SMOOTHING DOES NOT GIVE A SELECTION PRINCIPLE FOR TRANSPORT EQUATIONS WITH BOUNDED AUTONOMOUS FIELDS

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In honor of Sasha Shnirelman on occasion of his 75th birthday

ABSTRACT. We give an example of a bounded divergence free autonomous vector field in $\mathbb{R}^3$ (and of a nonautonomous bounded divergence free vector field in $\mathbb{R}^2$) and of a smooth initial data for which the Cauchy problem for the corresponding transport equation has 2 distinct solutions. We then show that both solutions are limits of classical solutions of transport equations for appropriate smoothings of the vector fields and of the initial data.

Nous donnons un exemple de champ vectoriel autonome, borné et à divergence nulle dans $\mathbb{R}^3$ (resp. d'un champ vectoriel nonautonome dans $\mathbb{R}^2$) et d'une donnée initiale réguliere pour lesquels il y a deux solutions différentes du probème de Cauchy pour l'équation de transport correspondant. Nous prouvons que chaque solution est la limite d'une suite de solutions d'équations de transport avec des champs régulieres opportunes et la même donnée initiale.

1. Introduction

In this note, we consider the classical Cauhy problem for a transport equation of type

$$\begin{cases} \partial_t \theta + (v \cdot \nabla_x)\theta = 0 \\ \theta(0, x) = \theta_{in}(x) \end{cases} \tag{1}$$

on $[0, T] \times \mathbb{R}^{d+1}$ (with $d \geq 2$), where θ is the unknown, while v is a known vector field. The vector fields considered will be divergence free and thus (1) can be rewritten as

$$\partial_t \theta + \operatorname{div}(v\theta) = 0$$

(which is usually called continuity equation). Therefore, as it is customary in the literature, when v and θ are summable enough (i.e. $v \in L^p$ and $\theta \in L^{p'}$ for a pair of dual exponents p, p') we understand solutions in the distributional sense.

We will restrict our attention to initial data θ_{in} which are bounded, to solutions which are bounded and to vector fields which are bounded. Under such assumptions (1) is classically well-posed if v is a Lipschitz vector field. Moreover the solutions are stable for perturbations of the vector field v. The famous seminal paper [15] established a similar well-posedness and stability theory when $v \in L^1([0, T], W^{1,p}(\mathbb{R}^n))$ for any $p \in [1, \infty]$: this is commonly called DiPerna-Lions theory and it has far-reaching applications to very different problems. The DiPerna-Lions theory was extended by Ambrosio in [5] to $L^1([0, T], BV(\mathbb{R}^n))$ and it was then showed that the result is essentially optimal: weak solutions for vector fields $v \in W^{s,1}$ are in general not unique for $s < 1$ (cf. [1, 14]; nonetheless there are still several important open problems in the area and very recent interesting developments, see for instance [6, 3, 4, 7, 8, 9, 10, 12, 13, 17, 18]).

The next natural question in this regard is then whether there is a meaningful selection principle among these different weak solutions. For instance, do solutions of suitable regularizations have a unique limit? To our knowledge this question is specifically raised for the first time in [11], where the authors give a partial negative answer. The aim of this note is to show that, at least if we only require the regularizations to just enjoy (in a uniform way) the same regularity estimates of the vector field, then the answer is negative. The answer is negative even if we consider autonomous vector fields and if the initial data remains fixed, or anyway they are regularized by convolution with a classical kernel. Our main theorem is the following:

Theorem 1.1. *Let $d \geq 2$. Then there exist*

(i) *an autonomous compactly supported divergence-free vector field $v \in L^\infty(\mathbb{R}^{d+1};\mathbb{R}^{d+1})$,*
(ii) *a smooth initial data $\theta_{in} \in C^\infty_c(\mathbb{R}^{d+1};\mathbb{R})$ with compact support,*
(iii) *two sequences of divergence-free vector fields $\{v'_i\}_{i=1}^\infty, \{\tilde{v}_i\}_{i=1}^\infty \subset C^\infty_c(\mathbb{R}^{d+1};\mathbb{R}^{d+1})$, such that $v'_i \to v$, $\tilde{v}_i \to v$ strongly in L^1 as $i \to \infty$ and $\|v'_i\|_{L^\infty}, \|\tilde{v}_i\|_{L^\infty} \leq C$ for some constant C independent of i,*

with the following properties. If θ'_i and $\tilde{\theta}_i$ are the unique solutions to the transport equation (1) *with initial data θ_{in}, then:*

(*) *$\theta'_i \rightharpoonup \theta'$ and $\tilde{\theta}_i \rightharpoonup \tilde{\theta}$ weakly in L^1, where θ' and $\tilde{\theta}$ are 2 <u>distinct</u> solutions to* (1) *with field v and initial data θ_{in}.*

Moreover, the vector field v belongs to $W^{s,p}$ for every $s < 1$ and $p < \frac{1}{s}$ and the regularized fields $v'_i, \tilde{v}_i$ enjoy uniform estimates in the corresponding spaces.

Previous work of [11] has shown this theorem for a suitable field $v \in L^p$ for $p \in [1, \frac{4}{3}]$, with a completely different construction. If we drop the requirement that the field be autonomous, we can show the same conclusion for 2-dimensional fields.

