Cracking brittle thin films

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Abstract

We describe thin films as surfaces endowed with directors satisfying a non-degeneracy condition under large strain. We then consider through-the-thickness descriptors of the material microstructure taking values on a finite-dimensional, complete, differentiable, intrinsic manifold, a choice leading to general results, unifying models of different material classes because they are independent of specific microstructural geometric features. We look at brittle materials. We consider cracks as rectifiable sets supporting appropriate curvature varifolds and containing the jump sets of fields involved. The choice allows us to describe the circumstance in which along the deformation crack margins remain in contact at least partially. We prove existence of energy minima for different (constitutive) functional choices. In all of them we consider deformations as weak diffeomeorphisms modeled over the space of SBV maps satisfying an impenetrability condition.

Key words: Fracture, Varifolds, Ground States, Shells, Microstructures, Calculus of Variations

1 Introduction

In this paper we combine three topics in continuum mechanics:

- the representation of films as thin shells modeled as surfaces with directors,
- the variational description of crack nucleation,
- the multi-field (at times called phase-field to adopt a terminology more common in solid-state physics, although with more restricted meaning) depiction of microstructural effects on macroscopic motion.

The pertinent geometric and functional elements enter the energy governing elastic and elastic-brittle states. Requiring minimality of such an energy selects equilibrium configurations in a large set containing not only those free of cracks but also those involving all possible crack paths, which can be considered as appropriate rectifiable sets. Fields describing macroscopic and microstructural film morphology and a (potentially empty) crack path are pertinent unknowns.

We prove existence of fields and crack paths minimizing energy in large strain regimes with Dirichlet or Dirichlet-type (so called strong anchoring) conditions, under impenetrability constraint. The energy considered is surface polyconvex with respect to deformation gradient of film middle surface and convex in the derivative of film-thickness-section and microstructure descriptor field. It also incorporates crack surface energy including a Griffith's term, which is proportional to the crack area, and the power of a crack path generalized curvature tensor. This last term takes into account bending effects in breaking material bonds, which can be considered—in a lattice-type view at low-scale level—as springs or beams (and we opt for the second choice). A term taking into account possible tip energy (it appears prominently in cohesive scheme) completes the energy, which is modified with respect to Griffith's one, enriched in the sense that it accounts for a refined description of crack contribution to energy, being considered not just proportional to crack area. In this scheme we consider *separately* deformation and other fields describing the material morphology from the crack path; they are only constrained to have the jump set *contained* in the crack path *but not coinciding necessarily* with it.

Further details on the items listed above clarify the stage.

1.1 Thin films as surfaces with directors

In 1958, J. L. Ericksen and C. A. Truesdell III recovered the view on continuum mechanics presented systematically by the brothers E. and F. Cosserat in 1909, a view resting also on P. Duhem's 1893 proposal, and indicated it as a way to model structural elements as rods and shells as one-dimensional and two-dimensional bodies endowed with directors [26]. In that view each material element is considered as a *very small* rigid body, which can rotate independently of its neighbors. A vector field—the one of so-called *directors*—can describe such a behavior. In Ericksen-Truesdell's proposal those directors of unit length can be used to describe the behavior of cross sections. A scalar factor, multiplying such vectors, may represent variable thickness.

Such an approach opened the way to a massive body of work on shells, which is concretely difficult to present in a list of references even just with a semblance of completeness. Significant examples are references [6], [61], [62], [63], [54],

[55], and papers quoted therein. We are conscious that this list excludes many deep and highly erudite works; however, aim and scope of this introduction force us to make a choice, though incomplete.

We may look just at bending effects when we constrain the director to be coincident with the normal of the shell middle surface. Bending is a curvature effect. G. R. Kirchhoff's, T. von Karman's, P. Naghdi, W. Koiter's, and P. Ciarlet's plate and shell theories consider, in fact, shells as surfaces embedded in the three-dimensional space and endowed with different curvature-dependent energies [18], [67].

All these schemes appear as natural tools to model the mechanics of thin films, when shear and bending effect cannot be negligible in the film, be it solid or fluid [66], [65].

These models are convenient approximations, highlighting some characteristic features of the shell behavior. They naturally imply the question of their justification in terms of dimensional reduction from 3D space. A natural way to tackle the problem is via E. De Giorgi's Γ-convergence [12], [20]. According to this path, in 1999 K. Bhattacharya and R. D. James argued that if one considers a portion of an elastic cylinder between two cross sections at reciprocal finite distance, a cylinder resting over a substrate, and imagine to compute the Γ-limit of the energy (and related minimizers) as the thickness goes to zero, one may obtain the energy pertinent to a membrane or to a Cosserat's surface, depending on whether the cylinder is made of a simple elastic material or a second-grade one. In the latter case, energy dependence on the second deformation gradient generates a vector field over the surface on which we shrink the cylinder portion. This view opened some mathematical problems, which generated nontrivial analytical results [60], [31], [32].

Roughly speaking, membrane energy of different plate models emerge from separate scaling assumptions for the energy with respect to the plate thickness. This result comes from a top-to-bottom approach. On the other hand, if we reverse the view, as in reference [30], and we construct a film by superposing (in terms of energy) an atomic layer over the other, we get at continuum level a surface endowed at each point with as many directors as the number of atomic layers considered. When we consider films constituted by a mixture of atomic layers, the sequence of vectors at each point is known to within a permutation. Even in this case, however, we get the possibility to prove existence of minimizers for the elastic energy with density assumed to be surface-polyconvex in a sense a bit different from the one proposed in [19], as shown in [50]. Here, however, minimizers do not foresee cracks: the minimizing deformations are weak diffeomorphisms while directors are S^2-multi-valued Sobolev maps [50].

1.2 Crack paths as rectifiable sets supporting varifolds

A variational approach is also useful to describe elastic-brittle behavior, as suggested in 1998 by G. Francfort and J.-J. Marigo [29] (see also [11], [10], [16]), who considered A. A. Griffith's energy [38] and presumed that its minimality is a criterion to select displacements and crack paths compatible with assigned boundary conditions. Then, they approach crack evolution by partitioning the time interval and presuming also that minimality governs transitions between subsequent minimizers at discrete times. This scheme has its roots in De Giorgi's idea of minimizig movements [25].

Progressively onward in the course of cracking, two unknowns have to be determined: deformation and crack path. Energy minimization requires the selection of minimizing sequences. A problem then emerges: in general, in $3D$ space we cannot always control sequences of surfaces. A way to overcome such a difficulty is to look for minimizing deformations in the space of special bounded variation maps (SBV, see the treatise [5] for a detailed account of their properties). They admit a distributional derivative, which is a finite measure with $\mathscr{H}^{n-1}$-measurable jump set, where n is ambient space dimension and $\mathscr{H}^{n-1}$ the $n-1$ Hausdorff measure. Such a view offered the ground for advanced analytical work [24], even in the attempt to looking at the description of evolutionary behavior in terms of maps that satisfy both a stability condition and an energy inequality (see pertinent analyses in [23], [22]) involving dissipation as a version of the second law of thermodynamics, in accord to a view proposed first in 2002 by A. Mielke (see the treatise [53] and references therein). When we look at small strain regimes, instead of considering SBV maps, we can refer to SBD ones, those for which just the symmetric part of their derivative is a finite measure with $\mathscr{H}^{n-1}$-measurable jump set (see reference [21] in this case).

So doing, however, we describe just crack opening during a motion: margins move relatively and deformation fails to be continuous across the crack path. We may have, however, cases in which distant margins of a crack come back in contact without reformation of material bonds across them or, else, fracture propagates maintaining its margins in contact. To describe those circumstances, we should come back to Franctort-Marigo's idea of maintaining distinct—although connected—deformation and crack path. But, once again, we face the original problem: controlling sequences of surfaces in $3D$ space; elements of the sequence could, in fact, blow up to infinity. A way to avoid such a difficulty is to consider sequences of surfaces with bounded curvature. If we look at these sequences we can describe them through *rectifiable varifolds* [45], i.e., Radon measures on the Grassmanian constructed through the tangent hyperplanes to $\mathscr{H}^{n-1}$-rectifiable sets. They are weaker entities with respect to locally rectifiable currents; as such they do not admit orientation but one can

define a generalized notion of *curvature* tensor for the set supporting such a measure. Looking at cracks in this sense (as proposed first in references [36], [47], [35]) implies the introduction of an energy which is more general than Griffith's one because it includes curvature terms. They account for bending effects in breaking material bonds (for them see pertinent remarks in reference [64]). The analysis of minimizers [36] shows crack paths which include the jump set of the minimizing deformation. Also, in $3D$ space, line defects beyond the crack front appear as a consequence of minimization analysis; in the case of crystalline materials they are dislocations nucleated in front of the crack tip.

The generalized curvature tensor appearing in the extension of Griffith's energy proposed by M. Giaquinta, P. M. Mariano, G. Modica, and D. Mucci (GM^3-*energy*) is a second-grade term; as such it is a form of regularization emerging from microstructural latent effects.

