Regularity of Solutions to the Fractional Cheeger-Laplacian on Domains in Metric Spaces of Bounded Geometry

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Abstract

We study existence, uniqueness, and regularity properties of the Dirichlet problem related to fractional Dirichlet energy minimizers in a complete doubling metric measure space (X, d_X, μ_X) satisfying a 2-Poincaré inequality. Given a bounded domain $\Omega \subset X$ with $\mu_X(X \setminus \Omega) > 0$, and a function f in the Besov class $B_{2,2}^{\theta}(X) \cap L^2(X)$, we study the problem of finding a function $u \in B_{2,2}^{\theta}(X)$ such that $u = f$ in $X \setminus \Omega$ and $\mathcal{E}_\theta(u, u) \leq \mathcal{E}_\theta(h, h)$ whenever $h \in B_{2,2}^{\theta}(X)$ with $h = f$ in $X \setminus \Omega$. We show that such a solution always exists and that this solution is unique. We also show that the solution is locally Hölder continuous on Ω, and satisfies a non-local maximum and strong maximum principle. Part of the results in this paper extend the work of Caffarelli and Silvestre in the Euclidean setting and Franchi and Ferrari in Carnot groups.

Contents

1   Introduction

The development of analysis on metric measure spaces in recent decades has provided a fruitful study of upper gradient p-energy minimizers in complete metric measure spaces equipped with a doubling measure supporting a p-Poincaré inequality. Here the notion of upper gradient is the metric space generalization of the norm of the derivative from the seminal work of Heinonen and Koskela [23]. An application of the subsequent work of Cheeger [11] gave a differential structure on such a metric measure space with respect to which every Lipschitz function enjoys a first order Taylor approximation property, and the differential structure can be equipped with a measurable inner product so that the induced norm on the differential of a Lipschitz function is comparable to the minimal p-weak upper gradient. Therefore, one can, instead of minimizing the upper gradient energy, minimize the energy given by integrating the p-th power of the norm of the differential. Such energy minimizers are upper gradient p-energy quasiminimizers in the sense of [27], and hence have regularity properties such as local Hölder continuity and the Harnack inequality that the upper gradient p-energy minimizers also satisfy.

In considering the Cheeger differential structure D_X on the metric measure space (X, d_X, μ_X), thanks to the inner product on this structure we have an induced Dirichlet form (corresponding to $p = 2$) in the sense of [17]. Thus the theory of Dirichlet forms yields a Cheeger Laplacian operator Δ_X. The fractional Laplace operator $(-\Delta_X)^\theta$, defined via spectral theory by using Δ_X and the associated Dirichlet forms, is a non-local operator on X. The goal of this paper is to study existence and regularity properties of the solution to the non-local equation $(-\Delta_X)^\theta u = 0$ on a bounded domain $\Omega \subset X$ with Dirichlet data $u = f$ on $X \setminus \Omega$ for f in the suitable function class on X. The suitable function class here is the inhomogeneous Besov class $B_{2,2}^\theta(X) \cap L^2(X)$. The permissible range of θ is $0 < \theta < 1$.

To achieve the goals described above, we use the line of investigation implemented by Caffarelli and Silvestre in [9]. Additional tools and structures from [2, 3, 19, 20, 27] are also key components in our proofs. The following are the main results of this paper. The first theorem below establishes the existence of a solution to the fractional Laplacian problem with given Dirichlet data. Here, with Δ_X a choice of

the Cheeger Laplacian on X, and $0 < \theta < 1$, we set

$$\mathcal{E}_\theta(f, f) := \int_X ((-\Delta_X)^{\theta/2} f)^2 \, d\mu_X. \tag{1.1}$$

Theorem 1.1. *Let $f \in B_{2,2}^\theta(X) \cap L^2(X)$, and Ω be a bounded domain in X with $\mu_X(X \setminus \Omega) > 0$. Then there is a unique $u \in B_{2,2}^\theta(X)$ with $u = f$ in $X \setminus \Omega$ such that whenever $h \in B_{2,2}^\theta(X)$ with $h = f$ in $X \setminus \Omega$, we have*

$$\mathcal{E}_\theta(u, u) \leq \mathcal{E}_\theta(h, h). \tag{1.2}$$

Equivalently, we have

$$\mathcal{E}_\theta(u, h) = 0$$

whenever $h \in B_{2,2}^\theta(X)$ such that h has compact support in Ω.

In the Euclidean setting the existence and uniqueness results for the fractional Dirichlet problem were obtained in [14, 25, 33]. The notion related to (1.2) is given in the next section, see Definition 2.7 below. The proof of the above theorem also shows that if f is in $B_{2,2}^\theta(X)$ but not necessarily in $L^2(X)$, then the solution still exists provided we can make sense of $\mathcal{E}_\theta(f, f)$. Indeed, if f is a non-zero constant, or a perturbation of a nonzero constant by a function in $B_{2,2}^\theta(X) \cap L^2(X)$, then $\mathcal{E}_\theta(f, f)$ should make sense.

The next theorem discusses the regularity properties of the solution.

Theorem 1.2. *Let $0 < \theta < 1$. Suppose (X, d_X, μ_X) is a complete and doubling metric measure space that satisfies a 2-Poincaré inequality, and that $\Omega \subset X$ is a bounded domain with $\mu_X(X \setminus \Omega) > 0$. Suppose further that $f \in B_{2,2}^\theta(X) \cap L^2(X)$ is a solution to (1.2). Then f is locally Hölder continuous on Ω. Moreover, if $f \geq 0$ on X, then u satisfies a Harnack inequality on balls $B \subset X$ for which $2B \subset \Omega$.*

The pioneering work related to this problem in the setting of Euclidean domains is due to Cafarelli and Silvestre [9]. They proved a Harnack inequality for functions $u : \mathbb{R}^n \to [0, \infty)$ which satisfy

$$(-\Delta)^\theta u(x) = 0, \qquad x \in \Omega$$

for a given Euclidean domain $\Omega \subset \mathbb{R}^n$. To do this they first consider extensions of Besov functions on $\mathbb{R}^n$ to Sobolev functions on a suitably weighted $\mathbb{R}^n \times \mathbb{R}$, solve a corresponding Dirichlet problem for the weighted analogue of the standard Laplacian on $\mathbb{R}^n \times (0, \infty)$ with boundary data the suitable Besov function on $\mathbb{R}^n$, and then study the boundary behavior on $\partial(\mathbb{R}^n \times (0, \infty)))$ of such a solution when the boundary datum itself is a Besov energy minimizer. They show that in this case, the solution on $\Omega_0 := \mathbb{R}^n \times (0, \infty)$ has an extension to all of $\Omega \times \mathbb{R}$ that is (weighted) harmonic on $\Omega \times \mathbb{R}$. Then the knowledge that the harmonic functions are locally Hölder continuous and satisfy a Harnack inequality can be used to verify the corresponding property for the Besov energy minimizer on Ω. This approach was extended in [15] to Carnot groups and in [2] to the parabolic setting. For a related

3

non-local problem in the manifold setting, see [10]. We follow the prescription of [9] and consider the metric space $Z = X \times \mathbb{R}$, equipped with the metric

$$d_Z((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2)^2 + (y_1 - y_2)^2},$$

and, for $a = 1 - 2\theta$, the measure μ_a given by $d\mu_a(x, y) = |y|^a \, dy \, d\mu_X(x)$. The next section describes the setting of this paper in greater detail.

The solution obtained in the proof of Theorem 1.2 was via an extension of the function $f \in B_{2,2}^\theta(X) \cap L^2(X)$ from $X \times \{0\}$ to $Z_+ := X \times (0, \infty)$ using a modified heat extension given in [3]. The final main theorem of this paper is that one can achieve this extension also by solving the Dirichlet problem on $U_\Omega := Z_+ \cup Z_- \cup (\Omega \times \{0\})$ where $Z_- = X \times (-\infty, 0)$.

Theorem 1.3. *With the hypotheses given in Theorem 1.2, a function $f \in B_{2,2}^\theta(X) \cap L^2(X)$ is a solution to (1.2) if and only if f is the trace on $X \times \{0\} = \partial Z_+$ of the solution, from the homogeneous Newton-Sobolev class $D^{1,2}(Z)$, to the Dirichlet problem related to the equation $\Delta_a u = 0$ on U_Ω with boundary data f. Moreover, such a solution is unique in that if h is another solution from $D^{1,2}(Z)$ with $h = f$ in ∂U_Ω, then $h = f$ in X. Furthermore, a maximum principle and a strong maximum principle hold:*

$$esssup_{x \in \Omega} f(x) \leq esssup_{w \in X \setminus \Omega} f(w),$$

and if there is $x_0 \in \Omega$ such that $esssup_{x \in X} f(x) = f(x_0)$, then f is constant on X.

Here, by referring to $h \in D^{1,2}(Z)$ with $h = f$ in ∂U_Ω we mean that the trace of h on ∂U_Ω is μ_X-almost everywhere equal to f. To make sense of this, we do develop the notion of trace in the setting here, see Section 4 below. Moreover, when we say that $esssup_{x \in X} f(x) = f(x_0)$ for some $x_0 \in \Omega$, we consider f to be the continuous representative in Ω obtained from Theorem 1.2, with the understanding that in $X \setminus \Omega$ the function f is well-defined only μ_X-almost everywhere.

The non-local nature of the fractional Laplacian is reflected in the non-local nature of the maximum and strong maximum principle. In the Euclidean setting, the maximum principle was obtained in [8, 12, 33].

The work [2] studied scale-invariant Harnack inequalities for fractional powers of smooth parabolic and elliptic operators on Euclidean spaces, extending the result of [9] to this generality. Indeed, the smoothness assumption seems to be cosmetic there, and it is not difficult to see that the work of [2] extends also to the setting of sub-Riemannian manifolds. It was pointed out in [2] that their methods extend to a general class of Dirichlet forms and associated infinitessimal generator as the elliptic operator.

During the preparation of this manuscript, we became aware of the concurrent work by Baudoin, Lang, and Sire [3], which established the Harnack principle of solutions to fractional Laplace problems in the context of strongly local Dirichlet forms that satisfy a 2-Poincaré inequality, and further studied an analog of the boundary Harnack principle for the case that Ω is an inner uniform domain in X. Their approach, as well as that of [2], is based on spectral theory and gave us valuable tools to use in the study undertaken here. In our setting, we consider a specific Dirichlet

form given by the measurable inner product structure on a choice of Cheeger differential structure available on the metric space. Given the quasiconvexity of X (a consequence of the measure being doubling and supporting a Poincaré inequality), the Dirichlet form of interest here satisfies the hypotheses of [3]. Hence the Harnack inequality of the above theorem follows directly from [3]. In Theorem 1.2 we combine the tools developed in [3] with additional tools related to potential theory in metric setting to add to the results of [3] in our context. We also connect the domain of the fractional Laplacians explicitly to Besov spaces, traces and the upper gradient approach. In particular, we use their results related to the explicit Poisson-type extension (2.15), see also [2].

There is a rich collection of mathematical literature on fractional orders of operators in smooth setting, and it is not possible to list them all here. We direct the interested reader to the references cited above as well as the papers cited in them.

Acknowledgements

2   Background and Notation

2.1   Newton-Sobolev spaces and related notions

In this paper we are concerned with a metric measure space (X, d_X, μ_X). We first start with the notion of 2-modulus of a family of curves in X. Given a family Γ of curves in X, the 2-modulus of this family is the number

$$\mathrm{Mod}_2(\Gamma) := \inf_\rho \int_X \rho^2 \, d\mu_X,$$

where the infimum is over all non-negative Borel measurable functions ρ on X that satisfy $\int_\gamma \rho \, ds \geq 1$ for each locally rectifiable curve $\gamma \in \Gamma$.

The notion of upper gradients from [23] forms the foundation of first order analysis in metric measure spaces. Given a metric space (X, d_X), a non-negative Borel

function g on X is an upper gradient of a map $u\colon X \to \mathbb{R} \cup \{-\infty, \infty\}$ if

$$|u(\gamma(b)) - u(\gamma(a))| \le \int_{\gamma} g \, ds$$

for every rectifiable curve $\gamma\colon [a,b] \to X$. The right-hand side of the above is required to be infinite when at least one of $u(\gamma(b))$, $u(\gamma(a))$ is not finite. We say that g is a 2-weak upper gradient (or weak upper gradient for short) if there is a family Γ of curves in X such that (u, g) satisfies the above inequality for each non-constant compact rectifiable curve in X that does not belong to Γ and $\mathrm{Mod}_2(\Gamma) = 0$.

In this paper, we extend the study of potential theory associated with the fractional Laplacian to a complete doubling metric measure space (X, d_X, μ_X) supporting a 2-Poincaré inequality.

Definition 2.1. The Newton-Sobolev space $N^{1,2}(X)$ of all functions $f : X \to \mathbb{R}$ with the property that $\int_X |f|^2 \, d\mu_X < \infty$ and with $\inf_g \int_X g^2 \, d\mu_X < \infty$, where the infimum is over all 2-weak upper gradients g of f, is a Banach space (see [24, 36]).

Following [24], by $D^{1,2}(X)$ we mean the class of functions $f \in L^1_{loc}(X)$ with an upper gradient $g \in L^2(X)$.

The space $N^{1,2}(X)$ is also called the *inhomogeneous* Newton-Sobolev space, while the space $D^{1,2}(X)$ is also called the *homogeneous* Newton-Sobolev space as it will contain nonzero constant functions as well.

Just as sets of measure zero play a role in the study of L^p-spaces, the sets of 2-capacity zero play a role in the study of Sobolev spaces.

Definition 2.2. Given a set $E \subset X$, the 2-capacity of the set is the number

$$\mathrm{Cap}_2(E) := \inf_{(u,g)} \int_X |u|^2 \, d\mu_X + \int_X g^2 \, d\mu_X,$$

where the infimum is over all pairs of functions (u, g) with $u \in N^{1,2}(X)$ satisfying $u \ge 1$ on E and g a 2-weak upper gradient of u.

We assume in this paper that the measure μ_X is doubling, namely there is a constant $C \ge 1$ such that $\mu_X(B(x, 2r)) \le C\,\mu_X(B(x, r))$ for all $x \in X$ and $r > 0$. If X is connected and μ_X is doubling, there exist constants $c, C > 0$ and $b_l, b_u > 0$ for which

$$c\left(\frac{r}{R}\right)^{b_l} \le \frac{\mu_X(B(x, r))}{\mu_X(B(x, R))} \le C\left(\frac{r}{R}\right)^{b_u} \tag{2.1}$$

for each $0 < r < R < \infty$. If $b_u = b_l$ then the space is Ahlfors b_u-regular.

The assumption of connectedness of X is not a loss of generality. Indeed, the assumption that X supports a 2-Poincaré inequality (see the next paragraph) immediately implies that X is connected.

We also assume that (X, d_X, μ_X) supports a 2-Poincaré inequality, that is, there are constants $C > 0$ and $\lambda \ge 1$ such that whenever $B(x, r)$ is a ball in X and $f \in N^{1,2}(X)$ and g is an upper gradient of f, we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu_X \le C\,r \left(\fint_{B(x,\lambda r)} g^2 \, d\mu_X\right)^{1/2}. \tag{2.2}$$

When the doubling space X supports a 2-Poincaré inequality, it also supports an a priori stronger $(2, 2)$-Poincaré inequality:

$$\fint_{B(x,r)} |f - f_{B(x,r)}|^2 \, d\mu_X \leq C \, r^2 \fint_{B(x,\lambda r)} g^2 \, d\mu_X.$$

See [21] for more information.

Remark 2.3. Recall that we are interested in the weighted measure $|y|^a \, dy$ with $-1 < a < 1$. By [22, page 10] the measure $|y|^a dy$ is an A_2-weight on $\mathbb{R}$, and hence we know that both $\mathbb{R}$ and $(0, \infty)$, equipped with the Euclidean metric and the measure $|y|^a \, dy$, supports a 2-Poincaré inequality and this weighted measure is doubling. Hence the Cartesian product $Z = X \times \mathbb{R}$ as well as the Cartesian product $Z_+ = X \times (0, \infty)$, equipped with the metric d_Z and the measure μ_a, also supports a 2-Poincaré inequality with μ_a doubling, see [5, Remark 4] and [7] (where we use the fact that Z_+ is a uniform domain, see Proposition 4.1 below).

There is a strengthening of (2.2) under the assumption that $\{f = 0\} \cap B(x, r)$ is large:

$$\int_{B(x,r)} |f|^2 \, d\mu_X \leq C \, (r^2 + 1) \frac{\mu_X(B(x,r))}{\mathrm{Cap}_2(B(x,r) \cap N_f)} \int_{B(x,\lambda r)} g^2 \, d\mu_X, \qquad (2.3)$$

where $N_f := \{w \in X \; : \; f(w) = 0\}$. This inequality is known as the Maz'ya capacitary inequality, see [31]. For the setting of doubling metric measure spaces supporting a $(2, 2)$-Poincaré inequality, a good reference is [4, Theorem 5.53].

2.2 Cheeger Differential Structure

In this subsection we describe the Cheeger differential structure.

Definition 2.4. A *system of Lipschitz charts* $\{(U_i, \varphi_i) : i \in \mathbb{N}\}$ for a metric measure space (X, d_X, μ_X) is a collection of countably many measurable sets $U_i \subset X$ and Lipschitz maps $\varphi_i \colon X \to \mathbb{R}^{n_i}$ for each $i \in \mathbb{N}$, so that

$$\mu_X \left(X \setminus \bigcup_{i \in \mathbb{N}} U_i \right) = 0$$

and for any Lipschitz function $f \colon X \to \mathbb{R}$ and for each i the following holds. For almost every $x \in U_i$, there exists a unique $D_X f(x) \in \mathbb{R}^{n_i}$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - D_X f(x) \cdot (\varphi_i(y) - \varphi_i(x))}{d(x, y)} = 0. \qquad (2.4)$$

A space with a system of Lipschitz charts is said to admit a differentiable structure and is called a *Lipschitz differentiability space*.

