# $W^{1, p}$ REGULARITY ON THE SOLUTION OF THE BV LEAST GRADIENT PROBLEM WITH DIRICHLET CONDITION ON A PART OF THE BOUNDARY 

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#### Abstract

In this paper, we consider the BV least gradient problem with Dirichlet condition imposed on a part $\Gamma$ of the boundary $\partial \Omega$. In 2 D , we show that this problem is equivalent to an optimal transport problem with Dirichlet region $\partial \Omega \backslash \Gamma$. Thanks to this equivalence, we show existence and uniqueness of a solution $u$ to this least gradient problem. Then, we prove $W^{1, p}$ regularity on this solution $u$ by studying the $L^{p}$ summability of the transport density in the corresponding equivalent optimal transport formulation.


## 1. Introduction

The BV least gradient problem consists of minimizing the total variation of the vector measure $D u$ among all BV functions $u$ on an open convex domain $\Omega$ such that the trace of $u$ on the boundary is $u_{\mid \partial \Omega}=g$, where $g$ is a given $L^{1}$ function on $\partial \Omega$ (see $[5,12,13,23]$ ):

$$
\begin{equation*}
\inf \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega}=g\right\} \tag{1.1}
\end{equation*}
$$

In [12], the author proves existence of a solution to Problem (1.1) in the case where the boundary datum $g$ is in $B V(\partial \Omega)$. Moreover, the authors of [22] show by a counter-example that Problem (1.1) may have no solutions if $g$ is not a BV function. In [23], the authors prove existence and uniqueness of a solution $u$ to Problem (1.1) provided that $g \in C(\partial \Omega)$. In all these works, the domain $\Omega$ was assumed to be strictly convex. In fact, it is clear that a solution may not exist if $\Omega$ is not strictly convex; assume that $\Omega=[0,1]^{2}$ with $g\left(x_{1}, x_{2}\right)=x_{1}$ on $\left[0, \frac{1}{2}\right] \times\{0\}, g\left(x_{1}, x_{2}\right)=1-x_{1}$ on $\left[\frac{1}{2}, 1\right] \times\{0\}$ and $g\left(x_{1}, x_{2}\right)=0$ otherwise, then we see that the level sets (which are line segments; see [11, Chapter 10]) of a solution $u$ to Problem (1.1) are contained in the segment $[0,1] \times\{0\}$, which means that $u$ does not satisfy $u_{\mid \partial \Omega}=g$ and so, the problem (1.1) does not attain a minimum. However, the authors of $[18,19]$ considered the problem (1.1) in the case where $\Omega$ is convex, but not strictly convex. More precisely, they proved that under some admissibility conditions on the behavior of the boundary datum on the flat parts of $\partial \Omega$, Problem (1.1) reaches a minimum. In [8], the authors also provide a set of admissibility conditions under which they proved existence and uniqueness of solutions to Problem (1.1) in the case where $\Omega$ is an annulus.

On the other hand, the authors of [5,13] proved that for $g \in B V(\partial \Omega)$, the problem (1.1) is equivalent to the Beckmann problem (see [1]):

$$
\begin{equation*}
\inf \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f:=\partial_{\tau} g \text { on } \partial \Omega\right\} \tag{1.2}
\end{equation*}
$$

where $\partial_{\tau} g$ denotes the tangential derivative of $g$ and the divergence condition $\nabla \cdot v=0$ and $v \cdot n=$ $f$ on $\partial \Omega$ should be understood in the weak form $\int_{\bar{\Omega}} \nabla \phi \cdot \mathrm{d} v=\int_{\partial \Omega} \phi \mathrm{d} f$, for all $\phi \in C^{1}(\bar{\Omega})$. In fact, there is a one-to-one correspondence between vector measures $D u$ on $\bar{\Omega}$ (so we include the part of the derivative of $u$ which is on the boundary, i.e. the possible jump from $u_{\mid \partial \Omega}$ to $g$ ) and vector measures $v$ satisfying $\nabla \cdot v=0$ and $v \cdot n=f$ on $\partial \Omega$. In particular, if $v$ is an optimal flow for the Beckmann problem (1.2) such that $|v|$ gives zero mass to the boundary, then the function $u$ such that $v=R_{\frac{\pi}{2}} D u$ turns out to be a solution for the BV least gradient problem (1.1). Moreover, it is well known (see, for instance, [21]) that the Beckmann problem (1.2) is equivalent to the

[^0]Monge-Kantorovich problem [15, 17] with source and target measures located on the boundary $\partial \Omega$ :

$$
\begin{equation*}
\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=f^{+} \text {and }\left(\Pi_{y}\right)_{\#} \gamma=f^{-}\right\} \tag{1.3}
\end{equation*}
$$

where $f^{+}$and $f^{-}$are the positive and negative parts of $f$. We note that Problem (1.3) has also a dual formulation, which is the following:

$$
\begin{equation*}
\sup \left\{\int_{\bar{\Omega}} \phi \mathrm{d}\left(f^{+}-f^{-}\right): \phi \in \operatorname{Lip}_{1}(\bar{\Omega})\right\} . \tag{1.4}
\end{equation*}
$$

From this duality, one can prove that if $\phi$ is a maximizer of the dual problem (1.4) and if $\gamma$ is an optimal transport plan of the problem (1.3), then we have $\phi(x)-\phi(y)=|x-y|$ for $\gamma-$ a.e. $(x, y)$ and so, $\nabla \phi(z)=\frac{x-y}{|x-y|}$ for all $\left.z \in\right] x, y[$. On the other hand, if $\gamma$ is an optimal transport plan for Problem (1.3) then the vector measure $v_{\gamma}$ defined as follows

$$
<v_{\gamma}, \xi>=\int_{\bar{\Omega} \times \bar{\Omega}} \int_{0}^{1} \xi((1-t) x+t y) \cdot(y-x) \mathrm{d} t \mathrm{~d} \gamma(x, y), \text { for all } \xi \in C\left(\bar{\Omega}, \mathbb{R}^{2}\right)
$$

is a minimizer for the Beckmann problem (1.2). In addition, one can prove that any minimizer $v$ of Problem (1.2) is of this form $v=v_{\gamma}$, for some optimal transport plan $\gamma$ of Problem (1.3) (see [21]). If $\phi$ maximizes Problem (1.4), then we have $v_{\gamma}=-\left|v_{\gamma}\right| \nabla \phi$, thanks to the fact that $\nabla \phi(z)=\frac{x-y}{|x-y|}$ for all $z \in] x, y\left[\right.$ and $\gamma$-a.e. $(x, y)$. This means that the optimal flows $v_{\gamma}$ share the same directions (called transport rays); these are the rays along which the optimal transport plans $\gamma$ move the mass $f^{+}$to $f^{-}$. The measure $\sigma_{\gamma}:=\left|v_{\gamma}\right|$ plays a special role in the optimal transport theory; it is called transport density and it represents the amount of transport taking place in each region of $\Omega$ :

$$
<\sigma_{\gamma}, \varphi>=\int_{\bar{\Omega} \times \bar{\Omega}} \int_{0}^{1} \varphi((1-t) x+t y)|x-y| \mathrm{d} t \mathrm{~d} \gamma(x, y), \text { for all } \varphi \in C(\bar{\Omega})
$$

or equivalently,

$$
\begin{equation*}
\sigma_{\gamma}(A)=\int_{\bar{\Omega} \times \bar{\Omega}} \mathcal{H}^{1}(A \cap[x, y]) \mathrm{d} \gamma(x, y), \text { for every Borel set } A \subset \bar{\Omega} \tag{1.5}
\end{equation*}
$$

Many authors have already studied the properties (in particular, the $L^{p}$ summability) of $\sigma_{\gamma}$. In [9,20], the authors proved that the transport density $\sigma_{\gamma}$ is unique (i.e., it does not depend on the choice of the optimal transport plan $\gamma$ ) and it is in $L^{1}(\Omega)$ as soon as $f^{+}$or $f^{-}$is absolutely continuous with respect to the Lebesgue measure. On the other hand, the authors of [2-4] proved that the transport density $\sigma$ belongs to $L^{p}(\Omega)$ as soon as $f^{+}$and $f^{-}$are both in $L^{p}(\Omega)$, for all $p \in[1, \infty]$.

In order to prove existence of a solution $u$ to the BV least gradient problem (1.1), we need to show that the transport density $\sigma_{\gamma}$ gives zero mass to the boundary since in this case, the boundary part of the vector measure $D u$, where $u$ is the BV function such that $v_{\gamma}=R_{\frac{\pi}{2}} D u$, will be zero which means that $u_{\mid \partial \Omega}=g$ and so, thanks to the equivalence between Problems (1.1) \& (1.2), we infer that $u$ is a minimizer for Problem (1.1). If $\Omega$ is strictly convex, one can see that $\sigma_{\gamma}(\partial \Omega)=0$ and so, the least gradient problem (1.1) reaches a minimum. Moreover, one can show that Problem (1.3) has a unique solution $\gamma$ provided that $f^{+}$is atomless and so, Problem (1.2) has a unique optimal flow $v$ which is also equivalent to say that $\operatorname{Problem}$ (1.1) has a unique solution $u$ (see [5]). On the other hand, we see that the $W^{1, p}$ regularity of the solution $u$ of Problem (1.1) follows immediately from the $L^{p}$ summability of the transport density $\sigma$ between $f^{+}$and $f^{-}$; we note that this $L^{p}$ summability on $\sigma$ does not follow directly from the $L^{p}$ bounds on $\sigma$ in [2-4] as the source and target measures here are located on the boundary. But in [5], the authors have already studied the $L^{p}$ summability of the transport density $\sigma$ between two singular measures $f^{+}$ and $f^{-}$. More precisely, they proved that under the assumption that $\Omega$ is uniformly convex, the transport density $\sigma$ between $f^{+} \in L^{p}(\partial \Omega)$ with $p<2$ and any $f^{-} \in \mathcal{M}^{+}(\partial \Omega)$ is in $L^{p}(\Omega)$, while $\sigma$ belongs to $L^{2}(\Omega)$ as soon as both $f^{+}$and $f^{-}$are in $L^{2}(\partial \Omega)$; this implies that the solution $u$ of the BV least gradient problem (1.1) is in $W^{1, p}(\Omega)$ as soon as the boundary datum $g \in W^{1, p}(\partial \Omega)$, $p \leq 2$ and $\Omega$ is uniformly convex. Moreover, they show by a counter-example that in general the solution $u$ of Problem (1.1) is not in $W^{1, p}(\Omega)$, for $p>2$, even if $g \in \operatorname{Lip}(\partial \Omega)$. However, in order to obtain $W^{1, p}$ regularity on $u$ (or equivalently, $L^{p}$ summability on $\sigma$ ) for $p>2$, we need to assume that $g \in C^{1, \alpha}(\partial \Omega)$, with $\alpha=1-\frac{2}{p}$.