1.3 Microstructures and thin films

Thin films show microstructures emerging from their thinness. Tents and tunnels appear in film deposition [39] on a substrate as a consequence of martensite-austenite transitions [9], [43], [44], [41]. In polycristalline thin films, there may be grain growth larger than the film thickness in a way that grains traverse the film. In this case film surface and grain boundary energies become comparable. Their interplay may pin the boundaries against further migration, or may enhance them [33]. However, the presence of further active material microstructure may influence drastically the film behavior, as in the presence of magnetization [8]. Resting on the general model building framework for the mechanics of those bodies that we call *complex* to remind the essential influence of active microstructural effects [13], [14], [46], [48], [49], [28], we may construct a general theory describing the influence of active microstructures on thin film equilibrium configurations and their possible cracked states. To account directly for active microstructures, in representing the body morphology one embeds a body (intended as an abstract set of material elements) not only in the physical point space, identifying it with the fit region it occupies, but also into an abstract finite-dimensional manifold $\mathscr{M}$—the so-called *manifold of microstructural shapes*—which is taken here to be Riemannian and complete. Formally, on the fit region $\mathscr{B}$, occupied by the body in physical space, we define a mapping from $\mathscr{B}$ to $\mathscr{M}$, which attributes to every point in $\mathscr{B}$ a descriptor $\nu \in \mathscr{M}$ of some characteristic microstructural features. We consider ν as an *observable* entity, not merely a non-observable internal variable measuring just the removal from thermodynamic equilibrium. As an observable entity, the descriptor ν of contributes to the analysis of ground states.

The picture of shells as surfaces with directors not necessarily coinciding with the normal vector field falls within the above mentioned general scheme. When we consider such a kind of structure as made of a complex material, we have another descriptor for it, which is additional to the vector representing through-the-thickness behavior. This is the scheme that we investigate here. For it, we include the varifold-based description of cracks and propose an appropriate version of GM^3-energy, by analyzing existence of its minimizers, as already declared at the beginning of this introduction.

The multi-field approach to the mechanics of complex materials is also a way to approximately describe fractures. In fact, we can choose ν to be as a scalar damage indicator, which localizes in a small neighborhood of the crack path [52], [11]. Phenomena of localization, in fact, appear also for vector choices of ν and even in linear elasticity setting, as harbingers (or precursors) of fracture [51]. In linearized setting, also, if one considers a crack to be coincident with the jump set of SBD functions, it is possible to prove rigorously that the energy can be approximated in terms of Γ-convergence by a sequence of energies for complex materials, by choosing—as above—ν as a damage indicator [16], even with the constraint of normal jump non-negativeness. An open question is whether we can approximate GM^3-*energy* with the Γ-limit of some energies involving descriptors of the material morphology (phase-fields to adopt a common syntagma to the physics of second-order phase transitions, although a bit less general in its meaning). A conjecture could be that an energy including the second-gradient of ν should be involved, but the question is still unexplored.

1.4 The main energy under investigation

With progressive extensions, in this paper we end up to the following energy functional:

$$\begin{aligned}\mathscr{F}(u,\zeta,V,\nu) := \int_\Omega (\tilde{e}(x,u,\zeta,\nu,\nabla u) + \beta_1|\nabla\zeta|^q + f(x,\nu,\nabla\nu))\,dx \\ + \gamma\|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p\,dV + \alpha\|\partial V\|\,,\end{aligned} \tag{1.1}$$

with Ω a domain in the plane, ζ representing an out-of-plane vector field, ν the morphological descriptor of further material microstructure, $\tilde{e}$ a surface polyconvex function with respect to ∇u, f a non-negative, continuous, and quasiconvex $W^{1,s}(\Omega,\mathscr{M})$ function, V a one-dimensional varifold with mass $\|V\|$ (the area term in Griffith's energy) and generalized curvature A. The boundary mass term $\|\partial V\|$ accounts for possible tip energy. The density $\tilde{e}$ is polyconvex in ∇u and continuous in the remaining variables. For it we prove existence of minimizers in terms of deformations u, vector fields ζ, descriptors ν, varifold V.

Specifically, we take u in the space of weak diffeomorphisms constructed on SBV maps—a space defined in [35]—which satisfy an impenetrability condition. Also, ν is taken as a Sobolev map taking values on an intrinsic, finite-dimensional, complete, differentiable manifold. When ν is itself in SBV with values in the same intrinsic manifold, in the list of energy entries, we replace the distributional derivative of ν with its total variation (or seminorm).

2 Background material

2.1 Special functions with bounded variation

A summable function $u \in L^1(\Omega)$, defined on a bounded domain $\Omega \subset \mathbb{R}^2$, is said to be *of bounded variation* if the distributional derivative Du is a finite measure in Ω. In this case, the function u is approximately differentiable $\mathscr{L}^2$-a.e. in Ω, and the approximate gradient ∇u agrees with the density of Radon-Nikodym's derivative of Du with respect to the Lebesgue measure $\mathscr{L}^2$. Therefore, the decomposition $Du = \nabla u\,\mathscr{L}^2 + D^s u$ holds true, where the component $D^s u$ is singular with respect to $\mathscr{L}^2$. Also, the *jump set* $S(u)$ of u is a countably 1-rectifiable subset of Ω that agrees $\mathscr{H}^1$-essentially with the complement of u Lebesgue's set. If, in addition, the singular component $D^s u$ is concentrated on the Jump set $S(u)$, we say that u is a *special function of bounded variation*, and write in short $u \in SBV(\Omega)$.

A vector valued function $u : \Omega \to \mathbb{R}^3$ belongs to the class $SBV(\Omega, \mathbb{R}^3)$ if the components u^j are in $SBV(\Omega)$. By denoting by $\mathscr{H}^k$ the k-dimensional Hausdorff measure, one has:

$$|Du|(B) = \int_B |\nabla u|\, dx + \mathscr{H}^1(B \cap S(u))$$

for each Borel set $B \subset \Omega$, where the approximate gradient $\nabla u \in L^1(\Omega, \mathbb{M}^{3\times 2})$ and the jump set $S(u)$ is defined as in the scalar case, or componentwise. (The treatise [5] offers an accurate analysis of SBV functions properties.)

2.2 Currents carried by approximately differentiable maps

Let Ω be a bounded domain in $\mathbb{R}^2$. For $u : \Omega \to \mathbb{R}^3$ an a.e. approximately differentiable map, we denote by ∇u its approximate gradient. The map u has a Lusin representative on the subset $\widetilde{\Omega}$ of Lebesgue points pertaining to both u and ∇u. Also, we have $\mathscr{L}^2(\Omega \setminus \widetilde{\Omega}) = 0$. We shall thus denote $M(F) :=$

$(F, \mathrm{adj}_2 F) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$, where $\mathrm{adj}_2 F$ is the 3-vector given by 2×2 minors of the matrix $F \in \mathbb{M}^{3\times 2}$. We say that $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ if $\nabla u \in L^1(\Omega, \mathbb{M}^{3\times 2})$ and the 3-vector of the adjoints $\mathrm{adj}_2 \nabla u \in L^1(\Omega, \mathbb{R}^3)$.

The *graph* of a map $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is defined by

$$\mathscr{G}_u := \Big\{ (x, y) \in \Omega \times \mathbb{R}^2 \mid x \in \widetilde{\Omega},\ y = \widetilde{u}(x) \Big\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u. It turns out that $\mathscr{G}_u$ is a countably 2-rectifiable set of $\Omega \times \mathbb{R}^3$, with $\mathscr{H}^2(\mathscr{G}_u) < \infty$. The approximate tangent plane at $(x, u(x))$ is generated by the vectors $\mathbf{t}_1(x) = (1, 0, \partial_1 u(x))$ and $\mathbf{t}_2 = (0, 1, \partial_2 u(x))$, where the partial derivatives are the column vectors of the gradient ∇u, and we take $\nabla u(x)$ as the Lebesgue value of ∇u at $x \in \widetilde{\Omega}$. Therefore, the 2-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)}{|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)|}$$

provides an orientation to the graph $\mathscr{G}_u$.

As a consequence, by integrating compactly supported smooth 2-forms $\omega \in \mathscr{D}^2(\Omega \times \mathbb{R}^3)$ on $\mathscr{G}_u$, one defines the current G_u carried by the graph of u, namely

$$\langle G_u, \omega \rangle := \int_{\mathscr{G}_u} \langle \omega, \xi \rangle \, d\mathscr{H}^2$$

and it turns out that G_u is an *integer multiplicity* (in short i.m.) *rectifiable current* in $\mathscr{R}_2(\Omega \times \mathbb{R}^3)$, with finite mass $\mathbf{M}(G_u)$ equal to the area $\mathscr{H}^2(\mathscr{G}_u)$ pertaining to the graph of u. In fact, since the Jacobian of $x \mapsto (x, u(x))$, which is the graph map, is equal to $|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)|$, by the area formula we get

$$\mathbf{M}(G_u) = \mathscr{H}^2(\mathscr{G}_u) = \int_\Omega \sqrt{1 + |\nabla u|^2 + |\mathrm{adj}_2 \nabla u|^2}\, dx < \infty \ .$$

By duality the *boundary* of G_u is the 1-current ∂G_u acting on $\mathscr{D}^1(\Omega \times \mathbb{R}^3)$, the space of compactly supported smooth 1-forms η in $\Omega \times \mathbb{R}^3$, as

$$\langle \partial G_u, \eta \rangle := \langle G_u, d\eta \rangle, \qquad \eta \in \mathscr{D}^1(\Omega \times \mathbb{R}^3)\,,$$

where $d\eta$ is the differential of η. By Stokes theorem we get $\partial G_u = 0$ on $\mathscr{D}^1(\Omega \times \mathbb{R}^3)$ if u is of class C^2. Such a relation holds true also for Sobolev maps $u \in W^{1,2}(\Omega, \mathbb{R}^3)$, by approximation. However, in general, the boundary ∂G_u does not vanish and may not have finite mass in $\Omega \times \mathbb{R}^3$. On the other hand, if ∂G_u has finite mass, the boundary rectifiability theorem states that ∂G_u is an i.m. rectifiable current in $\mathscr{R}_1(\Omega \times \mathbb{R}^3)$. An extended treatment of currents is in the two-volume treatise [37].