The equation (2.4) gives the first order Taylor expansion of f near x with respect to the basis φ_i. This should not be confused with the notion of weak, or distributional, derivative that is usually considered with Sobolev spaces in the Euclidean setting.

7

Recall that any doubling metric measure space admitting a $(1,2)$-Poincaré inequality admits a differential structure by [11], with each $n_i \leq N$ for some positive integer N that depends solely on the doubling constant of the measure μ_X. By embedding each $\mathbb{R}^{n_i}$ into $\mathbb{R}^N$ if necessary, we may therefore assume that each $n_i = N$. In this case, the uniqueness of D_X is preserved by ensuring that the entries in the vector $D_X f(x)$ corresponding to the components $n_i + 1, \cdots, N$ are all zero when $n_i < N$. Hence, in our setting of (X, d_X, μ_X), we have a linear map

$$D_X : N^{1,2}(X) \to L^2(X)^N$$

for some fixed positive integer N that is determined by the doubling property of the measure μ_X. It was also shown in [11] that there is a measurable inner product structure related to the differential structure D_X, that is, for Lipschitz functions f, h (and then by extension, to functions $f, h \in N^{1,2}(X)$), for μ_X-a.e. $x \in X$ we have $\langle D_X f(x), D_X h(x) \rangle_x$ such that as a function of x this is measurable, and there is a constant $C > 0$ that is independent of f such that for μ_X-a.e. $x \in X$,

$$\frac{1}{C} g_f(x)^2 \leq \langle D_X f(x), D_X f(x) \rangle_x \leq C g_f(x)^2.$$

2.3 Besov Classes, Cheeger Harmonicity, and Dirichlet Forms

In our context the replacement for the standard Laplacian Δ_X is the infinitesimal generator Δ_X associated with the Dirichlet form

$$\mathcal{E}_X(f, g) = \int_X \langle D_X f(x), D_X g(x) \rangle_x \, d\mu_X(x)$$

as described in [17]. While $N^{1,2}(X)$ need not be a Hilbert space under the norm $\|f\|_{L^2(X)} + \inf_g \|g\|_{L^2(X)}$ where the infimum is over all upper gradients g of f, it does turn into a Hilbert space under the norm

$$\|f\|_{L^2(X)} + \sqrt{\int_X \langle D_X f(x), D_X f(x) \rangle_x \, d\mu_X(x)},$$

as seen from [11] or [16, Theorem 10].

From Remark 2.3 above, we know that Z also comes equipped with a choice of a Cheeger differential structure. We are interested in considering a particular Cheeger structure on Z. This structure is obtained as a Cartesian tensorization of the (choice of) Cheeger differential structure D_X on X and the standard Euclidean differential structure on $\mathbb{R}$, as explained in Subsection 3.2 below, see Theorem 3.6.

Definition 2.5. When considering the above-mentioned Dirichlet form associated with the metric space Z as described at the end of Section 1 above, the Dirichlet form obtained on (Z, d_Z, μ_a) from the Cartesian tensor product of the Dirichlet form $\mathcal{E}_X$ and the natural Dirichlet form on $(\mathbb{R}, d_{Euc}, |y|^a \, dy)$ is denoted by $\mathcal{E}_a^Z$.

It was shown in [19] that the interpolation of $L^2(X)$ with $N^{1,2}(X)$ yields Besov classes $B_{2,2}^\theta(X) \cap L^2(X)$ of functions $f \in L^2(X)$ for which the (non-local) energy

semi-norm $\|f\|_{B^\theta_{2,2}(X)}$ given by

$$\|f\|^2_{B^\theta_{2,2}(X)} := \int_X \int_X \frac{|f(z) - f(w)|^2}{d(z,w)^\theta \mu_X(B(z,d(z,w)))} \, d\mu_X(z) \, d\mu_X(w) \qquad (2.5)$$

is finite. This energy is comparable to the one given by $\mathcal{E}_\theta$ in Equation (1.1). The comparability was proved in [20, Corollary 5.5] in the Ahlfors regular case. However, that variants of the Gaussian bounds used in [20] apply also in the doubling case (see e.g. [34]). This leads to the comparability of the $B^\theta_{2,2}-$ and $\mathcal{E}_{\theta}-$ energies on doubling spaces satisfying a 2-Poincaré inequality. We give a direct proof of this for the readers convenience in Proposition 2.11.

Yet another advantage of considering the Besov space is its identity as the trace space of a Newton-Sobolev space, namely, $B^\theta_{2,2}(X)$ is the trace space of the homogeneous Newton-Sobolev space $D^{1,2}(Z_+)$ consisting of all functions $f \in L^2_{loc}(Z, \mu_a)$ such that f has an upper gradient g in Z_+ with $\int_{Z_+} g^2 \, d\mu_a$ finite. Here, a and θ are related by the equation $a = 1 - 2\theta$, see [29] for related results on traces of weighted Sobolev spaces and Besov spaces. For Z_+ as a domain in the weighted space (Z, d_Z, μ_a) this trace result is established in Proposition 4.2 below. By trace class we mean that every function $u \in D^{1,2}(Z_+)$ has a trace $Tu : X \to \mathbb{R}$ given by

$$Tu(x) = \lim_{r \to 0^+} \fint_{B((x,0),r) \cap Z_+} u \, d\mu_a \qquad (2.6)$$

for μ_X-a.e. $x \in X$. We show that $Tu \in B^\theta_{2,2}(X)$ and that for each $f \in B^\theta_{2,2}(X)$ there exists a function $Ef \in D^{1,2}(Z_+)$ such that $TEf = f$. As a consequence, we know that whenever $f \in B^\theta_{2,2}(X)$, there is at least one function in $D^{1,2}(Z_+)$ whose trace is f, and we use this to establish the existence of Cheeger harmonic functions in Z_+ with trace f, see the discussion in Section 5 below.

Definition 2.6. Given a domain $U \subset Z$, we say that a function u on U is Cheeger harmonic (or Cheeger 2-harmonic) in U if $u \in N^{1,2}_{loc}(U, \mu_a)$ and whenever $v \in N^{1,2}(U, \mu_a)$ with compact support contained in U we have $\mathcal{E}^Z_a(u,v) = 0$.

Definition 2.7. Fix a bounded domain $\Omega \subset X$ such that $\mu_X(X \setminus \Omega) > 0$ and a function $f \in B^\theta_{2,2}(X) \cap L^2(X)$. Then we call a function $u \in B^\theta_{2,2}(X)$ a *solution to the Dirichlet problem* $(-\Delta_X)^\theta u = 0$ *on* Ω *with boundary data* f if u satisfies (1.2), that is,

(1) $u = f$ almost everywhere on $X \setminus \Omega$, and

(2) for all $h \in B^\theta_{2,2}(X)$ satisfying $h = f$ almost everywhere in $X \setminus \Omega$, we have

$$\mathcal{E}_\theta(u,u) \le \mathcal{E}_\theta(h,h).$$

A direct argument using calculus of variations gives the weak formulation of the Euler-Lagrange equation associated with the above minimization property; namely, u is a minimizer in the sense of (2) above if and only if for each $h \in B^\theta_{2,2}(X)$ with compact support in Ω,

$$\mathcal{E}_\theta(u,h) = 0.$$

9

Observe that if $u \in B_{2,2}^{\theta}(X)$ with $u = f$ on $X \setminus \Omega$ and Ω is bounded, then $u \in L^2(X)$ whenever $f \in L^2(X)$.

The non-local nature of the energy $\mathcal{E}_{\theta}$ means that we cannot replace the global energy $\mathcal{E}_{\theta}$ with a local energy $\mathcal{E}_{\theta}^{\Omega}$ adapted to the smaller set that is Ω. Hence the classical approaches of De Giorgi and Nash-Moser (see [6, 27]) are not applicable in the study of these solutions. We instead adapt the method set out in the Euclidean setting by Caffarelli and Silvestre [9].

In addition to the construction of Cheeger harmonic functions as in Section 5, we will also use the explicit extension of $f \in B_{2,2}^{\theta}(X) \cap L^2(X)$ to Z_+ given in [3, Lemma 3.1 and bottom of page 8]. We show that their extension is Cheeger harmonic, and by uniqueness of Cheeger harmonic extensions, we obtain that both solutions coincide. To do so, we show that the trace of their extension in the sense of (2.6) above coincides with f and that their extension also belongs to $D^{1,2}(Z_+)$.

Then, we show that the reflection of u along ∂Z_+ given by

$$u^*(x, y) := \begin{cases} u(x, y) & \text{if } y \geq 0, \\ u(x, -y) & \text{if } y < 0 \end{cases}$$

gives a function that is Cheeger harmonic in $\Omega \times \mathbb{R}$ and hence is a quasiminimizer in the sense of [27]. The Cheeger harmonicity follows from uniqueness and considering a Dirichlet problem on $Z \setminus (X \setminus \Omega \times \{0\})$ with boundary data f and observing symmetry. Finally, application of the regularity results from [27] yields the regularity results for f referred to in Theorem 1.2 above.

2.4 Uniform Domains and Co-Dimension Hausdorff Measures

In this subsection we will gather together the geometric and measure-theoretic notions needed in discussing traces of functions in $D^{1,2}(Z_+)$. For a domain $U \subset Z$ denote the distance to the complement by $\delta_U(z) = d(z, Z \setminus U)$.

Definition 2.8. A domain $U \subset Z$ is A-uniform for some $A \geq 1$ if for every pair $x, y \in U$ there is a curve $\gamma \in U$ connecting x and y so that its length $\ell(\gamma)$ satisfies $\ell(\gamma) \leq A d_Z(x, y)$ and for all $z \in \gamma$,

$$\delta_U(z) \geq A^{-1} \min\{\ell(\gamma_{x,z}), \ell(\gamma_{y,z})\}.$$

Here $\gamma_{x,z}$ and $\gamma_{y,z}$ are the subcurves of γ connecting z to x and y respectively. Such a curve is called an A-uniform curve.

Let $\tau > 0$ and $K \subset Z$. The co-dimension τ Hausdorff measure of K is

$$\mathcal{H}^{*,\tau}(K) := \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_{i \in I \subset \mathbb{N}} \frac{\mu_a(B_i)}{\text{rad}(B_i)^{\tau}} : A \subset \bigcup_{i \in I} B_i \text{ and rad}(B_i) < \varepsilon \right\}. \qquad (2.7)$$

2.5 Heat-Kernel Associated with a Dirichlet Form

Corresponding to the Dirichlet form $\mathcal{E}_X$, there is a heat kernel $p_t : X \times X \to [0, \infty)$, $t > 0$, see [3, 17, 20, 34, 37, 38, 39, 40]. While [20] studied the structure of heat

10

kernels in the setting of Riemannian manifolds, the series of papers [37, 38, 39, 40] first studied them in the setting of metric measure spaces equipped with a strongly local Dirichlet form such that the intrinsic metric obtained from such a Dirichlet form gives a doubling measure supporting a 2-Poincaré inequality. From [28] we know that the Dirichlet form obtained from a Cheeger differential structure as explained above fits the hypotheses given in [39], and hence the results of [39] apply here.

The heat kernel helps us construct solutions to the heat equation with initial data $f \in L^2(X)$. With

$$P_t f(x) := \int_X f(w) p_t(x, w) d\mu_X(w),$$

we know that whenever v is a Lipschitz function on $X \times [0, \infty)$ with compact support in $X \times (0, \infty)$,

$$\int_0^\infty \mathcal{E}_X(P_t f, v) \, dt + \int_X \int_0^\infty v(w, t) \, \partial_t P_t f(w) \, d\mu_X(w) \, dt = 0.$$

Embedded in the above claim is also the property that $t \mapsto P_t f(x)$ is differentiable with respect to almost every t such that for each $x \in X$, the map $t \mapsto P_t f(x)$ is absolutely continuous on compact subintervals of $(0, \infty)$. In [17] the heat operator P_t is denoted by T_t, but we follow the notation from [3, 20] here. The following lemma gathers together results from [39].

Lemma 2.9. *We have the following properties for the heat kernel:*

(1) *Markovian property:*
$$\int_X p_t(x, z) \, d\mu_X(z) = 1.$$

(2) *Sub-Gaussian bounds:*
$$\frac{1}{C_1 \mu_X(B(x, \sqrt{t}))} e^{-\frac{c_1 d(x,z)^2}{t}} \leq p_t(x, z) \leq \frac{C_2}{\mu_X(B(x, \sqrt{t}))} e^{-\frac{c_2 d(x,z)^2}{t}}. \qquad (2.8)$$

(3) *Symmetry: $p_t(x, z) = p_t(z, x)$, for each x, z in X.*

(4) *The semigroup property holds:*
$$p_{t+s}(x, z) = \int_X p_t(x, w) p_s(w, z) \, d\mu_X(w). \qquad (2.9)$$

2.6 Spectral Theory

The spectral approach to fractional Laplacians seems to have been first formulated in [35], see for instance the discussion in [3, page 2]. Our presentation also closely follows [20], but a nice description of the relationship between Dirichlet forms and spectral theory is also given in [17, page 17]. We note that the results at the end of this section are covered in the Ahlfors regular case in [20]. Specifically, we slightly simplify the proof of [20, Corollary 5.5] and demonstrate that their work also applies in the doubling context without Ahlfors regularity.

The Dirichlet form $\mathcal{E}_X(u, v)$ as described at the beginning of this section defines a closed regular Dirichlet form with domain $\mathcal{F}_X = N^{1,2}(X)$. Here, the equality is in the sense that each function $f \in \mathcal{F}_X$ has an almost everywhere representative in $N^{1,2}$. While, as described above, Z also is equipped with a Dirichlet form, we will apply spectral theory solely to $\mathcal{E}_X$. The associated Laplacian is denoted Δ_X with $\mathcal{D}(\Delta_X) \subset N^{1,2}(X) \subset L^2(X)$. This operator is self-adjoint and has a spectral decomposition

$$-\Delta_X = \int_0^\infty \lambda dE_\lambda,$$

where dE_λ is a projection valued measure, i.e. the spectral resolution. This integral can be made sense of through pairing, namely $\int (-\Delta_X) uv \, d\mu_X = \int_0^\infty \lambda \langle dE_\lambda(u), v \rangle$ for $u, v \in L^2(X)$. See e.g. [3, 17] or [41, Chapter XI] for further discussion. In particular, if u is in the domain of Δ_X and $v \in L^2(X)$, then

$$-\int_X v(x) \Delta_X u(x) \, d\mu_X(x) = \int_0^\infty \lambda dE_\lambda(u, v).$$

Note that if v also belongs to $N^{1,2}(X)$, then

$$\int_X v(x) \Delta_X u(x) \, d\mu_X(x) = -\int_X \langle D_X v(x), D_X u(x) \rangle_x \, d\mu_X(x).$$

The domain of the Laplacian Δ_X, denoted $\mathcal{D}(\Delta_X)$, can also be described by the equation $\mathcal{D}(\Delta_X) = \{u \in L^2(X) : \int_0^\infty \lambda^2 dE_\lambda(u, u) < \infty\}$. The domain of the Dirichlet form, $\mathcal{D}(\mathcal{E}_X) = N^{1,2}(X)$, is identifiable with $\{u \in L^2(X) : \int_0^\infty \lambda dE_\lambda(u, u) < \infty\}$. The heat semigroup $\{P_t\}_{t>0}$ is given by

$$P_t = \int_0^\infty e^{-\lambda t} dE_\lambda,$$

which is a contraction on $L^2(X)$. For $0 < \theta < 1$ the fractional Laplacian can be expressed as

$$(-\Delta_X)^\theta = \int_0^\infty \lambda^\theta dE_\lambda,$$

and so when u is in the domain of Δ_X and $v \in L^2(X)$, we have

$$\int_X ((-\Delta_X)^\theta u) \, v \, d\mu_X = \int_0^\infty \lambda^\theta dE_\lambda(u, v).$$

We will also use

$$\mathcal{F}_\theta = \{u \in L^2(X) : \int_0^\infty \lambda^\theta dE_\lambda(u, u) < \infty\} = \mathcal{D}((-\Delta_X)^{\frac{\theta}{2}}).$$

The energy $\mathcal{E}_X$ can be recovered through a regularization process by defining

$$\mathcal{E}_{X,t}(f, f) = \frac{1}{t} \int_X (f - P_t f) f \, d\mu_X = \frac{1}{2t} \int_X \int_X |f(x) - f(w)|^2 p_t(x, w) d\mu_X(x) d\mu_X(w).$$

For the last equality, see [20, Section 4]. Then sending $t \to 0$ gives

$$\lim_{t \to 0} \mathcal{E}_{X,t}(f, f) = \mathcal{E}_X(f, f),$$

12

with $\mathcal{F}_X = \{f \in L^2(X): \sup_{t>0} \mathcal{E}_{X,t}(f,f) < \infty\}$. The benefit of the regularization is that $\mathcal{E}_{X,t}(f,f)$ is defined and finite for all $f \in L^2(X)$ and a fixed $t > 0$.

There is a non-negative continuous function $\eta_t^\theta(s): (0,\infty) \to (0,\infty)$ for which

$$e^{-t\lambda^\theta} = \int_0^\infty \eta_t^\theta(s)e^{-\lambda s}ds.$$

Thus the heat kernel corresponding to the infinitesimal generator $(-\Delta_X)^\theta$ is given by

$$q_t(x,y) = \int_0^\infty \eta_t^\theta(s)p_s(x,y) \ ds. \tag{2.10}$$

For these facts see [20, Section 5] or [41, Chapter IX, Section 11].