Theorem 1.2. *There exist*

(i) *a compactly supported divergence-free vector field $b \in L^\infty([0,2] \times \mathbb{R}^2;\mathbb{R}^2)$,*
(ii) *a smooth initial data $\rho_{in} \in C^\infty_c(\mathbb{R}^2;\mathbb{R})$ with compact support,*
(iii) *two sequences of divergence-free vector fields $\{b'_i\}_{i=1}^\infty, \{\tilde{b}_i\}_{i=1}^\infty \subset C^\infty_c([0,T] \times \mathbb{R}^2;\mathbb{R}^2)$, such that $b'_i \to b$, $\tilde{b}_i \to b$ strongly in $L^1([0,2] \times \mathbb{R}^2)$ as $i \to \infty$ and $\|b'_i\|_{L^\infty}, \|\tilde{b}_i\|_{L^\infty} \leq C$,*

with the following properties. If ρ'_i and $\tilde{\rho}_i$ are the unique solutions to the transport equation (1) *with fields b'_i and $\tilde{b}_i$ and initial data ρ_{in}, then:*

(**) *$\rho'_i \rightharpoonup \rho'$ and $\tilde{\rho}_i \rightharpoonup \tilde{\rho}$ weakly in L^1, where ρ' and $\tilde{\rho}$ are 2 <u>distinct</u> solutions to* (1) *with field b and initial data ρ_{in}.*

Indeed, since there is a simple way to pass from a non-autonomous example to an autonomous one in one dimension higher, we will mainly focus on how to build the example of Theorem 1.2. The construction is similar to other ones present in the literature, starting from the work of DePauw [14]: the contribution of this note is to show how it can be arranged so that the corresponding distinct solutions are limits of solutions of appropriate regularizations.

Remark 1.3. *The vector field b constructed in Theorem 1.2 has the symmetry $b(t,x) = -b(2-t,x)$. If we were to dispense with the smoothness of the initial data and with the compactness of the supports of the various objects, the corresponding two distinct solutions $\tilde\rho$ and ρ' behave in a way that can be described quite accurately (cf. Sections 2.1-2.2). In this simplified setting, the corresponding solutions (denoted by ρ'_* and $\tilde\rho_*$ in Sections 2.2) agree for times $t<1$ and as t converges to 1 they both converge (weakly* in L^∞) to the constant function $\frac{1}{2}$, even though the initial data is not constant. What then distinguishes $\tilde\rho_*$ and ρ'_* is that for $t \geq 1$ $\tilde\rho_*$ inherits the natural symmetry of the problem, i.e. satisfies $\tilde\rho_*(t,x) = \tilde\rho_*(2-t,x)$, while the solution ρ' on the contrary violates the symmetry, but remains constantly equal to $\frac{1}{2}$.*

In the course of the proof of Theorem 1.2 we will see also the following corollary on Regular Lagrangian Flows (cf. [5, 13] for the definition).

Corollary 1.4. *Let $b, b'_i, \tilde b_i$ be as in Theorem 1.2 and let $\Phi'_i, \tilde\Phi_i$ be the unique Regular Lagrangian Flows corresponding to $b'_i, \tilde b_i$ respectively with $\Phi'_i(0,x) = \tilde\Phi_i(0,x) = x$. Then:*

(i) *$\tilde\Phi_i \to \tilde\Phi$ strongly in $L^1([0,2]\times\mathbb{R}^2;\mathbb{R}^2)$ where $\tilde\Phi$ is a Regular Lagrangian Flow corresponding to the field b.*

(ii) *No subsequence of Φ'_i converges strongly in L^1_{loc}.*

Even though a "closure" of classical solutions does not provide a selection mechanism to single out one preferred solution to the final transport equation, our construction does not rule out the possibility that some "canonical" regularization (like smoothing by convolution with some specific kernel) still selects only one preferred solution in the limit. This seems however not likely. In fact in our construction any solution would have to coincide with $\tilde\rho$ and ρ' up to the time $t=1$, because the vector field is sufficiently smooth on $[0,1-\sigma]\times\mathbb{R}^2$ to apply the DiPerna-Lions theory. It then turns out that any canonical selection would have to give up one of the following two principles:

- the solution inherits the natural symmetry of b;
- the solution propagates a constant initial data as a constant.

Moreover while alternative (i) in Corollary 1.4 might suggest that $\tilde b_i$ is a more reasonable regularization, we caution the reader that it too has some drawbacks. In fact, while it is true that the symmetric solution $\tilde\rho$ would be recovered from the initial data at time $t=0$ through the flow $\tilde\Phi$, the analogous property fails if we regard $\tilde\rho$ as a solution of the Cauchy problem with initial data at time $t=1$, cf. Remark 3.1

We finally comment on the possibility of lowering the dimension of Theorem 1.1 to $d+1=2$. While the problem of uniqueness of solutions to (1) for 2-dimensional autonomous dinvergence-free fields has been completely solved by Alberti, Bianchini and Crippa in [2], [3], our construction sheds no light on whether smoothing is a selection principle for 2-dimensional autonomous fields. As we recover Theorem 1.1 by trading time for one space dimension, in order to follow the same pattern we would need an analog of Theorem 1.2 in one space dimension. This is however not possible: even dispensing with the divergence-free condition (which in 1 space dimension would imply constancy of the field), the ordering of the real line, which is preserved by the flow of any smooth 1-dimensional vector field, is an obvious obstruction.

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2. Construction of the core nonautonomous example

In this section we detail the construction of vector fields and of initial data which exhibit the same behavior as those in Theorem 1.1, but we will:

- drop the requirement that the vector fields (and the initial data) have compact support;
- drop the requirement that the initial data is smooth;
- postpone the estimates on their $W^{s,p}$ norms.

All three aspects are minor and will be addressed in the Section 3, where we also show how to pass to the autonomous example in one dimension higher, i.e. how to prove Theorem 1.2.

2.1. **Step 1. Definition of b, ρ_{in}, ρ', and $\tilde{\rho}$.** We first introduce the following two standard lattices on $\mathbb{R}^2$, namely $\mathcal{L}^1 := \mathbb{Z}^2 \subset \mathbb{R}^2$ and $\mathcal{L}^2 := \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2}) \subset \mathbb{R}^2$. To both of them we can associate a corresponding subdivision of the plane into squares which have vertices lying in the corresponding lattices, which we denote by $\mathcal{S}^1$ and $\mathcal{S}^2$. We then consider the rescaled lattices $\mathcal{L}^1_i := 2^{-i}\mathbb{Z}^2$ and $\mathcal{L}^2_i := (2^{-i-1}, 2^{-i-1}) + 2^{-i}\mathbb{Z}^2$ and the corresponding square subdivision of $\mathbb{Z}^2$, respectively $\mathcal{S}^1_i$ and $\mathcal{S}^2_i$. Observe that

(D) The centers of the squares $\mathcal{S}^1_i$ are elements of $\mathcal{L}^2_i$ and viceversa.

