Crack nucleation in shells with through-the-thickness microstructure

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Abstract

We refer to the common low-dimensional description of shells and thin films: surfaces endowed with directors satisfying a non-degeneracy condition under large strain. We consider in addition through-the-thickness material microstructure described by elements of a complete and intrinsic Riemannian manifold. We look at brittle materials. Among all possible cracked and uncracked admissible configurations, the one reached under Dirichlet-type boundary conditions realizes the minimum of a regularized Griffith's energy that includes curvature terms. For it we prove existence of minimizers for different constitutive functional choices and geometric structures with or without active through-the-thickness microstructure. Deformations are taken as SBV maps with jump set included in the support of a curvature varifold with boundary. Through-the-thickness descriptors of the material microstructure are taken first as manifold-valued Sobolev maps. Then, we consider pertinent SBV versions.

Key words: Fracture, Varifolds, Ground States, Shells, Microstructures, Calculus of Variations

1 Introduction

Three topics in continuum mechanics interact in the analysis that we propose here:

- the representation of films such as thin shells modeled as surfaces with directors,

- the variational description of crack nucleation,
- the multi-field (at times called phase-field to adopt a terminology more common in solid-state physics, although with more restricted meaning) depiction of microstructural effects on macroscopic motion.

Pertinent geometric and functional elements enter the energy. A requirement of minimality selects equilibrium configurations among those with all possible crack paths, depicted as rectifiable subsets of a domain Ω containing the shell middle surface. Fields describing macroscopic and microstructural through-the-thickness morphology and crack path are pertinent unknowns under large strain regime and Dirichlet or Dirichlet-type (so called strong anchoring) conditions, assuming a local strong non-degeneracy requirement.

The energy considered is surface polyconvex with respect to the shell middle surface deformation gradient, and convex with respect to the derivative of thickness and microstructure descriptor field. It also incorporates surface energy depending on crack area and a generalized curvature tensor. This last term has a regularizing nature: it accounts for surface non-local interactions (precisely of second-neighbor-type, thus of curvature-type) and bending effects in material bonds (for the latter ones see [53]). A term taking into account possible tip energy (it appears prominently in cohesive schemes) completes the energy. Deformations and microstructural fields are only constrained to have their jump set *contained* in the crack path *but not coinciding necessarily* with it.

Our analyses apply to several circumstances ranging from shells considered as structural elements to thin films, to the description of biological tissues in which one dimension is definitely smaller than the other two and a through-the-thickness microstructure implies actions hardly representable in terms of standard stresses.

1.1 Structural shells and thin films as surfaces with directors

In 1958, J. L. Ericksen and C. A. Truesdell adopted E. and F. Cosserat's 1909 scheme (which rests on a 1893 proposal by P. Duhem) to model structural elements as rods or shells as one-dimensional or two-dimensional bodies endowed, respectively, with an out-of-line or out-of-middle-surface vector field, with values, so-called *directors*, selected in the unit sphere [17]. They describe the behavior of cross-sections, considered as rigid bodies. An additional scalar factor may be used to represent variable thickness.

A massive body of pertinent work followed (see, for example, [6], [50], [51], [52], [43], [44], and references therein; this is an incomplete list limited just by the need to make a choice).

We may look just at bending – a curvature effect, indeed – when we constrain the director to be coincident with the normal of the shell middle surface. G. R. Kirchhoff's, T. von Karman's, P. Naghdi, W. Koiter's, and P. Ciarlet's plate and shell theories are based, in fact, on different curvature-dependent energies [13], [56] (see also [55], [54]).

These models imply the question of their justification in terms of dimensional reduction from 3D space. A natural way to tackle the problem is via E. De Giorgi's Γ-convergence. In this view, in 1999 K. Bhattacharya and R. D. James argued that if one considers a portion of an elastic cylinder between two cross sections at reciprocal finite distance, and compute the Γ-limit of the energy (and related minimizers) as the thickness goes to zero, one may obtain the energy pertinent to a membrane or to a Cosserat's surface, depending on whether the cylinder is made of a simple elastic material or a second-grade one, i.e., one including in the list of energy entries the second gradient of deformation. In the latter case, the limit generates a vector field over the surface on which we shrink the cylinder portion (see [49], [21], [22]).

Roughly speaking, membrane energy of different plate models emerges from separate scaling assumptions for the energy with respect to the plate thickness. This result comes from a top-to-bottom approach. On the other hand, if we reverse the view [20] and construct a film by superposing an atomic layer over the other, we get at continuum level a surface endowed at each point with as many directors as the number of atomic layers considered. When we consider films constituted by a mixture of atomic layers, the sequence of vectors at each point is known to within a permutation [40].

1.2 Crack paths as rectifiable sets supporting varifolds

A variational approach to the elastic-brittle behavior has been suggested in 1998 [19], based on minimality required for A. A. Griffith's energy [29]. Crack evolution is then considered by partitioning the time interval and presuming that minimality governs transitions between subsequent minimizers at discrete times. This scheme has its roots in De Giorgi's idea of minimizing movements [16].

At first there is distinction between deformations and crack path, although they are connected because the deformation jumps when crack margins detach. In two dimensions we can control sequences of curves so that we can maintain the previous distinction. To allow jumps across the crack path, deformations can be naturally considered as special bounded variation maps (SBV). Their distributional derivative is a finite measure with $\mathscr{H}^{n-1}$-measurable jump set, where n is the ambient space dimension and $\mathscr{H}^{n-1}$ the $(n-1)$-dimensional

Hausdorff measure (see the treatise [5]). In addition, considering a cracking process in the time-step sense above mentioned, in $2D$ environment we can describe steps in which the crack margins come back at least partially in a contact without shear with respect to the original uncracked configuration. Material bonds are however not restored and a crack persists. The approach requires to keep track of the previous steps (see [15] and subsequent work of the same authors).

However, in general, in $3D$ space we cannot always control sequences of surfaces. A way to overcome such a difficulty is to look for minimizing SBV deformations and to consider the crack to be coincident with the deformation jump set (see once again [15] and subsequent works of the same authors). Also, progressively onward in the course of cracking, appropriate choices in looking at a crack as coinciding with the jump set of a SBV deformation might allow us to describe steps in which the crack margins come back at least partially in a contact without shear with respect to the original uncracked configuration. The material bonds are however not restored, so the crack persists. In so doing, we should look at a time-step-process such that if at a time t_i the crack margins restore the original contact they were detached at time t_j with $j < i$, according to the SBV setting. However, there are cases in which material bonds break to allow the matter to reach a lower energy level but the margins of pertinent cracks do not detach as it occurs in some cracking events in windscreens of cars. Besides these specific cases, if we do not look at processes and consider a unique minimizing step – as we do here – we do not have at disposal a "previous step" in which crack margins are detached before coming back in contact.

To maintain the distinctions between deformations and crack paths in $3D$ environment, we could consider sequences of surfaces with bounded curvature. They can be described by means of *rectifiable varifolds*, i.e., Radon measures on the Grassmannian constructed by using the tangent planes to $\mathscr{H}^2$-rectifiable sets (see for a general treatment [1], [2], [3], [36]). They admit a generalized notion of *curvature* related to their support. The approach (proposed first in references [27], [38], [25]) implies the introduction of an energy which is a regularization of Griffith's one, as already mentioned above.

Here we apply to shells the varifold-based description of crack paths. The underlying two-dimensional setting notwithstanding (at least in terms of the reference domain where fields are defined) allows us to consider in a unique step cases in which crack margins remain or even "come back" at least partially in contact.

1.3 Through-the-thickness microstructures

Thin films show microstructures emerging from their thinness. Tents and tunnels appear in film deposition [30] on a substrate as a consequence of martensite-austenite transitions [9], [34], [35], [32]. In polycristalline thin films, there may be grain growth larger than the film thickness in a way that grains traverse the film. In this case film surface and grain boundary energies become comparable. Their interplay may pin the boundaries against further migration, or may enhance them [23]. However, the presence of further active material microstructure may influence drastically the film behavior, as in the presence of magnetization [8]. Besides thin films, in general shells may be made of materials with active microstructure determining effects hardly representable in terms of standard stresses. To account directly for them, in representing the body morphology we exploit not only a fit region in the physical point space but also introduce descriptors of the low-spatial-scale texture of the matter and consider them to be observable. With the aim of having a general model-building framework, we take such descriptors (say phase-fields) as elements of a finite-dimensional complete Riemannian manifold $\mathscr{M}$ not embedded into a linear space (see [11], [37], [48], [39]). The picture of shells as surfaces with directors not necessarily coinciding with the normal vector field falls within the above mentioned general scheme. When we consider structural shells and thin films endowed with active microstructures in the sense above specified, we have at least a pertinent descriptor, say ν, which complements the vector representing the cross-section behavior. This is the scheme that we investigate here accounting for the energy dependence on ν and its gradient. Such an approach is also a way to approximately describe fractures. In fact, we can choose ν to be as a scalar damage indicator, which localizes in a small neighborhood of the crack path [42], [10]. Phenomena of localization, in fact, appear also for vector choices of ν and even in linear elasticity setting, as harbingers (or precursors) of fracture [41].

1.4 The main energy under investigation

With progressive extensions, in this paper we end up to the following energy functional:

$$\mathscr{F}(u,\zeta,V,\nu) := \int_\Omega (\tilde{e}(x,u,\zeta,\nu,\nabla u) + \beta_1 |\nabla\zeta|^q + f(x,\nu,\nabla\nu))\, dx \\ + \gamma\|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p \, dV + \alpha\|\partial V\|\,,$$

with Ω a domain in the plane, ζ an out-of-plane vector field describing the shell cross-section (cf. equation (3.1)), ν the morphological descriptor of a through-the-thickness (active) material microstructure (cf. Sec. 6), $\tilde{e}$ a surface polyconvex function with respect to ∇u (cf. equation (6.3)), f a non-negative, continuous, and quasiconvex $W^{1,s}(\Omega, \mathscr{M})$ function (namely, f is bounded and such that $f(x, \nu, N)$, with N a linear map from $\mathbb{R}^2$ onto the tangent space of $\mathscr{M}$ at $\nu \in \mathscr{M}$, is lesser or equal to the integral over unit square of $f(x, \nu, N + d\varphi)$, with φ a compactly supported smooth function; for a precise general definition see Definition 6.1), V a one-dimensional varifold with mass $\|V\|$ (the term including $\|V\|$ corresponds to the area term in Griffith's energy) and generalized curvature tensor $A := A(V)$. The boundary mass term $\|\partial V\|$ accounts for possible tip energy (cf. Sec. 2.3 for the notation concerning curvature varifolds with boundary). We prove existence of minimizers in terms of deformations u, vector fields ζ, descriptors ν, varifolds V.