In this paper, we consider the BV least gradient problem with Dirichlet condition imposed on a relatively open connected part $\Gamma$ of the boundary (see [14]):

$$
\begin{equation*}
\inf \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega} \in B V(\partial \Omega), u_{\mid \Gamma}=g\right\} \tag{1.6}
\end{equation*}
$$

where $g$ is a given $B V$ function on $\Gamma$ and $u_{\mid \Gamma}=g$ is in the sense that there is an $\left(L^{1}\right)$ extension $\tilde{g}$ of $g$ to $\partial \Omega$ such that $u_{\mid \partial \Omega}=\tilde{g}$. We note that this is a constrained least gradient problem, since we assume that the trace of $u$ is a BV function on the boundary, which is not necessarily the case for any BV function over $\Omega$ such that $u_{\mid \Gamma}=g$. However, we will show in Section 5 that the classical least gradient problem with Dirichlet condition on $\Gamma$ (i.e. when we remove the condition $\left.u_{\mid \partial \Omega} \in B V(\partial \Omega)\right)$ is completely equivalent to this constrained least gradient problem (1.6). So, we prove existence and uniqueness of a solution $u$ to Problem (1.6) via an optimal transport approach and, we study the $W^{1, p}$ regularity of this solution $u$ by proving $L^{p}$ estimates on the transport density in the transport problem with Dirichlet region $\partial \Omega \backslash \Gamma$. To be more precise, we will show that Problem (1.6) is equivalent to the following variant of the Beckmann problem:

$$
\begin{equation*}
\min \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f:=\partial_{\tau} g \text { on } \Gamma\right\} \tag{1.7}
\end{equation*}
$$

where $\nabla \cdot v=0$ and $v \cdot n=f$ on $\Gamma$ is equivalent to say that there is a measure $\chi \in \mathcal{M}(\partial \Omega \backslash \Gamma)$ such that $\int_{\bar{\Omega}} \nabla \phi \cdot \mathrm{d} v=\int_{\partial \Omega} \phi \mathrm{d}[f+\chi]$, for all $\phi \in C^{1}(\bar{\Omega})$. Moreover, one can show that Problem (1.7) is equivalent to the following optimal transport problem with Dirichlet region $\partial \Omega \backslash \Gamma$ :
$\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}), \operatorname{spt}(\gamma) \subset \partial \Omega \times \partial \Omega,\left[\left(\Pi_{x}\right)_{\# \gamma} \gamma\right]_{\mid \Gamma}=f^{+}\right.$and $\left.\left[\left(\Pi_{y}\right)_{\#} \gamma\right]_{\mid \Gamma}=f^{-}\right\}$.
In [7], the authors studied the transport problem to the boundary. More generally, the import/export transport problem from/to the boundary has been considered in [6, 16]. Here, we study a mass transportation problem between two masses $f^{+}$and $f^{-}$(which do not have a priori the same total mass) with the possibility of transporting some mass from/to the arc $\partial \Omega \backslash \Gamma$, paying the transport cost $|x-y|$ for each unit of mass that moves from a point $x$ to another one $y$. This means that we can use $\partial \Omega \backslash \Gamma$ as an infinite reserve/repository, we can take as much mass as we wish from $\partial \Omega \backslash \Gamma$, or send back as much mass as we want. On the other hand, it is not difficult to show (see [6]) that Problem (1.8) has a dual formulation, which is the following:

$$
\begin{equation*}
\sup \left\{\int_{\bar{\Omega}} \phi \mathrm{d}\left(f^{+}-f^{-}\right): \phi \in \operatorname{Lip}_{1}(\bar{\Omega}), \phi=0 \text { on } \partial \Omega \backslash \Gamma\right\} . \tag{1.9}
\end{equation*}
$$

Coming back to Problem (1.7), one can show that $v$ is a solution for Problem (1.7) if and only if $v=v_{\gamma}$ for some optimal transport plan $\gamma$ of Problem (1.8). Now, let $v$ be such a solution and let $u$ be the BV function such that $v=R_{\frac{\pi}{2}} D u$. If $\Omega$ is strictly convex, then we have by (1.5) that $|v|(\partial \Omega)=0$ and so, $u$ solves Problem (1.6). However, we will refine this existence result by showing that $|v|$ gives zero mass to $\partial \Omega$ as soon as $\Gamma$ is strictly convex. We note that this is not obvious as $|v|$ is the transport density between $f^{+}+\chi^{+}$and $f^{-}+\chi^{-}$, where $\chi^{ \pm}$encode the import/export masses on $\partial \Omega \backslash \Gamma$. By the way, it is easy to see that this flow $v$ minimizes

$$
\begin{equation*}
\min \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f+\chi \text { on } \partial \Omega\right\} \tag{1.10}
\end{equation*}
$$

Let us denote by $\tilde{g}$ the trace of $u$ on $\partial \Omega$ (so, we have $\partial_{\tau} \tilde{g}=f+\chi$ on $\partial \Omega$ ). Then, it is clear that $u$ minimizes the problem:

$$
\begin{equation*}
\min \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega}=\tilde{g}\right\} . \tag{1.11}
\end{equation*}
$$

Yet, this boundary datum $\tilde{g}$ is a priori not continuous on $\partial \Omega$ and so, it is not clear if Problem (1.11) has a unique solution or not, even if $g \in C(\Gamma)$. Notice that even if Problem (1.11) has a unique solution $u$, this does not imply that the solution of Problem (1.6) is unique, since Problem (1.7) may have many different solutions. Yet, we will prove that Problem (1.8) has a unique optimal transport plan $\gamma$ provided that $f^{ \pm}$are atomless. This implies that Problem (1.7) has a unique optimal flow $v$ and so, the solution $u$ of Problem (1.6) is unique as soon as $g \in C(\Gamma)$. In addition, it is not clear if this unique solution $u$ of Problem (1.6) (or equivalently, Problem (1.11)) is in $W^{1, p}(\Omega)$ or not, since the $W^{1, p}$ regularity of $u$ is equivalent to the $L^{p}$ summability of $\sigma=|v|$, while the transport density $\sigma$ here is between $f^{+}+\chi^{+}$and $f^{-}+\chi^{-}$and so, we cannot use [5, Proposition
3.3] to infer that $\sigma \in L^{p}(\Omega)$ as the source and target measures $f^{+}+\chi^{+}$and $f^{-}+\chi^{-}$are a priori not in $L^{p}(\partial \Omega)$; we recall that $\chi^{ \pm}$are two unknown measures on $\partial \Omega \backslash \Gamma$ which are a priori not in $L^{p}(\partial \Omega \backslash \Gamma)$. Yet, we will prove that the transport density in Problem (1.8) is in $L^{p}(\Omega)$ as soon as $f^{ \pm} \in L^{p}(\Gamma), p<2$ and $\Gamma$ is uniformly convex (we note that the uniform convexity of the whole boundary $\partial \Omega$ is not required here to get $L^{p}$ summability on $\sigma$ ). Moreover, it is possible to prove $L^{p}$ estimates on the transport density $\sigma$, for $p \geq 2$, under the assumptions that $\Gamma$ is uniformly convex, $\operatorname{dist}(\operatorname{spt}(f), \partial \Omega \backslash \Gamma)>0$, and the projection of a.e. point $x \in \operatorname{spt}(f)$ onto $\partial \Omega \backslash \Gamma$ is not an endpoint of $\partial \Omega \backslash \Gamma$. In terms of $W^{1, p}$ regularity on the solution $u$ of Problem (1.6), we infer that under these geometric assumptions on $\Gamma$ and $\operatorname{spt}(f)$, the following statements hold:

$$
\begin{gathered}
g \in W^{1, p}(\Gamma) \Rightarrow u \in W^{1, p}(\Omega), \text { for all } p \leq 2 \\
g \in C^{1, \alpha}(\Gamma) \Rightarrow u \in W^{1, \frac{2}{1-\alpha}}(\Omega), \text { for all } \alpha \in(0,1) \\
g \in C^{1,1}(\Gamma) \Rightarrow u \in \operatorname{Lip}(\Omega)
\end{gathered}
$$

The paper is organized as follows. In Section 2, we prove that Problems (1.6), (1.7) \& (1.8) are completely equivalent. In Section 3, we study in details the transport problem (1.8) and we prove under the assumptions that $f^{ \pm}$are atomless and $\Gamma$ is strictly convex, that Problem (1.8) has a unique optimal transport plan $\gamma$ and that the corresponding transport density $\sigma$ gives zero mass to $\partial \Omega$. In Section 4, we prove $L^{p}$ estimates on the transport density $\sigma$ in Problem (1.8). More precisely, we show that $\sigma$ is in $L^{p}(\Omega)$ as soon as $f^{ \pm} \in L^{p}(\Gamma), \Gamma$ is uniformly convex and $p<2$. Moreover, $\sigma \in L^{2}(\Omega)$ provided that $f^{ \pm} \in L^{2}(\Gamma), \Gamma$ is uniformly convex, $\operatorname{dist}(\operatorname{spt}(f), \partial \Omega \backslash \Gamma)>0$, and the projection of a.e. $x \in \operatorname{spt}(f)$ onto $\partial \Omega \backslash \Gamma$ is not an endpoint of $\partial \Omega \backslash \Gamma$ (one can also obtain $L^{p}$ estimates on $\sigma$, for $p>2$, provided that $f^{ \pm}$are smooth enough). Finally, Section 5 summarizes the applications of these results to the BV least gradient problem with Dirichlet condition (1.6).

## 2. On the equivalence between the BV least gradient problem with Dirichlet CONDITION ON A PART OF THE BOUNDARY AND THE OPTIMAL TRANSPORT PROBLEM WITH Dirichlet region

In this Section, we prove that Problems (1.6), (1.7) and (1.8) are equivalent. Throughout the paper, $\Omega \subset \mathbb{R}^{2}$ is assumed to be an open convex set and $\Gamma$ is an open connected subset of $\partial \Omega$. Let $g$ be a BV function on $\Gamma$ and set $f=\partial_{\tau} g$ (the tangential derivative of $g$ ). Let $f^{+}$and $f^{-}$be the positive and negative parts of $f$. Then, we consider the following problems:

$$
\begin{gather*}
\inf \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega} \in B V(\partial \Omega), u_{\mid \Gamma}=g\right\},  \tag{2.1}\\
\min \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f \text { on } \Gamma\right\} \tag{2.2}
\end{gather*}
$$

and
$\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}), \operatorname{spt}(\gamma) \subset \partial \Omega \times \partial \Omega,\left[\left(\Pi_{x}\right)_{\#} \gamma\right]_{\mid \Gamma}=f^{+}\right.$and $\left.\left[\left(\Pi_{y}\right)_{\#} \gamma\right]_{\mid \Gamma}=f^{-}\right\}$.
Notice that as $f$ is the tangential derivative of a $B V$ function on an open set $\Gamma$, then the mass balance condition is not necessarily satisfied, i.e. $f^{+}$and $f^{-}$do not have a priori the same total mass. First, we prove that Problems (2.1) \& (2.2) share the same minimal value and, we also show a relationship between the minimizers of these two problems (see also [13]):

Proposition 2.1. We have $\inf (2.1)=\inf (2.2)$. Moreover, if $u$ is a solution for Problem (2.1) then $v:=R_{\frac{\pi}{2}} D u$ solves Problem (2.2). On the other hand, if $v$ is an optimal flow for Problem (2.2) with $|v|(\partial \Omega)=0$ then there exists a $B V$ function $u$ such that $v:=R_{\frac{\pi}{2}} D u$ and $u$ is a solution for Problem (2.1).