We let $\rho_{in,*}(x) = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor \; mod \; 2$. This is a 'chessboard' pattern based on the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$: if we index the squares of $\mathcal{S}^1$ with (k, j), where $(k + \frac{1}{2}, j + \frac{1}{2}) \in \mathcal{L}^2$ is the center of the corresponding square, then $\rho_{in,*}$ vanishes on the squares for which $k + j$ is even, while it is identically equal to 1 on squares for which $k + j$ is odd.

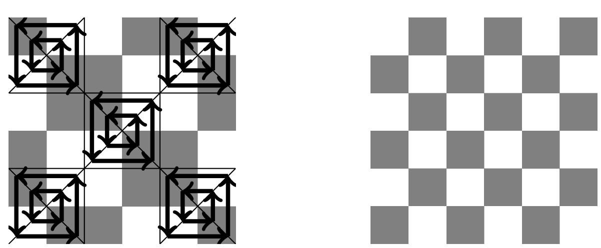

Figure 1. Action of the flow from $t = 0$ to $t = 1/2$. The shaded region denotes the set $\{\rho = 1\}$

Next we define the following 2-dimensional vector field:

$$w(x) = \begin{cases} (0, 4x_1) \text{ , if } 1/2 > |x_1| > |x_2| \\ (-4x_2, 0) \text{ , if } 1/2 > |x_2| > |x_1| \\ (0,0) \text{ , otherwise.} \end{cases}$$

Thus w is a weakly divergence free 'vortex'. (c.f. Section 7 of [16]). Periodise w (as in Figure 1) by defining $\Lambda = \{(y_1, y_2) \in \mathbb{Z}^2 : y_1 + y_2 \text{ is even}\}$ and setting

$$u(x) = \sum_{y \in \Lambda} w(x - y) .$$

Note that that w is supported in one square of $\mathcal{S}^2$ and thus the periodization consists of filling half the squares of $\mathcal{S}^2$ with copies of w, while leaving the field identically equal to 0 in the remaining squares. The "filled" and "empty" squares form likewise a chessboard pattern.

Even though u is irregular, it has locally bounded variation and it is piecewise linear. There is thus a unique solution ρ of (TE) with vector field u and similarly the flux Φ of u is well-defined. Its relevant property for our construction is that the map $\Phi(t, \cdot)$ is Lipschitz on each square S of $\mathcal{S}^2$ and $\Phi(\frac{1}{2}, \cdot)$ is a clockwise rotation of 90 degrees of the "filled" S, while it is the identity on the "empty ones". Each $S \in \mathcal{S}^2$ is formed precisely by its intersection with 4 squares of $\mathcal{S}^1_1$: in the case of "filled" S the 4 squares are permuted in a 4-cicle clockwise, while in the case of "empty" S the 4 squares are kept fixed. At any rate $\Phi(\frac{1}{2}, \cdot)$ maps rigidly one square of $\mathcal{S}^1_1$ onto another square of $\mathcal{S}^1_1$. Since for every $j \geq 2$ every square in $\mathcal{S}^1_j$ is contained in some square of $\mathcal{S}^1_1$, we obviously conclude the following

Lemma 2.1. *For every $j \geq 1$, $\Phi(\frac{1}{2}, \cdot)$ maps any element of $\mathcal{S}^1_j$ rigidly onto another element of $\mathcal{S}^1_j$.*

Using Lemma 2.1 it is therefore easy to see that

$$\rho(\tfrac{1}{2}, x) = 1 - \rho_{in,*}(2x) . \tag{2}$$

Likewise it is simple to use Lemma 2.1 to prove

Lemma 2.2. *Let ρ be a solution of the transport equation* (1) *with an arbitrary bounded initial data $\bar{\rho}$ and the specific vector field u described above. Assume $j \geq 2$ and $\alpha \in \mathbb{R}$ are constants such that $\bar{\rho}$ has average α on every $S \in \mathcal{S}^1_j$. Then $\rho(\frac{1}{2}, \cdot)$ has also average α on $S \in \mathcal{S}^1_j$.*

We now define b on $\mathbb{R}^2 \times [0, 2]$ in the following fashion. First of all $b(t, x) = u(x)$ for $0 < t < \frac{1}{2}$ and $b(t, x) = u(2^n x)$ for $1 - 2^{-n} < t < 1 - 2^{-(n+1)}$. For $1 < t < 2$, we let $b(t, x) = -b(2 - t, x)$. Note that (1) has a unique solution ρ on $[0, 1 - 2^{-n}]$ because b is a function of bounded variation. In particular this yields a unique solution on $[0, 1]$. Moreover, using recursively the appropriately scaled version of (2) we can readily check that $\rho(1-2^{-2k},x)=\rho_{in,*}(2^{2k}x)$ and $\rho(1-2^{-(2k+1)},x)=1-\rho_{in,*}(2^{2k+1}x)$. In particular

$$\rho(t,\cdot)\rightharpoonup^\star \frac{1}{2} \qquad \text{in } L^\infty \text{ as } t\to 1.$$

Moreover, considering that for times $t\geq 1$ the flow of b is just a time-reversal of the flow for $t\leq 1$, an obvious rescaling of Lemma 2.2 gives the following conclusion, which will be useful in the sequel.

