180 degrees, and each of the other two at an angle of 90 degrees. Then there exists a cross K passing through x_0 such that for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0, t_0]$ such that for all $r \in (0, \delta]$,

$$d_H(\Sigma \cap \overline{B}_r(x_0), K \cap \overline{B}_r(x_0)) \leq \varepsilon r. \tag{6.3}$$

It is also worth noting that each C^1 arc is Ahlfors regular of dimension 1 (see [18, Definition 18.9, p.108]), which can be easily seen using its local parameterization. This implies that $\Sigma \cap \overline{B}_{t_0}(x_0)$ is Ahlfors regular of dimension 1. Therefore, without loss of generality, we can also assume that there exists a positive constant C_0 such that

$$\mathcal{H}^1(\Sigma \cap B_r(x_0)) \leq C_0 r \quad \text{for all } r \in (0, t_0]. \tag{6.4}$$

Let us now fix $r \in (0, t_0/2]$. By the coarea inequality (see [33, Theorem 2.1]),

$$\mathcal{H}^1(\Sigma \cap B_{2r}(x_0)) \geq \int_0^{2r} \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \; d\varrho > \int_r^{2r} \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \; d\varrho,$$

where the latter estimate comes from the fact that $\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \geq 1$ for all $\varrho \in (0, 2r]$, since $x_0 \in \Sigma$, Σ is arcwise connected and $2r < \operatorname{diam}(\Sigma)/2$. Then there exists $\varrho \in [r, 2r]$ such that

$$\frac{1}{r}\mathcal{H}^1(\Sigma \cap B_{2r}(x_0)) \geq \mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)).$$

This, together with (6.4), implies that

$$\mathcal{H}^0(\Sigma \cap \partial B_\varrho(x_0)) \leq 2C_0. \tag{6.5}$$

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of radii such that: $2r_{n+1} < r_n$ for each $n \in \mathbb{N}$; $r_n \to 0$ as $n \to +\infty$; $2r_0 < \delta = \delta(\varepsilon)$, where $\varepsilon \in (0,1)$ to be determined. By virtue of (6.5), there exists $\varrho_n \in [r_n, 2r_n]$ such that $\mathcal{H}^0(\Sigma \cap \partial B_{\varrho_n}(x_0)) \leq 2C_0$. Following [11], for each $n \in \mathbb{N}$, we define the set $D_n = K \cap \partial B_{\varrho_n}(x_0)$ which consists of exactly four points. Denote by $S_4(D_n) \subset \overline{B}_{\varrho_n}(x_0)$ a closed set of minimum $\mathcal{H}^1$-measure in the ball $\overline{B}_{\varrho_n}(x_0)$ which connects the all four points of D_n (as in [11], we call it a *Steiner connection* of these points; for more details on Steiner connections, see, for instance [22, 34, 20]). For each point $z_i \in \Sigma \cap \partial B_{\varrho_n}(x_0)$, denote by $\gamma_{i,n}$ the geodesic in $\partial B_{\varrho_n}(x_0)$ connecting z_i with the point of the set D_n closest to z_i. For each $n \in \mathbb{N}$, let G_n denote the union of all arcs $\gamma_{i,n}$, and let us define the competitor Σ_n by

$$\Sigma_n = (\Sigma \backslash B_{\varrho_n}(x_0)) \cup G_n \cup S_4(D_n).$$

Due to the condition (6.3), each arc $\gamma_{i,n}$ has $\mathcal{H}^1$-measure less than or equal to $\arcsin(\varepsilon)\, \varrho_n$. On the other hand,

$$\mathcal{H}^1(\Sigma \cap B_{\varrho_n}(x_0)) \geq 4\varrho_n \;\text{ and }\; \mathcal{H}^1(S_4(D_n)) = \sqrt{2}(\sqrt{3}+1)\varrho_n,$$