Specifically, we take u in the space of weak diffeomorphisms constructed on SBV maps – a space defined in [25] – which satisfy a local strong non-degeneracy condition. Also, ν is taken as a Sobolev map valued on an intrinsic, finite-dimensional, complete, differentiable Riemannian manifold. When ν is itself in SBV with values in the same intrinsic manifold, in the list of energy entries, we replace the distributional derivative of ν with its total variation (or semi-norm).

2 Background material

2.1 Special functions of bounded variation

A summable function $u \in L^1(\Omega)$, defined on a bounded domain $\Omega \subset \mathbb{R}^2$, is said to be *of bounded variation* if the distributional derivative Du is a finite measure in Ω. In this case, the function u is approximately differentiable $\mathscr{L}^2$-a.e. in Ω, and the approximate gradient ∇u agrees with the density of Radon-Nikodym's derivative of Du with respect to the Lebesgue measure $\mathscr{L}^2$. Therefore, the decomposition $Du = \nabla u \mathscr{L}^2 + D^s u$ holds true, where the component $D^s u$ is singular with respect to $\mathscr{L}^2$. Also, denoting by $\mathscr{H}^k$ the k-dimensional Hausdorff measure, the *jump set* $S(u)$ of u is a countably 1-rectifiable subset of Ω that agrees $\mathscr{H}^1$-essentially with the complement of u Lebesgue's set. If, in addition, the singular component $D^s u$ is concentrated on the jump set $S(u)$, we say that u is a *special function of bounded variation*, and write in short $u \in SBV(\Omega)$.

A vector valued function $u : \Omega \to \mathbb{R}^3$ belongs to the class $SBV(\Omega, \mathbb{R}^3)$ if its components u^j are in $SBV(\Omega)$. In that case, we get

$$|Du|(B) = \int_B |\nabla u| \, dx + \int_{B \cap S(u)} |u^+ - u^-| \, d\mathscr{H}^1$$

for each Borel set $B \subset \Omega$, where the approximate gradient $\nabla u \in L^1(\Omega, \mathbb{M}^{3\times 2})$, the jump set $S(u)$ is defined as in the scalar case, or componentwise, and $u^\pm$ are the one sided limits at $\mathscr{H}^1$-a.e. point $x \in S(u)$ [5].

2.2 Currents carried by approximately differentiable maps

Let Ω be a bounded domain in $\mathbb{R}^2$. For $u : \Omega \to \mathbb{R}^3$ an a.e. approximately differentiable map, we denote by ∇u its approximate gradient. The map u has a Lusin representative on the subset $\widetilde{\Omega}$ of Lebesgue points pertaining to both u and ∇u. Also, we have $\mathscr{L}^2(\Omega \setminus \widetilde{\Omega}) = 0$. We shall thus denote $M(F) := (F, \mathrm{adj}_2 F) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$, where $\mathrm{adj}_2 F$ is the 3-vector given by the 2×2 minors of the matrix $F \in \mathbb{M}^{3\times 2}$. We say that $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ if $\nabla u \in L^1(\Omega, \mathbb{M}^{3\times 2})$ and the adjoint vector $\mathrm{adj}_2 \nabla u \in L^1(\Omega, \mathbb{R}^3)$.

The *graph* of a map $u \in \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is defined by

$$\mathscr{G}_u := \Big\{ (x, y) \in \Omega \times \mathbb{R}^3 \mid x \in \widetilde{\Omega},\ y = \widetilde{u}(x) \Big\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u. It turns out that $\mathscr{G}_u$ is a countably 2-rectifiable set of $\Omega \times \mathbb{R}^3$, with $\mathscr{H}^2(\mathscr{G}_u) < \infty$. The approximate tangent plane at $(x, u(x))$ is generated by the vectors $\mathbf{t}_1(x) = (1, 0, \partial_1 u(x))$ and $\mathbf{t}_2 = (0, 1, \partial_2 u(x))$ in $\mathbb{R}^5$, where the partial derivatives are the column vectors of the gradient ∇u, and we take $\nabla u(x)$ as the Lebesgue value of ∇u at $x \in \widetilde{\Omega}$. Therefore, the 2-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)}{|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)|}$$

provides an orientation to the graph $\mathscr{G}_u$.

The current G_u carried by the graph of u is a functional taking values

$$G_u(\omega) = \langle G_u, \omega \rangle := \int_{\mathscr{G}_u} \langle \omega, \xi \rangle \, d\mathscr{H}^2$$

where ω belongs to the space $\mathscr{D}^2(\Omega \times \mathbb{R}^3)$ of compactly supported 2-forms on $\mathscr{G}_u$. It turns out that G_u is an *integer multiplicity* (in short i.m.) *rectifiable current* with finite mass $\mathbf{M}(G_u)$ equal to the area $\mathscr{H}^2(\mathscr{G}_u)$ of $\mathscr{G}_u$; we then write $G_u \in \mathscr{R}_2(\Omega \times \mathbb{R}^3)$. Since the Jacobian of $x \mapsto (x, u(x))$, which is the graph map, is equal to $|\mathbf{t}_1(x) \wedge \mathbf{t}_2(x)|$, by the area formula we get

$$\mathbf{M}(G_u) = \mathscr{H}^2(\mathscr{G}_u) = \int_\Omega \sqrt{1 + |\nabla u|^2 + |\mathrm{adj}_2 \nabla u|^2} \, dx < \infty$$

(compare Thm. 4 at p. 225 in [28, Vol. I, Sec. 3.1.5]).

By duality the *boundary* of G_u is the 1-current ∂G_u acting on $\mathscr{D}^1(\Omega\times\mathbb{R}^3)$, the space of compactly supported smooth 1-forms η in $\Omega\times\mathbb{R}^3$, as

$$\langle \partial G_u, \eta\rangle := \langle G_u, d\eta\rangle, \qquad \eta\in\mathscr{D}^1(\Omega\times\mathbb{R}^3)\,,$$

where $d\eta$ is the differential of η. By Stokes theorem we get $\partial G_u = 0$ on $\mathscr{D}^1(\Omega\times\mathbb{R}^3)$ if u is of class C^2. Such a relation holds true also for Sobolev maps $u\in W^{1,2}(\Omega,\mathbb{R}^3)$, by approximation. However, in general, the boundary ∂G_u does not vanish and may not have finite mass in $\Omega\times\mathbb{R}^3$. On the other hand, if ∂G_u has finite mass, the boundary rectifiability theorem states that ∂G_u is an i.m. rectifiable current in $\mathscr{R}_1(\Omega\times\mathbb{R}^3)$. In particular, it turns out that u is special function of bounded variation, $u\in SBV(\Omega,\mathbb{R}^3)$. Actually, u belongs to the class SBV_0 discussed in reference [5].

An extended treatment of currents is in the two-volume treatise [28].

2.3 Curvature varifolds with boundary

A 1-varifold over Ω, a domain in $\mathbb{R}^2$, is a non-negative Radon measure on the trivial bundle $\mathscr{G}_1(\Omega):=\Omega\times\mathscr{G}_{1,2}$, where $\mathscr{G}_{1,2}$ is the Grassmannian manifold of 1-planes Π (straight lines) through the origin in $\mathbb{R}^2$.

If $\mathfrak{b}$ is a 1-rectifiable subset of $\Omega\subset\mathbb{R}^2$, for $\mathscr{H}^1\llcorner\mathfrak{b}$ a.e. $x\in\Omega$ there exists the approximate tangent 1-space $T_x\mathfrak{b}$ to $\mathfrak{b}$ at x. We thus denote by $\Pi(x)$ the 2×2 matrix that identifies the orthogonal projection of $\mathbb{R}^2$ onto $T_x\mathfrak{b}$.

We define

$$V_{\mathfrak{b},\theta}(\varphi) := \int_{\mathscr{G}_1(\Omega)}\varphi(x,\Pi)\,dV_{\mathfrak{b},\theta}(x,\Pi) := \int_{\mathfrak{b}}\theta(x)\varphi(x,\Pi(x))\,d\mathscr{H}^1(x) \tag{2.1}$$

for any $\varphi\in C_c^0(\mathscr{G}_1(\Omega))$, where $\theta\in L^1(\mathfrak{b},\mathscr{H}^1)$ is a nonnegative density function. If θ is integer valued, $V=V_{\mathfrak{b},\theta}$ is said to be the *integer rectifiable varifold* associated with $(\mathfrak{b},\theta,\mathscr{H}^1)$.

The *weight measure* of V is the Radon measure in Ω given by $\mu_V:=\pi_\# V$, where $\pi:\mathscr{G}_1(\Omega)\to\Omega$ is the canonical projection. Therefore, we find $\mu_V = \theta\,\mathscr{H}^1\llcorner\mathfrak{b}$, and the *mass* of V is

$$\|V\| := V(\mathscr{G}_1(\Omega)) = \mu_V(\Omega) = \int_{\mathfrak{b}}\theta\,d\mathscr{H}^1\,.$$

Definition 2.1 *An integer rectifiable* 1*-varifold* $V=V_{\mathfrak{b},\theta}$ *is called a* curvature 1-varifold with boundary *if there exist a function* $A\in L^1(\mathscr{G}_1(\Omega),\mathbb{R}^{2*}\otimes\mathbb{R}^2\otimes$

$\mathbb{R}^{2*}$), $A = (A_j^{\ell i})$, *and a vector valued Radon measure* ∂V *with finite mass* $\|\partial V\|$, *such that*

$$\int_{\mathscr{G}_1(\Omega)} (\Pi D_x \varphi + A D_\Pi \varphi + \varphi\, {}^t\mathrm{tr}\,(AI))\, dV(x,\Pi) = -\int_{\mathscr{G}_1(\Omega)} \varphi\, d\partial V(x,\Pi)$$

for every $\varphi \in C_c^\infty(\mathscr{G}_1(\Omega))$. *We write in short* $\partial V \in \mathscr{M}(\mathscr{G}_1(\Omega), \mathbb{R}^2)$. *Moreover, for* $p \geq 1$ *the subclass of curvature* 1*-varifolds with boundary such that* $|A| \in L^p(\mathscr{G}_1(\Omega))$ *is indicated by* $CV_1^p(\Omega)$ (see [27, Ex. 1,2] for specific examples).