Lemma 2.3. *Let ρ be a solution transport equation* (1) *with any bounded initial data and vector field b. Assume $i\in\mathbb{N}$ and $\alpha\in\mathbb{R}$ are such that $\rho(1+\frac{1}{2^i},\cdot)$ has average α on every $S\in\mathcal{S}^1_j$ for some $j\geq i+1$. Then $\rho(1+\frac{1}{2^k},\cdot)$ has also average α on every $S\in\mathcal{S}^1_j$ and for all $0\leq k\leq i$.*

We can thus continue ρ for $t\in[1,2]$ in two fashions, namely we set

$$\rho'_*(t,x)=\begin{cases}\rho(t,x), \text{ for } 0<t<1\\ \frac{1}{2}, \text{ for } 1<t<2\end{cases}$$

$$\tilde{\rho}_*(t,x)=\begin{cases}\rho(t,x), \text{ for } 0<t<1\\ \rho(2-t,x), \text{ for } 1<t<2\end{cases} .$$

This is because we can 'glue' weak solutions of (1) to get another weak solution. More precisely

Lemma 2.4. *If ρ_1 and ρ_2 are two bounded weak solutions of* (1) *with vector field b defined for $0<t<1$ and $1<t<2$ respectively, and if both $\rho_1(t,\cdot)$ and $\rho_2(t,\cdot)$ converge weakly* to the same limit as $t\to 1$, then*

$$\rho(t,x):=\begin{cases}\rho_1(t,x), \textit{ for } 0<t<1\\ \rho_2(t,x), \textit{ for } 1<t<2\end{cases}$$

is also a weak solution of (1) *with vector field b.*

While the argument is a standard exercise in functional analysis, we give the proof for the reader's convenience.

Proof. Indeed, as b is weakly divergence-free, for any $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ with $spt(\phi) \subset \{t < 2\} \times \mathbb{R}^2$ and $0<\beta<1$ we have

$$\begin{aligned}
&\left|\int_0^2 \int_{\mathbb{R}^d} \rho(t,x)(\partial_t\phi + (b\cdot\nabla)\phi)dxdt + \int_{\mathbb{R}^d} \rho_{in,*}(x)\phi(0,x)dx\right| \\
&\leq \left|\int_0^{1-\beta} \int_{\mathbb{R}^d} \rho_1(t,x)(\partial_t\phi + (b\cdot\nabla)\phi)dxdt \right.\\
&\qquad \left. + \int_{1+\beta}^2 \int_{\mathbb{R}^d} \rho_2(t,x)(\partial_t\phi + (b\cdot\nabla)\phi)dxdt + \int_{\mathbb{R}^d} \rho_{in,*}(x)\phi(0,x)dx\right| \\
&\quad + 2\beta\|D\phi\|_{L^\infty}\|b\|_{L^\infty}\|\rho\|_{L^1} \\
&\leq \left|-\int_{\mathbb{R}^d} \rho_{in,*}(x)\phi(0,x)dx + \int_{\mathbb{R}^d} \rho_1(1-\beta,x)\phi(1-\beta,x)dx \right.\\
&\qquad \left. - \int_{\mathbb{R}^d} \rho_2(1+\beta,x)\phi(1+\beta,x)dx + \int_{\mathbb{R}^d} \rho_{in,*}(x)\phi(0,x)dx\right| + 2\beta\|D\phi\|_{L^\infty}\|b\|_{L^\infty}\|\rho\|_{L^1} \\
&\leq \left|\int_{\mathbb{R}^d} \rho_1(1-\beta,x)\phi(1-\beta,x)dx - \int_{\mathbb{R}^d} \rho_2(1+\beta,x)\phi(1-\beta,x)dx\right| \\
&\quad + 2\beta\|\partial_t\phi\|_{L^\infty}\|\rho_2(1+\beta,\cdot)\|_{L^1} + 2\beta\|D\phi\|_{L^\infty}\|b\|_{L^\infty}\|\rho\|_{L^1}
\end{aligned}$$

Thus, as $\beta \to 0$ and since both $\rho_1(t,\cdot)$ and $\rho_2(t,\cdot)$ weakly tend to the same limit as $t \to 1$, we get that ρ is a weak solution of (1). □

2.2. **Step 2. Truncations.** We next construct two sequences of BV vector fields converging to b. Both are simple and they are given by

$$b_i^1(t,x) = \begin{cases} b(t,x), \text{ for } 0<t<1-2^{-2i}, \\ 0, \text{ for } 1-1/2^{2i} < t < 1+2^{-2i+2}, \\ b(t,x), \text{ for } 1+2^{-2i+2} < t < 2 \end{cases}$$

$$b_i^2(t,x) = \begin{cases} b(t,x), \text{ for } 0<t<1-2^{-2i}, \\ 0, \text{ for } 1-1/2^{2i} < t < 1+2^{-2i}, \\ b(t,x), \text{ for } 1+2^{-2i} < t < 2 \end{cases}$$

Let now ρ_i^1 and ρ_i^2 be the corresponding unique weak solutions of (1) with initial data ρ_{in}. By construction both ρ^1 and ρ^2 coincide with $\rho = \rho_*' = \tilde\rho_*$ on the time interval $[0, 1-2^{-2i}]$. Moreover for both we have

$$\begin{aligned}
\rho_i^1(1+2^{-2i},x) &= \rho(1-2^{-2i},x) = \rho_{in,*}(2^{2i}x) \\
\rho_i^2(1+2^{-2i},x) &= \rho(1-2^{-2i},x) = \rho_{in,*}(2^{2i}x)\,.
\end{aligned}$$

