A formula for the minimal perimeter of clusters with density

Vincenzo Scattaglia

Abstract This paper deals with the isoperimetric problem for clusters in a Euclidean space with double density. In particular, we show that a limit of an isoperimetric minimizing sequence of clusters with volumes $\mathbf{V}$ is always isoperimetric for its own volumes (which may be smaller than $\mathbf{V}$). In particular, if it is strictly smaller, we provide an explicit formula.

1 Introduction

In this paper, we examine some aspects of the isoperimetric problem with density for clusters; this arises as a fusion of two well known problems which we are going to briefly recall, both readable as generalizations of the classical Euclidean isoperimetric problem.

The first one is the *minimal partitioning problem*. Given a positive integer N, we call N*-cluster* every family of N mutually disjoint (measure theoretically) sets of finite perimeter $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1,\dots,N}$ and we look for a N-clusters satisfying the volume constraints $|\mathcal{E}(h)| = V(h)$ for every $h = 1, \dots, N$ which minimizes the perimeter

$$P(\mathcal{E}) = \mathcal{H}^{n-1}\left(\bigcup_{h=1}^{N} \partial^* \mathcal{E}(h)\right).$$

There is a huge literature on properties of minimal clusters, starting from the founding work of Almgren and Taylor ([1, 19]), where existence and regularity, among the many other results, have been proved. For what concerns the classification of minimal clusters, much is known about minimal 2-clusters [7, 8, 17], planar 3-clusters [20], and planar 4-clusters with chambers with equal area [13, 14], while there are still open problems regarding the structure of minima for more than three chambers, though symmetry properties are known, under restrictions on dimension and number of chambers.

The other well studied generalization of the Euclidean isoperimetric problem is the so called *isoperimetric problem with (double) density*: given two lower semi-continuous and locally summable functions $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}_+$, which we will call the *density* functions, we measure the f-volume and the g-perimeter of a Borel subset $E \subseteq \mathbb{R}^n$ as

$$|E|_f = \int_E f(x)dx, \qquad P_g(E) = \int_{\partial^* E} g(x, \nu_E(x)) d\mathcal{H}^{n-1}(x), \tag{1.1}$$

and we ask if there exists a set E which minimizes the g-perimeter among all sets of fixed f-volume V. In the previous definitions of perimeter, we consider $\partial^* E$ the reduced boundary of E and $\nu_E(x)$ the outer unit normal at $x \in \partial^* E$; for sufficient regular subsets of $\mathbb{R}^n$, the reduced boundary precisely corresponds to the usual topological boundary. Along with the problem of existence (or the *non*-existence) of isoperimetric sets, usual properties which are examined are boundedness and regularity of the boundary; in particular, information about boundedness of isoperimetric sets may be decisive in order to prove existence.

The isoperimetric problem with density may be seen as a generalization of the isoperimetric problem on Riemannian manifolds, since the density functions which weight volume and perimeter may be more general than the ones given by those related to the Riemannian metric ([10, 11]). Moreover, we underline that the generalization is consistent as long as we allow the density for the perimeter to be different from the one on the volume and to depend on the normal on $\partial^* E$. As one expects, the existence of isoperimetric sets and their geometric properties are intimately related to the densities f and g; a partial list of results is [3, 4, 5, 12, 18] in case $f = g$ (single density), [15, 16] for the general case (double density).

As anticipated, the isoperimetric problem with density for clusters is a combination of the two: we look for a N-cluster which minimizes the g-perimeter among those having chambers of fixed f-volumes $\{V(h)\}_{h=1,\dots,N}$, that is, if we define respectively the g-perimeter of a cluster and the (f, g)*-isoperimetric* *profile function* by

$$P_g(\mathcal{E}) := \frac{1}{2}\Big(\sum_{h=1}^N P_g(\mathcal{E}(h)) + P_g\Big(\bigcup_{h=1}^N \mathcal{E}(h)\Big)\Big), \tag{1.2}$$

$$\mathbf{V} = \{V(h)\}_{h=1}^N \in \mathbb{R}_+^N \mapsto \mathcal{I}_{(f,g)}(\mathbf{V}) := \inf\left\{P_g(\mathcal{E}) : \mathcal{E}\ N\text{-cluster}, |\mathcal{E}(h)| = V(h), h = 1,\dots,N\right\}, \tag{1.3}$$

we ask if the infimum is reached.