With respect to Allard's approach (see [1], [2]), with definition (2.1) we gain more information. For example, if Ω is the unit disk centered at the origin and we take a $1D$ varifold given by three half-lines from 0, which form three angles of $120°$, by using Allard's definition we find zero mean curvature and zero boundary. At variance (compare with the results in [36]), with the view adopted here the boundary measure is the sum of three Dirac deltas supported at the points $(0, P_i)$ in the Grassmannian $\mathscr{G}_1(\Omega)$, where P_i is the $1D$ space determined by the $i-th$ half-line, with $i = 1, 2, 3$.

Varifolds in $CV_1^p(\Omega)$ have generalized curvature in L^p [36]. Therefore, Allard's compactness theorem applies (see [1], [2], [3]):

Theorem 2.1 *For* $1 < p < \infty$, *let* $\{V^{(h)}\} \subset CV_1^p(\Omega)$ *be a sequence of curvature* 1*-varifolds* $V^{(h)} = V_{\mathfrak{b}_h,\theta_h}$ *with boundary. Corresponding curvatures and boundaries are indicated by* $A^{(h)}$ *and* $\partial V^{(h)}$, *respectively. Assume that there exists a constant* $c > 0$ *such that for every* h

$$\mu_{V^{(h)}}(\Omega) + \|\partial V^{(h)}\| + \int_{\mathscr{G}_1(\Omega)} |A^{(h)}|^p \, dV^{(h)} \leq c\,.$$

Then, there exists a (not relabeled) subsequence of $\{V^{(h)}\}$ *and a* 1*-varifold* $V = V_{\mathfrak{b},\theta} \in CV_1^p(\Omega)$, *with curvature* A *and boundary* ∂V, *such that*

$$V^{(h)} \rightharpoonup V, \quad A^{(h)}\, dV^{(h)} \rightharpoonup A\, dV, \qquad \partial V^{(h)} \rightharpoonup \partial V,$$

in the sense of measures. Moreover, for any convex and l.s.c. function $f : \mathbb{R}^{2*} \otimes \mathbb{R}^2 \otimes \mathbb{R}^{2*} \to [0, +\infty]$, *we get*

$$\int_{\mathscr{G}_1(\Omega)} f(A)\, dV \leq \liminf_{h\to\infty} \int_{\mathscr{G}_1(\Omega)} f(A^{(h)})\, dV^{(h)}.$$

3 A skeletal model

We look first to the shell middle surface and take for it a planar reference configuration that is a two-dimensional smooth domain Ω in $\mathbb{R}^2$, where Cartesian coordinates $x = (x_1, x_2)$ are fixed. A map $u : \Omega \to \mathbb{R}^3$, say $u = (u^1, u^2, u^3)$, represents a *deformation.*

When smoothness and non-singularity are assured, the map u determines an immersion of Ω into $\mathbb{R}^3$; the tangent plane to the deformed film middle plane does not degenerate. Formally, it is tantamount to impose $|\mathrm{adj}_2 \nabla u(x)| > 0$ for any $x \in \Omega$. In other words, if $\partial_i u$ denotes a column vectors of the gradient matrix ∇u, we are imposing that the vector product $\partial_1 u \times \partial_2 u$ does not vanish at every point. Therefore, the normal to $u(\Omega)$, the deformed middle surface, is the unit vector

$$\mathbf{n}(x) = \frac{\partial_1 u(x) \times \partial_2 u(x)}{|\partial_1 u(x) \times \partial_2 u(x)|}\,. \tag{3.1}$$

It can be considered as a descriptor of out-of-middle-surface film behavior. However, such an information can be carried out by an $\mathbb{S}^2$-valued vector field $x \mapsto \zeta(x)$ defined over Ω and constrained to be at every $x \in \Omega$ such that

$$(\partial_1 u(x) \times \partial_2 u(x)) \bullet \zeta(x) > 0 \tag{3.2}$$

where $\bullet$ is the scalar product in $\mathbb{R}^3$; in the absence of out-of-middle-surface shear, we get $\zeta(x) = \mathbf{n}(x)$ (the scheme is standard, see [17], [50], [52], [51], [6] and references therein).

Given $q > 2$ and $r > 1$, consider the energy

$$\mathscr{G}(u, \zeta) := \int_\Omega \Big(|\nabla u|^q + |\nabla \zeta|^r + \Phi(\nabla u, \zeta) \Big)\, dx$$

acting on couples of Sobolev functions (u, ζ) in $[W^{1,1}(\Omega, \mathbb{R}^3)]^2$, with $|\zeta| = 1$ $\mathscr{L}^2$-a.e. in Ω. Impose also the uniform bound $\|u\|_{L^\infty(\Omega)} \leq K$. Eventually, we may consider Dirichlet-type conditions for u and ζ over $\partial\Omega$ in the sense of traces.

As a matter of notation, according to reference [14], we define the set

$$\widetilde{Y} := \{(F, \zeta) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3 \mid \det(F|\zeta) > 0\}$$

where $(F|\zeta) \in \mathbb{M}^{3\times 3}$. Therefore, according to (3.2), if $F = \nabla u$ one has

$$\det(\nabla u|\zeta) = (\partial_1 u \times \partial_2 u) \bullet \zeta\,.$$

Definition 3.1 *We shall denote by* $\widetilde{\mathscr{F}}$ *the class of non-negative functions* $\Phi : \widetilde{Y} \to [0, +\infty)$ *such that*

(1) $\Phi(F, \cdot)$ *is continuous in* $\mathbb{R}^3$ *for all* $F \in \mathbb{M}^{3\times 2}$*;*

(2) $\Phi(\cdot,\zeta)$ *is polyconvex (precisely,* $\Phi(\cdot,\zeta)$ *is the restriction to* $\{F \in \mathbb{M}^{3\times 2} \mid \det(F|\zeta) > 0\}$ *of a polyconvex function* $g : \mathbb{M}^{3\times 2} \to \mathbb{R}$*, i.e., a convex functions of all the minors of* F*), for all* $\zeta \in \mathbb{R}^3$*;*
(3) $\Phi(F,\zeta) \to +\infty$ *if* $\det(F|\zeta) \to 0^+$.

We assume that the integrand Φ in $\mathscr{G}(u,\zeta)$ be in the class $\widetilde{\mathscr{F}}$. An example is given by $\Phi(F,\zeta) = -\log(\det(F|\zeta)) \vee 0$ if $\det(F|\zeta) > 0$, and $\Phi(F,\zeta) = +\infty$ otherwise in $\mathbb{M}^{3\times 2} \times \mathbb{R}^3$. We also fix $q > 2$ and $r > 1 \vee 2q/(3q-4)$.

Theorem 3.1 *With the previous assumptions, minima of* $\mathscr{G}(u,\zeta)$ *exist among couples in* $W^{1,q}(\Omega,\mathbb{R}^3) \times W^{1,r}(\Omega,\mathbb{R}^3)$*, with* $\|u\|_\infty \le K$ *and* $|\zeta| = 1$*. They are such that* (3.2) *holds true* $\mathscr{L}^2$*-a.e. on* Ω*.*

Proof. By using compactness arguments, we shall repeatedly pass to not relabeled subsequences. Taking a minimizing sequence $\{(u_h,\zeta_h)\}$, we infer that $u_h \rightharpoonup u_\infty$ weakly in $W^{1,q}(\Omega,\mathbb{R}^3)$ and $\zeta_h \rightharpoonup \zeta_\infty$ weakly $W^{1,r}(\Omega,\mathbb{R}^3)$ for some functions $(u_\infty,\zeta_\infty) \in W^{1,q}(\Omega,\mathbb{R}^3) \times W^{1,r}(\Omega,\mathbb{R}^3)$. Moreover, the a.e. convergences $u_h \to u_\infty$ and $\zeta_h \to \zeta_\infty$ imply that $\|u_\infty\|_{L^\infty(\Omega)} \le K$ and $|\zeta_\infty| = 1$ a.e. in Ω, whereas by lower semicontinuity

$$\int_\Omega (|\nabla u_\infty|^q + |\nabla \zeta_\infty|^r)\, dx \le \liminf_{h\to\infty} \int_\Omega (|\nabla u_h|^q + |\nabla \zeta_h|^r)\, dx \,.$$

The bound $\sup_h \int_\Omega \Phi(\nabla u_h,\zeta_h)\, dx < \infty$ implies that

$$(\partial_1 u_h(x) \times \partial_2 u_h(x)) \bullet \zeta_h(x) > 0 \tag{3.3}$$

holds true for each $h \in \mathbb{N}$ and $\mathscr{L}^2$-a.e. in Ω. We claim that condition (3.3) is preserved (with possibly the equality sign instead of $>$) when passing to the limit. In fact, by the parallelogram inequality we get for every h and for a.e. $x \in \Omega$ the bound

$$|\partial_1 u_h \times \partial_2 u_h|^{q/2} \le C \cdot |\nabla u_h|^q \,, \quad q > 2 \tag{3.4}$$

for some absolute constant C. Moreover, if $1 \vee 2q/(3q-4) < r \le 2$ we also have strong convergence $\zeta_h \to \zeta_\infty$ in $L^p(\Omega)$ for $p = 2r/(2-r)$, where p is greater than the conjugate exponent to $q/2$. Therefore, we infer the existence of a function $H \in L^1(\Omega)$ such that $(\partial_1 u_h \times \partial_2 u_h) \bullet \zeta_h \rightharpoonup H$ weakly in $L^1(\Omega)$. On the other hand, by (3.4) and a standard density argument it turns out that in the distributional sense we have, e.g.,

$$\mathrm{Div}(u_h^1\, \partial_2 u_h^2, -u_h^1\, \partial_1 u_h^2) = (\partial_1 u_h^1\, \partial_2 u_h^2 - \partial_2 u_h^1\, \partial_1 u_h^2)\, \mathscr{L}^2 \llcorner \Omega \quad \forall\, h \in \bar{\mathbb{N}}$$

where $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, i.e., for each test function $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega (\partial_1 u_h^1\, \partial_2 u_h^2 - \partial_2 u_h^1\, \partial_1 u_h^2)\, \varphi\, dx = \int_\Omega u_h^1\, (\partial_1 u_h^2\, \partial_2 \varphi - \partial_2 u_h^2\, \partial_1 \varphi)\, dx \,.$$