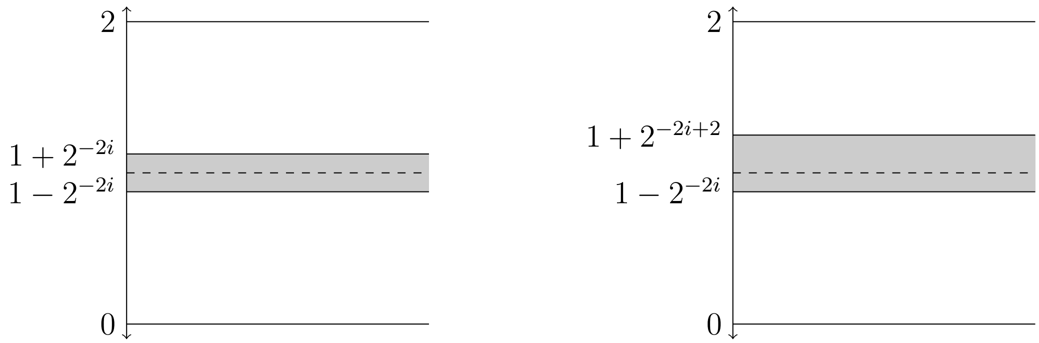

Figure 2. Schematic representations of the 2 flows where time is shown on the vertical axis. The shaded region corresponds to the times where the vector field is trivial.

Now, $b_i^2(t,x) = b(t,x)$ for $t \geq 1+2^{-2i}$. Since $\rho_i^2(1+2^{-2i},x) = \rho_{in,*}(2^{2i}x) = \tilde{\rho}_*(1+2^{-2i},x)$, we conclude that $\rho_i^2(t,x) = \tilde{\rho}_*(t,x)$ for $t \geq 1+2^{-2i}$. In particular we infer

$$\rho_i^2 \rightharpoonup^\star \tilde{\rho}_* \qquad \text{in } L^\infty .$$

We claim next that

$$\rho_i^1 \rightharpoonup^\star \rho_*' \qquad \text{in } L^\infty .$$

The overall intuition is illustrated schematically in Figure 2. The shadowed region on the left diagram is the time-region where b_i^2 vanishes identically. Since we have $b_i^2(t,\cdot) = -b_i^2(2-t,\cdot)$ on the complement, we see a complete reversal of the flow when $t \geq 1+2^{-2i}$, restoring the initial condition at the final time $t=2$. On the other hand, the diagram on the right highlights the time-region where the field b_i^1 vanishes. Very much like b_i^2, the field b_i^1 'mixes' the initial data up to the fine scale '2^{-2i}' at the time $t = 1-2^{-2i}$. Then the flow is trivial in the shaded region, which goes two dyadic scales more than the shaded region of the dyagram on the left. Thus ρ_i^1 is still 'finely mixed' at $t = 1+2^{-2i+2}$. At that point, when the vector field b_i^1 starts acting again, its flow will however only affect coarser scales (i.e. dyadic scales larger than 2^{-2i+2}). Therefore the solution remains 'finely mixed' at $t=2$.

In order to make such intuition rigorous, observe first that, since $b(t,x) = 0$ for $t \in [1+2^{-2i}, 1+2^{-2i+2}]$, we indeed have $\rho_i^1(1+2^{-2i+2},x) = \rho_{in,*}(2^{2i}x)$. We see that $\rho_i^1(1+2^{-2i+2},x)$ has average $\frac{1}{2}$ on every square of $\mathcal{S}^1_{2i-1}$. We can now use Lemma 2.3 to conclude that

$$\frac{1}{|S|}\int \rho_i^1(1+2^{-j},x)\,dx = \frac{1}{2} \qquad \forall S \in \mathcal{S}^1_{2i-1} \quad \text{and} \quad \forall j \leq 2i-2\,. \tag{3}$$

In particular we conclude the same for $j=0$. It is easy to see $\rho_i^1(2,\cdot) \rightharpoonup^\star \frac{1}{2}$ because of (3). Indeed having fixed $\varphi \in C_c(\mathbb{R}^2)$, we can write

$$\int \varphi(x)\rho_i^1(2,x)\,dx = \sum_{S\in\mathcal{S}^1_{2i-1}: S\cap \text{spt}\,(\varphi)\neq\emptyset} \int_S \varphi(x)\rho_i^1(2,x)\,dx$$

and hence estimate

$$
\begin{aligned}
&\left|\int \varphi(x)\rho_i^1(2,x)\,dx - \frac{1}{2}\int \varphi(x)\,dx\right| \\
=&\left|\sum_{S\in\mathcal{S}^1_{2i-1}:S\cap \mathrm{spt}\,(\varphi)\neq\emptyset} \int_S \varphi(x)\left(\rho_i^1(2,x) - \frac{1}{|S|}\int_S \rho_i^1(2,y)dy\right)dx\right| \\
=&\left|\sum_{S\in\mathcal{S}^1_{2i-1}:S\cap \mathrm{spt}\,(\varphi)\neq\emptyset} \int_S (\varphi(x)-\varphi(x_S))\left(\rho_i^1(2,x) - \frac{1}{|S|}\int_S \rho_i^1(2,y)dy\right)dx\right| \\
\leq& \sum_{S\in\mathcal{S}^1_{2i-1}:S\cap \mathrm{spt}\,(\varphi)\neq\emptyset} \|D\varphi\|_{C^0} 2^{-2i+1}2^{-2(2i-1)} \leq CR^2\|D\varphi\|_{C^0}2^{-2i+1}\,,
\end{aligned}
$$

where x_S denotes the center of the square S, R is a fixed radius such that $spt(\varphi)\subset B(0,R)$ and C is a geometric constant. Now since for any $\epsilon>0$ any $f\in L^1$ can be written as $\varphi+h$ with $\varphi\in C^1_c$, $spt(\varphi)\subset B(0,R)$ for some R, and $\|h\|_{L^1}<\epsilon$, we get that

$$
\begin{aligned}
\left|\int f(x)\rho_i^1(2,x)\,dx - \frac{1}{2}\int f(x)\,dx\right| &= \left|\int \varphi(x)\rho_i^1(2,x)\,dx - \frac{1}{2}\int \varphi(x)\,dx\right| \\
&\quad + \left|\int h(x)\rho_i^1(2,x)\,dx - \frac{1}{2}\int h(x)\,dx\right| \\
&\leq CR^2\|D\varphi\|_{C^0}2^{-2i+1} + \|\rho_i^1(2,x)-\tfrac{1}{2}\|_{L^\infty}\|h\|_{L^1} \\
&\leq CR^2\|D\varphi\|_{C^0}2^{-2i+1} + \epsilon
\end{aligned}
$$