This question inherits the difficulties of both problems it generalizes, in particular the existence of isoperimetric clusters is strictly related to the density; nevertheless, we can take advantage of strategies already working for the case of single sets. The basic idea, as customary in the Calculus of Variations, is to consider a minimizing sequence $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$, that is $|\mathcal{E}_j(h)|_f = V(h)$ for each $h \in \{1,\dots,N\}$ and $P_g(\mathcal{E}_j) \to \mathcal{I}_{(f,g)}(\mathbf{V})$, and apply a standard compactness–semi-continuity argument: by compactness properties of BV functions, up to subsequences we can assume $\mathcal{E}_j \stackrel{j\to\infty}{\longrightarrow} \mathcal{E}$, and by semi-continuity of the perimeter we have $P_g(\mathcal{E}) \le \liminf_{j\to\infty} P_g(\mathcal{E}_j)$. Actually, the limit cluster may not have the right f-volume, since there may be loss of mass at infinity for one or more than one of the chambers. This cannot happen if $f \in L^1(\mathbb{R}^n)$, since obviously $|\mathcal{E}|_f = \lim_{j\to\infty} |\mathcal{E}_j|_f$; this means that $\mathcal{E}$ is a competitor for the isoperimetric problem, and by lower semi-continuity $P_g(\mathcal{E}) \le \liminf_{j\to\infty} P_g(\mathcal{E}_j) = \mathcal{I}_{(f,g)}(\mathbf{V})$, thus we have that isoperimetric clusters exist for every volume $\mathbf{V}$. For general $f \in L^1_{loc} \setminus L^1$, we only have the inequality $|\mathcal{E}(h)|_f \le V(h)$.

Let us focus for a moment on the single-set case (i.e. $N = 1$). As already shown in [5, 15], even in the case of loss of volume at infinity, limits of minimizing sequences are *isoperimetric sets for their own volumes.* Moreover, if the density f and g converge at infinity both to a finite positive value a, the following formula holds:

$$\mathcal{I}_{(f,g)}(V) := \inf\{P_g(F) : |F|_f = V\} = P_g(E) + n\,(a\omega_n)^{\frac{1}{n}}\left(V - |E|_f\right)^{\frac{n-1}{n}},$$

that is, the optimal profile is obtained as the union of E and a *ball at infinity* of volume $V - |E|_f$, where the density is constantly equal to a.

As we are going to prove in the article, limit points of minimizing sequences of clusters behave in a similar fashion; moreover, we can notice some extra structure if the densities f and g are converging to positive limits at infinity.

Theorem A. *Let f and g be L^1_{loc} and lower semicontinuous functions and assume them to be bounded from above and below (away from 0) away from the origin. Define*

$$g^+(x) := \sup_{\nu\in\mathbb{S}^{n-1}} g(x,\nu) \tag{1.4}$$

and assume it is locally integrable in $\mathbb{R}^n$. Let also $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ be an isoperimetric sequence of clusters with volume $\mathbf{V}$ which converges to a cluster $\mathcal{E}$ in the L^1_{loc} sense. Then:

i) $\mathcal{E}$ is a cluster of minimal g-perimeter for its own volume;

ii) If in addition $\lim_{|x|\to\infty} f(x) = a \in (0,\infty)$ and $\lim_{|x|\to\infty} g(x,\nu) = b \in (0,\infty)$ uniformly in ν, then

$$\mathcal{I}_{(f,g)}(\mathbf{V}) = P_g(\mathcal{E}) + ba^{-\frac{n-1}{n}}\mathcal{I}_{eucl}\left(\mathbf{V} - |\mathcal{E}|_f\right). \tag{1.5}$$

For converging densities, as in [5, 15] for single sets, formula (1.5) suggests that an isoperimetric cluster for volumes $\mathbf{V}$ is given by the union of the limit $\mathcal{E}$ and a "cluster at infinity" which has precisely the missing f-volume.

The article is structured in the following way. In Section 2 we introduce the main definitions and we recall the basic properties of finite perimeter sets we will need. Section 3 covers the proof of Theorem A.

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2 Definitions and basic properties of finite perimeter sets

In this section, we introduce the definitions, the notation and the basic results on finite perimeter sets we will need in the proof of Theorem A; for more information on definitions and results, the reader should refer to [2], [6], [9].

Let $E \subseteq \mathbb{R}^n$ be a set of (locally) finite measure; we say this is a *(locally) finite perimeter set* if its characteristic function χ_E is a BV function (resp. BV_{loc} function), i.e. it is summable (resp. locally summable) and its distributional derivative $D\chi_E$ is a Radon measure, and we will put $\mu_E := -D\chi_E$. For any Borel subset $A \subseteq \mathbb{R}^n$, we define the *relative perimeter of E in A* by

$$P(E; A) := |\mu_E|(A),$$

and we define the *perimeter of E* by $P(E) = P(E; \mathbb{R}^n)$.

For a locally finite perimeter set E, the *reduced boundary* is

$$\partial^* E := \left\{ x \in \operatorname{spt}(\mu_E) \,\middle|\, \exists \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} =: \nu_E(x), \quad |\nu_E(x)| = 1 \right\},$$

and we define $\nu_E(x)$ the exterior normal to $\partial^* E$ at x.

