Origami and Partial Differential Equations

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Abstract

Origami is the ancient Japanese art of folding paper and it has well known algebraic and geometrical properties, but it also has unexpected relations with partial differential equations. In this note we describe these relations for a large audience, leaving the technical aspects to other specialized papers.

Introduction

Origami is the ancient Japanese art of folding paper. One of the most known origami is the *crane*, represented on the right-hand side of Figure 1. Other than their artistic interest, why and how to associate origami with mathematics?

A motivation comes from the properties of origami. Many mathematicians interested in geometry or algebra (for example in group theory, Galois theory, graph theory) studied origami constructions.

An important issue is the geometrical construction of numbers. In some aspect origami turns out to be more powerful than the classical *rule and compass* construction. In fact, in order to determine what can be constructed through origami, it is important to formalize the rules. These are known as *Huzita axioms* and have been proposed by Hatori, Huzita, Justin and Lang, see [1].

On the contrary, in this exposition we present an analytic approach to origami, based on maps which satisfy a suitable system of partial differential equations. We remain here to a non-technical level of exposition. The interested reader might refer to the papers [5, 7] obtained by the authors in collaboration with Bernard Dacorogna (*École Polytechnique Fédérale de Lausanne*). In these papers fractal constructions of origami are shown to solve a special class of Dirichlet problems arising in nonlinear elasticity (see [4]).

1 Axiomatic construction of origami

As already said in the introduction, origami constructions can be considered from an axiomatic point of view in a similar way as rule and compass construction. We give here few details about this geometric approach to origami (the interested reader may refer to [1].

Here are the *seven axioms*.

- *Axiom 1:* given two points P_1 and P_2, there is a unique fold passing through both of them.
- *Axiom 2:* given two points P_1 and P_2, there is a unique fold placing P_1 onto P_2.
- *Axiom 3:* given two lines L_1 and L_2, there is a fold placing L_1 onto L_2.
- *Axiom 4:* given a point P and a line L, there is a unique fold perpendicular to L passing through P.
- *Axiom 5:* given two points P_1 and P_2 and a line L, there is a fold placing P_1 onto L and passing through P_2.
- *Axiom 6:* given two points P_1 and P_2 and two lines L_1 and L_2, there is a fold placing P_1 onto L_1 and P_2 onto L_2.
- *Axiom 7:* given a point P and two lines L_1 and L_2, there is a fold placing P onto L_1 and perpendicular to L_2.

However this is not the only possible mathematical motivation and in the following we propose a different approach. We will present a mathematical model of origami which has a double purpose. In one hand we give an *analytical approach* which provides a new perspective to the existing algebraic and geometrical models. In the other hand we use origami as a tool to exhibit explicit solutions to some systems of partial differential equations.

2 A global definition of origami as a map

Instead of listing a set of properties, we identify an origami with a mathematical object, i.e., we give a mathematical model. We skip the *overlapping* and *interpenetration* problems (see [5]).

If we denote by $\Omega \subset \mathbb{R}^2$ a two dimensional domain (usually Ω is a rectangle), then an origami is a *suitable* immersion of the sheet of paper in the three dimensional space. Hence it can be identified with a map u

$$u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3.$$

Since origami is a folded paper, the map u cannot be everywhere smooth; it is only piecewise smooth. In fact *folding creates discontinuities in the gradient:* we do not allow cutting the sheet of paper. Thus u is a continuous map.

The *singular set* $\Sigma = \Sigma_u$ is the set of discontinuities of the gradient Du. This set represents the union of curves where the paper is folded and hence it is also called *crease pattern* in the origami context. Usually this set is composed by straight segments.

In the model – at the same time – we construct the origami and we unfold it. We now explain in which sense.

Let's consider the crane origami represented in Figure 1. If we unfold the origami we see the crease pattern Σ impressed in the sheet of paper. Clearly *the singular set* Σ is *uniquely determined* by the origami. In this case Σ is *the set of segments* represented on the left in Figure 1. What we really consider is a function, an application, a map u from *the sheet of paper* to *the three-dimensional space.*

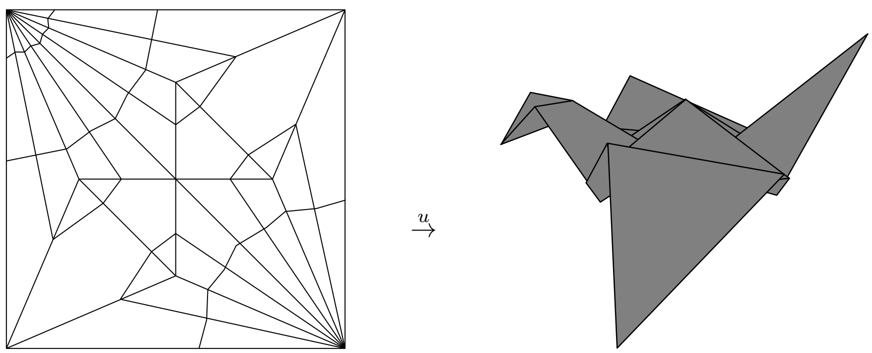

Figure 1: On the right: the crane is the most famous origami. On the left: the corresponding singular set