Thus, as ϵ was arbitrary, we see that $\rho_i^1(2,\cdot)\rightharpoonup^\star \frac{1}{2}$. So, any weak* limit of a convergent subsequence of ρ_i^1 converges to a (backward) solution of the transport equation which is identically equal to $\frac{1}{2}$ at time 2. We can now use the backward uniqueness for the transport equation with vector field b on intervals $[1+\sigma,2]$ for $\sigma>0$ (such uniqueness is guaranteed by the fact that the vector field b is BV on $[1+\sigma,2]\times\mathbb{R}^2$), to conclude that such weak* limit is identically equal to $\frac{1}{2}$ on $[1+\sigma,2]$. In particular we conclude that ρ_i^1 converges weakly* to ρ'_*.

2.3. **Step 3. Regularization of vector fields.** We now extend the vector fields b_i^2 and b_i^1 to times $t\notin[0,2]$ by setting them identically 0. We fix i and a space-time compactly supported convolution kernel φ and regularize both b_i^1 and b_i^2 to $b_{i,j}^1$ and $b_{i,j}^2$ setting $b_{i,j}^k := b_i^k*\varphi_{2^{-j}}$. Since each vector field b_i^k belongs to $L^\infty([0,2],BV\cap L^\infty(\mathbb{R}^2))$, we can use Ambrosio's extension of the DiPerna-Lions theory to conclude that, for each fixed i and k, the corresponding solutions $\rho_{i,j}^k$ of the transport equations with vector fields $b_{i,j}^k$ and initial data $\rho_{in,*}$ converge strongly in L^1_{loc} to ρ_i^k. In particular we can select $j(i)$ such that

$$
\sum_{k=1}^2 \|\rho^k_{i,j(i)}-\rho_i^k\|_{L^1([0,2]\times[-2^i,2^i]^2)} \leq 2^{-i}\,.
$$

We then set $b'_i = b^1_{i,j(i)}$, $\tilde{b}_i = b^2_{i,j(i)}$, $\rho'_{i,*} = \rho^1_{i,j(i)}$ and $\tilde{\rho}_{i,*} = \rho^2_{i,j(i)}$. Clearly, $\tilde{\rho}_{i,*} \rightharpoonup \tilde{\rho}_*$ and $\rho'_{i,*} \rightharpoonup \rho'_*$ in $L^1([0,T] \times U)$ for every bounded open set U.

3. Proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.4

3.1. **Step 4. Compact supports.** Thus far the vector fields, the initial data and the solutions do not have compact support. However, in order to make them have compact supports we just proceed as follows. We modify $\rho_{in,*}$ to $\rho_{in,*}\mathbf{1}_{Q_N}$, where $Q_N = [-N, N]^2$ for some large natural number N. In order to truncate appropriately b we need to act more carefully. For $t \in [1-2^{-i}, 1-2^{-i-1}]$ we substitute $b(t,x)$ with $b(t,x)\mathbf{1}_{Q_{N+2^{-i-1}}}(x)$. Observe that the choice of the sidelength of the square is made so to guarantee that the vector field remains divergence-free. We then keep the symmetric structure $b(t,x) = b(2-t,x)$ for the truncated field and we follow the same procedures of the previous steps. Note that the new regularized fields coincide with the old (nontruncated) ones in, say, $[0,2] \times Q_{N/2}$ and the initial data coincide with the old (nontruncated) ones in $Q_{N/2}$. Moreover the L^∞ norm of all the fields is bounded uniformly by an absolute constant independent of N. In particular, for N sufficiently large, the solutions of the transport equations with the truncated fields with truncated initial data coincide with the ones for the nontruncated fields and nontruncated initial data. We thus infer the same conclusions.

3.2. **Step 5. Regularization of initial data.** We let ρ_{in} be a smooth approximation of $\rho_{in,*}$ such that $\|\rho_{in} - \rho_{in,*}\|_{L^1(\mathbb{R}^2)} \leq c\|\tilde{\rho}_* - \rho'_*\|_{L^1([0,2]\times\mathbb{R}^2)} =: cA$ where c is a very small absolute constant to be determined later. Let $\rho'_i, \tilde{\rho}_i$ be the unique solutions of (1) with initial data ρ_{in} and vector fields b'_i, $\tilde{b}_i$ respectively. Since, $b'_i, \tilde{b}_i \in C^\infty([0,2] \times \mathbb{R}^2)$, the flow generated by them preserves the L^1 norm of the initial data. Thus, we have that

$$\|\rho'_i - \rho'_{i,*}\|_{L^1_{t,x}} \leq 2\|\rho'_i - \rho'_{i,*}\|_{L^\infty_t(L^1_x)} = 2\|\rho_{in} - \rho_{in,*}\|_{L^1(\mathbb{R}^2)} \leq 2cA$$
$$\|\tilde{\rho}_i - \tilde{\rho}_{i,*}\|_{L^1_{t,x}} \leq 2\|\tilde{\rho}_i - \tilde{\rho}_{i,*}\|_{L^\infty_t(L^1_x)} = 2\|\rho_{in} - \rho_{in,*}\|_{L^1(\mathbb{R}^2)} \leq 2cA$$