We recall a fundamental result on finite perimeter sets.

Theorem (Blow-Up, Structure)**.** *Assume E is a set of locally finite perimeter. Then:*

- *For any $x \in \partial^* E$, define $E_{x,r} := (E - x)/r$; then, we have the L^1_{loc} convergence*

$$E_{x,r} \overset{r \to 0^+}{\longrightarrow} H_{\nu_E(x)} = \{ y \in \mathbb{R}^n : y \cdot \nu_E(x) \le 0 \},$$

  and if we put $\pi_{\nu_E(x)} = \partial H_{\nu_E(x)}$, we have

$$\mu_{E_{x,r}} \overset{*}{\rightharpoonup} \nu_E(x) \mathcal{H}^{n-1} \llcorner \pi_{\nu_E(x)}, \qquad |\mu_{E_{x,r}}| \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \pi_{\nu_E(x)}.$$

- *The reduced boundary $\partial^* E$ is a $(n-1)$-dimensional rectifiable set, and the measure μ_E satisfies*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E. \tag{2.1}$$

In particular, this allows to rewrite the perimeter of E in the equivalent form $P(E) = \mathcal{H}^{n-1}(\partial^* E)$.

We say that a point $x \in \mathbb{R}^n$ is *of density $d \in [0,1]$ for the set E* if

$$\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = d, \tag{2.2}$$

where $|\cdot| := \mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$; we define the *essential boundary* of E

$$\partial^e E := \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)} \right).$$

By Federer's Theorem, we have that

$$\partial^* E = E^{(1/2)} = \partial^e E, \tag{2.3}$$

up to $\mathcal{H}^{n-1}$-negligible sets.

Given a positive integer N, a *N-cluster* is a family of finite perimeter sets $\{\mathcal{E}(h)\}_{h=1}^N$, called *chambers*, such that

$$|\mathcal{E}(h)| \in (0, \infty), \quad h = 1, \dots, N, \tag{2.4}$$

$$|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0 \quad h, k = 1, \dots, N, \quad h \ne k. \tag{2.5}$$

If we put $\partial^* \mathcal{E} := \bigcup_{h=1}^N \partial^* \mathcal{E}(h)$, the Euclidean perimeter of the cluster is defined as

$$P(\mathcal{E}) := \mathcal{H}^{n-1}(\partial^* \mathcal{E}). \tag{2.6}$$

We can think to the perimeter of a cluster as given by the sum of the perimeter of each chamber, counting only once each interface, meaning the non-empty intersection of two chambers. If we define $\mathcal{E}(0) := \mathbb{R}^n \setminus \bigcup_{h=1}^N \mathcal{E}(h)$ the *exterior chamber* of $\mathcal{E}$, we can equivalently define the perimeter of the cluster as

$$P(\mathcal{E}) := \sum_{0 \le h < k \le N} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(h) \cap \partial^* \mathcal{E}(k)) = \frac{1}{2} \sum_{h=0}^N P(\mathcal{E}(h)), \tag{2.7}$$

where the second equality is a consequence of Federer's Theorem.

We define the g-perimeter of a cluster as in (1.2). In the following, we will use $|\cdot|_{eucl}$ and P_{eucl} to define the Euclidean volume and perimeter, while we will use $|\cdot|_f$ and P_g for the weighted volume and perimeter.

For every cluster $\mathcal{E}$ and every Borel set B, we define the relative g-perimeter of $\mathcal{E}$ in B by

$$P_g(\mathcal{E};B) := \frac{1}{2}\left(\sum_{h=1}^{N}\int_{\partial^*\mathcal{E}(h)\cap B} g(x,\nu_{\mathcal{E}(h)}(x))d\mathcal{H}^{n-1}(x) + \int_{\partial^*(\cup\mathcal{E})\cap B} g(x,\nu_{\cup\mathcal{E}}(x))d\mathcal{H}^{n-1}(x)\right),$$

where we put $\cup\mathcal{E} := \bigcup_{h=1}^{N}\mathcal{E}(h)$.

3 Proof of Theorem A

3.1 Proof of statement (i)

Let us consider a minimizing sequence of clusters $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ for $\mathcal{I}_{(f,g)}(\mathbf{V})$, that is

$$\mathcal{I}_{(f,g)}(\mathbf{V}) = \lim_{j\to\infty} P_g(\mathcal{E}_j).$$

By assumption, we consider $\mathcal{E}$ such that $\mathcal{E}_j \xrightarrow{L^1_{loc}} \mathcal{E}$.

If $|\mathcal{E}|_f = \mathbf{V}$, there is nothing to prove; the cluster is isoperimetric for its own volume, thanks to the lower semicontinuity of perimeter.