Proof. For every $h \in B V(\partial \Omega \backslash \Gamma)$, we denote by $\tilde{g}_{h}$ a BV extension of $g$ to $\partial \Omega$ such that $\tilde{g}_{h}=h$ on $\partial \Omega \backslash \Gamma$. Then, we have obviously

$$
\begin{gathered}
\inf \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega} \in B V(\partial \Omega), u_{\mid \Gamma}=g\right\} \\
=\inf \left\{\inf \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega}=\tilde{g}_{h}\right\}: h \in B V(\partial \Omega \backslash \Gamma)\right\} .
\end{gathered}
$$

Thanks to [5,13], we have

$$
\begin{gathered}
\inf \left\{\int_{\Omega}|D u|: u \in B V(\Omega), u_{\mid \partial \Omega}=\tilde{g}_{h}\right\} \\
=\inf \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=\tilde{f}_{h} \text { on } \partial \Omega\right\},
\end{gathered}
$$

where $\tilde{f}_{h}:=\partial_{\tau} \tilde{g}_{h}$ is the tangential derivative of $\tilde{g}_{h}$. Yet, it is clear that $\tilde{f}_{h}=f+\chi$, where $\chi$ is a measure on $\partial \Omega \backslash \Gamma$. Then, we get that

$$
\inf (2.1)
$$

$$
\begin{gathered}
=\inf \left\{\inf \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f+\chi \text { on } \partial \Omega\right\}: \chi \in \mathcal{M}(\partial \Omega \backslash \Gamma)\right\} \\
=\inf \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f \text { on } \Gamma\right\} .
\end{gathered}
$$

Now, assume that $u$ is a minimizer of Problem (2.1) and set $v:=R_{\frac{\pi}{2}} D u$. Let $\left(u_{k}\right)_{k} \subset C^{\infty}(\bar{\Omega})$ with $u_{k} \rightarrow u$ strictly in $\operatorname{BV}(\Omega)$. As $\nabla \cdot\left[R_{\frac{\pi}{2}} \nabla u_{k}\right]=0$, then we have
$\int_{\Omega} R_{\frac{\pi}{2}} \nabla u_{k} \cdot \nabla \phi \mathrm{~d} x=\int_{\partial \Omega}\left[R_{\frac{\pi}{2}} \nabla u_{k} \cdot n\right] \phi \mathrm{d} \mathcal{H}^{1}=\int_{\partial \Omega} \partial_{\tau} u_{k} \phi \mathrm{~d} \mathcal{H}^{1}=-\int_{\partial \Omega} u_{k} \partial_{\tau} \phi \mathrm{d} \mathcal{H}^{1}, \forall \phi \in C^{1}(\bar{\Omega})$.
Passing to the limit when $k \rightarrow \infty$, we get

$$
\int_{\Omega} \nabla \phi \cdot \mathrm{d}\left[R_{\frac{\pi}{2}} D u\right]=-\int_{\partial \Omega} u \partial_{\tau} \phi \mathrm{d} \mathcal{H}^{1}=\int_{\partial \Omega} \phi \mathrm{d}\left[\partial_{\tau} u\right], \quad \forall \phi \in C^{1}(\bar{\Omega})
$$

Yet, $u_{\mid \Gamma}=g$ which means that there is a measure $\chi$ on $\partial \Omega \backslash \Gamma$ such that $\partial_{\tau} u=f+\chi$. Then, we have

$$
\int_{\Omega} \nabla \phi \cdot \mathrm{d}\left[R_{\frac{\pi}{2}} D u\right]=\int_{\partial \Omega} \phi \mathrm{d}[f+\chi], \text { for all } \phi \in C^{1}(\bar{\Omega}) .
$$

This implies that $v$ is admissible in Problem (2.2) (i.e., $\nabla \cdot v=0$ and $v \cdot n=f$ on $\Gamma$ ). On the other hand, we have

$$
\int_{\bar{\Omega}}|v|=\int_{\Omega}|D u|=\min (2.1)=\min (2.2)
$$

Then, $v$ solves Problem (2.2). In the other direction, let $v$ be a solution for Problem (2.2) with $|v|(\partial \Omega)=0$ (we extend $v$ by 0 outside $\Omega$ ). Let $\rho_{\varepsilon}$ be a sequence of mollifiers and set $v^{\varepsilon}$ to be the mollification of $v$, i.e. $v^{\varepsilon}=v * \rho_{\varepsilon}:=\left(v_{1}^{\varepsilon}, v_{2}^{\varepsilon}\right)$. As $\nabla \cdot v=0$, we also have $\nabla \cdot v^{\varepsilon}=0$. We define a sequence of mollified differential forms $v_{2}^{\varepsilon} \mathrm{d} x_{1}-v_{1}^{\varepsilon} \mathrm{d} x_{2}$. It is clear that this differential 1 -form is closed and so, it is exact. Then, there is a smooth function $u_{\varepsilon}$ such that $\nabla u_{\varepsilon}=R_{-\frac{\pi}{2}} v^{\varepsilon}$. Up to adding a constant, one can assume that the mean value of $u_{\varepsilon}$ on $\Omega$ is 0 and so, we have

$$
\int_{\Omega}\left|u_{\varepsilon}\right| \mathrm{d} x \leq C \int_{\Omega}\left|\nabla u_{\varepsilon}\right| \mathrm{d} x=C \int_{\Omega}\left|v^{\varepsilon}\right| \mathrm{d} x
$$

Then, we get

$$
\left\|u_{\varepsilon}\right\|_{W^{1,1}(\Omega)} \leq(C+1)\left\|v^{\varepsilon}\right\|_{L^{1}(\Omega)} \leq(C+1) \int_{\Omega}|v|
$$

Hence, up to a subsequence, $\left(u_{\varepsilon}\right)_{\varepsilon}$ converges weakly* in $B V(\Omega)$ to some function $u$ (moreover, $u_{\varepsilon} \rightarrow u$ strictly in BV since $\left.\left|v_{\varepsilon}\right| \rightharpoonup|v|\right)$. This implies directly that $D u=R_{-\frac{\pi}{2}} v$. On the other hand, we have

$$
\int_{\bar{\Omega}} v \cdot \nabla \phi \mathrm{~d} x=\int_{\partial \Omega} \phi \mathrm{d}[f+\chi], \text { for all } \phi \in C^{1}(\bar{\Omega})
$$

Yet,

$$
\begin{gathered}
\int_{\Omega} \nabla \phi \cdot \mathrm{d}\left[R_{\frac{\pi}{2}} D u\right]=\lim _{\varepsilon \rightarrow 0} \int_{\Omega} R_{\frac{\pi}{2}} \nabla u_{\varepsilon} \cdot \nabla \phi \mathrm{d} x=\lim _{\varepsilon \rightarrow 0} \int_{\partial \Omega}\left[R_{\frac{\pi}{2}} \nabla u_{\varepsilon} \cdot n\right] \phi \mathrm{d} \mathcal{H}^{1}=\lim _{\varepsilon \rightarrow 0} \int_{\partial \Omega} \partial_{\tau} u_{\varepsilon} \phi \mathrm{d} \mathcal{H}^{1} \\
=-\lim _{\varepsilon \rightarrow 0} \int_{\partial \Omega} u_{\varepsilon} \partial_{\tau} \phi \mathrm{d} \mathcal{H}^{1}=-\int_{\partial \Omega} u \partial_{\tau} \phi \mathrm{d} \mathcal{H}^{1}=\int_{\partial \Omega} \phi \mathrm{d}\left[\partial_{\tau} u\right], \text { for all } \phi \in C^{1}(\bar{\Omega}) .
\end{gathered}
$$

This implies that $\partial_{\tau} u=f+\chi$. Consequently, there is a BV function $u$ such that $v=R_{\frac{\pi}{2}} D u$ and $u_{\mid \Gamma}=g$. In addition, this function $u$ solves Problem (2.1) thanks to the fact that

$$
\int_{\Omega}|D u|=\int_{\bar{\Omega}}|v|=\min (2.2)=\min (2.1)
$$

On the other hand, one can show that the variant of the Beckmann problem (2.2) is equivalent to the optimal transport problem with Dirichlet region (2.3) in the sense that these two problems have the same minimal value and, every solution of Problem (2.2) comes from an optimal transport plan of Problem (2.3).

Proposition 2.2. We have $\min (2.2)=\min (2.3)$. Moreover, $v$ is a solution for Problem (2.2) if and only if $v=v_{\gamma}$, for some optimal transport plan $\gamma$ of Problem (2.3).
Proof. Thanks to the equivalence between the Beckmann problem (1.2) and the Monge-Kantorovich problem (1.3) (see [21]), we have

$$
\begin{gathered}
\min \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f \text { on } \Gamma\right\} \\
=\min \left\{\min \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f+\chi \text { on } \partial \Omega\right\}: \chi \in \mathcal{M}(\partial \Omega \backslash \Gamma)\right\} \\
=\min _{\chi \in \mathcal{M}(\partial \Omega \backslash \Gamma)}\left\{\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=f^{+}+\chi^{+} \text {and }\left(\Pi_{y}\right)_{\#} \gamma=f^{-}+\chi^{-}\right\}\right\} \\
=\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}), \operatorname{spt}(\gamma) \subset \partial \Omega \times \partial \Omega,\left[\left(\Pi_{x}\right)_{\#} \gamma\right]_{\mid \Gamma}=f^{+} \text {and }\left[\left(\Pi_{y}\right)_{\# \gamma}\right]_{\mid \Gamma}=f^{-}\right\} .
\end{gathered}
$$

Let $v$ be a solution of Problem (2.2) and let us denote by $\chi$ the measure on $\partial \Omega \backslash \Gamma$ such that $v \cdot n=f+\chi$ on $\partial \Omega$. So, it is clear that $v$ solves

$$
\begin{equation*}
\min \left\{\int_{\bar{\Omega}}|v|: v \in \mathcal{M}\left(\bar{\Omega}, \mathbb{R}^{2}\right), \nabla \cdot v=0 \text { and } v \cdot n=f+\chi \text { on } \partial \Omega\right\} \tag{2.4}
\end{equation*}
$$

From [21, Theorem 4.13], there is a transport plan $\gamma$ which minimizes the following Kantorovich problem

$$
\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=f^{+}+\chi^{+} \text {and }\left(\Pi_{y}\right)_{\#} \gamma=f^{-}+\chi^{-}\right\}
$$

such that $v=v_{\gamma}$. Yet, we see obviously that this transport plan $\gamma$ is admissible in Problem (2.3). Moreover, we have the following equalities:

$$
\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma=\int_{\bar{\Omega}}|v|=\min (2.2)=\min (2.3)
$$

This implies that $\gamma$ is an optimal transport plan for Problem (2.3) as well. In the other direction, let $\gamma$ be an optimal transport plan for Problem (2.3). Then, we have

$$
\int_{\bar{\Omega}}\left|v_{\gamma}\right|=\sigma_{\gamma}(\bar{\Omega})=\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma=\min (2.3)=\min (2.2)
$$