Since, $\tilde{\rho}_* \neq \rho'_*$, we can choose a compactly supported $g(t,x) \in L^\infty$ such that

$$\left|\int (\tilde{\rho}_* - \rho'_*)g\right| \geq 1\,.$$

Now let $\rho', \tilde{\rho}$ be any L^1-weak limits of the sequences $\{\rho'_i\}_{i=1}^\infty, \{\tilde{\rho}_i\}_{i=1}^\infty$, respectively (which exist as the sequences are bounded). Then

$$\int (\tilde{\rho} - \rho')g = \lim_i \int (\tilde{\rho}_i - \rho'_i)g$$

But now,

$$\begin{aligned}\left|\int (\tilde{\rho}_i - \rho'_i)g\right| &\geq \left|\int (\tilde{\rho}_{i,*} - \rho'_{i,*})g\right| - \|\rho'_i - \rho'_{i,*}\|_{L^1}\|g\|_{L^\infty} - \|\tilde{\rho}_i - \tilde{\rho}_{i,*}\|_{L^1}\|g\|_{L^\infty}\\ &\geq \left|\int (\tilde{\rho}_{i,*} - \rho'_{i,*})g\right| - 4cA\|g\|_{L^\infty}\,.\end{aligned}$$

Choosing $c \leq (8A\|g\|_{L^\infty})^{-1}$ and sending $i \to$ infinity we conclude

$$\left|\int (\rho' - \rho_*) g\right| \geq \left|\int (\tilde{\rho}_* - \rho'_*) g\right| - \frac{1}{2} \geq \frac{1}{2}\,,$$

which implies that $\tilde{\rho}$ and ρ' are distinct.

3.3. **Step 6.** $W^{s,p}$ **estimates.** We now show that our vector field b of the previous section is in $W^{s,1}_{loc}([0,T]\times\mathbb{R}^2)$ for every $s<1$. We'll make all our estimates on $B := [-1/2, 1/2]^2$ and $\Omega := [0,2]\times B$. Recall $b(t,x) = \pm u(2^{i+1}x)$ on $\mathcal{I}_i := (1-2^{-i}, 1-2^{-(i+1)}) \cup (1+2^{-(i+1)}, 1+2^{-i})$ and is identically 0 elsewhere. Thus as,

$$\begin{aligned}
\|u(2^{i+1}\cdot)\|_{BV(B)} \lesssim 2^{di}\|w(2^{i+1}\cdot)\|_{BV(B)} &= 2^{di}(\|w(2^{i+1}\cdot)\|_{L^1(B)} + \|Dw(2^{i+1}\cdot)\|_{TV(B)}) \\
&= 2^{di}(2^{-di}\|w\|_{L^1(B)} + 2^{-(d-1)i}\|Dw\|_{TV(B)}) \\
&\leq 2^{di}2^{-(d-1)i}\|w\|_{BV(B)} = 2^i\|w\|_{BV(B)}
\end{aligned}$$

The first inequality follows because there are approximately 2^{di} 'little' vortices in B. Now,

$$\|b\chi_{\mathcal{I}_i}\|_{BV(\Omega)} \leq C + \int_{\mathcal{I}_i} \|u(2^{i+1}x)\|_{BV(B)} dt \lesssim 1$$

The constant C comes from the 'horizontal' jump part of the measure at $\partial\mathcal{I}_i$. Note this constant is indeed independent of i. By Gagliardo-Nirenberg we get for $0<s<1$,

$$\|b\chi_{\mathcal{I}_i}\|_{W^{s,1}(\Omega)} \lesssim \|b\chi_{\mathcal{I}_i}\|^{1-s}_{L^1(\Omega)}\|b\chi_{\mathcal{I}_i}\|^{s}_{BV(\Omega)} \lesssim 2^{-i(1-s)}$$

Thus,

$$\|b\|_{W^{s,1}(\Omega)} = \left\|\sum_{i=1}^{\infty} b\chi_{\mathcal{I}_i}\right\|_{W^{s,1}(\Omega)} \lesssim \sum_{i=1}^{\infty} 2^{-i(1-s)} < +\infty\,.$$

We leave to the reader the obvious modifications to deal with the truncation of b.

3.4. **Step 7. Making the field autonomous.** For any $a(t,x) \in L^\infty(\mathbb{R}_t \times \mathbb{R}^2; \mathbb{R}^2)$, we can define

$$f(a)(y) = (1, a(y_0, y_1, y_2)),\ f(a) \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$$

If we apply this transformation to the nonautonomous field defined in Step 1 of the previous section, we reach an autonomous field $v = f(b)$ and an initial density

$$\theta_{in}(y_0, y_1, y_2) = \begin{cases} \rho_{in}(y_1, y_2)\ , \text{ for } -1 \leq y_0 \leq 0 \\ 0\ , \text{ otherwise} \end{cases}$$

Note that we again get 2 solutions for (1), namely:

i. $\theta'(t,y)$ such that $\theta'(t,y) = 1/2$ for $2<t<3$ and $2<y_0<t$
ii. $\tilde{\theta}(t,y)$ such that $\tilde{\theta}(t,y) = \rho_{in}(y_1,y_2)$ for $2<t<3$ and $2<y_0<t$

We then apply the same procedure to all the nonautonomous fields constructed in the previous section to get an example which satisfies the requirements of Theorem 1.1 with $d = 2$. The extension to higher dimension is simple: in $\mathbb{R}^{d+1} = \mathbb{R}^3 \times \mathbb{R}^{d-2}$ we just set the components v_j with $j \geq 3$ identically equal to 0, while the remaining three components are made constant in the directions $y_3, \ldots, y_d$. Similarly the initial data is assumed constant along the directions $y_3, \ldots, y_d$. This then gives a noncompactly supported example: to pass to a compactly supported proceed as in the previous section. We can have the initial data be smooth by arguing similarly as in Step 5.