where we have used that $\mathcal{H}^1(S_4(D_n)) = \mathcal{H}^1(S_4(K_0 \cap \partial B_1))\varrho_n = \sqrt{2}(\sqrt{3}+1)\varrho_n$. Thanks to (6.5) and the fact that $\mathcal{H}^1(\gamma_{i,n}) \leq \arcsin(\varepsilon)\varrho_n$, $\mathcal{H}^1(G_n) \leq 2C_0 \arcsin(\varepsilon)\varrho_n$. Next, choosing $\varepsilon \in (0, \varepsilon_0/2)$ small enough and observing that $\sqrt{2}(\sqrt{3}+1) \approx 3.86$, we can conclude that there is a constant $\widetilde{C} > 0$ independent of n such that for each $n \in \mathbb{N}$,

$$\mathcal{H}^1(\Sigma \cap \overline{B}_{\varrho_n}(x_0)) - \mathcal{H}^1(\Sigma_n \cap \overline{B}_{\varrho_n}(x_0)) \geq \widetilde{C}\varrho_n. \tag{6.6}$$

Now we want to apply Lemma 6.2 to Σ_n. If $\varrho_n \leq \varepsilon_0 r/2$ and $r \in (0, \delta]$, then

$$\begin{aligned}
&d_H(\Sigma_n \cap \overline{B}_r(x_0), K \cap \overline{B}_r(x_0)) \\
&\leq d_H(\Sigma_n \cap \overline{B}_r(x_0), \Sigma \cap \overline{B}_r(x_0)) + d_H(\Sigma \cap \overline{B}_r(x_0), K \cap \overline{B}_r(x_0)) \\
&\leq \varrho_n + \frac{\varepsilon_0 r}{2} \leq \frac{\varepsilon_0 r}{2} + \frac{\varepsilon_0 r}{2} = \varepsilon_0 r,
\end{aligned}$$

where we have used (6.3) and the fact that $\varepsilon \in (0, \varepsilon_0/2)$. So we can apply Lemma 6.2 to Σ_n, for the interval $[2\varrho_n/\varepsilon_0, \delta]$, provided that $2\varrho_n/\varepsilon_0 \le \delta/2$, and we obtain that

$$\int_{B_r(x_0)} |\nabla u_{\Sigma_n}|^p \, dx \le C\left(\frac{r}{\delta}\right)^{1+b} \int_{B_\delta(x_0)} |\nabla u_{\Sigma_n}|^p \, dx + Cr^{1+b} \quad \text{for all } r \in \left[\frac{2\varrho_n}{\varepsilon_0}, \delta\right].$$

Hereinafter in this proof, C denotes a positive constant independent of n, which can be different from line to line. Applying the above estimate for $r = 2\varrho_n/\varepsilon_0$ and using (2.11), we have

$$\int_{B_{\frac{2\varrho_n}{\varepsilon_0}}(x_0)} |\nabla u_{\Sigma_n}|^p \, dx \le C\varrho_n^{1+b} \tag{6.7}$$

for all $n \in \mathbb{N}$ such that $2\varrho_n/\varepsilon_0 \le \delta/2$. Recall that the exponent b given by Lemma 6.2 is positive provided $q > q_1$. Now, using the fact that Σ is a minimizer and Σ_n is a competitor for Σ, the estimate (6.6), Corollary 2.21 and the estimate (6.7), we deduce the following

$$\begin{aligned}
0 &\le \mathcal{F}_{\lambda,f,\Omega}(\Sigma_n) - \mathcal{F}_{\lambda,f,\Omega}(\Sigma) \le E_{f,\Omega}(u_\Sigma) - E_{f,\Omega}(u_{\Sigma_n}) - \lambda\widetilde{C}\varrho_n \\
&\le C\int_{B_{2\varrho_n}(x_0)} |\nabla u_{\Sigma_n}|^p \, dx + C\varrho_n^{N+p'-\frac{Np'}{q}} - \lambda\widetilde{C}\varrho_n \\
&\le C\int_{B_{\frac{2\varrho_n}{\varepsilon_0}}(x_0)} |\nabla u_{\Sigma_n}|^p \, dx + C\varrho_n^{N+p'-\frac{Np'}{q}} - \lambda\widetilde{C}\varrho_n \\
&\le C\varrho_n^{1+b} + C\varrho_n^{N+p'-\frac{Np'}{q}} - \lambda\widetilde{C}\varrho_n
\end{aligned}$$

for all $n \in \mathbb{N}$ such that $2\varrho_n/\varepsilon_0 \le \delta/2$. Notice that $N+p'-Np'/q > 1$ if and only if $q > Np/(Np-N+1)$, which is fulfilled under the assumption $q > q_1$. Finally, letting n tend to $+\infty$, we arrive to a contradiction. This completes our proof of Proposition 1.5. $\square$

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Appendix A. Auxiliary results

Recall that we write points of $\mathbb{R}^N$ as $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$.