A general 1-varifold in Ω, a domain in $\mathbb{R}^2$, is a non-negative Radon measure on the trivial bundle $\mathscr{G}_1(\Omega) := \Omega \times \mathscr{G}_{1,2}$, where $\mathscr{G}_{1,2}$ is the Grassmanian manifold of 1-planes Π through the origin in $\mathbb{R}^2$.

If $\mathfrak{b}$ is a 1-rectifiable subset of $\Omega \subset \mathbb{R}^2$, for $\mathscr{H}^1 \llcorner \mathfrak{b}$ a.e. $x \in \Omega$ there exists the approximate tangent 1-space $T_x\mathfrak{b}$ to $\mathfrak{b}$ at x. We thus denote by $\Pi(x)$ the 2×2 matrix that identifies the orthogonal projection of $\mathbb{R}^2$ onto $T_x\mathfrak{b}$.

We define

$$V_{\mathfrak{b},\theta}(\varphi) := \int_{\mathscr{G}_1(\Omega)} \varphi(x,\Pi)\, dV_{\mathfrak{b},\theta}(x,\Pi) := \int_{\mathfrak{b}} \theta(x)\varphi(x,\Pi(x))\, d\mathscr{H}^1(x) \qquad (2.1)$$

for any $\varphi \in C^0_c(\mathscr{G}_1(\Omega))$, where $\theta \in L^1(\mathfrak{b}, \mathscr{H}^1)$ is a nonnegative density function. If θ is integer valued, then $V = V_{\mathfrak{b},\theta}$ is said to be the *integer rectifiable varifold* associated with $(\mathfrak{b}, \theta, \mathscr{H}^1)$.

The *weight measure* of V is the Radon measure in Ω given by $\mu_V := \pi_{\#}V$, where $\pi : \mathscr{G}_1(\Omega) \to \Omega$ is the canonical projection. Therefore, one has $\mu_V = \theta\, \mathscr{H}^1 \llcorner \mathfrak{b}$, and the *mass* of V is

$$\|V\| := V(\mathscr{G}_1(\Omega)) = \mu_V(\Omega) = \int_{\mathfrak{b}} \theta\, d\mathscr{H}^1 .$$

Definition 2.1 *An integer rectifiable* 1*-varifold* $V = V_{\mathfrak{b},\theta}$ *is called a* curvature 1-varifold with boundary *if there exist a function* $A \in L^1(\mathscr{G}_1(\Omega), \mathbb{R}^* \otimes \mathbb{R} \otimes \mathbb{R}^*)$, $A = (A^{\ell i}_j)$, *and a* $\mathbb{R}^2$*-valued measure* ∂V *in* $\mathscr{G}_1(\Omega)$ *with finite mass* $\|\partial V\|$, *such that*

$$\int_{\mathscr{G}_1(\Omega)} (\Pi D_x\varphi + AD_\Pi\varphi + \varphi\ {}^t\mathrm{tr}\,(AI))\, dV(x,\Pi) = -\int_{\mathscr{G}_1(\Omega)} \varphi\, d\partial V(x,\Pi)$$

for every $\varphi \in C^\infty_c(\mathscr{G}_1(\Omega))$. *Moreover, for* $p \geq 1$ *the subclass of curvature* 1*-varifolds with boundary such that* $A \in L^p(\mathscr{G}_1(\Omega))$ *is indicated by* $CV^p_k(\Omega)$.

Varifolds in $CV^p_1(\mathscr{B})$ have generalized curvature in L^p [45]. Therefore, Allard's compactness theorem applies (see [1], [2], but also [3] as an essential reference on varifolds):

Theorem 2.1 *For* $1 < p < \infty$, *let* $\{V^{(h)}\} \subset CV^p_1(\Omega)$ *be a sequence of curvature* 1*-varifolds* $V^{(h)} = V_{\mathfrak{b}_h,\theta_h}$ *with boundary. The corresponding curvatures and boundaries are indicated by* $A^{(h)}$ *and* $\partial V^{(h)}$, *respectively. Assume that there*

exists a constant $c > 0$ such that for every h

$$\mu_{V^{(h)}}(\Omega) + \|\partial V^{(h)}\| + \int_{\mathscr{G}_1(\Omega)} |A^{(h)}|^p \, dV^{(h)} \leq c.$$

Then, there exists a (not relabeled) subsequence of $\{V^{(h)}\}$ and a 1-varifold $V = V_{\mathfrak{b},\theta} \in CV_1^p(\Omega)$, with curvature A and boundary ∂V, such that

$$V^{(h)} \rightharpoonup V, \quad A^{(h)} \, dV^{(h)} \rightharpoonup A \, dV, \qquad \partial V^{(h)} \rightharpoonup \partial V,$$

in the sense of measures. Moreover, for any convex and l.s.c. function $f : \mathbb{R}^* \otimes \mathbb{R} \otimes \mathbb{R}^* \to [0, +\infty]$, one gets

$$\int_{\mathscr{G}_1(\Omega)} f(A) \, dV \leq \liminf_{h\to\infty} \int_{\mathscr{G}_1(\Omega)} f(A^{(h)}) \, dV^{(h)}.$$

3 A first model

We look first to the film middle surface and take for it a planar reference configuration that is a two-dimensional smooth domain Ω in $\mathbb{R}^2$, where Cartesian coordinates $x = (x_1, x_2)$ are fixed. A map $u : \Omega \to \mathbb{R}^3$, say $u = (u^1, u^2, u^3)$, represents a *deformation*.

When smoothness and non-singularity are assured, the map u determines an immersion of Ω into $\mathbb{R}^3$; the tangent plane to the deformed film middle plane does not degenerate. Formally, it is tantamount to impose $|\mathrm{adj}_2 \nabla u(x)| > 0$ for any $x \in \Omega$. In other words, if $\partial_i u$ denotes a column vectors of the gradient matrix ∇u, we are imposing that the vector product $\partial_1 u \times \partial_2 u$ does not vanish at every point. Therefore, the normal to $u(\Omega)$, the deformed middle surface, is the unit vector

$$\mathbf{n}(x) = \frac{\partial_1 u(x) \times \partial_2 u(x)}{|\partial_1 u(x) \times \partial_2 u(x)|}\,. \tag{3.1}$$

It can be considered as a descriptor of out-of-middle-surface film behavior. However, such an information can be carried out by an $\mathbb{S}^2$-valued vector field $x \mapsto \zeta(x)$ defined over Ω and constrained to be at every $x \in \Omega$ such that

$$(\partial_1 u(x) \times \partial_2 u(x)) \bullet \zeta(x) > 0 \tag{3.2}$$

where $\bullet$ is the scalar product in $\mathbb{R}^3$ ([61], [26], [63], [62], [6] and references therein). When shear is absent, we get $\zeta(x) = \mathbf{n}(x)$.

We consider u to be a special function with bounded variation, with jump set *coinciding* with a crack. For the moment we avoid considering deformations along which the film breaks but, at least partially, crack margins remain in contact. Precisely, we assume that the components u^j of u are L^∞-functions

in the class $SBV(\Omega)$. Since the unit vector ζ does not make sense along the discontinuity set of the deformation map u, we shall assume that $\zeta : \Omega \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a special function of bounded variation in $SBV(\Omega, \mathbb{R}^3)$, with $|\zeta| = 1$ almost everywhere. Furthermore, we assume that the Jump set of ζ is contained in the jump set of u, namely

$$S(\zeta) \subseteq S(u)\,.$$

We assume that the constraint (3.2) holds true $\mathscr{L}^2$-a.e. in $\Omega \setminus S(u)$. For analytical reasons, the above inclusion is replaced by the condition:

$$\mathscr{H}^1 \llcorner S(\zeta) \leq \mathscr{H}^1 \llcorner S(u)\,. \tag{3.3}$$

3.1 A first minimization problem

For a given exponent $q > 1$, consider a model energy functional of the type

$$\mathscr{G}(u, \zeta) := \int_\Omega \Big(|\nabla u|^q + |\nabla \zeta|^q + \Phi(\nabla u, \zeta) \Big)\, dx + \mathscr{H}^1(S(u))$$

acting on couples of SBV-functions (u, ζ) so that the constraint (3.2) holds true $\mathscr{L}^2$-a.e. on $\Omega \setminus S(u)$ and (3.3) is satisfied. Assume also the validity of a uniform bound $\|u\|_{L^\infty(\Omega)} \leq K$ under a Dirichlet condition on u and a clamping one for ζ at the boundary $\partial\Omega$ of Ω.