Therefore, from now on we assume $|\mathcal{E}|_f < \mathbf{V}$, meaning that there exists $h \in \{1,\dots,N\}$ such that $|\mathcal{E}(h)|_f < V(h)$. Without loss of generality, we may assume that $|\mathcal{E}(h)|_f > 0$ for every $h = 1,\dots,N$; if $\mathcal{E}$ does not verify this, we simply consider it as a cluster with a smaller number of chambers.

We assume by contradiction that there exists a cluster $\mathcal{F}$ such that

$$|\mathcal{F}|_f = |\mathcal{E}|_f, \qquad \frac{P_g(\mathcal{E}) - P_g(\mathcal{F})}{6} =: \eta > 0. \tag{3.1}$$

We can find points $x_1,\dots,x_N$ of density 1 respectively for $\mathcal{F}(1),\dots,\mathcal{F}(N)$ and which are Lebesgue points for f and g^+ so that $f(x_h) > 0$ for every $h = 1,\dots,N$; hence, there exists $\bar{r} > 0$ such that for every $h = 1,\dots,N$:

$$\frac{1}{2}\omega_n f(x_h) r^n \le |B(x_h,r)\cap\mathcal{F}(h)|_f \le |B(x_h,r)|_f \le 2\omega_n f(x_h) r^n, \tag{3.2}$$

$$P_g(\mathbb{R}^n \setminus B(x_h,r)) \le 2n\omega_n g^+(x_h) r^{n-1}, \tag{3.3}$$

where (3.2) holds true for every $0 < r < \bar{r}$, (3.3) holds true for arbitrarily many r smaller than $\bar{r}$.

Since $f \notin L^1$ (otherwise we would have had $|\mathcal{E}|_f = \mathbf{V}$), we can find points $y_1,\dots,y_N$ of density 0 for $\cup\mathcal{F} := \cup_{h=1}^{N}\mathcal{F}(h)$ which are Lebesgue points for f and g^+ and verifying $f(y_h) > 0$ for every $h = 1,\dots,N$, far enough from the origin to assume $1/M \le f, g \le M$ as by assumptions on f and g; hence, we obtain the estimates

$$|B(y_h,\rho)\setminus(\cup\mathcal{F})|_f \ge \frac{f(y_h)}{2}\omega_n\rho^n, \tag{3.4}$$

$$P_g(B(y_h,\rho)) \le Mn\omega_n\rho^{n-1}, \tag{3.5}$$

both inequalities being true for every $\rho \in (0,\bar{\rho})$.

Let us define a constant $\delta > 0$ and $\bar{\rho}$ so small that

$$M^2\delta < \eta, \qquad Mn\omega_n\bar{\rho}^{n-1} < \frac{\eta}{N}, \qquad \frac{f(y_h)}{2}\omega_n\bar{\rho}^n > \delta \text{ for each } h = 1,\dots,N. \tag{3.6}$$

We claim that there exists a N-cluster $\mathcal{F}'$ and $R > 0$ big enough such that $\mathcal{F}' \subseteq B_R$ and

$$P_g(\mathcal{F}') < P_g(\mathcal{E}) - 5\eta, \tag{3.7}$$

$$0 < \delta'_h := |\mathcal{E}(h)|_f - |\mathcal{F}'(h)|_f < \frac{\delta}{2}, \tag{3.8}$$

for every $h = 1,\dots,N$.

Case 1: the cluster $\mathcal{F}$ is bounded.
For every $h = 1, \dots, N$, choose $r_h < \bar{r}$ so small that all balls $B_h := B(x_h, r_h)$ are mutually disjoint and transversally intersect all the chambers of $\mathcal{F}$ (i.e., $\mathcal{H}^{n-1}(\partial^* \mathcal{F} \cap \partial B_h) = 0$). Define the new cluster

$$\mathcal{F}' := \mathcal{F} \setminus \left(\bigcup_{h=1}^{N} B_h \right) = \left(\mathcal{F}(h) \setminus \left(\bigcup_{j=1}^{N} B_j \right) \right)_{h=1,\dots,N},$$

which is obviously bounded.

We easily notice that, for a given open locally finite perimeter set $B \subseteq \mathbb{R}^n$ transversal to each chamber:

$$P_g(\mathcal{F} \setminus B) = P_g(\mathcal{F}; \bar{B}^c) + \sum_{h=1}^{N} \int_{\partial^* B \cap \mathcal{F}(h)^{(1)}} g(x, -\nu_B(x)) d\mathcal{H}^{n-1}(x) \tag{3.9}$$

By the previous relations (3.2) and (3.3), up to possibly decreasing the $r_h' s$, we have that

$$P_g(\mathcal{F}') \leq P_g(\mathcal{F}) + \sum_{h=1}^{N} P_g(\mathbb{R}^n \setminus B_h) < P_g(\mathcal{E}) - 5\eta.$$

and (3.8) holds as well.