As we said, usually this set is composed by straight segments, but it is also possible to make origami with curved folds: this happens for instance in the representation of a map $u\colon \mathbb{R}^2 \to \mathbb{R}^3$, which has as singular set Σ_u along a circular curve, as in Figure 2.

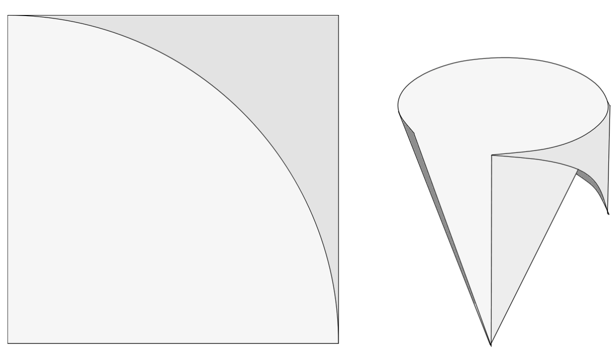

Figure 2: A non-flat origami with a curved singular set.

A sheet of paper Ω (again recall that usually Ω is a rectangle) is rigid in tangential directions. If a sheet of paper is constrained on a plane, it would only be possible to achieve rigid motions, i.e., rotations and translations of the whole sheet. On the other hand, in the normal direction it can be easily folded.

This property can be expressed in analytic form either with *local isometries* or with *orthogonality*.

That is, where the gradient of the map u exists (where the paper is not folded) angles and distances must be respected, cannot change in the image of the map. The map must be a *local isometry* and its gradient matrix must be *orthogonal*.

In more details, the origami $u = \left(u^j\left(x_1, x_2\right)\right)_{j=1,2,3}$ is a *vector-valued map* in two variables

$$u \colon \mathbb{R}^2 \to \mathbb{R}^3 .$$

That is, u is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ and its *gradient* $Du = \left(\frac{\partial u^j}{\partial x_i}\right)$ is a 3×2 *matrix*

$$u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}, \qquad Du = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \frac{\partial u^1}{\partial x_2} \\ \frac{\partial u^2}{\partial x_1} & \frac{\partial u^2}{\partial x_2} \\ \frac{\partial u^3}{\partial x_1} & \frac{\partial u^3}{\partial x_2} \end{pmatrix}.$$

The gradient $Du(x)$ has to be an *orthogonal* 3×2 matrix, i.e.,

$$Du^t \cdot Du = I.$$

This orthogonality condition is equivalent to the *differential system*

$$\sum_{i=1}^{3} \frac{\partial u^i}{\partial x_h} \cdot \frac{\partial u^i}{\partial x_k} = \delta_{hk} , \qquad \forall\ h, k = 1, 2,$$

which, in explicit form, means

$$\begin{cases} \left(\frac{\partial u^1}{\partial x_1}\right)^2 + \left(\frac{\partial u^2}{\partial x_1}\right)^2 + \left(\frac{\partial u^3}{\partial x_1}\right)^2 = 1 \\ \left(\frac{\partial u^1}{\partial x_2}\right)^2 + \left(\frac{\partial u^2}{\partial x_2}\right)^2 + \left(\frac{\partial u^3}{\partial x_2}\right)^2 = 1 \\ \frac{\partial u^1}{\partial x_1}\frac{\partial u^1}{\partial x_2} + \frac{\partial u^2}{\partial x_1}\frac{\partial u^2}{\partial x_2} + \frac{\partial u^3}{\partial x_1}\frac{\partial u^3}{\partial x_2} = 0 . \end{cases}$$

As we already said, we do not allow cutting the sheet of paper $u(\Omega)$. Thus $u \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a continuous map, more precisely a *Lipschitz-continuous map*. The singular set $\Sigma = \Sigma_u$, i.e., the set of discontinuities of the gradient Du, may have a very complicated structure, even no-structure, for a general Lipschitz-continuous map.

If we limit ourselves to *piecewise smooth maps*, precisely to *piecewise C^1 rigid maps*, then we have a more readable situation. For instance, for the map whose graph is represented in Figure 3 the singular set $\Sigma = \Sigma_u$ is empty.