## 3. Transport problem with Dirichlet region

In this section, we study the problem (2.3). More precisely, we will decompose Problem (2.3) into three subproblems: the first transport problem is going from $\Gamma$ to $\Gamma$, the second one (the export transport problem) from $\Gamma$ to $\partial \Omega \backslash \Gamma$ and the third one (the import transport problem) from $\partial \Omega \backslash \Gamma$ to $\Gamma$. In this way, we can write the optimal transport plan $\gamma$ of Problem (2.3) as a sum of three transport plans $\gamma(\Gamma, \Gamma)$ (which transports mass from $\Gamma$ to $\Gamma$ ), $\gamma(\Gamma, \partial \Omega \backslash \Gamma)$ (which transports mass from $\Gamma$ to $\partial \Omega \backslash \Gamma$ ) and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$ (which transports mass from $\partial \Omega \backslash \Gamma$ to $\Gamma$ ). Let $\sigma(\Gamma, \Gamma), \sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ be the transport densities associated with $\gamma(\Gamma, \Gamma), \gamma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$, respectively. Then, the transport density $\sigma$ associated with $\gamma$ is $\sigma(\Gamma, \Gamma)+\sigma(\Gamma, \partial \Omega \backslash \Gamma)+\sigma(\partial \Omega \backslash \Gamma, \Gamma)$. We will prove that under the assumption that $\Gamma$ is strictly convex, the transport densities $\sigma(\Gamma, \Gamma)$, $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ give zero mass to $\partial \Omega$ and so, $\sigma(\partial \Omega)=0$. On the other hand, thanks to this decomposition, we will also show that Problem (2.3) has a unique optimal transport plan $\gamma$. First, we have the following existence result:
Proposition 3.1. Problem (2.3) reaches a minimum.
Proof. Let $\left(\gamma_{k}\right)_{k}$ be a minimizing sequence in Problem (2.3). Then, it is not difficult to see that one can assume that $\gamma_{k}(\partial \Omega \backslash \Gamma \times \partial \Omega \backslash \Gamma)=0$, for every $k$. As $\gamma_{k} \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega})$ with $\operatorname{spt}\left(\gamma_{k}\right) \subset$ $\partial \Omega \times \partial \Omega,\left[\left(\Pi_{x}\right)_{\#} \gamma_{k}\right]_{\mid \Gamma}=f^{+}$and $\left[\left(\Pi_{y}\right)_{\#} \gamma_{k}\right]_{\mid \Gamma}=f^{-}$, then we have

$$
\gamma_{k}(\bar{\Omega} \times \bar{\Omega}) \leq \gamma_{k}(\Gamma \times \bar{\Omega})+\gamma_{k}(\bar{\Omega} \times \Gamma)=f^{+}(\Gamma)+f^{-}(\Gamma)
$$

Hence, there is a subsequence $\left(\gamma_{k_{i}}\right)_{k_{i}}$ and a transport plan $\gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}), \operatorname{spt}(\gamma) \subset \partial \Omega \times \partial \Omega$ with $\left[\left(\Pi_{x}\right)_{\# \gamma}\right]_{\mid \Gamma}=f^{+}$and $\left[\left(\Pi_{y}\right)_{\# \gamma}\right]_{\mid \Gamma}=f^{-}$such that $\gamma_{k_{i}} \rightharpoonup \gamma$. And so, $\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma_{k_{i}} \rightarrow$ $\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma$, which implies that $\gamma$ minimizes Problem (2.3).

Let $\gamma$ be an optimal transport plan for Problem (2.3) (we recall that $\gamma(\partial \Omega \backslash \Gamma \times \partial \Omega \backslash \Gamma)=0$ ) and let us denote by $\chi^{+}$and $\chi^{-}$the two nonnegative measures such that $\left(\Pi_{x}\right)_{\# \gamma}=f^{+}+\chi^{+}$and $\left(\Pi_{y}\right)_{\#} \gamma=f^{-}+\chi^{-}$. In other words, $\chi^{+}$is the mass to be imported from $\partial \Omega \backslash \Gamma$ while $\chi^{-}$is the mass to be exported to $\partial \Omega \backslash \Gamma$. So, we have clearly the following:

Proposition 3.2. $\gamma$ is an optimal transport plan for the Kantorovich problem (1.3) between $f^{+}+$ $\chi^{+}$and $f^{-}+\chi^{-}$.

Proof. Let $\Lambda$ be a transport plan between $f^{+}+\chi^{+}$and $f^{-}+\chi^{-}$. Then, we have obviously $\left[\left(\Pi_{x}\right)_{\#} \Lambda\right]_{\mid \Gamma}=f^{+}$and $\left[\left(\Pi_{y}\right)_{\#} \Lambda\right]_{\mid \Gamma}=f^{-}$. Yet, $\gamma$ minimizes Problem (2.3). Then, we have

$$
\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma \leq \int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \Lambda .
$$

Set

$$
\gamma(\Gamma, \Gamma)=\gamma_{\mid \Gamma \times \Gamma}, \quad \gamma(\Gamma, \partial \Omega \backslash \Gamma)=\gamma_{\mid \Gamma \times \partial \Omega \backslash \Gamma}, \quad \gamma(\partial \Omega \backslash \Gamma, \Gamma)=\gamma_{\mid \partial \Omega \backslash \Gamma \times \Gamma},
$$

and

$$
\nu^{+}=\left(\Pi_{x}\right)_{\#}[\gamma(\Gamma, \partial \Omega \backslash \Gamma)], \quad \nu^{-}=\left(\Pi_{y}\right)_{\#}[\gamma(\partial \Omega \backslash \Gamma, \Gamma)] .
$$

Then, we consider the following transport problems:

$$
\begin{align*}
& \min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=f^{+}-\nu^{+} \text {and }\left(\Pi_{y}\right)_{\#} \gamma=f^{-}-\nu^{-}\right\},  \tag{3.1}\\
& \quad \min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=\nu^{+} \text {and } \operatorname{spt}\left(\left(\Pi_{y}\right)_{\#} \gamma\right) \subset \partial \Omega \backslash \Gamma\right\},  \tag{3.2}\\
& \min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}), \operatorname{spt}\left(\left(\Pi_{x}\right)_{\#} \gamma\right) \subset \partial \Omega \backslash \Gamma \text { and }\left(\Pi_{y}\right)_{\#} \gamma=\nu^{-}\right\} . \tag{3.3}
\end{align*}
$$

So, we have the following:
Proposition 3.3. $\gamma(\Gamma, \Gamma), \gamma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$ minimize Problems (3.1), (3.2) and (3.3), respectively.

Proof. It is easy to see that $\gamma(\Gamma, \Gamma), \gamma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$ are admissible in Problems (3.1), (3.2) and (3.3), respectively. Now, let $\gamma_{1}, \gamma_{2}$ and $\gamma_{3}$ be the optimal transport plans for Problems (3.1), (3.2) and (3.3), respectively. Then, the transport plan $\gamma_{1}+\gamma_{2}+\gamma_{3}$ minimizes Problem (2.3), since the functional to be minimized is linear in $\gamma$ and $\gamma(\Gamma, \Gamma)+\gamma(\Gamma, \partial \Omega \backslash \Gamma)+\gamma(\partial \Omega \backslash \Gamma, \Gamma)$ is in fact an optimal transport plan for Problem (2.3). This concludes the proof.

Set

$$
\tilde{P}(x)=\operatorname{argmin}\{|x-y|: y \in \partial \Omega \backslash \Gamma\}, \text { for every } x \in \Gamma
$$

Lemma 3.4. $\tilde{P}(x)$ is a singleton at every point $x \in \Gamma$, except possibly at a countable set $A \subset \Gamma$.
Proof. Let us denote by $A \subset \Gamma$ the set of points $x$ such that $\tilde{P}(x)$ is not a singleton. For each $x \in A$, let $P_{1}(x)$ and $P_{2}(x)$ be two different points in $\tilde{P}(x)$. Let $\Delta_{x} \subset \Omega$ be the region delimited by $\left[x, P_{1}(x)\right],\left[x, P_{2}(x)\right]$ and $\partial \Omega \backslash \Gamma$. It is easy to check that these sets $\left\{\Delta_{x}\right\}_{x \in A}$ are disjoint with $\mathcal{L}^{2}\left(\Delta_{x}\right)>0$. Hence, the set $A$ is at most countable.

In the sequel, we will denote by $P$ the Borel selector function of this projection map to the arc $\partial \Omega \backslash \Gamma$. Then, we have the following:

Proposition 3.5. The transport plans $(I d, P)_{\#} \nu^{+}$and $(P, I d)_{\# \nu^{-}}$minimize Problems (3.2) \& (3.3), respectively. Moreover, for $\gamma(\Gamma, \partial \Omega \backslash \Gamma)-$ a.e. $(x, y), y \in \tilde{P}(x)$ and for $\gamma(\partial \Omega \backslash \Gamma, \Gamma)-$ a.e. $(x, y), x \in \tilde{P}(y)$. If $f^{ \pm}$are atomless (i.e., $f^{ \pm}(\{x\})=0$, for every $x \in \Gamma$ ), then $\gamma(\Gamma, \partial \Omega \backslash \Gamma)=$ $(I d, P)_{\#} \nu^{+}$and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)=(P, I d)_{\#} \nu^{-}$.

Proof. Let us prove that for $\gamma(\Gamma, \partial \Omega \backslash \Gamma)$ - a.e. $(x, y), y \in \tilde{P}(x)$ (in the same way, we prove that for $\gamma(\partial \Omega \backslash \Gamma, \Gamma)-$ a.e. $(x, y), x \in \tilde{P}(y))$. Assume that this statement does not hold, then we get

$$
\begin{aligned}
\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d}[\gamma(\Gamma, \partial \Omega \backslash \Gamma)] & >\int_{\bar{\Omega} \times \bar{\Omega}}|x-P(x)| \mathrm{d}[\gamma(\Gamma, \partial \Omega \backslash \Gamma)]=\int_{\Gamma}|x-P(x)| \mathrm{d} \nu^{+}(x) \\
& =\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d}\left[(I d, P)_{\#} \nu^{+}\right]
\end{aligned}
$$

This is a contradiction since $\gamma(\Gamma, \partial \Omega \backslash \Gamma)$ minimizes Problem (3.4) while $(I d, P)_{\# \nu^{+}}$is admissible in Problem (3.4). This shows at the same time that $(I d, P)_{\#} \nu^{+}$minimizes Problem (3.4). On the other hand, it is clear that $\nu^{+}=\left(\Pi_{x}\right)_{\#}[\gamma(\Gamma, \partial \Omega \backslash \Gamma)] \leq\left[\left(\Pi_{x}\right)_{\#} \gamma\right]_{\Gamma}=f^{+}$. Now, assume that $f^{+}$ is atomless. Then, by Lemma 3.4 and thanks to the fact that $\nu^{+}$is atomless, we infer that for $\gamma(\Gamma, \partial \Omega \backslash \Gamma)$ - a.e. $(x, y)$, we have $y \in \tilde{P}(x)=\{P(x)\}$. Hence, $\gamma(\Gamma, \partial \Omega \backslash \Gamma)=(I d, P)_{\#} \nu^{+}$.
Corollary 3.6. Assume $f^{ \pm}$are atomless. Then, the transport plans $\gamma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$ minimize the following problems, respectively

$$
\begin{equation*}
\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=\nu^{+} \text {and }\left(\Pi_{y}\right)_{\#} \gamma=P_{\#} \nu^{+}\right\} \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\min \left\{\int_{\bar{\Omega} \times \bar{\Omega}}|x-y| \mathrm{d} \gamma: \gamma \in \mathcal{M}^{+}(\bar{\Omega} \times \bar{\Omega}),\left(\Pi_{x}\right)_{\#} \gamma=P_{\#} \nu^{-} \text {and }\left(\Pi_{y}\right)_{\#} \gamma=\nu^{-}\right\} \tag{3.5}
\end{equation*}
$$

Let $\sigma(\Gamma, \Gamma), \sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ be the transport densities associated with the transport plans $\gamma(\Gamma, \Gamma), \gamma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$, respectively. From (1.5), for every Borel set $A \subset \bar{\Omega}$, we have
and

$$
\begin{gather*}
\sigma(\Gamma, \Gamma)[A]=\int_{\Gamma \times \Gamma} \mathcal{H}^{1}(A \cap[x, y]) \mathrm{d}[\gamma(\Gamma, \Gamma)](x, y),  \tag{3.6}\\
\sigma(\Gamma, \partial \Omega \backslash \Gamma)[A]=\int_{\Gamma \times \partial \Omega \backslash \Gamma} \mathcal{H}^{1}(A \cap[x, y]) \mathrm{d}[\gamma(\Gamma, \partial \Omega \backslash \Gamma)](x, y), \tag{3.7}
\end{gather*}
$$

$$
\begin{equation*}
\sigma(\partial \Omega \backslash \Gamma, \Gamma)[A]=\int_{\partial \Omega \backslash \Gamma \times \Gamma} \mathcal{H}^{1}(A \cap[x, y]) \mathrm{d}[\gamma(\partial \Omega \backslash \Gamma, \Gamma)](x, y) . \tag{3.8}
\end{equation*}
$$

The aim now is to prove that under a geometric assumption on $\Gamma$, these transport densities $\sigma(\Gamma, \Gamma), \sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ give zero mass to the boundary $\partial \Omega$. More precisely, we have the following:

Proposition 3.7. Assume that $\Gamma$ is strictly convex. Then, the transport density $\sigma$ associated with the optimal transport plan $\gamma$ of Problem (2.3) gives zero mass to $\partial \Omega$.