We thus infer the weak convergence

$$(\partial_1 u_h^1\, \partial_2 u_2^h - \partial_2 u_h^1\, \partial_1 u_h^2) \rightharpoonup (\partial_1 u_\infty^1\, \partial_2 u_\infty^2 - \partial_2 u_\infty^1\, \partial_1 u_\infty^2)$$

in $L^{q/2}(\Omega)$, and hence a.e. in Ω. This implies $H = (\partial_1 u_\infty \times \partial_2 u_\infty) \bullet \zeta_\infty$ and also the pointwise convergence a.e. in Ω

$$(\partial_1 u_h \times \partial_2 u_h) \bullet \zeta_h \to (\partial_1 u_\infty \times \partial_2 u_\infty) \bullet \zeta_\infty \,.$$

Therefore, the lower semicontinuity

$$\int_\Omega \Phi(\nabla u_\infty, \zeta_\infty)\, dx \le \liminf_{h\to\infty} \int_\Omega \Phi(\nabla u_h, \zeta_h)\, dx < \infty$$

holds, whence the couple of functions (u_∞, ζ_∞) satisfies the strict inequality in (3.2) for $\mathscr{L}^2$-a.e. $x \in \Omega$, as required. ∎

It may happen that the deformed surface $u(\Omega)$ has a crease. This is described, e.g., when the unit normal $\mathbf{n}(x)$ is smooth outside a 1-rectifiable set $\mathfrak{J}$ of Ω. In this case, ζ satisfies the condition $\zeta(x) \bullet \mathbf{m} > 0$ at $x \in \mathfrak{J}$, for some unit vector $\mathbf{m}$ that lies in the cone between the one-sided limits of $\mathbf{n}$ at $x \in \mathfrak{J}$.

Therefore, a second order theory (yielding e.g. to a regularity of the unit normal $\mathbf{n}(x)$) should be applied in order to describe in a precise way the above mentioned angle condition along creases over the deformed surface.

4 Cracks and jump sets

We assume here that the components u^j of u are L^∞-functions in the class $SBV(\Omega)$. Again, we assume a uniform bound $\|u\|_{L^\infty(\Omega)} \le K$.

We then choose the descriptor $\zeta : \Omega \to \mathbb{S}^2 \subset \mathbb{R}^3$ of the out-of-middle-surface film behavior as a special function of bounded variation in $SBV(\Omega, \mathbb{R}^3)$, with $|\zeta| = 1$ almost everywhere, in such a way that the constraint (3.2) holds true $\mathscr{L}^2$-a.e. in $\Omega \setminus S(u)$. Since ζ describes the behavior of shell cross-sections, the continuity of matter allows us to consider the discontinuity set of ζ as included in the one of u, namely

$$S(\zeta) \subseteq S(u) \,.$$

The strict inclusion occurs when the crack margins are in contact across the whole shell thickness; in this case we would have no deformation jump along the crack portion where the margins remain in contact. If one thinks of a cracking process, there could be circumstances already above mentioned in which margins remain, at least in part, always in contact, although the energetic content of the material bond is such that they become unstable and prefer, energetically, to break. Accounting for this aspect – i.e., a permanent contact in a portion of a crack – is a peculiarity of the present model. For analytical reasons, we replace the above set inclusion by the inequality:

$$\mathscr{H}^1 \llcorner S(\zeta_h) \leq \mathscr{H}^1 \llcorner S(u_h) \qquad \forall\, h, \tag{4.1}$$

along minimizing sequences $\{(u_h, \zeta_h)\}$.

In principle, the limit maps (u, ζ) may not satisfy condition (4.1). In fact, it may happen that the jump of u_h on $S(u_h)$ goes to zero somewhere, when passing to the limit through the compactness theorem. In order to preserve inequality (4.1), we shall introduce a condition ensuring that the jump $u_h^+ - u_h^-$ on $S(u_h)$ cannot "decrease" or "disappear", so that one has $\mathscr{H}^1(S(u_h)) \to \mathscr{H}^1(S(u))$.

Over the reference domain Ω we consider a curvature 1-varifold with boundary $V = V_{\mathfrak{b},\theta}$ such that the discontinuity set $S(u)$ of the deformation map u is contained in the 1-rectifiable set $\mathfrak{b}$. A crack in which the margins remain at least partially in contact is characterized by $\mathscr{H}^1(\mathfrak{b} \setminus S(u)) > 0$.

4.1 Membranes with cracks

At a first glance we do not consider the out-of-middle-surface descriptor ζ.

Definition 4.1 *A macroscopic configuration of a cracked membrane is a pair composed by the bounded connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and a curvature 1-varifold with boundary, namely $V = V_{\mathfrak{b},\theta} \in CV_1^p(\Omega)$ for some $p > 1$.*

We consider bounded deformation maps u in $\mathscr{A}^1(\Omega, \mathbb{R}^3)$, i.e., a.e. approximately differentiable maps u with $\nabla u \in L^1(\Omega, \mathbb{M}^{3\times 2})$ and $|\mathrm{adj}_2 \nabla u| \in L^1(\Omega)$, so that the current G_u carried by the rectifiable graph of u is i.m. rectifiable in $\mathscr{R}_2(\Omega \times \mathbb{R}^3)$ and with finite mass, $\mathbf{M}(G_u) < \infty$.

In general, the boundary current $\partial G_u \in \mathscr{D}_1(\Omega \times \mathbb{R}^3)$ is non zero and it may also have unbounded mass. Therefore, we assume furthermore that fractures and holes in the graph of u are controlled by the crack V, namely

$$\pi_\# |\partial G_u| \leq \mu_V \tag{4.2}$$

where μ_V is the weight measure in Ω of the varifold V, $\pi : \Omega \times \mathbb{R}^3 \to \Omega$ the projection onto the first two coordinates, and $|\cdot|$ the total variation of the vector-valued measure ∂G_u, so that $\pi_\# |\partial G_u|(B) = |\partial G_u|(B \times \mathbb{R}^3)$ for each Borel set $B \subset \Omega$.

If $\mu_V(\Omega) < \infty$ and the bound (4.2) holds true, $|\partial G_u|$ is a finite measure, and actually $\mathbf{M}(\partial G_u) < \infty$. Therefore, the boundary rectifiability theorem yields that ∂G_u is an i.m. rectifiable current – in short $\partial G_u \in \mathscr{R}_1(\Omega \times \mathbb{R}^3)$ – and hence u is a special function of bounded variation, namely $u \in SBV(\Omega, \mathbb{R}^3)$.

More precisely, since $|D^j u| \leq \|u\|_\infty \, \pi_\# |\partial G_u|$, recalling that $\mu_V = \theta \, \mathscr{H}^1 \llcorner \mathfrak{b}$, by the assumption (4.2) it turns out that the jump set of u is contained in the support $\mathfrak{b}$ of V where the positive multiplicity function θ is integer-valued, and we actually have

$$\mathscr{H}^1 \llcorner S(u) \leq \mathscr{H}^1 \llcorner \mathfrak{b} \, . \tag{4.3}$$

Example 1 *Recalling that $\mu_V = \theta \, \mathscr{H}^1 \llcorner \mathfrak{b}$, the validity of inequality* (4.2) *relies on the presence of the positive integer θ, that actually accounts for the multiplicity of the projection of ∂G_u. Taking, e.g., $\Omega = B^2$, the unit open disk centered at the origin, and $u : B^2 \to \mathbb{R}^3$ given by*

$$u(x_1, x_2) := \begin{cases} (x_1,\, x_2,\, 0) & \text{if} \quad x_1 < 0 \\ (1 + x_1,\, x_2,\, 0) & \text{if} \quad x_1 > 0 \end{cases}$$

we get $\partial G_u = \gamma_{0\#}[\![I]\!] - \gamma_{1\#}[\![I]\!]$ on $\mathscr{D}^1(B^2 \times \mathbb{R}^3)$, where $I = (-1, 1)$ and $\gamma_\alpha : I \to \mathbb{R}^5$ is defined by $\gamma_\alpha(t) := (0,\, t,\, \alpha,\, t, 0)$, for $\alpha = 0, 1$. Therefore, $u \in SBV(B^2, \mathbb{R}^3)$, with jump set $S(u) = \{0\} \times I$, so that inequality (4.2) *holds true with, e.g., $\theta = 2$ and $\mathfrak{b} = S(u)$.*

We have already defined $M(F) = (F, \mathrm{adj}_2 F) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$. Here we assume that a sequence $\{u_h\}_h \subset \mathscr{A}^1(\Omega, \mathbb{R}^3)$ is such that

$$\sup_h \|u_h\|_{L^\infty(\Omega)} \leq K \, , \quad \sup_h \|M(\nabla u_h)\|_{L^1(\Omega)} < \infty \, ,$$

where the sequence $\{|M(\nabla u_h)|\}$ is equi-integrable on Ω. In addition, we assume that $\pi_\# |\partial G_{u_h}| \leq \mu_{V_h}$ for each h. If we have $\sup_h \mu_{V_h}(\Omega) < \infty$, by compactness, possibly passing to a (not relabeled) subsequence, we have $V_h \rightharpoonup V$ weakly as measures, and we find a deformation map u as above, such that G_u is i.m. rectifiable and with finite mass, $G_{u_h} \rightharpoonup G_u$ weakly as currents, $\pi_\# |\partial G_u| \leq \mu_V < \infty$, whence ∂G_u is i.m. rectifiable, too.

