On Serrin's Lower Semicontinuity Problem

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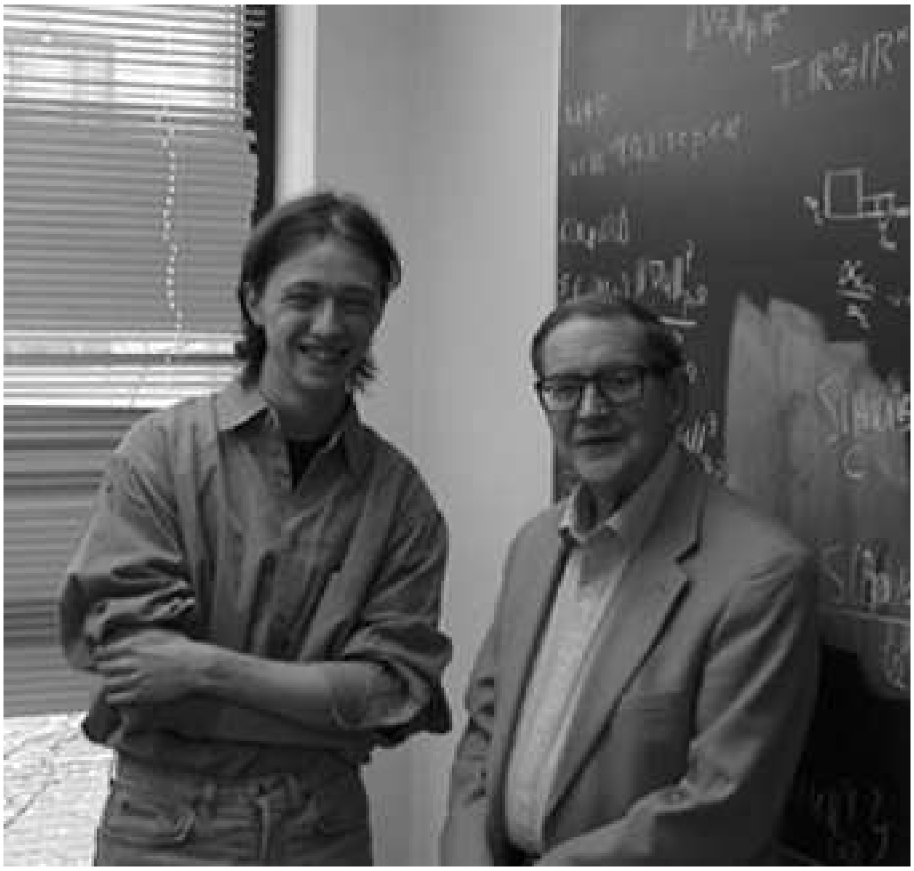
Contents



Me and James Serrin
in the office of Giovanni Leoni,
April 2003, Pittsburgh.

Chapter 1

Overview

Abstract: This thesis is intended to be an account of the research work I have carried out during the three years of my PhD programme. On one hand I have focused my attention ([46], [44], [55], [37]) on some natural questions arising from a classical theorem by James Serrin [70] on certain lower semicontinuity properties of integral functionals; on the other hand I have worked also on different problems ([45], [57], [15], [58]), all coming from the area of the Calculus of Variations, but not strictly related to the study of Serrin's theorem. For this reason it seemed to me that reporting here the details of all these studies would have generated a much too dispersive work.

Thus, in this first chapter, we have presented all the studied problems and the achieved results. A special effort has been made, in particular, to introduce in depth Serrin's problem. Then, in the subsequent chapters, the detailed proofs from the papers [46], [44], [55] and [37] are furnished.

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1.1 Serrin's lower semicontinuity theorem

The main problem we are interested in here was proposed, under inspiration of Lebesgue's work, by James Serrin [70] in 1961 in connection with some basic questions in the theory of the direct methods for the Calculus of Variation. We postpone for the moment the description of these motivations and state immediately the problem.

Let Ω be an open subset in $\mathbb{R}^n$ and let n be an integer, $n \geq 1$. Consider a function f, briefly referred to as the *integrand*,

$$f = f(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},$$

such that the following conditions hold:

$$f \text{ is non negative and continuous;} \tag{1.1}$$

$$f = f(x, s, \cdot) \text{ is convex on } \mathbb{R}^n \text{ for every } (x, s) \in \Omega \times \mathbb{R}. \tag{1.2}$$

For every $u \in C^1(\Omega)$ and every measurable set $E \subset \Omega$ we can consider the integral

$$F[f](u, E) := \int_E f(x, u(x), \nabla u(x))dx.$$

What we refer to as **Serrin's problem** is the search for conditions on the integrand f sufficient to imply the following lower semicontinuity inequality:

$$F[f](u, \Omega) \leq \liminf_{h\to\infty} F[f](u_h, \Omega) \tag{1.3}$$

whenever

$$u_h, u \in C^1(\Omega), \quad u_h \to u \text{ in } L^1_{\text{loc}}(\Omega). \tag{1.4}$$

Of course, in our formulation of the problem it is implicit the knowledge of the fact that, without imposing further conditions in addition to (1.1) and (1.2), in general, this lower semicontinuity property is not satisfied. This is the content of the following counterexample due to Aronszajn [67], dated 1941 (note that this example works in dimension $n = 2$: some revised versions of it working when $n = 1$ will be discussed in detail later).

Theorem 1 (Aronszajn's counterexample [67]). *There exists a continuous function $\nu : (0, 1)^2 \to S^1$ such that, the functional $F[f]$ generated by the integrand*

$$f(x, \xi) = |\nu(x) \cdot \xi|, \quad (x, \xi) \in (0, 1)^2 \times \mathbb{R}^2,$$

does not satisfy the lower semicontinuity inequality (1.3) *on a certain sequence of functions u_h meeting the requirements in* (1.4).

This fact is particularly interesting since the hypotheses (1.1) and (1.2) are the natural ones under which one proves the lower semicontinuity inequality (1.3) for sequences u_h converging in a stronger sense than (1.4), for example when further to (1.4) we assume that

$$\{\nabla u_h\}_{h\in\mathbb{N}} \text{ is bounded in } L^1_{\mathrm{loc}}, \tag{1.5}$$

(see Ambrosio [3] for the state of the art on this problem). In this respect Serrin's problem is really degenerate, since nothing is assumed on the behaviour of the gradients ∇u_h: we just know they are the gradients of functions strongly converging in L^1. For this reason one could say that it is not the existence of a counterexample to be surprising, but rather the validity of the following theorem!

Theorem 2 (Serrin's lower semicontinuity Theorem [70]). *If f satisfies* (1.1), (1.2) *and* **at least one** *of the following conditions:*

(A) the integrand f is **normal** *in the gradient variable ξ, that is*

$$\lim_{|\xi|\to\infty} f(x,s,\xi) = \infty \quad \forall (x,s) \in \Omega \times \mathbb{R}; \tag{1.6}$$

(B) the integrand f is **strictly convex** *in the gradient variable ξ, that is*

$$f(x,s,\cdot) : \mathbb{R}^n \to [0,\infty) \textit{ is strictly convex on } \mathbb{R}^n, \quad \forall (x,s) \in \Omega \times \mathbb{R}; \tag{1.7}$$

(C) the integrand f is **regular** *in the variables (x,ξ), that is*

$$\nabla_x f,\ \nabla_\xi f,\ \nabla_x(\nabla_\xi f) \textit{ exist and are continuous on } \Omega\times\mathbb{R}\times\mathbb{R}^n; \tag{1.8}$$

then the lower semicontinuity inequality (1.3) *is satisfied by $F[f]$ for every u_h and u as in* (1.4).

In connection with the discussion above note that the following corollary follows immediately by applying part A of Serrin's Theorem to $f_\varepsilon(x,s,\xi) = f(x,s,\xi) + \varepsilon|\xi|$ and then by letting $\varepsilon \to 0$:

Corollary 1. *If f satisfies* (1.1) *and* (1.2) *then the lower semicontinuity inequality* (1.3) *is satisfied by $F[f]$ for every u_h and u as in* (1.4) *such that* (1.5) *holds true.*

The natural need in front of this situation is to understand better which is the borderline between the failure and the validity of lower semicontinuity, and this is the aim of our research. The program of inquiry passes through the answer to the following three questions:

(I) What is really hidden inside the *regularity condition* C?

(II) Is there a link between the *geometric conditions* A and B?

(III) How can it be that the geometric conditions look so different from the regularity condition, but that they are all sufficient to the same lower semicontinuity property?

We have obtained the answers to these three questions in a series of papers written in collaboration with Michele Gori, Nicola Fusco and Paolo Marcellini, [46], [44], [55], [37]. The rest of this section is devoted to introduce and motivate these results that we shall prove in the following chapters.

In subsection 1.1.1 we introduce the reader deeper into Serrin's theorem by discussing some classical results due to Serrin [70], Gavioli [38] and De Giorgi, Buttazzo and Dal Maso [24].

In subsection 1.1.2 we explain the results found by Gori and Marcellini in [47] and later extended and completed by Gori, Marcellini and the author in [46] in order to answer to question (I) (the proofs of these theorems are contained in Chapter 2). In that subsection we also try to give an account of the further extensions due to De Cicco and Leoni [22], Fusco, Giannetti and Verde [36] and De Cicco, Fusco and Verde [21].

In subsection 1.1.3 we discuss the answer to question (II) due to Gori and the author [44]. The proofs of the corresponding theorems are contained in Chapter 3.

In subsection 1.1.4 we recall the motivation for studying Serrin's problem arising from the relaxation theory on BV. This allows us to introduce a more general lower semicontinuity problem for which we have obtained [55] some new results, that are proved in Chapter 4.

In the end, in subsection 1.1.5 we address question (III) by describing a recent result obtained jointly with Gori and Fusco [37], that will be proved in Chapter 5.

1.1.1 Some warm up remarks on Serrin's problem

The aim of this section is to give to the reader the feeling of what are the real questions we are facing in studying Serrin's problem.

Lower semicontinuity and approximation

Here we want to point out the fundamental fact that the class of integrands $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ for which the thesis of Serrin's theorem holds true, call it Φ, is closed under countable supremum, that is,

$$\{f_k\}_{k\in\mathbb{N}} \subset \Phi \quad \text{implies} \quad f := \sup_{k\in\mathbb{N}} f_k \in \Phi$$

(one has to be careful in stating this requirement: what we mean is that $f(x, s, \xi)$ is equal to $\sup_{k\in\mathbb{N}} f_k(x, s, \xi)$ for a.e. $x \in \Omega$ and *for every* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. In this way $f(x, u(x), \nabla u(x))$ is the supremum of $f_k(x, u(x), \nabla u(x))$ for a.e. $x \in \Omega$ and for every $u \in W^{1,1}_{\text{loc}}(\Omega)$). Then $f \in \Phi$ follows immediately by the monotone convergence theorem if f_k is an *increasing* sequence. In general one can argue as follows: on one hand, if $\{A_k\}_{k=1}^N$ are **disjoint** open subsets of Ω we have that

$$\begin{aligned}
\sum_{k=1}^{N} \int_{A_k} f_k(x, u, \nabla u)dx \quad &\leq \quad \liminf_{h\to\infty} \sum_{k=1}^{N} \int_{A_k} f_k(x, u_h, \nabla u_h)dx \\
&\leq \quad \liminf_{h\to\infty} \int_{\Omega} f(x, u_h, \nabla u_h)dx \ ;
\end{aligned}$$

on the other hand, for every $N \in \mathbb{N}$, one can find disjoint measurable sets $\{B_k\}_{k=1}^N$ in Ω such that

$$\begin{aligned}
&\int_{\Omega} \sup_{1\leq k\leq N} f_k(x, u, \nabla u)dx = \sum_{k=1}^{N} \int_{B_k} f_k(x, u, \nabla u)dx \\
= \quad &\sup \left\{ \sum_{k=1}^{N} \int_{K_k} f_k(x, u, \nabla u)dx : \{K_k \subset \Omega\}_{k=1}^N \text{ compact and disjoint} \right\} \\
= \quad &\sup \left\{ \sum_{k=1}^{N} \int_{A_k} f_k(x, u, \nabla u)dx : \{A_k \subset \Omega\}_{k=1}^N \text{ open and disjoint} \right\}
\end{aligned}$$

where we have used both the inner and the outer regularity of Lebesgue measure. This proves that the class Φ is closed under the operation of countable supremum.

We have chosen to mention this fact here because it is the basic principle on which the proofs of lower semicontinuity theorems are based on. Indeed the usual way in which the convexity condition in the gradient variable (1.2) comes into play is in that, by the Hahn-Banach theorem, it furnishes a natural decomposition of f as a countable supremum of integrands f_k that are *affine* in the gradient variable:

$$\begin{aligned} &f(x,s,\xi) = \sup_{k\in\mathbb{N}} f_k(x,s,\xi), \quad f_k(x,s,\xi) = a_k(x,s) + b_k(x,s)\cdot\xi, \quad (1.9)\\ &a_k : \Omega\times\mathbb{R}\to\mathbb{R}, \quad b_k : \Omega\times\mathbb{R}\to\mathbb{R}^n. \end{aligned}$$

A crucial problem is then to understand how much of the regularity of f in the variables (x,s) is transferred to the coefficients $a_k(x,s)$ and $b_k(x,s)$. If the desired regularity is measurability or continuity we can apply the basic theorems of existence of selections for multi-valued maps: this is the case, for example, if one considers the lower semicontinuity problem with respect to the convergence (1.4) strengthened by (1.5), see [3]. On the other hand, it is of course an issue behind statement C of Serrin's theorem how to transfer stronger regularity properties, like differentiability in the x variable. We shall come back to this point in subsection 1.1.2. However, note how the geometric conditions A and B do not manifestly contain any special information about selecting smoothly varying supporting hyperplanes to the epigraph of $f(x,s,\cdot)$. This remark seems to suggest that behind these lower semicontinuity results there should be an approximation technique different from the usual one by means of supporting hyperplanes, as it will be confirmed and explained in subsection 1.1.3.

The special case $f = f(\xi)$

The next fact that we would like to put in light is that the rich structure of Serrin's Theorem is present only if we look at integrands depending on the full set of variables $f = f(x,s,\xi)$, i.e., to integrands f depending on *both* the *independent variable* x and the *dependent variable* s (further than, obviously, the *gradient variable* ξ). To convince the reader of this, we describe briefly what happens in the cases $f = f(\xi)$, $f = f(x,\xi)$ and $f = f(s,\xi)$.

We start of course with the case $f = f(\xi)$. We have the following:

Lemma 1 (Serrin's Lemma [70]). *Let $f : \mathbb{R}^n \to [0,\infty]$ be lower semicontinuous and convex and let $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $u_h \to u$ in $\mathcal{D}'(\Omega)$. Then*

$$F[f](u,\Omega) \le \liminf_{h\to\infty} F[f](u_h,\Omega).$$

The proof is really instructive and essential, so that we report it here.

Proof. Step one: For every $u \in BV_{\rm loc}(\Omega)$ let us consider the ε-regularized of u, defined as

$$u_\varepsilon(x) := \int \frac{1}{\varepsilon^n}\,\alpha\left(\frac{x-y}{\varepsilon}\right) u(y)dy = \int \alpha_\varepsilon(x-y)u(y)dy, \quad x \in [\Omega]_\varepsilon,$$

where $[\Omega]_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ and where $\alpha \in \mathcal{D}(B^n; [0,\infty))$, $\int \alpha = 1$. If we denote by Du the distributional derivative of u in Ω then we have

$$\begin{aligned}\nabla u_\varepsilon(x) &= \int \nabla\alpha_\varepsilon(x-y)u(y)dy \\ &= \int \alpha_\varepsilon(x-y)dDu(y) =: [Du]_\varepsilon(x), \quad \forall x \in [\Omega]_\varepsilon.\end{aligned}$$

Let $u_h \in BV_{\rm loc}(\Omega)$ and $u_h \to u$ in $\mathcal{D}'(\Omega)$: then, for every fixed ε and $x \in [\Omega]_\varepsilon$ we have

$$\lim_{h\to\infty} |\nabla[u_h]_\varepsilon - \nabla u_\varepsilon|(x) = \lim_{h\to\infty}\left|\int \nabla\alpha_\varepsilon(x-y)(u_h(y)-u(y))dy\right| = 0.$$

Step two: Since f is convex and lower semicontinuous there are two sequences $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\{b_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that $f(\xi) = \sup_{k\in\mathbb{N}}\{a_k + b_k\cdot\xi\}$. For every fixed $x \in [\Omega]_\varepsilon$ and $u \in W^{1,1}_{\rm loc}(\Omega)$ we have

$$\begin{aligned}f(\nabla u_\varepsilon(x)) &= \sup_{k\in\mathbb{N}}\{a_k + b_k\cdot\nabla u_\varepsilon(x)\} \\ &= \sup_{k\in\mathbb{N}}\int \alpha_\varepsilon(x-y)\{a_k + b_k\cdot\nabla u(y)\}\,dy \\ &\le \int \alpha_\varepsilon(x-y)f(\nabla u(y))dy.\end{aligned}$$

Of course this is just Jensen inequality. The reason why we have decided to write its proof in this particular case will become apparent in subsection 1.1.4.

Step three: Let us fix $\Omega' \subset\subset \Omega$ (this means that the closure of Ω' is a compact set contained in Ω) and consider $u_h \in W^{1,1}_{\rm loc}(\Omega)$. We apply step two to u_h and integrate on Ω'. By Fubini's theorem we find that

$$\begin{aligned}\int_{\Omega'} f(\nabla[u_h]_\varepsilon(x))dx &\leq \int_{\Omega'} dx \int \alpha_\varepsilon(x-y) f(\nabla u_h(y))dy \\ &\leq \int_{\Omega} f(\nabla u_h(y))dy.\end{aligned}$$

If $u_h \to u$ in $\mathcal{D}'(\Omega)$ for $u \in BV_{\text{loc}}(\Omega)$, by Fatou's lemma ($f \geq 0$) and the lower semicontinuity of f we find, according to step one,

$$\liminf_{h\to\infty} F[f](u_h, \Omega) \geq \int_{\Omega'} f(\nabla u_\varepsilon(x))dx. \tag{1.10}$$

If we further have $u \in W^{1,1}_{\text{loc}}(\Omega)$ then $\nabla u_\varepsilon(x) \to \nabla u(x)$ for a.e. $x \in \Omega$, and, by using again the lower semicontinuity and the non negativity of f we deduce that

$$\liminf_{\varepsilon\to 0} \int_{\Omega'} f(\nabla u_\varepsilon(x))dx \geq F[f](u, \Omega').$$

Since Ω' was arbitrary we recover the thesis. □

The special case $f = f(x, \xi)$

Next, let $f = f(x, \xi)$. This is still a very special situation because the crucial feature of the case $f = f(\xi)$ is still alive, namely we have that:

$$F[f](\cdot, \Omega) \text{ is a } \textit{convex} \text{ function on } C^1(\Omega).$$

Since the lower semicontinuous envelope of a convex function F (with respect to a certain topology) is equal to its Fenchel-Moreau biconjugate F^{**} (with respect to a duality pairing suitable for the topology under consideration), the problem can be studied in a nice functional analytic setting. This approach in relation to Serrin's problem has been explored by Gavioli [38] and leads to a *characterization* of the lower semicontinuity of $F[f]$ in terms of a property of its integrand f. It turns out that this condition is rather abstract, and thus the problem of determining reasonably simple conditions on the integrand f that are still sufficient to the lower semicontinuity of $F[f]$ is of interest also in this case. Another lesson that one can maybe learn from Gavioli's result is that, if a characterization of lower semicontinuity is possible in the general case $f = f(x, s, \xi)$, then it has necessarily to be *very* abstract.

We close by saying that the special features of the case $f = f(x,\xi)$ allow for a detailed treatment of lower semicontinuity and relaxation issues along the lines depicted above in a much more general framework than the one considered here. A very complete reference on this approach is the paper by Bouchitté and Dal Maso [8].

The special case $f = f(s,\xi)$

The case $f = f(s,\xi)$ has been treated by De Giorgi, Buttazzo and Dal Maso [24].

Theorem 3 (De Giorgi, Buttazzo and Dal Maso [24]). *Let $f = f(s,\xi) : \mathbb{R}\times\mathbb{R}^n \to [0,\infty)$ be such that $f(\cdot,\xi)$ is measurable on $\mathbb{R}$ for all $\xi \in \mathbb{R}^n$, $f(\cdot,0)$ is lower semicontinuous on $\mathbb{R}$, $f(s,\cdot)$ is convex on $\mathbb{R}^n$ for all $s \in \mathbb{R}$ and the function $\alpha_f : \mathbb{R} \to [0,\infty)$ defined by*

$$\alpha_f(s) := \limsup_{\xi\to 0} \frac{\{f(s,0) - f(s,\xi)\}^+}{|\xi|} \tag{1.11}$$

is in $L^1_{\mathrm{loc}}(\mathbb{R})$. Then the lower semicontinuity inequality (1.3) is satisfied by $F[f]$ for every u_h and u as in (1.4).

It is really worth to sketch the proof of this theorem.

Proof. Step one: Let $\varphi \in \mathcal{D}(\Omega;[0,1])$. If $M \geq 0$ and $a : \mathbb{R} \to [-M,0]$ is lower semicontinuous then

$$\int_\Omega \varphi(x)a(u(x))dx \leq \liminf_{h\to\infty} \int_\Omega \varphi(x)a(u_h(x))dx$$

whenever $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. This is an obvious consequence of Fatou's Lemma.

Step two: Let $\varphi \in \mathcal{D}(\Omega;[0,1])$. If $b \in L^1(\mathbb{R};\mathbb{R}^n)$ then

$$\lim_{h\to\infty} \int_\Omega \varphi(x)b(u_h(x))\cdot\nabla u_h(x)dx = \int_\Omega \varphi(x)b(u(x))\cdot\nabla u(x)dx,$$

whenever $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ with

$$\int_\Omega \{b(u_h(x))\cdot\nabla u_h(x)\}^+ + \{b(u(x))\cdot\nabla u(x)\}^+dx < \infty. \tag{1.12}$$

If $b \in L^\infty(\mathbb{R};\mathbb{R}^n)$ by the chain rule formula for the composition of the Lipschitz map $A(t) = \int_0^t b(s)ds$ with the Sobolev function u_h (see [59]) we have that

$$\begin{aligned}\int_\Omega \varphi(x)\, b(u_h(x)) \cdot \nabla u_h(x) dx &= -\int_\Omega \nabla\varphi(x) \cdot \left\{ \int_0^{u_h(x)} b(s)ds \right\} dx \\ &= \int_\Omega \nabla\varphi(x) \cdot B(u_h(x)) dx. \qquad (1.13)\end{aligned}$$

One proves (1.13) for $b \in L^1(\mathbb{R};\mathbb{R}^n)$ by truncation (i.e., by taking the limit $k \to \infty$ in (1.13) for $1_{\{s\in\mathbb{R}:|b(s)|\le k\}}(s)b(s)$, and here (1.12) is needed). Then the continuity assertion follows by applying the dominated convergence theorem (note that $B(s)$ is bounded and continuous, while φ has compact support).

Step three: Without loss of generality we are in the case

$$\liminf_{h\to\infty} F[f](u_h,\Omega) = \lim_{h\to\infty} F[f](u_h,\Omega) < \infty. \qquad (1.14)$$

Let us consider $g(s,\xi) := f(s,\xi) - f(s,0) - b(s)\cdot\xi$ where $b : \mathbb{R} \to \partial f(s,0)$ is a measurable selection of the subdifferential of $f(s,\cdot)$ at $\xi = 0$. Clearly $g(s,0) \equiv 0$: thus

$$g(s,\xi) = \sup_{k\in\mathbb{N}} \{a_k(s) + b_k(s)\cdot\xi\}^+, \quad \text{with } a_k \le 0, \qquad (1.15)$$

where $a_k : \mathbb{R} \to \mathbb{R}$ and $b_k : \mathbb{R} \to \mathbb{R}^n$ are measurable functions. Since a_k is measurable, there exists an increasing sequence of compact sets K_j exhausting in measure $\mathbb{R}$ and such that a_k is continuous on each K_j. Thus, for every $u \in W^{1,1}_{\rm loc}(\Omega)$,

$$F[g](u,\Omega) = \sup_{k,j\in\mathbb{N}} \int_\Omega \{\chi_{K_j}(u)a_k(u) + \chi_{K_j}(u)b_k(u)\cdot\nabla u\}^+,$$

(note that here one has to invoke the following property of Sobolev functions: $\nabla u = 0$ a.e. on $u^{-1}(E)$ for every $E \subset \mathbb{R}$ of measure zero). Since $\chi_{K_j}(s)a_k(s)$ is lower semicontinuous (by the compactness of K_j and by the continuity of $a_k \le 0$ on K_j), we can directly assume, up to relabeling the indexes, that (1.15) holds with a_k lower semicontinuous and with a_k and b_k having compact support in $\mathbb{R}$, and, as in step two, up to truncating, we can further assume that the b_k's are bounded. Since

$$\begin{aligned}\int_\Omega \{a_k(u) + b_k(u)\cdot\nabla u\}^+ &= \sup\left\{ \int_\Omega \varphi\{a_k(u) + b_k(u)\cdot\nabla u\}dx \right. \\ &\qquad \left. : 0 \le \varphi \le 1, \varphi \in \mathcal{D}(\Omega)\right\},\end{aligned}$$

by step one and step two we easily see that $F[g]$ is lower semicontinuous with respect to the L^1_{loc} convergence. By approximating φ with piecewise constant functions, it follows that the integrand $\varphi(x)g(s,\xi)$ generates a lower semicontinuous functional too, whenever $\varphi \in \mathcal{D}(\Omega;[0,1])$. Thus, by using also the lower semicontinuity of $f(\cdot,0)$, we find that

$$\begin{aligned}\liminf_{h\to\infty}\int_\Omega f(u_h,\nabla u_h) &\geq \int_\Omega \varphi\,\{g(u,\nabla u)+f(u,0)\} \\ &+ \liminf_{h\to\infty}\int_\Omega \varphi\; b(u_h)\cdot\nabla u_h dx.\end{aligned}$$

Let us assume for a moment that $b \in L^1(\mathbb{R};\mathbb{R}^n)$. Then we can apply step two to prove that the last term in the previous inequality is actually equal to $\int_\Omega \varphi\, b(u)\cdot\nabla u dx$ (indeed (1.12) can be deduced by (1.14)), and thus conclude the proof of the theorem.

Now, who is b? It is a measurable selection from the subdifferential of $f(s,\cdot)$ at zero. Since we want this selection to be in L^1 it is natural to try to select for each s the element $b(s) \in \partial f(s,0)$ with *minimal norm*: and this norm, by some basic convex analysis, turns out to be equal to $\alpha_f(s)$, that we have assumed to be in $L^1_{\mathrm{loc}}(\mathbb{R})$. Then lower semicontinuity follows as f is the monotone increasing limit of $f_k = f_k(s,\xi) := \sigma_k(s)f(s,\xi)$, if $\sigma_k \in \mathcal{D}([-k,k];[0,1])$, $\sigma_k \uparrow 1$: indeed, for every $k \in \mathbb{N}$, $\alpha_{f_k} = \sigma_k\alpha_f \in L^1(\mathbb{R})$. $\square$

Thus, in the case $f = f(s,\xi)$ lower semicontinuity holds under very mild (and, as the proof above should have clarified, very natural) assumptions. Let us also recall that if α_f is not in L^1_{loc} then one can construct examples where the lower semicontinuity fails: consider for instance $n=1$, $\Omega=(0,1)$,

$$f(s,\xi) = \begin{cases} \{1+\xi/s\}^+ & \text{if } s \neq 0 \\ 1 & \text{else}\end{cases}$$

and $u_h(x) = h^{-1}(1-x)$. Thus this condition is really sharp and one would be interested to bring it into the framework of Serrin's problem $f = f(x,s,\xi)$ (see the end of the next subsection for more on this).