The following lemma gives the required estimates for q_t in doubling metric measure spaces.

Lemma 2.10. *Suppose (X, d_X, μ_X) is connected and measure doubling and the conclusions of Lemma 2.9 hold. Then there are constants $C_1, C_2 > 0$ for which*

$$q_t(x,y) \leq C_1 \frac{t}{d(x,y)^\theta \mu_X(B(x,d(x,y)))} \quad \text{when } t, s > 0 \tag{2.11}$$

$$q_t(x,y) \geq C_2 \frac{t}{d(x,y)^\theta \mu_X(B(x,d(x,y)))} \quad \text{when } d(x,y)^\theta \geq t > 0 \tag{2.12}$$

Proof. From [20, Equations 5.32 and 5.33] we have for some constants B_1, B_2

$$\eta_t^\theta(s) \leq B_1 \frac{t}{s^{1+\theta}} \quad \text{when } t, s > 0 \tag{2.13}$$

$$\eta_t^\theta(s) \geq B_2 \frac{t}{s^{1+\theta}} \quad \text{when } s^\theta \geq t > 0. \tag{2.14}$$

The claim follows by substituting the bounds from Estimate (2.1) and Part 2 of Lemma 2.9 to the Equation (2.10). Indeed, both estimates follow by noting that the main contribution for the integral comes from $s \sim d(x,y)^{2\theta}$. For the upper bound, the remaining scales are bounded by a geometric sum and its largest term, and for the lower bound we can restrict to the interval with $s \in [d(x,y)^\theta/2, d(x,y)^\theta]$. $\square$

Similarly, we can define

$$\mathcal{E}_{\theta,t}(f,f) = \frac{1}{t}\int_X (f - T_{t,\theta}f)f \ d\mu_X = \frac{1}{2t}\int_X\int_X |f(x) - f(y)|^2 q_t(x,y)d\mu_X d\mu_X$$

where $T_{t,\theta}$, $t > 0$, is the semigroup related to $(-\Delta_X)^\theta$, given by $T_{t,\theta} = \int_0^\infty e^{-t\lambda^\theta} \ dE_\lambda$. We also note that $\mathcal{E}_{\theta,t}$ is monotone decreasing in t and obtain a description of the domain of the fractional Laplacian by

$$\mathcal{D}((-\Delta_X)^{\frac{\theta}{2}}) = \mathcal{F}_\theta = \{f \in L^2(X): \lim_{t\to 0} \mathcal{E}_{t,\theta}(f,f) = \sup_{t>0} \mathcal{E}_{t,\theta}(f,f) < \infty\}$$

These claims follow from [20, Section 4], when applied to the heat semigroup $T_{t,\theta}$.

Using this we can identify the Domain with the Besov space.

Proposition 2.11. Let $f \in B^\theta_{2,2}(X) \cap L^2(X)$. Then there is a constant C so that

$$\frac{1}{C} \mathcal{E}_\theta(f,f) \leq \|f\|^2_{B^\theta_{2,2}(X)} \leq C \mathcal{E}_\theta(f,f)$$

and moreover $B^\theta_{2,2}(X) \cap L^2(X) = \mathcal{F}_\theta$.

Proof. We follow the proof in [20, Theorem 5.2]. We have by Estimate (2.11) that

$$
\begin{aligned}
\mathcal{E}_{\theta,t}(f,f) &= \frac{1}{2t} \int_X \int_X |f(x) - f(y)|^2 q_t(x,y) d\mu_X d\mu_X \\
&\lesssim \int_X \int_X \frac{1}{d(x,y)^\theta \mu_X(B(x,d(x,y)))} |f(x) - f(y)|^2 \, d\mu_X \, d\mu_X.
\end{aligned}
$$

This holds for any $f \in L^2(X)$, and all $t > 0$. Thus, by the remark before the statement, we obtain $\frac{1}{C} \mathcal{E}_\theta(f,f) \leq \|f\|_{B^\theta_{2,2}}$ and that $B^\theta_{2,2}(X) \cap L^2(X) \subset \mathcal{F}_\theta$.

Define $X^2_t = \{(x,y) \in X \times X : d(x,y)^{2\theta} \geq t > 0\}$. The other direction follows by

$$
\begin{aligned}
\mathcal{E}_{\theta,t}(f,f) &\gtrsim \frac{1}{2t} \int_{X^2_t} |f(x) - f(y)|^2 q_t(x,y) d\mu_X d\mu_X \\
&\gtrsim \int_{X^2_t} \frac{1}{d(x,y)^{2\theta} \mu_X(B(x,d(x,y)))} |f(x) - f(y)|^2 \, d\mu_X \, d\mu_X.
\end{aligned}
$$

Here, the comparability constants are independent of t. Sending $t \to 0$, we obtain

$$\mathcal{E}_\theta(f,f) \gtrsim \int_X \int_X \frac{1}{d(x,y)^{2\theta} \mu_X(B(x,d(x,y)))} |f(x) - f(y)|^2 \, d\mu_X \, d\mu_X,$$

and thus $C \mathcal{E}_\theta(f,f) \geq \|f\|_{B^\theta_{2,2}}$ for some constant C and $\mathcal{F}_\theta \subset B^\theta_{2,2}(X) \cap L^2(X)$. This concludes the proof. $\square$

2.7 Explicit Solution

In this section we give a kernel for the solutions of the Dirichlet problem (5.1). Let $\mathcal{P}_a((x,y),z)$ be the kernel given by

$$\mathcal{P}_a((x,y),z) = C_a y^{1-a} \int_0^\infty s^{\frac{a-3}{2}} e^{-\frac{y^2}{4s}} p_s(x,z) \, ds,$$

where $p_s(\cdot,\cdot)$ is the heat kernel associated to Δ and

$$\frac{1}{C_a} = \int_0^\infty \tau^{\frac{a-3}{2}} e^{-1/4\tau} \, d\tau.$$

For each $f \in B^\theta_{2,2}(X) \cap L^2(X) = \mathcal{F}_\theta(X)$ with $\theta = \frac{1-a}{2}$, $x \in X$, and $y > 0$, we set

$$\Pi_a f(x,y) := u(x,y) = \int_X f(z) \mathcal{P}_a((x,y),z) \, d\mu_X(z). \tag{2.15}$$

14

In [3, Lemma 3.1] and [2, (3.21)] this function is denoted by $U(x, y)$. In the following lemma we collect some properties for this function, which are mostly contained in [3], see also [2].

Bochner differentiation is the analog of weak derivative for Banach space-valued functions on intervals. A function $v : (0, \infty) \to N^{1,2}(X)$ is differentiable if there is a function $g : (0, \infty) \to N^{1,2}(X)$ such that whenever $\varphi : (0, \infty) \to \mathbb{R}$ is compactly supported and smooth, we have the following integration by parts formula:

$$\int_0^\infty \varphi'(t) v(t) \, dt = - \int_0^\infty \varphi(t) g(t) \, dt,$$

with the above integrals taken as the Bochner integrals, see [13]. Such a function g is said to be the derivative of f, denoted $\partial_t v$.

Lemma 2.12. *Suppose that $f \in B_{2,2}^\theta(X) \cap L^2(X)$ and $u(x, y) = \Pi_a f(x, y)$.*

  (1) *The map $(x, y) \mapsto u(x, y)$ is continuous and $y \mapsto u(x, y)$ is smooth in y for each $x \in X$.*

  (2) *The map $y \mapsto u(\cdot, y)$ is Bochner measurable and defines a three times continuously Bochner differentiable function from $(0, \infty)$ to $L^2(X)$.*

  (3) *$u(\cdot, y) \in \mathcal{D}(\Delta_X)$ for every $y \in (0, \infty)$ and*

$$\begin{cases} \Delta_a u = \left(\Delta_X + \dfrac{\partial^2}{\partial y^2} + \dfrac{a}{y} \dfrac{\partial}{\partial y} \right) u = 0 \quad in \quad Z_+ \\ u(\cdot, 0) = f(\cdot). \end{cases} \tag{2.16}$$

  (4) *$u(\cdot, y) \in N^{1,2}(X)$ for every $y \in (0, \infty)$*

  (5) *$u \in N_{loc}^{1,2}(Z_+)$.*

Here, by the statement $u(\cdot, 0) = f(\cdot)$ we mean that $u(\cdot, y) \to f(\cdot)$ in $L^2(X)$ as $y \to 0^+$.

Proof. The continuity and differentiality follow from the definition together with Lemma 2.9 and dominated convergence.

The second and third claims follow from [3, Lemma 3.1]. Indeed, in Part 3 of the proof of [3, Lemma 3.1] it is shown that $u(\cdot, 0) \to f(\cdot)$ in $L^2(X)$ as $y \to 0^+$. There the terminology is not explained in depth, but the techniques of dominated convergence yield that the map from $[0, \infty)$ to L^2 given by $y \mapsto u(\cdot, y)$ is continuous, and thus Bochner measurable. In [3] only the first two derivatives are explicitly computed, but further derivatives are simple to compute by the same techniques.

Since $u(x, y)$ is continuous and $u(\cdot, y) \in \mathcal{D}(\Delta_X) \subset \mathcal{F}_X$, we have $u(\cdot, y) \in N^{1,2}(X)$ for every $y \in (0, \infty)$.

By Remark 2.13 below we have that the pointwise y-derivatives and the Bochner derivatives coincide. Now, for $[T_1, T_2]$ with $0 < T_1 < T_2$ using the equation from the

third claim and integration by parts (using the Bochner derivatives) and the third claim of the lemma, we get

$$\int_{X\times[T_1,T_2]}\int_X (\partial_y u)^2 + \langle D_X u, D_X u\rangle d\mu_a \tag{2.17}$$

$$= \int_{T_1}^{T_2}\int_X (\partial_y u)^2 + \langle -\Delta_X u, u\rangle y^a dy d\mu_X$$

$$= \int_{T_1}^{T_2}\int_X \partial_y(y^a u\,\partial_y u) - u\,\partial_y(y^a\partial_y u) - \langle\Delta_X u, u\rangle y^a dy d\mu_X$$

$$= \int_{T_1}^{T_2}\int_X \partial_y(y^a u\,\partial_y u)dy d\mu_X$$

$$= \int_X y^{T_2}\partial_y u(x,T_2)u(x,T_2) - y^{T_1}\partial_y u(x,T_1)u(x,T_1)d\mu_X. \tag{2.18}$$

Since $\partial_y u(\cdot,y), u(\cdot,y) \in L^2(X)$ by [3], we have $(\partial_y u)^2 + \langle D_X u, D_X u\rangle \in L^2_{loc}(Z_+)$. Further, we have that for every $y > 0$ by the third part,

$$\int_X \langle D_X u, D_X u\rangle d\mu_a = \int_X (-\Delta_X u)u\,d\mu_a = \int_X u\,(\partial_y^2 + a^{-1}\partial_y)u\,d\mu_a.$$

Therefore $y \to \int_X \langle D_X u(\cdot,y), D_X u(\cdot,y)\rangle d\mu_a$ is now differentiable in y, and moreover, is continuous, since $y \to u(\cdot,y)$ is of class C^3 in the Bochner sense.

In order to conclude from these the final claim that $u \in N^{1,2}_{loc}(Z_+)$, we would like to apply Lemma 3.1. However, in order to do this, we need that u has measurable and $L^2_{loc}(Z)$-integrable (weak) upper gradients in the $\mathbb{R}$ and the X directions. Note, that the lemma is applied with $Y = [T_1, T_2]$ with the weighted measure $y^a d\lambda$ and X as is. For us $\partial_y u$ is continuous, and thus measurable.

Since $y \to u(\cdot,y)$ is a continuous curve in L^2, we have that if we define, for $k \in \mathbb{N}$, $u_k(x,y) = u(x, \lfloor y\cdot k\rfloor/k)$, that $u_k(\cdot,y) \to u(\cdot,y)$ as $k \to \infty$ in $L^2(X)$ for each $y > 0$. We have that $u_k(\cdot,y) \in N^{1,2}(X)$, and $u_k(\cdot,y)$ is constant for $y \in [l/k, (l+1)/k)$ and any $l \in \mathbb{N}$. Thus, u_k has a piecewise constant, and thus measurable, minimal upper gradient in the X-direction. Call it $g_{u_k^x}$. Now, by construction of the Cheeger differential structure,

$$\int_{T_1}^{T_2}\int_X g_{u_k^x}^2 d\mu_X y^a dt \le C\int_{T_1}^{T_2}\int_X \langle D_X u_k, D_X u_k\rangle.$$

Since $y \to \int_X \langle D_X u, D_X u\rangle d\mu_a$ is continuous, we get that $g_{u_k^x}$ is uniformly bounded in $L^2(X\times[T_1,T_2],\mu_a)$. By taking convex combinations of tails of u_k, we get a sequence of $v_k \to u$ in $L^2(X\times[T_1,T_2],\mu_a)$ where $g_{v_k^x}$ converges in $L^2(X\times[T_1,T_2],\mu_a)$ to a function $g \in L^2(X\times[T_1,T_2],\mu_a)$. Then, finally, as $v_k \to u$ in L^2, we obtain for almost every $y > 0$, that $g(\cdot,y)$ is 2-weak upper gradient for u. $\qquad\square$

Remark 2.13. Note that if for μ_X-a.e. $x \in X$ the real-valued function $y \mapsto u(x,y)$ is absolutely continuous on $[0,\infty)$, then $\partial_y u(x,y)$ exists for almost every y such that the above integration by parts formula holds. It follows then that the Bochner derivative coincides with this real derivative.

Finally, for functions in the Besov class we can strengthen and show that the extension $u \in D^{1,2}(Z_+)$.

Proposition 2.14. If $f \in B^\theta_{2,2}(X) \cap L^2(X)$, then $u = \Pi_a f \in D^{1,2}(Z_+)$, and

$$\|u\|_{D^{1,2}(Z_+)} = \int_0^\infty \int_X (\partial_y u)^2 + \langle D_X u, D_X u \rangle d\mu_a = \frac{2^{2\theta-1}\Gamma(\theta)}{\Gamma(1-\theta)} \mathcal{E}_\theta(f,f).$$

Proof. By Proposition 2.11 we have that $B^\theta_{2,2}(X) \cap L^2(X) = \mathcal{F}_\theta$ with comparable norms. It is easy to show that $\mathcal{D}((-\Delta_X)^\theta)$ is dense in $\mathcal{F}_\theta$ in the norm $\mathcal{E}_\theta(f,f)$, and thus we may assume that $f \in \mathcal{D}((-\Delta_X)^\theta)$. From [3, Lemma 3.2], we have that

$$\lim_{y \to 0^+} -\frac{2^{2\theta-1}\Gamma(\theta)}{\Gamma(1-\theta)} y^a \partial_y u = (-\Delta_X)^\theta f,$$

where $\Gamma(\cdot)$ is the standard gamma function.

Moreover, $\lim_{y \to \infty} -y^a \partial_y u = 0$ weakly in $L^2(X)$. Indeed, for any $v \in L^2(X)$, by the proof of [3, Lemma 3.1] we have that

$$\int_X -y^a \partial_y u \, v \, d\mu_X = \frac{1}{\Gamma(s)} \int_0^\infty \frac{y^{1+a} e^{-\frac{y^2}{4t}}}{2t} \int_X P_t(-\Delta_X)^\theta f v d\mu_X \frac{dt}{t^{1-\theta}}.$$

Then, sending $y \to \infty$ easily proves the claim together with dominated convergence, and since $\| \int_X P_t(-\Delta_X)^\theta f v \, d\mu_X \| \le \|(-\Delta_X)^\theta f\|_{L^2(X)} \|v\|_{L^2(X)}$.

The claim then follows from these combined with the calculation at the end of the proof of Lemma 2.12 and sending $T_1 \to 0$ and $T_2 \to \infty$. $\qquad\square$

3 Tensorization

In this section we collect some results regarding tensorization that will be useful in working with $Z = X \times \mathbb{R}$. Recall the discussion from Remark 2.3 above. We treat a more general Cartesian product here by considering the product of two doubling metric measure spaces (X, d_X, μ_X), (Y, d_Y, μ_Y), both supporting a 2-Poincaré inequality. The product $Z = X \times Y$ is equipped with the metric d_Z given by

$$d_Z((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

and the measure μ_Z given by $d\mu_Z(x,y) = d\mu_X(x)\, d\mu_Y(y)$. While in our application we will consider $Y = \mathbb{R}$ with $d\mu_Y(y) = y^a\, dy$, we formulate the results in the section in this generality as they are of independent interest and add to the results in [1].

3.1 Tensorization of Newton-Sobolev Energies

Lemma 3.1. *If $f \in L^2(Z)$, then there is a modification of f on a $\mu_X \times \mu_Y$-measure zero subset of Z that is in $N^{1,2}(Z)$ if and only if, after modification on a set of $\mu_X \times \mu_Y$-measure zero if necessary, we have*

(1) for μ_X-almost every $x \in X$ we have that $f^x := f(x, \cdot) \in N^{1,2}(Y)$ with $(x,y) \mapsto g_{f^x}(y) \in L^2(Z)$ where g_{f^x} is an upper gradient of f^x in X, and

17

(2) for μ_Y-almost every $y \in Y$ we have that $f^y := f(\cdot, y) \in N^{1,2}(X)$ with $(x, y) \mapsto g_{f^y}(x) \in L^2(Z)$.

If either (X, μ_X) or (Y, μ_Y) does not support s 2-Poincaré inequality, we do not know the validity of the conclusion in the above lemma, see for example the discussion in [18].