The estimates obtained in Step 6 imply that $v \in W^{s,1}$ for every $s < 1$. Fix now $s < 1$ and select $\sigma \in (s, 1)$. By interpolation we have

$$\|v\|_{W^{s,p(s,\sigma)}} \leq C\|v\|_{L^\infty}^{1-\frac{s}{\sigma}} \|v\|_{W^{\sigma,1}}^{\frac{s}{\sigma}}$$

for $\frac{1}{p(s,\sigma)} = \frac{s}{\sigma}$. Since we can take σ arbitrarily close to 1 we conclude that $v \in W^{s,p}$ for every $p < \frac{1}{s}$.

Observe next that, the vector fields b_i^1 and b_i^2 enjoy similar estimates, uniformly in i. Since the $\tilde{v}_i$ and v_i' are obtained from the latter through convolution with standard kernels and an application of f, the same uniform estimates are inherited by them.

3.5. **Proof of Corollary 1.4.** First of all, let Φ_i^1 and Φ_i^2 be the unique regular Lagrangian flows of the truncated versions of the vector fields b_i^1 and b_i^2 (as outlined in Section 3.1. Since $b_i^2 = -b_i^2(2-t,x)$, $\Phi_i^2(2,x) = x$. Recall that for every $\sigma > 0$ $b \in L^\infty([0, 1-\sigma] \cup [1+\sigma, 2], BV(\mathbb{R}^2))$. In particular there is a unique regular Lagrangian flow $\Phi_\sigma^f : [0, 1-\sigma] \times \mathbb{R}^2 \to \mathbb{R}^2$ with $\Phi_\sigma^f(0,x) = x$ and a unique regular Lagrangian flow $\Phi_\sigma^b : [1+\sigma, 2] \times \mathbb{R}^2 \to \mathbb{R}^2$ with $\Phi_\sigma^b(2,x) = x$. Moreover $\Phi_\sigma^f(t,x) = \Phi_s^f(t,x)$ for every $t \leq 1-\sigma \leq 1-s$ and likewise $\Phi_\sigma^b(t,x) = \Phi_s^b(t,x)$ for every $t \geq 1+\sigma \geq 1+s$. We then identify a regular Lagrangian flow $\tilde{\Phi}(t,x)$ for b by setting $\tilde{\Phi}(t,x) = \Phi_t^f(t,x)$ for $t < 1$ and $\tilde{\Phi}(t,x) = \Phi_{t-1}^b(t,x)$ for $t > 1$. It is obvious that $\Phi_i^2(t,x) = \Phi(t,x)$ for every $t \leq 1 - 2^{-2i}$ and every $t \geq 1 + 2^{-2i}$, while $\Phi_i^2(t,x) = \Phi(1-2^{-2i}, x) = \Phi(1+2^{-2i}, x)$ for every $1 - 2^{2i} \leq t \leq 1 + 2^{-2i}$. The strong L^1_{loc} convergence of Φ_i^2 to Φ is thus obvious. As for the truncation procedure applied in Section 3.1 observe that the flows of the corresponding fields is given by Φ_i^2 for $x \in [-N, N]^2$, while it is the identity for any x outside the box at any time t. A simple analogous modification of Φ yields the desired limit. With a slight abuse of notation we keep denoting the fields by b_i^2 and the flows by Φ_i^2 and we denote $\tilde{\Phi}$ the limit. Note that the truncation procedure implies now the strong L^1 convergence on $[0,2] \times \mathbb{R}^2$ (i.e. without need of localizing).

For each fixed i consider now the regularization $b_{i,j}^2$ of Section 2.3 and let $\Phi_{i,j}^2$ be the corresponding regular Lagrangian flows. Using the DiPerna-Lions theory, for j sufficiently large we have $\|\Phi_{i,j}^2 - \Phi_i^2\|_{L^1([0,2]\times\mathbb{R}^2)} \leq (i+1)^{-1}$. In particular, by possibly choosing $j(i)$ even larger than in Section 2.3, we can ensure that $\Phi_{i,j(i)}^2$ converges strongly to $\tilde{\Phi}$.

Consider next the flows Ψ_i for the vector fields $b_i' = b_{i,j(i)}$, as produced in Section 2.3, after the trunction of Section 3.1. Recall that

$$\int \varphi(\Psi_i(t,x))\rho_{in,*}(x)\,dx = \int \varphi(x)\rho_{i,*}'(t,x)\,dx\,.$$

In particular we conclude

$$\lim_{i\to\infty}\int\int\varphi(t,\Psi_i(t,x))\rho_{in,*}(x)\,dx\,dt=\frac{1}{2}\int\int\varphi(t,x)\,dx\,dt$$

for every continuous test function $\varphi\in C_c((1,2)\times B(0,1))$ (assuming the integer N in Section 3.1 is chosen so that $[-\frac{N}{2},\frac{N}{2}]^2\supset B(0,1)$). If a subsequence of Ψ_i were to converge strongly to some Ψ, we would then have

$$\int\int\varphi(t,\Psi(t,x))\rho_{in,*}(x)\,dx\,dt=\frac{1}{2}\int\int\varphi(t,x)\,dx\,dt$$

for every such test function. It is easy to see that such a map Ψ cannot exist.

Remark 3.1. *Corollary 1.4 seems to suggest that from the point of view of flows the approximation b_i^2 is more reasonable. Note however that, while the function $\tilde\rho(1,\cdot)$ is identically equal to $\frac{1}{2}$ on $[-\frac{N}{2},\frac{N}{2}]^2$, for any time $t\in[0,1)\cup(1,2]$ it takes in fact the values 1 and 0 a.e. on $[-\frac{N}{2},\frac{N}{2}]^2$. In particular it is easy to see that there is no regular Lagrangian flow Λ of b with $\Lambda(1,x)=x$ for which we would have the identity $\tilde\rho(t,\Lambda(t,x))=\tilde\rho(1,x)$ for a.e. (t,x).*

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