1.1.2 The answer to the first question

As explained at the beginning of the previous subsection, a relevant issue in dealing with lower semicontinuity problems is to have at hand good approximation theorems for convex functions depending on parameters. The crucial point is being able to transfer the regularity of $f(x, s, \xi)$ with respect to the variables (x, s) to the coefficients $a_k(x, s)$, $b_k(x, s)$ defining the approximating supporting hyperplanes to $f(x, s, \cdot)$, see (1.9). Of course the theory of selections can work if one is interested in transferring properties like measurability, or continuity. A more flexible tool was developed by De Giorgi during a course on lower semicontinuity theorems he lectured in Rome in the Sixties [23]. His result, that will be proved here in section 2.1, gives explicit expressions for the coefficients a_k and b_k used in (1.9). This approximation technique has been applied for the first time in the context of Serrin's problem by Gori and Marcellini [47], see also [42], thus leading to the following sufficient condition for lower semicontinuity, that represents a generalization of condition C:

Theorem 4 (Gori and Marcellini [47]). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ satisfy (1.1) and (1.2). Assume that, for every $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L = L(\Omega', H, K)$ such that*

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in \Omega',\ (s, \xi) \in H \times K. \tag{1.16}$$

Then the lower semicontinuity inequality (1.3) is satisfied by $F[f]$ for every u_h and u as in (1.4).

This theorem clarifies that the fundamental aspect of condition C in Serrin's theorem is in fact the existence of a bounded derivative in the x variable and opens a new question: how far is (1.16) to be sharp? For example is it the theorem still true if one replaces the Lipschitz condition with an analogous condition of Hölderian type? Some answers to these questions have been furnished in [46] through a careful analysis of the methods developed in [47].

The first result is a counterexample to lower semicontinuity under Hölder conditions in the x variable. This counterexample (that will be proved in section 2.3) is based on a previous attempt contained in [47], that in turn was inspired by some hand-written notes due to Dal Maso about Aronszajn's counterexample.

Theorem 5 (Gori, Maggi and Marcellini [46]). *For every $\alpha \in (0, 1)$ there exists a continuous integrand $f : (-1, 1) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ and a sequence $u_h \in W^{1,\infty}(-1, 1)$ uniformly converging to $u \equiv 0$ such that*

$$\int_{-1}^{1} f(x, u, u')dx > \liminf_{h \to \infty} \int_{-1}^{1} f(x, u_h, u_h')dx,$$

even if, for every $K \subset\subset \mathbb{R}^n$, there exists a constant $L = L(K)$ such that

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L|x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in (-1, 1),\ (s, \xi) \in \mathbb{R} \times K.$$

As a consequence, condition (1.16) is sharp, at least in the comparison scale of "qualified continuity assumptions". However we can look at it under a different light, since it is equivalent to asking that: $f(\cdot, s, \xi) \in W^{1,\infty}(\Omega)$ and for every $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L = L(\Omega', H, K)$ such that

$$\|\nabla_x f(\cdot, s, \xi)\|_{L^\infty(\Omega')} \leq L, \quad \forall (s, \xi) \in H \times K. \tag{1.17}$$

Inspired by this remark, we look for an extension of Theorem 4 where the L^∞ norm is replaced by the L^1 norm. Note that this extension poses the question of proving a lower semicontinuity result in Serrin's framework under a mere *measurability* assumption in the x variable! This is the second main result we take from [46], that will be proved in section 2.2:

Theorem 6 (Gori, Maggi and Marcellini [46]). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ be measurable, locally bounded, such that $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$ and (1.2) holds true. If $f(\cdot, s, \xi) \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and for every $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L = L(\Omega', H, K)$ such that*

$$\int_{\Omega'} |\nabla_x f(x, s, \xi)|\, dx \leq L, \quad \forall (s, \xi) \in H \times K,$$

then the lower semicontinuity inequality (1.3) is satisfied by $F[f]$ for every u_h and u as in (1.4).

In fact, as it is clear from the proof of the theorem, one does not need the whole distributional gradient of f in the x variable to be in L^1, but just its distributional divergence.

We conclude this paragraph by mentioning that this technique based on De Giorgi's approximation theorem, and the line of inquiry initiated in [47] and [46] (that is, working under an assumption of weak derivability in the x variable with a local bound on the distributional gradient), have recently interested also other authors. De Cicco and Leoni [22] have studied with care the validity of the chain rule formula for a divergence structure (as the one that we have met in the proof of Theorem 3), and then have been able to prove the lower semicontinuity inequality for integrands that are not continuous in the s variable, of course in presence of the "summability of the minimal norm element of the subdifferential at zero"-condition (1.11). Further research in this direction is due to Fusco, Giannetti and Verde [36] and to De Cicco, Fusco and Verde [21] in order to: (1) extend Theorem 6 by "replacing $W^{1,1}$ with BV"; (2) export this kind of hypotheses to the framework of the lower semicontinuity problem for the "parametric extension" functional $H[f]$ (see section 1.1.4 below).

1.1.3 The answer to the second question

A good way for introducing our next result is by looking at the explicit form of the integrand considered in the counterexample described in Theorem 5. There we have $n = 1$, $a : (-1, 1) \times \mathbb{R} \to \mathbb{R}$ and

$$f(x, s, \xi) = |a(x, s)\xi - 1|.$$

The function $a(x, s)$ and the sequence u_h on which lower semicontinuity fails are linked through (see also section 2.3)

$$a(x, u_h(x)) = \frac{1}{u_h'(x)} \text{ where } u_h'(x) \neq 0.$$

In this way

$$F[f](u_h, (-1, 1)) = |\{x \in (-1, 1) : u_h'(x) = 0\}| =: |\{u_h' = 0\}|.$$

The sequence u_h is uniformly converging to $u \equiv 0$, so that $F[f](u, (-1, 1)) = 1$, and we shall have failure of lower semicontinuity if, for example, $|\{u_h' = 0\}| \to 0$. A wise remark that this counterexample can suggest is that the property

$$f(x_0, s_0, \xi) \equiv 1 > \inf_{\Omega\times\mathbb{R}\times\mathbb{R}^n} f \text{ for many pairs } (x_0, s_0)$$

is crucial to the failure of lower semicontinuity (compare also with the example at the end of subsection 1.1.1).

In order to understand if this is really a good remark we can try to guess an argument sufficient to prove the lower semicontinuity theorem for one dimensional integrands $f = f(x, s, \xi)$ satisfying something like: for every fixed $(x, s) \in \Omega \times \mathbb{R}$

$$\begin{aligned} &f(x, s, \cdot) \text{ is not a constant function, or} \\ &f(x, s, \xi) = \inf_{\Omega\times\mathbb{R}\times\mathbb{R}^n} f \text{ for all } \xi \in \mathbb{R}. \end{aligned} \tag{1.18}$$

Since we are just trying to guess what can happen, we further specialize our working condition to: for every fixed $(x,s) \in \Omega \times \mathbb{R}$,

$$f(x,s,\cdot) \text{ is not a constant function.} \tag{1.19}$$

As usual we try to approximate from below our integrand by simpler objects. Instead of looking at supporting hyperplanes we shall consider the "maximal cones" associated to $f(x,s,\cdot)$ as follows. Let us fix $\xi_0 \in \mathbb{R}$ and define $K[f(x,s,\cdot);\xi_0] : \mathbb{R} \to [-1,\infty)$ as the greatest function everywhere less or equal to $f(x,s,\cdot)$ that can be represented as

$$\begin{aligned} K[f(x,s,\cdot);\xi_0](\xi) &= -1 + \max\{a(x,s)(\xi-\xi_0), b(x,s)(\xi-\xi_0)\}, \\ &\quad \text{where} \quad a(x,s) \le 0 \le b(x,s). \end{aligned}$$

Note that the epigraph of this function is indeed the smallest cone in $\mathbb{R}^n \times \mathbb{R}$ with vertex in $(\xi_0,-1)$ that contains the epigraph of $f(x,s,\cdot)$. The reason why we are interested in such functions is that if (1.19) holds true then

$$a(x,s)^2 + b(x,s)^2 > 0,$$

and in turn this condition ensures that each maximal cone can be approximated from below by an increasing family of strictly convex functions! Then lower semicontinuity for integrands satisfying (1.19) will follow from part B of Serrin's theorem.

The next question is: which is the right formulation of this condition in the general n-dimensional case, $n \ge 2$? To understand this point is useful to consider the original Aronszajn's counterexample, that was in the form $f(x,\xi) = |\nu(x)\cdot\xi|$ for $n=2$ and $\nu : (0,1)^2 \to S^{n-1}$ continuous. Then $f(x,\cdot)$, for all values of x, is not constant: but its one dimensional restriction to $\nu(x)^\perp$ is constant, preventing the possibility of approximating f from below with integrands that are strictly convex in the gradient variable (indeed, no strictly convex function can lie below $f(x,\cdot)$). A way to recast (1.19) in dimension $n \ge 2$ is then the following: for every fixed $(x,s,\nu) \in \Omega\times\mathbb{R}\times S^{n-1}$, we ask that

$$\rho \mapsto f(x,s,\rho\nu) \text{ is not a constant function of } \rho \in \mathbb{R}. \tag{1.20}$$

Then we can prove

Theorem 7 (Gori and Maggi [44]). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ satisfy* (1.1) *and* (1.2)*. Then* (1.20) *holds true if and only if there exists a sequence of continuous integrands $f_k : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [-1, \infty)$ that are strictly convex in the gradient variable and are such that*

$$f = \sup_{k \in \mathbb{N}} f_k .$$

In particular, if f satisfies (1.20)*, then the lower semicontinuity inequality* (1.3) *is satisfied by $F[f]$ for every u_h and u as in* (1.4)*.*

Note that (1.20) is implied by both condition A and B in Serrin's theorem, so that we have found the common root of these two conditions. Furthermore the examples discussed above should have convinced the reader that this condition is optimal in the class of "geometric conditions". Thinking to (1.18) the following theorem should be almost sharp:

Theorem 8 (Gori and Maggi [44], Maggi [55]). *Let f be locally bounded, lower semicontinuous and convex in the gradient variable. Assume that for every $(x, s) \in \Omega \times \mathbb{R}$ the function $f(x, s, \cdot)$ satisfies* (1.20) **or** *is constantly equal to $\inf_{\Omega \times \mathbb{R} \times \mathbb{R}^n} f$. Then the lower semicontinuity inequality* (1.3) *is satisfied by $F[f]$ for every u_h and u as in* (1.4)*.*

The proof of Theorem 7 is presented in Chapter 3, while Theorem 8 will be an immediate corollary of the results proved in Chapter 4.

As we have seen, condition (1.20) arises naturally from a careful study of Serrin's problem, and in a first moment it seemed to us something really peculiar of it. It has been surprising then to discover that this condition has already appeared in (at least) two other instances.

Anzellotti, Buttazzo and Dal Maso [5] call *demicoercive* a function $f : \mathbb{R}^n \to \mathbb{R}$ if there exist $a > 0$, $b \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}^n$ such that

$$f(\xi) + \xi_0 \cdot \xi \geq a|\xi| + b \quad \forall \xi \in \mathbb{R}^n .$$

In the class of convex functions, demicoercivity is equivalent to (1.20). Of course, examples of demicoercive functions are given by normal functions or by strictly convex functions, but there is much more room: for example, if $\nu_i : \Omega \times \mathbb{R} \to S^{n-1}$, $i = 1, ..., n$, are arbitrary lower semicontinuous functions with $\{\nu_i(x, s)\}_{i=1}^n$ orthonormal in $\mathbb{R}^n$ for every fixed (x, s), then

$$f(x, s, \xi) := \sum_{i=1}^n \{\nu_i(x, s) \cdot \xi\}^+$$

is lower semicontinuous, locally bounded and convex and demicoercive in the gradient variable. The reason why in [5] these functions are introduced is that even if they are not coercive, the integral functionals $H[f]$ they generate (compare with subsection 1.1.4) are coercive on functions u in $BV_{\mathrm{loc}}(\Omega)$ sharing the same boundary data (see Lemma 4 in Chapter 4). This fact allows to prove existence of minimizers in BV under a prescribed Dirichlet condition for $H[f]$ when f is demicoercive.

The other instance in which we have seen this condition appearing is in a paper by Benoist and Hiriart-Urruty [7] on the regularity of convex envelopes, where it was introduced through a third equivalent formulation and called *epi-pointedness* (see also [51]).

1.1.4 Motivation to Serrin's problem, relaxation in BV

Serrin's original motivation was to understand which is the right definition of the usual integral functional of the Calculus of Variations when we evaluate it on particularly non smooth functions. For example let us consider the functional that represents the area of the graph of a function $u : \Omega \to \mathbb{R}$,

$$F(u, \Omega) = \int_{\Omega} \sqrt{1 + |\nabla u(x)|^2} dx.$$

If $u \in C^1(\Omega)$ this functional is exactly what we want it to be; but how to extend it to wilder u's? For example, we can construct continuous functions u that are almost everywhere differentiable with gradient equal to zero, and that anyway are not constant. If we plug the almost everywhere gradient of such an u inside the integral above the result is $|\Omega|$ and of course this is not a reasonable value for the area of the graph of u since u is not constant.

Having in mind such examples, i.e. following the approach initiated by Lebesgue for the area functional, in [70] Serrin proposed to consider the following extension of $F[f]$ from $C^1(\Omega)$ to $BV_{\mathrm{loc}}(\Omega)$: given a Borel measurable $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ we define for every $u \in BV_{\mathrm{loc}}(\Omega)$

$$\mathcal{F}[f](u, \Omega) \ := \ \inf \left\{ \liminf_{h\to\infty} F[f](u_h, \Omega) : u_h \in C^1(\Omega),\ u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \right\},$$

We call $\mathcal{F}[f]$ the *relaxed functional* of $F[f]$. The fundamental problem is then to understand if $\mathcal{F}[f]$ is still a functional of integral form and, if it is so, to find the most explicit possible formula to represent it.

What does Serrin's theorem say in this respect? First of all let us say that we have stated it in a weaker form than the one actually proved by Serrin: in fact, in all the lower semicontinuity theorems proved so far, the inequality (1.3) holds true on every u_h and u such that

$$u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega), \;\; u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega). \tag{1.21}$$

Then the interest of Serrin's theorem and its extensions is that they furnish conditions on the integrand f sufficient to ensure that

$$F[f](u, \Omega) \leq \mathcal{F}[f](u, \Omega), \quad \forall u \in W^{1,1}_{\mathrm{loc}}(\Omega). \tag{1.22}$$

Also these partial results are of interest in the relaxation problem since in general there is no hope to find a complete and explicit description of the relaxed functional.

In the case $f = f(\xi)$ the relaxed functional can be explicitly characterized as explained in the following theorem.

Theorem 9 (Goffman and Serrin [41]). *If $f = f(\xi) : \mathbb{R}^n \to [0, \infty]$ is convex and lower semicontinuous then*

$$\mathcal{F}[f](u, \Omega) = H[f](u, \Omega), \quad \forall u \in BV_{\mathrm{loc}}(\Omega),$$

where

$$H[f](u, \Omega) := \int_\Omega f(\nabla u(x))dx + \int_\Omega f^\infty \left(\frac{D^s u}{|D^s u|}(x) \right) d|D^s u|(x),$$

and

$$f^\infty(\xi) := \lim_{t \to \infty} \frac{f(t\xi)}{t}, \quad \xi \in \mathbb{R}^n,$$

is called the recession function of f.

The best way to explain this theorem is to prove it. We can do this through a careful study of the proof of Lemma 1.

Proof. It is easy to adapt the argument in the proof of (1.10) in Serrin's Lemma to show that

$$\liminf_{h \to \infty} F[f](u_h, \Omega) \geq \int_{[\Omega]_\varepsilon} f(\nabla u_\varepsilon(x))dx,$$

whenever $u_h \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ with $u \in BV_{\mathrm{loc}}(\Omega)$, and that in fact

$$\mathcal{F}[f](u,\Omega) = \lim_{\varepsilon\to 0}\int_{[\Omega]_\varepsilon} f(\nabla u_\varepsilon(x))dx.$$

Thus we have reduced to prove that

$$\lim_{\varepsilon\to 0}\int_{[\Omega]_\varepsilon} f(\nabla u_\varepsilon(x))dx = H[f](u,\Omega),$$

for every $u \in BV_{\mathrm{loc}}(\Omega)$. To this end we remark that when in the proof of Serrin's Lemma we applied Jensen inequality to show that

$$f(\nabla u_\varepsilon(x)) \le \int \alpha_\varepsilon(x-y)f(\nabla u(y))dy, \quad x \in [\Omega]_\varepsilon$$

the assumption $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ was essential. To understand what goes on if $u \in BV_{\mathrm{loc}}(\Omega)$ we can argue as follows: clearly

$$\text{if } f(\xi) = \sup_{k\in\mathbb{N}}\{a_k + b_k\cdot\xi\}, \quad \text{then} \quad f^\infty(\xi) = \sup_{k\in\mathbb{N}}\{b_k\cdot\xi\};$$

then, in place of Jensen inequality, we find that

$$\begin{aligned}
& a_k + b_k\cdot\nabla u_\varepsilon(x) = a_k + b_k\cdot\int \alpha_\varepsilon(x-y)dDu(y) \\
= & \int \alpha_\varepsilon(x-y)\{a_k + b_k\cdot\nabla u(y)\}dy + \int \alpha_\varepsilon(x-y)b_k\cdot\frac{D^s u}{|D^s u|}(y)d|D^s u|(y) \\
\le & \int \alpha_\varepsilon(x-y)f(\nabla u(y))dy + \int \alpha_\varepsilon(x-y)f^\infty\left(\frac{D^s u}{|D^s u|}(y)\right)d|D^s u|(y).
\end{aligned}$$

By taking the supremum as $k \in \mathbb{N}$ on the left hand side, by performing an integration over $[\Omega]_\varepsilon$ and by applying Fubini's theorem we deduce that, if $u \in BV_{\mathrm{loc}}(\Omega)$,

$$\int_{[\Omega]_\varepsilon} f(\nabla u_\varepsilon(x))dx \le H[f](u,\Omega).$$

On the other hand, let us fix $\varphi \in \mathcal{D}(\Omega;[0,1])$ and recall that

$$\nabla u_\varepsilon d(\mathcal{L}^n\lfloor[\Omega]_\varepsilon) \overset{*}{\rightharpoonup} Du, \quad \text{as } \varepsilon \to 0$$

in the sense of Radon measure, with $|\nabla u_\varepsilon|([\Omega]_\varepsilon) \to |Du|(\Omega)$. Then

$$\begin{aligned} H[f](u,\Omega) &\geq \liminf_{\varepsilon\to 0} \int_{[\Omega]_\varepsilon} \varphi(x) f(\nabla u_\varepsilon(x))dx \\ &\geq \liminf_{\varepsilon\to 0} \int_{[\Omega]_\varepsilon} \varphi(x)\{a_k + b_k \cdot \nabla u_\varepsilon(x)\}dx \\ &= \int_\Omega \varphi(x)\{a_k + b_k \cdot \nabla u(x)\}dx + \int_\Omega \varphi(x) b_k \cdot \frac{D^s u}{|D^s u|}(x) dD^s u(x). \end{aligned}$$

To achieve the proof it suffices to take the supremum over φ and k. $\square$

This theorem says that in extending $F[f]$ to BV we have to measure the absolutely continuous part ∇u of the distributional derivative Du of u with $F[f]$ and to take into account the "vertical parts" of Du, encoded in $D^s u$, by means of the recession function of the integrand f. This is intuitively very natural since f^∞ describes the linear behaviour of f at infinity on the various possible directions.

The adaptation of this theorem to the case $f = f(x,s,\xi)$ is somewhat subtle. The analogous representation result to the one by Goffman and Serrin holds true under very natural, but in a sense also very restrictive, hypotheses. Let us justify better this claim by introducing an important result due to Dal Maso [18].

The central idea is that the functional $H[f]$ that we have proved being equal to $\mathcal{F}[f]$ is nothing but the "parametric extension" of $F[f]$. The consideration of the parametric extension when studying a non parametric integral functional defined on curves with finite total variation has been classically known for a long time, see for example the 1934 paper by McShane [54].

One can bring the idea of the parametric extension in the context of multi dimensional integrals $f = f(x,s,\xi)$ in the following way. Given $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,\infty]$ with $f(x,s,\cdot)$ convex, we define

$$f^\#(x,s,\xi,t) := \begin{cases} tf(x,s,\xi/t) & t > 0, \\ f^\infty(x,s,\xi) & t = 0. \end{cases}$$

Then $f^\#(x,s,\cdot,\cdot)$ is a positively homogeneous of degree one, convex function on the cone $\mathbb{R}^n \times [0,\infty) =: \mathcal{K}$. We can use it to define an integral functional on the class of all $\mathcal{K}$-vector valued Radon measures on $\Omega \times \mathbb{R}$, by

$$\Phi[f^\#](\mu) := \int_{\Omega\times\mathbb{R}} f^\# \left(x, s, \frac{d\mu}{d|\mu|}(x,s) \right) d|\mu|(x,s).$$

If $u \in C^1(\Omega)$ and $\Sigma(u) := \{(x,s) : x \in \Omega, s \leq u(x)\}$ is the subgraph of u, then the outer normal derivative to $\Sigma(u)$ defines a $\mathcal{K}$-valued Radon measure on $\Omega \times \mathbb{R}$ that can be represented as

$$\mu[u] := \frac{(\nabla u(x), 1)}{\sqrt{1+|\nabla u(x)|^2}}\, d\left(\mathcal{H}^n \lfloor \partial\Sigma(u)\right)(x,s).$$

Then

$$\begin{aligned}
\Phi[f^\#](\mu[u]) &= \int_{\Omega\times\mathbb{R}} f^\# \left(x, s, \frac{(\nabla u(x), 1)}{\sqrt{1+|\nabla u(x)|^2}}\right) d\left(\mathcal{H}^n \lfloor \partial\Sigma(u)\right)(x,s) \\
&= \int_\Omega f^\# \left(x, u(x), \frac{(\nabla u(x), 1)}{\sqrt{1+|\nabla u(x)|^2}}\right) \sqrt{1+|\nabla u(x)|^2} dx \\
&= F[f](u,\Omega).
\end{aligned}$$

If now $u \in BV_{\mathrm{loc}}(\Omega)$, the natural way to define the subgraph of u is to consider the *approximate upper limit of* u *at* x,

$$u^+(x) = \inf\{t \in \mathbb{R} : \{y \in \Omega : u(y) > t\} \text{ has zero density at } x\},$$

and then define $\Sigma[u] := \{(x,s) : x \in \Omega,\ s \leq u^+(x)\}$. Then $\Sigma[u]$ is a set of locally finite perimeter in $\Omega \times \mathbb{R}$, that is to say, its characteristic function $1_{\Sigma[u]}$ belongs to $BV_{\mathrm{loc}}(\Omega \times \mathbb{R})$ (see [61]). Thus for $u \in BV_{\mathrm{loc}}(\Omega)$ is natural to define

$$\mu[u] := D1_{\Sigma[u]}.$$

With this definition one can prove (see Lemma 2.1 and 2.2 in [18] for the non trivial measure theoretic details) that

$$\Phi[f^\#](\mu[u]) = H[f](u,\Omega), \quad \text{for every } u \in BV_{\mathrm{loc}}(\Omega), \tag{1.23}$$

where we have defined

$$\begin{aligned}
H[f](u,\Omega) &:= \int_\Omega f(x, u(x), \nabla u(x)) dx \qquad (1.24)\\
&+ \int_\Omega f^\infty \left(x, u(x), \frac{D^c u}{|D^c u|}(x)\right) d|D^c u| \\
&+ \int_{J(u)} \frac{d\mathcal{H}^{n-1}(x)}{u^+(x) - u^-(x)} \left\{ \int_{u^-(x)}^{u^+(x)} f^\infty(x, s, \nu(x))\, ds \right\}.
\end{aligned}$$

The meaning of this functional is understood in the light of the canonical decomposition of Du for $u \in BV$,

$$Du = \nabla u dx + D^c u + (u^+(x) - u^-(x))\nu_u(x) d\mathcal{H}^{n-1} \lfloor J(u)\,;$$

here $u^-(x)$ is the *approximate lower limit of* u *at* x

$$u^-(x) = \sup\{t \in \mathbb{R} : \{y \in \Omega : u(y) < t\} \text{ has zero density at } x\},$$

and $J(u) := \{x \in \Omega : u^-(x) < u^+(x)\}$ is called the *jump set of* u, and turns out to be an $(n-1)$-rectifiable set in Ω, thus admitting a well defined normal vector $\nu_\Omega(x)$ at $\mathcal{H}^{n-1}$ a.e. $x \in J(u)$ (the versus of $\nu_\Omega(x)$ is determined according to $u^-(x) < u^+(x)$). $D^c u$ is called the *Cantorian part*[1] *of* Du, and we recall that u is approximately continuous on the support of this measure, so that we can unambiguously write $u(x)$ in the second integral in the definition of $H[f]$. Of course when $f = f(\xi)$ the new definition coincides with the one in Goffman and Serrin Theorem. The main result about the parametric extension $H[f]$ is the following theorem by Dal Maso, that, for clarity of exposition, we state at the *minimum* level of detail sufficient to discuss its main features: the interested reader will consult [18] for the sharp statement.

Theorem 10 (Dal Maso [18]). *Let* $f = f(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,\infty)$ *be locally bounded, lower semicontinuous and such that* $f(x,s,\cdot)$ *is convex for every* $(x,s) \in \Omega \times \mathbb{R}$. *Then* $f^\sharp$ *is lower semicontinuous. Furthermore:*

Part one: *If there exist* $a > 0$ *and* $b \in \mathbb{R}$ *such that*

$$a|\xi| - b \le f(x,s,\xi), \tag{1.25}$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, *then the functional* $H[f]$ *satisfies the lower semicontinuity inequality*

$$H[f](u,\Omega) \le \liminf_{h\to\infty} H[f](u_h,\Omega) \tag{1.26}$$

whenever

$$u_h, u \in BV_{\text{loc}}(\Omega), \quad u_h \to u \text{ in } L^1_{\text{loc}}(\Omega). \tag{1.27}$$

[1]Note that $D^c u$ is uniquely defined as difference between Du and the sum of the absolutely continuous part $\nabla u dx$ and the jump part $(u^+(x) - u^-(x))\nu_u(x) d\mathcal{H}^{n-1} \lfloor J(u)$.

In particular, since $H[f](u,\Omega) = F[f](u,\Omega)$ *for every* $u \in C^1(\Omega)$*, it follows that*

$$H[f](u,\Omega) \le \mathcal{F}[f](u,\Omega), \quad \forall u \in BV_{\mathrm{loc}}(\Omega). \tag{1.28}$$

Part two: *If* $f^\sharp$ *is continuous and there exists* $C > 0$ *such that*

$$f(x,s,\xi) \le C(1+|\xi|), \tag{1.29}$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$*, then*

$$H[f](u,\Omega) \ge \mathcal{F}[f](u,\Omega), \quad \forall u \in BV(\Omega).$$

The basic idea of the proof is very instructive and is the following. First of all one notes that if (1.27) holds true **and**

$$\sup_{h\in\mathbb{N}} |Du_h|(\Omega') < \infty, \quad \forall \Omega' \subset\subset \Omega,$$

then (see [61]) $\mu[u_h] \overset{*}{\rightharpoonup} \mu[u]$ in the sense of Radon measures on $\Omega \times \mathbb{R}$. Furthermore the additional information $|Du_h|(\Omega') \to |Du|(\Omega')$ implies that

$$|\mu[u_h]|(\Omega' \times \mathbb{R}) \to |\mu[u]|(\Omega' \times \mathbb{R}).$$

Note that if $u \in BV_{\mathrm{loc}}(\Omega)$ we can always find such a sequence u_h just by considering its ε-regularization for $\varepsilon = \varepsilon_h \to 0$. By using these remarks and (1.23) it is then clear that Theorem 10 is a natural consequence of the Reshetnyak lower semicontinuity theorem (for a modern proof, see Chapter 2 in [4]):

Theorem 11 (Reshetnyak lower semicontinuity Theorem, [69]). *Let* $\Sigma \subset \mathbb{R}^d$ *be an open set.*

Part one: *If* $\Psi = \Psi(t,p) : \Sigma \times \mathbb{R}^m \to [0,\infty]$ *is lower semicontinuous with* $\Psi(t,\cdot)$ *convex and positively homogeneous of degree one in* $\mathbb{R}^m$*, then*

$$\int_\Sigma \Psi\left(t, \frac{d\nu}{d|\nu|}(t)\right) d|\nu|(t) \le \liminf_{h\to\infty} \int_\Sigma \Psi\left(t, \frac{d\nu_h}{d|\nu_h|}(t)\right) d|\nu_h|(t) \tag{1.30}$$

whenever $\nu_h \overset{*}{\rightharpoonup} \nu$ *as Radon measures in* Σ.