As a matter of notation, according to reference [19], we define the set

$$\widetilde{Y} := \{(F, \zeta) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3 \mid \det(F|\zeta) > 0\}$$

where $(F|\zeta) \in \mathbb{M}^{3\times 3}$. Therefore, according to (3.2), if $F = \nabla u$ one has

$$\det(\nabla u|\zeta) = (\partial_1 u \times \partial_2 u) \bullet \zeta\,.$$

Definition 3.1 *We shall denote by $\widetilde{\mathscr{F}}$ the class of non-negative functions $\Phi : \widetilde{Y} \to [0, +\infty)$ such that*

(1) $\Phi(F, \cdot)$ is continuous in $\mathbb{R}^3$ for all $F \in \mathbb{M}^{3\times 2}$;
(2) $\Phi(\cdot, \zeta)$ is polyconvex (precisely, $\Phi(\cdot, \zeta)$ is the restriction to $\{F \in \mathbb{M}^{3\times 2} \mid \det(F|\zeta) > 0\}$ of a polyconvex function $g : \mathbb{M}^{3\times 2} \to \mathbb{R}$) for all $\zeta \in \mathbb{R}^3$;
(3) $\Phi(F, \zeta) \to +\infty$ if $\det(F|\zeta) \to 0^+$.

We assume that the integrand Φ in $\mathscr{G}(u, \zeta)$ be in the class $\widetilde{\mathscr{F}}$. An example is given by $\Phi(F, \zeta) = -\log(\det(F|\zeta)) \vee 0$.

Theorem 3.1 *With the previous assumptions, minima of $\mathscr{G}(u,\zeta)$ exist among couples in $SBV(\Omega,\mathbb{R}^3)$, with $\|u\|_\infty \le K$ and $|\zeta| = 1$. They satisfy the boundary conditions already described and are such that* (3.2) *holds true $\mathscr{L}^2$-a.e. on $\Omega \setminus S(u)$.*

Proof. Taking a minimizing sequence $\{(u_h,\zeta_h)\}$, and possibly passing to not relabeled subsequences, by compactness we infer that $u_h \to u$ in $L^1(\Omega,\mathbb{R}^3)$, for some function $u \in SBV(\Omega,\mathbb{R}^3)$ satisfying $\|u\|_{L^\infty(\Omega)} \le K$. Moreover, ∇u_h weakly converges to ∇u in $L^q(\Omega,\mathbb{M}^{3\times 2})$ and $\mathscr{H}^1 \llcorner S(u_h)$ weakly converges in Ω to a measure $\widetilde{\mu}$ greater than $\mathscr{H}^1 \llcorner S(u)$, whence we get the lower semicontinuity inequalities

$$\int_\Omega |\nabla u|^q\,dx \le \liminf_{h\to\infty} \int_\Omega |\nabla u|^q\,dx \quad \text{and} \quad \mathscr{H}^1(S(u)) \le \liminf_{h\to\infty} \mathscr{H}^1(S(u_h))\,.$$

Moreover, since

$$\mathscr{H}^1 \llcorner S(\zeta_h) \le \mathscr{H}^1 \llcorner S(u_h) \qquad \forall\, h, \tag{3.4}$$

we similarly obtain the existence of a function $\zeta \in SBV(\Omega,\mathbb{R}^3)$ satisfying $|\zeta| = 1$ a.e. in Ω, such that $\zeta_h \to \zeta$ in $L^1(\Omega,\mathbb{R}^3)$, $\nabla\zeta_h$ weakly converges to $\nabla\zeta$ in $L^q(\Omega,\mathbb{M}^{3\times 2})$, and $\mathscr{H}^1 \llcorner S(\zeta_h)$ weakly converges in Ω to a measure greater than $\mathscr{H}^1 \llcorner S(\zeta)$, so that

$$\int_\Omega |\nabla\zeta|^q\,dx \le \liminf_{h\to\infty} \int_\Omega |\nabla\zeta|^q\,dx\,.$$

Condition

$$(\partial_1 u_h(x) \times \partial_2 u_h(x)) \bullet \zeta_h(x) > 0 \quad \forall\, h,$$

which is assumed to hold true $\mathscr{L}^2$-a.e. in $\Omega\setminus S(u_h)$, is preserved (with possibly the equality sign instead of $>$) when passing to the limit $\nabla u_h \to \nabla u$ of the approximate gradients by a.e. convergence, and by the strong L^1-convergence of ζ_h to ζ. We also have

$$\int_\Omega \Phi(\nabla u,\zeta)\,dx \le \liminf_{h\to\infty} \int_\Omega \Phi(\nabla u_h,\zeta_h)\,dx < \infty\,,$$

whence strict equality holds true in (3.2) for $\mathscr{L}^2$- a.e. $x \in \Omega \setminus S(u)$. ■

In principle, condition (3.4) fails to be satisfied by the limit maps (u,ζ). In fact, it may happen that the Jump of u_h on $S(u_h)$ goes to zero somewhere, when passing to the limit through the compactness theorem.

In order to preserve the inequality (3.3), we need a condition ensuring that the jump $u_h^+ - u_h^-$ on $S(u_h)$ cannot "decrease" or "disappear", so that $\mathscr{H}^1(S(u_h)) \to \mathscr{H}^1(S(u))$. We shall return to this feature in the next section.

Also, if (3.3) holds true, it may happen that the deformed surface $u(\Omega)$ has a crease. This is described, e.g., when the unit normal $\mathbf{n}(x)$ is smooth outside

a 1-rectifiable set $\mathfrak{J}$ of Ω. In this case, ζ satisfies the condition $\zeta(x) \bullet \mathbf{m} > 0$ at $x \in \mathfrak{J}$, for some unit vector $\mathbf{m}$ that lies in the cone between the one-sided limits of $\mathbf{n}$ at $x \in \mathfrak{J}$.

Therefore, a second order theory (yielding e.g. to a regularity of the unit normal $\mathbf{n}(x)$ in the BV or SBV sense) should be applied in order to describe in a precise way the above mentioned angle condition along the creases of the deformed surface.

4 Cracks beyond jump sets

There are circumstances where, although a crack occurs along the deformation, its margins remain in contact at least partially. Consequently, the map u may not jump across the contact, although material bonds are there broken. The following definition opens the way to a pertinent analysis.

Definition 4.1 *A macroscopic configuration of a cracked membrane is a pair composed by the bounded connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and a curvature 1-varifold with boundary, namely $V = V_{\mathfrak{b},\theta} \in CV_1^p(\Omega)$ for some $p > 1$.*

As before, we consider bounded deformation maps u in $SBV(\Omega, \mathbb{R}^3)$ and we assume, in addition, that $|\mathrm{adj}_2 \nabla u| \in L^1(\Omega)$, so that the current G_u carried by the rectifiable graph of u is i.m. rectifiable in $\mathscr{R}_2(\Omega \times \mathbb{R}^3)$ and with finite mass, $\mathbf{M}(G_u) < \infty$.

In general, the boundary current $\partial G_u \in \mathscr{D}_1(\Omega \times \mathbb{R}^3)$ is non zero and it may also have unbounded mass. Therefore, we assume in addition that fractures and holes in the graph of u are controlled by the crack V, namely

$$\pi_{\#}|\partial G_u| \leq \mu_V \tag{4.1}$$

where μ_V is the weight measure in Ω of the varifold V, $\pi : \Omega \times \mathbb{R}^3 \to \Omega$ the projection onto the first two coordinates, and $|\cdot|$ the total variation of the vector-valued measure ∂G_u, so that $\pi_{\#}|\partial G_u|(B) = |\partial G_u|(B \times \mathbb{R}^3)$ for each Borel set $B \subset \Omega$.

If $\mu_V(\Omega) < \infty$ and the bound (4.1) holds true, $|\partial G_u|$ is a finite measure, and actually $\mathbf{M}(\partial G_u) < \infty$. Therefore, the boundary rectifiability theorem yields that ∂G_u is an i.m. rectifiable current in $\mathscr{R}_1(\Omega \times \mathbb{R}^3)$ and hence u is a special function of bounded variation, $u \in SBV(\Omega, \mathbb{R}^3)$, with jump set contained in the support $\mathfrak{b}$ of V. Actually, u belongs to the class SBV_0 discussed in reference [5].

In particular, recalling that $\mu_V = \theta\,\mathscr{H}^1 \llcorner \mathfrak{b}$, where the positive multiplicity function θ is integer-valued, we get

$$\mathscr{H}^1 \llcorner S(u) \le \mathscr{H}^1 \llcorner \mathfrak{b}\ .$$

We have already defined $M(F) = (F, \mathrm{adj}_2 F) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$. Here we assume that a sequence $\{u_h\}_h \subset \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is such that

$$\sup_h \|u_h\|_{L^\infty(\Omega)} \le K\,, \quad \sup_h \|M(\nabla u_h)\|_{L^1(\Omega)} < \infty$$

where the sequence $\{|M(\nabla u_h)|\}_h$ is equi-integrable on Ω. In addition, assume that $\pi_\# |\partial G_{u_h}| \le \mu_{V_h}$, where $V_h \rightharpoonup V$ in the sense of varifolds. If we have $\sup_h \mu_{V_h}(\Omega) < \infty$, by compactness, we find a deformation map u as above, such that G_u is i.m. rectifiable and with finite mass, $G_{u_h} \rightharpoonup G_u$ weakly as currents, $\pi_\# |\partial G_u| \le \mu_V < \infty$, whence ∂G_u is i.m. rectifiable, too.