Proof. Since the two metrics d_Z and $\max\{d_X, d_Y\}$ are biLipschitz equivalent, and the space $N^{1,2}(Z)$ is biLipschitz invariant, in this proof we will assume that

$$d_Z((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

For $r > 0$ and $(x_0, y_0) \in Z$, consider $B_Z := B_Z((x_0, y_0), r) = B_X(x_0, r) \times B_Y(y_0, r)$. Suppose that f satisfies the hypotheses of the two conditions of the lemma. Then by the respective Poincaré inequalities of X and Y, we obtain

$$\fint_{B_Z}\fint_{B_Z} |f(x_1, y_1) - f(x_2, y_2)| \, d\mu_Z(x_1, y_1) d\mu_Z(x_2, y_2)$$

$$= \fint_{B_X}\fint_{B_X}\fint_{B_Y}\fint_{B_Y} |f(x_1, y_1) - f(x_2, y_2)| \, d\mu_Y(y_2) d\mu_Y(y_1) \, d\mu_X(x_2) d\mu_X(x_1)$$

$$\leq \fint_{B_X}\fint_{B_X}\fint_{B_Y}\fint_{B_Y} |f(x_1, y_1) - f(x_1, y_2)|$$
$$\quad + \; |f(x_1, y_2) - f(x_2, y_2)| \, d\mu_Y(y_2) d\mu_Y(y_1) d\mu_X(x_2) d\mu_X(x_1)$$

$$\leq 2C_Y r \fint_{B_X} \left(\fint_{\lambda_Y B_Y} g_{f^{x_1}}^2 \, d\mu_Y \right)^{1/2} d\mu_X(x_1)$$

$$\quad + \; 2C_Y r \fint_{B_Y} \left(\fint_{\lambda_X B_X} g_{f^{y_2}}^2 \, d\mu_X \right)^{1/2} d\mu_Y(y_2)$$

$$\leq C_0 r \left(\fint_{\lambda_0 B_X}\fint_{\lambda_0 B_Y} [g_{f^x}(y) + g_{f^y}(x)]^2 \, d\mu_X(x) d\mu_Y(y) \right)^{1/2}.$$

In the above, $\lambda_0 = \max\{\lambda_X, \lambda_Y\}$ with C_X, λ_X the constants related to the Poincaré inequality on X and C_Y, λ_Y the corresponding constants for Y, and C_0 is a constant depending on $C_X, C_Y, \lambda_X, \lambda_Y$ and doubling. It follows that with $g(x, y) := g_{f^x}(y) + g_{f^y}(x)$, we have that $g \in L^p(Z)$ and that the pair (f, g) satisfies the p-Poincaré inequality with constants C_0 and λ_0. Hence, by [24, Theorem 10.3.4], we know that there is a constant $C \geq 1$ (with C depending only on the Poincaré constants of Z and on C_0, λ_0) such that Cg is a p-weak upper gradient of a function f_0 in Z such that $f_0 = f$ μ_a-a.e. in Z. Let E consist of points $(x, y) \in Z$ for which $f_0(x, y) \neq f(x, y)$. Then by Fubini's theorem we have that for μ_X-a.e. $x \in X$, we have that $\mu_Y(E \cap \{x\} \times Y) = 0$, and so for such $x \in X$ we have that $f(x, \cdot) = f_0(x, \cdot)$ μ_Y-a.e. in Y. Combining this with the assumptions on f, we have that for μ_X-a.e. $x \in X$, the 2-capacity (with respect to Y) of $E \cap \{x\} \times Y$ is zero. Similarly, for μ_Y-a.e. $y \in Y$, the 2-capacity (with respect to X) of $E \cap X \times \{y\}$ is zero. If $Y = \mathbb{R}$, then we can say more; indeed, only the empty set is of zero 2-capacity with respect to $(\mathbb{R}, |y|^a \, dy)$. It follows that for μ_X-a.e. $x \in X$ we have that $f(x, y) = f_0(x, y)$ whenever $y \in \mathbb{R}$.

Conversely, suppose that $f \in N^{1,2}(Z)$, and let g be an upper gradient of f such that $g \in L^2(Z)$. Then by Fubini's theorem, for μ_X-a.e. $x \in X$ we know that $g(x, \cdot), f(x, \cdot) \in L^2(Y)$ and for μ_Y-a.e. $y \in Y$ we have $g(\cdot, y), f(\cdot, y) \in L^2(X)$. For such x and y we know that $f^x \in N^{1,2}(Y)$ with $g(x, \cdot)$ acting as an upper gradient of f^x, and that $f^y \in N^{1,2}(X)$ with $g(\cdot, y)$ acting as an upper gradient of f^y. That is, f satisfies the two conditions listed in the claim of the lemma. $\qquad\square$

Remark 3.2. In the above, we only require that f^y have an upper gradient g_{f^y} such that the map $(x, y) \mapsto g_{f^x}(y)$ belongs to $L^2(Z)$, and that f^x satisfies a similar condition. We do not require g_{f^y} to be a *minimal weak upper gradient* if f^y, since we would not in practice be able to guarantee the measurability of such function in Z. However, if $Y = \mathbb{R}$ is equipped with the weighted measure $|y|^a \, dy$, then we have the following improvement, see Corollary 3.4 below.

To prove the corollary mentioned above, we need the following lemma.

Lemma 3.3. *For $A \subset X$ that is μ_X-measurable, and $h > 0$, consider the family $\Gamma(A, h)$ of curves in Z of the form $\gamma_x : [0, h] \to Z$ given by $\gamma_x(t) = (x, t)$; $x \in A$. Then*

$$Mod_2(\Gamma(A, h)) \simeq \frac{\mu_X(A)}{h^{1-a}}.$$

Moreover, if $u \in N^{1,2}_{loc}(Z_+)$ and $0 \le t_1 < t_2 < \infty$, then for μ_X-a.e. $x \in X$ we have that $u \circ \gamma_x : [t_1, t_2] \to \mathbb{R}$ is absolutely continuous with $\partial_t u \circ \gamma_x \le g_u \circ \gamma_x$, where $\gamma_x : [t_1, t_2] \to Z_+$ is given by $\gamma_x(t) = (x, t)$. In particular, if $Cap_2(A \times \{0\}) = 0$, then $\mu_X(A) = 0$.

Proof. Setting $\rho := h^{-1} \chi_{A \times [0,h]}$ we see that for each $\gamma \in \Gamma(A, h)$, $\int_\gamma \rho \, ds = 1$. It follows that

$$\mathrm{Mod}_2(\Gamma(A, h)) \le \int_Z \rho^2 \, d\mu_a = \frac{\mu_X(A)}{(a+1)h^{1-a}}.$$

On the other hand, if ρ is a non-negative Borel measureable function on Z such that for each $\gamma \in \Gamma(A, h)$ we have $\int_\gamma \rho \, ds \ge 1$, then

$$1 \le \int_0^h \rho(x, t) \, dt \le \left(\int_0^h \rho(x, t)^2 t^a \, dt \right)^{1/2} \left(\int_0^h t^{-a} \, dt \right)^{1/2}$$

$$= \left(\int_0^h \rho(x, t)^2 t^a \, dt \right)^{1/2} \left(\frac{h^{1-a}}{1-a} \right)^{1/2}.$$

It follows that

$$\frac{1-a}{h^{1-a}} \le \int_0^h \rho(x, t)^2 t^a \, dt.$$

Integrating over $x \in A$ gives

$$\frac{(1-a)\mu_X(A)}{h^{1-a}} \le \int_Z \rho^2 \, d\mu_a.$$

Taking the infimum over all such ρ gives

$$\frac{(1-a)\mu_X(A)}{h^{1-a}} \le \mathrm{Mod}_2(\Gamma(A, h)).$$

As a consequence of the above, we know that $\mathrm{Mod}_2(\Gamma(A, h)) = 0$ if and only if $\mu_X(A) = 0$. The final statement about capacity follows from the fact that Mod_2 of the family of all curves intersecting a set of measure zero is null if and only if the capacity of that set is zero, see [24, 36]. $\qquad\square$

Corollary 3.4. *Let $Y = \mathbb{R}$ or $Y = (0, \infty)$ be equipped with the Euclidean metric and the measure μ_Y given by $d\mu_Y = y^a\, dy$. Let $u \in N^{1,2}(Z)$. Then for μ_X-almost every $x \in X$ we have that $u(x, \cdot)$ is absolutely continuous on Y with $y \mapsto \partial_y u(x, y) \in L^2(Z)$. In particular, $y \mapsto \partial_y u(x, y)$ is measurable on Z.*

Proof. Note that as $u \in N^{1,2}(Z)$, u is absolutely continuous on 2-modulus almost every non-constant compact rectifiable curve in Z, see for instance [24, 36]. Hence by Lemma 3.3 we know that for μ_X-almost every $x \in X$ the map $y \mapsto u(x, y)$ is absolutely continuous on compact subintervals of Y. As u has an upper gradient g in Z, we know that $|\partial_y u(x, y)| \leq g(x, y)$, and so as $g \in L^2(Z)$, it suffices to show that $\partial_y u$ is measurable on Z. To this end we argue as follows.

Suppose $Y = \mathbb{R}$ so that $u \colon X \times \mathbb{R} \to \mathbb{R}$. The proof is similar if $Y = (0, \infty)$. Let

$$\widetilde{X} = \{x \in X \; : \; y \mapsto u(x, y) \text{ is continuous}\}.$$

Then $\widetilde{X}$ is a measurable subset of X and $\mu(X \setminus \widetilde{X}) = 0$. We show that the domain of $\partial_y u(x, y)$ is a measurable subset of $\widetilde{X} \times \mathbb{R}$ and that $\partial_y u(x, y)$ is a measurable function of (x, y) on that set. It suffices to prove that for every closed bounded interval of positive length $B \subset \mathbb{R}$, the set

$$D_B := \{(x_0, y_0) \in \widetilde{X} \times \mathbb{R} \; : \; \partial_y u(x_0, y_0) \text{ exists and belongs to } B\}$$

is measurable.

Fix a closed bounded interval B of positive length. For $(x, y) \in X \times \mathbb{R}$ it follows from compactness of B that $\partial_y u(x_0, y_0)$ exists and belongs to B if and only if for every $\varepsilon \in \mathbb{Q}^+$ there exists $\delta \in \mathbb{Q}^+$ and $q \in B \cap \mathbb{Q}$ such that

$$\sup_{0 < |t| < \delta} \frac{|u(x_0, y_0 + t) - u(x_0, y_0) - qt|}{|t|} \leq \varepsilon.$$

Observe that if $x_0 \in \widetilde{X}$ it is equivalent to take the above supremum only for $t \in (-\delta, \delta) \cap \mathbb{Q}$. We claim that

$$D_B = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{\delta \in \mathbb{Q}^+} \bigcup_{q \in B \cap \mathbb{Q}} \bigcap_{t \in \mathbb{Q} \cap (-\delta, \delta) \setminus \{0\}} D_B(\varepsilon, \delta, q, t), \qquad (3.1)$$

where

$$D_B(\varepsilon, \delta, q, t) = \left\{(x_0, y_0) \in \widetilde{X} \times \mathbb{R} \; : \; \frac{|u(x_0, y_0 + t) - u(x_0, y_0) - qt|}{|t|} \leq \varepsilon\right\}.$$

That the left side of (3.1) is contained in the right side follows from routine approximation by rationals. To show the right side is contained in the left side, we take

20

$\varepsilon = 1/n$ for $n \in \mathbb{N}$ to find a sequence of rational $\delta_n > 0$ and $q_n \in B \cap \mathbb{Q}$ such that for all $t \in \mathbb{Q} \cap (-\delta, \delta) \setminus \{0\}$ we have

$$\frac{|u(x_0, y_0 + t) - u(x_0, y_0) - q_n t|}{|t|} \leq 1/n.$$

This inequality necessarily holds for all $t \in (-\delta, \delta) \setminus \{0\}$ because the left side is a continuous function of t away from 0. Since B is compact, the sequence $\{q_n\}$ has a subsequence converging to some $q \in B$. It is then routine to show that $\partial_y u(x_0, y_0)$ exists and equals q.

Since u is measurable on $X \times \mathbb{R}$ and $\widetilde{X}$ is a measurable subset of X, it follows that each set $D_B(\varepsilon, \delta, q, t)$ is measurable. Hence, by (3.1), D_B is a measurable subset of $X \times \mathbb{R}$. $\qquad \square$

3.2 Tensorization of Differentiable Structures

Recall the definition of a Cheeger differentiable structure from Definition 2.4. Given two doubling metric measure spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y), in this section we construct a differential structure on $Z = X \times Y$.

Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be metric measure spaces, each equipped with a doubling measure and admitting a differentiable structure. Denote the charts in the differentiable structure by (U_i, φ_i) and (V_j, ψ_j) where $\varphi_i \colon X \to \mathbb{R}^{m_i}$ and $\psi_j \colon X \to \mathbb{R}^{n_j}$ respectively for $i, j \in \mathbb{N}$.

Definition 3.5. Suppose $f \colon X \times Y \to \mathbb{R}$ is Lipschitz with respect to $d_X \times d_Y$ and $(x_0, y_0) \in U_i \times V_j$ for some $i, j \in \mathbb{N}$.

We say f is differentiable at the point (x_0, y_0) in the X direction with derivative $D_X f(x_0, y_0)$ (with respect to φ_i) if $D_X f(x_0, y_0)$ is the unique element $v \in \mathbb{R}^{m_i}$ such that

$$\lim_{x \to x_0} \frac{|f(x, y_0) - f(x_0, y_0) - v \cdot (\varphi_i(x) - \varphi_i(x_0))|}{d_X(x, x_0)} = 0.$$

We say f is differentiable at (x_0, y_0) in the Y direction with derivative $D_Y f(x_0, y_0)$ (with respect to ψ_j) if $D_Y f(x_0, y_0)$ is the unique element $w \in \mathbb{R}^{n_j}$ such that

$$\lim_{x \to x_0} \frac{|f(x_0, y) - f(x_0, y_0) - w \cdot (\psi_j(y) - \psi_j(y_0))|}{d_Y(y, y_0)} = 0.$$

For each $i, j \in \mathbb{N}$, let $\varphi_i \times \psi_j \colon X \times Y \to \mathbb{R}^{m_i} \times \mathbb{R}^{n_j}$ denote the map

$$(x, y) \mapsto (\varphi_i(x), \psi_j(y)).$$

In this section we will prove the following theorem.

Theorem 3.6. *Suppose (X, d_X, μ_X) and (Y, d_Y, μ_Y) are doubling metric measure spaces each admitting a differentiable structure. Then $(X \times Y, d_X \times d_Y, \mu_X \times \mu_Y)$ is also a doubling metric measure space and admits a differentiable structure.*

More precisely, the charts in the differentiable structure for $(X \times Y, d_X \times d_Y, \mu_X \times \mu_Y)$ can be chosen to be $(U_i \times V_j, \varphi_i \times \psi_j)$ for $i, j \in \mathbb{N}$, where $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^{m_i} \times \mathbb{R}^{n_j}$ is defined by $(\varphi_i \times \psi_j)(x, y) = (\varphi_i(x), \psi_j(y))$.

Given $f\colon X\times Y\to\mathbb{R}$ and $i,j\in\mathbb{N}$, for almost every $(x_0,y_0)\in U_i\times V_j$, the deriva-tive of f with respect to $\varphi_i\times\psi_j$ is $(D_Xf(x_0,y_0),D_Yf(x_0,y_0))$, where $D_Xf(x_0,y_0)$ and $D_Yf(x_0,y_0)$ are the derivatives of f with respect to X and φ_i or Y and ψ_j.

We now proceed to prove Theorem 3.6. To prove Theorem 3.6, we first establish some needed lemmas. We denote the Lipschitz constant of any Lipschitz map g by L_g. Fix a Lipschitz map $f\colon X\times Y\to\mathbb{R}$.

Recall that a set P in a metric space M is porous if there exists $\lambda>0$ such that for every $x\in P$ the following property holds. There exists $x_n\in M$ such that $x_n\to x$ and $B(x_n,\lambda d(x_n,x))\cap P=\varnothing$. A set is σ-porous if it is a countable union of porous sets.

Lemma 3.7. *Given $m,n\in\mathbb{N}$, $v\in\mathbb{R}^m$, $w\in\mathbb{R}^n$, $\varepsilon,\delta\in(0,1)$, let $P(v,w,\varepsilon,\delta,f)$ be the set of $(x_0,y_0)\in X\times Y$ with the following properties.*

(1) For every chart U_i containing x_0 with $m_i=m$, $0<d_X(x,x_0)<\delta$ implies

$$|f(x,y_0)-f(x_0,y_0)-v\cdot(\varphi_i(x)-\varphi_i(x_0))|<\varepsilon d_X(x,x_0).$$

(2) For every chart V_j containing y_0 with $n_j=n$, $0<d_Y(y,y_0)<\delta$ implies

$$|f(x_0,y)-f(x_0,y_0)-w\cdot(\psi_j(y)-\psi_j(y_0))|<\varepsilon d_Y(y,y_0).$$

(3) There exist charts U_i containing x_0 with $m_i=m$ and V_j containing y_0 with $n_j=n$ for which there is (x,y) arbitrarily close to (x_0,y_0) with

$$|f(x,y)-f(x_0,y_0)-v\cdot(\varphi_i(x)-\varphi_i(x_0))-w\cdot(\psi_j(y)-\psi_j(y_0))| \qquad (3.2)$$
$$>\varepsilon(2L_f+2+|w|L_{\psi_j})d_X(x,x_0)+\varepsilon d_Y(y,y_0).$$

Then the set $P(v,w,\varepsilon,\delta,f)$ is porous.