Part two: *If* $\Psi = \Psi(t,p) : \Sigma \times S^{m-1} \to [0,\infty)$ *is continuous and bounded and* $|\nu_h|(\Sigma) \to |\nu|(\Sigma)$*, then have equality in* (1.30).

Let us comment first on the second part of Dal Maso's Theorem. The continuity requirement on $f^{\#}$ arises very naturally from this strategy of proof. It is a strong requirement on the behaviour at infinity of f with respect to (x, s). For example, a sufficient condition to it is that for every $(x_0, s_0) \in \Omega \times \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x_0, s_0, \xi) - f(x, s, \xi)| \leq \varepsilon(1 + |\xi|), \tag{1.31}$$

for every $(x, s, \xi) \in B_\delta(x_0) \times B_\delta(s_0) \times \mathbb{R}^n$. On the other hand is not that difficult to construct examples of integrands f such that this condition is violated and, in general, $H[f] < \mathcal{F}[f]$ (see Example 4.4 in [18] and some of variants of it [56]). This is not a limit of Theorem 10, but is just an unavoidable fact of life: *relaxation does not coincide with parametric extension* even if f is smooth, convex in the gradient variable and such that $c|\xi| \leq f(x, s, \xi) \leq C(1 + |\xi|)$.

About the first part of Dal Maso's Theorem it is particularly interesting to note that it furnishes a stronger lower bound for $\mathcal{F}[f]$ than the one given in Serrin's Theorem, compare (1.22) with (1.28). The main result in [55] is the extension of this lower bound from the coercive case (1.25) considered in Dal Maso's theorem to the demicoercive case:

Theorem 12 (Maggi [55]). *Let f be as in part one of Theorem 10 with the only exception that* (1.25) *is replaced by*

$$f(x, s, \cdot) \quad \textit{is demicoercive}$$

for every $(x, s) \in \Omega \times \mathbb{R}$. Then

$$H[f](u, \Omega) \leq \liminf_{h \to \infty} F[f](u_h, \Omega),$$

whenever $u_h \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u \in BV_{\mathrm{loc}}(\Omega)$ and $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. In particular (1.28) *holds true.*

In Dal Maso's proof the hypothesis (1.25) seems really crucial in order to have the right convergence of the $\mu[u_h]$ to $\mu[u]$ and to apply Reshetnyak theorem. To prove Theorem 12 we have thus followed a different strategy relying on the powerful blow up method first developed by Fonseca and Müller in [33], [34] to deal with the relaxation problem for quasiconvex integrands with linear growth, and later applied by Fonseca and Leoni [31] and [32] in the context of Serrin's problem. In section 4.1 we shall describe the blow up method, and in particular we refer the reader to the end of that section for an account of the results in [31] and [32] and their relation with the other theorems discussed so far.

Of course the blow up method is not the only tool in the proof of Theorem 12, that is based also on the approximation result in Theorem 7. There is a nice interaction between the two, since the blow up method, roughly speaking, allows us to recast the lower semicontinuity problem in such a way that we can focus on the behaviour of f with respect to (x, s) just in a small neighborhood of a fixed point (x_0, s_0). This fact naturally calls for a sort of stability property of the approximatng maximal cones, see Lemma 5 in Chapter 4, that in turn is the crucial step in the proof of Theorem 12. It is even more interesting to note that this stability property plays a crucial role also in answering to our third question.

1.1.5 The answer to the third question

What can have in common the three properties in Serrin's statement? The following theorem should explain what happens:

Theorem 13 (Fusco, Gori and Maggi [37]). *If $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ is lower semicontinuous and such that $f(x, s, \cdot)$ is convex and demicoercive for every $(x, s) \in \Omega \times \mathbb{R}$, then there exists a sequence of* **smooth** *functions $f_k : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ such that $f_k(x, s, \cdot)$ convex for every $(x, s) \in \Omega \times \mathbb{R}$ and*

$$f = \sup_{k \in \mathbb{N}} f_k.$$

That is to say, parts A and B of Serrin's theorem, as well as their unification, Theorem 7, can be seen as corollaries of part C! For further insight on this theorem we refer the reader to its proof in Chapter 5. Here we just limit ourselves to note that Theorem 13 does not really furnish a *new* proof of the lower semicontinuity results in [44] and [55] since it is in fact based on the fundamental ingredients by which that theorems are proved. In our opinion its real interest is in that it is a bridge between the realm of "geometric conditions" A and B, that have been completely understood and generalized into demicoercivity, and the realm of "smoothness conditions" like C and its various possible generalizations.

1.2 On the lower semicontinuity of supremal functionals.

In the last ten years there has been an increasing interest in the variational theory of supremal functionals (we refer to the two recent PhD thesis by Gori [43] and by Prinari [68] for a complete bibliography on the subject). In [45], under suggestion of Giuseppe Buttazzo, we have addressed the study of what could be called the "Serrin's problem" for supremal functionals.

A *supremal functional* $S[f](\cdot, \Omega) : W^{1,\infty}(\Omega) \to \mathbb{R}$ is a functional of the form

$$S[f](u, \Omega) := \text{ess-sup}\{f(x, u(x), \nabla u(x)) : x \in \Omega\},$$

where $f = f(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, called *supremand*. Let us define

$$\{f(x, s, \cdot) \leq t\} := \{\xi \in \mathbb{R}^n : f(x, s, \xi) \leq t\},$$

and consider a convergence $\to$ on $W^{1,\infty}$ and a sequence $u_h \in W^{1,\infty}(\Omega)$ such that

$$u_h \to u, \quad \text{for some } u \in W^{1,\infty}(\Omega).$$

The lower semicontinuity inequality

$$S[f](u, \Omega) \leq \liminf_{h \to \infty} S[f](u_h, \Omega), \tag{1.32}$$

is equivalent to asking that for every $t \in \mathbb{R}$ such that

$$\nabla u_h(x) \in \{f(x, u_h(x), \cdot) \leq t\},$$

for a.e. $x \in \Omega$ and for infinitely many h, then one has

$$\nabla u(x) \in \{f(x, u(x), \cdot) \leq t\},$$

for a.e. $x \in \Omega$. It is thus clear that in the study of the lower semicontinuity properties of $S[f]$ what is really crucial is the convexity of the level sets of $f(x, s, \cdot)$, rather than the one of the function itself. A *level convex supremand* is a supremand such that $\{f(x, s, \cdot) \leq t\}$ is convex in $\mathbb{R}^n$ for every $(x, s) \in \Omega \times \mathbb{R}$ and $t \in \mathbb{R}$.

When the convergence $\to$ is the weak star convergence of $W^{1,\infty}$ and $f \geq 0$, one can prove the lower semicontinuity inequality by simply approximating

$S[f]$ with $F[f^p]^{1/p}$ (as $p \to \infty$ of course) and by using the lower semicontinuity theory for integral functionals. In [45] the considered convergence is

$$u_h \to u \quad \text{in } L^\infty(\Omega), \tag{1.33}$$

so that nothing is assumed on the behaviour of the gradients of the u_h's, in the spirit of Serrin's theorem for integral functionals. Interestingly enough, with this kind of convergence it does not seem possible to invoke the theory for the integral functionals as explained above. The following theorem is the main result we have produced on this problem (however the continuity assumption here can be replaced by mere lower semicontinuity, see [43]):

Theorem 14 (Gori and Maggi [45]). *If f is a continuous, level convex supremand, then* (1.32) *holds true whenever u_h and u are as in* (1.33). *In fact, if the u_h are of class C^1 then* (1.32) *holds true even if f is not level convex.*

In particular it seems that there is no "Aronszajn's counterexample" in the theory of supremal functionals. The basic fact used in the proof of this theorem is the following sort of Mazur's Lemma (for gradients of uniformly converging sequences of Lipschitz functions), that could be, maybe, of independent interest:

Lemma 2. *Let u_h, u be as in* (1.33) *and let Ω_0 be a subset of full measure in Ω such that $\nabla u_h(x)$ and $\nabla u(x)$ exist for every $x \in \Omega_0$. Then for every $\rho_h \to 0$ and every $x_0 \in \Omega_0$ there exists*

$$x_h \to x_0, \quad y_h^i \in B(x_h, \rho_h), \quad 0 \le \lambda_h^i \le 1, \quad i = 1, ..., n+1,$$

such that

$$\lim_{h\to\infty} \sum_{i=1}^{n+1} \lambda_h^i \nabla u_h(y_h^i) = \nabla u(x_0).$$

We refer the reader to [45] for further results and examples about the sharpness of these theorems.

1.3 Γ-convergence of variational integrators generated by Lagrangians with quadratic growth

In [64] Müller and Ortiz have introduced the study of convergence properties of discrete dynamics and variational integrators by using the tool of Γ-convergence. The theory of *discrete dynamics* is a formulation of Lagrangian mechanics in which time is considered as a discrete variable. This point of view leads to studying discrete trajectories, the *variational integrators*, generated by a discrete version of Hamilton's principle in which the classical integral action is replaced by a sum, called *discrete action*. The interest of this construction lies in the fact that, compared to the trajectories generated by more usual approximation schemes, variational integrators converge to classical solutions in a more stable way. We refer to Marsden and West [60] for a complete account on the topic.

We say that a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f = f(s, \xi)$, is a *Lagrangian with quadratic behaviour* if

$$f \in C^2(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{with} \quad \sup\{|D^2 f(s,\xi)| : (s,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\} < \infty, \tag{1.34}$$

where $D^2 f$ is the matrix of all the second order partial derivatives of f. Thus, for some $C > 0$,

$$|f(s,\xi)| \leq C(1 + |s|^2 + |\xi|^2), \quad \forall (s,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The *action* is the functional $F : H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{E} \to \mathbb{R}$, defined by

$$F(u, A) := \int_A f(u, \dot{u}) dt, \quad \forall (u, A) \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{E},$$

where $\mathcal{E}$ is the family of all open bounded intervals of $\mathbb{R}$. A function u is said to be *stationary* for the action if,

$$\delta F(u, \varphi, A) = 0, \quad \forall A \in \mathcal{E}, \forall \varphi \in C^\infty_c(A; \mathbb{R}^n),$$

where the *first variation* of $F(\cdot, A)$ on u with respect to φ is given by

$$\delta F(u, \varphi, A) := \lim_{t \to 0} \frac{F(u + t\varphi, A) - F(u, A)}{t}.$$

For every $h > 0$ we fix the grid $\mathcal{T}_h = \{hi\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ of step h and define the set X_h of all $\mathcal{T}_h$-piecewise affine continuous functions with values in $\mathbb{R}^n$. A function $u_h \in X_h$ is said to be *h-stationary* for F if

$$\delta I(u_h, \varphi, A) = 0, \quad \forall A \in \mathcal{E}, \forall \varphi \in X_h, \text{ with spt } \varphi \subset A.$$

In [57], together with Massimiliano Morini, we have proved the following theorem that represents an extension of the previous result by Müller and Ortiz [64] dealing with the case $f(s,\xi) = |\xi|^2/2 - V(s)$ (in the end, for the sake of honesty, we have to say that this extension just amounts to tackle the technical complications arising from the full non linearity of the problem):

Theorem 15 (Maggi and Morini [57]). *Let f be a Lagrangian with quadratic behavior, see (1.34), and let $f(s,\cdot)$ be uniformly convex, that is we assume there exists $\nu > 0$ such that*

$$\left[\frac{\partial^2 f}{\partial \xi^2}(s,\xi)\eta\right]\cdot \eta \geq \nu|\eta|^2, \quad \forall s, \xi, \eta \in \mathbb{R}^n.$$

Then there exists $h_0 = h_0(C,\nu)$ such that

a) for every $(u_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and $h < h_0$, there exists an h-stationary u_h with $u_h(0) = u_0$ and $(u_h(h) - u_h(0))/h = \xi_0$. Furthermore,

$$\sup_{h<h_0} \|u_h\|_{W^{1,\infty}(A;\mathbb{R}^n)} \leq K < \infty, \quad \forall A \in \mathcal{E},$$

where K is a constant depending only on $|u_0|, |\xi_0|, C, \nu, \mathcal{L}^1(A)$.

b) every sequence $\{u_h\}_h$, with u_h h-stationary and such that,

$$\sup_h \max\{|u_h(0)|, |(u_h(h) - u_h(0))/h|\} < \infty$$

has a subsequence that converges weakly-$*$ star in $W^{1,\infty}_{loc}(\mathbb{R};\mathbb{R}^n)$ to a stationary u.

The main tool in the proof of this theorem is a sort of discrete version of Gronwall's Lemma. We close by noticing that this kind of statement is not true when the Lagrangian f has different growths with respect to the s and to the ξ variables, as can be seen by inspection of the simple case $f(s,\xi) = \xi^2/2 + |s|^{2+\varepsilon}/(2+\varepsilon)$, $\varepsilon > 0$, where, in general, stationary functions blow up in a finite time.

1.4 A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity and regularity for gradients of separately convex functions

In the classical framework of Continuum Mechanics, see for example Gurtin's book [48], the equilibrium configurations of an hyperelastic material body undergoing a certain prescribed boundary deformation u_0 in presence of a field f of external body forces can be determined by minimizing

$$\min\left\{\int_\Omega W(\nabla u) - f\cdot u\; dx : u:\Omega\subset\mathbb{R}^3\to\mathbb{R}^3,\; u_{|\partial\Omega} = (u_0)_{|\partial\Omega}\right\},$$

where the invariance under change of observer of the problem translates into asking that

$$W(F) = W(OF),\quad \text{for every}\quad F\in M(\mathbb{R}^3;\mathbb{R}^3),\, O\in SO(3).$$

If we renormalize the problem by choosing the identity deformation to have energy zero then $W = 0$ on $SO(3)$ and the prototype *stored energy function* is given by

$$W(F) = \mathrm{dist}^2(F, SO(3)).$$

This fact could cause lack of compactness for minimizing sequences independently from the growth condition assumed on W at infinity. This is indeed the case if we minimize $\int_\Omega W(V(x))dx$ where V is a generic vector field. However the constraint that V is a gradient allows oscillations only along rank one directions, and this is the key to the so called "rigidity theorems" that are of fundamental importance in proving existence of minimizers in elasticity. The prototype statement is the classical theorem by Liouville stating that if $\nabla u\in SO(n)$ and u is sufficiently regular then in fact ∇u is constantly equal to a fixed rotation (here Ω is assumed to be connected). A more quantitative version of Liouville's theorem, dealing with the linearization of this problem, is the well known Korn's inequality

$$\min_{\{S=-S^T\}}\int_\Omega |\nabla u - \mathrm{Id} - S|^2 dx \le C\int_\Omega |\nabla u + (\nabla u)^T - \mathrm{Id}|^2 dx,$$

that corresponds to linearize the constraint $\nabla u\in SO(n)$ near the identity matrix. The quantitative version of Liouville's theorem in the nonlinear setting has been proved by Friesecke, James and Müller [35] (previous complementary results are due to John [49] and Kohn [53]):

$$\min_{\{S\in SO(n)\}}\int_\Omega |\nabla u - S|^2 dx \le C\int_\Omega \mathrm{dist}^2(\nabla u, SO(n))dx.$$

These fundamental rigidity estimates, both in the linear and the nonlinear case, are based on estimates for linear partial differential equations that in turn are linked to estimates for suitable singular integral operators [71]. Thus they can be extended from the case of a stored energy function W with quadratic growth at infinity ($p = 2$) to the case of a generic polynomial growth $p \in (1,\infty)$ (these extensions are however not so immediate, see [14] for a complete account on this point).

The cases $p = 1$ and $p = \infty$ are critical for the theory of singular integral operators and in turn are critical also for the validity of the rigidity theorems. The failure of Korn's inequality in the case $p = 1$ is a matter of particular interest since it implies the non equivalence between the space of functions with bounded deformation

$$BD(\Omega) = \{u : \Omega \to \mathbb{R}^n : \nabla u + (\nabla u)^T \text{ is a Radon measure}\},$$

and the corresponding space of functions with bounded variation, $BV(\Omega;\mathbb{R}^n)$. We recall that the importance of BD relies in that it is the natural ambient space for the variational study of linearized plasticity, see Temam [74]. The failure of Korn's inequality for $p = 1$ was first shown by Ornstein with a remarkable counterexample [66]. This example in fact implies something stronger than the failure of Korn's inequality for $p = 1$, since what is proved is the existence, for every $k \in \mathbb{N}$, of a function $f_k : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_{(0,1)^2}\left|\frac{\partial^2 f_k}{\partial x_1 \partial x_2}\right| dx \ge k\int_{(0,1)^2}\left|\frac{\partial^2 f_k}{\partial^2 x_1}\right| + \left|\frac{\partial^2 f_k}{\partial^2 x_2}\right| dx.$$

Then one has only to consider $u_k = (\partial f_k/\partial x_1, -\partial f_k/\partial x_2)$ to contradict the possibility of a Korn's inequality in L^1. Another important application of Ornstein's counterexample is found by means of a duality argument that implies the non existence of solution for the prescribed divergence equation $\mathrm{div}\, v = f$ when $f \in L^\infty$ (see [10], [17]).

A little drawback of Ornstein's counterexample is its elevated complexity. A convincing evidence about this assertion can be found in noticing that

Temam, in his monograph on BD spaces, does not report its proof, despite the great theoretic importance of this counterexample in that subject!

In the joint paper with Sergio Conti and Daniel Faraco [15], we have realized how to use the simple concept of laminate in order to furnish a new construction leading to the same implications that can be deduced by using Ornstein's counterexample and in fact addressing further questions that are out of its reach. The main advantage of this method is in that it separates the difficulty of explicitly constructing a deformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ from the difficulty of imposing some constraint on the values of its gradient. The first difficulty is completely absorbed by the simplest lemmas in the theory of convex integration; the second one is transferred in considering some explicit geometric problem in low dimensional Euclidean spaces. For further insight we have to refer the reader to [15], but let us mention that these ideas are strictly related to previous work by Faraco [27], [28] and by Faraco and Székelyhidi [29]. For the general framework in which one should put this result, see [50], [63], [65], [52]...

As said, beside the fact that we believe this method to furnish an intrinsically simpler way to construct examples having the same properties of Ornstein's one, we are able to prove by it some new results, that we are going to introduce.

In the general theory of vectorial variational problems (that is, in minimization problems over classes of vector valued maps) a central role is played by the well known notion of quasiconvexity of W. The understaning of quasiconvexity is a formidable problem and, in a sense, it is a problem we cannot avoid to consider as soon as we minimize a non convex integral functional on vector valued maps (and, needless to say, all the interesting vectorial problems are ruled by non convex W's!). Very little is known on quasiconvex functions: for example, an open problem is to understand if a quasiconvex function $W : \mathbb{R}^{m\times n} \to \mathbb{R}$ has gradient of locally bounded variation. It is well known that this is true for convex functions, since in this case the gradient is locally bounded and monotone [2]. Since quasiconvexity implies (and, in general, is strictly stronger than) rank one convexity, that is, convexity of W along rank one directions, one would like to understand if, for example, rank one convex functions have gradient in BV. Since the gradient of a rank one convex function is always locally bounded we could say that, in a sense, we are asking if rank one convexity contains enough "monotonocity" to force bounded variation. This seems still a too hard problem, so we start to look to the simplest possible model of rank one convexity that we can imagine (see [73]): a separately convex function in the plane, that is $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x_1, \cdot) \text{ and } f(\cdot, x_2) \text{ are convex on } \mathbb{R} \text{ for very fixed } x_1 \text{ and } x_2.$$

It is relatively simple to see that f is locally Lipschitz, that is that $\nabla f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and here we are asking if $\nabla f \in BV_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$. This problem has been known by specialists since a long time and have been solved in [15]. Namely, by the method inspired initially from the study of Ornstein counterexample, one can prove the following theorem:

Theorem 16 (Conti, Faraco and Maggi [15]). *There exists a function $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ such that*

$$\begin{aligned} &\frac{\partial^2 f}{\partial^2 x_i} \in BV_{\mathrm{loc}}(\mathbb{R}^2), \;\; i = 1, 2, \\ &\frac{\partial^2 f}{\partial^2 x_i} \geq 0, \;\; \textit{in the sense of distributions}, \;\; i = 1, 2, \end{aligned}$$

but such that the distribution $(\partial^2 f)/(\partial x_1 \partial x_2)$ has infinite total variation on every open subset of $(0, 1)^2$: that is,

$$\int_A \left| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right| = \infty,$$

for every open set $A \subset (0, 1)^2$. Furthermore $f(x) = |x|^2$ outside $(0, 1)^2$.

1.5 On sharp Sobolev inequalities with trace terms

We close this overview by describing some results from the joint work (still in progress) with Cédric Villani [58]. We confine ourselves to a very brief introduction, concluded by an interesting statement that can be deduced as a corollary of the main theorems in the paper.

The starting point is the **Sobolev inequality on** $\mathbb{R}^n$,

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)} \leq S_n(p)\|\nabla f\|_{L^p(\mathbb{R}^n)},$$

where $n \geq 2$ is an integer, $p \in [1, n)$ is a real number, $p^\star := np/(n-p)$, $f : \mathbb{R}^n \to \mathbb{R}$ is a function vanishing at infinity, that is

$$|\{x \in \mathbb{R}^n : f(x) \geq a\}| < \infty \quad \text{for every } a > 0,$$

and where $S_n(p)$ is the optimal (smaller) constant for which the inequality is valid. The Sobolev inequality has been studied in depth by many authors. In particular, the problem of determining the value of the sharp constant and the explicit form of the optimizers has been solved independently by Aubin [6] and Talenti [72]. The optimizers, up to translations, have the form

$$f_{a,b}(x) := \frac{1}{(a + b|x|^{p'})^{1-(p/n)}},\ a > 0, b > 0, x \in \mathbb{R}^n.$$

The problem of how to recast the Sobolev inequality on a reasonably regular domain $\Omega \subset \mathbb{R}^n$, let us say Ω open, connected and with Lipschitz boundary, can be attacked in several ways, the most common one being, maybe, to consider Poincaré type inequalities like

$$\|f - f_\Omega\|_{L^{p^\star}(\Omega)} \leq P_\Omega(p)\|\nabla f\|_{L^p(\Omega)},$$

where $f_\Omega = |\Omega|^{-1}\int_\Omega f$. In the celebrated paper [11], Brézis and Lieb propose many alternatives to Poincaré type inequalities, *one* of them being to consider what we call the **homogeneous Sobolev inequality**

$$\|f\|_{L^{p^\star}(\Omega)} \leq A\|\nabla f\|_{L^p(\Omega)} + B\|f\|_{L^{p^\sharp}(\partial\Omega)}, \tag{1.35}$$

where the exponent $p^\sharp := (n-1)p/(n-p)$ is the only one we can use for the norm of the trace of f at the boundary of Ω if we want the inequality to be valid with constants *independent from* Ω. Brézis and Lieb treat this inequality in the case of smooth bounded domains, and prove it when $p=2$ with constants

$$A = S_n(2), \quad B = B(\Omega).$$

Note that the first constant is sharp. Among the several problems mentioned in their paper there is the following one: is the stronger variant of (1.35)

$$\|f\|^p_{L^{p^\star}(\Omega)} \le S_n(p)^p \|\nabla f\|^p_{L^p(\Omega)} + C(\Omega)\|f\|^p_{L^{p^\sharp}(\partial\Omega)}, \tag{1.36}$$

true? This question has been answered in the affermative by Zhu [76] when $p=2$ and Ω is a ball. Both Brézis and Lieb [11] and Zhu's [76] papers are based on PDE techniques and thus take great advantage from the assumption $p=2$.

In the recent paper [16], Cordero-Erausquin, Nazaret and Villani have found a new proof of the Sobolev inequality on $\mathbb{R}^n$ in sharp form based on mass transportation techniques (see [75] for a general reference). Their approach produces a remarkably elegant proof, valid for all p and whatever the norm on $\mathbb{R}^n$ we are using to evaluate the gradient in the integral norm $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ (this is a concrete issue since we are dealing with the sharp form of the inequality). In [58] we adapt this method of inquiry to the study of inequalities like (1.35) or (1.36) and prove several results. A corollary of these theorems that we quote here to give an example of the typical result that can be found in [58] is the following:

Theorem 17 (Maggi and Villani [58]). *Let Ω be an open Lipschitz domain in $\mathbb{R}^n$. Then*

$$\|f\|_{L^{p^\star}(\Omega)} \le S_n(p)\|\nabla f\|_{L^p(\Omega)} + S_n(1)^{1/p^\sharp}\|f\|_{L^{p^\sharp}(\partial\Omega)}, \tag{1.37}$$

for every $p\in[1,n)$. Furthermore there exists a constant $C=C(n,p)$ such that

$$\|f\|^p_{L^{p^\star}(\Omega)} \le S_n(p)^p \|\nabla f\|^p_{L^p(\Omega)} + C(n,p)\|f\|^p_{L^{p^\sharp}(\partial\Omega)}. \tag{1.38}$$

About (1.37) we note that it is sharp with respect to both the first and the second constant, in the latter case if we concur to search this constant among the ones that make the inequality valid independently from the the domain under consideration. About (1.38), we note that it answers to Brézis and Lieb question in the affirmative for every $p\in[1,n)$, for every Lipschitz domain (bounded or not) and for every norm. It also furnishes a (non optimal) constant in front of the second term that is independent from the domain.

Chapter 2

Lower semicontinuity and summability conditions

In this chapter we present the results contained in the paper [46]. In section 2.1 we prove the approximation theorem by De Giorgi that is crucial in our method of proof, since reference [23] is not published. For an introduction to the subject of this chapter we refer the reader to subsection 1.1.2.

2.1 A theorem by De Giorgi

Let $f : \Sigma \times \mathbb{R}^n \to \mathbb{R}$, $f = f(t, \xi)$ be a function with $f(t, \cdot)$ convex, where $\Sigma \subset \mathbb{R}^d$ is an open set. By the Hahn-Banach Theorem we have that

$$f(t, \xi) = \sup_{k \in \mathbb{N}} f_k(t, \xi), \quad \text{where } f_k(t, \xi) = a_k(t) + b_k(t) \cdot \xi,$$

where $a_k(t) \in \mathbb{R}$ and $b_k(t) \in \mathbb{R}^n$ for every $t \in \Sigma$. The problem we consider in this section is if it is possible a choice of the a_k's and of the b_k's reflecting the various type of regularity one can wish to consider on $f(\cdot, \xi)$. As already mentioned, when dealing with measurability or continuity this problem can be tackled by the theory of selections for multi-valued maps. However, in the present case we shall need to transfer finer regularity properties.

For the moment let us suppress the dependence from the t variable, so that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. If $f \in C^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$ we have that

$$f(\eta) \geq A[f](\xi, \eta) := f(\xi) + \nabla f(\xi) \cdot (\eta - \xi), \tag{2.1}$$

for every $\eta \in \mathbb{R}^n$, since $A[f](\xi, \cdot)$ is the function whose graph is the supporting hyperplane to the epigraph of f at $(\xi, f(\xi))$. Let us consider for a moment the slightly more degenerate case in which f is differentiable everywhere except that at ξ. Then we can replace ξ by fixing a test function $\alpha \in \mathcal{D}(B^n; [0, \infty))$ with $\int \alpha = 1$ and by averaging through it the well defined supporting hyperplanes associated to the points ζ near to ξ. More precisely, we multiply (2.1) by $\varepsilon^{-n}\alpha((\zeta - \xi)/\varepsilon)) =: \alpha_\varepsilon(\zeta - \xi)$ and the integrate over ζ to find that

$$f(\eta) \geq \int \alpha_\varepsilon(\zeta - \xi) A[f](\zeta, \eta) d\zeta, \tag{2.2}$$

for every $\eta, \xi \in \mathbb{R}^n$ and $\varepsilon > 0$. The right hand side of (2.2) is an affine function of the η variable and, as $\eta = \xi$, it is equal to the average of f with respect to $\alpha_\varepsilon(\cdot - \xi)$, thus converging to $f(\xi)$ as $\varepsilon \to 0$. Note also that since convex functions are always a.e. differentiable, for every fixed η the function $A[f](\zeta, \eta)$ is well defined for a.e. $\zeta \in \mathbb{R}^n$, and thus the right hand side of (2.2) has a meaning even for a generic convex f. However we can avoid to use these regularity results and operate directly at a more elementary level by remarking that, whenever $f, \beta \in C^1(\mathbb{R}^n)$,

$$\begin{aligned} \operatorname{div}_\zeta(\beta(\zeta) f(\zeta)(\eta - \zeta)) &= \beta(\zeta)\nabla f(\zeta) \cdot (\eta - \zeta) \\ &\quad + f(\zeta)\nabla\beta(\zeta) \cdot (\eta - \zeta) - n\beta(\zeta) f(\zeta), \end{aligned}$$

a relation that, applied to $\beta(\zeta) = \alpha_\varepsilon(\zeta - \xi)$, gives us

$$\begin{aligned} &\int \alpha_\varepsilon(\zeta - \xi) A[f](\zeta, \eta) d\zeta \\ &= \int f(\zeta) \left\{(n+1)\alpha_\varepsilon(\zeta - \xi) - \nabla\alpha_\varepsilon(\zeta - \xi) \cdot (\eta - \zeta)\right\} d\zeta, \end{aligned} \tag{2.3}$$

where the right hand side has a meaning for every measurable f. Then we can prove the following theorem.