In order to recover the weak L^1 convergence of $M(\nabla u_h)$ to $M(\nabla u)$, the starting point is the following special case of the closure theorem proven in reference [24]. On account of the compactness theorem in SBV, it extends a classical result proved in reference [7] for Sobolev maps, where the divergence form of gradient minors is exploited.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\{u_h\}$ a sequence of functions from $SBV(\Omega, \mathbb{R}^3)$ converging in $L^1(\Omega, \mathbb{R}^3)$ to a summable function*

$u:\Omega\to\mathbb{R}^3$. *Assume that for some real exponents* $q>2$ *and* $r>1$

$$\sup_h\Big\{\|u_h\|_\infty+\int_\Omega\Big(|\nabla u_h|^q+|\mathrm{adj}_2\nabla u_h|^r\Big)\,dx+\mathscr{H}^1(S(u_h))\Big\}<\infty\,.$$

Then, $u\in SBV(\Omega,\mathbb{R}^3)$*, and the sequence* $\mathscr{H}^1\llcorner S(u_h)$ *weakly converges in* Ω *to a measure* μ *greater than* $\mathscr{H}^1\llcorner S(u)$*. Moreover,* ∇u_h *weakly converges to* ∇u *in* $L^q(\Omega,\mathbb{M}^{3\times 2})$*, and* $\mathrm{adj}_2\nabla u_h$ *weakly converges to* $\mathrm{adj}_2\nabla u$ *in* $L^r(\Omega,\mathbb{R}^3)$*.*

4.2 Accounting for the thickness

We fix $q>2$, $r,p>1$, $K>0$, and introduce the class $\mathscr{A}_{q,r,p,K}$ of triplets (u,ζ,V) where

- *(i)* $u:\Omega\to\mathbb{R}^3$ is a special function of bounded variation, $u\in SBV(\Omega,\mathbb{R}^3)$, with $\mathscr{H}^1(S(u))<\infty$, such that $\|u\|_{L^\infty(\Omega)}\le K$, $\partial_1 u\times\partial_2 u\neq 0$ $\mathscr{L}^2$-a.e. on Ω, $\nabla u\in L^q(\Omega,\mathbb{M}^{3\times 2})$, and $|\mathrm{adj}_2\nabla u|\in L^r(\Omega)$;
- *(ii)* $\zeta:\Omega\to\mathbb{R}^3$ is a special function of bounded variation, with $\mathscr{H}^1(S(\zeta))<\infty$, such that $|\zeta|=1$ $\mathscr{L}^2$-a.e. on Ω, and $\nabla\zeta\in L^q(\Omega,\mathbb{M}^{3\times 2})$;
- *(iii)* $V\in CV_1^p(\Omega)$, i.e. V is a integer rectifiable curvature 1-varifold with boundary and second fundamental form $A\in L^p(\mathscr{G}_1(\Omega),\mathbb{R}^{2*}\otimes\mathbb{R}^2\otimes\mathbb{R}^{2*})$, with $\mu_V=\theta\mathscr{H}^1\llcorner\mathfrak{b}$;
- *(iv)* $\pi_\#|\partial G_u|\le\mu_V$;
- *(v)* $\mathscr{H}^1\llcorner S(\zeta)\le\pi_\#|\partial G_u|$;
- *(vi)* $(\partial_1 u(x)\times\partial_2 u(x))\bullet\zeta(x)>0$ for $\mathscr{L}^2$-a.e. $x\in\Omega\setminus\mathfrak{b}$.

Then, we consider on the class of triplets $(u,\zeta,V)\in\mathscr{A}_{q,r,p,K}$ the energy

$$\mathscr{F}(u,\zeta,V):=\int_\Omega e(x,u,\zeta,\nabla u,\nabla\zeta)\,dx+\|V\|+\int_{\mathscr{G}_1(\Omega)}\|A\|^p\,dV+\|\partial V\|\,, \quad (4.4)$$

where $e:\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{M}^{3\times 2}\times\mathbb{M}^{3\times 2}\to\mathbb{R}\cup\{+\infty\}$ is the sum

$$e(x,u,\zeta,F,G)=\widetilde{e}(x,u,\zeta,F)+\beta_1\,|G|^q$$

for every $(x,u,\zeta,F,G)\in\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{M}^{3\times 2}\times\mathbb{M}^{3\times 2}$, with $\beta_1>0$ and the first addendum a non-negative Carathéodory function satisfying the following properties:

(a) $\widetilde{e}(x,u,\zeta,F)$ is polyconvex with respect to F, namely

$$\widetilde{e}(x,u,\zeta,F)=g(x,u,\zeta,M(F))\qquad\forall F\in\mathbb{M}^{3\times 2},$$

where $g:\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times(\mathbb{M}^{3\times 2}\times\mathbb{R}^3)\to[0,+\infty]$ is a Carathéodory function, with $g(x,u,\zeta,\cdot)$ convex for $\mathscr{L}^2$-a.e. $x\in\Omega$, and for all $(u,\zeta)\in\mathbb{R}^3\times\mathbb{R}^3$;

(b) for $\mathscr{L}^2$-a.e. $x \in \Omega$ and every (u, ζ, F) such that $\det(F|\zeta) > 0$,

$$\widetilde{e}(x, u, \zeta, F) \geq \beta_2 \left(|F|^q + |\mathrm{adj}_2 F|^r\right) + \Phi(F, \zeta),$$

where $\beta_2 > 0$ and $\Phi \in \widetilde{\mathscr{F}}$, see Definition 3.1.

Theorem 4.2 *The energy minimum of $\mathscr{F}(u, V, \zeta)$ is attained in the class $\mathscr{A}_{q,r,p,K}$.*

Proof. As before, we repeatedly pass to not relabeled subsequences. Choose a minimizing sequence $\{(u_h, \zeta_h, V^{(h)})\} \subset \mathscr{A}_{q,r,p,K}$. On account of assumptions (iv) and (v), by the energy lower bounds we infer that Ambrosio's compactness theorem in SBV [5, Thm. 4.8] applies to both sequences $\{u_h\}$ and $\{\zeta_h\}$. Therefore, $u_h \to u$ in $L^1(\Omega, \mathbb{R}^3)$ to some function $u \in SBV(\Omega, \mathbb{R}^3)$ with $\|u\|_{L^\infty(\Omega)} \leq K$, whereas ∇u_h weakly converges to ∇u in $L^q(\Omega, \mathbb{M}^{3\times 2})$ and $\mathscr{H}^1 \llcorner S(u_h)$ weakly converges in Ω to a measure greater than $\mathscr{H}^1 \llcorner S(u)$. In a similar way, we prove existence of a function $\zeta \in SBV(\Omega, \mathbb{R}^3)$ satisfying $|\zeta| = 1$ a.e. in Ω, such that $\zeta_h \to \zeta$ in $L^1(\Omega, \mathbb{R}^3)$, $\nabla \zeta_h$ weakly converges to $\nabla \zeta$ in $L^q(\Omega, \mathbb{M}^{3\times 2})$, and $\mathscr{H}^1 \llcorner S(\zeta_h)$ weakly converges in Ω to a measure greater than $\mathscr{H}^1 \llcorner S(\zeta)$. Furthermore, Allard's compactness theorem 2.1 applies to the sequence $\{V^{(h)}\}$ in $CV^p(\Omega)$. Also, Federer-Fleming's closure theorem applies to the sequence G_{u_h} (see [28]).

On account of Theorem 4.1, we obtain a triplet (u, ζ, V) satisfying properties (i)–(v). In fact, using (a), the lower-semicontinuity result in reference [24] yields

$$\int_\Omega \widetilde{e}(x, u, \nabla u, \zeta)\, dx \leq \liminf_{h\to\infty} \int_\Omega \widetilde{e}(x, u_h, \nabla u_h, \zeta_h)\, dx$$

and

$$\int_\Omega |\nabla \zeta|^q\, dx \leq \liminf_{h\to\infty} \int_\Omega |\nabla \zeta_h|^q\, dx$$

whereas

$$\|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p\, dV + \|\partial V\| \leq \liminf_{h\to\infty} \Big(\|V^{(h)}\| + \int_{\mathscr{G}_1(\Omega)} \|A^{(h)}\|^p\, dV^{(h)} + \|\partial V^{(h)}\| \Big)$$

so that by equation (4.4) we get

$$\mathscr{F}(u, \zeta, V) \leq \liminf_{h\to\infty} \mathscr{F}(u_h, \zeta_h, V^{(h)})\,.$$

Also, by the weak convergence $\mathrm{adj}_2 \nabla u_h \rightharpoonup \mathrm{adj}_2 \nabla u$ in $L^1(\Omega, \mathbb{R}^3)$, using the bound $\sup_h |\nabla \zeta_h|^q\, dx < \infty$ for $q > 2$ and the embedding theorem, we obtain the weak convergence $(\partial_1 u_h \times \partial_2 u_h) \bullet \zeta_h \rightharpoonup (\partial_1 u \times \partial_2 u) \bullet \zeta$ in $L^1(\Omega)$. As a consequence, the lower semicontinuity inequality

$$\int_\Omega \Phi(\nabla u, \zeta)\, dx \leq \liminf_{h\to\infty} \int_\Omega \Phi(\nabla u_h, \zeta_h)\, dx$$

holds true. Hence, by the lower bound (b), property (vi) is satisfied by the triplet (u, ζ, V), whence $(u, \zeta, V) \in \mathscr{A}_{q,r,p,K}$, as required. ■

In fact, we could follow the view suggested in reference [14], considering energy functionals $\widetilde{\mathscr{F}}(u, \zeta, V)$ in which we mix together the gradients of u and ζ. Namely, in (4.4) we could assume that $e(x, u, \zeta, F, G) = \widetilde{e}(x, h, H)$, where $h := (u, \zeta)$, $H \in \mathbb{M}^{6\times 2}$ is the matrix with rows (F, G), while $\widetilde{e}(x, h, H)$ is a non negative Carathéodory function, which is polyconvex with respect to H and such that for $\mathscr{L}^2$-a.e. $x \in \Omega$ and every (h, H), with $\det(F|\zeta) > 0$

$$\widetilde{e}(x, h, H) \geq \beta_2 \left(|H|^q + |\mathrm{adj}_2 H|^r\right) + \Phi(F, \zeta)$$

where $\beta_2 > 0$ and $\Phi \in \widetilde{\mathscr{F}}$.