Proof. Fix a set $P=P(v,w,\varepsilon,\delta,f)$ as in the statement and let $(x_0,y_0)\in P$. Fix charts U_i and V_j as in (3). Let (x,y) satisfy the estimates $0<d_X(x,x_0)<\delta/2$ and $0<d_Y(y,y_0)<\delta/2$ such that the inequality (3.2) in (3) holds. Notice that U_i is a chart containing x_0 with $m_i=m$ and V_j is a chart containing y_0 with $n_j=n$. Notice

$$d_{X\times Y}((x_0,y_0),(x,y_0))=d_X(x,x_0).$$

Hence to show that P is porous it suffices to show that

$$P\cap B_{X\times Y}((x,y_0),\varepsilon d_X(x,x_0))=\varnothing. \qquad (3.3)$$

Suppose not and fix $(a,b)\in P\cap B_{X\times Y}((x,y_0),\varepsilon d_X(x,x_0))$. We show how to obtain a contradiction.

By using $(x_0,y_0)\in P$ and applying property (1) from the definition of P, we see

$$|f(x,y_0)-f(x_0,y_0)-v\cdot(\varphi_i(x)-\varphi_i(x_0))|<\varepsilon d_X(x,x_0). \qquad (3.4)$$

' Next notice that $d(b,y)\le d(b,y_0)+d(y_0,y)<\delta$. Hence we can use $(a,b)\in P$ and property (2) from the definition of P to obtain

$$|f(a,y)-f(a,b)-w\cdot(\psi_j(y)-\psi_j(b))|<\varepsilon d_Y(y,b). \qquad (3.5)$$

Using the fact that $(a, b) \in B_{X \times Y}((x, y_0), \varepsilon d_X(x, x_0))$, it is straightforward to derive the following estimates

$$|f(a, y) - f(x, y)| \le L_f \varepsilon d_X(x, x_0), \tag{3.6}$$

$$|f(x, y_0) - f(a, b)| \le L_f \varepsilon d_X(x, x_0), \tag{3.7}$$

$$|\psi_j(b) - \psi_j(y_0)| \le L_{\psi_j} \varepsilon d_X(x, x_0). \tag{3.8}$$

$$d_Y(y, b) \le d_Y(y, y_0) + d_Y(y_0, b) \le d_Y(y, y_0) + d_X(x, x_0). \tag{3.9}$$

Combining (3.4)–(3.9) and applying the triangle inequality, it can be shown that

$$|f(x, y) - f(x_0, y_0) - v \cdot (\varphi_i(x) - \varphi_i(x_0)) - w \cdot (\psi_j(y) - \psi_j(y_0))|$$
$$\le \varepsilon(2L_f + 2 + |w|L_{\psi_j}) d_X(x, x_0) + \varepsilon d_Y(y, y_0).$$

This contradicts the fact that (x_0, y_0) together with (x, y) satisfies the inequality (3.2) of (3) in the definition of P. Therefore (3.3) is true. This shows that P is porous as claimed. □

Now define

$$P(f) = \bigcup_{\substack{m \in \mathbb{N} \\ n \in \mathbb{N}}} \bigcup_{\substack{v \in \mathbb{Q}^m \\ w \in \mathbb{Q}^n}} \bigcup_{\substack{\varepsilon \in (0,1) \cap \mathbb{Q} \\ \delta \in (0,1) \cap \mathbb{Q}}} P(v, w, \varepsilon, \delta, f).$$

Clearly $P(f) \subset X \times Y$ is a σ-porous set. Since $\mu_X \times \mu_Y$ is doubling, it follows that $\mu_Z(P(f)) = 0$. This is because doubling measures assign measure zero to porous sets, which is well known and follows from the fact that the Lebesgue differentiation theorem holds whenever the underlying measure is doubling. For an explicit proof one could follow the steps in [32], which do not depend on the Carnot group structure in that paper.

Lemma 3.8. *Fix $(x_0, y_0) \in X \times Y$ such that*

(1) For a chart (U_i, φ_i), f is differentiable at (x_0, y_0) in the X direction with unique derivative $D_X f(x_0, y_0)$.

(2) For a chart (V_j, ψ_j), f is differentiable at (x_0, y_0) in the Y direction with unique derivative $D_Y f(x_0, y_0)$.

(3) $(x_0, y_0) \notin P$.

Then f is differentiable with respect to the chart $(U_i \times V_j, \varphi_i \times \psi_j)$ on $X \times Y$ with unique derivative $(D_X f(x_0, y_0), D_Y f(x_0, y_0))$.

Proof. Let $\varepsilon > 0$ be rational. By (1) and (2) from the statement of the lemma, we can choose $\delta > 0$ rational such whenever $0 < d_X(x, x_0) < \delta$ and $0 < d_Y(y, y_0) < \delta$,

$$|f(x, y_0) - f(x_0, y_0) - D_X f(x_0, y_0) \cdot (\varphi_i(x) - \varphi_i(x_0))| < \varepsilon d_X(x, x_0)/2,$$

$$|f(x_0, y) - f(x_0, y_0) - D_Y f(x_0, y_0) \cdot (\psi_j(y) - \psi_j(y_0))| < \varepsilon d_Y(y, y_0)/2.$$

Fix $v \in \mathbb{Q}^{m_i}$ and $w \in \mathbb{Q}^{n_j}$ such that

$$|v - D_X f(x_0, y_0)| \le \varepsilon / 2L_{\varphi_i}$$

and

$$|w - D_Y f(x_0, y_0)| \le \varepsilon / 2L_{\psi_j}.$$

It is easy to check using the triangle inequality that (1) and (2) from the definition of $P(v, w, \varepsilon, \delta, f)$ hold. Since $(x_0, y_0) \notin P$ it follows that (3) cannot hold. Hence

$$|f(x, y) - f(x_0, y_0) - v \cdot (\varphi_i(x) - \varphi_i(x_0)) - w \cdot (\psi_j(y) - \psi_j(y_0))|$$
$$\le \varepsilon(2L_f + 2 + |w|L_{\psi_j})d_X(x, x_0) + \varepsilon d_Y(y, y_0).$$

Using again the triangle inequality gives for $0 < d(x, x_0) < \delta$ and $0 < d(y, y_0) < \delta$

$$|f(x, y) - f(x_0, y_0) - D_X f(x_0, y_0) \cdot (\varphi_i(x) - \varphi_i(x_0))$$
$$- D_Y f(x_0, y_0) \cdot (\psi_j(y) - \psi_j(y_0))|$$
$$\le \varepsilon(2L_f + 3 + |w|L_{\psi_j})d_X(x, x_0) + 2\varepsilon d_Y(y, y_0).$$
$$\le \varepsilon(3L_f + 4 + |D_Y f(x_0, y_0)|L_{\psi_j})d_X(x, x_0) + 2\varepsilon d_Y(y, y_0).$$

This shows that f is differentiable at (x_0, y_0) with derivative with respect to the given chart given by $(D_X f(x_0, y_0), D_Y f(x_0, y_0))$.

To show uniqueness of the derivative, suppose $v \in \mathbb{R}^{m_i}$ and $w \in \mathbb{R}^{n_j}$ such that

$$\lim_{(x,y) \to (x_0, y_0)} \frac{|f(x, y) - f(x_0, y_0) - v \cdot (\varphi_i(x) - \varphi_i(x_0)) - w \cdot (\psi_j(y) - \psi_j(y_0))|}{d_X(x, x_0) + d_Y(y, y_0)} = 0.$$

Substituting $y = y_0$ gives

$$\lim_{x \to x_0} \frac{|f(x, y_0) - f(x_0, y_0) - v \cdot (\varphi_i(x) - \varphi_i(x_0))|}{d_X(x, x_0)} = 0.$$

Uniqueness of the derivative $D_X f(x_0, y_0)$ in the X direction yields $v = D_X f(x_0, y_0)$. Similarly substituting $x = x_0$ yields $w = D_Y f(x_0, y_0)$. This proves uniqueness of the derivative $(D_X f(x_0, y_0), D_Y f(x_0, y_0))$. $\square$

Lemma 3.9. *Let (U_i, φ_i) and (V_j, ψ_j) be charts on X and Y respectively. Then both $D_X f(x_0, y_0)$ and $D_Y f(x_0, y_0)$ exist for $\mu_X \times \mu_Y$ almost every $(x_0, y_0) \in U_i \times V_j$.*

Proof. For each fixed $y_0 \in V_j$, $D_X f(x_0, y_0)$ exists for μ_X almost every $x_0 \in U_i$ because $x \mapsto f(x, y_0)$ is a Lipschitz map $X \to \mathbb{R}$. The claimed result will follow for $D_X f(x_0, y_0)$ by Fubini's theorem once we show that the set where $D_X f(x_0, y_0)$ exists is a measurable subset of $X \times Y$ and that $(x_0, y_0) \mapsto D_X f(x_0, y_0)$ is a measurable function on that set. To that end, first notice that we can write

$$\{(x_0, y_0) \in U_i \times V_j : D_X f(x_0, y_0) \text{ exists with respect to } \varphi_i\} = \bigcup_{N=1}^{\infty} A_N,$$

where A_N is the set of $(x_0, y_0) \in U_i \times V_j$ for which $D_X f(x_0, y_0)$ exists with respect to φ_i and belongs to the closed ball $\overline{B}(0, N) \subset \mathbb{R}^{m_i}$. We claim $A_N = \widetilde{A}_N$, where

$$\tilde{A}_N = \bigcap_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} \bigcup_{\substack{\delta > 0 \\ \delta \in \mathbb{Q}}} \bigcup_{\substack{q \in \mathbb{Q}^{m_i} \\ |q| \leq N}} \{ (x_0, y_0) \in U_i \times V_j :$$

$$\sup_{0 < d_X(x, x_0) < \delta} \frac{|f(x, y_0) - f(x_0, y_0) - q \cdot (\varphi_i(x) - \varphi_i(x_0))|}{d_X(x, x_0)} \leq \varepsilon \}.$$

Clearly $A_N \subset \widetilde{A}_N$. For the opposite inclusion, suppose $(x_0, y_0) \in \widetilde{A}_N$. Then there exists $\delta_n \downarrow 0$ and a sequence $q_n \in \overline{B}(0, N)$ such that

$$\sup_{0 < d_X(x, x_0) < \delta_n} \frac{|f(x, y_0) - f(x_0, y_0) - q_n \cdot (\varphi_i(x) - \varphi_i(x_0))|}{d_X(x, x_0)} < \frac{1}{n}.$$

By taking a subsequence if necessary, we may assume that $q_n \to q \in \overline{B}(0, N)$. Then the triangle inequality yields

$$\sup_{0 < d_X(x, x_0) < \delta_n} \frac{|f(x, y_0) - f(x_0, y_0) - q \cdot (\varphi_i(x) - \varphi_i(x_0))|}{d_X(x, x_0)} < \frac{1}{n} + |q_n - q| L_{\varphi_i}.$$

Since the right side converges to 0 as $n \to \infty$, this shows $D_X f(x_0, y_0)$ exists and equals $q \in \overline{B}(0, N)$. Hence $(x_0, y_0) \in A_N$, which shows that $A_N = \widetilde{A}_N$. The fact that $\widetilde{A}_N$ is measurable follows because X is separable, so one can equivalently consider the supremum over a countable dense set of x with $0 < d_X(x, x_0) < \delta$.

A similar argument with $\overline{B}(0, N)$ replaced by other balls shows that $D_X f(x_0, y_0)$ is measurable on its domain. The argument for $D_Y f(x_0, y_0)$ is analogous with X replaced by Y. $\qquad \square$

We can now prove Theorem 3.6.

Proof of Theorem 3.6. The measure $\mu_X \times \mu_Y$ is doubling, so porous sets have measure zero. Hence for any Lipschitz $f \colon X \times Y \to \mathbb{R}$ we have $\mu_X \times \mu_Y(P(f)) = 0$. Fix a chart (U_i, φ_i) and (V_j, ψ_j). By applying Lemma 3.9, we see that almost every point $(x_0, y_0) \in U_i \times V_j$ has the following properties:

(1) Both $D_X f(x_0, y_0)$ and $D_Y f(x_0, y_0)$ exist and are unique.

(2) $(x_0, y_0) \notin P$.

Hence, by Lemma 3.8, f is differentiable at (x_0, y_0) with respect to the chart $(U_i \times V_j, \varphi_i \times \psi_j)$ with unique derivative $(D_X f(x_0, y_0), D_Y f(x_0, y_0))$. $\qquad \square$

Remark 3.10. When $Y = \mathbb{R}$ equipped with the Euclidean metric and the measure $y^a \, dy$, the corresponding tensorization of the differential structure D_X on X and the Euclidean differential structure on Y yields the differential structure on Z denoted by D_a and the corresponding Dirichlet form is denoted $\mathcal{E}_a^Z$. This is not to be confused with the fractional Dirichlet form $\mathcal{E}_\theta$ referred to in (1.1) above.

Lemma 3.11. *With the differential structure D_a as in Remark 3.10, we consider the inner product on this structure given by*

$$\langle D_a u(x,y), D_a v(x,y) \rangle_{(x,y)} := \langle D_X u(x,y), D_X v(x,y) \rangle_x + \partial_y u(x,y) \, \partial_y v(x,y).$$

Then there is a constant $C > 0$ such that whenever u is a Lipschitz function on Z, we have

$$\frac{1}{C} g_u(x,y)^2 \leq \langle D_a u(x,y), D_a u(x,y) \rangle_{(x,y)} \leq C \, g_u(x,y)^2$$

where g_u is the minimal weak upper gradient of u (equivalently, $g_u = \operatorname{Lip} u \ \mu_a$-a.e. in Z).

Proof. From [1, Theorem 3.4] we know that

$$\operatorname{Lip} u(x,y)^2 = \limsup_{w \to x} \frac{|u(w,y) - u(x,y)|^2}{d_X(w,x)^2} + |\partial_y u(x,y)|^2$$

for μ_a-a.e. $(x,y) \in Z$ when u is locally Lipschitz continuous. Combining this with the results from [11] completes the proof. $\square$

4 Potential Theory in Relation to the Domain $Z_+ \subset Z$

We fix $-1 < a < 1$ and set $\theta = \frac{1-a}{2}$. The central function space on X from our point of view is the Besov space $B_{2,2}^\theta(X)$. Recall from (2.5) that this space consists of functions $f \in L^1_{loc}(X)$ such that

$$\|f\|_{B_{2,2}^\theta(X)}^2 := \int_X \int_X \frac{|f(y) - f(x)|^2}{d_X(y,x)^{2\theta} \mu_X(B(y, d_X(y,x)))} \, d\mu_X(x) \, d\mu_X(y)$$

is finite.

Recall Remark 2.3. It is known from the work of Maly [30] that when Ω is a bounded domain in a metric measure space equipped with a doubling measure supporting a Poincaré inequality and $\partial\Omega$ is equipped with a co-dimension τ-Hausdorff measure induced by the measure on Ω, and if Ω is a uniform domain, then the trace space of the Sobolev space $N^{1,2}(\Omega)$ is $B_{2,2}^\theta(\partial\Omega) \cap L^2(X)$ for a suitable choice of θ. In our setting, the domain we are interested in is Z_+, but even when X is bounded, Z_+ is unbounded. We will show in this section that Z_+ is a uniform domain in Z and that a homogeneous version of Maly's theorem [30] holds true. This gives the desired link between the Dirichlet space $D^{1,2}(Z_+)$ and $B_{2,2}^\theta(X)$ where $\theta = (1-a)/2$. We first need the following proposition, which also demonstrates the co-dimensionality between the measures μ_a on Z_+ and μ_X on X.

Proposition 4.1. *Fix $-1 < a < 1$. Then the domain $Z_+ := X \times (0, \infty)$ is a uniform domain in Z, and for each $x \in X$ and $r > 0$, we have*

$$\mu_X(B(x,r)) \simeq \frac{\mu_a(B_Z((x,0), r))}{r^{1+a}}. \tag{4.1}$$

Proof. The metric d_∞ on Z given by $d_\infty((x_1, y_2), (x_2, y_2)) = \max\{d_X(x_1, x_2), |y_1 - y_2|\}$ is biLipschitz with respect to the metric d_Z, and it is convenient to consider this metric here. So in this proof d_Z will denote d_∞.

Note that Z_+ cannot be a John domain even if X is itself bounded, for there is no reasonable point acting as the John center. However, we show that Z_+ is a uniform domain. Since μ_X is doubling on X and supports a 2-Poincaré inequality, we know that there is a constant $C_q \geq 1$ such that X is C_q-quasiconvex.

Let $(x_1, y_1), (x_2, y_2) \in Z_+$. Without loss of generality, we assume that $0 < y_1 \leq y_2$. Let $\beta_1 : [0, y_2 - y_1 + d(x_1, x_2)] \to Z_+$ be given by $\beta_1(t) = (x_1, y_1 + t)$, and $\beta_3 : [0, d(x_1, x_2)] \to Z_+$ be given by $\beta_3(t) = (x_2, d(x_1, x_2) + y_2 - t)$. We also fix a C_q-quasiconvex curve $\widehat{\beta_2} : [0, L] \to X$ in X that starts from x_1 and ends at x_2, and we set $\beta_2 : [0, L] \to Z_+$ by $\beta_2(t) = (\widehat{\beta_2}(t), d(x_1, x_2) + y_2)$. The concatenation of these three curves, $\beta_1 + \beta_2 + \beta_3$, is denoted by γ. Then the trajectory of γ lies entirely in Z_+, starts at (x_1, y_1), and ends at (x_2, y_2). We wish to show that γ is a uniform curve.