Theorem 18 (De Giorgi [23]). *Let $\alpha \in \mathcal{D}(B^n; [0, \infty))$ with $\int \alpha = 1$, $\varepsilon > 0$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Let us define*

$$\mathcal{A}[f](\varepsilon, \xi, \eta) := \int f(\zeta) \left\{(n+1)\alpha_\varepsilon(\zeta - \xi) - \nabla\alpha_\varepsilon(\zeta - \xi) \cdot (\eta - \zeta)\right\} d\zeta, \tag{2.4}$$

for every $\xi, \eta \in \mathbb{R}^n$. Then there exist two sequences $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, 1)$ such that

$$f(\eta) = \sup_{k\in\mathbb{N}} \left\{\mathcal{A}[f](\varepsilon_k, \xi_k, \eta)\right\}, \quad \forall \eta \in \mathbb{R}^n.$$

Let $f = f(t,\xi) : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ be a function with $f(t,\cdot)$ convex on $\mathbb{R}^n$ for every $t \in \Sigma$. By Theorem 18 we have $f = \sup_{k\in\mathbb{N}} f_k$ for $f_k(t,\eta) = a_k(t) + b_k(t)\cdot\eta$,

$$\begin{aligned} a_k(t) &= \int f(t,\zeta)\,\{(n+1)\alpha_k(\zeta) + \nabla\alpha_k(\zeta)\cdot\zeta\}\,d\zeta, \\ b_k(t) &= -\int f(t,\zeta)\nabla\alpha_k(\zeta)d\zeta. \end{aligned}$$

Here $\{\alpha_k\}_{k\in\mathbb{N}}$ is a suitable sequence in $\mathcal{D}(\mathbb{R}^n;[0,\infty))$ with $\int \alpha_k = 1$. It is then clear that if f is measurable or continuous or differentiable in t then so are a_k and b_k and thus f_k.

Proof. For every $\delta > 0$

$$f_\delta(\eta) := \int \alpha_\delta(\eta-\zeta)f(\zeta)d\zeta,$$

is convex and smooth on $\mathbb{R}^n$. Since f_δ is smooth we can consider $A[f_\delta](\zeta,\eta)$ and write down (2.1) to find that, for every $\zeta,\eta \in \mathbb{R}^n$,

$$f_\delta(\eta) \geq A[f_\delta](\zeta,\eta).$$

We fix $\varepsilon > 0$, $\xi \in \mathbb{R}^n$ and multiply this inequality by $\alpha_\varepsilon(\zeta-\xi)$ and then integrate it in ζ. By (2.3)

$$f_\delta(\eta) \geq \mathcal{A}[f_\delta](\varepsilon,\xi,\eta).$$

Since f is continuous, as $\delta \to 0$, $f_\delta \to f$ uniformly on compact subsets of $\mathbb{R}^n$. On the other hand is easily seen from (2.4) that

$$f \mapsto \mathcal{A}[f](\varepsilon,\xi,\eta)$$

is continuous with respect to the local uniform convergence. Thus we can pass to the limit as $\delta \to 0$ to find that

$$f(\eta) \geq \mathcal{A}[f](\varepsilon,\xi,\eta),$$

for every $\eta,\xi \in \mathbb{R}^n$ and $\varepsilon > 0$. By the affine change of variables $\overline{\zeta} = \varepsilon^{-1}(\zeta-\xi)$, $\zeta = \xi + \varepsilon\overline{\zeta}$,

$$\begin{aligned} &\mathcal{A}[f](\varepsilon,\xi,\eta) \\ = &\int f(\xi+\varepsilon\overline{\zeta})\left\{(n+1)\varepsilon^{-n}\alpha(\overline{\zeta}) - \varepsilon^{-(n+1)}\nabla\alpha(\overline{\zeta})\cdot(\eta-\xi-\varepsilon\overline{\zeta})\right\}\varepsilon^n d\overline{\zeta} \\ = &\int f(\xi+\varepsilon\overline{\zeta})\left\{(n+1)\alpha(\overline{\zeta}) - \nabla\alpha(\overline{\zeta})\cdot\left(\frac{\eta-\xi}{\varepsilon} - \overline{\zeta}\right)\right\} d\overline{\zeta}. \end{aligned}$$

Thus, if $\eta = \xi$,

$$\mathcal{A}[f](\varepsilon, \eta, \eta) = \int f(\eta + \varepsilon\overline{\zeta}) \left\{(n+1)\alpha(\overline{\zeta}) - \nabla\alpha(\overline{\zeta}) \cdot \overline{\zeta}\right\} d\overline{\zeta}.$$

If we let $\varepsilon \to 0$, since $\int \nabla\alpha(\overline{\zeta}) \cdot \overline{\zeta} d\overline{\zeta} = -n$, we deduce that

$$\lim_{\varepsilon \to 0} \mathcal{A}[f](\varepsilon, \eta, \eta) = f(\eta).$$

In conclusion

$$f(\eta) = \sup\{\mathcal{A}[f](\varepsilon, \xi, \eta) : \varepsilon > 0, \xi \in \mathbb{R}^n\},$$

and we are done by the following lemma. □

Lemma 3. *Let $\mathcal{G}$ be a family of lower semicontinuous functions on some Euclidean space E and let*

$$G(x) := \sup\{g(x) : g \in \mathcal{G}\},$$

for every $x \in E$. Then there exists a sequence $\{g_k\}_{k \in \mathbb{N}} \subset \mathcal{G}$ such that

$$G = \sup_{k \in \mathbb{N}} g_k.$$

Of course this lemma works as well in any σ-compact metric space E.

Proof. For every $k \in \mathbb{N}$ and every $x \in E$ there exists a function $g_{[k,x]} \in \mathcal{G}$ such that $G(x) \leq g_{[k,x]}(x) + 2^{-k}$ and, relatively to $g_{[k,x]}$, there is a $\rho_{[k,x]} > 0$ such that $g_{[k,x]}(x) \leq g(y) + 2^{-k}$ whenever $y \in B(x, \rho_{[k,x]})$. Of course we can always suppose $\rho_{[k,x]} < 2^{-k}$. Thus for every $k \in \mathbb{N}$ we have an open covering of E in the form

$$\{B(x, \rho_{[k,x]})\}_{x \in E}.$$

Since E is union of countable many compact sets, for every $k \in \mathbb{N}$ we find

$$x_h^k \in E, \quad \rho_h^k \in (0, 2^{-k}), \quad g_h^k \in \mathcal{G}, \quad h \in \mathbb{N},$$

such that E is covered by $\{B(x_h^k, \rho_h^k)\}_{k,h \in \mathbb{N}}$ and

$$\begin{aligned} G(x_h^k) &\leq g_h^k(x_h^k) + 2^{-k}, \\ g_h^k(x_h^k) &\leq g_h^k(x) + 2^{-k}, \quad \forall x \in B(x_h^k, \rho_h^k). \end{aligned}$$

Let $H(x) := \sup_{h,k\in\mathbb{N}} g_h^k(x)$ for every $x \in E$ and let us prove that $H = G$. Of course $G \geq H$. On the other hand, let us fix $x \in E$, $k \in \mathbb{N}$ and consider $h = h[x,k]$ such that $x \in B(x_h^k, \rho_h^k)$: then,

$$\begin{aligned} H(x) \geq g_h^k(x) &\geq g_h^k(x_h^k) - 2^{-k} \\ &\geq G(x_h^k) - 2^{-k+1} \geq \inf_{B(x,2^{-k})} G - 2^{-k+1}, \end{aligned}$$

where in the last inequality we have used $x_h^k \in B(x, \rho_h^k)$ and $\rho_h^k < 2^{-k}$. Since G is lower semicontinuous (as it is the supremum of lower semicontinuous functions), we let $k \to \infty$ to recover $H(x) \geq G(x)$ and conclude the proof of the lemma. □

2.2 A lower semicontinuity theorem

Theorem 19 (Gori, Maggi and Marcellini [46]). *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ *be such that*

(i) $f \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$*;*

(ii) $f(x, \cdot, \cdot)$ *is continuous for a.e.* $x \in \Omega$*;*

(iii) $f(x, s, \cdot)$ *is convex for every* $(x, s) \in \Omega \times \mathbb{R}$*;*

(iv) $f(\cdot, s, \xi) \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *and for every* $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ *there exists a constant* $L = L(\Omega', H, K)$ *such that*

$$\int_{\Omega'} |\nabla_x f(x, s, \xi)| dx \leq L, \quad \forall (s, \xi) \in H \times K.$$

Then

$$F[f](u, \Omega) \leq \liminf_{h\to\infty} F[f](u_h, \Omega)$$

whenever $u_h \to u$ *in* $L^1_{\mathrm{loc}}(\Omega)$*,* $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$*.*

Proof. In the first three steps of the proof we repeatedly approximate f from below in order to simplify the starting problem. In the fourth step we fix the sequence u_h on which we want to prove the lower semicontinuity inequality and perform a further approximation of f, this time induced by the limit function u. We end up by facing a limit relation that is treated by means of the chain rule and hypothesis (iv).

Step one: Let $\gamma_k \in \mathcal{D}(\Omega \times \mathbb{R}; [0,1])$ with $\gamma_k \uparrow 1$, then

$$\gamma_k(x,s) f(x,s,\xi) \uparrow f(x,s,\xi)$$

and we can reduce to prove the lower semicontinuity relation for the functional $F[\gamma_k f]$. Thus, without loss of generality, we can assume the existence of an open set $\Omega' \times H \subset\subset \Omega \times \mathbb{R}$ such that

(v) $f(x,s,\xi) = 0$ for every $(x,s,\xi) \in (\Omega \setminus \Omega') \times (\mathbb{R} \setminus H) \times \mathbb{R}^n$;

and we can also assume that

(vi) $f(\cdot,s,\xi) \in W^{1,1}(\Omega)$ and for every $K \subset\subset \mathbb{R}^n$ there exists a constant $L = L(K)$ such that

$$\int_\Omega |\nabla_x f(x,s,\xi)| dx \leq L, \quad \forall (s,\xi) \in \mathbb{R} \times K.$$

Step two: According to Theorem 18 and the subsequent remarks we have that

$$f(x,s,\xi) = \sup_{k \in \mathbb{N}} f_k(x,s,\xi) = \sup_{k \in \mathbb{N}} \{a_k(x,s) + b_k(x,s) \cdot \xi\}^+,$$

where

$$\begin{aligned} a_k(x,s) &= \int f(x,s,\zeta) \left\{ (n+1)\alpha_k(\zeta) + \nabla \alpha_k(\zeta) \cdot \zeta \right\} d\zeta, \\ b_k(x,s) &= -\int f(x,s,\zeta) \nabla \alpha_k(\zeta) d\zeta, \end{aligned}$$

for a suitable sequence $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n; [0,\infty))$ with $\int \alpha_k = 1$. It is really simple to check that each f_k satisfies properties (i), (ii), (iii), (iv), (v) and (vi). We note in particular that (v) is true with the same $\Omega' \times H$ used for f and that the bounds for f_k will depend also from the C^1 norm of α_k, further than from the constants in the corresponding bounds for f. By these properties and since $f_k(x,s,\cdot)$ is the positive part of an affine function, we also see that for each k there exists a constant M_k such that

$$|f_k(x,s,\xi_1) - f_k(x,s,\xi_2)| \leq M_k |\xi_1 - \xi_2|,$$

for every $(x, s, \xi_i) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $i = 1, 2$. As explained in subsection 1.1.1, by $f = \sup_{k\in\mathbb{N}} f_k$, we can reduce to prove the lower semicontinuity for $F[f_k]$. Thus, without loss of generality, we can assume directly that f, further than (i), (ii), (iii), (iv), (v) and (vi) also satisfies

(vii) $|f(x, s, \xi_1) - f(x, s, \xi_2)| \leq M|\xi_1 - \xi_2|$ for every $(x, s, \xi_i) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $i = 1, 2$,

for a suitable constant M.

Step three: We now perform a regularization of f in the ξ variable. To this end let $\alpha \in \mathcal{D}(B^n; [0, \infty))$ with $\alpha(\xi) = \alpha(-\xi)$, $\int \alpha = 1$ and define as usual $\alpha_\varepsilon(\xi) := \varepsilon^{-n}\alpha(\xi/\varepsilon)$. Let $\varepsilon_k := (kM)^{-1}$ and consider

$$f_k(x, s, \xi) := \int \alpha_{\varepsilon_k}(\eta) f(x, s, \xi - \eta) d\eta.$$

By property (vii) of f we immediately deduce that

$$\begin{aligned} |f_k(x, s, \xi) - f(x, s, \xi)| &\leq \int |f(x, s, \xi - \eta) - f(x, s, \xi)|\alpha_{\varepsilon_k}(\eta) d\eta \\ &\leq M \int |\eta|\alpha_{\varepsilon_k}(\eta) d\eta \leq k^{-1}, \end{aligned}$$

so that for every $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$

$$f(x, s, \xi) - 2k^{-1} \leq f_k(x, s, \xi) - k^{-1} \leq f(x, s, \xi).$$

In particular, if we have the lower semicontinuity inequality for $F[f_k]$ then we have it for $F[f]$ since, by (v),

$$\begin{aligned} F[f](u, \Omega) - 2k^{-1}|\Omega'| &= F[f](u, \Omega') - 2k^{-1}|\Omega'| \\ &\leq F[f_k](u, \Omega') - k^{-1}|\Omega'| \\ &\leq \liminf_{h\to\infty} F[f_k](u_h, \Omega') - k^{-1}|\Omega'| \\ &\leq \liminf_{h\to\infty} F[f](u_h, \Omega') = \liminf_{h\to\infty} F[f](u_h, \Omega). \end{aligned}$$

Thus it suffices to consider the lower semicontinuity problem for each f_k rather than for f. In passing from f to f_k the properties (i), (ii), (iii), (iv),

(v) and (vi) are of course still in force (with bounds depending also on k, as remarked in step two). Since f_k is smooth in the ξ variable, let us look at

$$\nabla_\xi f_k(x,s,\xi) = \int \nabla_\xi \alpha_{\varepsilon_k}(\xi-\eta) f(x,s,\eta) d\eta.$$

Of course $\nabla_\xi f_k$ is measurable in x and continuous in (s,ξ). By (vii) $f(x,s,\cdot)$ is uniformly Lipschitz in $\mathbb{R}^n$ (with the same constant for every $(x,s)\in\Omega\times\mathbb{R}^n$), and thus

$$|\nabla_\xi f_k(x,s,\xi)| \leq M_k,$$

for every $(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$, where the constant M_k depends also on the C^1 norm of α_{ε_k}. It is also clear that $\nabla_\xi f_k(\cdot,s,\xi)$ is in $L^1(\Omega)$ and is weakly differentiable with

$$\nabla_x\nabla_\xi f_k(x,s,\xi) = \int \nabla_\xi \alpha_{\varepsilon_k}(\xi-\eta)\nabla_x f(x,s,\eta) d\eta.$$

Thus

$$\begin{aligned}\int_\Omega |\nabla_x\nabla_\xi f_k(x,s,\xi)|dx &\leq \int_\Omega dx \int |\nabla_x f(x,s,\xi-\eta)||\nabla_\xi\alpha_{\varepsilon_k}(\eta)|d\eta \\ &\leq \int_{\varepsilon_k B^n} |\nabla_\xi \alpha_{\varepsilon_k}(\eta)| \left\{\int_\Omega |\nabla_x f(x,s,\xi-\eta)|dx\right\} d\eta.\end{aligned}$$

By (vi), for every fixed $K\subset\subset\mathbb{R}^n$, this double integral is uniformly bounded by a constant (depending also on the C^1 norm of α_k) as $(s,\xi)\in\mathbb{R}\times K$. In conclusion we can assume, without loss of generality, that our starting f satisfies (i), (ii), (iii), (iv), (v), (vi), (vii) and the following three properties:

(viii) for a.e. $x\in\Omega$ and every $s\in\mathbb{R}$, the classical derivative $\nabla_\xi f(x,s,\cdot)$ exists and $\nabla_\xi f(x,\cdot,\xi)$ is continuous on $\mathbb{R}$ for a.e. $x\in\Omega$ and for every $\xi\in\mathbb{R}^n$;

(ix) $\nabla_\xi f(\cdot,s,\xi)\in W^{1,1}(\Omega)$ for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$ and furthermore, for every $K\subset\subset\mathbb{R}^n$, there exists a constant $L=L(K)$ such that

$$\int_\Omega |\nabla_x\nabla_\xi f(x,s,\xi)|dx \leq L,$$

for every $(s,\xi)\in\mathbb{R}\times K$;

(x) $|\nabla_\xi f(x,s,\xi)| \le M$ for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

where M is a suitable constant.

Step four: Let $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. We want to prove that

$$F[f](u,\Omega) \le \liminf_{h\to\infty} F[f](u_h,\Omega).$$

Of course we can reduce to consider the case in which

$$\liminf_{h\to\infty} F[f](u_h,\Omega) = \lim_{h\to\infty} F[f](u_h,\Omega) < \infty.$$

Let us fix $\Omega'' \subset \Omega$ with Lipschitz boundary such that $\Omega' \subset\subset \Omega$ and for every $\varepsilon > 0$ let us define $v_\varepsilon \in W^{1,\infty}_0(\Omega'')$ such that

$$\lim_{\varepsilon\to 0} \int_{\Omega'} |\nabla u - \nabla v_\varepsilon| = 0, \tag{2.5}$$

and there exists a finite partition of Ω'' is disjoint open Lipschitz subsets $\{\Omega_j\}_{j=1}^{N(\varepsilon)}$ of Ω'' such that

$$\nabla v_\varepsilon \equiv \xi_j \quad \text{in} \quad \Omega_j.$$

This is the standard approximation theorem for $W^{1,1}$ functions by means of piece-wise affine functions, see for example [26].

In order to localize the lower semicontinuity problem in domains in which the limit function is affine we consider, for the fixed $\varepsilon > 0$ under consideration, a sequence of functions $\{\beta_k^\varepsilon\}_{k=1}^\infty \subset \mathcal{D}(\mathbb{R}^n;[0,1])$ such that

$$0 \le \beta_k^\varepsilon(x) \le \beta_{k+1}^\varepsilon(x) \uparrow 1 \quad \text{as } k\to\infty, \quad \text{for a.e. } x \in \Omega',$$

and $1_{\Omega_j\cap\Omega'}\,\beta_k^\varepsilon \in \mathcal{D}(\Omega_j \cap \Omega')$, for every $j = 1, ..., N(\varepsilon)$.

By the convexity of f (recall that f is smooth in ξ) we thus find that

$$\begin{aligned}
&\int_{\Omega'} \beta_k^\varepsilon \left\{ f(x,u_h,\nabla u_h) - f(x,u,\nabla v_\varepsilon)\right\} dx \\
\ge\ &\int_{\Omega'} \beta_k^\varepsilon \left\{ f(x,u_h,\nabla v_\varepsilon) - f(x,u,\nabla v_\varepsilon)\right\} dx \\
+\ &\int_{\Omega'} \beta_k^\varepsilon \nabla_\xi f(x,u_h,\nabla v_\varepsilon)\cdot(\nabla u_h - \nabla v_\varepsilon) dx.
\end{aligned}$$

By (i), (ii), (v), the fact that ∇v_ε takes its values in a compact set and that, up to extracting a not relabeled subsequence, $u_h(x) \to u(x)$ for a.e. $x \in \Omega$, we can say that the first term on the right hand side vanishes by dominated convergence as $h \to \infty$, for every $\varepsilon > 0$ and $k \in \mathbb{N}$ fixed. Let us further consider the decomposition

$$\begin{aligned}
& \int_{\Omega'} \beta_k^\varepsilon \nabla_\xi f(x, u_h, \nabla v_\varepsilon) \cdot (\nabla u_h - \nabla v_\varepsilon) dx \\
= & \int_{\Omega'} \beta_k^\varepsilon \Big\{ [\nabla_\xi f(x, u_h, \nabla v_\varepsilon) \cdot \nabla u_h] - [\nabla_\xi f(x, u, \nabla v_\varepsilon) \cdot \nabla u] \Big\} dx \\
+ & \int_{\Omega'} \beta_k^\varepsilon \nabla_\xi f(x, u, \nabla v_\varepsilon) \cdot (\nabla u - \nabla v_\varepsilon) dx \\
+ & \int_{\Omega'} \beta_k^\varepsilon (\nabla_\xi f(x, u, \nabla v_\varepsilon) - \nabla_\xi f(x, u_h, \nabla v_\varepsilon)) \cdot \nabla v_\varepsilon dx.
\end{aligned}$$

For every fixed $\varepsilon > 0$ and $k \in \mathbb{N}$, the last term on the right hand side is vanishing as $h \to \infty$ by dominated convergence, that we can apply in virtue of (viii), (x) and the fact that ∇v_ε is bounded. We have thus proved that

$$\begin{aligned}
& \liminf_{h \to \infty} F[f](u_h, \Omega) \\
\geq & \int_{\Omega'} \beta_k^\varepsilon f(x, u, \nabla v_\varepsilon) dx + \int_{\Omega'} \beta_k^\varepsilon \nabla_\xi f(x, u, \nabla v_\varepsilon) \cdot (\nabla u - \nabla v_\varepsilon) dx \\
+ & \limsup_{h \to \infty} \int_{\Omega'} \beta_k^\varepsilon \Big\{ [\nabla_\xi f(x, u_h, \nabla v_\varepsilon) \cdot \nabla u_h] - [\nabla_\xi f(x, u, \nabla v_\varepsilon) \cdot \nabla u] \Big\} dx.
\end{aligned}$$

Let us denote by $\mathcal{I}(h, \varepsilon, k)$ the last term on the right hand side: we claim that

$$\lim_{h \to \infty} \mathcal{I}(h, \varepsilon, k) = 0 \quad \text{for every fixed } \varepsilon > 0 \text{ and } k \in \mathbb{N}. \tag{2.6}$$

We shall prove this claim in step five and assume it for the moment. On the other hand, by Fatou's Lemma (note that $f \geq 0$ and recall (2.5))

$$\begin{aligned}
\lim_{\varepsilon \to 0} \lim_{k \to \infty} \int_{\Omega'} \beta_k^\varepsilon f(x, u, \nabla v_\varepsilon) dx &= \lim_{\varepsilon \to 0} \int_{\Omega'} f(x, u, \nabla v_\varepsilon) dx \\
&\geq F[f](u, \Omega') = F[f](u, \Omega),
\end{aligned}$$

and, by (x) and (2.5),

$$\begin{aligned}
& \limsup_{\varepsilon \to 0} \sup_{k \in \mathbb{N}} \left| \int_{\Omega'} \beta_k^\varepsilon \nabla_\xi f(x, u, \nabla v_\varepsilon) \cdot (\nabla u - \nabla v_\varepsilon) dx \right| \\
\leq \; & M \lim_{\varepsilon \to 0} \int_{\Omega'} |\nabla u - \nabla v_\varepsilon| dx = 0,
\end{aligned}$$

so that the lower semicontinuity inequality will follow at once.

Step five. We aim to prove (2.6). Let us define

$$G(x,s) := \beta_k^\varepsilon(x)\nabla_\xi f(x,s,\nabla v_\varepsilon(x))$$

so that

$$\mathcal{I}(h,\varepsilon,k) = \int_{\Omega'} G(x,u_h(x))\cdot\nabla u_h(x) - G(x,u(x))\cdot\nabla u(x)dx.$$

Note that $G:\Omega\times\mathbb{R}\to\mathbb{R}^n$ is uniformly bounded, has compact support in $\Omega\times\mathbb{R}$ with $G(x,\cdot)$ continuous for a.e. $x\in\Omega$. Furthermore there exists a constant L such that, for every $s\in\mathbb{R}$ and $j=1,...,N(\varepsilon)$, we have

$$G(\cdot,s)\in W^{1,1}(\Omega'\cap\Omega_j;\mathbb{R}^n),$$

with

$$\int_{\Omega'\cap\Omega_j}|\nabla_x G(x,s)|\,dx \le L<\infty.$$

We claim that, for every $j=1,...,N(\varepsilon)$,

$$\int_{\Omega'\cap\Omega_j} G(x,u_h(x))\cdot\nabla u_h(x) - G(x,u(x))\cdot\nabla u(x)dx = -\int_{\Omega'\cap\Omega_j}\left\{\int_{u(x)}^{u_h(x)}\operatorname{div}{}_x G(x,s)ds\right\}dx. \tag{2.7}$$

This will be justified in step six. For the moment we assume it and go on with the proof of (2.6). Indeed by (2.7) and by (x) it follows that

$$\begin{aligned}|\mathcal{I}(h,\varepsilon,k)| &\le \sum_{j=1}^{N(\varepsilon)}\int_{\Omega'\cap\Omega_j}\left|\int_{u(x)}^{u_h(x)}|\operatorname{div}{}_x(\beta_k^\varepsilon(x)\nabla_\xi f(x,s,\xi_j))|ds\right|dx\\ &\le \sum_{j=1}^{N(\varepsilon)}\int_{\Omega'\cap\Omega_j}\left|\int_{u(x)}^{u_h(x)}|\operatorname{div}{}_x(\nabla_\xi f(x,s,\xi_j))|ds\right|dx\\ &\quad +M\|\nabla\beta_k^\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u-u_h\|_{L^1(\Omega')}.\end{aligned}$$

Obviously, as $\varepsilon>0$ and $k\in\mathbb{N}$ are fixed, the second term is vanishing when $h\to\infty$. On the other hand, for every $j=1,...,N(\varepsilon)$,

$$\begin{aligned}&\int_{\Omega'\cap\Omega_j}\left|\int_{u(x)}^{u_h(x)}|\operatorname{div}{}_x(\nabla_\xi f(x,s,\xi_j))|ds\right|dx\\ &\qquad\qquad = \int_{D_{j,h}}|\operatorname{div}{}_x(\nabla_\xi f(x,s,\xi_j))|dxds,\end{aligned}$$

where

$$D_{j,h} := \{(x,s) \in (\Omega' \cap \Omega_j) \times H : \min\{u(x), u_h(x)\} \le s \le \max\{u(x), u_h(x)\}\}.$$

Since $|D_{j,h}| \le \|u_h - u\|_{L^1(\Omega')} \to 0$, $K = \{\xi_j\}_{j=1}^{N(\varepsilon)}$ is bounded and, of course, $|\mathrm{div}\ | \le n|\nabla|$, the thesis follows by (ix).