Lemma A.1. *Let $N \ge 2$, $p \in (N-1, +\infty)$, $\beta = (p-N+1)/(p-1)$ and $\gamma \in (0, \beta)$. There exists $\delta \in (0,1)$, depending only on N, p and γ, such that $\hat{u}(x) = |x'|^\gamma + x_N^2$ is a supersolution to the p-Laplace equation in $\{0 < |x'| < \delta\} \cap \{|x_N| < 1\}$.*

Proof. To simplify the notation, we denote $\{0 < |x'| < \delta\} \cap \{|x_N| < 1\}$ by $C^o_{\delta,1}$. We need to prove that there exists $\delta = \delta(N, p, \gamma) \in (0,1)$ such that

$$\Delta_p \hat{u} = \Delta\hat{u}|\nabla\hat{u}|^{p-2} + (p-2)|\nabla\hat{u}|^{p-4}\Delta_\infty\hat{u} \le 0 \text{ in } C^o_{\delta,1}, \tag{A.1}$$

where $\Delta\hat{u} = \Delta_2\hat{u} := \sum_{i=1}^N \hat{u}_{x_i,x_i}$ and $\Delta_\infty\hat{u} := \sum_{i,j=1}^N \hat{u}_{x_i}\hat{u}_{x_j}\hat{u}_{x_i,x_j}$. Since $|\nabla\hat{u}| \ne 0$ in $C^o_{\delta,1}$, (A.1) is equivalent to the following

$$\hat{\Delta}_p\hat{u} := \Delta\hat{u}|\nabla\hat{u}|^2 + (p-2)\Delta_\infty\hat{u} \le 0 \text{ in } C^o_{\delta,1}.$$

Calculating the partial derivatives of $\hat{u}$ in $C^o_{\delta,1}$, we have: $\hat{u}_{x_i} = \gamma x_i |x'|^{\gamma-2}$, $i \in \{1, ..., N-1\}$; $\hat{u}_{x_N} = 2x_N$; $\hat{u}_{x_i,x_j} = \gamma(\gamma-2)x_i x_j |x'|^{\gamma-4} + \delta_{ij}\gamma|x'|^{\gamma-2}$, where $i, j \in \{1, ..., N-1\}$ and $\delta_{i,j}$ is the Kronecker delta; $\hat{u}_{x_N,x_N} = 2$. Next, we deduce that

$$\Delta\hat{u} = \gamma(\gamma+N-3)|x'|^{\gamma-2} + 2, \quad |\nabla\hat{u}|^2 = \gamma^2|x'|^{2\gamma-2} + 4x_N^2 \text{ and } \Delta_\infty\hat{u} = \gamma^3(\gamma-1)|x'|^{3\gamma-4} + 8x_N^2$$

in $C^o_{\delta,1}$. This yields the following

$$\hat{\Delta}_p\hat{u} = \gamma^3(\gamma p - p - \gamma + N - 1)|x'|^{3\gamma-4} + 4\gamma(\gamma+N-3)|x'|^{\gamma-2}x_N^2 + 2\gamma^2|x'|^{2\gamma-2} + (p-1)8x_N^2 \quad \text{(A.2)}$$

in $C^o_{\delta,1}$. Since $0 < \gamma < \beta$, $\gamma^3(\gamma p - p - \gamma + N - 1) < 0$ and $3\gamma - 4 < \gamma - 2 < 0$. Thus, analyzing (A.2), we deduce that there exists $\delta = \delta(N, p, \gamma) \in (0, 1)$ such that $\hat{\Delta}_p\hat{u} \leq 0$ in $C^o_{\delta,1}$. This completes the proof. □

The following lemma will be used to prove Lemma A.3.