Case 2: the cluster $\mathcal{F}$ is unbounded.
Without loss of generality, let us assume that the chambers $\mathcal{F}(1), \dots, \mathcal{F}(L)$ are unbounded, for a certain $1 \leq L \leq N$.

We choose $R_0 > 0$ big enough so that $\mathcal{F}(h) \subset B_{R_0}$, for all $h = L+1, \dots, N$, $1/M \leq f, g \leq M$ with $M > 0$, and

$$|\mathcal{F}(h) \setminus B_{R_0}|_f < \frac{\delta}{2} \tag{3.10}$$

for all $h = 1, \dots, L$.

We define the new cluster

$$\mathcal{F}' := \mathcal{F} \cap B_R = (\mathcal{F}(h) \cap B_R)_{h=1,\dots,N}$$

and we notice that for every open locally finite perimeter set B transversal to each chamber:

$$P_g(\mathcal{F} \cap B) = P_g(\mathcal{F}; B) + \sum_{h=1}^{N} \int_{\partial B \cap \mathcal{F}(h)^{(1)}} g(x, \nu_B(x)) d\mathcal{H}^{n-1}(x). \tag{3.11}$$

We need to find a $R > R_0$ such that

$$P_g(\mathcal{F}') < P_g(\mathcal{E}) - \left(5 + \frac{1}{2}\right) \eta \tag{3.12}$$

By contradiction, let us assume that for every $R > R_0$ the inequality (3.12) does not hold. By (3.1) and (3.11), we obtain

$$\sum_{h=1}^{L} \int_{\partial B \cap \mathcal{E}(h)^{(1)}} g(x, \nu_B(x)) d\mathcal{H}^{n-1}(x) \geq \frac{\eta}{2}.$$

and so

$$\begin{aligned} +\infty > \sum_{h=1}^{L} |\mathcal{F}(h) \setminus B_R|_f &\geq \int_{R_0}^{+\infty} \sum_{h=1}^{L} \int_{\partial B_R \cap \mathcal{F}(h)^{(1)}} f(x) d\mathcal{H}^{n-1}(x) dR \\ &\geq \int_{R_0}^{+\infty} \frac{1}{M^2} \sum_{h=1}^{L} \int_{\partial B \cap \mathcal{E}(h)^{(1)}} g(x, \nu_B(x)) d\mathcal{H}^{n-1}(x) \geq \int_{R_0}^{+\infty} \frac{\eta}{2M^2} \, dR = +\infty, \end{aligned}$$

contradiction. Thus, there exists $R > R_0$ for which (3.12) holds.

Now, we want to reduce the volume of the bounded chambers and obtain the complete estimate (3.7). We apply the same strategy of *case 1* to $\mathcal{F}'$; we call $\mathcal{F}''$ the new cluster and we require that

$$P_g(\mathcal{F}'') < P_g(\mathcal{F}') + \frac{1}{2}\eta,$$
$$0 < \delta_h' := |\mathcal{E}(h)|_f - |\mathcal{F}''(h)|_f < \frac{\delta}{2}, \quad h = L+1, \dots, N.$$

Putting together the estimates and renaming $\mathcal{F}''$ in $\mathcal{F}'$, we obtain (3.7) and (3.8) also in this case.

The leading idea in the proof is to construct a new sequence of competitors $\{\mathcal{E}'_j\}_{j\in\mathbb{N}}$ with the right volumes $\mathbf{V}$, but converging to the "wrong" perimeter, that is a little smaller than the minimum.

For every $R' > R$ large enough (say, more than $\max|y_h| + \bar{\rho}$), we have that

$$|\mathcal{E}(h) \setminus B_{R'}|_f < \frac{\delta'_h}{2}, \quad h = 1, ..., N, \tag{3.13}$$

$$P_g(\mathcal{E}; B_{R'}) > P_g(\mathcal{E}) - \eta. \tag{3.14}$$

By the L^1_{loc} convergence of $\mathcal{E}_j$ to $\mathcal{E}$, for j big enough and by lower semicontinuity of P_g:

$$|\mathcal{E}(h)|_f - \frac{\delta'_h}{N} < |\mathcal{E}_j(h) \cap B_{R'}|_f \leq |\mathcal{E}_j(h) \cap B_{R'+1}|_f < |\mathcal{E}(h)|_f + \frac{\delta'_h}{N} \tag{3.15}$$