Step six: To conclude the proof of the theorem we just need to show the validity of (2.7), i.e. we need to prove that if $G \in L^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n)$ has compact support in $\Omega \times \mathbb{R}$, with $G(x,\cdot)$ continuous for a.e. $x \in \Omega$, with $G(\cdot, s) \in W^{1,1}(\Omega; \mathbb{R}^n)$ and

$$\int_\Omega |\nabla_x G(x,s)|\, dx \le L < \infty, \tag{2.8}$$

for every $s \in \mathbb{R}$, then

$$\int_\Omega G(x,v(x))\cdot\nabla v(x) - G(x,u(x))\cdot\nabla u(x) dx = -\int_\Omega \left\{ \int_{u(x)}^{v(x)} \mathrm{div}\ _x G(x,s) ds \right\} dx, \tag{2.9}$$

whenever $u, v \in W^{1,1}_{\mathrm{loc}}(\Omega)$. To this end, let $\sigma > 0$, $\kappa \in \mathcal{D}(B^n; [0,\infty))$ with $\int \kappa = 1$, define $\kappa_\sigma(x) = \sigma^{-n}\kappa(x/\sigma)$ and consider

$$G_\sigma(x,s) := \int G(y,s)\kappa_\sigma(x-y) dy,$$

for every $(x,s) \in \Omega \times \mathbb{R}$. By the chain rule for the composition of a Lipschitz function with a Sobolev function we find that for every $u, v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $\sigma > 0$

$$\int_\Omega G_\sigma(x,v(x))\cdot\nabla v(x) - G_\sigma(x,u(x))\cdot\nabla u(x) dx = -\int_\Omega \left\{ \int_{u(x)}^{v(x)} \mathrm{div}\ _x G_\sigma(x,s) ds \right\} dx. \tag{2.10}$$

We intend to recover (2.9) by passing to the limit as $\sigma \to 0$. This is reasonable but not completely obvious. The first thing we have to prove is that

$$\lim_{\sigma \to 0} G_\sigma(x, v(x)) = G(x, v(x)) \text{ for a.e. } x \in \Omega\,; \tag{2.11}$$

note that if (2.11) holds true then we can pass to the limit in the left hand side of (2.10) by dominated convergence since G is bounded. We know that

$$\lim_{\sigma \to 0} G_\sigma(x,s) = G(x,s), \quad \text{for every } x \in \Omega[s], \tag{2.12}$$

where $\Omega[s]$ is that set of all the $x \in \Omega$ such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x,r)} |G(x,s) - G(y,s)| dy = 0.$$

By Lebesgue density theorem, $|\Omega \setminus \Omega[s]| = 0$ for every $s \in \mathbb{R}$. The problem is that for having (2.11) we need to know the validity of (2.12) for $x \in \Omega_0$ with $|\Omega \setminus \Omega_0| = 0$ and the *same* Ω_0 *for every* $s \in \mathbb{R}$. This can be done as follows. By Scorza-Dragoni Theorem (a variant of the classical Lusin theorem, see [40]) we find a sequence of disjoint compact set K_i such that $|\Omega \setminus (\bigcup_{i=1}^\infty K_i)| = 0$ and G is uniformly continuous on each $K_i \times \mathbb{R}$ (recall that G has compact support also in s). Then we consider

$$C = \bigcup_{i=1}^\infty C_i, \quad C_i := \left\{ x \in K_i : \lim_{r\to 0} \frac{|K_i \cap B(x,r)|}{r^n |B^n|} = 1 \right\},$$

By Lebesgue density theorem $|K_i \setminus C_i| = 0$ and so $|\Omega \setminus C| = 0$. We finish the proof by showing that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x,r)} |G(x,s) - G(y,s)| dy = 0,$$

for every $(x,s) \in C \times \mathbb{R}$. Indeed, assume that $x \in C_i$ and that λ_i is the modulus of uniform continuity for G in $K_i \times \mathbb{R}$. Then

$$\begin{aligned} \frac{1}{r^n} \int_{B(x,r)} |G(x,s) - G(y,s)| dy &\le \frac{1}{r^n} \int_{B(x,r)\cap K_i} |G(x,s) - G(y,s)| dy \\ &+ \frac{1}{r^n} \int_{B(x,r)\setminus K_i} |G(x,s) - G(y,s)| dy \\ &\le \frac{|B(x,r)\cap K_i|}{r^n} \lambda(r) + 2M \frac{|B(x,r)\setminus K_i|}{r^n}, \end{aligned}$$

since, as $x \in C_i$, we have that $|B(x,r) \setminus K_i| r^{-n} \to 0$ when $r \to 0$.

It remains to show that

$$\lim_{\sigma\to 0} \int_\Omega \left\{ \int_{v(x)}^{w(x)} \operatorname{div}{}_x G_\sigma(x,s) ds \right\} dx = \int_\Omega \left\{ \int_{v(x)}^{w(x)} \operatorname{div}{}_x G(x,s) ds \right\} dx.$$

To this end let us note that for every $i = 1, ..., n$,

$$\int_\Omega \left| \int_{v(x)}^{w(x)} \frac{\partial G^i_\sigma}{\partial x_i}(x,s) - \frac{\partial G^i}{\partial x_i}(x,s) ds \right| dx \leq \int_H ds \int_\Omega \left| \frac{\partial G^i_\sigma}{\partial x_i}(x,s) - \frac{\partial G^i}{\partial x_i}(x,s) \right| dx.$$

By standard properties of convolutions, for fixed $s \in \mathbb{R}$,

$$t(\sigma, s) = \int_\Omega \left| \frac{\partial G^i_\sigma}{\partial x_i}(x,s) - \frac{\partial G^i}{\partial x_i}(x,s) \right| dx \to 0,$$

as $\sigma \to 0$. By (2.8) we can apply the dominated convergence theorem and prove our assertion. This remark concludes the proof of (2.9) and of the theorem. □

2.3 A counterexample to lower semicontinuity

Theorem 20 (Gori, Maggi and Marcellini [46]). *For every $\alpha \in (0,1)$ there exists a continuous integrand $f : (-1,1) \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ and a sequence $u_h \in W^{1,\infty}(-1,1)$ uniformly converging to $u \equiv 0$ such that*

$$\int_{-1}^{1} f(x,u,u')dx > \liminf_{h\to\infty} \int_{-1}^{1} f(x,u_h,u'_h)dx, \tag{2.13}$$

even if, for every $K \subset\subset \mathbb{R}$, there exists a constant $L = L(K)$ such that

$$|f(x_1,s,\xi) - f(x_2,s,\xi)| \leq L|x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in (-1,1),\ (s,\xi) \in \mathbb{R} \times K. \tag{2.14}$$

Proof. Outline of the proof: The integrand f will have the form:

$$f(x,s,\xi) = |a(x,s)\xi - 1|, \quad (x,s,\xi) \in (-1,1) \times \mathbb{R} \times \mathbb{R},$$

for a continuous $a : (-1,1) \times \mathbb{R} \to \mathbb{R}$. By

$$|f(x_1,s,\xi) - f(x_2,s,\xi)| \leq |a(x_1,s) - a(x_2,s)||\xi|,$$

(2.14) will follow provided there exists a constant $C \in (0, \infty)$ such that

$$|a(x_1, s) - a(x_2, s)| \leq C|x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in (-1,1), s \in \mathbb{R}. \tag{2.15}$$

The construction of a goes as follows. Assume we have a sequence of functions $u_h \in W^{1,\infty}(-1,1)$ uniformly converging to zero and such that the graphs of the u_h's, call them

$$\Gamma_h := \{(x, u_h(x)) : x \in (-1,1)\},$$

are contained in disjoint boxes R_h, that is

$$\Gamma_h \subset R_h := (-1,1) \times (A_h, B_h), \quad R_h \cap R_k = \emptyset \;\; \forall h \neq k. \tag{2.16}$$

We consider the decomposition of $(-1,1)$ into the two parts $\{x \in (-1,1) : u_h'(x) = 0\} =: \{u_h' = 0\}$ and $\{x \in (-1,1) : u_h'(x) \neq 0\} =: \{u_h' \neq 0\}$, and correspondingly we consider

$$\Gamma_h^* := \{(x, u_h(x)) \in \Gamma_h : x \in \{u_h' \neq 0\}\} \subset \Gamma_h.$$

By (2.16) it makes sense to define

$$a(x,s) := \frac{1}{u_h'(x)} \quad \forall (x,s) \in \Gamma_h^*. \tag{2.17}$$

Then we have

$$F[f](u_h, (-1,1)) = |\{u_h' = 0\}|$$

and (2.13) follows provided that

$$\lim_{h \to \infty} |\{u_h' = 0\}| < 2.$$

The proof of the theorem will be then completed if we show that a can be extended, starting from the definition given in (2.17), to a continuous function on all of $(-1,1) \times \mathbb{R}$ that obeys (2.15).

Step one: We start defining our sequence of functions $\{u_h\}_{h\in\mathbb{N}}$. Let four sequences of real numbers a_h, b_h, c_h, d_h be given. We define $u_h : (-1,1) \to \mathbb{R}$ by

$$u_h(x) := a_h + \frac{1}{b_h} v_h(c_h x), \quad x \in (-1,1)$$

where $v_h : \mathbb{R} \to \mathbb{R}$ is defined by

$$v_h(x) := \begin{cases} 0 & \text{if } x \in (0, d_h), \\ x - d_h & \text{if } x \in (d_h, 1 - d_h), \\ 1 - 2d_h & \text{if } x \in (1 - d_h, 1), \end{cases}$$

if $x \in (0,1)$, by $v_h(x) = v_h(-x)$ if $x \in (-1,0)$ and by periodicity for a generic $x \in \mathbb{R}$. On the four sequences a_h, b_h, c_h and d_h we shall assume the following:

$$a_h \to 0 \text{ as } h \to \infty; \tag{2.18}$$

$$b_h \to \infty \text{ as } h \to \infty,\ \ b_h \in \mathbb{N}; \tag{2.19}$$

$$c_h \to \infty \text{ as } h \to \infty,\ \ c_h \in \mathbb{N}; \tag{2.20}$$

$$d_h \to 0 \text{ as } h \to \infty,\ \ d_h \in (0, 1/2). \tag{2.21}$$

This already suffices to have $u_h \to 0$ uniformly in $(-1,1)$. Since

$$a_h \le u_h(x) \le a_h + \frac{1 - 2d_h}{b_h} \le a_h + \frac{1}{b_h},$$

we find that (2.16) is satisfied if we define

$$R_h := (-1,1) \times \left(\frac{a_h}{2}, \frac{3}{2}a_h + \frac{1}{b_h} \right) \tag{2.22}$$

and further ask that

$$\frac{3}{2}a_{h+1} + \frac{1}{b_{h+1}} < \frac{a_h}{2}, \quad \forall h \in \mathbb{N}. \tag{2.23}$$

In the end let us remark that

$$\{x \in (-1,1) : v_h'(x) = 0\} = (-1, -1 - d_h) \cup (-d_h, d_h) \cup (1 - d_h, 1)$$

so that, by $c_h \in \mathbb{N}$, we have

$$|\{u_h' = 0\}| = |\{v_h' = 0\}| = 4d_h \to 0.$$

The theorem will be proved if we show that we can find four sequences as in (2.18), (2.19), (2.20), (2.21) and (2.23) such that the function a defined as in (2.17) on $\bigcup_{h=1}^{\infty} \Gamma_h^*$ can be extended to a continuous function on all of $(-1,1) \times \mathbb{R}$ satisfying (2.15).

Step two: In this step we discuss the problem of the extension of a. For the moment, by (2.17), a is defined on a countable union of disjoint compact sets in $(-1,1)\times\mathbb{R}$, namely the Γ_h^*'s, with

$$a(x,s) := \pm\frac{b_h}{c_h}, \qquad \forall (x,s)\in\Gamma_h^*. \tag{2.24}$$

The first remark consists in estimating the Lipschitz constant of a on each one of the Γ_h^* as follows:

$$|a(x_1,s_1)-a(x_2,s_2)|\le\frac{b_h}{d_h}|x_1-x_2|, \qquad \forall (x_1,s_1),(x_2,s_2)\in\Gamma_h^*. \tag{2.25}$$

Indeed, if (x_1,s_1) and (x_2,s_2) are in the same connected component of Γ_h^* then $a(x_1,s_1)=a(x_2,s_2)$. On the other hand, if they lie in two distinct connected components of Γ_h^*, necessarily $c_h|x_1-x_2|>2d_h$ and by (2.24) it follows

$$|a(x_1,s_1)-a(x_2,s_2)|\le 2\frac{b_h}{c_h}\le\frac{b_h}{d_h}|x_1-x_2|,$$

so that (2.25) is proved.

Now we define

$$a(x,s)=0, \qquad \forall (x,s)\in\partial R_h, \tag{2.26}$$

where R_h is as in (2.22). By (2.24) and by

$$|s_1-s_2| \;\ge\; \frac{a_h}{2}, \qquad \forall (x_1,s_1)\in\partial R_h,(x_2,s_2)\in\Gamma_h^*,$$

we find that, if $(x_1,s_1)\in\partial R_h$ and $(x_2,s_2)\in\Gamma_h^*$,

$$|a(x_1,s_1)-a(x_2,s_2)|\le\frac{b_h}{c_h}\le 2\frac{b_h}{c_h}\frac{|s_1-s_2|}{a_h}\le\frac{b_h}{d_h}|s_1-s_2|$$

provided

$$\frac{2d_h}{c_h}\le a_h, \quad \forall h\in\mathbb{N}. \tag{2.27}$$

In conclusion, by the usual extension theorem for scalar Lipschitz functions, we can consider a to be defined on each R_h in such a way that (2.24), (2.26) and

$$|a(x_1,s_1)-a(x_2,s_2)|\le\frac{b_h}{d_h}(|x_1-x_2|+|s_1-s_2|), \tag{2.28}$$

for every $(x_i, s_i) \in R_h$, $i = 1, 2$. Furthermore, since this Lipschitz extensions not only does not increase the sup norm of the gradient but neither the sup norm of the function, we also have that

$$|a(x,s)| \leq \frac{b_h}{c_h} \quad \forall (x,s) \in R_h. \tag{2.29}$$

Up to now we have defined a on the set $\bigcup_{h=1}^{\infty} R_h$. We put

$$a(x,s) = 0 \quad \forall (x,s) \in ((-1,1)\times\mathbb{R}) \setminus \bigcup_{h=1}^{\infty} R_h.$$

Since each R_h is disjoint from the other, $a \in \text{Lip}\,(R_h)$ and since $a = 0$ on ∂R_h it follows that $a \in C^0((-1,1)\times\mathbb{R})$. Let us test the Hölder continuity of a. We start by proving that

$$|a(x_1,s) - a(x_2,s)| \leq 2\frac{b_h}{(c_h)^{1-\alpha}(d_h)^{\alpha}}|x_1 - x_2|^{\alpha}, \quad \forall (x_1,s),(x_2,s) \in R_h. \tag{2.30}$$

Indeed let us fix $t > 0$ and consider $(x_1,s),(x_2,s) \in R_h$. If $|x_1 - x_2| \geq t$, by (2.29), we have trivially

$$|a(x_1,s) - a(x_2,s)| \leq 2\frac{b_h}{c_h}\frac{|x_1 - x_2|^{\alpha}}{t^{\alpha}}.$$

If $|x_1 - x_2| \leq t$, by (2.28), we find

$$|a(x_1,s) - a(x_2,s)| \leq \frac{b_h}{d_h}|x_1 - x_2| \leq \frac{b_h t^{1-\alpha}}{d_h}|x_1 - x_2|^{\alpha}$$

and then (2.30) follows since

$$\min_{t>0}\left\{\max\left\{\frac{1}{c_h t^{\alpha}}, \frac{t^{1-\alpha}}{d_h}\right\}\right\} = \frac{1}{(c_h)^{1-\alpha}(d_h)^{\alpha}}.$$

If we assume that for some constant C

$$\frac{b_h}{(c_h)^{1-\alpha}(d_h)^{\alpha}} \leq C < \infty, \quad \forall h \in \mathbb{N}, \tag{2.31}$$

then, by (2.30), a is α Hölder continuous on each R_h with the same constant independent of h, and thus (2.15) is satisfied.

Step three: To conclude the proof of the theorem it suffices to show that there exist four sequences a_h, b_h, c_h and d_h satisfying (2.18), (2.19), (2.20), (2.21), (2.23), (2.27) and (2.31).

Let us fix $\alpha \in (0,1)$ and select $k \in \mathbb{N}$, $k > (1-\alpha)^{-1}$. We define

$$b_h := h, \quad c_h := h^k.$$

Note that then (2.19) and (2.20) (since $k > 1$) are valid. Next we define

$$d_h := \frac{b_h^{1/\alpha}}{c_h^{1/\alpha - 1}} = \left(\frac{h}{h^{k(1-\alpha)}}\right)^{1/\alpha}$$

so that (2.31) holds with $C = 1$ and (2.21) holds since $k(1-\alpha) > 1$. By defining

$$a_h := \frac{2d_h}{c_h} = 2\left(\frac{b_h}{c_h}\right)^{1/\alpha} = \frac{1}{h^{(k-1)/\alpha}},$$

we also have that (2.27) is satisfied with equality and that (2.18) is true as $k > 1$. In conclusion it remains to check if

$$\frac{3}{2}a_{h+1} + \frac{1}{b_{h+1}} < \frac{a_h}{2}, \quad \forall h \in N,$$

and of course this can always be assured up to extracting a subsequence. $\square$

Chapter 3

Lower semicontinuity and geometric conditions

In section 3.1 we prove the main result from [44], namely the approximation theorem for demicoercive functions by means of strictly convex functions that is used to unify the conditions A and B in Serrin's Theorem. In doing this we pass from an approximation theorem for demicoercive functions by means of "maximal cones", see Theorem 22 below, that will be applied again in the next two chapters. For an introduction to the results presented here we refer the reader to subsection 1.1.3.

3.1 Approximation by strictly convex functions

Theorem 21 (Gori and Maggi [44]). *Let $\Sigma \subset \mathbb{R}^d$ be an open set and $f = f(t,\xi) : \Sigma \times \mathbb{R}^n \to [0,\infty)$ be a continuous function such that, for every $t \in \Sigma$, $f(t,\cdot)$ is convex and demicoercive. Then there exists a family of continuous functions $f_k : \Sigma \times \mathbb{R}^n \to [-2,\infty)$ such that*

$$f(t,\xi) = \sup_{k\in\mathbb{N}} f_k(t,\xi) \quad \forall (t,\xi) \in \Sigma \times \mathbb{R}^n \tag{3.1}$$

$$f_k(t,\cdot) \text{ is strictly convex on } \mathbb{R}^n. \tag{3.2}$$

Note that it immediately follows from this theorem that a function is convex and demicoercive on $\mathbb{R}^n$ if and only if it is the countable supremum of strictly convex functions.

In order to explain clearly the proof of this theorem it seems necessary to recall some definitions and simple facts from convex analysis.

Notation for sets: For $K \subset \mathbb{R}^m$ the set cl K is the topological closure of K. If $z_0 \in \mathbb{R}^m$ we define *the positive conic envelope of K with respect to z_0* as

$$\text{pos}\,[K, z_0] := \{z_0 + \lambda(z - z_0) : z \in K, \lambda \geq 0\}.$$

This is just the smallest cone with vertex z_0 containing K. Note that if K is convex then pos $[K, z_0]$ is convex too. In general, however, pos $[K, z_0]$ is not closed even if K is (example: $m = 2$, $z_0 = (0, 0)$, $K = \{z = (x, y) : xy \geq 1, y > 0\}$, then pos $[K, z_0] = \{z : x > 0, y > 0\} \cup \{z_0\}$).

Notation for functions: Let us consider a function $g : \mathbb{R}^m \to \mathbb{R}$. The *epigraph* of g is

$$\text{epi}\,[g] := \{(z, \alpha) \in \mathbb{R}^m \times \mathbb{R} : \alpha \geq g(z)\}.$$

This set is closed if and only if g is lower semicontinuous and is convex in $\mathbb{R}^m \times \mathbb{R}$ if and only if g is a convex function on $\mathbb{R}^m$.

The *subdifferential at $z_0 \in \mathbb{R}^m$* of g is the set

$$\partial g(z_0) := \{w \in \mathbb{R}^m : g(z) \geq g(z_0) + w \cdot (z - z_0) \text{ for every } z \in \mathbb{R}^m\},$$

i.e., it is the set of the slopes w such that the corresponding affine functions taking the value $g(z_0)$ at z_0 lie below g on all $\mathbb{R}^m$. In case g is convex, the subdifferential is a non empty closed convex set at every $z_0 \in \mathbb{R}^m$, and satisfies:

$$\text{if } \begin{cases} z_h \to z_0, \\ w_h \in \partial g(z_h), \end{cases} \quad \text{then } \cap_{k=1}^{\infty} \text{cl } (\cup_{h=k}^{\infty} \{w_h\}) \subset \partial g(z_0). \tag{3.3}$$

The *recession function* of g with respect to $z_0 \in \mathbb{R}^m$ is the function $g_{z_0}^{\infty} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$g_{z_0}^{\infty}(z) := \lim_{\lambda \to 0} \lambda\, g\left(z_0 + \frac{z - z_0}{\lambda}\right).$$

If g is convex then for every z_0 we always find the same recession function (this is consequence of the fact that if K is a closed convex set and it contains a half line, then it contains all the half lines parallel to and with the same versus that the first one, starting from any other of its points) and in turn g^{∞}

is convex too. In this case (since convex functions have linear growth from below) the value $-\infty$ is never attained, the recession function is positively homogeneous of degree one and it takes the value $+\infty$ on one direction if and only if g grows more than linearly at infinity in that direction.

Proof of Theorem 21. We will follow the path outlined in subsection 1.1.3. The proof will be divided in several steps.

Step one: We want to define the "maximal cones" that will approximate f from below and that in turn will be approximated by strictly convex functions. In doing this we don't need to consider the t variable and so we drop this dependency for the moment. So we are considering $f : \mathbb{R}^n \to [0, \infty)$ convex and such that for every $\nu \in S^{n-1}$ the function of one real variable

$$f_\nu(\rho) := f(\rho\nu) \text{ is not constant on } \rho \in \mathbb{R}.$$

Let us fix a point $\xi_0 \in \mathbb{R}^n$ and consider, in $\mathbb{R}^{n+1}$, the positive conic envelope of epi $[f]$ with respect to $(\xi_0, -1)$. We take this set to be the epigraph of the "maximal cone" related to f and generated by ξ_0. So we define a function $K[f, \xi_0] : \mathbb{R}^n \to \mathbb{R}$ by

$$\text{epi } [K[f, \xi_0]] = \text{cl } \text{ pos } [\text{epi } [f], (\xi_0, -1)],$$

or equivalently

$$K[f, \xi_0](\xi) = \inf\{\alpha \in \mathbb{R} : (\xi, \alpha) \in \text{pos } [\text{epi } [f], (\xi_0, -1)]\}.$$

We start by claiming that:

i) $K[f, \xi_0]$ is convex, $K[f, \xi_0] \leq f$ and

$$\xi \mapsto 1 + K[f, \xi_0](\xi + \xi_0) \text{ is positively homogeneous of degree one;}$$

ii) $K[f, \xi_0]$ is the greatest function that satisfies i);

Since the epigraph of $K[f, \xi_0]$ is the closure of a positive conic envelope of the convex set epi $[f]$, it follows that it is a convex set in $\mathbb{R}^{n+1}$ and so $K[f, \xi_0]$ is a convex function. Also, by definition of positive conic envelope and of epigraph, we know that $(\xi,\alpha) \in \text{pos}\,[\text{epi}\,[f],(\xi_0,-1)]$ if and only if there exist $\lambda \geq 0$ and $\zeta \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ such that $\beta \geq f(\zeta)$ and

$$(\xi,\alpha) = (\xi_0 + \lambda(\zeta - \xi_0), -1 + \lambda(\beta+1)).$$

Thus $1 + K[f,\xi_0](\xi)$ equals

$$\inf\{\lambda(f(\zeta)+1) : \exists(\zeta,\lambda) \in \mathbb{R}^n \times [0,\infty) \text{ such that } \xi = \xi_0 + \lambda(\zeta - \xi_0)\}$$

and thus it is easily seen (one just have to distinguish the case $\xi = \xi_0$ and $\xi \neq \xi_0$) that

$$K[f,\xi_0](\xi) = -1 + \inf\left\{\lambda + \lambda f\left(\xi_0 + \frac{\xi - \xi_0}{\lambda}\right) : \lambda > 0\right\}. \tag{3.4}$$

By taking $\lambda = 1$ in the minimization problem we see that $K[f,\xi_0] \leq f$. On the other hand, for $\mu > 0$, we have

$$\begin{aligned}
1 + K[f,\xi_0](\mu\xi + \xi_0) &= \inf\left\{\lambda + \lambda f\left(\xi_0 + \frac{\mu\xi}{\lambda}\right) : \lambda > 0\right\} \\
&= \mu \inf\left\{\frac{\lambda}{\mu} + \frac{\lambda}{\mu} f\left(\xi_0 + \frac{\xi}{\lambda/\mu}\right) : \lambda > 0\right\} \\
&= \mu \inf\left\{\lambda + \lambda f\left(\xi_0 + \frac{\xi}{\lambda}\right) : \lambda > 0\right\} \\
&= \mu(1 + K[f,\xi_0](\xi + \xi_0)),
\end{aligned}$$

so that claim i) is proved.

To prove claim ii) let us remark that, if $g : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous and less or equal to f then $\text{epi}\,[f] \subset \text{epi}\,[g]$ and thus

$$\begin{aligned}
\text{pos}\,[\text{epi}\,[f],(\xi_0,-1)] &\subset \text{pos}\,[\text{epi}\,[g],(\xi_0,-1)] \\
(\text{ if } \xi \mapsto 1 + g(\xi+\xi_0) \text{ is positively homogeneous}) &= \text{epi}\,[g] = \text{cl }\,\text{epi}\,[g].
\end{aligned}$$

In particular, $\text{epi}\,[K[f,\xi_0]] \subset \text{epi}\,[g]$ for every g that satisfies claim i), and thus claim ii) is proved.

Step two: We are going to prove that

iii) $f(\xi) = \sup_{\xi_0 \in \mathbb{R}^n} K[f,\xi_0](\xi)$;

We can reduce to prove (iii) in the case $n = 1$:

$$f_\nu(\rho) = \sup_{\rho_0 \in \mathbb{R}} K[f_\nu, \rho_0](\rho). \tag{3.5}$$

Indeed, by (3.4), we have always

$$\begin{aligned}
&K[f, \rho_0\nu](\rho\nu) \\
= \quad & -1 + \inf\left\{\lambda + \lambda f\left(\rho_0\nu + \frac{\rho\nu - \rho_0\nu}{\lambda}\right) : \lambda > 0\right\} \\
= \quad & -1 + \inf\left\{\lambda + \lambda f_\nu\left(\rho_0 + \frac{\rho - \rho_0}{\lambda}\right) : \lambda > 0\right\} = K[f_\nu, \rho_0](\rho),
\end{aligned}$$

and thus, if (3.5) holds true, we conclude

$$\begin{aligned}
f(\rho\nu) = f_\nu(\rho) \quad &= \quad \sup_{\rho_0 \in \mathbb{R}} K[f_\nu, \rho_0](\rho) = \sup_{\rho_0 \in \mathbb{R}} K[f, \rho_0\nu](\rho\nu) \\
&\leq \quad \sup_{\xi_0 \in \mathbb{R}^n} K[f, \xi_0](\rho\nu) = f(\rho\nu).
\end{aligned}$$

We prove (3.5). Let $n = 1$ and consider the set of all the slopes $v \in \mathbb{R}$ such that the corresponding affine function that takes the value -1 at $\xi_0 \in \mathbb{R}$ lies below f:

$$\Lambda[f, \xi_0] := \{v \in \mathbb{R} : v(\xi - \xi_0) - 1 \leq f(\xi) \quad \forall \xi \in \mathbb{R}\}.$$

A non zero slope v is in the subdifferential of f at ξ_0 if and only if it belongs to $\Lambda[f, \xi^*(\xi_0, v)]$ where $\xi^*(\xi_0, v) = \xi_0 - v^{-1}(1 + f(\xi_0))$, since, when $v \neq 0$ so that $\xi^*(\xi_0, v)$ is defined, we have

$$v(\xi - \xi^*(\xi_0, v)) - 1 = v(\xi - \xi_0) + f(\xi_0). \tag{3.6}$$

We are led to consider

$$\beta[f, \xi_0] := \sup \Lambda[f, \xi_0], \quad \gamma[f, \xi_0] := \inf \Lambda[f, \xi_0].$$

Then

$$K[f, \xi_0](\xi) = \max\{\beta[f, \xi_0](\xi - \xi_0) - 1, \gamma[f, \xi_0](\xi - \xi_0) - 1\}. \tag{3.7}$$

Indeed, the right hand side of this equation is, by construction, the greatest function g that lies below f and is such that $\xi \mapsto (1 + g(\xi + \xi_0)$ is positively homogeneous of degree one: by property ii) of $K[f,\xi_0]$, (3.7) follows. Note also that, trivially, $0 \in \Lambda[f,\xi_0]$ for every $\xi_0 \in \mathbb{R}$, so that $\beta[f,\xi_0] \geq 0$ and $\gamma[f,\xi_0] \leq 0$. Furthermore, since f is not constant and convex, it is easy to check that for every $\xi_0 \in \mathbb{R}$ it must always be $\Lambda[f,\xi_0] \setminus \{0\} \neq \emptyset$. These two remarks implies that

$$\beta[f,\xi_0] \geq 0 \geq \gamma[f,\xi_0],$$

with at least one of the two $\geq$ signs that in fact is a $>$. In particular, by (3.7), $K[f,\xi_0]$ is not constant on $\mathbb{R}$, for every $\xi \in \mathbb{R}$ (**Digression:** in passing let us note that, coming back to the case $n \geq 2$ for a moment, by

$$K[f,\rho_0\nu](\rho\nu) = K[f_\nu,\rho_0](\rho),$$

and by property ii) of $K[f,\xi_0]$ it follows

$$K[f,\rho_0\nu](\xi) \geq K[f_\nu,\rho_0](\xi \cdot \nu), \quad \forall(\xi,\nu,\rho_0) \in \mathbb{R}^n \times S^{n-1} \times (0,\infty),$$

so that we have proved, for every n,

iv) for every $\nu \in S^{n-1}$, $(K[f,\xi_0])_\nu(\rho) := K[f,\xi_0](\rho\nu)$ is not constant on $\mathbb{R}$,

or, equivalently, that $K[f,\xi_0]$ is demicoercive. This is of course a necessary property in order to approximate $K[f,\xi_0]$ from below by strictly convex functions.)