Since the closure and semicontinuity properties continue to hold, the minimum of $\widetilde{\mathscr{F}}(u, \zeta, V)$ is attained in the class $\widetilde{\mathscr{A}}_{q,r,p,K}$ of triplets (u, ζ, V) in $\mathscr{A}_{q,r,p,K}$, which satisfy the additional condition $|\mathrm{adj}_2 \nabla(u, \zeta)| \in L^r(\Omega)$, so that we should presume in this case what follows:

(1) $u \in SBV(\Omega, \mathbb{R}^3)$, with $\mathscr{H}^1(S(u)) < \infty$, $\|u\|_{L^\infty(\Omega)} \leq K$, $\partial_1 u \times \partial_2 u \neq 0$ $\mathscr{L}^2$-a.e. on Ω, $\nabla u \in L^q(\Omega, \mathbb{M}^{3\times 2})$;
(2) $\zeta \in SBV(\Omega, \mathbb{R}^3)$, with $\mathscr{H}^1(S(\zeta)) < \infty$, $|\zeta| = 1$ $\mathscr{L}^2$-a.e. on Ω, and $\nabla \zeta \in L^q(\Omega, \mathbb{M}^{3\times 2})$;
(3) $|\mathrm{adj}_2 \nabla(u, \zeta)| \in L^r(\Omega)$;
(4) $V \in CV_1^p(\Omega)$, with $\mu_V = \theta \mathscr{H}^1 \llcorner \mathfrak{b}$;
(5) $\pi_\# |\partial G_u| \leq \mu_V$;
(6) $\mathscr{H}^1 \llcorner S(\zeta) \leq \pi_\# |\partial G_u|$;
(7) $(\partial_1 u(x) \times \partial_2 u(x)) \bullet \zeta(x) > 0$ for $\mathscr{L}^2$-a.e. $x \in \Omega \setminus \mathfrak{b}$.

However, we prefer to maintain separate membrane and out-of-middle-surface behavior, so that we refer to surface polyconvexity as considered in reference [40].

5 Boundary conditions

We consider two types of boundary conditions. As a first choice we prescribe Dirichlet-type data, namely

$$u = u_0\,, \quad \zeta = \frac{\partial_1 u_0 \times \partial_2 u_0}{|\partial_1 u_0 \times \partial_2 u_0|} \quad \text{on } \partial\Omega \tag{5.1}$$

in the sense of traces, respectively, for some given a.e. injective function $u_0 \in W^{1,q}(\Omega, \mathbb{R}^3)$ with $\|u_0\|_\infty \leq K$ and $|\mathrm{adj}_2 \nabla u_0| \in L^r(\Omega, \mathbb{R}^+)$.

Then, we consider those that we call *strong anchoring conditions*, determined by assigning the boundary current, i.e., in terms of smooth and bounded 1-forms in $\overline{\Omega} \times \mathbb{R}^3$,

$$\partial G_u \llcorner (\partial\Omega \times \mathbb{R}^3) = \partial G_{u_0} \llcorner (\partial\Omega \times \mathbb{R}^3)$$

a condition that clearly implies the trace equality $u = u_0$. As already mentioned, from a physical viewpoint such a condition means that we are assigning the work performed in all possible strain modes, all considered at first to be independent and then reconciled in the limit to be compatible (see [26]). This boundary condition is generally not preserved in the minimization process because along the boundary open cracks may have optimal placement.

A confinement condition prescribing the existence of a compact set $\mathscr{C}$ contained in the open set Ω and such that

$$\operatorname{spt} \mu_V \subset \mathscr{C} \qquad \forall\, (u, \zeta, V) \in \mathscr{A}_{q,r,p,K} \tag{5.2}$$

avoids the problem.

The chain of inequalities $\mathscr{H}^1 \llcorner S(\zeta) \leq \pi_\# |\partial G_u| \leq \mu_V$ implies that the restriction to $\Omega \setminus \mathscr{C}$ of both u and ζ belongs to the Sobolev class $W^{1,q}(\Omega \setminus \mathscr{C}, \mathbb{M}^{3\times 2})$. Therefore, the prescribed Dirichlet or strong anchoring conditions are preserved in the limit process, due to the weak convergence in $W^{1,q}(\Omega \setminus \mathscr{C}, \mathbb{M}^{3\times 2})$ of both $\{u_h\}$ and $\{\zeta_h\}$.

Corollary 5.1 *Let $\mathscr{C}$ and u_0 as above. Assume that competitors (u, V, ζ) in $\mathscr{A}_{q,r,p,K}$ have finite energy $\mathscr{F}(u, V, \zeta)$ and satisfy the prescribed boundary, clamping, and confinement conditions, with $\mathscr{C}$ is a non-empty set. Then, the energy minimum of $\mathscr{F}(u, V, \zeta)$ is attained in the same class.*

Remark 5.1 *Condition* (5.2) *excludes circumstances in which a crack path may go to the boundary, breaking part of a link with the environment.*

6 Shells made of complex materials

To account for active microstructures (e.g., polarization in ferroelectric films) we introduce over Ω another map ν taking values on a connected, complete, n-dimensional Riemannian differentiable manifold $\mathscr{M}$ of class C^2. At every $x \in \Omega$, the map ν summarizes at gross scale geometric information on material microstructure in the film thickness. For this reason, we call $\mathscr{M} = (\mathscr{M}^n, g)$ as the *manifold of microstructural shapes*. The generality adopted in choosing $\mathscr{M}$ is a way to furnish unified results, which are independent of specific microstructural features.

First we assume the layer-descriptor map $\nu : \Omega \to \mathscr{M}$ to be an *intrinsic Sobolev map* in $W^{1,s}(\Omega, \mathscr{M})$ for some real exponent $s > 1$. Then, we'll discuss the case in which ν can be considered as a special function of bounded variation.

The essential point is that we do not use any embedding of $\mathscr{M}$ in some Euclidean space. Even choosing it to be isometric, it would not be unique (as Nash's theorems indicate). Thus, its choice would become part of the model, while the common effort is to offer a description of the phenomenological world as much as possible free of non intrinsic elements.

6.1 The Sobolev case

Under previous assumptions, with $d_{\mathscr{M}}$ the geodesic distance in $\mathscr{M}$, by the Hopf-Rinow theorem $(\mathscr{M}, d_{\mathscr{M}})$ is a complete metric space. Consequently, we keep referring to results in such spaces, summarizing those aspects that we need for the analysis developed here from essential references [4], [45], [46], [47] on this topic.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain and $s \geq 1$. A Borel map $\nu : \Omega \to \mathscr{M}$ is said to be an intrinsic Sobolev map in $W^{1,s}(\Omega, \mathscr{M})$, if there exists a non-negative function $\phi \in L^s(\Omega)$ such that for every $\nu_0 \in \mathscr{M}$

(1) $x \mapsto d_{\mathscr{M}}(\nu(x), \nu_0)$ is in $L^s(\Omega)$, i.e., $\nu \in L^s(\Omega, \mathscr{M})$;
(2) the distributional gradient map $x \mapsto Dd_{\mathscr{M}}(\nu(x), \nu_0)$ satisfies the inequality $|Dd_{\mathscr{M}}(\nu(x), \nu_0)| \leq \phi(x)$ for $\mathscr{L}^m$-a.e. $x \in \Omega$.

In this setting, the "norm" $|D\nu|(x)$, which is, in essence, the optimal function $\phi \in L^s(\Omega)$ satisfying the inequality $|Dd_{\mathscr{M}}(\nu(x), \nu_0)| \leq \phi(x)$, is well-defined for $\mathscr{L}^m$-a.e. $x \in \Omega$ by

$$|D\nu|(x) := \sup_{k \in \mathbb{N}} |D(d_{\mathscr{M}}(\nu(x), \nu_k))|$$

where $\{\nu_k\}_{k\in\mathbb{N}}$ forms a dense and enumerable set in $\mathscr{M}$.

Weak convergence in $W^{1,s}$ of a sequence $\{\nu_h\} \subset W^{1,s}(\Omega, \mathscr{M})$ to some map $\nu \in W^{1,s}(\Omega, \mathscr{M})$, when $s > 1$, is defined by requiring that $\|d_{\mathscr{M}}(\nu_h, \nu)\|_{L^s(\Omega)} \to 0$ as $h \to \infty$ and $\sup_h \| \, |D\nu_h| \, \|_{L^s(\Omega)} < \infty$. When $s = 1$, one assumes in addition that the sequence $\{|D\nu_h|\}$ is equi-integrable.

Also, the trace operator $\mathrm{Tr} : W^{1,s}(\Omega, \mathscr{M}) \to L^s(\partial\Omega, \mathscr{M})$ is well-defined in such a way that for continuous maps $\nu \in W^{1,s}(\Omega, \mathscr{M}) \cap C^0(\overline{\Omega}, \mathscr{M})$ it agrees with the restriction $\nu_{|\partial\Omega}$. Moreover, if $\{\nu_h\} \subset W^{1,s}(\Omega, \mathscr{M})$ weakly converges to some map $\nu \in W^{1,s}(\Omega, \mathscr{M})$, then $\mathrm{Tr}(\nu_h)$ converges to $\mathrm{Tr}(\nu)$ strongly in $L^s(\partial\Omega, \mathscr{M})$.

Finally, traces of maps in $W^{1,s}(\Omega,\mathscr{M})$ have a $W^{1-1/s,s}$-regularity, when $s>1$ (see [33] and [12]).

We endow the tangent bundle $T\mathscr{M}$ with the metric $d_{T\mathscr{M}}$ induced by $d_{\mathscr{M}}$ and consider the vector bundle with base space $\mathscr{M}$ and typical fiber the space of linear homomorphisms $\mathrm{Hom}(\mathbb{R}^m,T\mathscr{M})$. Points of such a bundle are couples (ν,N), where $\nu\in\mathscr{M}$ and $N:\mathbb{R}^m\to T_\nu\mathscr{M}$ is a linear map. For any fixed $\nu\in\mathscr{M}$, we can identify $N\in\mathrm{Hom}(\mathbb{R}^m,T_\nu\mathscr{M})$ with the m-tuple $(v_1,\dots,v_m)\in(T_\nu\mathscr{M})^m$, where $v_i=Ne_i$ and $(e_1,\dots,e_m)$ is the canonical basis in $\mathbb{R}^m$. Therefore, a metric structure on the vector bundle $\mathrm{Hom}(\mathbb{R}^m,T\mathscr{M})$ is defined through the distance

$$D((\nu,N),(\widetilde{\nu},\widetilde{N})):=\left\{\sum_{i=1}^m d_{T\mathscr{M}}\Big((\nu,v_i),(\widetilde{\nu},w_i)\Big)^2\right\}^{1/2}$$

if $\widetilde{N}\in\mathrm{Hom}(\mathbb{R}^m,T_{\widetilde{\nu}}\mathscr{M})$ and $w_i=\widetilde{N}e_i$ for each i.