In order to recover the weak L^1 convergence of $M(\nabla u_h)$ to $M(\nabla u)$, the starting point is the following special case of the closure theorem proven in reference [34]. On account of the compactness theorem in SBV, it extends a classical result proved in reference [7] for Sobolev maps, where the divergence form of gradient minors is exploited.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\{u_h\}$ a sequence of functions from $SBV(\Omega, \mathbb{R}^3)$ converging in $L^1(\Omega, \mathbb{R}^3)$ to a summable function $u : \Omega \to \mathbb{R}^3$. Assume that for some real exponents $q > 2$ and $r > 1$*

$$\sup_h \left\{ \|u_h\|_\infty + \int_\Omega \left(|\nabla u_h|^q + |adj_2 \nabla u_h|^r \right) dx + \mathscr{H}^1(S(u_h)) \right\} < \infty\,.$$

Then, $u \in SBV(\Omega, \mathbb{R}^3)$, the sequence $\mathscr{H}^1 \llcorner S(u_h)$ weakly converges in Ω to a measure μ greater than $\mathscr{H}^1 \llcorner S(u)$, and $\{M(\nabla u_h)\}$ converges to $M(\nabla u)$ weakly in $L^1(\Omega, \mathbb{M}^{3\times 2} \times \mathbb{R}^3)$.

We thus fix $q > 2$, $r, p > 1$, $K > 0$, and introduce the class $\mathscr{A}_{q,r,p,K}$ of triplets (u, ζ, V), where:

- (i) $u : \Omega \to \mathbb{R}^3$ is a special function with bounded variation, $u \in SBV(\Omega, \mathbb{R}^3)$, with $\mathscr{H}^1(S(u)) < \infty$, such that $\|u\|_{L^\infty(\Omega)} \le K$, $\partial_1 u \times \partial_2 u \neq 0$ $\mathscr{L}^2$-a.e. on Ω, $\nabla u \in L^q(\Omega, \mathbb{M}^{3\times 2})$, and $|adj_2 \nabla u| \in L^r(\Omega)$;
- (ii) $\zeta : \Omega \to \mathbb{R}^3$ is a special function with bounded variation, with $\mathscr{H}^1(S(\zeta)) < \infty$, such that $|\zeta| = 1$ $\mathscr{L}^2$-a.e. on Ω, and $\nabla \zeta \in L^q(\Omega, \mathbb{M}^{3\times 2})$;
- (iii) $V \in CV_1^p(\Omega)$, i.e. V is a integer rectifiable curvature 1-varifold with boundary and second fundamental form $A \in L^p(\mathscr{G}_1(\Omega), \mathbb{R}^* \otimes \mathbb{R} \otimes \mathbb{R}^*)$, with $\mu_V = \theta \mathscr{H}^1 \llcorner \mathfrak{b}$;
- (iv) $\pi_\# |\partial G_u| \le \mu_V$;
- (v) $\mathscr{H}^1 \llcorner S(\zeta) \le \pi_\# |\partial G_u|$;
- (vi) $(\partial_1 u(x) \times \partial_2 u(x)) \bullet \zeta(x) > 0$ for $\mathscr{L}^2$-a.e. $x \in \Omega \setminus \mathfrak{b}$.

Remark 4.1 *Concerning property (*v*), previous assumptions imply that*

$$\mathscr{H}^1 \llcorner S(\zeta) \leq \mathscr{H}^1 \llcorner \mathfrak{b} \ .$$

We then consider on the class of triplets $(u, \zeta, V) \in \mathscr{A}_{q,r,p,K}$ an energy functional of the type

$$\mathscr{F}(u, \zeta, V) := \int_\Omega e(x, u, \zeta, \nabla u, \nabla \zeta)\, dx + \|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p \, dV + \|\partial V\| \,. \quad (4.2)$$

The energy density $e : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \to \mathbb{R} \cup \{+\infty\}$ is the sum of two terms, namely

$$e(x, u, \zeta, F, G) = \widetilde{e}(x, u, \zeta, F) + \beta_1 |G|^q,$$

for every $(x, u, \zeta, F, G) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2}$, where $\beta_1 \geq 0$ and the first addendum is a non-negative Carathéodory function satisfying the following properties:

(a) $\widetilde{e}(x, u, \zeta, F)$ is polyconvex with respect to F, namely

$$\widetilde{e}(x, u, \zeta, F) = g(x, u, \zeta, M(F)) \qquad \forall F \in \mathbb{M}^{3\times 2},$$

where $g : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times (\mathbb{M}^{3\times 2} \times \mathbb{R}^3) \to [0, +\infty]$ is a Carathéodory function, with $g(x, u, \zeta, \cdot)$ convex for $\mathscr{L}^2$-a.e. $x \in \Omega$, and for all $(u, \zeta) \in \mathbb{R}^3 \times \mathbb{R}^3$;

(b) for $\mathscr{L}^2$-a.e. $x \in \Omega$ and every (u, ζ, F) such that $\det(F|\zeta) > 0$,

$$\widetilde{e}(x, u, \zeta, F) \geq \beta_2 \left(|F|^q + |adj_2 F|^r\right) + \Phi(F, \zeta),$$

where $\beta_2 > 0$ and $\Phi \in \widetilde{\mathscr{F}}$, see Definition 3.1.

As a Dirichlet condition on the triplets (u, ζ, V) in $\mathscr{A}_{q,r,p,K}$, we assume that

$$u = u_0 \,, \quad \zeta = \frac{\partial_1 u_0 \times \partial_2 u_0}{|\partial_1 u_0 \times \partial_2 u_0|} \quad \text{on } \partial\Omega$$

in the sense of traces, for some given function $u_0 \in W^{1,q}(\Omega, \mathbb{R}^3)$ with $\|u_0\|_\infty \leq K$ and $|adj_2 \nabla u_0| \in L^r(\Omega)$.

Referring to *strong anchoring conditions*, i.e., assignment of the boundary current, we assume that, in terms of smooth and bounded 1-forms in $\overline{\Omega} \times \mathbb{R}^3$, we have

$$\partial G_u \llcorner (\partial\Omega \times \mathbb{R}^3) = \partial G_{u_0} \llcorner (\partial\Omega \times \mathbb{R}^3) \,.$$

With the previous notation, the following existence theorem holds true.

Theorem 4.2 *Under prescribed Dirichlet or strong anchoring conditions, if the class of competitors (u, V, ζ) in $\mathscr{A}_{q,r,p,K}$ with finite energy is non-empty, the energy minimum of $\mathscr{F}(u, V, \zeta)$ is attained in the same class.*

Proof. Choose a minimizing sequence $\{(u_h, \zeta_h, V^{(h)}\} \subset \mathscr{A}_{q,r,p,K}$, which satisfies prescribed Dirichlet or strong anchoring conditions.

As above, on account of assumptions (iv) and (v), by the energy lower bounds the compactness theorem in SBV applies to the sequences $\{u_h\}$ and $\{\zeta_h\}$. Furthermore, Allard's compactness theorem 2.1 applies to the sequence $\{V^{(h)}\}$ in $CV^p(\Omega)$. Also, Federer-Fleming's closure theorem applies to the sequence G_{u_h} (see [37]).

On account of Theorem 4.1, possibly passing to a subsequence we obtain a triplet (u, ζ, V) satisfying properties (i)–(v). In fact, using (a), the lower-semicontinuity result in reference [34] yields

$$\int_\Omega \widetilde{e}(x, u, \nabla u, \zeta)\, dx \le \liminf_{h\to\infty} \int_\Omega \widetilde{e}(x, u_h, \nabla u_h, \zeta_h)\, dx$$

and

$$\int_\Omega |\nabla\zeta|^q\, dx \le \liminf_{h\to\infty} \int_\Omega |\nabla\zeta_h|^q\, dx,$$

whereas

$$\|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p\, dV + \|\partial V\| \le \liminf_{h\to\infty} \Big(\|V^{(h)}\| + \int_{\mathscr{G}_1(\Omega)} \|A^{(h)}\|^p\, dV^{(h)} + \|\partial V^{(h)}\| \Big),$$

so that by equation (4.2) we get

$$\mathscr{F}(u, \zeta, V) \le \liminf_{h\to\infty} \mathscr{F}(u_h, \zeta_h, V^{(h)})\,.$$

Also, the bound

$$\int_\Omega \Phi(\nabla u, \zeta)\, dx \le \liminf_{h\to\infty} \int_\Omega \Phi(\nabla u_h, \zeta_h)\, dx < +\infty$$

holds true, so that, by the lower bound (b), the property (vi) is satisfied by the triplet (u, ζ, V), whence $(u, \zeta, V) \in \mathscr{A}_{q,r,p,K}$. Both Dirichlet's and strong anchoring conditions are preserved in the limit process, so that the proof is complete. ■

Actually, we could follow the view suggested in reference [19], considering energy functionals $\mathscr{F}(u, \zeta, V)$ in which we mix together the gradients of u and ζ. Namely, in (4.2) we could assume that $e(x, u, \zeta, F, G) = \widetilde{e}(x, h, H)$, where $h := (u, \zeta)$, $H \in \mathbb{M}^{6\times 2}$ is the matrix with rows (F, G), while $\widetilde{e}(x, h, H)$ is a non negative Carathéodory function, which is polyconvex with respect to H and such that for $\mathscr{L}^2$-a.e. $x \in \Omega$ and every (h, H), with $\det(F|\zeta) > 0$

$$\widetilde{e}(x, u, \zeta, F) \ge \beta_2 \left(|H|^q + |adj_2 H|^r\right) + \Phi(F, \zeta)$$

where $\beta_2 > 0$ and $\Phi \in \widetilde{\mathscr{F}}$.