Now that we have proved (3.7), we can argue as follows to prove (3.5). Since f is not constant and convex we necessarily in one of the following three cases: *Case one:* $f(\xi) \to \infty$ as $\xi \to \infty$ but not as $\xi \to -\infty$; *Case two:* $f(\xi) \to \infty$ as $\xi \to -\infty$ but not as $\xi \to \infty$; *Case three:* $f(\xi) \to \infty$ as $|\xi| \to \infty$. Case two is equivalent to case one up to considering $f(-\xi)$ in place of $f(\xi)$ while case three follows from the first two since, in that situation, there exists $\xi_m \in \mathbb{R}$ such that $f(\xi_m) = \inf_{\mathbb{R}} f$ and the two functions

$$f_1(\xi) = \begin{cases} f(\xi) & \xi \geq \xi_m \\ f(\xi_m) & \xi \leq \xi_m \end{cases}, \quad f_2(\xi) = \begin{cases} f(\xi_m) & \xi \geq \xi_m \\ f(\xi) & \xi \leq \xi_m \end{cases}$$

are, respectively, in case one and in case two: thus if (3.5) is proved in the first two cases we also have

$$\begin{aligned} f(\xi) = \max\{f_1(\xi), f_2(\xi)\} &= \max\left\{\sup_{\xi_0 \in \mathbb{R}} K[f_1,\xi_0](\xi), \sup_{\xi_0 \in \mathbb{R}} K[f_2,\xi_0](\xi)\right\} \\ (\text{ by } f_i \leq f,) &\leq K[f,\xi_0](\xi) \leq f(\xi). \end{aligned}$$

Thus we have reduced ourselves to prove (3.5) in case one. It must be $\partial f(\xi_0) \subset [0, \infty)$, so that $\gamma[f, \xi_0] = 0$ and we want to prove

$$f(\xi) = \sup_{\xi_0 \in \mathbb{R}} \{\beta[f, \xi_0](\xi - \xi_0)\} - 1, \quad \forall \xi \in \mathbb{R}. \tag{3.8}$$

By (3.6) we see that (3.8) holds for every $\xi \in \mathbb{R}$ such that there is $v \in \partial f(\xi) \setminus \{0\}$, with equality holding with attainment in the supremum for $\xi_0 = \xi^*(\xi, v)$. So it remains to prove (3.8) for

$$\xi \in A := \{\zeta \in \mathbb{R} : \partial f(\zeta) = \{0\}\}.$$

Of course $f(\xi) = \inf_{\mathbb{R}} f$ for all $\xi \in A$ and, by the assumption of being in case one, A has to be bounded from above: that is, $\xi_A = \sup A < \infty$. By continuity of f and by the (outer semi-)continuity property of the subdifferential (3.3) applied to a sequence $\xi_h \in A$, $\xi_h \to \xi_A$ we find that

$$f(\xi_A) = \inf_{\mathbb{R}} f, \quad 0 \in \partial f(\xi_A).$$

Let $\xi \in A$ be fixed. If $\xi_A \in A$, let us consider $\xi_h \to \xi_A$, $\xi_h > \xi_A$ so that if we choose $v_h \in \partial f(\xi_h)$, $v_h > 0$, then it must be $v_h \to 0$. By passing to the limit as $h \to \infty$ in

$$v_h(\xi - \xi_h) + f(\xi_h) \leq \beta[f, \xi^*(\xi_h, v_h)](\xi - \xi_h) - 1 \leq f(\xi) = \inf_{\mathbb{R}} f,$$

we deduce (3.8). If else $\xi_A \in \mathbb{R} \setminus A$ we consider $v_A \in \partial f(\xi_A) \setminus \{0\}$, and necessarily $v_A > 0$ so that $[0, v_A] \subset \partial f(\xi_A)$. Thus $h^{-1} v_A \in \partial f(\xi_A)$ and by

$$\begin{aligned} v_h(\xi - \xi_A) + f(\xi_A) &= v_h(\xi - \xi^*(\xi_A, v_h)) - 1 \\ &\leq \beta[f, \xi^*(\xi_A, v_h)](\xi - \xi^*(\xi_A, v_h)) - 1 \leq f(\xi) = \inf_{\mathbb{R}} f, \end{aligned}$$

we are done by letting $h \to \infty$.

Step three: In this step of the proof of Theorem 21 we state and prove the following intermediate approximation result:

Theorem 22. *Let $\Sigma \subset \mathbb{R}^d$ be an open set and $f = f(t, \xi) : \Sigma \times \mathbb{R}^n \to [0, \infty)$ be a function such that, for all $t \in \Sigma$, $f(t, \cdot)$ is convex and demicoercive. Then the family of functions $K[f, \xi_0] : \Sigma \times \mathbb{R}^n \to [-1, \infty)$, $\xi_0 \in \mathbb{R}^n$, defined by*

$$K[f, \xi_0](t, \xi) = -1 + \inf \left\{ \lambda + \lambda f\left(t, \xi_0 + \frac{\xi - \xi_0}{\lambda}\right) : \lambda > 0 \right\}, \tag{3.9}$$

satisfies the following properties:

i) for all $t \in \Sigma$, $K[f,\xi_0](t,\cdot)$ *is convex,* $K[f,\xi_0] \leq f$ *and*

$$\xi \mapsto 1 + K[f,\xi_0](t,\xi+\xi_0) \text{ is positively homogeneous of degree one;}$$

ii) $K[f,\xi_0]$ *is the greatest function that satisfies i);*

iii) $f(t,\xi) = \sup_{\xi_0 \in \mathbb{R}^n} K[f,\xi_0](t,\xi)$;

iv) for every $t \in \Sigma$, $K[f,\xi_0](t,\cdot)$ *is demicoercive;*

v) if f *is lower semicontinuous (respectively, continuous) on* $\Sigma \times \mathbb{R}^n$*, then* $K[f,\xi_0]$ *is lower semicontinuous (respectively, continuous) on* $\Sigma \times \mathbb{R}^n$*. Furthermore iii) holds with* $\xi_0 \in \mathbb{R}^n$ *replaced by* $\xi_0 \in \mathbb{Q}^n$*.*

The proof of the first four statements follows by the application of steps one and two to $f(t,\cdot)$ for t fixed. The proof of statement v) can go as follows. Since the infimum of a family of upper semicontinuous functions is upper semicontinuous, by (3.9) we just have to prove that if f is lower semicontinuous then so is $K[f,\xi_0]$. To this end, let us assume without loss of generality that $\xi_0 = 0$ and note that

$$K[f,0](t,\xi) = -1 + \inf_{\lambda \in [0,\infty)} G[f](\lambda,t,\xi),$$

where

$$G[f](\lambda,t,\xi) := \begin{cases} \lambda + \lambda f\left(t, \frac{\xi}{\lambda}\right) & \lambda > 0, \\ f^\infty(t,\xi) & \lambda = 0. \end{cases}$$

As shown in the proof of Theorem 3.1 in Dal Maso [18] the function $G[f]$ is lower semicontinuous on $[0,\infty) \times \Sigma \times \mathbb{R}^n$. Furthermore, by

$$\lambda \leq G[f](\lambda,t,\xi) \text{ and } G[f](1,t,\xi) = 1 + f(t,\xi) \tag{3.10}$$

it follows that

$$K[f,0](t,\xi) = -1 + \inf_{\lambda \in [0,1+f(t,\xi)]} G[f](\lambda,t,\xi).$$

By lower semicontinuity of $G[f]$ it follows that for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$ we can choose $\lambda(t,\xi) \in [0, 1+f(t,\xi)]$ such that

$$K[f,0](t,\xi) = -1 + G[f](\lambda(t,\xi),t,\xi) \leq -1 + G[f](\lambda,t,\xi).$$

Let us consider $(t_h, \xi_h) \to (t_1, \xi_1)$ and denote for brevity

$$\lambda_h := \lambda(t_h, \xi_h), \quad \lambda_1 := \lambda(t_1, \xi_1).$$

If $\lambda_h \to \infty$, by the first relation in (3.10), the lower semicontinuity of $K[f, 0]$ is trivially verified. If else λ_h is bounded we can extract a subsequence such that

$$\lambda_h \to \lambda^* \in [0, \infty), \quad \liminf_{h\to\infty} G[f](\lambda_h, t_h, \xi_h) = \lim_{h\to\infty} G[f](\lambda_h, t_h, \xi_h),$$

and deduce, by the second relation in (3.10) and by the lower semicontinuity of $G[f]$, that

$$\begin{aligned} K[f, 0](t_1, \xi_1) &= -1 + G[f](\lambda_1, t_1, \xi_1) \leq -1 + G[f](\lambda^*, t_1, \xi_1) \\ &\leq -1 + \lim_{h\to\infty} G[f](\lambda_h, t_h, \xi_h) \leq \lim_{h\to\infty} K[f, 0](t_h, \xi_h). \end{aligned}$$

The second part of statement v) follows by Lemma 3.

Step four: By Theorem 22, in order to conclude the proof of Theorem 21, we can further suppose that f satisfies

$$f(t, \cdot) \text{ is positively homogeneous of degree one on } \mathbb{R}^n$$

for all $t \in \Sigma$. We want to prove the existence of a family of continuous functions $f_m : \Sigma \times \mathbb{R}^n \to [-1, \infty)$ such that

$$f(t, \xi) = \sup_{m\in\mathbb{N}} f_m(t, \xi) \quad \forall (t, \xi) \in \Sigma \times \mathbb{R}^n \tag{3.11}$$

$$f_m(t, \cdot) \text{ is strictly convex on } \mathbb{R}^n. \tag{3.12}$$

For $n = 1$ this is trivial. Indeed, in this case, we can write $f(t, \xi) = \max\{a(t)\xi, b(t)\xi\}$ for $a(t) \leq 0 \leq b(t)$, $a(t) < b(t)$ and it suffices to define

$$f_\delta(t, \xi) := f(t, \xi) - \delta + \delta \exp\left\{ -\frac{b(t) - a(t)}{2\delta} |\xi| \right\}.$$

In the general case case the geometry behind the problem seems much more complicated and we to achieve the proof we have been led to consider the following long construction.

We start by proving that there exists a continuous function $N : \Sigma \to \mathbb{R}^{n+1}$, such that, for every $t \in \Sigma$,

$$\text{epi }(f(t,\cdot))\backslash\{0\} \subset \{M \in \mathbb{R}^{n+1} : M \cdot N(t) > 0\}. \tag{3.13}$$

To this end let us define the sets $K(t) = \text{epi }(f(t,\cdot)) \cap B^{n+1}$ and $S(t) = \text{epi }(f(t,\cdot)) \cap \{(\xi,\alpha) \in \mathbb{R}^{n+1} : |\xi| \le 1, \alpha \ge 0\}$: clearly we have $S(t) \cap B^{n+1} = K(t)$ by which it's easy to compute that, for every $t_1, t_2 \in \Sigma$, $K(t_1)\triangle K(t_2) = (S(t_1)\triangle S(t_2)) \cap B^{n+1}$. In particular, $K(t_1)\triangle K(t_2) \subseteq S(t_1)\triangle S(t_2)$.

Let us now define a vector field $N(t) : \Sigma \to \mathbb{R}^{n+1}$ by

$$N(t) := \int_{K(t)} N dN.$$

Note that $N(t)$ is continuous on Σ. Indeed, whenever $t_h, t_0 \in \Sigma$, $t_h \to t_0$, we find

$$\begin{aligned} |N(t_h) - N(t_0)| &\le \mathcal{L}^{n+1}(K(t_h)\triangle K(t_0)) \le \mathcal{L}^{n+1}(S(t_h)\triangle S(t_0)) \\ &= \int_{B^n} |f(t_h,\xi) - f(t_0,\xi)| d\xi \to 0, \end{aligned}$$

as h tends to infinity.

Let us prove the inclusion (3.13). For every $M \in S^n$ let us ϕ_M be the reflection of $\mathbb{R}^{n+1}$ with respect to the hyperplane $M^\perp$, i.e.,

$$\phi_M(N) = N - 2(N \cdot M)M, \quad \forall N \in \mathbb{R}^{n+1}.$$

The existence of $M \in S^n \cap K(t) \cap N(t)^\perp$ would imply

$$0 = \int_{K(t)} (N \cdot M) dN = \int_{\mathbb{R}^{n+1}} \chi_{K(t)}(N)(N \cdot M) dN.$$

If now $H_M^+ := \{N : (N \cdot M) > 0\}$ and $H_M^- := \{N : (N \cdot M) < 0\}$, it would follow

$$0 = \int_{H_M^+} \chi_{K(t)}(N)(N \cdot M) dN + \int_{H_M^-} \chi_{K(t)}(N)(N \cdot M) dN.$$

By applying the change of variables $N = \phi_M(L)$ to the second integral (note that $\phi_M(H_M^+) = H_M^-$, and that ϕ_M preserves $(n+1)$-dimensional volumes, so that its Jacobian is equal to 1) we would find

$$\int_{H_M^-} \chi_{K(t)}(N)(N \cdot M) dN = \int_{H_M^+} \chi_{K(t)}(\phi_M(L))(\phi_M(L) \cdot M) dL.$$

Since $(\phi_M(L)\cdot M) = -(L\cdot M)$ for all $L\in\mathbb{R}^{n+1}$ we conclude that

$$0 = \int_{H_M^+} \left\{\chi_{K(t)}(N) - \chi_{K(t)}(\phi_M(N))\right\}(N\cdot M)dN. \tag{3.14}$$

Let us remark that,

$$\chi_{K(t)}(N) - \chi_{K(t)}(\phi_M(N)) \geq 0, \quad \forall N\in H_M^+. \tag{3.15}$$

Indeed this is trivial if $N\in K(t)$ or $|N|>1$ (note that ϕ_M is norm preserving); if $N\in(H_M^+\cap B^{n+1})\backslash K(t)$ then $\phi_M(N)\in\mathbb{R}^{n+1}\backslash K(t)$, since, if otherwise $\phi_M(N)\in K(t)$, from $M\in K(t)$ and $N=2(N\cdot M)M+\phi_M(N)$, it would be $N\in \text{epi}\,[f(t,\cdot)]$ (the convex cones are closed under addition with nonnegative coefficients) and then in particular, by $|N|\leq 1$, $N\in K(t)$. From (3.14), (3.15) and the definition of H_M^+, it is necessarily

$$\chi_{K(t)}(N) - \chi_{K(t)}(\phi_M(N)) = 0, \quad \text{for a.e.} \quad N\in H_M^+.$$

Then there exists a sequence $(M_k)_{k\in\mathbb{N}}\subset K(t)$, converging to M and such that $\phi_M(M_k)\in K(t)$. By the continuity of ϕ_M and the closedness of $K(t)$, it is $\phi_M(M) = -M\in K(t)$, which contradicts the assumption that $f(t,\cdot)$ is demicoercive, since $M\in K(t)$. We have proved (3.13).

Since it must be $N(t) = (\nu(t), a(t))$ with $a(t)>0$, inclusion (3.13) implies

$$g(t,\xi) := f(t,\xi) + \frac{\nu(t)\cdot\xi}{a(t)} > 0, \quad \forall\xi\neq 0. \tag{3.16}$$

The function $c(t) := \min\{g(t,v) : v\in S^{n-1}\} > 0$ is continuous on Σ and $c(t)|\xi|\leq g(t,\xi)$.

For every $\delta\in(0,1)$, let us consider

$$h_\delta(t,\xi) := \sqrt{\delta^2 + c(t)^2|\xi|^2} - \delta, \quad (t,\xi)\in\Sigma\times\mathbb{R}^n.$$

Obviously, for every $\delta\in(0,1)$, $h_\delta(t,\xi)\leq c(t)|\xi|$ and $h_\delta(t,\xi)\to c(t)|\xi|$ as $\delta\to 0$. Moreover for every $(t,\xi)\in\Sigma\times\mathbb{R}^n$, $\delta\in(0,1)$,

$$\frac{dh_\delta}{d\delta}(t,\xi) = -1 + \frac{\delta}{\sqrt{\delta^2 + c(t)^2|\xi|^2}} \leq 0,$$

so that it follows $h_\delta(t,\xi)\uparrow c(t)|\xi|$ as $\delta\to 0$. Let us also note that, for $t\in\Sigma$, $\delta\in(0,1)$ fixed, $h_\delta(t,\cdot)$ is strictly convex. To prove this, we consider

$\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$, $\lambda \in (0,1)$; if $|\xi| \neq |\eta|$ then

$$\begin{aligned}\sqrt{\delta^2 + c(t)^2|\lambda\xi + (1-\lambda)\eta|^2} &\leq \sqrt{\delta^2 + c(t)^2(\lambda|\xi| + (1-\lambda)|\eta|)^2} \\ &< \lambda\sqrt{\delta^2 + c(t)^2|\xi|^2} + (1-\lambda)\sqrt{\delta^2 + c(t)^2|\eta|^2};\end{aligned}$$

if $|\xi| = |\eta|$ then $|\lambda\xi + (1-\lambda)\eta| < \lambda|\xi| + (1-\lambda)|\eta|$ and so

$$\begin{aligned}\sqrt{\delta^2 + c(t)^2|\lambda\xi + (1-\lambda)\eta|^2} &< \sqrt{\delta^2 + c(t)^2(\lambda|\xi| + (1-\lambda)|\eta|)^2} \\ &= \lambda\sqrt{\delta^2 + c(t)^2|\xi|^2} + (1-\lambda)\sqrt{\delta^2 + c(t)^2|\eta|^2}.\end{aligned}$$

Now we define, for $\delta \in (0,1)$, $f_\delta : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ as

$$f_\delta(t,\xi) := (1-\delta)g(t,\xi) + \delta h_\delta(t,\xi) - \frac{(\nu(t)\cdot\xi)}{a(t)}, \quad (t,\xi) \in \Sigma \times \mathbb{R}^n.$$

This is a continuous function, strictly convex in the variable ξ (as it is a sum of a convex function and a strictly convex one). Clearly $f_\delta(t,\xi) \to f(t,\xi)$ as $\delta \to 0$. Since $h_\delta(t,\xi) \leq c(t)|\xi| \leq g(t,\xi)$ we have also

$$f_\delta(t,\xi) \leq g(t,\xi) - \frac{(\nu(t)\cdot\xi)}{a(t)} = f(t,\xi).$$

For every $(t,\xi) \in \Sigma \times \mathbb{R}^n$, $\delta \in (0,1)$,

$$\frac{df_\delta}{d\delta}(t,\xi) = -g(t,\xi) + h_\delta(t,\xi) + \delta\frac{dh_\delta}{d\delta}(t,\xi) \leq 0,$$

and so the family $\{f_\delta\}_{\delta\in(0,1)}$ is increasing in δ as $\delta \to 0$. Note however that, in general, f_δ is not necessarily bounded from below, as required in the statement: indeed, defining the open set $\Sigma_+ = \{t \in \Sigma : \nu(t) \neq 0\}$, we have that, for every $t \in \Sigma \setminus \Sigma_+$, surely $f_\delta(t,\cdot) \geq 0$, while, when $t \in \Sigma_+$, $f_\delta(t,\cdot)$ could tend to $-\infty$ as $|\xi| \to \infty$ and $f(t,\xi) = 0$.

To fix this problem is sufficient to prove the existence of a continuous function $g^* : \Sigma \times \mathbb{R}^n \to [-1,\infty)$ such that, for every $t \in \Sigma_+$, $g^*(t,\cdot)$ is strictly convex, for every $t \in \Sigma \setminus \Sigma_+$, $g^*(t,\cdot) = 0$, and $g^* \leq f$. Indeed, if such a g^* exists then we can define

$$f^*_\delta(t,\xi) := \max\{f_\delta(t,\xi), g^*(t,\xi)\}, \quad (t,\xi) \in \Sigma \times \mathbb{R}^n,$$

and note that f^*_δ is continuous as the maximum between two continuous functions, $f \geq f^*_\delta \geq g^* \geq -1$, and for every $t \in \Sigma$, $f^*_\delta(t,\cdot)$ is strictly convex:

indeed, for every $t \in \Sigma_+$, it is the maximum of two strictly convex functions while, for every $t \in \Sigma \setminus \Sigma_+$, $f_\delta^*(t,\cdot) = f_\delta(t,\cdot)$ that is strictly convex. Moreover $f_\delta^*(t,\xi) \to f(t,\xi)$ as $\delta \to 0$ since $f_\delta(t,\xi) \to f(t,\xi)$ as $\delta \to 0$ and $g^* \leq f$.

For what concerns the existence of g^* we can argue as follows. Let us define

$$\varphi(s) := \max\{0, s\} - 1 + \exp\left\{-\frac{|s|}{2}\right\}, \quad \forall s \in \mathbb{R}.$$

This function is strictly convex and of class C^2 on $\mathbb{R}$ and $-1 \leq \varphi(s) \leq \max\{0, s\}$.

For every $t \in \Sigma_+$, we make the following construction. In relation to (3.16), let $b(t) := \left|\frac{\nu(t)}{a(t)}\right| > 0$, $\mu(t) := -\frac{\nu(t)}{a(t)b(t)} \in S^{n-1}$. Thus

$$f(t,\xi) \geq c(t)|\xi| + b(t)(\mu(t)\cdot\xi)$$

and, for $d(t) := \min\{b(t), 1\}\min\{c(t), 1\} > 0$,

$$f(t,\xi) \geq d(t)|\xi| + b(t)(\mu(t)\cdot\xi) \geq d(t)(v\cdot\xi) + b(t)(\mu(t)\cdot\xi),$$

for every $v \in S^{n-1}$ so that

$$f(t,\xi) \geq \max\{0, e(t,v)\cdot\xi\}, \quad \forall v \in S^{n-1},$$

where $e(t,v) := d(t)v + b(t)\mu(t)$. Then we have

$$\begin{aligned} f(t,\xi) &\geq \frac{1}{\sigma_{n-1}}\int_{S^{n-1}} \max\{0, e(t,v)\cdot\xi\} d\mathcal{H}^{n-1}(v) \\ &\geq \frac{1}{\sigma_{n-1}}\int_{S^{n-1}} \varphi\left(e(t,v)\cdot\xi\right) d\mathcal{H}^{n-1}(v) \\ &=: g^*(t,\xi). \end{aligned}$$

Thus $-1 \leq g^* \leq f$ on $\Sigma_+ \times \mathbb{R}^n$ and, by Lebesgue dominated convergence Theorem, it can be verified that g^* is continuous on $\Sigma_+ \times \mathbb{R}^n$. If now $(t_0, \xi_0) \in (\Sigma \setminus \Sigma_+) \times \mathbb{R}^n$ then $\lim_{t\to t_0} b(t) = \lim_{t\to t_0} d(t) = 0$ (we remark that $a(t)$ is locally uniformly positive on Σ) and so $\lim_{t\to t_0} e(t,v) = 0$ uniformly as $v \in S^{n-1}$. As a consequence $\lim_{(t,\xi)\to(t_0,\xi_0)} g^*(t,\xi) = 0$. Thus, if we define $g^* : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ as

$$g^*(t,\xi) := \begin{cases} \frac{1}{\sigma_{n-1}}\int_{S^{n-1}} \varphi(e(t,v)\cdot\xi) d\mathcal{H}^{n-1}(v), & \text{if } t \in \Sigma_+, \\ 0, & \text{if } t \in \Sigma \setminus \Sigma_+, \end{cases} \tag{3.17}$$

we have that g^* is continuous and $-1 \leq g^* \leq f$ on $\Sigma \times \mathbb{R}^n$. It remains to check that $g^*(t, \cdot)$ is strictly convex for every $t \in \Sigma_+$. By the fact that $\varphi \in C^2(\mathbb{R})$ and by Lebesgue dominated convergence Theorem, we have that $g^*(t, \cdot) \in C^2(\mathbb{R}^n)$ for every $t \in \Sigma_+$, and

$$D^2_\xi g^*(t, \xi) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} \varphi''(e(t, v) \cdot \xi) e(t, v) \otimes e(t, v) d\mathcal{H}^{n-1}(v).$$

Let us fix $t \in \Sigma_+$ and $w \in S^{n-1}$: to achieve the proof it suffices to show that $(D^2_\xi g^*(t, \xi) w) \cdot w > 0$. Since

$$(D^2_\xi g^*(t, \xi) w) \cdot w = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} \varphi''(e(t, v) \cdot \xi)(e(t, v) \cdot w)^2 d\mathcal{H}^{n-1}(v),$$

we only have to prove that it cannot be

$$e(t, v) \cdot w = (d(t) v + b(t) \mu(t)) \cdot w = 0, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } v \in S^{n-1}.$$

If this holds, by continuity of $e(t, \cdot)$, we also have

$$(d(t) v + b(t) \mu(t)) \cdot w = 0, \quad \forall v \in S^{n-1}.$$

But for $v \in w^\perp$ we deduce that $w \in \mu(t)^\perp$ and then, for $v = w$, we get $w = 0 \notin S^{n-1}$. The proof of Theorem 21 is now completed. $\square$

Chapter 4

A relaxation result without coercivity assumptions

In this chapter, see section 4.2, we prove the lower bound $H[f] \leq \mathcal{F}[f]$ for the relaxed functional of $F[f]$ under the assumption that f is demicoercive in the gradient variable [55]. As explained in the introduction we have proved this result in the framework provided by the blow up method due to Fonseca and Müller [33], [34]. In section 4.1 we review the features of this method in relation to Serrin's problem, along the lines depicted in the papers by Fonseca and Leoni [31] and [32]. We warn the reader that we limit ourselves to present a qualitative discussion, referring to the original papers quoted above for the (sometimes considerably involved) technical details. This will be also an occasion to quote some important results by Fonseca and Leoni [31], [32] on Serrin's problem. For an introduction to the problems considered here the reader is referred to subsection 1.1.4.

4.1 The blow up method

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ be a Borel function, so that

$$x \mapsto f(x, u(x), \nabla u(x)) \text{ is measurable in } \Omega, \quad \forall u \in W^{1,1}_{\mathrm{loc}}(\Omega).$$

Let us consider a sequence u_h such that

$$u_h \in W^{1,1}_{\mathrm{loc}}(\Omega),\ u \in BV_{\mathrm{loc}}(\Omega),\ u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega). \tag{4.1}$$

We would like to understand if

$$H[f](u,\Omega) \leq \liminf_{h\to\infty} F[f](u_h,\Omega), \tag{4.2}$$

where $H[f]$ is the parametric extension functional introduced in subsection 1.1.4, see (1.24). As usual, to this end, one can extract a (not relabeled) subsequence such that

$$\liminf_{h\to\infty} F[f](u_h,\Omega) = \lim_{h\to\infty} F[f](u_h,\Omega) < \infty,$$

thus having a non trivial problem and gaining the possibility to extract further subsequences without altering the value of the inferior limit. Under this condition the sequence of non negative Radon measures

$$\mu_h := f(x,u_h(x),\nabla u_h(x))d(\mathcal{L}^n \lfloor \Omega)$$

admits a subsequence converging to some non negative Radon measure μ, and in particular, for every $\varphi \in C_c(\Omega)$ with $0 \leq \varphi \leq 1$, we have

$$\lim_{h\to\infty} F[f](u_h,\Omega) \geq \lim_{h\to\infty} \int_\Omega \varphi d\mu_h = \int_\Omega \varphi d\mu.$$

If we show that

$$\begin{aligned} &\mu \geq f(x,u(x),\nabla u(x))d(\mathcal{L}^n \lfloor \Omega)(x) \\ &+f^\infty\left(x,u(x),\frac{D^c u}{|D^c u|}(x)\right) d|D^c u|(x) \\ &+\frac{1}{u^+(x)-u^-(x)}\left\{\int_{u^-(x)}^{u^+(x)} f^\infty\left(x,s,\nu(x)\right) ds\right\} d(\mathcal{H}^{n-1}\lfloor J(u))(x), \end{aligned} \tag{4.3}$$

(where the $\geq$ is in the sense of Radon measures!) we have the lower semicontinuity theorem by taking the supremum over all the considered φ. The three measures on the right hand side of (4.3) are each one singular to the other: thus we can apply Besicovitch differentiation theorem for Radon measures and isolate the following three inequalities, that if proved, will imply (4.3):

$$\frac{d\mu}{dx}(x) \geq f(x,u(x),\nabla u(x)) \quad \text{for } dx\text{-a.e. } x\in\Omega; \tag{4.4}$$

$$\frac{d\mu}{d|D^c u|}(x) \geq f^\infty\left(x,u(x),\frac{D^c u}{|D^c u|}(x)\right) \quad \text{for } |D^c u|\text{-a.e. } x\in\Omega; \tag{4.5}$$

$$\frac{d\mu}{d\mathcal{H}^{n-1}}(x) \geq \frac{1}{u^+(x)-u^-(x)}\left\{\int_{u^-(x)}^{u^+(x)} f^\infty\left(x,s,\nu(x)\right) ds\right\} d\mathcal{H}^{n-1}(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in J(u). \tag{4.6}$$

Now we show that the validity of each one of these three inequalities is in turn implied by the validity of a special lower semicontinuity inequality. The gain with respect to (4.2) is that these new lower semicontinuity problems will have a sort of "fundamental character" and will allow us to separate the difficulties that are present and mixed together in the starting problem. All these considerations will be true under the only assumptions of non negativity and Borel regularity on f.