$$P_g(\mathcal{E}; B_{R'}) \leq P_g(\mathcal{E}_j; B_{R'}) + \eta. \tag{3.16}$$

Combining (3.15), (3.13) and (3.6), we notice that

$$\begin{aligned}\int_{R'}^{R'+1} \sum_{h=1}^{N} \int_{\partial B \cap \mathcal{E}(h)^{(1)}} g(x, -\nu_B(x)) d\mathcal{H}^{n-1}(x) &\leq \int_{R'}^{R'+1} M^2 \sum_{h=1}^{N} \mathcal{H}^{n-1}_f (\mathcal{E}_j(h) \cap \partial B_R)\, dR \\ = M^2 \sum_{h=1}^{N} |\mathcal{E}_j(h) \cap (B_{R'+1} \setminus B_{R'})|_f &< 2M^2 \max_{h=1,\dots,N} \delta'_h < M^2\delta < \eta,\end{aligned}$$

and so for each j big enough there exists $R_j \in (R', R'+1)$ such that

$$\sum_{h=1}^{N} \int_{\partial B \cap \mathcal{E}(h)^{(1)}} g(x, -\nu_{B_{R_j}}(x)) d\mathcal{H}^{n-1}(x) < \eta; \tag{3.17}$$

moreover, for each chamber we have the estimate on the volume

$$V(h) - |\mathcal{E}(h)|_f - \delta'_h < \left|\mathcal{E}_j(h) \setminus B_{R_j}\right|_f < V(h) - |\mathcal{E}(h)|_f + \delta'_h. \tag{3.18}$$

We define the new sequence of clusters $\mathcal{G}_j := \big(\mathcal{F}'(h) \cup \big(\mathcal{E}_j(h) \setminus B_{R_j}\big)\big)_{h=1,\dots,N}$. By (3.18) and since $|\mathcal{E}_j(h)|_f = V(h)$ for every $h = 1, \dots, N$, we notice that

$$|\mathcal{G}_j(h)|_f \in \big(V(h) - \delta, V(h)\big), \tag{3.19}$$

and by (3.7), (3.17), (3.14), (3.16), we have the estimate on the perimeter

$$\begin{aligned} P_g(\mathcal{G}_j) &\leq P_g(\mathcal{F}') + P_g(\mathcal{E}_j \setminus B_{R_j}) \\ &< P_g(\mathcal{E}) - 5\eta + P_g(\mathcal{E}_j; \bar{B}^c_{R_j}) + \sum_{h=1}^{N} \int_{\partial B_{R_j} \cap \mathcal{E}(h)^{(1)}} g(x, -\nu_{B_{R_j}}(x)) d\mathcal{H}^{n-1}(x) \\ &< P_g(\mathcal{E}; B_{R'}) - 3\eta + P_g(\mathcal{E}_j, \bar{B}^c_{R_j}) \leq P_g(\mathcal{E}_j; B_{R'}) - 2\eta + P_g(\mathcal{E}_j, \bar{B}^c_{R_j}) \\ &\leq P_g(\mathcal{E}_j) - 2\eta. \end{aligned}$$

Finally, we define the new sequence

$$\widetilde{\mathcal{E}}_j := (\mathcal{G}_j(h) \cup (B(y_h, \rho_h) \setminus \cup\mathcal{G}))_{h=1,\dots,N},$$

where we choose each $\rho_h \in (0, \bar{\rho})$ so that $\left|\widetilde{\mathcal{E}}_j(h)\right|_f = V(h)$ for each $h = 1, \dots, N$ (we can have this, because of (3.4), (3.19) and the condition $R' > \max|y_h| + \bar{\rho}$ implies that $B(y_h, \rho) \setminus \cup\mathcal{G} = B(y_h, \rho) \setminus \cup\mathcal{F}$ for each $h = 1, \dots, N$). Each $\widetilde{\mathcal{E}}_j$ is a N-cluster of volume $\mathbf{V}$; putting together the preceding estimates and (3.5), we have

$$P_g(\widetilde{\mathcal{E}}_j) < P_g(\mathcal{E}_j) - \eta,$$

that is, we have built a sequence of competitors for the problem having perimeters strictly smaller than the infimum if j is large enough. the contradiction concludes the proof of statement (i). □

3.2 Proof of statement (ii)

From now on, we assume that the densities f and g are converging to finite positive limits a and b at infinity. Our goal is to prove that

$$\mathcal{I}_{(f,g)}(\mathbf{V}) = P_g(\mathcal{E}) + ba^{-\frac{n-1}{n}}\mathcal{I}_{eucl}\left(\mathbf{V} - |\mathcal{E}|_f\right), \tag{3.20}$$

being $\mathcal{E}$ limit of a minimizing sequence $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ with $|\mathcal{E}(h)|_f = V(h)$ for each $h = 1,\dots,N$.