Lemma A.2. *Let $p \in (N-1, +\infty)$. Then there exists a positive integer $q = q(N, p)$ such that the following holds. Let $x_0 \in \mathbb{R}^N$, $r > 0$ and $L \subset \mathbb{R}^N$ be an affine line passing through x_0. Then for any nonnegative p-harmonic function u in $B_r(x_0)\backslash L$, continuous in $\overline{B}_r(x_0)$ and satisfying $u = 0$ on $L \cap B_r(x_0)$, the following estimate holds*

$$\max_{x\in\overline{B}_{2^{-q}r}(x_0)} u(x) \leq \frac{1}{2} \max_{x\in\overline{B}_r(x_0)} u(x).$$

Proof. Since the p-Laplacian is invariant under scalings, rotations and translations, we can assume that $B_r(x_0) = B_1$ and $L \cap B_r(x_0) = \{0\}^{N-1} \times (-1, 1)$. To lighten the notation, we denote $\{0\}^{N-1} \times (-1, 1)$ by S. Let $\gamma = (p - N + 1)/(2p - 2)$. Then, by Lemma A.1, there exists $\delta = \delta(N, p) \in (0, 1/2)$ such that $\hat{u}(x) = |x'|^\gamma + x_N^2$ is a weak supersolution to the p-Laplace equation in $\{0 < |x'| < 2\delta\} \cap \{|x_N| < 1\}$, and is continuous in $\mathbb{R}^N$. Hereinafter in this proof, C denotes a positive constant that can only depend on N, p and can be different from line to line. Since

$$\hat{u}(x) = \delta^\gamma + x_N^2 \geq \delta^\gamma \text{ if } |x'| = \delta \text{ and } \hat{u}(x) = |x'|^\gamma + \delta^2 \geq \delta^2 \text{ if } |x_N| = \delta,$$

the estimate

$$u \leq C\left(\max_{\overline{B}_1} u\right)\hat{u}$$

holds on $\partial\left(\{|x'| < \delta\} \cap \{|x_N| < \delta\}\right)$. Furthermore,

$$u(x) \leq C\left(\max_{\overline{B}_1} u\right)\hat{u}(x) \text{ if } x \in S.$$

Then the comparison principle (see [24, Theorem 7.6]) says that

$$u \leq C\left(\max_{\overline{B}_1} u\right)\hat{u} \text{ in } \{|x'| \leq \delta\} \cap \{|x_N| \leq \delta\}.$$

This implies that $u(x) \leq C\left(\max_{\overline{B}_1} u\right)|x|^\gamma$ for all $x \in \overline{B}_\delta$, since $|x'|^\gamma + x_N^2 \leq 2|x|^\gamma$ for all $x \in \overline{B}_\delta$. Next, choosing $q = q(N, p) \in \mathbb{N}$ such that $2^{-q} \in (0, \delta)$ and $C2^{-q\gamma} \leq 1/2$, we obtain the following

$$\max_{\overline{B}_{2^{-q}}} u \leq \frac{1}{2}\max_{\overline{B}_1} u,$$

which concludes the proof of Lemma A.2. □

We prove the following Carleson estimate.

Lemma A.3. *Let $p \in (N-1, +\infty)$. Then there exist $\varepsilon = \varepsilon(N,p) \in (0,1)$ and $C = C(N,p) > 0$ such that the following holds. Let $x_0 \in \mathbb{R}^N$, $r > 0$ and $L \subset \mathbb{R}^N$ be an affine line passing through x_0. Then for any nonnegative p-harmonic function u in $B_r(x_0)\backslash L$, continuous in $B_r(x_0)$ and satisfying $u = 0$ on $L \cap B_r(x_0)$, the following estimate holds*

$$\max_{x\in \overline{B}_{\varepsilon r}(x_0)} u(x) \leq C u(A_{\varepsilon r}(x_0)),$$

where $A_{\varepsilon r}(x_0)$ denotes a point such that $\mathrm{dist}(A_{\varepsilon r}(x_0), L) = \varepsilon r$ *and* $A_{\varepsilon r}(x_0) \in \partial B_{\varepsilon r}(x_0)$.