We start by looking at (4.4). By Besicovitch differentiation theorem we know that, if ν is a non negative Radon measure on Ω and $|\mu| << \nu$ then μ is recovered by integrating in $d\nu$ the following ν-summable function:

$$\frac{d\mu}{d\nu}(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \quad \text{for } \nu\text{-a.e. } x \in \Omega.$$

Let us fix such an $x \in \Omega$. By elementary properties of weak convergence of Radon measures we know that

$$\mu(B(x,r)) = \lim_{h \to \infty} \mu_h(B(x,r)) \quad \text{whenever } r \text{ is such that } \mu(\partial B(x,r)) = 0.$$

This condition, in the worst case, has to hold for every $r \in (0,\infty) \setminus I(x)$, where $I(x)$ is at most countable: thus we can safely write

$$\frac{d\mu}{d\nu}(x) = \lim_{h \to \infty} \frac{\int_{B(x,r_h)} f(y, u_h(y), \nabla u_h(y))dy}{\nu(B(x,r_h))},$$

where we can select $r_h \to 0$ as we wish in $(0,\infty) \setminus I$.

Let us apply these arguments to $\nu = d(\mathcal{L}^n \lfloor \Omega)$: in order to study the validity (4.4) we need to understand when, for a.e. $x \in \Omega$,

$$\lim_{h \to \infty} \frac{\int_{B(x,r_h)} f(y, u_h(y), \nabla u_h(y))dy}{|B(x,r_h)|} \geq f(x, u(x), \nabla u(x)).$$

The natural thing to do now is to rescale the integrals (from this step the name of "blow up method") by the change of variables $y = x + r_h z$, $z \in B^n$: then we have

$$\begin{aligned} &\frac{\int_{B(x,r_h)} f(y, u_h(y), \nabla u_h(y))dy}{|B(x,r_h)|} \\ &= \frac{1}{|B^n|} \int_{B^n} f(x + r_h z, u_h(x + r_h z), \nabla u_h(x + r_h z))dz. \end{aligned}$$

To make it look even more like a lower semicontinuity problem we rewrite the term in the "dependent variables" s by noticing that $u_h(x+r_h z)$ is converging to $u(x)$ for a.e. $z \in B^n$: that is, we consider the functions $v_h \in W^{1,1}(B^n)$ defined by

$$v_h(z) := \frac{u_h(x + r_h z) - u(x)}{r_h},$$

remark that $\nabla v_h(z) = \nabla u_h(x + r_h z)$ and thus rewrite our integrals as

$$\begin{aligned} &\int_{B^n} f(x + r_h z, u_h(x + r_h z), \nabla u_h(x + r_h z))dz \\ &= \int_{B^n} f(x + r_h z, u(x) + r_h v_h(z), \nabla v_h(z))dz. \end{aligned}$$

This is particular appealing since

$$v_h(z) \to \nabla u(x) \cdot z \quad \text{strongly in } L^1(B^n),$$

if x is selected in such a way that $u_h(x) \to u(x)$ and it is a point of approximate differentiability of u, that is if we assume that

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{r} dz = 0. \tag{4.7}$$

Since these two properties are true for a.e. $x \in \Omega$, see [4], we can safely assume them and since we are interested in proving (4.4) at these points, we can assume that this is the case and state that to prove (4.4) is sufficient to show

$$\liminf_{h\to\infty} \int_{B^n} f(x + r_h z, s + r_h v_h(z), \nabla v_h(z))dz \geq |B^n| f(x, s, \xi),$$

for every[1] $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and whenever $v_h \in W^{1,1}(B^n)$ is such that $v_h \to v$ strongly in $L^1(B^n)$ for $v(z) := \xi \cdot z$.

The story repeats with minor changes in the case of the jump part, (4.6). We want to prove that for a.e. $x \in J(u)$,

$$\begin{aligned} &\lim_{h\to\infty} \frac{\int_{B(x,r_h)} f(y, u_h(y), \nabla u_h(y))dy}{\mathcal{H}^{n-1}(B(x, r_h) \cap J(u))} \\ &\geq \frac{1}{u^+(x) - u^-(x)} \int_{u^-(x)}^{u^+(x)} f^\infty(x, s, \nu_u(x))ds. \end{aligned}$$

[1] Note that we need this property for every possible (x, s, ξ) if we want to have the lower semicontinuity theorem on arbitrary u.

By the rectifiability of the jump set $J(u)$, for $\mathcal{H}^{n-1}$ a.e. $x \in J(u)$ we can replace $\mathcal{H}^{n-1}(B(x,r_h) \cap J(u))$ with $|S^{n-1}|r_h^{n-1}$, and thus, by the change of variables $y = x + r_h z$, $z \in B^n$, deduce that

$$\lim_{h\to\infty} \frac{\int_{B(x,r_h)} f(y, u_h(y), \nabla u_h(y))dy}{\mathcal{H}^{n-1}(B(x,r_h) \cap J(u))}$$
$$= \frac{1}{|S^{n-1}|} \int_{B^n} r_h f(x + r_h z, u_h(x + r_h z), \nabla u_h(x + r_h z))dz.$$

If now $v_h(z) := u_h(x + r_h z)$, as $z \in B^n$, we have

$$\int_{B^n} r_h f(x + r_h z, u_h(x + r_h z), \nabla u_h(x + r_h z))dz$$
$$= \int_{B^n} r_h f\left(x + r_h z, v_h(z), \frac{\nabla v_h(z)}{r_h}\right) dz$$

with

$$v_h(z) \to u^+(x)\{\nu_u(x) \cdot z\}^+ + u^-(x)\{\nu_u(x) \cdot z\}^- \quad \text{strongly in } L^1(B^n).$$

This last fact happens at $\mathcal{H}^{n-1}$ a.e. $x \in J(u) \cap \Omega$ since $u_h \to u$ in $L^1(\Omega)$ and since, by the blow up theorem for BV functions at jump points, see [4], we have:

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\{y:\ \nu_u(x)\cdot(y-x)>0\}} |u(y) - u^+(x)|dy = 0, \tag{4.8}$$
$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\{y:\ \nu_u(x)\cdot(y-x)<0\}} |u(y) - u^-(x)|dy = 0,$$

for $\mathcal{H}^{n-1}$ a.e. $x \in J(u)$. In conclusion we have proved that (4.6) follows from

$$\liminf_{h\to\infty} \int_{B^n} r_h f\left(x + r_h z, v_h(z), \frac{\nabla v_h(z)}{r_h}\right) dz \geq \frac{|S^{n-1}|}{b-a} \int_a^b f^\infty(x, s, \nu)ds,$$

for every $(x, s, \xi) \in \Omega \times \mathbb{R} \times S^{n-1}$, $a < b$, whenever $v_h \in W^{1,1}(B^n)$ is such that $v_h \to v$ strongly in $L^1(B^n)$ for $v(z) := b\{\nu \cdot z\}^+ + a\{\nu \cdot z\}^-$.

The story becomes a little bit more serious when we pass to consider the Cantorian term. The idea, from the general point of view, is just the same, but what really changes is the effort one needs to do in order to prove the blow up result for the Cantor part (due to Alberti [1]) corresponding to (4.7) and to (4.8) in the cases of the absolutely continuous part and of the jump part respectively. We do not insist more on this point and pass directly to state the theorem we are interested in. In the statement we use cubes instead of spheres and denote by Q the unit cube of unitary side length. Since the problem is invariant under rotations, instead of considering generic vectors $\nu \in S^{n-1}$, as we have done in the statement reported above about the jump part, we can limit ourselves to consider $e_n = (0, ..., 1)$. As said, a detailed proof can be found, for example, in [31].

Theorem 23 (The blow up method). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a Borel function. A sufficient condition for the lower semicontinuity inequality*

$$H[f](u, \Omega) \leq \liminf_{h \to \infty} F[f](u_h, \Omega)$$

to hold true whenever

$$u_h \in W^{1,1}_{\text{loc}}(\Omega),\ u \in BV_{\text{loc}}(\Omega),\ u_h \to u \text{ in } L^1_{\text{loc}}(\Omega),$$

is the following:

Lebesgue part: *If*

$$\begin{cases} (x_0, s_0, \xi_0) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \\ \varepsilon_h \to 0^+, \\ u_h \to v, \quad \text{in } L^1(Q), \end{cases} \tag{4.9}$$

where $v(y) := \xi_0 \cdot y$ for every $y \in Q$, then

$$\liminf_{h \to \infty} \int_Q f(x_0 + \varepsilon_h y, s_0 + \varepsilon_h u_h(y), \nabla u_h(y)) dy \geq f(x_0, s_0, \xi_0). \tag{4.10}$$

Jump part: *If*

$$\begin{cases} (x_0, a, b) \in \Omega \times \mathbb{R} \times S^{n-1},\ a < b, \\ \varepsilon_h \to 0^+, \\ u_h \in W^{1,1}(Q),\ u_h \to v, \quad \text{in } L^1(Q), \end{cases} \tag{4.11}$$

where $v(y) := b\{e_n \cdot y\}^+ + a\{e_n \cdot y\}^-$ for every $y \in Q$, then

$$\liminf_{h \to \infty} \int_Q \varepsilon_h f\left(x_0 + \varepsilon_h y, u_h(y), \frac{\nabla u_h(y)}{\varepsilon_h}\right) dy \geq \int_a^b f^\infty(x_0, s, e_n) ds. \tag{4.12}$$

Cantor part: *If*

$$\begin{cases} (x_0, s_0) \in \Omega \times \mathbb{R}, \\ \varepsilon_h \to 0^+,\ s_h \to s_0,\ \lambda_h \to \lambda_0,\ t_h := \lambda_h/\varepsilon_h \to \infty, \\ \alpha : (-1/2, 1/2) \to \mathbb{R}, \ \textit{is non decreasing,} \\ \ \textit{with}\ \alpha((1/2)^-) - \alpha((-1/2)^+) = 1,\ \int_{-1/2}^{1/2} \alpha = 0; \\ u_h \in W^{1,1}(Q),\ u_h \to v \ \textit{in}\ L^1(Q), \end{cases} \tag{4.13}$$

where $v(y) := \alpha(e_n \cdot y)$ *for every* $y \in Q$*, then*

$$\liminf_{h\to\infty} \int_Q \frac{1}{t_h} f(x_0 + \varepsilon_h y, s_h + \lambda_h u_h(y), t_h \nabla u_h(y))dy \geq f^\infty(x_0, s_0, e_n). \tag{4.14}$$

The above argument, albeit based on non elementary considerations on BV functions, is in the end a very natural approach to the study of lower semicontinuity in BV. This impression seems confirmed by the fact that the hypothesis assumed on f are really minimal. The gain is evident, since we have reduced the study of the general lower semicontinuity problem to the one of the three "archetypical" problems corresponding to the canonical decomposition of the distributional derivative of the target function $u \in BV$.

A remarkably natural hypothesis on f (further to the convexity of $f(x, s, \cdot)$ on $\mathbb{R}^n$) under which one can prove (4.10), (4.12) and (4.14) is the following one, introduced in Serrin's problem by Fonseca and Leoni in [31], and already met in the first papers on the blow up method by Fonseca and Müller [33], [34]: for every $(x_0, s_0) \in \Omega \times \mathbb{R}$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x_0, s_0, \xi) \ \leq\ \varepsilon + (1+\varepsilon) f(x, s, \xi), \tag{4.15}$$

for every $(x, s, \xi) \in B((x_0, s_0), \delta) \times \mathbb{R}^n$. Let us consider, for example, the case of the absolutely continuous part, (4.10). We have

$$(1+\varepsilon) \int_Q f(x + r_h z, s + r_h v_h(z), \nabla v_h(z))dz \geq \int_Q f(x, s, \nabla v_h(z))dz - \varepsilon,$$

provided h is big enough and we can truncate v_h so that they are uniformly bounded in L^∞ (and it is always possible to reduce to this case since the v_h's converge in L^1 to a limit v that is in $L^\infty(Q)$). We pass to the limit as $h \to \infty$: if we assume that $f(x, s, \cdot)$ is convex, by Lemma 1 we find

$$\int_Q f(x, s, \nabla v_h(z))dz \geq f(x, s, \xi),$$

so that (4.10) is proved in the limit $\varepsilon \to 0$. If we forget for a moment of $H[f]$ and of BV, and assume that $u \in W^{1,1}$ we have the following theorem related to Serrin's problem:

Theorem 24 (Fonseca and Leoni [31]). *Assume f is Borel integrable, non negative, convex in the gradient variable and such that* (4.15) *holds: then the lower semicontinuity inequality* (1.3) *is satisfied by $F[f]$ for every u_h and u as in* (1.21).

This could be disturbing after all our discussions on the regularity of f in the x variable as a crucial condition for lower semicontinuity! In fact this "paradox" can be explained by means of Lemma 6, asserting that, if f satisfies (4.15) then it is the supremum of a sequence of smooth functions, in particular it is the supremum of functions that satisfy the hypotheses of part C of Serrin's theorem. For completeness let us also quote the following result, of which the previous theorem is just a corollary, and that represents the analogous statement to part one of Dal Maso's theorem where the coercivity assumption (1.25) is replaced by (4.15).

Theorem 25 (Fonseca and Leoni [31]). *Assume f is Borel integrable, non negative, convex in the gradient variable and such that* (4.15) *holds: then*

$$H[f](u,\Omega) \leq \liminf_{h\to\infty} F[f](u_h,\Omega),$$

for every $u_h \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u \in BV_{\mathrm{loc}}(\Omega)$ with $u_h \to u$ in $L^1(\Omega)$. In particular

$$H[f](u,\Omega) \leq \mathcal{F}[f](u,\Omega), \quad \forall u \in BV_{\mathrm{loc}}(\Omega).$$

For further results that one can obtain by applying the blow up method to the study of Serrin's problem we refer the reader to [31].

We conclude by mentioning that the blow up method has been used by Fonseca and Leoni also in connection with another interesting problem related to Serrin's theorem, namely its validity when we pass to consider sequences of vector valued functions $u_h : \Omega \to \mathbb{R}^m$, $m \geq 2$. In [32] it is proved that under condition A Serrin's theorem is still valid in the vector valued setting. This is not the case under condition B, as showed by a counterexample due to Černy and Malý [13], or under condition C, as a consequence of a classic counterexample by Eisen [25] (see also [45]). Thus the "vector-valued counterpart" of Serrin's problem has not the same characteristic structure of the original one. A very convincing reason for this phenomenon can be understood if one tries to see what happens to the chain rule when passing from the scalar to the vector valued case.

4.2 Relaxation and lower semicontinuity of demicoercive functionals

Theorem 26 (Maggi [55]). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ be locally bounded and lower semicontinuous with $f(x, s, \cdot)$ demicoercive and convex for every $(x, s) \in \Omega \times \mathbb{R}$. Then*

$$H[f](u, \Omega) \leq \mathcal{F}[f](u, \Omega), \quad \forall.$$

for every $u_h \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u \in BV_{\mathrm{loc}}(\Omega)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$.

We shall use the following result:

Lemma 4 (Anzellotti, Buttazzo and Dal Maso [5]). *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a demicoercive and convex function and let Ω be a domain with Lipschitz boundary. Then there exists $\alpha > 0$, $\beta, \gamma \geq 0$ such that*

$$H[g](u, \Omega) \geq \alpha |Du|(\Omega) - \beta \int_{\partial\Omega} |u| - \gamma |\Omega|,$$

for every $u \in BV(\Omega)$.

Proof. By demicoercivity there exist $a > 0$, $b \geq 0$ and $\xi_0 \in \mathbb{R}^n$ such that $g(\xi) \geq a|\xi| - b - \xi_0 \cdot \xi$. Then

$$\begin{aligned} H[g](u, \Omega) &\geq a \int_\Omega |\nabla u| dx - b|\Omega| - \xi_0 \cdot \int_\Omega \nabla u dx \\ &\quad + a \int_\Omega d|D^s u| - \xi_0 \cdot \int_\Omega dD^s u \\ &= a|Du|(\Omega) - b|\Omega| - \xi_0 \cdot \int_\Omega dDu. \end{aligned}$$

On the other hand, by the divergence theorem

$$\int_\Omega dDu = \int_{\partial\Omega} u \nu_\Omega d\mathcal{H}^{n-1}.$$

□

Proof of Theorem 26. By Theorem 23 we can reduce ourselves to consider the three archetypical problems (4.10), (4.12) and (4.14).

Step one: We note that in all the three cases the limit function v is in $L^\infty(Q)$, that is there exists $a, b \in \mathbb{R}$ with $a \leq b$ such that $a \leq v(y) \leq b$ for every $y \in Q$. Let us fix $\tau > 0$, and define

$$\begin{aligned} E_h &:= \{y \in Q : a' := a - \tau \leq u_h(y) \leq b' := b + \tau\}, \\ E_h^+ &:= \{y \in Q : u_h(y) > b'\}, \\ E_h^- &:= \{y \in Q : a' < u_h(y)\}. \end{aligned}$$

Then the sequence

$$v_h(y) = \begin{cases} u_h(y), & y \in E_h, \\ b', & y \in E_h^+, \\ a', & y \in E_h^-, \end{cases} \tag{4.16}$$

belongs to $W^{1,1}(Q) \cap L^\infty(Q)$, with

$$|Q \setminus E_h| \leq |\{y \in Q : |u_h(y) - v(y)| > \tau\}| \leq \frac{1}{\tau}\|u_h - v\|_{L^1(Q)} \to 0,$$

and in particular $v_h \to v$ in $L^1(Q)$.

Step two: We also set $\Sigma := \Omega \times \mathbb{R}$, $t := (x, s)$ and consider the maximal cones f_k associated to f according to Theorem 22. Let us recall that the functions f_k are lower semicontinuous, demicoercive and convex in the gradient variable and that, for a suitable sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, we have:

$$f_k(t, \xi) = -1 + \inf\left\{\lambda + \lambda f\left(t, \xi_k + \frac{\xi - \xi_k}{\lambda}\right) : \lambda > 0\right\}. \tag{4.17}$$

Let us note that for every $\Sigma' \subset\subset \Sigma$ there exist constants $C_k \in (0, \infty)$ such that, for every $(t, \xi) \in \Sigma' \times \mathbb{R}^n$ and $\lambda > 0$,

$$-1 \leq f_k(t, \xi) \leq C_k(1 + |\xi|), \tag{4.18}$$

$$\lambda f_k(t, \xi/\lambda) \geq f_k^\infty(t, \xi) - C_k\lambda. \tag{4.19}$$

Indeed, by (4.17) it follows that, for every $(t, \xi) \in \Sigma \times \mathbb{R}^n$ and $\lambda > 0$,

$$\lambda f_k\left(t, \frac{\xi}{\lambda}\right) = 1 - \lambda + f_k(t, \xi + (1 - \lambda)\xi_k).$$

That is, f_k is transformed into its recession function just by a translation of its epigraph along by the vector joining $(\xi_k, -1)$ to the origin $(0, 0)$. Thus

$f_k(t,\cdot)$ and $f_k^\infty(t,\cdot)$ have the same Lipschitz constant $L_k(t)$. Since $f_k(t, cdot)$ is positively homogeneous of degree one

$$L_k(t) = \mathrm{Lip}(f_k^\infty(t,\cdot)) = \mathrm{Lip}(f_k^\infty(t,\cdot); B^n) \leq \max_{2B^n} f_k^\infty(t,\cdot) - \min_{B^n} f_k^\infty(t,\cdot),$$

where the last inequality is consequence of convexity. Being $f_k^\infty \geq 0$ and f_k^∞ locally bounded (since f_k is and f_k^∞ is obtained from f_k by the translation described above). Thus, from this inequality we deduce that $L_k(t) \leq L_k < \infty$ for $t \in \Sigma'$ and (4.18) follows. Furthermore

$$\begin{aligned} \left|\lambda f_k\left(t, \frac{\xi}{\lambda}\right) - f_k^\infty(t,\xi)\right| &= |f_k(t, \xi + (1-\lambda)\xi_k) - f_k(t, \xi+\xi_k) - \lambda| \\ &\leq \lambda(1 + L_k(t)|\xi_k|) \leq C_k \lambda, \end{aligned}$$

as $t \in \Sigma'$ and $\xi \in \mathbb{R}^n$, that is (4.19).

Step three: In the economy of this proof the following lemma plays a fundamental role.

Lemma 5 (Maggi, [55]). *If $f : \Sigma \times \mathbb{R}^n \to [0,\infty)$ is lower semicontinuous with $f(t,\cdot)$ convex, demicoercive and positively homogeneous of degree one on $\mathbb{R}^n$, then for every $t_0 \in \Sigma$ there exist $\delta > 0$ and a convex demicoercive function $\Phi : \mathbb{R}^n \to [0,\infty)$ such that*

$$\Phi(\xi) \leq f(t,\xi), \tag{4.20}$$

for every $(t,\xi) \in B(t_0,\delta) \times \mathbb{R}^n$.

Proof of Lemma 5. The set $F(t) := \mathrm{epi}\,[f(t,\cdot)]$ is a closed convex cone in $\mathbb{R}^{n+1}$ that does not contain straight lines (by demicoercivity of $f(t,\cdot)$). For this reason we can find a half space in $\mathbb{R}^{n+1}$ that intersects $F(t_0)$ only at the origin, that is there exists $\varepsilon > 0$ such that

$$F^\varepsilon(t_0) := \{(\xi,\alpha) \in \mathbb{R}^{n+1} : \mathrm{dist}((\xi,\alpha), F(t_0)) \leq \varepsilon|(\xi,\alpha)|\},$$

does not contain any straight line. We claim that for $\delta > 0$ suitably small

$$\bigcup_{|t-t_0|<\delta} F(t) \subset F^\varepsilon(t_0), \tag{4.21}$$

Then we consider $\Phi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathrm{epi}\,[\Phi] := \overline{co}\left(\bigcup_{|t-t_0|<\delta} F(t)\right).$$

Since epi $[\Phi]$ is closed, convex and, by (4.21), does not contain straight lines, we have that Φ is convex and demicoercive. Furthermore, since epi $[\Phi]$ will contain by construction every $F(t)$ for $t \in B(t_0, \delta)$, automatically (4.20) holds true. In order to conclude the proof of the lemma it remains to prove (4.21). By contradiction, let us assume there exist $t_h \to t_0$ and $p_h := (\xi_h, \alpha_h) \in F(t_h)$ such that

$$\operatorname{dist}(p_h, F(t_0)) > \varepsilon |p_h|. \tag{4.22}$$

Since $F(t_0)$ contains the origin, it must be $p_h \neq 0$ for every h and thus $\pi_h = p_h/|p_h| \to \pi_0 =: (\eta, \beta) \in S^n$, $\pi_h \in F(t_h)$. Being f lower semicontinuous and positively homogeneous of degree one we would find then

$$f(t_0, \eta) \leq \liminf_{h\to\infty} f\left(t_h, \frac{\xi_h}{|p_h|}\right) \leq \liminf_{h\to\infty} \frac{\alpha_h}{|p_h|} = \beta,$$

i.e., $\pi_0 \in F(t_0)$. On the other hand by (4.22)

$$\varepsilon < \operatorname{dist}(\pi_h, F(t_0)) \to \operatorname{dist}(\pi_0, F(t_0)),$$

and thus we have a contradiction. □

Step four: We deal with the Lebesgue part and with the Cantor part, that is, we are going to prove (4.10) and (4.14).

To this end, let us fix $k \in \mathbb{N}$ and put $t_0 := (x_0, s_0)$. We can apply Lemma 5 to $1 + f_k(t, \xi + \xi_k)$ to find $\delta = \delta[t_0, k] > 0$ and a convex, demicoercive function $\Phi_k : \mathbb{R}^n \to [-1, \infty)$ such that

$$\Phi_k(\xi) \leq f_k(t, \xi), \tag{4.23}$$

for every $(t, \xi) \in B(t_0, \delta) \times \mathbb{R}^n$. We claim that for every $\varepsilon, \sigma > 0$, up to considering a smaller δ, we also have

$$f_k(t_0, \xi) + \sigma|\xi| \leq \varepsilon + (1 + \varepsilon)[f_k(t, \xi) + \sigma|\xi|], \tag{4.24}$$

for every $(t, \xi) \in B(t_0, \delta) \times \mathbb{R}^n$. Indeed, by the lower semicontinuity of f_k^∞, we can assume for δ small that

$$f_k^\infty(t_0, \nu) \leq f_k^\infty(t, \nu) + \frac{\varepsilon\sigma}{\min\{|\xi_k|, 1\}},$$

for every $(t, \nu) \in B(t_0, \delta) \times S^{n-1}$. Thus, by positive homogeneity and by $f_k^\infty(t, \xi) = 1 + f_k(t, \xi + \xi_k)$, we find that, for every $(t, \xi) \in B(t_0, \delta) \times \mathbb{R}^n$,

$$f_k^\infty(t_0, \xi) \leq f_k^\infty(t, \xi) + \frac{\varepsilon\sigma}{\min\{|\xi_k|, 1\}}\, |\xi|,$$

and

$$f_k(t_0,\xi) \leq f_k(t,\xi) + \frac{\varepsilon\sigma}{\min\{|\xi_k|,1\}}\ |\xi - \xi_k|.$$

We just have to add to both sides the term $\sigma|\xi|$ to deduce the validity of (4.24).

In corrispondence of this δ for which (4.23) and (4.24) hold true, we choose $\Sigma' = B(t_0, \delta/2)$ and apply to it (4.18) and (4.19). In conclusion:

$$f(x,s,\xi) = \sup_{k\in\mathbb{N}} f_k(x,s,\xi),$$

for every $(x,s,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^n$ and

$$\begin{aligned}
-1 \leq f_k(x,s,\xi) &\leq C_k(1+|\xi|),\\
\lambda f_k\left(x,s,\frac{\xi}{\lambda}\right) &\geq f_k^\infty(x,s,\xi) - C_k\lambda,\\
\Phi_k(\xi) &\leq f_k(x,s,\xi),\\
f_k(x_0,s_0,\xi) + \sigma|\xi| &\leq \varepsilon + (1+\varepsilon)[f_k(x,s,\xi)+\sigma|\xi|],
\end{aligned}$$

for every $(x,s,\xi) \in B((x_0,s_0),\delta/2)\times\mathbb{R}^n$.