If $(x, y) \in \beta_1$, then $x = x_1$ and

$$\text{dist}_Z((x, y), X) = y \geq y - y_1 = \ell(\beta_1|_{[0,t]}) = \ell(\gamma_{(x_1, y_1), (x, y)}).$$

If $(x, y) \in \beta_3$, then $x = x_2$ and

$$\text{dist}_Z((x, y), X) = y \geq y - y_2 = \ell(\gamma_{(x_2, y_2), (x, y)}).$$

If $(x, y) \in \beta_2$, then $y = d(x_1, x_2) + y_2$ and so

$$\text{dist}_Z((x, y), X) = d(x_1, x_2) + y_2.$$

On the other hand,

$$\ell(\gamma_{(x, y), (x_2, y_2)}) = \ell((\widehat{\beta_2})_{x, x_2}) + d(x_1, x_2) \leq (1 + C_q) d(x_1, x_2).$$

Therefore in this case,

$$(1 + C_q) \, \text{dist}_Z((x, y), X) \geq \ell(\gamma_{(x, y), (x_2, y_2)}).$$

Moreover,

$$\begin{aligned}
\ell(\gamma) = \ell(\beta_1) + \ell(\widehat{\beta_2}) + \ell(\beta_3) &= (y_2 - y_1 + d(x_1, x_2)) + \ell(\widehat{\beta_2}) + d(x_1, x_2) \\
&\leq (y_2 - y_1) + (2 + C_q) d(x_1, x_2) \\
&\leq 2(2 + C_q) d_Z((x_1, y_1), (x_2, y_2)).
\end{aligned}$$

As $1 + C_q \leq 2(2 + C_q)$, it follows that γ is a $2(2 + C_q)$-uniform curve in Ω with end points (x_1, y_1) and (x_2, y_2). Thus Z_+ is a uniform domain.

Finally, we prove the codimensionality condition that is the last part of the proposition.

For $x_0 \in X$ and $r > 0$ note that $B_Z((x_0, 0), r) \cap Z_+ = B_X(x_0, r) \times (0, r)$, and so

$$\mu_a(B_Z((x_0, 0), r) \cap Z_+) = \frac{r^{1+a}}{1 + a} \mu_X(B_X(x_0, r)). \tag{4.2}$$

27

If $A \subset X = X \times \{0\} = \partial Z_+$ is a Borel set and $\tau > 0$, then from (2.7) we know that the co-dimension τ Hausdorff measure $\mathcal{H}^{*,\tau}(A)$ is the number

$$\lim_{\varepsilon \to 0^+} \inf \left\{ \sum_i \frac{\mu_a(B_i)}{\mathrm{rad}(B_i)^\tau} \ : \ A \subset \bigcup_i B_i \text{ and } \mathrm{rad}(B_i) < \varepsilon \right\}.$$

Here, B_i are balls in $\overline{Z_+}$, centered at points in $X \times \{0\}$; this is the "co-dimension τ Hausdorff measure on X. Then by (4.2), we have

$$\mathcal{H}^{*,1+a}(A) = \frac{1}{1+a} \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_i \mu_X(B_i^X) \ : \ A \subset \bigcup_i B_i^X \text{ and } \mathrm{rad}(B_i^X) < \varepsilon \right\}$$

$$= \frac{1}{1+a} \mu_X(A)$$

because μ_X is a Borel regular measure on X. Thus μ_X is a codimension $1 + a$ Hausdorff measure on $X = \partial Z_+$ satisfying (4.2). $\qquad \square$

Recall the definition of $D^{1,2}(Z_+)$ from Definition 2.1, and that with $-1 < a < 1$ fixed, we set $\theta = \frac{1-a}{2}$.

Proposition 4.2. Trace space of $D^{1,2}(Z_+)$ is the class $B_{2,2}^\theta(X)$ where $\theta = \frac{1-a}{2}$. That is, the operator T as defined in (2.6) forms a bounded linear operator

$$T : D^{1,2}(Z_+) \to B_{2,2}^\theta(X).$$

In addition, there is a bounded linear operator

$$E : B_{2,2}^\theta(X) \to D^{1,2}(Z_+)$$

such that $T \circ E$ is the identity map on $B_{2,2}^\theta(X)$. Furthermore, for functions u in $D^{1,2}(Z_+)$ that have continuous extensions, also denoted by u, to $X \times \{0\}$, we have $Tu = u|_{X \times \{0\}}$.

The proof of this proposition follows along the lines of the proof of [30, Theorem 1.1]; however, our uniform domain Z_+ is not bounded and the functions u in the above proposition are not necessarily in the global Sobolev class $N^{1,2}(Z_+)$ as non-zero constant functions have globally finite energy but are not in $N^{1,2}(Z_+)$. Therefore the proof in [30] is not directly applicable here. Instead, for the convenience of the reader, we provide a complete proof below.

Proof. For $x \in X$ we set $\gamma_x : [0, \infty) \to \overline{Z_+}$ by $\gamma_x(t) = (x, t)$. Let $u \in D^{1,2}(Z_+)$ with a 2-weak upper gradient $g \in L^2(Z_+)$. As Z_+ is a uniform domain, by [7] we know that u has an extension, also denoted u, that belongs to $D^{1,2}(\overline{Z_+})$, and so Cap_2-almost every point in $\overline{Z_+}$ is a μ_a-Lebesgue point of u. Let $x, w \in X$ be two distinct points such that $u \circ \gamma_x$ and $u \circ \gamma_w$ are absolutely continuous and have limits along γ_x, γ_w at x and w respectively, see Lemma 3.3. By Lemma 3.3 and by the connection between $\mathrm{Cap}_2(E) = 0$ and 2-modulus of the family of all curves in $\overline{Z_+}$ intersecting E being zero (see [24] or [36]), we know that if $E \subset \partial Z_+ = X$ has $\mathrm{Cap}_2(E) = 0$, then $\mu_X(E) = 0$. Thus the trace defined by (2.6) exists μ_X-a.e. Hence, we may

28

assume also that x, w are both Lebesgue points of u (with respect to the measure μ_a).

From the proof of Proposition 4.1, the concatenation of the three curves γ_1, γ_2, and γ_3 is a uniform curve in Z_+ with end points $(x, 0)$ and $(w, 0)$, where

$$\gamma_1(t) := (x, t), \quad 0 \leq t \leq d_X(x, w),$$
$$\gamma_3(t) := (w, d_X(x, w) - t), \quad 0 \leq t \leq d_X(x, w),$$
$$\gamma_2(t) := (\beta(t), d_X(x, w)), \quad 0 \leq t \leq \ell_X(\beta),$$

where β is a C_q-quasiconvex curve in X arc-length parametrized to be beginning at x and ending at w. We cover γ by Whitney-type balls as follows. Let N_0 be the smallest positive integer such that $N_0 d_Z(x, w)/2 \geq \ell_X(\beta)$. Because β is C_q-quasiconvex, it follows that $N_0 \leq 2C_q$.

To construct the cover, we provide three groups of balls, one for each of γ_1, γ_2, and γ_3. The first group, associated with γ_1, consists of balls B_k for negative integers k. The middle group, B_k for $k = 0, \cdots, N_0$, is associated with γ_2, while the balls B_k for positive integers $k > N_0$ are associated with γ_3. This is done as follows.

For $k = 0, \cdots, N_0 - 1$ let $B_k := B(\gamma_2(k d_X(x, w)), d_X(x, w)/2)$ the ball centered a length-distance $k d_X(x, w)/2$ along γ_2 from $(x, d_X(z, w))$, with radius $d_X(w, x)/2$, and let $B_{N_0} := B(\gamma_2(\ell(\beta)), d_X(w, x)/2)$. For positive integers $k > N_0$ we set $B_k := B(\gamma_3(t_k), 2^{N_0-k} d_X(x, w))$ for $t_k := (1 - 2^{N_0-k}) d_X(x, w)$. For negative integers k we set $B_k := B(\gamma_1(\tau_k), 2^k d_X(x, w))$ with $\tau_k := 2^k d_X(x, w)$. For $k \in \mathbb{Z}$ we set $r_k = 2^{-|k|} d_X(x, w)$. Note that the radius of B_k is comparable to r_k.

Strictly speaking, the balls B_k with $k \leq 0$ or $k \geq N_0$ are not balls centered at x, w respectively; however, balls centered at x, w respectively and twice the radius of B_k will contain B_k, and so as long as $\lim_{\rho \to 0^+} \rho \left(\fint_{B(x,\rho)} g^2 \, d\mu_a \right)^{1/2} = 0$, by the Poincaré inequality we still have that $\lim_{k \to -\infty} u_{B_k} = u((x, 0))$ and $\lim_{k \to \infty} u_{B_k} = u((w, 0))$. Hence, by a telescoping argument (see for example [24, 23]), we have (we denote $u(x, 0)$ by an abuse of notation as $u(x)$) and for a choice of $\varepsilon > 0$,

$$|u(x) - u(w)| \leq \sum_{k \in \mathbb{Z}} |u_{B_k} - u_{B_{k+1}}| \leq C \sum_{k \in \mathbb{Z}} \fint_{2B_k} |u - u_{2B_k}| \, d\mu_a$$

$$\leq C \sum_{k \in \mathbb{Z}} r_k \left(\fint_{2B_k} g^2 \, d\mu_a \right)^{1/2}$$

$$\leq C \sum_{k \in \mathbb{Z}} r_k^{\frac{1-a-\varepsilon}{2}} \left(r_k^{1+a+\varepsilon} \fint_{2B_k} g^2 \, d\mu_a \right)^{1/2},$$

where we use the doubling property of μ_a together with the fact that $B_{k+1} \subset 2B_k$ for $k \in \mathbb{Z}$, followed by the 2-Poincaré inequality on Z_+. Now by Hölder's inequality,

29

we obtain

$$|u(x) - u(w)| \leq C \left(\sum_{k \in \mathbb{Z}} r_k^{1-(a+\varepsilon)} \right)^{1/2} \left(\sum_{k \in \mathbb{Z}} r_k^{1+a+\varepsilon} \fint_{2B_k} g^2 \, d\mu_a \right)^{1/2}$$

$$\leq C d_X(x,w)^{\frac{1-a-\varepsilon}{2}} \left(\sum_{k \in \mathbb{Z}} \frac{r_k^{1+a+\varepsilon}}{\mu_a(B_k)} \int_{2B_k} g^2 \, d\mu_a \right)^{1/2}$$

$$\leq C d_X(x,w)^{\frac{1-a-\varepsilon}{2}} \left(\sum_{k \in \mathbb{Z}} \frac{r_k^\varepsilon}{\mu_X(B(z_k, r_k))} \int_{2B_k} g^2 \, d\mu_a \right)^{1/2},$$

where $z_k = x$ when $k \leq 0$ and $z_k = w$ when $k > 0$. Following [30], we set

$$C_1[x,w] := \bigcup_{k=0}^\infty 2B_k, \quad C_2[x,w] := \bigcup_{k=1}^\infty 2B_{-k}.$$

Then we have

$$|u(x) - u(w)| \leq C d_X(x,w)^{\frac{1-a-\varepsilon}{2}} \left(\int_{C_1[x,w]} \frac{d_Z((x,0),z)^\varepsilon}{\mu_X(B(x, d_Z((x,0),z)))} g(z)^2 \, d\mu_a(z) \right.$$

$$\left. + \int_{C_2[x,w]} \frac{d_Z((w,0),z)^\varepsilon}{\mu_X(B(w, d_Z((w,0),z)))} g(z)^2 \, d\mu_a(z) \right)^{1/2}.$$

Recalling that $\theta = \frac{1-a}{2}$, we obtain

$$\frac{|u(x) - u(w)|^2}{d_X(x,w)^{2\theta}} \leq C d_Z(x,w)^{-\varepsilon} \left(\int_{C_1[x,w]} \frac{d_Z((x,0),z)^\varepsilon}{\mu_X(B(x, d_Z((x,0),z)))} g(z)^2 \, d\mu_a(z) \right.$$

$$\left. + \int_{C_2[x,w]} \frac{d_Z((w,0),z)^\varepsilon}{\mu_X(B(w, d_Z((w,0),z)))} g(z)^2 \, d\mu_a(z) \right). \tag{4.3}$$

Now, we estimate the Besov energy seminorm of u on $\partial Z_+ = X \times \{0\} \simeq X$.

By the results in [26, Theorem 1.11], we know that u is Borel measureable in $\overline{Z_+}$, and hence its restriction to $\partial Z_+ = X$ is measurable with respect to the Borel measure μ_X. Recall that

$$\|u\|_{B_{2,2}^\theta(X)}^2 = \int_X \int_X \frac{|u(x) - u(w)|^2}{d_X(x,w)^{2\theta} \mu_X(B(x, d_X(x,w)))} \, d\mu_X(w) \, d\mu_X(x).$$

By (4.3), we obtain

$$\|u\|_{B_{2,2}^\theta(X)}^2 \lesssim \int_X \int_X \left(\int_{C_1[x,w]} \frac{d_Z((x,0),z)^\varepsilon d_X(x,w)^{-\varepsilon}}{\mu_X(B(x, d_Z((x,0),z)))\mu_X(B(x, d_X(x,w)))} g(z)^2 d\mu_a(z) \right.$$

$$\left. + \int_{C_2[x,w]} \frac{d_Z((w,0),z)^\varepsilon d_X(x,w)^{-\varepsilon}}{\mu_X(B(w, d_Z((w,0),z)))\mu_X(B(x, d_X(x,w)))} g(z)^2 \, d\mu_a(z) \right) d\mu_X(w) \, d\mu_X(x)$$

$$=: C(I_1 + I_2), \tag{4.4}$$

30

where, by Tonelli's theorem,

$$I_1 := \int_{X^2} \int_{C_1[x,w]} \frac{g(z)^2 \, d_Z((x,0),z)^\varepsilon \, d_X(x,w)^{-\varepsilon}}{\mu_X(B(x,d_Z((x,0),z)))\mu_X(B(x,d_X(x,w)))} \, d\mu_Z(z) \, d\mu_X(w) \, d\mu_X(x)$$

$$= \int_{Z_+} g(z)^2 \int_{X^2} \frac{\chi_{C_1[x,w]}(z) \, d_Z((x,0),z)^\varepsilon \, d_X(x,w)^{-\varepsilon}}{\mu_X(B(x,d_Z((x,0),z)))\mu_X(B(x,d_X(x,w)))} \, d\mu_X(w) \, d\mu_X(x) \, d\mu_a(z),$$

(4.5)

$$I_2 := \int_{X^2} \int_{C_2[x,w]} \frac{g(z)^2 \, d_Z((w,0),z)^\varepsilon \, d_X(x,w)^{-\varepsilon}}{\mu_X(B(w,d_Z((w,0),z)))\mu_X(B(x,d_X(x,w)))} \, d\mu_Z(z) \, d\mu_X(w) \, d\mu_X(x)$$

$$= \int_{Z_+} g(z)^2 \int_{X^2} \frac{\chi_{C_2[x,w]}(z) \, d_Z((w,0),z)^\varepsilon \, d_X(x,w)^{-\varepsilon}}{\mu_X(B(w,d_Z((w,0),z)))\mu_X(B(x,d_X(x,w)))} \, d\mu_X(w) \, d\mu_X(x) \, d\mu_a(z).$$

(4.6)

Here, we abbreviated $\int_X \int_X$ by $\int_{X^2}$.

Observe that if $z \in C_1[x,w]$, then $\frac{d_Z((x,0),z)}{4} \le \text{dist}_Z(z,\partial Z_+) \le 4d_X(x,w)$, and if $z \in C_2[x,w]$, then $\frac{d_Z((w,0),z)}{4} \le \text{dist}_Z(z,\partial Z_+) \le 4d_X(x,w)$. To simplify notation, we set $\delta(z) := \text{dist}_Z(z,\partial Z_+)$ for $z \in Z_+$. Then,

$$\chi_{C_1[x,w]}(z) \le \chi_{B(z,4\delta(z))}((x,0))\chi_{X \setminus B(x,\delta(z)/4)}(w).$$

Thus, for each $z \in Z_+$, setting $B_k := B(x,2^k\delta(z)/4) \subset X$ for $k = 0,1,\cdots$, we have

$$\int_X \int_X \frac{\chi_{C_1[x,w]}(z) \, d_Z((x,0),z)^\varepsilon \, d_X(x,w)^{-\varepsilon}}{\mu_X(B(x,d_Z((x,0),z)))\mu_X(B(x,d_X(x,w)))} \, d\mu_X(w) \, d\mu_X(x)$$

$$\lesssim \int_{B(z,4\delta(z)) \cap \partial Z_+} \int_{X \setminus B(x,\delta(z)/4)} \frac{\delta(z)^\varepsilon d_X(x,w)^{-\varepsilon}}{\mu_X(B(x,\delta(z))) \, \mu_X(B(x,d_X(x,w)))} \, d\mu_X(w) \, d\mu_X(x)$$

$$\lesssim \int_{B(z,4\delta(z)) \cap \partial Z_+} \frac{\delta(z)^\varepsilon}{\mu_X(B(x,\delta(z)))} \sum_{k=0}^\infty \frac{(2^k\delta(z))^{-\varepsilon}}{\mu_X(B(x,2^k\delta(z)/4))} \mu_X(B_{k+1} \setminus B_k) \, d\mu_X(x)$$

$$\lesssim \int_{B(z,4\delta(z)) \cap \partial Z_+} \frac{\delta(z)^\varepsilon}{\mu_X(B(x,\delta(z)))} \sum_{k=0}^\infty 2^{-k\varepsilon}\delta(z)^{-\varepsilon} \, d\mu_X(x) \lesssim 1.$$

A similar argument shows that

$$\int_X \int_X \frac{\chi_{C_2[x,w]}(z) \, d_Z((w,0),z)^\varepsilon \, d_X(x,w)^{-\varepsilon}}{\mu_X(B(w,d_Z((w,0),z)))\mu_X(B(x,d_X(x,w)))} \, d\mu_X(w) \, d\mu_X(x) \lesssim 1.$$

Therefore, from (4.5) and (4.6) we obtain $I_1 + I_2 \le C \int_{Z_+} g(z)^2 \, d\mu_Z(z)$, from whence we obtain by (4.4) that

$$\|u\|^2_{B^\theta_{2,2}(X)} \le C \int_{Z_+} g^2 \, d\mu_Z \tag{4.7}$$

as desired.