Since the closure and semicontinuity properties continue to hold, the minimum of $\mathscr{F}(u, \zeta, V)$ under Dirichlet or strong anchoring condition is attained in the class $\widetilde{\mathscr{A}}_{q,r,p,K}$ of triplets (u, ζ, V) in $\mathscr{A}_{q,r,p,K}$, which satisfy the additional condition $|adj_2 \nabla(u, \zeta)| \in L^r(\Omega)$, so that we should presume in this case what follows:

(1) $u \in SBV(\Omega, \mathbb{R}^3)$, with $\mathscr{H}^1(S(u)) < \infty$, $\|u\|_{L^\infty(\Omega)} \leq K$, $\partial_1 u \times \partial_2 u \neq 0$ $\mathscr{L}^2$-a.e. on Ω, $\nabla u \in L^q(\Omega, \mathbb{M}^{3\times 2})$;
(2) $\zeta \in SBV(\Omega, \mathbb{R}^3)$, with $\mathscr{H}^1(S(\zeta)) < \infty$, $|\zeta| = 1$ $\mathscr{L}^2$-a.e. on Ω, and $\nabla \zeta \in L^q(\Omega, \mathbb{M}^{3\times 2})$;
(3) $|adj_2 \nabla(u, \zeta)| \in L^r(\Omega)$;
(4) $V \in CV_1^p(\Omega)$, with $\mu_V = \theta \mathscr{H}^1 \llcorner \mathfrak{b}$;
(5) $\pi_\# |\partial G_u| \leq \mu_V$;
(6) $\mathscr{H}^1 \llcorner S(\zeta) \leq \pi_\# |\partial G_u|$;
(7) $(\partial_1 u(x) \times \partial_2 u(x)) \bullet \zeta(x) > 0$ for $\mathscr{L}^2$-a.e. $x \in \Omega \setminus \mathfrak{b}$.

However, we prefer to maintain separate membrane and out-of-middle-surface behavior, so that we refer to surface polyconvexity as considered in reference [50].

5 Films made of complex materials

To account for active microstructures (e.g., polarization in ferroelectric thin films) we introduce over Ω another map ν taking values on a connected, complete, n-dimensional Riemannian differentiable manifold $\mathscr{M}$ of class C^2. At every $x \in \Omega$, the map ν summarizes at gross scale geometric information on material microstructure in the film thickness. For this reason, we call $\mathscr{M} = (\mathscr{M}^n, g)$ as the *manifold of microstructural shapes*. The generality adopted in choosing $\mathscr{M}$ is a way to furnish unified results, which are independent of specific microstructural features.

First we assume the layer-descriptor map $\nu : \Omega \to \mathscr{M}$ to be an *intrinsic Sobolev map* in $W^{1,s}(\Omega, \mathscr{M})$ for some real exponent $s > 1$. Then, we'll discuss the case in which ν can be considered as a special function of bounded variation.

The essential point is that we do not use any embedding of $\mathscr{M}$ in some Euclidean space. Even choosing it to be isometric, it would not be unique (as Nash's theorems indicate). Thus, its choice would become part of the model, while the common effort is to offer a description of the phenomenological world as much as possible free of non intrinsic elements. Essentially for the same reason we commonly ask objectivity, i.e., independence under classes of

changes in observers.

5.1 The Sobolev case

Under previous assumptions, with $d_{\mathscr{M}}$ the geodesic distance in $\mathscr{M}$, by the Hopf-Rinow theorem $(\mathscr{M}, d_{\mathscr{M}})$ is a complete metric space. Consequently, we keep referring to results in such spaces, summarizing those aspects that we need for the analysis developed here from essential references [4], [56], [57], [58] on this topic.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain and $s \geq 1$. A Borel map $\nu : \Omega \to \mathscr{M}$ is said to be an intrinsic Sobolev map in $W^{1,s}(\Omega, \mathscr{M})$, if there exists a non-negative function $\phi \in L^s(\Omega)$ such that for every $\nu_0 \in \mathscr{M}$

(1) $x \mapsto d_{\mathscr{M}}(\nu(x), \nu_0)$ is in $L^s(\Omega)$, i.e., $\nu \in L^s(\Omega, \mathscr{M})$;
(2) the distributional gradient map $x \mapsto Dd_{\mathscr{M}}(\nu(x), \nu_0)$ satisfies the inequality $|Dd_{\mathscr{M}}(\nu(x), \nu_0)| \leq \phi(x)$ for $\mathscr{L}^m$-a.e. $x \in \Omega$.

In this setting, the "norm" $|D\nu(x)|$, which is, in essence, the optimal function $\phi \in L^s(\Omega)$ satisfying the inequality $|Dd_{\mathscr{M}}(\nu(x), \nu_0)| \leq \phi(x)$ is well-defined for $\mathscr{L}^m$-a.e. $x \in \Omega$, by

$$|D\nu(x)| := \sup_{k \in \mathbb{N}} d_{\mathscr{M}}(\nu_k(x), \nu_0)\,,$$

where $\{\nu_k\}_{k \in \mathbb{N}}$ form a dense and enumerable set in $\mathscr{M}$.

Weak convergence in $W^{1,s}$ of a sequence $\{\nu_h\} \subset W^{1,s}(\Omega, \mathscr{M})$ to some map $\nu \in W^{1,s}(\Omega, \mathscr{M})$, when $s > 1$, is defined by requiring that $\|d_{\mathscr{M}}(\nu_h, \nu)\|_{L^s(\Omega)} \to 0$ as $h \to \infty$ and $\sup_h \| \, |D\nu_h| \, \|_{L^s(\Omega)} < \infty$. When $s = 1$, one assumes in addition that the sequence $\{|D\nu_h(x)|\}_h$ is equi-integrable.

Also, the trace operator $\mathrm{Tr} : W^{1,s}(\Omega, \mathscr{M}) \to L^s(\partial\Omega, \mathscr{M})$ is well-defined in such a way that for continuous maps $\nu \in W^{1,s}(\Omega, \mathscr{M}) \cap C^0(\overline{\Omega}, \mathscr{M})$ it agrees with the restriction $\nu_{|\partial\Omega}$. Moreover, if $\{\nu_h\} \subset W^{1,s}(\Omega, \mathscr{M})$ weakly converges to some map $\nu \in W^{1,s}(\Omega, \mathscr{M})$, then $\mathrm{Tr}(\nu_h)$ converges to $\mathrm{Tr}(\nu)$ strongly in $L^s(\partial\Omega, \mathscr{M})$. Finally, traces of maps in $W^{1,s}(\Omega, \mathscr{M})$ have a $W^{1-1/s,s}$-regularity, when $s > 1$ (see [42] and [17]).

We endow the tangent bundle $T\mathscr{M}$ with the metric $d_{T\mathscr{M}}$ induced by $d_{\mathscr{M}}$ and consider the vector bundle with base space $\mathscr{M}$ and typical fiber the space of linear homomorphisms $\mathrm{Hom}(\mathbb{R}^m, T\mathscr{M})$. Points of such a bundle are couples (ν, N), where $\nu \in \mathscr{M}$ and $N : \mathbb{R}^m \to T_\nu\mathscr{M}$ is a linear map. For any fixed $\nu \in \mathscr{M}$, we can identify $N \in \mathrm{Hom}(\mathbb{R}^m, T_\nu\mathscr{M})$ with the m-tuple $(v_1, \ldots, v_m) \in (T_\nu\mathscr{M})^m$, where $v_i = Ne_i$ and $(e_1, \ldots, e_m)$ is the canonical basis in $\mathbb{R}^m$. Therefore, a metric structure on the vector bundle $\mathrm{Hom}(\mathbb{R}^m, T\mathscr{M})$ is

defined through the distance

$$D((\nu,N),(\widetilde{\nu},\widetilde{N})) := \Big\{\sum_{i=1}^{m} d_{T\mathscr{M}}\Big((\nu,v_i),(\widetilde{\nu},w_i)\Big)^2\Big\}^{1/2}$$

if $\widetilde{N} \in \mathrm{Hom}(\mathbb{R}^m, T_{\widetilde{\nu}}\mathscr{M})$ and $w_i = \widetilde{N}e_i$ for each i.