We are in the position to prove (4.10). Indeed, if we take h sufficiently large, by definition of v_h and the fact that they are bounded, and by using the above stated properties of the f_k's we deduce that

$$\begin{aligned}
&\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy\\
\geq\ &\int_{E_h} f(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h)dy\\
\geq\ &\int_{E_h} f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h)dy\\
\geq\ &\int_Q f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h)dy - C_k|Q\setminus E_h| \qquad (4.25)\\
\geq\ &\int_Q \Phi_k(\nabla v_h)dy - C_k|Q\setminus E_h|.
\end{aligned}$$

By Lemma 4 we deduce

$$\begin{aligned}
&\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy\\
\geq\ &\alpha_k\int_Q|\nabla v_h|dy - \beta_k\int_{\partial Q}|v_h|d\mathcal{H}^{n-1} - \gamma_k,
\end{aligned}$$

where the left hand side is bounded on h and where v_h are uniformly bounded on ∂Q by $\max\{|a'|,|b'|\}$. Since we can always assume that

$$\begin{aligned} &\liminf_{h\to\infty}\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy \\ = &\lim_{h\to\infty}\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy < \infty, \end{aligned}$$

it follows that the uniform bound

$$\sup_{h\in\mathbb{N}}\|\nabla v_h\|_{L^1(Q;\mathbb{R}^n)} \le C < \infty, \tag{4.26}$$

holds true. The proof can now be concluded by using the Goffman-Serrin's theorem, for if we define

$$l_k(x,s,\xi) = l_k(\xi) := f_k(x_0,s_0,\xi) + \sigma|\xi|,$$

then by (4.25), (4.26) and the properties of the f_k's we have

$$\begin{aligned} &\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy \\ \ge\ &\int_Q f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h) + \sigma|\nabla v_h|dy - C_k|Q\setminus E_h| - \sigma C \\ \ge\ &\frac{1}{1+\varepsilon}\int_Q f_k(x_0,s_0,\nabla v_h) + \sigma|\nabla v_h|dy - \varepsilon - C_k|Q\setminus E_h| - \sigma C \\ =\ &\frac{1}{1+\varepsilon}H[l_k](v_h,Q) - \varepsilon - C_k|Q\setminus E_h| - \sigma C. \end{aligned}$$

and thus, as $h\to\infty$,

$$\begin{aligned} \liminf_{h\to\infty}\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy &\ge \frac{1}{1+\varepsilon}F[l_k](v,Q) - \varepsilon - \sigma C \\ &= \frac{1}{1+\varepsilon}l_k(x_0,s_0,\xi_0) - \varepsilon - \sigma C. \end{aligned}$$

As $\varepsilon,\sigma\to 0$ the last term tends to $f_k(x_0,s_0,\xi_0)$. Since this is true for all $k\in\mathbb{N}$ we have achieved the proof of (4.10).

The proof of (4.14) is similar. Once again, for h sufficiently large, by using the properties of the f_k's we have that

$$\begin{aligned}
&\int_Q \frac{1}{t_h} f(x_0+\varepsilon_h y, s_h+\lambda_h u_h, t_h\nabla u_h)dy \\
\geq\ & \int_{E_h} \frac{1}{t_h} f_k(x_0+\varepsilon_h y, s_h+\lambda_h v_h, t_h\nabla v_h)dy \\
\geq\ & \int_Q \frac{1}{t_h} f_k(x_0+\varepsilon_h y, s_0+\lambda_h v_h, t_h\nabla v_h)dy - \frac{C_k}{t_h}|Q\setminus E_h| \\
\geq\ & \int_Q f_k^\infty(x_0+\varepsilon_h y, s_0+\lambda_h v_h, \nabla v_h)dy - \frac{C_k}{t_h} - \frac{C_k}{t_h}|Q\setminus E_h|. \quad (4.27)
\end{aligned}$$

From $f_k \geq \Phi_k$ it follows $f_k^\infty \geq \Phi_k^\infty$, and hence by combining (4.27) with Lemma 4 we find again that

$$\sup_{h\in\mathbb{N}} \|\nabla v_h\|_{L^1(Q;\mathbb{R}^n)} \leq C < \infty, \tag{4.28}$$

provided we are in the non trivial case

$$\begin{aligned}
&\liminf_{h\to\infty} \int_Q \frac{1}{t_h} f(x_0+\varepsilon_h y, s_h+\lambda_h u_h, t_h\nabla u_h)dy \\
=\ & \lim_{h\to\infty} \int_Q \frac{1}{t_h} f(x_0+\varepsilon_h y, s_h+\lambda_h u_h, t_h\nabla u_h)dy < \infty,
\end{aligned}$$

Let us also note that we have

$$f_k^\infty(x_0,s_0,\xi)+\sigma|\xi| \leq (1+\varepsilon)[f_k^\infty(x,s,\xi)+\sigma|\xi|], \tag{4.29}$$

for every $(x,s,\xi) \in B((x_0,s_0),\delta/2)\times\mathbb{R}^n$. Once again the proof can be concluded by using the Goffman-Serrin's theorem, for if we consider

$$l_k(x,s,\xi) = l_k(\xi) := f_k^\infty(x_0,s_0,\xi)+\sigma|\xi|,$$

then, by (4.27), (4.28) and (4.29) we have

$$\begin{aligned}
&\liminf_{h\to\infty} \int_Q \frac{1}{t_h} f(x_0+\varepsilon_h y, s_h+\lambda_h u_h, t_h\nabla u_h)dy \\
\geq\ & \liminf_{h\to\infty} \int_Q f_k^\infty(x_0+\varepsilon_h y, s_0+\lambda_h v_h, \nabla v_h)+\sigma|\nabla v_h|dy - \sigma C \\
\geq\ & \frac{1}{1+\varepsilon} \liminf_{h\to\infty} H[l_k](v_h,Q) - \sigma C \geq \frac{1}{1+\varepsilon} H[l_k](v,Q) - \sigma C.
\end{aligned}$$

Since

$$\lim_{\sigma\to 0} H[l_k](v,Q) = f_k^\infty(x_0,s_0,e_n),$$

by letting $\varepsilon,\sigma\to 0$ and then taking the supremum on $k\in\mathbb{N}$, as

$$f^\infty = \sup_{k\in\mathbb{N}} f_k^\infty,\ \text{since}\ f = \sup_{k\in\mathbb{N}} f_k,$$

we conclude the proof of (4.14).

Step five. To conclude the proof of the theorem it remains to discuss the case of the jump part, namely to prove (4.12). This is just a little bit more involved than the discussion of the Lebesgue part and of the Cantor part.

Let us fix $\varepsilon,\sigma>0$, $x_0\in\Omega$ and $k\in\mathbb{N}$. For every $s'\in[a',b']$, by Lemma 5 and by the argument at the very beginning of step four, we can find a positive number $\delta[x_0,s',k]=\delta[s']>0$ and a convex, demicoercive function $\Phi_k[s']:\mathbb{R}^n\to[-1,\infty)$ such that

$$\Phi_k[s'](\xi)\le f_k(x,s,\xi)$$

and

$$f_k(x_0,s,\xi)+\sigma|\xi|\le\varepsilon(1+\varepsilon)[f_k(x,s,\xi)+\sigma|\xi|],$$

for every $(x,s,\xi)\in B(x_0,\delta[s'])\times B(s',\delta[s'])\times\mathbb{R}^n$. Then we come back to step two and prove (4.18) and (4.19) taking in place of Σ' the set

$$\Sigma[s'] = B(x_0,\delta[s']/2)\times B(s',\delta[s']/2).$$

If we denote with $C_k[s']\in(0,\infty)$ the corresponding constants we have thus proved that

$$-1\le f_k(x,s,\xi)\le C_k[s'](1+|\xi|),$$
$$\lambda f_k(x,s,\xi/\lambda)\ge f_k^\infty(x,s,\xi)-C_k[s']\lambda,$$

whenever $(x,s,\xi)\in\Sigma[s']\times\mathbb{R}^n$.

Since $[a,b]\subset\bigcup_{s'\in[a,b]}B(s',\delta[s'])$ we can find $\{s_i\}_{i=1}^N\subset[a,b]$ such that

$$[a,b]\subset\bigcup_{i=1}^N B(s_i,\delta[s_i]/2)\subset[a',b'].$$

Then if we put,

$$\delta:=\min\{\delta[s_i]:1\le i\le N\},\quad C_k=\max\{C_k[s_i]:1\le i\le N\},$$

the following properties hold true:

$$f(x,s,\xi) = \sup_{k\in\mathbb{N}} f_k(x,s,\xi),$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$\begin{array}{rcl} -1 \leq f_k(x,s,\xi) & \leq & C_k(1+|\xi|), \\ \lambda f_k\left(x,s,\dfrac{\xi}{\lambda}\right) & \geq & f_k^\infty(x,s,\xi) - C_k\lambda, \\ f_k(x_0,s,\xi)+\sigma|\xi| & \leq & \varepsilon(1+\varepsilon)[f_k(x,s,\xi)+\sigma|\xi|] \end{array}$$

for every $(x,s,\xi) \in B(x_0,\delta/2) \times [a',b'] \times \mathbb{R}^n$ and, in the end,

$$\Phi_k[s_i](\xi) \quad \leq \quad f_k(x,s,\xi),$$

for every $(x,s,\xi) \in B(x_0,\delta) \times B(s_i,\delta[s_i]) \times \mathbb{R}^n$ and $i = 1,...,N$.

We can repeat this construction for every $j \in \{1,...,k\}$ ans define in $\Omega \times \mathbb{R} \times \mathbb{R}^n$

$$l_k(x,s,\xi) = \max\{f_j(x,s,\xi) : 1 \leq j \leq k\},$$

to find that

$$l_k(x,s,\xi) \uparrow f(x,s,\xi),$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$\begin{array}{rcl} -1 \leq l_k(x,s,\xi) & \leq & M_k(1+|\xi|), \\ \lambda l_k\left(x,s,\dfrac{\xi}{\lambda}\right) & \geq & l_k^\infty(x,s,\xi) - M_k\lambda, \\ l_k(x_0,s,\xi)+\sigma|\xi| & \leq & \varepsilon+(1+\varepsilon)[l_k(x,s,\xi)+\sigma|\xi|] \end{array}$$

for every $(x,s,\xi) \in B(x_0,\delta/2) \times [a',b'] \times \mathbb{R}^n$ and, in the end,

$$\Phi_k[s_i](\xi) \quad \leq \quad l_k(x,s,\xi),$$

for every $(x,s,\xi) \in B(x_0,\delta) \times B(s_i,\delta[s_i]) \times \mathbb{R}^n$ and $i = 1,...,N$.

Thus, for h big enough,

$$\begin{array}{rl} & \displaystyle\int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right) dy \\ \geq & \displaystyle\int_Q \varepsilon_h l_k\left(x_0+\varepsilon_h y, v_h(y), \frac{\nabla v_h}{\varepsilon_h}\right) dy - M_k\varepsilon_h!Q \setminus E_h| \\ \geq & \displaystyle\int_Q l_k^\infty(x_0+\varepsilon_h y, v_h(y), \nabla v_h) dy - M_k\varepsilon_h - M_k\varepsilon_h|Q\setminus E_h|. \end{array} \tag{4.30}$$

Let us prove that, if we are in the non trivial case

$$\begin{aligned}
&\liminf_{h\to\infty}\int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right)dy \\
\geq\ &\lim_{h\to\infty}\int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right)dy<\infty,
\end{aligned}$$

then $\{\nabla v_h\}_{h\in\mathbb{N}}$ is bounded in $L^1(Q;\mathbb{R}^n)$. For every $i=1,...,N$, $y\in Q$, we define

$$T_i v_h(y)=\begin{cases} v_h(y), & \text{if } s_i-\delta[s_i]\leq v_h(y)\leq s_i+\delta[s_i],\\ s_i+\delta[s_i], & \text{if } v_h(y)>s_i+\delta[s_i],\\ s_i-\delta[s_i], & \text{if } v_h(y)<s_i-\delta[s_i],\end{cases}$$

so that, by $l_k^\infty(x,s,0)=0$ and by Lemma 4, we have

$$\begin{aligned}
&\int_Q l_k^\infty(x_0+\varepsilon_h y, v_h(y),\nabla v_h)dy\\
\geq\ &\int_{\{s_i-\delta[s_i]\leq v_h(y)\leq s_i+\delta[s_i]\}} l_k^\infty(x_0+\varepsilon_h y, v_h(y),\nabla v_h)dy\\
=\ &\int_{\{s_i-\delta[s_i]\leq v_h(y)\leq s_i+\delta[s_i]\}} l_k^\infty(x_0+\varepsilon_h y, T_i v_h(y),\nabla T_i v_h)dy\\
=\ &\int_Q l_k^\infty(x_0+\varepsilon_h y, T_i v_h(y),\nabla T_i v_h)dy\\
\geq\ &\int_Q \Phi_k^\infty[s_i](\nabla T_i v_h)dy\\
\geq\ &\alpha_k\int_Q|\nabla T_i v_h|dy-\beta_k\int_{\partial Q}|T_i v_h|d\mathcal{H}^{n-1}-\gamma_k.
\end{aligned}$$

Since the left hand side is bounded on h and since

$$|T_i v_h|\leq|v_h|\leq\max\{|a'|,|b'|\},$$

we deduce that, for every $i=1,...,N$,

$$\sup_{h\in\mathbb{N}}\|\nabla T_i v_h\|_{L^1(Q;\mathbb{R}^n)}\leq C<\infty.$$

In particular

$$\alpha:=\sup_{h\in\mathbb{N}}\|\nabla v_h\|_{L^1(Q;\mathbb{R}^n)}\leq NC<\infty. \tag{4.31}$$

Then if we define

$$j_k(x,s,\xi) = j_k(s,\xi) := l_k^\infty(x_0,s,\xi) + \sigma|\xi|,$$

by (4.30), by (4.31), by the properties of l_k and by Theorem 10 we deduce

$$\begin{aligned}
& \liminf_{h\to\infty} \int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right) dy \\
\geq\ & \liminf_{h\to\infty} \int_Q l_k^\infty(x_0+\varepsilon_h y, v_h(y), \nabla v_h) + \sigma|\nabla v_h| dy - \sigma\alpha \\
\geq\ & \frac{1}{1+\varepsilon} \liminf_{h\to\infty} \int_Q l_k^\infty(x_0, v_h(y), \nabla v_h) + \sigma|\nabla v_h| dy - \sigma\alpha \\
=\ & \frac{1}{1+\varepsilon} \liminf_{h\to\infty} H[j_k](v_h, Q) - \sigma\alpha \\
\geq\ & \frac{1}{1+\varepsilon} H[j_k](v, Q) - \sigma\alpha.
\end{aligned}$$

We remark that

$$\lim_{\sigma\to 0} H[j_k](v,Q) = \int_a^b l_k^\infty(x_0,s,e_n)ds,$$

and, when $\varepsilon, \sigma \to 0$ we eventually find

$$\liminf_{h\to\infty} \int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right) dy \geq \int_a^b l_k^\infty(x_0,s,e_n)ds.$$

It suffices to apply Beppo Levi's Theorem as $k\to\infty$ and remark that,

$$l_k^\infty = \left(\max_{1\leq j\leq k} f_j\right)^\infty = \max_{1\leq j\leq k} f_j^\infty \uparrow \sup_{j\in\mathbb{N}} f_j^\infty = f^\infty, \text{ as } k\to\infty,$$

in order to achieve the proof of (4.12) and, as a consequence of step four, of Theorem 26. $\square$

Chapter 5

Serrin's theorem unified

In this chapter we present the proof of the approximation theorem that, as explained in subsection 1.1.5, allows to unify the three-parts-statement of Serrin's theorem, and that is based on Theorem 22 and Lemma 5.

5.1 The unifying approximation theorem

Theorem 27 (Fusco, Gori and Maggi, [37]). *Let $\Sigma \subset \mathbb{R}^d$ be an open set, and let $f : \Sigma \times \mathbb{R}^n \to [0, \infty)$ be a lower semicontinuous function with $f(t, \cdot)$ convex and demicoercive on $\mathbb{R}^n$ for every fixed $t \in \Sigma$. Then there exists a sequence of smooth functions $f_k : \Sigma \times \mathbb{R}^n \to [0, \infty)$ such that*

$$f = \sup_{k \in \mathbb{N}} f_k,$$

with $f_k(t, \cdot)$ convex for every fixed $t \in \Sigma$.

Proof. Step one: By Theorem 22 there exists a sequence of vectors $\xi_k \in \mathbb{R}^n$ and of lower semicontinuous functions $f_k : \Sigma \times \mathbb{R}^n \to [0, \infty)$ such that

$$f = \sup_{k \in \mathbb{N}} f_k, \tag{5.1}$$

and

$$f_k^o(t, \xi) := 1 + f_k(t, \xi + \xi_k) \tag{5.2}$$

is convex, demicoercive and positively homogeneous of degree one in $\xi \in \mathbb{R}^n$. For every $t_0 \in \Sigma$ and f_k^o we denote by $\Phi[k, t_0] : \mathbb{R}^n \to [0, \infty)$ the function constructed in Lemma 5. Thus $\Phi[k,t_0]$ is demicoercive on $\mathbb{R}^n$, non negative and such that

$$\Phi[k,t_0](\xi) \le f_k^o(t,\xi) = 1 + f(t,\xi+\xi_k),$$

for every $(t,\xi) \in B(t_0,\delta[k,t_0]) \times \mathbb{R}^n$.

Step two: By demicoercivity, for every $k \in \mathbb{N}$ and $t_0 \in \Sigma$ there exist

$$v[k,t_0] \in \mathbb{R}^n, \quad c[k,t_0] \in (0,\infty),$$

such that for every $\xi \in \mathbb{R}^n$

$$c[k,t_0]|\xi| \le \Phi[k,t_0](\xi) + v[k,t_0]\cdot\xi.$$

Thus the function $f_k^{oo} : B(t_0,\delta[k,t_0]) \times \mathbb{R}^n \to [0,\infty)$ defined by

$$f_k^{oo}(t,\xi) := f_k^o(t,\xi) + v[k,t_0]\cdot\xi, \tag{5.3}$$

is lower semicontinuous, positively homogeneous of degree one in the ξ variable and such that

$$c[k,t_0]|\xi| \le f_k^{oo}(t,\xi) \tag{5.4}$$

for every $(t,\xi) \in B(t_0,\delta[k,t_0]) \times \mathbb{R}^n$. By the lower semicontinuity, for every $\varepsilon > 0$ we can find

$$\delta[k,t_0,\varepsilon] \le \delta[k,t_0]$$

such that whenever $t \in B(t_0,\delta[k,t_0,\varepsilon])$ and $\xi \in S^{n-1}$

$$f_k^{oo}(t_0,\xi) \le \varepsilon c[k,t_0] + f_k^{oo}(t,\xi).$$

By positive homogeneity and by (5.4) we find that

$$f_k^{oo}(t_0,\xi) \le \varepsilon c[k,t_0]|\xi| + f_k^{oo}(t,\xi) \le (1+\varepsilon) f_k^{oo}(t,\xi),$$

for every $(t,\xi) \in B(t_0,\delta[k,t_0,\varepsilon]) \times \mathbb{R}^n$. Thus f_k^{oo} is in the hypothesis of the following lemma:

Lemma 6. *If $f : \Sigma \times \mathbb{R}^n \to [0,\infty)$ is lower semicontinuous, with $f(t,\cdot)$ convex and such that for every $t_0 \in \Sigma$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(t_0,\xi) \le \varepsilon + (1+\varepsilon) f(t,\xi)$$

whenever $(t,\xi) \in \Sigma \times \mathbb{R}^n$, then there exist two sequences of smooth functions $a_j : \Sigma \to [0,\infty)$ and $g_j : \mathbb{R}^n \to [0,\infty)$, with g_j convex, such that

$$f(t,\xi) = \sup_{j\in\mathbb{N}} a_j(t) g_j(\xi)$$

for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$.

Proof of Lemma 6. For every $t_0 \in \Sigma$ and every $\varepsilon > 0$ we find $\delta[t_0, \varepsilon] > 0$ such that

$$\frac{f(t_0, \xi) - \varepsilon}{1 + \varepsilon} \leq f(t, \xi), \tag{5.5}$$

for every $(t, \xi) \in B(t_0, \delta[t_0, \varepsilon]) \times \mathbb{R}^n$. Let $\alpha[\varepsilon, t_0] \in \mathcal{D}(B(t_0, \delta[t_0, \varepsilon]); [0, 1])$ with $\alpha[\varepsilon, t_0](t_0) = 1$ and consider

$$f[\varepsilon, t_0](t, \xi) = \alpha[\varepsilon, t_0](t) \int \frac{1}{\varepsilon^n} \, \kappa \left(\frac{\xi - \eta}{\varepsilon} \right) \frac{f(t_0, \eta) - \varepsilon}{1 + \varepsilon} \, d\eta,$$

where $\kappa \in \mathcal{D}(B^n)$ with $\kappa \geq 0$, $\int \kappa = 1$ and $\kappa(\xi) = \kappa(-\xi)$. If we put $\eta = \xi - \varepsilon\zeta$ we find

$$\begin{aligned} f[\varepsilon, t_0](t, \xi) &= \alpha[\varepsilon, t_0](t) \int \kappa(\zeta) \frac{f(t_0, \xi - \varepsilon\zeta) - \varepsilon}{1 + \varepsilon} \, d\zeta \\ &\leq \alpha[\varepsilon, t_0](t) \frac{f(t_0, \int \kappa(\zeta)(\xi - \varepsilon\zeta) d\zeta) - \varepsilon}{1 + \varepsilon} \\ &= \alpha[\varepsilon, t_0](t) \frac{f(t_0, \xi) - \varepsilon}{1 + \varepsilon} \leq f(t, \xi), \end{aligned}$$

for every $(t, \xi) \in \Sigma \times \mathbb{R}^n$. On the other hand, since $f(t_0, \cdot)$ is continuous, as $\varepsilon \to 0$ we find

$$f[\varepsilon, t_0](t_0, \xi) = \int \frac{1}{\varepsilon^n} \, \kappa \left(\frac{\xi - \eta}{\varepsilon} \right) \frac{f(t_0, \eta) - \varepsilon}{1 + \varepsilon} \, d\eta \to f(t_0, \xi),$$

for every $(t, \xi) \in \Sigma \times \mathbb{R}^n$. Thus

$$f(t, \xi) = \sup_{t_0 \in \Sigma} \sup_{\varepsilon > 0} f[\varepsilon, t_0](t, \xi).$$

Let us note $f[\varepsilon, t_0]$ is smooth, with $f[\varepsilon, t_0](t, \cdot)$ convex (indeed $\kappa \geq 0$). However $f[\varepsilon, t_0]$ in general will not be non negative, but just bounded from below:

$$f[\varepsilon, t_0](t, \xi) \geq - \frac{\varepsilon}{1 + \varepsilon}.$$

To fix this problem it suffices to consider an increasing sequence of smooth, non negative, increasing functions σ_m defined on $\mathbb{R}$ and converging to s^+ as $m \to \infty$, for example we can take

$$\sigma_m(s) := \int m \, \beta(m(s - r)) r^+ dr, \; m \in \mathbb{N}, \tag{5.6}$$

where $\beta \in \mathcal{D}((-1/2,1/2);[0,1])$, $\int \beta = 1$, $\beta(s) = \beta(-s)$. Then we pass to consider

$$g[\varepsilon,t_0](t,\xi) = \alpha[\varepsilon,t_0](t)\sigma_m \left[\int \frac{1}{\varepsilon^n} \, \kappa\left(\frac{\xi-\eta}{\varepsilon}\right) \frac{f(t_0,\eta)-\varepsilon}{1+\varepsilon} \, d\eta \right].$$

It is easily seen that these functions meet all the requirements of the lemma (note that the convexity in the ξ variable follows since σ_m is increasing). As usual, by continuity, we can achieve the supremum by using a countable family of these functions. □

As a consequence of the application of this lemma to f_k^{oo}, for each $t_0 \in \Sigma$ and $k \in \mathbb{N}$ we find two sequences of smooth functions

$$a[k,t_0,j] : B(t_0,\delta[k,t_0]) \to [0,\infty) \;\text{ and }\; g[k,t_0,j] : \mathbb{R}^n \to [0,\infty),$$

with $g[k,t_0,j]$ convex, such that

$$f_k^{oo}(t,\xi) = \sup_{j\in\mathbb{N}} a[k,t_0,j](t) g[k,t_0,j](\xi) \tag{5.7}$$

for every $(t,\xi) \in B(t_0,\delta[k,t_0]) \times \mathbb{R}^n$.

Step three: Let us fix a sequence of open set $\{\Sigma_h\}_{h\in\mathbb{N}}$ such that $\Sigma_h \subset\subset \Sigma$ and $\Sigma_h \uparrow \Sigma$. For every fixed $k \in \mathbb{N}$ the family of balls

$$B(t_0,\delta[k,t_0]), \quad t_0 \in \Sigma_h,$$

is an open covering of Σ_h. Thus for every k and h in $\mathbb{N}$ we can find points

$$t[k,h,i] \in \Sigma_h, \quad i \in \{1,...,N(h,k)\} =: I(h,k),\; N(h,k) \in \mathbb{N},$$

such that, for every $k \in \mathbb{N}$,

$$\overline{\Sigma_h} \subset \bigcup_{i\in I(h,k)} B(t[k,h,i],\delta[k,t[k,h,i]]) =: \bigcup_{i\in I(h,k)} B[k,h,i]. \tag{5.8}$$

For every fixed triplet (k,h,i) we consider an increasing sequence of non negative smooth functions with compact support in $B[k,h,i]$ converging to the characteristic function of $B[k,h,i]$: that is we consider $\{\varphi[k,h,i,l]\}_{l\in\mathbb{N}} \subset \mathcal{D}(B[k,h,i];[0,1])$ such that

$$\varphi[k,h,i,l] \uparrow 1 \quad \text{ in } B[k,h,i]. \tag{5.9}$$

As a consequence of (5.9), (5.8) and of $\Sigma_h \uparrow \Sigma$ we have that,

$$1 = \sup_{h\in\mathbb{N}} \sup_{i\in I(h,k)} \sup_{l\in\mathbb{N}} \varphi[k,h,i,l](t), \tag{5.10}$$

for every $t \in \Sigma$ and $k \in \mathbb{N}$.

Step four: We put all these remarks together. By $f \geq 0$, by (5.1) and by (5.10) we have

$$\begin{aligned} f(t,\xi) &= \sup_{k\in\mathbb{N}} \{f_k(t,\xi)\}^+ \\ &= \sup_{k\in\mathbb{N}} \sup_{h\in\mathbb{N}} \sup_{i\in I(h,k)} \sup_{l\in\mathbb{N}} \Big\{\varphi[k,h,i,l](t)\{f_k(t,\xi)\}^+\Big\} \end{aligned} \tag{5.11}$$

By (5.2), by (5.3) and by (5.7), for every fixed k, h, i and l we have that

$$\begin{aligned} \{f_k(t,\xi)\}^+ &= \{f_k^o(t,\xi-\xi_k)-1\}^+ \\ &= \{f_k^{oo}(t,\xi-\xi_k) - v[k,t[k,h,i]]\cdot(\xi-\xi_k)-1\}^+ \\ &= \sup_{j\in\mathbb{N}} \Big\{a[k,t[k,h,i],j](t)g[k,t[k,h,i],j](\xi-\xi_k) \\ &\qquad\qquad -v[k,t[k,h,i]]\cdot(\xi-\xi_k)-1\Big\}^+ \\ &= \sup_{j\in\mathbb{N}} \sup_{m\in\mathbb{N}} \sigma_m\Big(a[k,t[k,h,i],j](t)g[k,t[k,h,i],j](\xi-\xi_k) \\ &\qquad\qquad -v[k,t[k,h,i]]\cdot(\xi-\xi_k)-1\Big), \end{aligned} \tag{5.12}$$

if $\sigma_m : \mathbb{R} \to \mathbb{R}$ is a sequence of increasing, non negative smooth functions such that

$$s^+ := \sup_{m\in\mathbb{N}} \sigma_m(s),$$

that can be defined as in (5.6). By plugging (5.12) into (5.11) we deduce that $f(t,\xi)$ can be written as the supremum of functions of the form

$$\begin{aligned} \varphi[k,h,i,l](t)\sigma_m\Big[&a[k,t[k,h,i],j](t)g[k,t[k,h,i],j](\xi-\xi_k) \\ &-v[k,t[k,h,i]]\cdot(\xi-\xi_k)-1\Big], \end{aligned} \tag{5.13}$$

where the supremum is taken over

$$k\in\mathbb{N},\ h\in\mathbb{N},\ i\in I(k,h),\ l\in\mathbb{N},\ j\in\mathbb{N},\ m\in\mathbb{N}.$$

Since (5.13) defines a non negative, smooth function on $\Sigma \times \mathbb{R}^n$, that furthermore is convex in the gradient variable, up to relabeling the indexes we achieve the proof of the theorem. $\square$

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