As already seen in statement (i), we can choose N points $y_1,\dots,y_N$ far away from the origin which are Lebesgue points for f and 0-density points of $\cup\mathcal{E}$. It follows that there exist $\bar\rho > 0$ such that for each $\rho\in(0,\bar\rho)$ we have

$$\begin{gathered}\omega_n\frac{f(y_h)}{2}\rho^n \le |B(y_h,\rho)\setminus\mathcal{E}|_f \le |B(y_h,\rho)|_f \le \omega_n\rho^n 2a,\\ P_g\left(B(y_h,\rho)\right) \le n\omega_n\rho^{n-1}2b\end{gathered} \tag{3.21}$$

We choose $\varepsilon > 0$ so small that for each $h = 1,\dots,N$

$$\omega_n\frac{f(y_h)}{2}(\bar\rho)^n > \left(\frac{2V(h)}{a}+1\right)\varepsilon; \tag{3.22}$$

we define $\mathcal{F} := \mathcal{E}\cap B_R$, with $R\gg 1$, $R > \max_{h=1,\dots,N} y_h + \bar\rho$ such that

$$\begin{gathered}|(\mathcal{E}\cap B_R)(h)|_f \ge |\mathcal{E}(h)|_f - \varepsilon,\\ P_g(\mathcal{E}\cap B_R) \le P_g(\mathcal{E}) + \varepsilon.\end{gathered}$$

We choose an Euclidean minimal N-cluster $\mathcal{B}$, with *Euclidean* volumes $\left(\frac{V(h)-|\mathcal{E}(h)|_f}{a+\varepsilon}\right)_{h=1,\dots,N}$, so far from the origin that it does not intersect B_R and $a-\varepsilon < f < a+\varepsilon$, $b-\varepsilon < g < b+\varepsilon$. Clearly, we notice that

$$(a-\varepsilon)\,|\mathcal{B}(h)|_{eucl} \le |\mathcal{B}(h)|_f \le V(h) - |\mathcal{E}(h)|_f$$

for each $h = 1,\dots,N$. We define the cluster $\mathcal{G} := (\mathcal{F}(h)\cup\mathcal{B}(h))_{h=1,\dots,N}$ and we notice that

$$\begin{aligned}P_g(\mathcal{G}) &\le P_g(\mathcal{E}) + \varepsilon + P_g(\mathcal{B})\\ &\le P_g(\mathcal{E}) + \varepsilon + (b+\varepsilon)\mathcal{I}_{eucl}\left(\frac{\mathbf{V}-|\mathcal{E}|_f}{a+\varepsilon}\right)\\ &= P_g(\mathcal{E}) + \varepsilon + (b+\varepsilon)(a+\varepsilon)^{\frac{1}{n}-1}\mathcal{I}_{eucl}(\mathbf{V}-|\mathcal{E}|_f).\end{aligned}$$

This is not yet a competitor for the minimization problem with volumes $\mathbf{V}$; indeed, we have that

$$V(h) - \left(\frac{2V(h)}{a}+1\right)\varepsilon \le |\mathcal{E}(h)|_f - \varepsilon + \frac{a-\varepsilon}{a+\varepsilon}\left(V(h) - |\mathcal{E}(h)|_f\right) \le |\mathcal{G}(h)|_f \le V(h). \tag{3.23}$$

We define the new cluster $\mathcal{E}'$ chamber by chamber by

$$\mathcal{E}'(h) := \mathcal{G}(h)\cup B(y_h,\rho_h),$$

with $0 < \rho_h < \bar\rho$ so that $|\mathcal{E}'(h)|_f = V(h)$ for each $h = 1,\dots,N$, taking (3.21), (3.22) and (3.23) into account. Finally, we have that

$$\begin{aligned}\mathcal{I}_{(f,g)}(\mathbf{V}) \le P_g(\mathcal{E}') &= P_g(\mathcal{G}) + \sum_{h=1}^N P_g(B(y_h,\rho_h))\\ &\le P_g(\mathcal{E}) + \varepsilon + (b+\varepsilon)(a+\varepsilon)^{\frac{1}{n}-1}\mathcal{I}_{eucl}(\mathbf{V}-|\mathcal{E}|_f) + Nn\omega_n(\bar\rho)^{n-1}2b\end{aligned}$$

and since $\bar\rho \ll 1$ and $\varepsilon \ll \bar\rho$, by sending $\varepsilon,\rho\to 0$ we have one side of (3.20).

To get the reverse inequality, we need to act on a minimizing sequence converging to $\mathcal{E}$.

By the continuity of the Euclidean isoperimetric function, for a fixed $\varepsilon' > 0$ there exists $\delta > 0$ such that

$$\left|\mathcal{I}_{eucl}(\mathbf{V}') - \mathcal{I}_{eucl}(\mathbf{V} - |\mathcal{E}|_f)\right| \le \varepsilon', \tag{3.24}$$

if $\left|\mathbf{V}' - (\mathbf{V} - |\mathcal{E}|_f)\right| \le \delta$. Choose ε such that $0 < \varepsilon\left(N + \frac{|V-|\mathcal{E}|_f|+\varepsilon}{a-\varepsilon}\right) < \delta$.