A non-negative and continuous integrand $f : \Omega \times \mathrm{Hom}(\mathbb{R}^m, T\mathscr{M}) \to [0,+\infty)$ is said to be *admissible* in $W^{1,s}(\Omega,\mathscr{M})$ if for some fixed point $\nu_0 \in \mathscr{M}$ and some positive constant C the bounds

$$0 \le f(x,\nu,N) \le C\left(1 + d_{\mathscr{M}}(\nu,\nu_0)^s + \|N\|^s_{g(\nu)}\right)$$

hold true for all $(x,\nu,N) \in \Omega\times\mathrm{Hom}(\mathbb{R}^m, T_\nu\mathscr{M})$, where $\|\cdot\|_{g(\nu)}$ is the operatorial norm (see [27]).

A functional

$$\nu \mapsto \int_\Omega f(x,\nu(x),d\nu_x)\,dx \tag{5.1}$$

with an admissible integrand f is well-defined on maps $\nu \in W^{1,s}(\Omega,\mathscr{M})$, where $d\nu_x$ is the approximate differential. Precisely, if $\nu : \Omega \to \mathscr{M}$ is a Borel map, and $x \in \mathscr{M}$ is a point of approximate continuity of ν, a linear map $N \in \mathrm{Hom}(\mathbb{R}^m, T_{\nu(x)}\mathscr{M})$ is said to be an *approximate differential* of ν at x if, for all $\epsilon > 0$,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^m}\,\mathscr{L}^m\Big(\Big\{y \in B^m_\rho(x) \mid d_{\mathscr{M}}\Big(\nu(y), \exp_{\nu(x)}(N(y-x))\Big) \ge \epsilon\,|y-x|\Big\}\Big) = 0\,.$$

When it exists, the approximate differential of ν at x is unique. For smooth maps, it agrees with the classical differential (in geometric sense). For this reason, the notation $d\nu_x$ is used for it. A Sobolev map ν in $W^{1,s}(\Omega,\mathscr{M})$ is approximately differentiable at $\mathscr{L}^m$-a.e. $x \in \mathscr{M}$, whence the functional (5.1) makes sense [27].

We ask for sequential lower semicontinuity. Thus, we need a suitable notion of quasiconvexity.

A locally bounded function $f : \Omega \times \mathrm{Hom}(\mathbb{R}^m, T\mathscr{M}) \to \mathbb{R}$ is said to be *quasiconvex* if for every $(x,\nu,N) \in \Omega\times\mathrm{Hom}(\mathbb{R}^m, T_\nu\mathscr{M})$ and every test function $\varphi \in C^\infty_c(Q_1, T_\nu\mathscr{M})$, where $Q_1 := [-1/2,1/2]^m$, by looking at $d\varphi_y$ as an element of $\mathrm{Hom}(\mathbb{R}^m, T_\nu\mathscr{M})$, the following inequality holds true:

$$f(x,\nu,N) \le \int_{Q_1} f(x,\nu,N+d\varphi_y)\,dy\,.$$

Theorem 5.1 ([27]) *Take $s \geq 1$. Let $f : \Omega \times Hom(\mathbb{R}^m, T\mathscr{M}) \to [0, +\infty)$ be a non-negative and continuous admissible integrand in $W^{1,s}(\Omega, \mathscr{M})$. Then, the functional* (5.1) *is sequentially weakly lower semicontinuous in $W^{1,s}(\Omega, \mathscr{M})$ if and only if f is quasiconvex.*

Coming back to the physical dimension $m = 2$, a membrane with out-of-middle-surface vector field ζ, a crack represented by a 1-varifold V, a descriptor ν of the through-the-thickness material morphology, subjected to a deformation u, is modeled by a quadruplet (u, ζ, V, ν) belonging to the class $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ constructed as follows: we choose four exponents $q > 2$, $r, p, s > 1$, the constant $K > 0$, assume that the triplet (u, ζ, V) belongs to the class $\mathscr{A}_{q,r,p,K}$ introduced above, while we let $\nu \in W^{1,s}(\Omega, \mathscr{M})$, where $\mathscr{M} = (\mathscr{M}^n, g)$ is as above.

On the class $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ we consider the energy functional

$$\mathscr{F}(u, \zeta, V, \nu) := \int_\Omega e(x, u, \zeta, \nu, \nabla u, \nabla \zeta, d\nu)\, dx + \|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p\, dV + \|\partial V\|\,. \tag{5.2}$$

The energy density $e : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M}) \to \mathbb{R} \cup \{+\infty\}$ is presumed to be sum of three terms, namely

$$e(x, u, \zeta, \nu, F, G, N) = \widetilde{e}(x, u, \zeta, \nu, F) + \beta_1 |G|^q + f(x, \nu, N)$$

for every $(x, u, \zeta, \nu, F, G, N) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M})$, where $\beta_1 \geq 0$, and the first addendum is a non-negative Carathéodory function with the following properties:

(a') $\widetilde{e}(x, u, \zeta, \nu, F)$ is polyconvex with respect to F, namely

$$\widetilde{e}(x, u, \zeta, \nu, F) = g(x, u, \zeta, \nu, M(F)) \qquad \forall F \in \mathbb{M}^{3\times 2}$$

where $g : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times (\mathbb{M}^{3\times 2} \times \mathbb{R}^3) \to [0, +\infty]$ is a Carathéodory function, with $g(x, u, \zeta, \nu, \cdot)$ convex for $\mathscr{L}^2$-a.e. $x \in \Omega$ and for all $(u, \zeta, \nu) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M}$;

(b') for $\mathscr{L}^2$-a.e. $x \in \Omega$ and every (u, ζ, ν, F) such that $\det(F|\zeta) > 0$

$$\widetilde{e}(x, u, \zeta, \nu, F) \geq \beta_2 \left(|F|^q + |adj_2 F|^r\right) + \Phi(F, \zeta)$$

where $\beta_2 > 0$ and $\Phi \in \widetilde{\mathscr{F}}$, a function class specified in Definition 3.1.

Moreover, the third addendum $f : \Omega \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M}) \to [0, +\infty)$ is a non-negative, continuous, and quasiconvex admissible integrand in $W^{1,s}(\Omega, \mathscr{M})$. We presume it is such that for positive constants $C_1, C_2 > 0$ the inequality

$$C_1 \cdot \|N\|^s_{g(\nu)} \leq f(x, \nu, N) \leq C_2 \cdot \|N\|^s_{g(\nu)}$$

holds true for every $(x, \nu, N) \in \Omega \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M})$.

As a Dirichlet condition on the quadruplets (u, ζ, V, ζ) in $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$, we assume

$$u = u_0\,, \quad \zeta = \frac{\partial_1 u_0 \times \partial_2 u_0}{|\partial_1 u_0 \times \partial_2 u_0|}\,, \quad \nu = \nu_0 \quad \text{on } \partial\Omega$$

in the sense of traces, for given maps $\nu_0 \in W^{1,s}(\Omega, \mathscr{M})$ and $u_0 \in W^{1,q}(\Omega, \mathbb{R}^3)$, with $\|u_0\|_\infty \leq K$ and $|adj_2 \nabla u_0| \in L^r(\Omega)$. Strong anchoring conditions are correspondingly defined in terms of the graph boundary current ∂G_u as in the previous section.

Theorem 5.2 *Under prescribed Dirichlet or strong anchoring conditions, if the class of competitors (u, V, ζ, ν) in $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ with finite energy is non-empty, the energy minimum of $\mathscr{F}(u, V, \zeta, \nu)$ is attained in the same class.*

Proof. Choose a minimizing sequence $\{(u_h, \zeta_h, V^{(h)}, \nu_h)\} \subset \mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ satisfying Dirichlet's or strong anchoring boundary conditions. The growth assumptions on f imply

$$\sup_h \int_\Omega \|d(\nu_h)_x\|^s_{g(\nu_h(x))}\, dx < \infty\,.$$

Therefore, by compactness (see [42]), possibly passing to a (not relabeled) subsequence, $\{\nu_h\}$ weakly converges in $W^{1,s}$ to some $\nu \in W^{1,s}(\Omega, \mathscr{M})$, which satisfies $\mathrm{Tr}\nu = \mathrm{Tr}\nu_0$, and $d_{\mathscr{M}}(\nu_h(x), \nu(x)) \to 0$ for $\mathscr{L}^2$-a.e. $x \in \Omega$. Moreover, by exploiting Theorem 5.1, we get

$$\int_\Omega f(x, \nu(x), d\nu_x)\, dx \leq \liminf_{h\to\infty} \int_\Omega f(x, \nu_h(x), d(\nu_h)_x)\, dx\,.$$

The proof then follows the same path followed for proving Theorem 4.2. We omit further details. ■

5.2 The SBV case

The through-the-thickness descriptor of the material microstructure ν might jump across the crack margins. So it could be natural to consider $\nu : \Omega \to \mathscr{M}$ as a special function of bounded variation. However, in this case we do not have at disposal a lower semicontinuity result for manifold-valued BV-maps, analogous to the one in reference [27] on which we have based the analyses developed in the previous section. Consequently, we only restrict ourselves to consider just the total variation of ν distributional derivative among the energy entries and base the pertinent analyses on the closure-compactness theorem proven in reference [4].