We next show that if $u : X \to \mathbb{R}$ such that $\|u\|_{B^\theta_{2,2}(X)}$ is finite, then u has an extension to Z_+, also denoted u, such that u has a 2-weak upper gradient g in Z_+ with the property that $g \in L^2(Z_+)$. To this end, for each $n \in \mathbb{Z}$ we choose

$A_n \subset X$ to be a maximal 2^{-n}-separated set. We can ensure also that $A_n \subset A_{n+1}$ for each $n \geq 0$. As μ_X-almost every point in X is a Lebesgue point of u, we can also ensure that each point in A_n is a Lebesgue point of u. Then the balls $Q_{i,n} := B_X(x_i, 2^{-n}) \times (3 \cdot 2^{-n}, 7 \cdot 2^{-n}) \subset Z_+$, $x_i \in A_n$, forms a Whitney cover of Z_+ such that

(1) we have $Z_+ = \bigcup_{i,n} Q_{i,n}$,

(2) for each $\tau \geq 1$ there is a constant $C_\tau \geq 1$ such that for each $n \in \mathbb{N}$ we have $\sum_i \chi_{\tau Q_{i,n}} \leq C_\tau$, that is, we have the bounded overlap property,

(3) for each i_0, n_0 there are at most M number of balls $Q_{i,n}$ such that $Q_{i,n} \cap Q_{i_0,n_0} \neq \emptyset$, and moreover, if $Q_{i,n}$ intersects Q_{i_0,n_0}, then $|n - n_0| \leq 1$.

(4) The center $z_{i,n} := (x_i, 5 \cdot 2^{-n}) \in Z_+$ of $Q_{i,n}$ satisfies $\delta(z_{i,n}) = 5 \cdot 2^{-n}$ and $Q_{i,n}$ is at a distance $3 \cdot 2^{-n}$ from ∂Z_+.

Here, for $\tau > 0$, by $\tau Q_{i,n}$ we mean the scaled ball $B_X(x_i, \tau 2^{-n}) \times ((5 - 2\tau)2^{-n}, (5 + 2\tau)2^{-n})$. It is sometimes in our interest to make sure that $0 < \tau < 5/2$ so that $\overline{\tau Q_{i,n}} \subset Z_+$. However, when $\tau > 5/2$ we have that $\tau Q_{i,n} \supset B_X(x_i, 2^{-n}) \times \{0\}$. Let $\varphi_{i,n}$ be a Lipschtiz partition of unity subordinate to the cover $Q_{i,n}$, that is,

(1) we have $0 \leq \varphi_{i,n} \leq 1$ on Z,

(2) the support of $\varphi_{i,n}$ is contained in $Q_{i,n}$,

(3) the function $\varphi_{i,n}$ is $2^n C$-Lipschitz continuous,

(4) the sum $\sum_{i,n} \varphi_{i,n} = \chi_{Z_+}$.

We set $B_{i,n} := B_X(x_i, 2^{1-n})$. Now, for $u \in B^\theta_{2,2}(X)$ we set Eu to be the function on Z_+ given by
$$Eu(z) := \sum_{i,n} u_{B_{i,n}} \, \varphi_{i,n}(z).$$

If $z, w \in Q_{i_0,n_0}$, then

$$|Eu(z) - Eu(w)| = \left| \sum_{i,n} [u_{B_{i,n}} - u_{B_{i_0,n_0}}][\varphi_{i,n}(z) - \varphi_{i,n}(w)] \right|$$

$$\leq C \sum_{\substack{i,n: \\ Q_{i_0,n_0} \cap Q_{i,n} \neq \emptyset}} 2^n d_Z(z,w) \fint_{B_{i,n}} \fint_{B_{i_0,n_0}} |u(x_1) - u(x_2)| \, d\mu_X(x_1) \, d\mu_X(x_2)$$

$$\leq C \, 2^{n_0} d_Z(z,w) \fint_{2B_{i_0,n_0}} \fint_{2B_{i_0,n_0}} |u(x_1) - u(x_2)| \, d\mu_X(x_1) \, d\mu_X(x_2).$$

In the above, we used the bounded overlap property of $2B_{i_0,n_0}$. Using the fact that if $z \in Q_{i_0,n_0}$, then $\delta(z) \simeq 2^{-n_0}$, we obtain

$$|Eu(z) - Eu(w)| \lesssim \frac{d_Z(z,w)}{\delta(z)} \fint_{2B_{i_0,n_0}} \fint_{2B_{i_0,n_0}} |u(x_1) - u(x_2)| \, d\mu_X(x_1) \, d\mu_X(x_2).$$

Therefore, when $z \in Q_{i_0, n_0}$, we have

$$\mathrm{Lip} Eu(z) \lesssim \delta(z)^{-1} \fint_{2B_{i_0, n_0}} \fint_{2B_{i_0, n_0}} |u(x_1) - u(x_2)| \, d\mu_X(x_1) \, d\mu_X(x_2).$$

For integers $n \in \mathbb{Z}$, we set $A_n := \{z \in Z_+ \,:\, 2^{-(n+1)} \le \delta(z) \le 2^{1-n}\}$. Then we have $\delta(z) \simeq 2^{-n}$ when $z \in A_n$, and so

$$\int_{A_n} \mathrm{Lip} Eu(z)^2 \, d\mu_a(z) = \int_X \int_{2^{-(n+1)}}^{2^{1-n}} \delta((x,y))^a \mathrm{Lip} Eu((x,y))^2 \, dy \, d\mu_X(x)$$

$$\simeq \sum_i \int_{B_{i,n}} \int_{2^{-(n+1)}}^{2^{1-n}} \delta((x,y))^a \mathrm{Lip} Eu((x,y))^2 \, dy \, d\mu_X(x)$$

$$\lesssim \sum_i \int_{B_{i,n}} \int_{2^{-(n+1)}}^{2^{1-n}} \fint_{2B_{i,n}} \fint_{2B_{i,n}} \frac{|u(x_1) - u(x_2)|^2}{2^{n(a-2)}} \, d\mu_X(x_1) \, d\mu_X(x_2) \, dy \, d\mu_X(x)$$

$$\lesssim \sum_i \int_{B_{i,n}} \int_{2^{-(n+1)}}^{2^{1-n}} 2^n \fint_{2B_{i,n}} \fint_{2B_{i,n}} \frac{|u(x_1) - u(x_2)|^2}{2^{-n(1-a)}} \, d\mu_X(x_1) \, d\mu_X(x_2) \, dy \, d\mu_X(x)$$

$$\simeq \sum_i \int_{B_{i,n}} \fint_{2B_{i,n}} \fint_{2B_{i,n}} \frac{|u(x_1) - u(x_2)|^2}{2^{-n(1-a)}} \, d\mu_X(x_1) \, d\mu_X(x_2) \, d\mu_X(x)$$

$$\simeq \sum_i \int_{2B_{i,n}} \fint_{2B_{i,n}} \frac{|u(x_1) - u(x_2)|^2}{2^{-n(1-a)}} \, d\mu_X(x_1) \, d\mu_X(x_2).$$

The bounded overlap property of the balls $2B_{i,n}$ for a fixed $n \in \mathbb{Z}$ gives us

$$\int_{A_n} \mathrm{Lip} Eu(z)^2 \, d\mu_a(z) \lesssim \int_X \fint_{B(x_2, 2^{1-n})} \frac{|u(x_1) - u(x_2)|^2}{2^{-n(1-a)}} \, d\mu_X(x_1) \, d\mu_X(x_2).$$

It follows that

$$\int_{Z_+} \mathrm{Lip} Eu(z)^2 \, d\mu_a(z) \lesssim \sum_{n \in \mathbb{Z}} \int_X \fint_{B(x_2, 2^{1-n})} \frac{|u(x_1) - u(x_2)|^2}{2^{-n(1-a)}} \, d\mu_X(x_1) \, d\mu_X(x_2)$$

$$\simeq \|u\|_{B_{2,2}^\theta(X)}^2,$$

where the last comparison is from [19]. From the above we also get the reverse estimate of (4.7):

$$\int_{Z_+} g_{Eu}^2 \, d\mu_a \le C \|u\|_{B_{2,2}^\theta(X)}^2. \tag{4.8}$$

Recall that $\theta = \frac{1-a}{2}$. Finally, it remains to show that $TEu = u$ μ_X-a.e. in X. As from the discussion at the beginning of the proof, we know that for μ_X-a.e. $x \in X$ we have that $\lim_{y \to 0^+} Eu(x, y) = v(x)$ exists and is the trace TEu of Eu given by the condition that

$$\lim_{r \to 0^+} \fint_{B((x,0),r) \cap Z_+} |Eu - v(x)| \, d\mu_a = 0.$$

33

Moreover, by the fact that $u \in L^1_{loc}(X)$, we can also ensure that μ_X-a.e. $x \in X$ is a μ_X-Lebesgue point of u. Let $x \in X$ be both a μ_X-Lebesgue point as well as satisfying the above condition on $v(x)$. The goal here is to show that $v(x) = u(x)$. To this end, let $y > 0$; then

$$Eu((x, y)) - u(x) = \sum_{i,n} [u_{B_{i,n}} - u(x)] \, \phi_{i,n}(y).$$

Observe that $\phi_{i,n}(y) \neq 0$ only when the radius of $B_{i,n}$ is comparable to y and its center is at a comparable distance from (x, y) as well. Therefore

$$|Eu((x,y)) - u(x)| \leq C \sum_{i,n} \left(\fint_{B((x,0),Cy)} |u - u(x)| \, d\mu_X \right) \phi_{i,n}(y)$$

$$= C \fint_{B((x,0),Cy)} |u - u(x)| \, d\mu_X \to 0 \text{ as } y \to 0^+.$$

It follows that $v(x) = u(x)$, completing the proof. $\qquad\square$

The following is a type of gluing lemma that allows us to combine a Newton-Sobolev function on Z_+ with a Newton-Sobolev function on $Z_- := X \times (-\infty, 0)$ to obtain a Newton-Sobolev function on $Z = X \times \mathbb{R}$.

Lemma 4.3. *Let T_+ be the trace operator on $D^{1,2}(Z_+)$ and T_- be the trace operator on $D^{1,2}(Z_-)$. Suppose $u \in D^{1,2}(Z_+)$ and $v \in D^{1,2}(Z_-)$ such that $T_+u(x) = T_-v(x)$ for μ_X-a.e. $x \in X$. Define $w : Z \to \mathbb{R}$ by*

$$w(x, y) = \begin{cases} u(x, y) & \text{if } y > 0, \\ T_+u(x) & \text{if } y = 0, \\ v(x, y) & \text{if } y < 0. \end{cases}$$

Then w belongs to $D^{1,2}(Z)$.

Proof. Let $f = T_+u$. Then from Proposition 4.2 we know that $f \in B^{\theta}_{2,2}(X)$; we set w_0 on Z by

$$w_0(x, y) = \begin{cases} Ef(x, y) & \text{if } y > 0, \\ f(x) & \text{if } y = 0, \\ Ef(x, -y) & \text{if } y < 0. \end{cases}$$

Then from the properties of Ef obtained in the proof of Proposition 4.2 and from Lemma 3.1 we know that $w_0 \in D^{1,2}(Z)$. By the local nature of minimal 2-weak upper gradients, it follows that with g the minimal 2-weak upper gradient of Ef, we have that $g_0(x, y) = g(x, y)$ when $y > 0$ and $g_0(x, y) = g(x, -y)$ when $y < 0$ is the minimal 2-weak upper gradient of w_0 in Z.

Now, consider $\varphi := w - w_0$. Then $T_+\varphi = 0 = T_-\varphi$. Note that $\varphi|_{Z_+} \in D^{1,2}(Z_+)$ and $\varphi|_{Z_-} \in D^{1,2}(Z_-)$. Hence from [7], $\varphi|_{\overline{Z_+}} \in D^{1,2}(\overline{Z_+})$ and $\varphi|_{\overline{Z_-}} \in D^{1,2}(\overline{Z_-})$. We now show that the zero extension of $\varphi|_{\overline{Z_+}}$ to Z is in $D^{1,2}(Z)$. Indeed, if $g \in L^2(Z_+)$ is an upper gradient of $\varphi|_{\overline{Z_+}}$ and γ is a non-constant compact rectifiable curve in Z

with end points denoted x and y, then if both $x, y \in \overline{Z_-}$, we have that $\varphi|_{\overline{Z_+}}(y) = 0 = \varphi|_{\overline{Z_+}}(x)$ and so the upper gradient inequality is satisfied by the pair $\varphi|_{\overline{Z_+}}, g$ with respect to γ. Hence, without loss of generality, we may assume that $x \in Z_+$. If γ lies entirely in $\overline{Z_+}$, then again the above pair satisfies the upper gradient inequality with respect to γ. If γ intersects both Z_+ and Z_-, then by breaking γ into two subcurves if necessary, we may assume that $y \in \overline{Z_-}$. Now let β be the largest subcurve of γ beginning at x and lying entirely in Z_+, and let $z \in X \times \{0\}$ be the end-point of β. Then the closure of the path β has z as its terminal point, and this compact path, also denoted β, lies entirely in $\overline{Z_+}$. Therefore the pair $\varphi|_{\overline{Z_+}}, g$ satisfies the upper gradient inequality with respect to β. Since $\varphi|_{\overline{Z_+}}(z) = \varphi_+(z) = 0 = \varphi_+(y)$, with g_+ the zero-extension of g to Z, it also follows that φ_+, g_+ satisfy the upper gradient inequality on the subcurve of γ left-over from β. Hence φ_+, g_+ satisfy the upper gradient inequality on γ itself. It follows from the arbitrariness of γ that $g_+ \in L^2(Z)$ is an upper gradient of φ_+, that is, $\varphi_+ \in D^{1,2}(Z)$. A similar argument gives that $\varphi_- \in D^{1,2}(Z)$. Therefore $\varphi = \varphi_+ + \varphi_-$ lies in $D^{1,2}(Z)$. Finally it follows that $w = \varphi + w_0 \in D^{1,2}(Z)$, completing the proof. $\qquad \square$

5 Existence and uniqueness of Cheeger-Harmonic Extensions

Given a bounded domain $\Omega \subset X$, let us denote

$$U_\Omega = Z_+ \cup Z_- \cup (\Omega \times \{0\}).$$

Given a function f on X, we want to consider the Cheeger harmonic solutions to the Dirichlet problem in U_Ω with boundary values f on $\partial U_\Omega = (X \setminus \Omega) \times \{0\}$. That is, we want to find a function $u \in D^{1,2}(U_\Omega)$ such that

$$\Delta_a u = 0 \text{ on } U_\Omega \quad \text{with} \quad Tu = f \text{ on } (X \setminus \Omega) \times \{0\}. \tag{5.1}$$

Here, Δ_a is the infinitesimal generator on Z associated with the Cheeger differential structure constructed in Subsection 3.2 above. Observe that the first condition above is equivalent to knowing that whenever $h \in D^{1,2}(U_\Omega)$ with compact support in the domain U_Ω, then as $\mu_a(Z \setminus U_\Omega) = 0$ and by Lemma 4.3, we have

$$\mathcal{E}_a^Z(u, h) = \int_{U_\Omega} \langle D_a u(x,y), D_a h(x,y) \rangle_{(x,y)} \, d\mu_a(x,y) = 0. \tag{5.2}$$

Here D_a is the tensor product of the Cheeger differential form D_X on X and the Euclidean differential form on $\mathbb{R}$, see Remark 3.10 above.

Observe that a function u that satisfies (5.1) as above for all compactly supported $h \in D^{1,2}(U_\Omega)$ also is a minimizer of the Cheeger energy in the following sense: for each $v \in D^{1,2}(Z)$ such that $v = u$ outside a compact subset of U_Ω, we have

$$\mathcal{E}_a^Z(u, u) \leq \mathcal{E}_a^Z(v, v). \tag{5.3}$$

Remark 5.1. Combining the construction of the Cheeger differential structure D_a from Subsection 3.2 and Lemma 3.11 tell us that a Cheeger harmonic function u in U is a quasiminimizer in the sense of [27], and so is necessarily locally Hölder continuous on U, and if it is non-negative, then satisfies a Harnack inequality also.

Theorem 5.2. *Suppose that $\Omega \subset X$ is a bounded domain with $\mu_X(X \setminus \Omega) > 0$. Then, there exists a linear extension operator $H : B_{2,2}^\theta(X) \to D^{1,2}(Z)$ such that $u_f = Hf$ is the unique Cheeger harmonic function in U_Ω with $Tu = f$ on ∂U_Ω for every $f \in B_{2,2}^\theta(X)$. Moreover $\|Hf\|_\infty \leq \|f\|_\infty$ whenever f is bounded.*

Proof. Fix $f \in B_{2,2}^\theta(X)$. Let us consider the minimization problem

$$E_{min} = \inf_{v \in \mathcal{A}} \mathcal{E}_a^Z(v, v), \tag{5.4}$$

where the class of admissible functions $\mathcal{A}$ consists of functions $u \in D^{1,2}(Z)$ such that $Tu = f$ on ∂U_Ω. We know that the infimum above is finite as we can extend symmetrically the extension Ef given by Proposition 4.2 to Z_-, see also Lemma 4.3. Let $\{u_k\}_k$ be a minimizing sequence for (5.4). Let us start by considering the case $|f| \leq 1$ at every point. In this case, we can assume without loss of generality that also $|u_k| \leq 1$ everywhere as truncation to interval $[-1, 1]$ can only decrease the energy $\mathcal{E}_a(u_k, u_k)$.