By means of formulae (3.9) and (3.11), we find R big enough so that $a-\varepsilon<f<a+\varepsilon$, $b-\varepsilon<g<b+\varepsilon$ out of B_R, and for every $h=1,\dots,N$

$$\begin{aligned}|\mathcal{E}(h)|_f-\varepsilon<|\mathcal{E}(h)\cap B_R|_f\,,\\ P_g(\mathcal{E}\setminus B_R)\leq\varepsilon.\end{aligned}\tag{3.25}$$

We claim that, for every j big enough there exists $R_j\in(R,R+1)$ so that

$$|\mathcal{E}(h)|_f-\varepsilon\leq\left|\mathcal{E}_j(h)\cap B_{R_j}\right|_f\leq|\mathcal{E}(h)|_f+\varepsilon,\qquad\text{for every } h=1,\dots,N\tag{3.26}$$

$$\sum_{h=1}^N\int_{\partial B_{R_j}\cap\mathcal{E}(h)^{(1)}}g(x,\nu_{B_{R_j}}(x))+g(x,-\nu_{B_{R_j}}(x))d\mathcal{H}^{n-1}(x)\leq 2\varepsilon,\tag{3.27}$$

$$P_g\left(\mathcal{E}\right)\leq P_g\left(\mathcal{E}_j\cap B_{R_j}\right)+2\varepsilon.\tag{3.28}$$

Indeed, estimates (3.26) and (3.27) are perfectly analogous to what already seen in statement (i); by (3.11), (3.9), (3.25) and the lower semicontinuity of P_f, for $j\gg1$ we get

$$\begin{aligned}P_g(\mathcal{E})&=P_g(\mathcal{E};B_R)+P_g(\mathcal{E};\bar{B_R}^c)\leq P_g(\mathcal{E};B_R)+P_g(\mathcal{E}\setminus B_R)\\&<P_g(\mathcal{E};B_R)+\varepsilon<P_g(\mathcal{E}_j;B_R)+2\varepsilon\leq P_g(\mathcal{E}_j;B_{R_j})+2\varepsilon.\end{aligned}$$

By (3.26), we notice that

$$\begin{aligned}\left|\mathbf{V}-|\mathcal{E}|_f-a\left|\mathcal{E}_j\setminus B_{R_j}\right|_{eucl}\right|&\leq\varepsilon\left(N+\left|\left|\mathcal{E}_j\setminus B_{R_j}\right|_{eucl}\right|\right)\leq\varepsilon\left(N+\frac{\left|\left|\mathcal{E}_j\setminus B_{R_j}\right|_f\right|}{a-\varepsilon}\right)\\&\leq\varepsilon\left(N+\frac{\left|\mathbf{V}-|\mathcal{E}|_f\right|+\varepsilon}{a-\varepsilon}\right)<\delta,\end{aligned}$$

by our choice of ε. Thanks to these estimates and by (3.24), we obtain

$$\begin{aligned}P_g\left(\mathcal{E}_j\setminus B_{R_j}\right)&\geq(b-\varepsilon)P_{eucl}\left(\mathcal{E}_j\setminus B_{R_j}\right)\geq\frac{b-\varepsilon}{a^{\frac{n-1}{n}}}\mathcal{I}_{eucl}\left(a\left|\mathcal{E}_j\setminus B_{R_j}\right|_{eucl}\right)\\&\geq\frac{b-\varepsilon}{a^{\frac{n-1}{n}}}\left[\mathcal{I}_{eucl}\left(\mathbf{V}-|\mathcal{E}|_f\right)-\varepsilon'\right].\end{aligned}$$

Finally, by (3.27) and (3.28) we can conclude

$$\begin{aligned}P_g(\mathcal{E}_j)&=P_g(\mathcal{E}_j\cap B_{R_j})+P_g(\mathcal{E}_j\setminus B_{R_j})-\sum_{h=1}^N\int_{\partial B_{R_j}\cap\mathcal{E}_j(h)}g(x,\nu_{B_{R_j}}(x))+g(x,-\nu_{B_{R_j}}(x))d\mathcal{H}^{n-1}(x)\\&\geq P_g(\mathcal{E})+\frac{b-\varepsilon}{a^{\frac{n-1}{n}}}\left[\mathcal{I}_{eucl}\left(\mathbf{V}-|\mathcal{E}|_f\right)-\varepsilon'\right]-6\varepsilon.\end{aligned}$$

Sending first $j\to\infty$ and then $\varepsilon'\to0$ (hence $\varepsilon\to0$ as well), we have that

$$\mathcal{I}_{(f,g)}(\mathbf{V})\geq P_g(\mathcal{E})+ba^{-\frac{n-1}{n}}\mathcal{I}_{eucl}\left(\mathbf{V}-|\mathcal{E}|_f\right),$$

thus concluding formula (3.20). □

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