Proof. We follow the same strategy as in the proof of [14, Theorem 1.1]. Since the p-Laplacian is invariant under scalings, rotations and translations, without loss of generality, we can assume that $B_r(x_0) = B_1$, $L \cap B_r(x_0) = \{0\}^{N-1} \times (-1,1)$. To simplify the notation, we denote the set $\{0\}^{N-1} \times (-1,1)$ by S. Let $q = q(N,p)$ be the positive integer of Lemma A.2. Define $\varepsilon = 2^{-q-2m}$ with $m \in \mathbb{N}$ to be determined (ε is small enough). Fix an arbitrary A_ε such that $\mathrm{dist}(A_\varepsilon, S) = \varepsilon$ and $A_\varepsilon \in \partial B_\varepsilon$. Notice that if $u(A_\varepsilon) = 0$, then by the Harnack inequality (see [24, Theorem 6.2]), $u(x) = 0$ for all $x \in B_1$ and the proof follows. Without loss of generality, we assume that $u(A_\varepsilon) = 1$. By Lemma A.2, for each $x_0 \in S$ and $r \in (0, \mathrm{dist}(x_0, \partial B_1))$,

$$\max_{\overline{B}_{2^{-q}r}(x_0)} u \leq \frac{1}{2} \max_{\overline{B}_r(x_0)} u. \tag{A.3}$$

On the other hand, by the Harnack inequality, there exists $M = M(N,p) > 1$ such that

$$u(x', x_N) \leq \begin{cases} Mu(2x', x_N) & \text{for } x \in \overline{B}_{1/4}\backslash S \\ Mu(A_\varepsilon) = M & \text{for } x \in \{|x'| \geq \varepsilon/2\} \cap \overline{B}_{2\varepsilon}. \end{cases}$$

Suppose that there exists $y_0 \in \overline{B}_\varepsilon$ such that $u(y_0) \geq M^{n+2}$ with $n \in \mathbb{N}$ to be determined. Then

$$\mathrm{dist}(y_0, S) \leq 2^{-n}\varepsilon,$$

because otherwise $u(y_0) \leq M^{n+1}u(A_\varepsilon) = M^{n+1}$. Let $\widetilde{y}_0$ be the projection of y_0 to S. Then, applying (A.3), we have

$$\max_{\overline{B}_{2^{-n+qm}\varepsilon}(\widetilde{y}_0)} u \geq 2^m \max_{\overline{B}_{2^{-n}\varepsilon}(\widetilde{y}_0)} u \geq 2^m M^{n+2},$$

where we have also used the facts $y_0 \in \overline{B}_{2^{-n}\varepsilon}(\widetilde{y}_0)$, $u(y_0) \geq M^{n+2}$. We now choose and fix m so that $2^m \geq M^2$. Hence

$$u(y_1) = \max_{\overline{B}_{2^{-n+qm}\varepsilon}(\widetilde{y}_0)} u \geq M^{n+4},$$

where $y_1 \in \overline{B}_{2^{-n+qm}\varepsilon}(\widetilde{y}_0)$. Therefore $\mathrm{dist}(y_1, S) \leq 2^{-n-2}\varepsilon$ and

$$\max_{\overline{B}_{2^{-n-2+qm}\varepsilon}(\widetilde{y}_1)} u \geq 2^m \max_{\overline{B}_{2^{-n-2}\varepsilon}(\widetilde{y}_1)} u \geq M^{n+6},$$

where $\widetilde{y}_1$ is the projection of y_1 to S. Clearly, there exists $y_2 \in \overline{B}_{2^{-n-2+qm}\varepsilon}(\widetilde{y}_1)$ such that $u(y_2) \geq M^{n+6}$. So $\mathrm{dist}(y_2, S) \leq 2^{-n-4}\varepsilon$ and

$$\max_{\overline{B}_{2^{-n-4+qm}\varepsilon}(\widetilde{y}_2)} u \geq 2^m \max_{\overline{B}_{2^{-n-4}\varepsilon}(\widetilde{y}_2)} u \geq M^{n+8},$$