Proof. From (3.6), we have

$$
\sigma(\Gamma, \Gamma)[\partial \Omega]=\int_{\Gamma \times \Gamma} \mathcal{H}^{1}(\partial \Omega \cap[x, y]) \mathrm{d}[\gamma(\Gamma, \Gamma)](x, y) .
$$

Thanks to the fact that $\Gamma$ is an open strictly convex part of $\partial \Omega$, we have obviously $\partial \Omega \cap] x, y[=\emptyset$, for all $(x, y) \in \Gamma \times \Gamma$ and so, $\sigma(\Gamma, \Gamma)[\partial \Omega]=0$. On the other hand, one has

$$
\sigma(\Gamma, \partial \Omega \backslash \Gamma)[\partial \Omega]=\int_{\Gamma \times \partial \Omega \backslash \Gamma} \mathcal{H}^{1}(\partial \Omega \cap[x, y]) \mathrm{d}[\gamma(\Gamma, \partial \Omega \backslash \Gamma)](x, y)
$$

Yet,

$$
\mathcal{H}^{1}(\partial \Omega \cap[x, y])=\mathcal{H}^{1}(\Gamma \cap[x, y])+\mathcal{H}^{1}(\partial \Omega \backslash \Gamma \cap[x, y])
$$

From the strict convexity of $\Gamma$, for all $(x, y) \in \Gamma \times \partial \Omega \backslash \Gamma$, we have $\Gamma \cap[x, y]=\{x\}$ and so, $\mathcal{H}^{1}(\Gamma \cap[x, y])=0$. In addition, it is clear that $\mathcal{H}^{1}(\partial \Omega \backslash \Gamma \cap[x, y])=0$, for all $(x, y) \in \Gamma \times \partial \Omega \backslash \Gamma$, since, by Proposition 3.5, $\partial \Omega \backslash \Gamma \cap[x, y]=\{y\}$, for $\gamma(\Gamma, \partial \Omega \backslash \Gamma)-$ a.e. ( $x, y$ ). This implies that $\sigma(\Gamma, \partial \Omega \backslash \Gamma)[\partial \Omega]=0$. In the same way, one can prove that $\sigma(\partial \Omega \backslash \Gamma, \Gamma)[\partial \Omega]=0$. This concludes the proof as $\sigma=\sigma(\Gamma, \Gamma)+\sigma(\Gamma, \partial \Omega \backslash \Gamma)+\sigma(\partial \Omega \backslash \Gamma, \Gamma)$.

Now, we will prove that the transport problem (2.3) has a unique optimal transport plan $\gamma$. This will imply that the solution of the minimal flow problem (2.2) is unique as well and then, Problem (2.1) has a unique solution. First, we introduce the following:

Lemma 3.8. Assume $f^{ \pm}$are atomless. Let $\gamma$ be an optimal transport plan for Problem (2.3) and consider the transport plans $\gamma(\Gamma, \Gamma), \gamma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)$. Let $\nu^{+}:=\left(\Pi_{x}\right) \#[\gamma(\Gamma, \partial \Omega \backslash \Gamma)]$ and $\nu^{-}:=\left(\Pi_{y}\right)_{\#}[\gamma(\partial \Omega \backslash \Gamma, \Gamma)]$. Then, there exists two sets $A^{ \pm} \subset \Gamma$ such that $A^{ \pm}$is a countable union of connected arcs and, $\nu^{ \pm}=f^{ \pm} \cdot 1_{A^{ \pm}}$.

Proof. Let us prove that there is a set $A^{+} \subset \Gamma$ such that $\nu^{+}=f^{+} \cdot 1_{A^{+}}$. Assume that this is not the case, i.e. there is some set $A$ such that $0<\nu^{+}(A)<f^{+}(A)$. This means that on $A$ we split the mass in two parts: one is going to $\Gamma$ and the second one is exported to $\partial \Omega \backslash \Gamma$. For each $x \in A$, let $R_{x}^{+}$and $R_{x}^{-}$be two transport rays from $x$ to $\Gamma$ and $\partial \Omega \backslash \Gamma$, respectively. Let $\Delta_{x} \subset \Omega$ be the region delimited by $R_{x}^{+}, R_{x}^{-}$and $\partial \Omega \backslash \Gamma$. Similarly to Lemma 3.4, one can see that these sets $\left\{\Delta_{x}\right\}_{x \in A}$ are disjoint with $\mathcal{L}^{2}\left(\Delta_{x}\right)>0$. Hence, $A$ is at most countable and so, $f^{+}(A)=0$. Hence, $\nu^{+}=f^{+} \cdot 1_{A^{+}}$, for some $A^{+} \subset \Gamma$ (one can see that $A^{+}$is a countable union of connected arcs).

Proposition 3.9. Assume that $f^{ \pm}$are atomless. Then, Problem (2.3) has a unique optimal transport plan $\gamma$.

Proof. First, we show that the transport plan $\gamma(\Gamma, \Gamma)$ is induced by a map $T$ (we note that this is similar to [5, Proposition 2.5], but we will introduce the proof for the sake of completeness). Let $\Lambda$ be an optimal transport plan for Problem (3.1). We denote by $D$ the set of points that belong to different transport rays. We note that two different transport rays can only intersect at an endpoint and so, $D \subset \Gamma$. Fix $x \in \operatorname{spt}\left(f^{+}-\nu^{+}\right) \cap D$ and let us denote by $R_{x}^{ \pm}$two different transport rays from $x$ to $\operatorname{spt}\left(f^{-}-\nu^{-}\right)$. Let $\Delta_{x} \subset \Omega$ be the region delimited by $R_{x}^{+}, R_{x}^{-}$and $\Gamma$. Then, it is clear that these sets $\left\{\Delta_{x}\right\}_{x}$ are disjoint with $\mathcal{L}^{2}\left(\Delta_{x}\right)>0$. This implies that the set $D$ is at most countable and so, thanks to the fact that $f^{+}$is atomless, $f^{+}(D)=0$. For every $x \in \operatorname{spt}\left(f^{+}-\nu^{+}\right) \backslash D$, there is a unique transport ray $R_{x}$ starting at $x$ and this ray intersects $\Gamma$ at exactly one point (say $T(x))$. This yields that $\gamma(\Gamma, \Gamma)=(I d, T)_{\#}\left[f^{+}-\nu^{+}\right]$. On the other hand, by Proposition 3.5, we have $\gamma(\Gamma, \partial \Omega \backslash \Gamma)=(I d, P)_{\#} \nu^{+}$and $\gamma(\partial \Omega \backslash \Gamma, \Gamma)=(P, I d)_{\#} \nu^{-}$. Now, assume that $\gamma_{1}$ and $\gamma_{2}$ minimize Problem (2.3). Set $\nu_{1}^{+}=\left(\Pi_{x}\right)_{\#}\left[\gamma_{1}(\Gamma, \partial \Omega \backslash \Gamma)\right], \nu_{1}^{-}=\left(\Pi_{y}\right)_{\#}\left[\gamma_{1}(\partial \Omega \backslash \Gamma, \Gamma)\right], \nu_{2}^{+}=$ $\left(\Pi_{x}\right)_{\#}\left[\gamma_{2}(\Gamma, \partial \Omega \backslash \Gamma)\right]$ and $\nu_{2}^{-}=\left(\Pi_{y}\right)_{\#}\left[\gamma_{2}(\partial \Omega \backslash \Gamma, \Gamma)\right]$. Let $T_{1}$ and $T_{2}$ be the two transport maps such that $\gamma_{1}(\Gamma, \Gamma)$ and $\gamma_{2}(\Gamma, \Gamma)$ are induced by $T_{1}$ and $T_{2}$, respectively. Then, for all $\varphi \in C(\bar{\Omega} \times \bar{\Omega})$, we have

$$
<\gamma_{1}, \varphi>=\int_{\Gamma} \varphi\left(x, T_{1}(x)\right) \mathrm{d}\left[f^{+}-\nu_{1}^{+}\right](x)+\int_{\Gamma} \varphi(x, P(x)) \mathrm{d} \nu_{1}^{+}(x)+\int_{\Gamma} \varphi(P(y), y) \mathrm{d} \nu_{1}^{-}(y)
$$

and

$$
<\gamma_{2}, \varphi>=\int_{\Gamma} \varphi\left(x, T_{2}(x)\right) \mathrm{d}\left[f^{+}-\nu_{2}^{+}\right](x)+\int_{\Gamma} \varphi(x, P(x)) \mathrm{d} \nu_{2}^{+}(x)+\int_{\Gamma} \varphi(P(y), y) \mathrm{d} \nu_{2}^{-}(y) .
$$

Yet, it is easy to see that $\gamma:=\frac{\gamma_{1}+\gamma_{2}}{2}$ minimize Problem (2.3) as well. But, this is a contradiction as, for all $\varphi \in C(\bar{\Omega} \times \bar{\Omega})$, one has

$$
\begin{aligned}
&\langle\gamma, \varphi\rangle \\
&= \int_{\Gamma} \varphi(x, T(x)) \mathrm{d}\left[f^{+}-\nu^{+}\right](x)+\int_{\Gamma} \varphi(x, P(x)) \mathrm{d} \nu^{+}(x)+\int_{\Gamma} \varphi(P(y), y) \mathrm{d} \nu^{-}(y) \\
&= \frac{1}{2}\left[\int_{\Gamma} \varphi\left(x, T_{1}(x)\right) \mathrm{d}\left[f^{+}-\nu_{1}^{+}\right](x)+\int_{\Gamma} \varphi(x, P(x)) \mathrm{d} \nu_{1}^{+}(x)+\int_{\Gamma} \varphi(P(y), y) \mathrm{d} \nu_{1}^{-}(y)\right] \\
&+ \frac{1}{2}\left[\int_{\Gamma} \varphi\left(x, T_{2}(x)\right) \mathrm{d}\left[f^{+}-\nu_{2}^{+}\right](x)+\int_{\Gamma} \varphi(x, P(x)) \mathrm{d} \nu_{2}^{+}(x)+\int_{\Gamma} \varphi(P(y), y) \mathrm{d} \nu_{2}^{-}(y)\right] .
\end{aligned}
$$