A non-negative and continuous integrand $f:\Omega\times\mathrm{Hom}(\mathbb{R}^m,T\mathscr{M})\to[0,+\infty)$ is said to be *admissible* in $W^{1,s}(\Omega,\mathscr{M})$ if for some fixed point $\nu_0\in\mathscr{M}$ and some positive constant C the bounds

$$0\le f(x,\nu,N)\le C\left(1+d_{\mathscr{M}}(\nu,\nu_0)^s+\|N\|^s_{g(\nu)}\right)$$

hold true for all $(x,\nu,N)\in\Omega\times\mathrm{Hom}(\mathbb{R}^m,T_\nu\mathscr{M})$, where $\|\cdot\|_{g(\nu)}$ is the operatorial norm.

A functional

$$\nu\mapsto\int_\Omega f(x,\nu(x),d\nu_x)\,dx \tag{6.1}$$

with an admissible integrand f is well-defined on maps $\nu\in W^{1,s}(\Omega,\mathscr{M})$, where $d\nu_x$ is the approximate differential. Precisely, if $\nu:\Omega\to\mathscr{M}$ is a Borel map, and $x\in\mathscr{M}$ is a point of approximate continuity of ν, a linear map $N\in\mathrm{Hom}(\mathbb{R}^m,T_{\nu(x)}\mathscr{M})$ is said to be an *approximate differential* of ν at x if, for all $\epsilon>0$,

$$\lim_{\rho\to0^+}\frac{1}{\rho^m}\mathscr{L}^m\Big(\Big\{y\in B^m_\rho(x)\mid d_{\mathscr{M}}\Big(\nu(y),\exp_{\nu(x)}(N(y-x))\Big)\ge\epsilon\,|y-x|\Big\}\Big)=0\,.$$

When it exists, the approximate differential of ν at x is unique. For smooth maps, it agrees with the classical differential (in geometric sense). For this reason, the notation $d\nu_x$ is used for it. A Sobolev map ν in $W^{1,s}(\Omega,\mathscr{M})$ is approximately differentiable at $\mathscr{L}^m$-a.e. $x\in\mathscr{M}$, whence the functional (6.1) makes sense.

We ask for sequential lower semicontinuity. Thus, we need a suitable notion of quasiconvexity.

Definition 6.1 *A locally bounded function* $f : \Omega \times Hom(\mathbb{R}^m, T\mathscr{M}) \to \mathbb{R}$ *is said to be* quasiconvex *if for every* $(x, \nu, N) \in \Omega \times Hom(\mathbb{R}^m, T_\nu\mathscr{M})$ *and every test function* $\varphi \in C_c^\infty(Q_1, T_\nu\mathscr{M})$*, where* $Q_1 := [-1/2, 1/2]^m$*, by looking at* $d\varphi_y$ *as an element of* $Hom(\mathbb{R}^m, T_\nu\mathscr{M})$*, the following inequality holds true:*

$$f(x, \nu, N) \leq \int_{Q_1} f(x, \nu, N + d\varphi_y)\, dy\,.$$

Theorem 6.1 ([18]) *Take* $s \geq 1$*. Let* $f : \Omega \times Hom(\mathbb{R}^m, T\mathscr{M}) \to [0, +\infty)$ *be a non-negative and continuous admissible integrand in* $W^{1,s}(\Omega, \mathscr{M})$*. Then, the functional* (6.1) *is sequentially weakly lower semicontinuous in* $W^{1,s}(\Omega, \mathscr{M})$ *if and only if* f *is quasiconvex.*

Coming back to the physical dimension $m = 2$, a membrane with out-of-middle-surface vector field ζ, a crack represented by a 1-varifold V, a descriptor ν of the through-the-thickness material morphology, subjected to a deformation u, is modeled by a quadruplet (u, ζ, V, ν) belonging to the class $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ constructed as follows: for $q > 2$, $r, p, s > 1$, $K > 0$, we assume that the triplet (u, ζ, V) belongs to the class $\mathscr{A}_{q,r,p,K}$ introduced above, while we let $\nu \in W^{1,s}(\Omega, \mathscr{M})$, where $\mathscr{M} = (\mathscr{M}^n, g)$ is as above.

On the class $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ we consider the energy functional

$$\mathscr{F}(u, \zeta, V, \nu) := \int_\Omega e(x, u, \zeta, \nu, \nabla u, \nabla\zeta, d\nu)\, dx + \|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p\, dV + \|\partial V\|\,. \tag{6.2}$$

The energy density $e : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times2} \times \mathbb{M}^{3\times2} \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M}) \to \mathbb{R} \cup \{+\infty\}$ is presumed to be

$$e(x, u, \zeta, \nu, F, G, N) = \widetilde{e}(x, u, \zeta, \nu, F) + \beta_1 |G|^q + f(x, \nu, N)$$

for every $(x, u, \zeta, \nu, F, G, N) \in \Omega\times\mathbb{R}^3\times\mathbb{R}^3\times\mathscr{M}\times\mathbb{M}^{3\times2}\times\mathbb{M}^{3\times2}\times\mathrm{Hom}(\mathbb{R}^2, T\mathscr{M})$, where $\beta_1 > 0$, and the first addendum is a non-negative Carathéodory function with the following properties:

(a') $\widetilde{e}(x, u, \zeta, \nu, F)$ is polyconvex with respect to F, namely

$$\widetilde{e}(x, u, \zeta, \nu, F) = g(x, u, \zeta, \nu, M(F)) \qquad \forall F \in \mathbb{M}^{3\times2} \tag{6.3}$$

where $g : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times (\mathbb{M}^{3\times2} \times \mathbb{R}^3) \to [0, +\infty]$ is a Carathéodory function, with $g(x, u, \zeta, \nu, \cdot)$ convex for $\mathscr{L}^2$-a.e. $x \in \Omega$ and for all $(u, \zeta, \nu) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M}$;

(b') for $\mathscr{L}^2$-a.e. $x \in \Omega$ and every (u, ζ, ν, F) such that $\det(F|\zeta) > 0$

$$\widetilde{e}(x, u, \zeta, \nu, F) \geq \beta_2 \left(|F|^q + |\mathrm{adj}_2 F|^r\right) + \Phi(F, \zeta)$$

where $\beta_2 > 0$ and $\Phi \in \widetilde{\mathscr{F}}$, a function class specified in Definition 3.1.

Moreover, the third addendum $f : \Omega \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M}) \to [0, +\infty)$ is a non-negative, continuous, and quasiconvex admissible integrand in $W^{1,s}(\Omega, \mathscr{M})$. We presume it is such that for positive constants $C_1, C_2 > 0$ the inequality

$$C_1 \cdot \|N\|^s_{g(\nu)} \le f(x, \nu, N) \le C_2 \cdot \|N\|^s_{g(\nu)}$$

holds true for every $(x, \nu, N) \in \Omega \times \mathrm{Hom}(\mathbb{R}^2, T\mathscr{M})$.

Theorem 6.2 *Under the previous assumptions, the minimum of the energy functional $\mathscr{F}(u, V, \zeta, \nu)$ is attained in the class $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$*

Proof. Choose a minimizing sequence $\{(u_h, \zeta_h, V^{(h)}, \nu_h)\} \subset \mathscr{A}_{q,r,p,K,s}(\mathscr{M})$. The growth assumptions on f imply

$$\sup_h \int_\Omega \|d(\nu_h)_x\|^s_{g(\nu_h(x))} \, dx < \infty \, .$$

Therefore, by compactness (see [33]), possibly passing to a (not relabeled) subsequence, $\{\nu_h\}$ weakly converges in $W^{1,s}$ to some $\nu \in W^{1,s}(\Omega, \mathscr{M})$, which satisfies $d_{\mathscr{M}}(\nu_h(x), \nu(x)) \to 0$ for $\mathscr{L}^2$-a.e. $x \in \Omega$. Moreover, by exploiting Theorem 6.1, we get

$$\int_\Omega f(x, \nu(x), d\nu_x) \, dx \le \liminf_{h \to \infty} \int_\Omega f(x, \nu_h(x), d(\nu_h)_x) \, dx \, .$$

The proof then follows the same path as that of Theorem 4.2. We omit further details. ■

As to the boundary conditions on the quadruplets (u, ζ, V, ν) in $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$, we first assume

$$u = u_0 \, , \quad \zeta = \frac{\partial_1 u_0 \times \partial_2 u_0}{|\partial_1 u_0 \times \partial_2 u_0|} \, , \quad \nu = \nu_0 \quad \text{on } \partial\Omega$$

in the sense of traces, for given maps $\nu_0 \in W^{1,s}(\Omega, \mathscr{M})$ and $u_0 \in W^{1,q}(\Omega, \mathbb{R}^3)$, with $\|u_0\|_\infty \le K$ and $|\mathrm{adj}_2 \nabla u_0| \in L^r(\Omega)$.

Then, we also prescribe a compact set $\mathscr{C}$ contained in Ω such that

$$\mathrm{spt}\, \mu_V \subset \mathscr{C} \qquad \forall \, (u, \zeta, V, \nu) \in \mathscr{A}_{q,r,p,K,s}(\mathscr{M}) \, .$$

In this case, strong anchoring conditions are correspondingly defined in terms of the graph boundary current ∂G_u as in the previous section.

Corollary 6.3 *Let $\mathscr{C}$, u_0, and ν_0 as above. Assume that competitors (u, V, ζ, ν) in $\mathscr{A}_{q,r,p,K,s}(\mathscr{M})$ have finite energy $\mathscr{F}(u, V, \zeta, \nu)$ and satisfy prescribed boundary, clamping, and confinement conditions, with $\mathscr{C}$ a non-empty set. Then, the energy minimum of $\mathscr{F}(u, V, \zeta, \nu)$ is attained in the same class.*

The through-the-thickness descriptor of the material microstructure ν might jump across the crack margins. So it could be natural to consider $\nu : \Omega \to \mathscr{M}$ as a special function of bounded variation. However, in this case we do not have at disposal a lower semicontinuity result for manifold-valued BV-maps. Then, we restrict ourselves to consider just the total variation of ν distributional derivative among the energy entries and base the pertinent analyses on the closure-compactness theorem proven in reference [4].