where $\widetilde{y}_2$ is the projection of y_2 to S. Once again there exists $y_3 \in \overline{B}_{2^{-n-4+qm}\varepsilon}(\widetilde{y}_2)$ satisfying the following: $u(y_3) \geq M^{n+8}$, $\mathrm{dist}(y_3, S) \leq 2^{-n-6}\varepsilon$ and

$$\max_{\overline{B}_{2^{-n-6+qm}\varepsilon}(\widetilde{y}_3)} u \geq 2^m \max_{\overline{B}_{2^{-n-6}\varepsilon}(\widetilde{y}_3)} u \geq M^{n+10}.$$

We obtain by induction a sequence of points (y_k) such that

$$\mathrm{dist}(y_k, S) \leq 2^{-n-2k}\varepsilon, \;\; y_k \in \overline{B}_{2^{-n-2(k-1)+qm}\varepsilon}(\widetilde{y}_{k-1})$$

and

$$u(y_k) \geq M^{n+2(k+1)}, \;\; k \geq 1.$$

We shall obtain a contradiction if we can make sure that each y_k belongs to some closed ball contained in B_1. For each $k \geq 1$,

$$\begin{aligned}|y_k| &\leq |y_k - \widetilde{y}_{k-1}| + |\widetilde{y}_{k-1} - y_{k-1}| + |y_{k-1}| \\ &\leq 2^{-n+qm}\varepsilon \sum_{l=1}^{k} 2^{-2(l-1)} + 2^{-n}\varepsilon \sum_{l=0}^{k-1} 2^{-2l} + |y_0| \\ &\leq (2^{-n+qm} + 2^{-n})\varepsilon \sum_{l=0}^{\infty} 2^{-2l} + \varepsilon.\end{aligned}$$

By choosing n large depending on N and p, we can make $|y_k| \leq \frac{3}{2}\varepsilon$. This completes the proof of Lemma A.3. $\square$

The next lemma is classical, however, we could not find a precise reference in the exact following form, thus we provide a proof for the reader's convenience.

Lemma A.4. *Let $N \geq 2$, $p \in (1, +\infty)$, $\Sigma \subset \mathbb{R}^N$ be a closed set and $u \in W^{1,p}(B_1)$ be a p-harmonic function in $B_1 \backslash \Sigma$, continuous in B_1 with $u = 0$ on $\Sigma \cap B_1$. Then $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ are p-subharmonic in B_1.*

Proof. Since $u^- = (-u)^+$ and $(-u)$ is p-harmonic in $B_1 \backslash \Sigma$, it is enough to prove that the function u^+ is p-subharmonic in B_1. Let us fix an arbitrary nonnegative function $\varphi \in C_0^\infty(B_1)$ and for all $\varepsilon, \eta \in (0,1)$ define $\varphi_{\eta,\varepsilon} = ((\eta + (u-\varepsilon)^+)^\varepsilon - \eta^\varepsilon)\varphi$. Since $u \in W^{1,p}(B_1)$ is p-harmonic in $B_1 \backslash \Sigma$ and $\varphi_{\eta,\varepsilon} \in W_0^{1,p}(B_1 \backslash \Sigma)$,

$$\int_{B_1} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi_{\eta,\varepsilon} \rangle \; dx = 0.$$

This implies that

$$\int_{B_1} ((\eta + (u-\varepsilon)^+)^\varepsilon - \eta^\varepsilon) \langle |\nabla u^+|^{p-2} \nabla u^+, \nabla \varphi \rangle \; dx + \varepsilon \int_{B_1 \cap \{u > \varepsilon\}} |\nabla u^+|^p (\eta + (u-\varepsilon)^+)^{\varepsilon-1} \varphi \; dx = 0$$

and hence

$$\int_{B_1} ((\eta + (u-\varepsilon)^+)^\varepsilon - \eta^\varepsilon) \langle |\nabla u^+|^{p-2} \nabla u^+, \nabla \varphi \rangle \; dx \leq 0. \tag{A.4}$$

Letting η and then ε tend to $0+$ in (A.4), by Lebesgue's dominated convergence theorem, we get

$$\int_{B_1} \langle |\nabla u^+|^{p-2} \nabla u^+, \nabla \varphi \rangle \; dx \leq 0,$$

which concludes the proof. $\square$

The next lemma is a refined version of [15, Lemma 5.14].