The class of Sobolev maps $\nu \in W^{1,s}(\Omega, \mathscr{M})$ can be equivalently defined in

terms of post-composition with Lipschitz functions $\varphi : \mathscr{M} \to \mathbb{R}$. By letting

$$\widehat{\mathscr{F}} := \{\varphi \in \mathrm{Lip}(\mathscr{M}, \mathbb{R}) \mid \mathrm{Lip}(\varphi) \leq 1\}$$

one requires $\nu \in L^s(\Omega, \mathscr{M})$ and the existence of a non-negative function $\phi \in L^s(\Omega)$ such that $\varphi \circ \nu \in W^{1,s}(\Omega)$ and $\|\nabla(\varphi \circ \nu)\|_{L^s(\Omega)} \leq \|\phi\|_{L^s(\Omega)}$ for every $\varphi \in \widehat{\mathscr{F}}$. The optimal function $\phi \in L^s(\Omega)$, which satisfies the previous inequality independently of $\varphi \in \widehat{\mathscr{F}}$, agrees $\mathscr{L}^m$-essentially with the function $|D\nu(x)|$ previously considered.

A summable map $\nu \in L^1(\Omega, \mathscr{M})$ is said to be a function of bounded variation in $BV(\Omega, \mathscr{M})$ if there exists a finite Borel measure μ in Ω such that the total variation of the distributional derivative $D(\varphi \circ \nu)$ is bounded by μ, i.e., $|D(\varphi \circ \nu)|(B) \leq \mu(B)$ for every Borel set $B \subset \Omega$ and every $\varphi \in \widehat{\mathscr{F}}$. The least measure μ satisfying such a property is denoted by $|D\nu|$ (see reference [4] for further details).

If $\nu \in BV(\Omega, \mathscr{M})$, the *countably* $(m-1)$*-rectifiable jump set* $S(\nu)$ is defined in terms of the Jump sets of the BV functions $\varphi \circ \nu$, for a suitable countable set of functions $\varphi \in \widehat{\mathscr{F}}$. The one-sided limits $\nu^{\pm}(x)$ are correspondingly defined at $\mathscr{H}^{m-1}$-a.e. $x \in S(\nu)$, once the $\mathscr{H}^{m-1}$-measurable unit normal $\mathbf{n}_\nu(x)$ to $S(\nu)$ is fixed, as the manifold elements $z^{\pm} \in \mathscr{M}$ such that the set $\{y \in \Omega \mid d_{\mathscr{M}}(\nu(y), z^{\pm}) > \epsilon\,,\ \langle y - x, \pm\mathbf{n}_\nu(x)\rangle > 0\}$ has 0-density at x.

In a similar way one defines a non-negative function $|\nabla\nu| \in L^1(\Omega)$ in terms of $\sup|\nabla(\varphi \circ \nu)|$, where $\nabla(\varphi \circ \nu)$ denotes the approximate gradient of the BV function $\varphi \circ \nu$. Then, $|\nabla\nu|$ agrees with the Radon-Nikodym derivative of $|D\nu|$ with respect to $\mathscr{L}^m \llcorner \Omega$ so that for $\mathscr{L}^m$-a.e. $x \in \Omega$ we compute the approximate limit

$$\operatorname{ap}\lim_{y \to x} \frac{d_{\mathscr{M}}(\nu(y), \nu(x))}{|y - x|} = |\nabla\nu|(x)\,.$$

A function $\nu \in BV(\Omega, \mathscr{M})$ is said to be a *special function of bounded variation* if $\varphi \circ \nu \in SBV(\Omega)$ for every $\varphi \in \widehat{\mathscr{F}}$. If $\nu \in SBV(\Omega, \mathscr{M})$, for every Borel set $B \subset \Omega$ we get the decomposition formula

$$|D\nu|(B) = \int_B |\nabla\nu|\, dx + \int_{B \cap S(\nu)} d_{\mathscr{M}}(\nu^+, \nu^-)\, d\mathscr{H}^{m-1}\,.$$

A compactness theorem holds true; it has been proven in reference [4].

Theorem 5.3 *For $\mathscr{K} \subset \mathscr{M}$ a compact set, consider $\{\nu_h\} \subset SBV(\Omega, \mathscr{M})$*

such that $\nu_h(\Omega) \subset \mathscr{K}$ *for every* h. *If for some exponent* $s > 1$

$$\sup_h \Big(\int_\Omega |\nabla \nu_h|^s \, dx + \int_{S(\nu_h)} d_{\mathscr{M}}(\nu_h^+, \nu_h^-) \, d\mathscr{H}^{m-1} \Big) < \infty, \tag{5.3}$$

there exists a (not relabeled) subsequence of $\{\nu_h\}$ *converging* $\mathscr{L}^m$*-a.e. in* Ω *to some* $\nu \in SBV(\Omega, \mathscr{M})$, *and such that*

$$\int_\Omega |\nabla \nu|^s \, dx \le \liminf_{h\to\infty} \int_\Omega |\nabla \nu_h|^s \, dx$$
$$\int_{S(\nu)} d_{\mathscr{M}}(\nu^+, \nu^-) \, d\mathscr{H}^{m-1} \le \liminf_{h\to\infty} \int_{S(\nu_h)} d_{\mathscr{M}}(\nu_h^+, \nu_h^-) \, d\mathscr{H}^{m-1} .$$

In the case of thin films considered here, we take first $q > 2$, $r, p, s > 1$, $K > 0$. Then, we consider a compact set $\mathscr{K}$ in $\mathscr{M}$, where $\mathscr{M} = (\mathscr{M}^n, g)$ is given as above. We say that $(u, \zeta, V, \nu) \in \widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$ if the triplet (u, ζ, V) belongs to the class $\mathscr{A}_{q,r,p,K}$ previously introduced and $\nu \in SBV(\Omega, \mathscr{M})$ satisfies the following relations:

(1) $\nu(\Omega) \subset \mathscr{K}$ with $|\nabla u| \in L^s(\Omega)$, and
(2) $\mathscr{H}^1 \llcorner S(\nu) \le \mu_V$.

Assumption (1) is of technical nature, whereas (2) means that ν jumps only where the film is fractured.

We then consider the energy to be

$$\mathscr{F}(u, \zeta, V, \nu) := \int_\Omega e(x, u, \zeta, \nu, \nabla u, \nabla \zeta, |\nabla \nu|) \, dx + \|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p \, dV + \|\partial V\| . \tag{5.4}$$

The density $e : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is the sum of three terms, namely

$$e(x, u, \zeta, \nu, F, G, N) = \tilde{e}(x, u, \zeta, \nu, F) + \beta_1 |G|^q + \beta_3 |N|^s \tag{5.5}$$

for every $(x, u, \zeta, \nu, F, G, N) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \times \mathbb{R}$, where $\beta_1 \ge 0$, $\beta_3 > 0$, and the first addendum is a non-negative Carathéodory function satisfying the properties (a') and (b') that we introduced in the case when $\nu \in W^{1,s}(\Omega, \mathscr{M})$.

The Dirichlet conditions are defined as in the case when $\nu \in W^{1,s}(\Omega, \mathscr{M})$, where this time $\nu_0 \in SBV(\Omega, \mathscr{M})$. Strong anchoring conditions are also as above.

Theorem 5.4 *Under prescribed Dirichlet's or strong anchoring conditions, if the class of competitors* (u, V, ζ, ν) *in* $\widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$ *with finite energy* (5.4) *is non-empty, the minimum of* $\mathscr{F}(u, \zeta, V, \nu)$ *is attained in the same class.*

Proof. Take a minimizing sequence $\{(u_h, \zeta_h, V^{(h)}, \nu_h)\} \subset \widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$ satisfying Dirichlet's or strong anchoring conditions. Since $\mathscr{H}^1 \llcorner S(\nu_h) \leq \mu_{V^{(h)}}$ and $\nu_h(\Omega) \subset \mathscr{K}$ for each h, we have $d_{\mathscr{M}}(\nu_h^+, \nu_h^-) \leq c(\mathscr{K}) < \infty$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_h)$ and for each h, where $c(\mathscr{K})$ is a real constant. Therefore, since $\beta_3 > 0$ in formula (5.5), the uniform bound (5.3) holds and we can apply Theorem 5.2, where the limit point $\nu \in SBV(\Omega, \mathscr{N})$ satisfies $\mathrm{Tr}\nu = \mathrm{Tr}\nu_0$. The proof then goes along the same path followed in Theorem 4.2. We omit further details. ■

Remark 5.1 *In order to consider cohesive effects instead of looking at brittle fracture, as we have done here so far, we could add to the energy $\mathscr{F}(u, \zeta, V, \nu)$, with density* (5.5)*, a term of the type*

$$\nu \mapsto \beta_4 \int_{S(\nu)} \delta_{\mathscr{M}}(\nu^+, \nu^-)\, d\mathscr{H}^1\,, \quad \beta_4 \geq 0\,.$$

On the other hand, a term of the type

$$\int_{S(\nu)} g(\nu^+, \nu^-, \mathbf{n}_\nu)\, d\mathscr{H}^{m-1}\,,$$

with $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{m-1} \to [0, +\infty)$ a suitable "jointly-convex" function is presently hard to be treated when we consider $\mathscr{M}$ as intrinsic manifold not necessarily coinciding with a linear space.

Remark 5.2 *Finally, a lower semicontinuity result necessary to analyze the SBV case when $D\nu$ enters the energy density instead of its total variation could perhaps be reached by adopting techniques presented in reference [40]. However, we do not tackle the problem here, leaving it open.*

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