Next we prove that

$$\mathcal{E}_a^Z(u_k - u_l, u_k - u_l) \to 0 \quad \text{as} \quad k, l \to \infty.$$

Let $u_{k,l} = \frac{1}{2}(u_k + u_l)$. Notice that $u_{k,l}$ satisfies also the boundary condition $Tu_{k,l} = f$ as $Tu_k = Tu_l = f$. By the triangle inequality,

$$\mathcal{E}_a^Z(u_{k,l}, u_{k,l}) \leq \frac{1}{2}\left(\mathcal{E}_a^Z(u_k, u_k) + \mathcal{E}_a^Z(u_l, u_l)\right).$$

Thus we can conclude that $\mathcal{E}_a^Z(u_{k,l}, u_{k,l}) \to E_{min}$ as $k, l \to \infty$. Now it follows by the parallelogram identity that

$$\lim_{k,l \to \infty} \mathcal{E}_a^Z(u_k - u_l, u_k - u_l) = \lim_{k,l \to \infty} \left(2\mathcal{E}_a^Z(u_k, u_k) + 2\mathcal{E}_a^Z(u_l, u_l) - 4\mathcal{E}_a(u_{k,l}, u_{k,l})\right) = 0.$$

By assumption, $\mu_X(X \setminus \Omega) > 0$. Hence for sufficiently large balls $B \subset Z$ centered at a point in $\Omega \times \{0\}$ we have $\mu_X(B \cap (X \setminus \Omega)) > 0$. If X is unbounded, we can in addition consider a ball $B \subset Z$ centered at $X \times \{0\}$ and with large enough radius so that $\mu_X(B \cap \Omega \times \{0\}) \leq \frac{1}{2}\mu_X(B \cap X \times \{0\})$. Either way, we see that for sufficiently large balls B we have from Lemma 3.3 that $\mathrm{Cap}_2(B \cap \{u_k - u_l = 0\}) \geq \widehat{C}_B$ for some $\widehat{C}_B > 0$ that is independent of k, l. Therefore by (2.3), we have

$$\int_B |(u_k - u_l)|^2 \mu_a \leq C_B \int_B |D_a u_k - D_a u_l|^2 d\mu_a \to 0 \quad \text{as } k, l \to \infty.$$

As the above inequality holds for all large enough balls, this implies that there exists a limiting function $u_f \in D^{1,2}(Z)$ such that $u_k \to u_f$ in $L_{loc}^2(Z)$. This limit is obtained as follows. For each ball $B \subset Z$ centered at a point in $(x, 0) \in Z$ with a fixed choice of $x \in \Omega$, we have $u_k \to u_f$ in $N^{1,2}(B)$, and as $u_k - Ef \in N_0^{1,2}(\overline{B} \cap (Z_\pm \cup \Omega \times \{0\}))$, it follows from the Banach space property of $N^{1,2}(\overline{B} \cap Z_\pm)$ that $u_f - Ef \in N^{1,2}(\overline{B} \cap Z_\pm)$. As this happens for all balls B, it follows that pointwise μ_X-a.e. we have $Tu_f = f$ as given by (2.6).

As $\{\mathcal{E}_a^Z(u_k, u_k)\}$ converges to the minimum and $Tu_k = f$ on ∂U_Ω, the function u_f is the unique global minimizer of (5.4). To see the uniqueness of the minimizer,

suppose that $v \in D^{1,2}(Z)$ is also a solution. Then, as $u = v = f$ on ∂U_Ω, it follows that $\mathcal{E}_a^Z(u_f, u_f) = \mathcal{E}_a^Z(v, v)$. Therefore,

$$\mathcal{E}_a^Z(u_f - v, u_f - v) = 2\mathcal{E}_a^Z(u_f, u_f) - 2\mathcal{E}_a^Z(u_f, v) = 2\mathcal{E}_a^Z(u_f, u_f - v) = 0,$$

the latter equality following from the Euler-Lagrange formulation that equate minimization of the energy $\mathcal{E}_a^Z$ with Definition 2.6. It follows from the Poincaré inequality that $u_f - v$ is a constant on Z. Since $Tu_f = Tv = f$ on $X \setminus \Omega$ and the 2-capacity of $X \setminus \Omega$ is positive (see Lemma 3.3), it follows that $u_f = v$, that is, the solution is unique.

We set $Hf = u_f$. As a minimizer of (5.4), u_f is a Cheeger harmonic function in the sense that it satisfies (5.2) in U_Ω. For unbounded and nonnegative f, we define the extension operator as

$$Hf = \sum_{k=0}^\infty Hf_k,$$

where $f_k = \max\{\min\{f - k, 1\}, 0\}$. Observe that

$$\mathcal{E}_a^{Z+}(Hf, Hf) = \sum_{k=0}^\infty \mathcal{E}_a^{Z+}(Hf_k, Hf_k).$$

Then for general $f \in B_\theta^{2,2}(X)$, we set $Ef = Ef^+ - Ef^-$, where f^+ and f^- are the positive and negative parts of f. As linear combinations of Cheeger harmonic functions are also Cheeger harmonic, this construction leads to Cheeger harmonic extensions of f, which are the unique global minimizers of the energy (5.4). Note that

$$\mathcal{E}_a^{Z+}(Hf, Hf) = \mathcal{E}_a^{Z+}(Hf^+, Hf^+) + \mathcal{E}_a^{Z+}(Hf^-, Hf^-).$$

$\square$

Proposition 5.3. The extension given by Theorem 5.2 is symmetric, namely we have $u(x, y) = u(x, -y)$ for every $(x, y) \in Z$.

Proof. Let u be a minimizer of (5.4). Let us define

$$\widetilde{u}(x, y) = \frac{1}{2}\left(u(x, y) + u(x, -y)\right).$$

As $u(x, -y)$ has the same trace as $u(x, y)$ on $X \setminus \Omega$, we have $T\widetilde{u} = f(x)$ for almost every $x \in X \setminus \Omega$ and thus $\widetilde{u}$ satisfies the trace condition for the minimization problem. By the inner product structure of the energy, we see that

$$\mathcal{E}_a(\widetilde{u}, \widetilde{u}) \leq \tfrac{1}{2}(\mathcal{E}_a(u(x, y), u(x, y)) + \mathcal{E}_a(u(x, -y), u(x, -y))) = \mathcal{E}_a(u, u)$$

and the equality holds only if $D_a u = D_a \widetilde{u}$ μ_a-almost everywhere, which implies that $\widetilde{u} = u$ and thus u is symmetric. $\square$

6 Cheeger Harmonic Solution in $U_\Omega \subset Z$

Theorem 6.1. *Suppose $f \in B_{2,2}^\theta(X) \cap L^2(X)$ such that f is a solution to the Dirichlet problem $(-\Delta_X)^\theta f = 0$ on Ω. Define the function $\tilde{u}$ by $\tilde{u}(x,0) = f(x)$ and $\tilde{u}(x,y) = \Pi_a f(x, |y|)$ when $y \neq 0$, where $\Pi_a f$ is defined in (2.15). Then $\tilde{u}$ is Cheeger harmonic in U_Ω.*

Proof. Since $u \in D^{1,2}(Z_+)$ by Proposition 2.14 and as $\tilde{u}$ is obtained by reflection with the same trace, then $\tilde{u} \in N_{loc}^{1,2}(Z)$. Fix a compactly supported Lipschitz function h in $D^{1,2}(U_\Omega)$. Let $\varepsilon > 0$ small enough such that $\{|y| < \varepsilon\} \cap \mathrm{supp}(h) \subset\subset \Omega \times \mathbb{R}$. By Lemma 3.11 and integration by parts, we have

$$\int_{U_\Omega} \langle D_a u(x,y), D_a h(x,y)\rangle_{(x,y)}\, d\mu_a = \int_{U_\Omega} \langle D_X \tilde{u}, D_X h\rangle + \langle \partial_y \tilde{u}, \partial_y h\rangle d\mu_a$$
$$\int_{U_\Omega} -\langle \Delta_X \tilde{u}, h\rangle + \langle \partial_y \tilde{u}, \partial_y h\rangle d\mu_a = \int_{U_\Omega \cap \{|y| \geq \varepsilon\}} -\langle \Delta_X \tilde{u}, h\rangle + \langle \partial_y \tilde{u}, \partial_y h\rangle d\mu_a$$
$$+ \int_{U_\Omega \cap \{|y| < \varepsilon\}} -\langle \Delta_X \tilde{u}, h\rangle + \langle \partial_y \tilde{u}, \partial_y h\rangle d\mu_a = I + J.$$

A straightforward computation shows that

$$I = \int_{U_\Omega \cap \{|y| \geq \varepsilon\}} -\langle \Delta_X \tilde{u}, h\rangle + \langle \partial_y \tilde{u}, \partial_y h\rangle |y|^a d\mu_a$$
$$= \int_{U_\Omega \cap \{|y| \geq \varepsilon\}} -\langle \Delta_X \tilde{u}, h\rangle - \langle (\frac{a}{y}\partial_y + \partial_{yy})\tilde{u}, h\rangle d\mu_a + \int_{\partial(U_\Omega \cap \{|y| \geq \varepsilon\})} |y|^a \partial_y u\, h\, d\mu_a$$
$$= 2 \int_{(\Omega \times \mathbb{R}) \cap \{y = \varepsilon\}} |y|^a \partial_y u\, h\, d\mu_X.$$

Thanks to [3, Lemma 3.1] we have

$$\lim_{y \to 0^+} -\frac{2^{2\theta - 1}\Gamma(\theta)}{\Gamma(1 - \theta)} \int_X y^a \partial_y u \phi\, d\mu_X = \mathcal{E}_\theta(f, \phi),$$

that vanishes by assumption, for each $\phi \in L^2(X) \cap B_{2,2}^\theta(X)$. The claim then follows for Lipschitz h since dominated convergence gives $h(\cdot, y) \to h(\cdot, 0)$ in $L^2(X) \cap B_{2,2}^\theta(X)$ as $y \to 0$, and since the argument in [3, Lemma 3.1] shows that $y^a \partial_y u$ is uniformly bounded in the dual of $L^2 \cap B_{2,2}^\theta$. Therefore $\lim_{\varepsilon \to 0} I = 0$. Moreover, by the proof of [3, Lemma 3.1], $\Delta_X \tilde{u}$ and $\tilde{u}_y$ are locally integrable in $Z \setminus \{t = 0\}$, then we have $\lim_{\varepsilon \to 0} J = 0$. $\qquad\square$

7 Proof of the Main Theorems

We now collect tools from earlier in the paper to prove the main theorems.

Proof of Theorem 1.1. Let $f \in B_{2,2}^\theta(X)$ such that $\mathcal{E}_\theta(f, f)$ is finite, and set $\mathcal{K}_f$ to be the collection of all functions $h \in B_{2,2}^\theta(X)$ such that $h = f$ μ_X-a.e. in $X \setminus \Omega$. Then $\mathcal{K}_f$ is a convex subset of $B_{2,2}^\theta(X)$. Moreover, if $(f_k)_k$ is a sequence from $\mathcal{K}_f$ such

that $f_k \to f_\infty$ in $L^2(\Omega)$, then the extension of f_∞ by f to $X \setminus \Omega$, also denoted f_∞, satisfies $\mathcal{E}_\theta(f,f) \leq \liminf_k \mathcal{E}_\theta(f_k, f_k)$. It follows that $f_\infty \in \mathcal{K}_f$ provided the limit inferior is finite. Let

$$I := \inf\{\mathcal{E}_\theta(h,h) \, : \, h \in \mathcal{K}_f\}.$$

If $I = 0$, then necessarily f is constant on $X \setminus \Omega$, and then the extension of f to Ω by that constant yields the desired solution. Hence without loss of generality we may assume that $I > 0$.

As $I \leq \mathcal{E}_\theta(f,f)$, it follows that I is finite. Let $h_k \in \mathcal{K}_f$ be a corresponding minimizing sequence. Without loss of generality we may assume that $\mathcal{E}_\theta(h_k, h_k) \leq 2I$. It follows that $\mathcal{E}_\theta(h_k - f, h_k - f) \leq 6I + \mathcal{E}_\theta(f,f) < \infty$. Then as $h_k - f = 0$ on $X \setminus \Omega$ and so belongs to $L^2(X)$ as well, by Proposition 2.11 we know that the sequence $(h_k - f)$ is bounded in the Besov seminorm as well; $\|h_k - f\|^2_{B^\theta_{2,2}(X)} \leq C(6I + \mathcal{E}_\theta(f,f))$. As $f \in B^\theta_{2,2}(X)$, it follows now that $\|h_k\|^2_{B^\theta_{2,2}(X)} \leq C(6I + \mathcal{E}_\theta(f,f)) + \|f\|^2_{B^\theta_{2,2}(X)}$.

For each positive integer k consider the function v_k on $X \times X$ given by $v_k(x,w) = h_k(x) - h_k(w)$. Then, equipping $X \times X$ by the measure ν given by

$$d\nu(x,w) = \frac{1}{d(x,w)^{2\theta} \mu_X(B(x, d_X(x,w)))} \, d\mu_X(x) \, d\mu_X(w),$$

we see that $v_k \in L^2(X \times X, \nu)$ is a bounded sequence. Hence there is a function $v_\infty \in L^2(X \times X, \nu)$ such that (a convex combination of) $v_k \to v_\infty$ in $L^2(X \times X, \nu)$. By passing to a subsequence if necessary, we may also assume that this convergence is ν-almost everywhere as well (and hence, $\mu_X \times \mu_X$-almost everywhere as well). Note that if both $x, w \in X \setminus \Omega$, then $v_\infty(x,w) = f(x) - f(w)$. If $x \in \Omega$ and $w \in X \setminus \Omega$ such that $v_\infty(x,w) = \lim_k v_k(x,w)$, then $v_\infty(x,w) = \lim_k h_k(x) - f(w)$, and so we set h_∞ to be the function on X given by $h_\infty(x) = f(x)$ when $x \in X \setminus \Omega$ and $h_\infty(x) = v_\infty(x,w) + f(w)$ with $w \in X \setminus \Omega$ as chosen above. With the aid of Fubini's theorem, we know that for μ_X-almost every $x \in \Omega$ we can find $w \in X \setminus \Omega$ such that $v_\infty(x,w) = \lim_k v_k(x,w)$. Thus h_∞ is well-defined. Moreover, the function $X \times X \ni (x,w) \mapsto h_\infty(x) - h_\infty(w)$ is the function v_∞, and so $h_\infty \in B^\theta_{2,2}(X)$, with $h_\infty = f$ μ_X-a.e. in $X \setminus \Omega$; that is, $h_\infty \in \mathcal{K}_f$. The above argument also shows that $h_k \to h_\infty$ in $L^2_{loc}(X)$ (and indeed, $h_k - h_\infty \to 0$ in $L^2(X)$ as Ω is bounded). Finally, by the lower semicontinuity and the bilinearity of $\mathcal{E}_\theta$, we see that

$$I \leq \mathcal{E}_\theta(h_\infty, h_\infty) \leq \liminf_k \mathcal{E}_\theta(h_k, h_k) = I,$$

and so h_∞ is the desired solution. $\qquad \square$

Proof of Theorem 1.2. Let f be as in the statement. Then $T\Pi_a f = f$ on X where $\Pi_a f$ is as in (2.15). We will denote the reflection of $\Pi_a f$ along X by setting $\Pi_a f(x, -y) := \Pi_a f(x, y)$ for $y > 0$. Then, by Theorem 6.1, we know that $\Pi_a f$ is Cheeger harmonic in U_Ω and therefore is a quasiminimizer in the sense of [27]. Hence $\Pi_a f$ is locally Hölder continuous on U_Ω and hence necessarily also on $\Omega \times \{0\}$. As $T\Pi_a f = f$, it follows that f is locally Hölder continuous on Ω. $\qquad \square$

Now we are ready to prove the final main theorem of the paper, Theorem 1.3.

Proof of Theorem 1.3. We know from Theorem 1.1 that given $f \in B_{2,2}^\theta(X) \cap L^2(X)$ that there is a function $f_0 \in B_{2,2}^\theta(X) \cap L^2(X)$ such that $f_0 = f$ in $X \setminus \Omega$ and f_0 is a solution to the problem (1.2). So let f simply denote such a solution.

Consider Hf, the solution to the Dirichlet problem on U_Ω with boundary data f, as constructed in Theorem 5.2, and the function $\Pi_a f$ on Z as in Theorem 6.1 above. Then both Hf and $\Pi_a f$ are Cheeger 2-harmonic in U_Ω with trace f on $\partial U_\Omega = (X \setminus \Omega) \times \{0\}$, and so by the uniqueness of such solution as shown in the proof of Theorem 5.2, we obtain that $\Pi_a f = Hf$, and so Hf is the unique solution to the Dirichlet problem studied.

The maximum principle follows immediately if $\text{esssup}_{w \in X \setminus \Omega} f(w) = \infty$, and so we may assume without loss of generality that this supremum is finite. Setting

$$M = \text{esssup}_{w \in X \setminus \Omega} f(w),$$

the Markov property of $\mathcal{E}_\theta$ implies that $\mathcal{E}_\theta(\min\{f, M\}, \min\{f, M\}) \leq \mathcal{E}_\theta(f, f)$ with $\min\{f, M\} = f$ μ_X-a.e. on $X \setminus \Omega$; thus $\min\{f, M\}$ is a solution. By the uniqueness of the solution, it follows that $\min\{f, M\} = f$ on X, thus proving the maximum principle. The strong maximum principle follows from the analogous principle for Hf in U_Ω upon noting that $(X \setminus \Omega) \times \{0\} = \partial U_\Omega$, see [27]. $\qquad \square$

The three main theorems together demonstrate the existence, uniqueness, and regularity of the solution to the Dirichlet problem related to the fractional Laplacian on X.

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