Lemma A.5. *Let $N \geq 2$ and let $\gamma : [0,1] \to \mathbb{R}^N$ be a curve such that $\Gamma := \gamma([0,1]) \subset \overline{B}_r(x_0)$. Assume that $\xi_1 = \gamma(0) \in \partial B_r(x_0)$ and $\xi_2 = \gamma(1) \in \partial B_r(x_0)$. Then*

$$\max_{y \in \Gamma} \operatorname{dist}(y, [\xi_1, \xi_2]) \leq (2r(\mathcal{H}^1(\Gamma) - |\xi_2 - \xi_1|))^{\frac{1}{2}}.$$

Proof. Let $z \in \operatorname{argmax}_{y \in \Gamma} \operatorname{dist}(y, [\xi_1, \xi_2])$. Assume that $h := \operatorname{dist}(z, [\xi_1, \xi_2]) > 0$ and $|\xi_1 - \xi_2| > 0$, otherwise the proof follows. Let $z' \in \mathbb{R}^N$ be a point making (ξ_1, z', ξ_2) an isosceles triangle such that $\operatorname{dist}(z', [\xi_1, \xi_2]) = h$. Notice that $h \leq 2r$, $|\xi_1 - \xi_2|/2 \leq r$ and hence

$$|z' - \xi_2| \leq h + \frac{|\xi_1 - \xi_2|}{2} \leq 3r.$$

On the other hand, $\mathcal{H}^1(\Gamma) \geq 2|z' - \xi_2|$. Then, using the Pythagorean theorem, we get

$$\begin{aligned} h^2 = |z' - \xi_2|^2 - \frac{|\xi_1 - \xi_2|^2}{4} &= \left(|z' - \xi_2| - \frac{|\xi_1 - \xi_2|}{2}\right)\left(|z' - \xi_2| + \frac{|\xi_1 - \xi_2|}{2}\right) \\ &\leq \left(\frac{\mathcal{H}^1(\Gamma)}{2} - \frac{|\xi_1 - \xi_2|}{2}\right)(3r + r) \\ &= 2r(\mathcal{H}^1(\Gamma) - |\xi_1 - \xi_2|). \end{aligned}$$

This completes the proof of Lemma A.5. $\square$

References

[1] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] T. Bagby. Quasi topologies and rational approximation. *Journal of Functional Analysis*, 10(3):259 – 268, 1972.

[4] L. Boccardo and F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.*, 19:581 – 597, 1992.

[5] A. Bonnet. On the regularity of edges in image segmentation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(4):485–528, 1996.

[6] M. Bonnivard, A. Lemenant, and F. Santambrogio. Approximation of length minimization problems among compact connected sets. *SIAM J. Math. Anal.*, 47(2):1489–1529, 2015.

[7] D. Bucur and P. Trebeschi. Shape optimisation problems governed by nonlinear state equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(5):945–963, 1998.

[8] B. Bulanyi. On the importance of the connectedness assumption in the statement of the optimal p-compliance problem. *J. Math. Anal. Appl.*, 499(2):125064, 2021.

[9] B. Bulanyi and A. Lemenant. Regularity for the planar optimal p-compliance problem. *ESAIM: COCV*, 27:35, 2021.

[10] G. Buttazzo, E. Mainini, and Stepanov E. Stationary configurations for the average distance functional and related problems. *Control Cybernet.*, 38(4A):1107–1130, 2009.

[11] G. Buttazzo, E. Oudet, and E. Stepanov. Optimal transportation problems with free Dirichlet regions. *Progress in Nonlinear Differential Equations and Their Applications*, 51:41–65, 2002.

[12] G. Buttazzo and F. Santambrogio. Asymptotical compliance optimization for connected networks. *Netw. Heterog. Media*, 2(4):761–777, 2007.