The class of Sobolev maps $\nu \in W^{1,s}(\Omega, \mathscr{M})$ can be equivalently defined in terms of post-composition with Lipschitz functions $\varphi : \mathscr{M} \to \mathbb{R}$. By letting

$$\widehat{\mathscr{F}} := \{\varphi \in \mathrm{Lip}(\mathscr{M}, \mathbb{R}) \mid \mathrm{Lip}(\varphi) \leq 1\}$$

one requires $\nu \in L^s(\Omega, \mathscr{M})$ and the existence of a non-negative function $\phi \in L^s(\Omega)$ such that $\varphi \circ \nu \in W^{1,s}(\Omega)$ and $\|\nabla(\varphi \circ \nu)\|_{L^s(\Omega)} \leq \|\phi\|_{L^s(\Omega)}$ for every $\varphi \in \widehat{\mathscr{F}}$. The optimal function $\phi \in L^s(\Omega)$, which satisfies the previous inequality independently of $\varphi \in \widehat{\mathscr{F}}$, agrees $\mathscr{L}^m$-essentially with the function $|D\nu|$ previously considered.

A summable map $\nu \in L^1(\Omega, \mathscr{M})$ is said to be a function of bounded variation in $BV(\Omega, \mathscr{M})$ if there exists a finite Borel measure μ in Ω such that the total variation of the distributional derivative $D(\varphi \circ \nu)$ is bounded by μ, i.e., $|D(\varphi \circ \nu)|(B) \leq \mu(B)$ for every Borel set $B \subset \Omega$ and every $\varphi \in \widehat{\mathscr{F}}$. The least measure μ satisfying such a property is denoted by $|D\nu|$ (see reference [4] for further details).

If $\nu \in BV(\Omega, \mathscr{M})$, the *countably* $(m-1)$*-rectifiable jump set* $S(\nu)$ is defined in terms of the jump sets of the BV functions $\varphi \circ \nu$, for a suitable countable set of functions $\varphi \in \widehat{\mathscr{F}}$. The one-sided limits $\nu^{\pm}(x)$ are correspondingly defined at $\mathscr{H}^{m-1}$-a.e. $x \in S(\nu)$, once the $\mathscr{H}^{m-1}$-measurable unit normal $\mathbf{n}_\nu(x)$ to $S(\nu)$ is fixed, as the manifold elements $z^{\pm} \in \mathscr{M}$ such that the set $\{y \in \Omega \mid d_{\mathscr{M}}(\nu(y), z^{\pm}) > \epsilon\,,\ \langle y - x, \pm\mathbf{n}_\nu(x)\rangle > 0\}$ has 0-density at x for any $\epsilon > 0$.

In a similar way, we define a non-negative function $|\nabla\nu| \in L^1(\Omega)$ in terms of $\sup|\nabla(\varphi \circ \nu)|$, where $\nabla(\varphi \circ \nu)$ denotes the approximate gradient of the BV function $\varphi \circ \nu$. Then, $|\nabla\nu|$ agrees with the Radon-Nikodym derivative of $|D\nu|$ with respect to $\mathscr{L}^m \llcorner \Omega$ so that for $\mathscr{L}^m$-a.e. $x \in \Omega$ we compute the approximate limit

$$\operatorname{ap}\lim_{y \to x} \frac{d_{\mathscr{M}}(\nu(y), \nu(x))}{|y - x|} = |\nabla\nu|(x)\,.$$

A function $\nu \in BV(\Omega, \mathscr{M})$ is said to be a *special function of bounded variation* if $\varphi \circ \nu \in SBV(\Omega)$ for every $\varphi \in \widehat{\mathscr{F}}$. If $\nu \in SBV(\Omega, \mathscr{M})$, for every Borel set $B \subset \Omega$ we get the decomposition formula

$$|D\nu|(B) = \int_B |\nabla \nu| \, dx + \int_{B \cap S(\nu)} d_{\mathscr{M}}(\nu^+, \nu^-) \, d\mathscr{H}^{m-1} .$$

Theorem 6.4 ([4]) *For $\mathscr{K} \subset \mathscr{M}$ a compact set, consider $\{\nu_h\} \subset SBV(\Omega, \mathscr{M})$ such that $\nu_h(\Omega) \subset \mathscr{K}$ for every h. If for some exponent $s > 1$*

$$\sup_h \Big(\int_\Omega |\nabla \nu_h|^s \, dx + \int_{S(\nu_h)} \Big(1 + d_{\mathscr{M}}(\nu_h^+, \nu_h^-) \Big) \, d\mathscr{H}^{m-1} \Big) < \infty, \tag{6.4}$$

there exists a (not relabeled) subsequence of $\{\nu_h\}$ converging $\mathscr{L}^m$-a.e. in Ω to some $\nu \in SBV(\Omega, \mathscr{M})$, and such that

$$\int_\Omega |\nabla \nu|^s \, dx \le \liminf_{h \to \infty} \int_\Omega |\nabla \nu_h|^s \, dx$$
$$\int_{S(\nu)} d_{\mathscr{M}}(\nu^+, \nu^-) \, d\mathscr{H}^{m-1} \le \liminf_{h \to \infty} \int_{S(\nu_h)} d_{\mathscr{M}}(\nu_h^+, \nu_h^-) \, d\mathscr{H}^{m-1} .$$

In the case considered here, we take first $q > 2$, $r, p, s > 1$, $K > 0$. Then, we consider a compact set $\mathscr{K}$ in $\mathscr{M}$, where $\mathscr{M} = (\mathscr{M}^n, g)$ is given as above. We say that $(u, \zeta, V, \nu) \in \widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$ if the triplet (u, ζ, V) belongs to the class $\mathscr{A}_{q,r,p,K}$ previously introduced and $\nu \in SBV(\Omega, \mathscr{M})$ satisfies the following relations:

(1) $\nu(\Omega) \subset \mathscr{K}$ with $|\nabla \nu| \in L^s(\Omega)$, and
(2) $\mathscr{H}^1 \llcorner S(\nu) \le \mu_V$.

Assumption (1) is of technical nature, whereas (2) means that ν jumps only over the fracture.

Thus, we consider the energy to be

$$\widetilde{\mathscr{F}}(u, \zeta, V, \nu) := \int_\Omega e(x, u, \zeta, \nu, \nabla u, \nabla \zeta, |\nabla \nu|) \, dx + \|V\| + \int_{\mathscr{G}_1(\Omega)} \|A\|^p \, dV + \|\partial V\| . \tag{6.5}$$

The density $e : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is such that

$$e(x, u, \zeta, \nu, F, G, N) = \widetilde{e}(x, u, \zeta, \nu, F) + \beta_1 |G|^q + \beta_3 |N|^s \tag{6.6}$$

for every $(x, u, \zeta, \nu, F, G, N) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathscr{M} \times \mathbb{M}^{3\times 2} \times \mathbb{M}^{3\times 2} \times \mathbb{R}$, where $\beta_1 > 0$, $\beta_3 > 0$, and the first addendum is a non-negative Carathéodory function satisfying the properties (a') and (b') above introduced when $\nu \in W^{1,s}(\Omega, \mathscr{M})$.

Theorem 6.5 *The minimum of the energy functional $\widetilde{\mathscr{F}}(u, \zeta, V, \nu)$ given by* (6.5) *is attained in the class $\widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$.*

Proof. Take a minimizing sequence $\{(u_h, \zeta_h, V^{(h)}, \nu_h)\} \subset \widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$. Since $\mathscr{H}^1 \llcorner S(\nu_h) \leq \mu_{V^{(h)}}$ and $\nu_h(\Omega) \subset \mathscr{K}$ for each h, we have $d_{\mathscr{M}}(\nu_h^+, \nu_h^-) \leq c(\mathscr{K}) < \infty$ for $\mathscr{H}^1$-a.e. $x \in S(\nu_h)$ and for each h, where $c(\mathscr{K})$ is a real constant. Therefore, since $\beta_3 > 0$ in formula (6.6), the uniform bound (6.4) holds and we can apply Theorem 6.2. The proof then goes along the same path followed in Theorem 4.2. Also, the limit point $\nu \in SBV(\Omega, \mathscr{M})$ satisfies conditions (1)-(2). We omit further details. ■

Under prescribed Dirichlet or strong anchoring conditions for the triplets (u, V, ζ) as given in previous section, if the corresponding class of competitors (u, V, ζ, ν) in $\widetilde{\mathscr{A}}_{q,r,p,K,s,\mathscr{K}}(\mathscr{M})$ with finite energy (6.5) is non-empty, the energy attains its minimum in the same class. Notice that a Dirichlet condition on the through-the-thickness descriptor ν is not preserved by the weak convergence in Theorem 6.2.

Remark 6.1 *In order to consider cohesive effects instead of looking at brittle fracture, as we have done here so far, we could add to the energy* $\widetilde{\mathscr{F}}(u, \zeta, V, \nu)$, *with density* (6.6), *a term of the type*

$$\nu \mapsto \beta_4 \int_{S(\nu)} \delta_{\mathscr{M}}(\nu^+, \nu^-)\, d\mathscr{H}^1 , \quad \beta_4 \geq 0 .$$

On the other hand, a term of the type

$$\int_{S(\nu)} g(\nu^+, \nu^-, \mathbf{n}_\nu)\, d\mathscr{H}^{m-1} ,$$

with $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{m-1} \to [0, +\infty)$ *a suitable "jointly-convex" function is presently hard to be treated when we consider* $\mathscr{M}$ *as intrinsic manifold not necessarily coinciding with a linear space.*

Remark 6.2 *Finally, a lower semicontinuity result necessary to analyze the* SBV *case when* $D\nu$ *enters the energy density instead of its total variation could perhaps be reached by adopting techniques presented in reference [31]. However, we do not tackle the problem here, leaving it open.*

Remark 6.3 *Our analysis does not prevent interpenetration between distant portions of the shell. Avoiding it would require to account for the shell thickness and to prescribe a pertinent bound involving it. We leave the analysis to a further ongoing work.*

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