From Lemma 3.8, there are subsets $A^{ \pm}, A_{1}^{ \pm}$and $A_{2}^{ \pm}$of $\Gamma$ such that $\nu^{ \pm}=f^{ \pm} \cdot 1_{A^{ \pm}}, \nu_{1}^{ \pm}=f^{ \pm} \cdot 1_{A_{1}^{ \pm}}$ and $\nu_{2}^{ \pm}=f^{ \pm} \cdot 1_{A_{2}^{ \pm}}$. Then, for all $\varphi \in C(\bar{\Omega} \times \bar{\Omega})$ such that $\varphi=0$ on $\partial \Omega \backslash \Gamma \times \partial \Omega$, we have

$$
\int_{\Gamma} \varphi\left(x, T^{+}(x)\right) \mathrm{d} f^{+}(x)=\frac{1}{2}\left[\int_{\Gamma} \varphi\left(x, T_{1}^{+}(x)\right) \mathrm{d} f^{+}(x)+\int_{\Gamma} \varphi\left(x, T_{2}^{+}(x)\right) \mathrm{d} f^{+}(x)\right],
$$

where

$$
T^{+}(x):= \begin{cases}P(x) & \text { if } x \in A^{+}, \\ T(x) & \text { if } x \in \operatorname{spt}\left(f^{+}\right) \backslash A^{+} .\end{cases}
$$

Yet, it is clear that this equality holds if and only if $T_{1}^{+}=T_{2}^{+}=T^{+}$, which is equivalent to say that $A_{1}^{+}=A_{2}^{+}=A^{+}$and $T_{1}=T_{2}=T$. On the other hand, we infer that, for all $\varphi \in C(\bar{\Omega} \times \bar{\Omega})$,

$$
\begin{equation*}
\int_{A^{-}} \varphi(P(y), y) \mathrm{d} f^{-}(y)=\frac{1}{2} \int_{A_{1}^{-}} \varphi(P(y), y) \mathrm{d} f^{-}(y)+\frac{1}{2} \int_{A_{2}^{-}} \varphi(P(y), y) \mathrm{d} f^{-}(y) \tag{3.9}
\end{equation*}
$$

If $1_{A^{-}}=\frac{1}{2}\left(1_{A_{1}^{-}}+1_{A_{2}^{-}}\right)$does not hold $f^{-}$-a.e., then there exists a function $\psi \in C(\bar{\Omega})$ such that $\int_{\Gamma} \psi\left[1_{A^{-}}-\frac{1}{2}\left(1_{A_{1}^{-}}+1_{A_{2}^{-}}\right)\right] \mathrm{d} f^{-}>0$. But, this contradicts (3.9) with $\varphi(x, y):=\psi(y)$, for all $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. This implies that $A_{1}^{-}=A_{2}^{-}=A^{-}$as well (or equivalently, $\nu^{-}=\nu_{1}^{-}=\nu_{2}^{-}$).

## 4. $L^{p}$ SUMMABILITY OF THE TRANSPORT DENSITY BETWEEN SINGULAR MEASURES

In this section, we study the $L^{p}$ summability of the transport density $\sigma$ in Problem (1.8) or equivalently, in the Kantorovich problem between $f^{+}+\chi^{+}$and $f^{-}+\chi^{-}$, where $\chi^{ \pm}$represent the import/export masses on $\partial \Omega \backslash \Gamma$. In [5], the authors have already studied the $L^{p}$ summability of the transport density $\sigma$ between two singular measures $f^{+}$and $f^{-}$on $\partial \Omega$. In particular, they proved that if $\Omega$ is uniformly convex, then $\sigma$ is in $L^{p}(\Omega)$ provided that $f^{+}$or $f^{-}$is in $L^{p}(\partial \Omega)$ and $p<2$. But, the $L^{2}$ summability of $\sigma$ requires that both $f^{+}$and $f^{-}$belong to $L^{2}(\partial \Omega)$ and, to go beyond $L^{2}$ summability we need extra regularity on $f^{+}$and $f^{-}$. The problem is that here $\chi^{+}$and $\chi^{-}$are two unknown measures on $\partial \Omega \backslash \Gamma$ and so, $\chi^{ \pm}$are a priori not in $L^{p}(\partial \Omega \backslash \Gamma)$. Then, the idea will be to study the $L^{p}$ summability of each transport density $\sigma(\Gamma, \Gamma), \sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ so that we get the summability of $\sigma$. More precisely, we will show that under some geometric assumptions, the following statements hold:

$$
\begin{gathered}
f^{ \pm} \in L^{p}(\Gamma) \Rightarrow \sigma \in L^{p}(\Omega), \text { for all } p \leq 2 \\
f^{ \pm} \in C^{0, \alpha}(\Gamma) \Rightarrow \sigma \in L^{\frac{2}{1-\alpha}}(\Omega), \text { for all } \alpha \in(0,1) \\
f^{ \pm} \in \operatorname{Lip}(\Gamma) \Rightarrow \sigma \in L^{\infty}(\Omega)
\end{gathered}
$$

First, we need to introduce the following
Definition 4.1. Let $\Omega$ be an open bounded domain and $\Gamma \subset \partial \Omega$. We say that $\Gamma$ is uniformly convex if there exists $R<\infty$ such that, for every $x \in \Gamma$ and every unit vector $-n$ in the exterior normal cone to $\Omega$ at $x$, we have $\Gamma \subset B(z, R)$ with $z=x+R n$.
Remark 4.2. If $\Gamma$ is smooth, then the notion of uniform convexity in Definition 4.1 is equivalent to saying that the curvature of $\Gamma$ is larger than $\frac{1}{R}$.
Proposition 4.3. The transport density $\sigma(\Gamma, \Gamma)$ belongs to $L^{p}(\Omega)$ provided that $f^{ \pm} \in L^{p}(\Gamma), p \leq 2$ and $\Gamma$ is uniformly convex.
Proof. $\sigma(\Gamma, \Gamma)$ is the transport density between $f^{+}-\nu^{+}$and $f^{-}-\nu^{-}$. Yet, by Lemma 3.8, $f^{+}-\nu^{+}=f^{+} \cdot 1_{B^{+}}$and $f^{-}-\nu^{-}=f^{-} \cdot 1_{B^{-}}$, for some sets $B^{ \pm} \subset \Gamma$. Hence, $f^{+}-\nu^{+}$and $f^{-}-\nu^{-}$ are in $L^{p}(\Gamma)$. Yet, in [5, Proposition 3.3], the authors show that the transport density $\sigma$ between two $L^{p}$ densities $g^{+}$and $g^{-}$on $\partial \Omega$ is in $L^{p}(\Omega)$ as soon as $\partial \Omega$ is uniformly convex (see Definition 4.1 with $\Gamma=\partial \Omega)$. However, it is not difficult to check that the proof of [5, Proposition 3.3] also works if we only have $\left[\operatorname{spt}\left(g^{+}\right) \cup \operatorname{spt}\left(g^{-}\right)\right] \subset \Gamma$ and $\Gamma$ is uniformly convex. Hence, thanks to the uniform convexity of $\Gamma$, we infer that $\sigma(\Gamma, \Gamma) \in L^{p}(\Omega)$.

Proposition 4.4. Suppose that $\Gamma$ is uniformly convex and $f^{ \pm} \in C^{0, \alpha}(\Gamma)$ with $0<\alpha \leq 1$. Then, the transport density $\sigma(\Gamma, \Gamma)$ is in $L^{p}(\Omega)$ for $p=\frac{2}{1-\alpha}$ (with $p=\infty$ for $\alpha=1$ ).
Proof. First, we recall that there exists two sets $B^{ \pm} \subset \Gamma$ such that $B^{ \pm}$is a countable union of connected $\operatorname{arcs}\left(B_{n}^{ \pm}\right)_{n} \subset \Gamma, f^{+}-\nu^{+}=f^{+} \cdot 1_{B^{+}}, f^{-}-\nu^{-}=f^{-} \cdot 1_{B^{-}}$and, $f^{+}-\nu^{+}$on $B_{n}^{+}$is transported to $f^{-}-\nu^{-}$on $B_{n}^{-}$, for all $n$. Hence, we have $f^{ \pm}-\nu^{ \pm} \in C^{0, \alpha}\left(B_{n}^{ \pm}\right)$. If $\operatorname{dist}\left(B_{n}^{+}, B_{n}^{-}\right)>0$, then by [5, Remark 5.10], the transport density between $\left[f^{+}-\nu^{+}\right] \cdot 1_{B_{n}^{+}}$and $\left[f^{-}-\nu^{-}\right] \cdot 1_{B_{n}^{-}}$is in $L^{\infty}(\Omega)$. Now, assume that $\overline{B_{n}^{+}} \cap \overline{B_{n}^{-}} \neq \emptyset$. We have that $f^{+}-\nu^{+}$and $f^{-}-\nu^{-}$are $C^{0, \alpha}$ on the $\operatorname{arc} B_{n}^{+} \cup B_{n}^{-}$. In [5, Proposition 3.5], the authors show that the transport density $\sigma$ between two $C^{0, \alpha}$ densities $g^{+}$and $g^{-}$on $\partial \Omega$ is in $L^{p}(\Omega)$ with $p=\frac{2}{1-\alpha}$ as soon as $\partial \Omega$ is uniformly convex, and we have the following estimate:

$$
\begin{equation*}
\|\sigma\|_{L^{p}}^{p} \leq C \mathcal{H}^{1}\left(\operatorname{spt}\left(g^{+}\right) \cup \operatorname{spt}\left(g^{-}\right)\right) \tag{4.1}
\end{equation*}
$$

where $C=C\left(\frac{1}{R},\left\|g^{ \pm}\right\|_{C^{0, \alpha}}\right)<\infty$. Again, it is not difficult to check that the proof of [5, Proposition 3.5] also works if $g^{ \pm} \in C^{0, \alpha}\left(\operatorname{spt}\left(g^{+}\right) \cup \operatorname{spt}\left(g^{-}\right)\right),\left[\operatorname{spt}\left(g^{+}\right) \cup \operatorname{spt}\left(g^{-}\right)\right] \subset \Gamma$ and $\Gamma$ is uniformly convex.

Then, the transport density $\sigma(\Gamma, \Gamma)$ between $B_{n}^{+}$and $B_{n}^{-}$is in $L^{\frac{2}{1-\alpha}}(\Omega)$, for all $n$, and so thanks to (4.1), we infer that $\sigma(\Gamma, \Gamma) \in L^{\frac{2}{1-\alpha}}(\Omega)$.