[13] G. Buttazzo and E. Stepanov. Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2(4):631–678, 2003.

[14] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa. Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana J. Math.*, 30(4):621–640, 1981.

[15] A. Chambolle, J. Lamboley, A. Lemenant, and E. Stepanov. Regularity for the optimal compliance problem with length penalization. *SIAM J. Math. Anal.*, 49(2):1166–1224, 2017.

[16] B. Dacorogna, W. Gangbo, and N. Subía. Sur une généralisation de l'inégalité de Wirtinger. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9(1):29–50, 1992.

[17] G. Dal Maso and F. Murat. Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24, 1997.

[18] G. David. *Singular Sets of Minimizers for the Mumford-Shah Functional*, volume 233 of *Progress in mathematics*. Birkhäuser-Verlag, Basel, 2005.

[19] G. David, C. Kenig, and T. Toro. Asymptotically optimally doubling measures and reifenberg flat sets with vanishing constant. *Communications on Pure and Applied Mathematics*, 54(4):385–449, 2001.

[20] D.Z. Du, F.K. Hwang, G.D. Song, and G.Y. Ting. Steiner minimal trees on sets of four points. *Discrete Comput. Geom.*, 2:401–414, 1987.

[21] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2001.

[22] E.N. Gilbert and H.O. Pollak. Steiner minimal trees. *SIAM J. Appl. Math.*, 16:1–29, 1968.

[23] L.I. Hedberg. Non-linear potentials and approximation in the mean by analytic functions. *Math. Z.*, 129:299–319, 1972.

[24] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc. Mineola, New York, 2006.

[25] A. Lemenant. About the regularity of average distance minimizers in $\mathbb{R}^2$. *J. Convex Anal.*, 18(4):949–981, 2011.

[26] P. Lindqvist. *Notes on the stationary p-Laplace equation*. SpringerBriefs in Mathematics. Springer, Cham, 2019.

[27] N. L. P. Lundström. Estimates for p-harmonic functions vanishing on a flat. *Nonlinear Anal.*, 74(18):6852 – 6860, 2011.

[28] N. L. P. Lundström. Phragmén-Lindelöf theorems and p-harmonic measures for sets near low-dimensional hyperplanes. *Potential Analysis*, 44:313–330, 2016.

[29] M. Miranda, E. Paolini, and E. Stepanov. On one-dimensional continua uniformly approximating planar sets. *Calc. Var.*, 27:287–309, 2006.

[30] A. Nayam. Asymptotics of an optimal compliance-network problem. *Netw. Heterog. Media*, 8(2):573–589, 2013.

[31] A. Nayam. Constant in two-dimensional p-compliance-network problem. *Netw. Heterog. Media*, 9(1):161–168, 2014.

[32] E. Paolini and E. Stepanov. Qualitative properties of maximum distance minimizers and average distance minimizers in $\mathbb{R}^N$. *Journal of Mathematical Sciences*, 122:3290–3309, 2004.

[33] E. Paolini and E. Stepanov. Existence and regularity results for the Steiner problem. *Calc. Var. Partial Differential Equations*, 46(3-4):837–860, 2013.

[34] H.O. Pollak. Some remarks on the Steiner problem. *J. Combin. Theor. Ser. A*, 24:278–295, 1978.

[35] F. Santambrogio and P. Tilli. Blow-up of optimal sets in the irrigation problem. *J. Geom. Anal.*, 15(2):343–362, 2005.

[36] D. Slepčev. Counterexample to regularity in average-distance problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(1):169–184, 2014.

[37] E. Stepanov. Partial geometric regularity of some optimal connected transportation networks. *J. Math. Sci. (N.Y.)*, 132(4):522–552, 2006.

[38] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl.*, 110:353–372, 1976.

[39] P. Tilli. Some explicit examples of minimizers for the irrigation problem. *J. Convex Anal.*, 17(2):583–595, 2010.

[40] K. Yosida. *Functional analysis (6th edn)*. Springer-Verlag, Berlin, 1980.

[41] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

[42] V. Šverák. On optimal shape design. *J. Math. Pures Appl.*, 72:537–551, 1993.