On the other hand, we have the following:
Proposition 4.5. The transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ (resp. $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ ) belongs to $L^{p}(\Omega)$ provided that $f^{+}$(resp. $f^{-}$) is in $L^{p}(\Gamma), p<2$ and $\Gamma$ is uniformly convex.
Proof. $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ is the transport density between $\nu^{+}$and $P_{\#} \nu^{+}$. So, we can decompose this transport density into two parts $\sigma_{1}$ and $\sigma_{2}$, where $\sigma_{1}$ is the transport density between a part of $\nu^{+}$and an endpoint of $\partial \Omega \backslash \Gamma$, while $\sigma_{2}$ is the transport density between the other part of $\nu^{+}$(supported on a set $\Gamma^{\prime} \subset \Gamma$ ) and its projection $P\left(\Gamma^{\prime}\right)$ onto $\partial \Omega \backslash \Gamma$. It is easy to see that $\operatorname{dist}\left(\Gamma^{\prime}, P\left(\Gamma^{\prime}\right)\right)>0$. Thanks to [5, Proposition 3.2], the transport density between $g^{+} \in L^{p}(\partial \Omega)$ and any $g^{-} \in \mathcal{M}^{+}(\partial \Omega)$ is in $L^{p}(\Omega)$ provided that $p<2$ and $\partial \Omega$ is uniformly convex. But, as we have already mentioned previously, one can prove the same result under the assumption that $\left[\operatorname{spt}\left(g^{+}\right) \cup \operatorname{spt}\left(g^{-}\right)\right] \subset \Gamma$ and $\Gamma$ is uniformly convex. Hence, $\sigma_{1} \in L^{p}(\Omega)$ provided that $f^{+} \in L^{p}(\Gamma)$ with $p<2$ and $\Gamma$ is uniformly convex. On the other hand, we recall that by [5, Remark 5.10], the transport density between $g^{+}$and $g^{-}$is in $L^{p}(\Omega)$, for all $p<2$, as soon as $g^{+} \in L^{p}(\partial \Omega)$ and $\operatorname{spt}\left(g^{+}\right) \cap \operatorname{spt}\left(g^{-}\right)=\emptyset$. Hence, as $\operatorname{dist}\left(\Gamma^{\prime}, P\left(\Gamma^{\prime}\right)\right)>0$, this yields that $\sigma_{2}$ is in $L^{p}(\Omega)$ as soon as $f^{+} \in L^{p}(\Gamma)$.

Consequently, we get the following $L^{p}$ summability on the transport density $\sigma$ for $p<2$ :
Proposition 4.6. The transport density $\sigma$ belongs to $L^{p}(\Omega)$ as soon as $f^{ \pm} \in L^{p}(\Gamma), p<2$ and $\Gamma$ is uniformly convex.

Proof. This follows immediately from Propositions $4.3 \& 4.5$ and the fact that $\sigma=\sigma(\Gamma, \Gamma)+$ $\sigma(\Gamma, \partial \Omega \backslash \Gamma)+\sigma(\partial \Omega \backslash \Gamma, \Gamma)$.

The aim now is to extend this $L^{p}$ summability result on the transport density $\sigma$ to the case $p \geq 2$. We recall that, for $p \geq 2$, the $L^{p}$ summability of the transport density $\sigma$ between two measures $f^{+}$and $f^{-}$on $\partial \Omega$ requires $L^{p}$ summability of both $f^{+}$and $f^{-}$. So, it is not clear if the transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ is in $L^{p}(\Omega)$ or not, since the target measure is the projection of $\nu^{+}$ onto $\partial \Omega \backslash \Gamma$. However, we will show that under some geometric assumptions, it is possible to prove $L^{p}$ estimates on the transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$, for $p \geq 2$. In the sequel, we will say that the assumption (A) holds if
(A) $\quad \operatorname{dist}(\operatorname{spt}(f), \partial \Omega \backslash \Gamma)>0$ and for a.e. $x \in \operatorname{spt}(f), P(x)$ is not an endpoint of $\partial \Omega \backslash \Gamma$.

Proposition 4.7. Assume that ( $A$ ) holds and $\Gamma$ is strictly convex. Then, the transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ is in $L^{p}(\Omega)$ provided that $f^{+} \in L^{p}(\Gamma)$, for all $p \in[1, \infty]$.
Proof. The transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ is between $\nu^{+}$and its projection to $\partial \Omega \backslash \Gamma$, i.e. we have

$$
<\sigma(\Gamma, \partial \Omega \backslash \Gamma), \phi>=\int_{\Gamma} \int_{0}^{1} \phi((1-t) x+t P(x))|x-P(x)| \mathrm{d} t \mathrm{~d} \nu^{+}(x), \text { for all } \phi \in C(\bar{\Omega})
$$

Let us find an explicit formula for this transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$. Assume that $\partial \Omega \backslash \Gamma$ is $C^{2}$. Fix a point $x_{0}$ on $\operatorname{spt}\left(\nu^{+}\right)$and let $\Gamma^{\prime} \subset \Gamma$ be a small arc around $x_{0}$. Let $\tilde{\alpha}(s):=(s, \alpha(s)), s \in(-\varepsilon, \varepsilon)$, be a parametrization of the arc $P\left(\Gamma^{\prime}\right)$ and $\beta(s):=\left(\beta_{1}(s), \beta_{2}(s)\right)$ be a parametrization of $\Gamma^{\prime}$ (we will show later that $\beta(s)$ is Lipschitz) such that $\alpha(0)=\alpha^{\prime}(0)=0$ and $P(\beta(s))=\tilde{\alpha}(s)$, for every $s \in[-\varepsilon, \varepsilon]$. Let $\Delta$ be the set of all transport rays $[x, P(x)], x \in \Gamma^{\prime}$. We see that, for all $y \in \Delta$, there exists $s \in[-\varepsilon, \varepsilon]$ and $t \in[0,1]$ such that

$$
y=\left((1-t) \beta_{1}(s)+t s,(1-t) \beta_{2}(s)+t \alpha(s)\right) .
$$

Then, we have

$$
<\sigma(\Gamma, \partial \Omega \backslash \Gamma), \phi>:=
$$

$$
\int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} \phi\left((1-t) \beta_{1}(s)+t s,(1-t) \beta_{2}(s)+t \alpha(s)\right) \tau(s)\left|\beta^{\prime}(s)\right| \nu^{+}(\beta(s)) \mathrm{d} t \mathrm{~d} s, \forall \phi \in C(\Delta)
$$

where

$$
\tau(s):=|\beta(s)-\tilde{\alpha}(s)|, \quad \forall s \in[-\varepsilon, \varepsilon] .
$$

Hence,

$$
<\sigma(\Gamma, \partial \Omega \backslash \Gamma), \phi>=\int_{\Omega} \phi(y) \frac{\tau(s)\left|\beta^{\prime}(s)\right| \nu^{+}(\beta(s))}{J(s, t)} \mathrm{d} y, \text { for all } \phi \in C(\Delta)
$$

where

$$
J(s, t):=\left|\operatorname{det}\left(D_{(s, t)}\left(y_{1}, y_{2}\right)\right)\right|=\left(\beta_{1}(s)-s, \beta_{2}(s)-\alpha(s)\right) \cdot\left[(1-t)\left(-\beta_{2}^{\prime}(s), \beta_{1}^{\prime}(s)\right)+t\left(-\alpha^{\prime}(s), 1\right)\right]
$$

Then,

$$
\sigma(\Gamma, \partial \Omega \backslash \Gamma)[y]=\frac{\tau(s)\left|\beta^{\prime}(s)\right| \nu^{+}(\beta(s))}{J(s, t)}, \text { for a.e. } y \in \Delta .
$$

The aim now is to prove a uniform upper bound on $\frac{\left|\beta^{\prime}(s)\right|}{J(s, t)}$. First, it is easy to see that the following holds

$$
\begin{equation*}
\left(\beta_{1}(s)-s, \beta_{2}(s)-\alpha(s)\right) \cdot\left(-\alpha^{\prime}(s), 1\right) \geq \operatorname{dist}(\operatorname{spt}(f), \partial \Omega \backslash \Gamma) \tag{4.2}
\end{equation*}
$$

Let $\tilde{\beta}(r), r \in(-\delta, \delta)$, be a regular parametrization of the $\underset{\tilde{\beta}}{\operatorname{arc}} \Gamma^{\prime}$ such that $\left|\tilde{\beta}^{\prime}\right|=1$ and $\tilde{\beta}_{1}^{\prime}>0$. For every $s \in(-\varepsilon, \varepsilon)$, let $r(s) \in(-\delta, \delta)$ be such that $P(\tilde{\beta}(r(s)))=\tilde{\alpha}(s)$ (we note that $r(s)$ is increasing). From the strict convexity of $\Gamma$ and the fact that $\operatorname{dist}(\operatorname{spt}(f), \partial \Omega \backslash \Gamma)>0$, we see that there is a uniform constant $c>0$ such that

$$
\begin{equation*}
\left(\beta_{1}(s)-s, \beta_{2}(s)-\alpha(s)\right) \cdot\left(-\tilde{\beta}_{2}^{\prime}(r(s)), \tilde{\beta}_{1}^{\prime}(r(s))\right) \geq c \tag{4.3}
\end{equation*}
$$

In particular, one has

$$
\tilde{\beta}_{1}^{\prime}(0) \geq c .
$$

On the other hand, we have

$$
\begin{equation*}
(\tilde{\beta}(r(s))-\tilde{\alpha}(s)) \cdot R_{\frac{\pi}{2}} D d_{\partial \Omega \backslash \Gamma}(\tilde{\alpha}(s))=0 . \tag{4.4}
\end{equation*}
$$

Yet,

$$
D d_{\partial \Omega \backslash \Gamma}(\tilde{\alpha}(s))=D d_{\partial \Omega \backslash \Gamma}(\tilde{\alpha}(0))+D^{2} d_{\partial \Omega \backslash \Gamma}(\tilde{\alpha}(0))(\tilde{\alpha}(s)-\tilde{\alpha}(0))+o(|\tilde{\alpha}(s)-\tilde{\alpha}(0)|) .
$$

Let us denote by $\kappa(\tilde{\alpha}(s))$ the curvature of $\partial \Omega \backslash \Gamma$ at the point $\tilde{\alpha}(s)$. Then, one has (see, for instance, [10, Lemma 14.17])

$$
D^{2} d_{\partial \Omega \backslash \Gamma}(\tilde{\alpha}(0))=-\kappa(\tilde{\alpha}(0))\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] .
$$

Hence, we get that

$$
D d_{\partial \Omega \backslash \Gamma}(\tilde{\alpha}(s))=\left[\begin{array}{c}
-\kappa(\tilde{\alpha}(0)) s+o(s) \\
1+o(s)
\end{array}\right] .
$$

Recalling (4.4), this yields that

$$
s-\tilde{\beta}_{1}(r(s))-\kappa(\tilde{\alpha}(0))\left(\tilde{\beta}_{2}(r(s))-\alpha(s)\right) s+o(s)=0 .
$$

But,

$$
\tilde{\beta}(r)=\tilde{\beta}(0)+\tilde{\beta}^{\prime}(0) r+o(r)=(0, \tau(0))+\tilde{\beta}^{\prime}(0) r+o(r) .
$$

Therefore,

$$
\begin{gathered}
s-\tilde{\beta}_{1}^{\prime}(0) r(s)-\kappa(\tilde{\alpha}(0))\left(\tau(0)+\tilde{\beta}_{2}^{\prime}(0) r(s)-\alpha(s)\right) s+o(s)+o(r(s)) \\
=(1-\kappa(\tilde{\alpha}(0)) \tau(0)) s-\left(\tilde{\beta}_{1}^{\prime}(0)+\kappa(\tilde{\alpha}(0)) \tilde{\beta}_{2}^{\prime}(0) s\right) r(s)+o(s)+o(r(s))=0 .
\end{gathered}
$$

Consequently, the map $s \mapsto r(s)$ is Lipschitz on $(-\varepsilon, \varepsilon)$ (and so, for $\beta(s))$ and, one has the following approximation:

$$
r(s)=\frac{1-\kappa(\tilde{\alpha}(0)) \tau(0)}{\tilde{\beta}_{1}^{\prime}(0)} s+o(s) .
$$

Combining (4.2) \& (4.3), we infer that

$$
J(s, t) \geq c\left[(1-t) r^{\prime}(s)+t\right]
$$

Hence,

$$
\frac{\left|\beta^{\prime}(s)\right|}{J(s, t)} \leq c^{-1} \frac{r^{\prime}(s)}{(1-t) r^{\prime}(s)+t} \leq 2 c^{-1} \max \left\{r^{\prime}(s), 1\right\} \leq 2 c^{-1}\left(\frac{1-\kappa(\tilde{\alpha}(0)) \tau(0)}{\tilde{\beta}_{1}^{\prime}(0)}+1\right)
$$

This implies that there is a uniform constant $C$ depending only on the geometry of $\Gamma$ and the distance between $\operatorname{spt}(f)$ and $\partial \Omega \backslash \Gamma$ such that

$$
\frac{\left|\beta^{\prime}(s)\right|}{J(s, t)} \leq C
$$

Therefore, we get

$$
\begin{gathered}
\|\sigma(\Gamma, \partial \Omega \backslash \Gamma)\|_{L^{p}(\Delta)}^{p} \\
=\int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} \frac{\tau(s)^{p}\left|\beta^{\prime}(s)\right|^{p} \nu^{+}(\beta(s))^{p}}{J(s, t)^{p-1}} \mathrm{~d} t \mathrm{~d} s \\
=\int_{-\varepsilon}^{\varepsilon} \int_{0}^{1} \tau(s)^{p}\left(\frac{\left|\beta^{\prime}(s)\right|}{J(s, t)}\right)^{p-1} \nu^{+}(\beta(s))^{p}\left|\beta^{\prime}(s)\right| \mathrm{d} t \mathrm{~d} s \\
\leq C^{p} \int_{-\varepsilon}^{\varepsilon} \nu^{+}(\beta(s))^{p}\left|\beta^{\prime}(s)\right| \mathrm{d} s \\
=C^{p}| | f^{+} \|_{L^{p}\left(\Gamma^{\prime}\right)}^{p} .
\end{gathered}
$$

Hence,

$$
\|\sigma(\Gamma, \partial \Omega \backslash \Gamma)\|_{L^{p}(\Omega)} \leq C\left\|f^{+}\right\|_{L^{p}(\Gamma)}
$$

Finally, we note that the constant $C$ in the estimates above does not depend on the regularity of $\partial \Omega \backslash \Gamma$ and then, by an approximation argument, it is standard to remove the assumption that $\partial \Omega \backslash \Gamma$ is $C^{2}$.

Then, we get the following:
Proposition 4.8. Suppose that (A) holds and $\Gamma$ is uniformly convex. Then, the transport density $\sigma$ is in $L^{2}(\Omega)$ as soon as $f^{ \pm} \in L^{2}(\Gamma)$. Moreover, $\sigma$ is in $L^{p}(\Omega)$ for $p=\frac{2}{1-\alpha}$ provided that $f^{ \pm} \in C^{0, \alpha}(\Gamma)$ with $\alpha \in(0,1)$. In particular, $\sigma$ belongs to $L^{\infty}(\Omega)$ if $f^{ \pm}$are Lipschitz on $\Gamma$.
Proof. This follows immediately from Propositions 4.3, 4.4\& 4.7 and the fact that $\sigma=\sigma(\Gamma, \Gamma)+$ $\sigma(\Gamma, \partial \Omega \backslash \Gamma)+\sigma(\partial \Omega \backslash \Gamma, \Gamma)$.

We finish this section by the following:
Remark 4.9. In fact, one can prove that the projection of $\nu^{+}$onto $\partial \Omega \backslash \Gamma$ is a $L^{p}$ density on $\partial \Omega \backslash \Gamma$ as soon as the assumption (A) is well satisfied, $\Gamma$ is strictly convex and $\nu^{+} \in L^{p}(\Gamma)$. In this way, we can use [5] to infer that $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ is in $L^{p}(\Omega)$ provided that $\nu^{+} \in L^{p}(\Gamma)$. But anyway, the $L^{p}$ estimates on $P_{\#} \nu^{+}$will be too similar to those in the proof of Proposition 4.7 on the transport density $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and so, we decided to introduce instead the $L^{p}$ estimates on $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$.

## 5. LEAST GRADIENT PROBLEM

In this section, we apply all the results of the previous sections to prove existence and uniqueness of a solution $u$ to the BV least gradient problem with Dirichlet condition (2.1) and to give $W^{1, p}$ estimates on this solution $u$.

Theorem 5.1. Let $g$ be a $B V$ function on $\Gamma \subset \partial \Omega$. Then, the $B V$ least gradient problem (2.1) with Dirichlet condition on $\Gamma$ has a solution as soon as $\Gamma$ is strictly convex.

Proof. From Proposition 3.1, Problem (2.3) has an optimal transport plan $\gamma$. By Proposition 2.2, the flow $v_{\gamma}$ minimizes Problem (2.2). Thanks to Proposition 3.7, $\left|v_{\gamma}\right|(\partial \Omega)=0$. So, thanks to Proposition 2.1, there exists a BV function $u$ such that $v_{\gamma}=R_{\frac{\pi}{2}} D u$ and this $u$ turns out to be a solution for Problem (2.1).

In fact, the authors of [13] prove existence of a solution $u$ to the classical least gradient problem with Dirichlet condition on $\Gamma$ (i.e. in the case where the Dirichlet condition $u_{\mid \Gamma}=g$ is in the sense that there is an $L^{1}$ extension $\tilde{g}$ of $g$ such that $u_{\mid \partial \Omega}=\tilde{g}$ ) under the assumptions that $\Omega$ is strictly convex with Lipschitz boundary and $g \in C(\Gamma)$. While in [14], the author proves existence of such a solution $u$ as soon as $\Gamma$ is strictly convex and $g \in B V(\Gamma)$. On the other hand, [13, Lemma 3.3] shows that if $u$ is a solution to the classical least gradient problem and if $\gamma$ is a connected component of $\partial\{u \geq t\}$ in $\Omega$ intersecting the interior of $\partial \Omega \backslash \Gamma$, then $\gamma$ is orthogonal to $\partial \Omega \backslash \Gamma$. But, this implies that $u_{\mid \partial \Omega \backslash \Gamma}$ changes monotonicity finitely many times. In particular, this means that $u_{\mid \partial \Omega} \in B V(\partial \Omega)$ and so, $u$ is a minimizer for the constrained Problem (2.1) as well. Consequently, there is no difference between the classical least gradient problem and the constrained least gradient problem (2.1) (where we additionally suppose that the trace of $u$ is a BV function on the boundary).

Moreover, [13, Theorem 3.2] shows some results about the uniqueness of the solution $u$ of Problem (2.1) but under very restrictive assumptions on the boundary data. Here, we show that the solution $u$ of the least gradient problem (2.1) is unique as soon as $g$ is continuous on $\Gamma$.

Theorem 5.2. Assume that $\Gamma$ is strictly convex and $g \in B V(\Gamma)$. Then, the $B V$ least gradient problem (2.1) with Dirichlet condition on $\Gamma$ has a unique solution provided that $g \in C(\Gamma)$.
Proof. As $g \in C(\Gamma)$, then its tangential derivative $f:=\partial_{\tau} g$ has no atoms. Thanks to Proposition 3.9, we infer that Problem (2.3) has a unique optimal transport plan $\gamma$ and then, by Proposition 2.2 , the problem (2.2) has a unique optimal flow $v$ as well. Finally, Proposition 2.1 yields that the solution of Problem (2.1) is unique.

On the other hand, we note that there are no results in the literature concerning the higher order regularity of the solution $u$ of Problem (2.1). Yet, thanks to the $L^{p}$ estimates on the transport densities in Section 4, we get the following $W^{1, p}$ regularity on the solution $u$ of the BV least gradient problem (2.1):

Theorem 5.3. Assume that $\Gamma$ is uniformly convex and $g \in W^{1, p}(\Gamma)$ with $p<2$. Then, the solution $u$ of Problem (2.1) is in $W^{1, p}(\Omega)$.
Proof. Set $f=\partial_{\tau} g$. The condition $g \in W^{1, p}(\Gamma)$ implies that $f \in L^{p}(\Gamma)$ and so, by Proposition 4.6, the transport density $\sigma$ belongs to $L^{p}(\Omega)$. Thanks to Proposition 2.1, this implies that $\nabla u \in$ $L^{p}\left(\Omega, \mathbb{R}^{2}\right)$.

We recall that assumption (A) holds if there is an arc $\Gamma^{\prime} \subset \Gamma$ such that $g$ is constant on $\Gamma \backslash \Gamma^{\prime}$, $\operatorname{dist}\left(\Gamma^{\prime}, \partial \Omega \backslash \Gamma\right)>0$ and, for a.e. $x \in \Gamma^{\prime}$, the projection of $x$ onto $\partial \Omega \backslash \Gamma$ is not an endpoint of $\partial \Omega \backslash \Gamma$. Then, we have the following:

Theorem 5.4. Assume that (A) holds, $\Gamma$ is uniformly convex and $g \in H^{1}(\Gamma)$. Then, the solution $u$ of Problem (2.1) is in $H^{1}(\Omega)$.
Proof. $g \in H^{1}(\Gamma)$ implies that $f \in L^{2}(\Gamma)$ and so, thanks to Proposition 4.8, the transport density $\sigma$ belongs to $L^{2}(\Omega)$. From Proposition 2.1, we infer that $\nabla u \in L^{2}\left(\Omega, \mathbb{R}^{2}\right)$.
Theorem 5.5. Assume that (A) holds, $\Gamma$ is uniformly convex and $g \in C^{1, \alpha}(\Gamma)$ with $0<\alpha<1$. Then, the solution $u$ of Problem (2.1) belongs to $W^{1, p}(\Omega)$ with $p=\frac{2}{1-\alpha}$. Moreover, the solution $u$ of Problem (2.1) is Lipschitz as soon as $g \in C^{1,1}(\Gamma)$.

Proof. This follows immediately from Propositions 4.8 \& 2.1.
We finish this paper by the following
Remark 5.6. In fact, one can prove $L^{p}$ estimates, for all $p$, on the transport density $\sigma$ without assuming that $f^{ \pm}$are smooth but instead under the assumption that $\operatorname{spt}\left(f^{+}\right)$and $\operatorname{spt}\left(f^{-}\right)$are disjoint. More precisely, assume that (A) holds, $\Gamma$ is strictly convex and $\operatorname{spt}\left(f^{+}\right) \cap \operatorname{spt}\left(f^{-}\right)=\emptyset$, then the transport density $\sigma$ is in $L^{p}(\Omega)$ provided that $f \in L^{p}(\Gamma)$, for all $p \in[1, \infty]$; this follows from the fact that $\sigma(\Gamma, \Gamma) \in L^{p}(\Omega)$ (see [5, Remark 5.10]), while Proposition 4.7 implies that $\sigma(\Gamma, \partial \Omega \backslash \Gamma)$ and $\sigma(\partial \Omega \backslash \Gamma, \Gamma)$ are in $L^{p}(\Omega)$. In terms of $W^{1, p}$ regularity on the solution $u$ of Problem (2.1), this means that if the boundary datum $g$ has flat parts separating those where $g$ is increasing or decreasing, then $u$ is in $W^{1, p}(\Omega)$ as soon as $g \in W^{1, p}(\Gamma)$, for all $p \in[1, \infty]$.

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