As we said the singular set $\Sigma = \Sigma_u$ is *uniquely determined* by the map u, but in general the reverse is not true; in fact many rigid maps u may have the same singular set. On the contrary a special attention will be given to the so-called flat origami. A *flat origami* is defined as a map whose image is contained

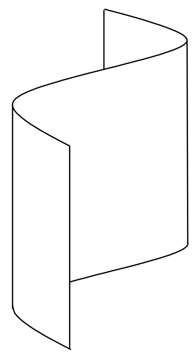

Figure 3: This sheet of paper $u(\Omega)$ is bended but not folded. The corresponding singular set Σ_u is empty (in correspondence to several (not folded) maps).

in a plane. It can be represented, up to a change of coordinates, as a map $u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. Let us consider a *flat origami*, i.e., instead of

$$u\colon \Omega \subset \mathbb{R}^2 \to u(\Omega) \subset \mathbb{R}^3,$$

we consider an application of the form

$$u\colon \Omega \subset \mathbb{R}^2 \to u(\Omega) \subset \mathbb{R}^2.$$

3 Analytic properties of flat origami

In the case of flat origami we have the possibility of reconstructing the map u from its singular set Σ_u. That is, if $u(\Omega) \subset \mathbb{R}^2$, it is possible to *uniquely reconstruct* a map, with orthogonal gradient, from a given set of singularities; i.e., from a given singular set. A fundamental ingredient in this reconstruction is a necessary and sufficient *compatibility condition* on the geometry of the singular set.

Following the terminology that can be found in the not numerous mathematical literature on origami (see for instance [2]), we call it *angle condition*. It was discovered by Kawasaki in the origami setting.

Let $\Sigma \subset \Omega \subset \mathbb{R}^2$ be a locally finite union of segments. Then Σ is the *singular set* of a *piecewise C^1 rigid map* if and only if the following *angle condition* holds at every *internal* vertex of Σ. If we let $\alpha_1, \ldots, \alpha_N$ be the *amplitude of the consecutive angles* determined by the N edges of Σ meeting in the vertex, then N is even and (see Figure 4)

$$\alpha_1 + \alpha_3 + \ldots + \alpha_{N-1} = \alpha_2 + \alpha_4 + \ldots + \alpha_N = \pi.$$

We prove that every polyhedral pattern Σ which satisfies the *angle condition* is the singular set Σ_u of some rigid map u. Precisely the following result holds (the result is valid in the general $n-$dimensional setting, with $\Omega \subset \mathbb{R}^n$).

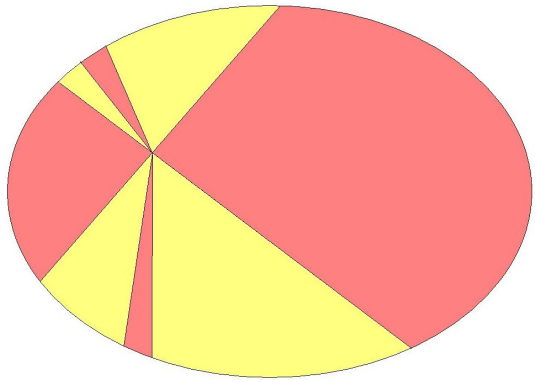

Figure 4: *the angle condition*: at every internal vertex an even number of angles meet. The alternating sum of angles is equal each other.

The following result has been proved in [5].

Theorem 1 (Recovery Theorem) *Let Ω be a* simply connected *open subset of $\mathbb{R}^2$. Let $\Sigma \subset \Omega$ be a locally finite polyhedral set satisfying the angle condition at every vertex. Then there exists a map u with orthogonal gradient* (flat origami) *such that $\Sigma = \Sigma_u$ is the singular set of u. Moreover u is uniquely determined once we fix the value $y_0 = u(x_0)$ and the Jacobian gradient $J_0 = Du(x_0)$ at a point $x_0 \in \Omega \setminus \Sigma$.*

For a *flat origami* $u \colon \Omega \subset \mathbb{R}^2 \to u(\Omega) \subset \mathbb{R}^2$, with components (with a little abuse of notation we identify $\mathbb{R}^2$ with a subset of $\mathbb{R}^3$)

$$u = \begin{pmatrix} u^1 \\ u^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \qquad Du = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \frac{\partial u^1}{\partial x_2} \\ \frac{\partial u^2}{\partial x_1} & \frac{\partial u^2}{\partial x_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \frac{\partial u^1}{\partial x_2} \\ \frac{\partial u^2}{\partial x_1} & \frac{\partial u^2}{\partial x_2} \end{pmatrix},$$

the *orthogonality condition* $Du^t \cdot Du = I$ is equivalent to the *differential system*

$$\begin{cases} \left(\frac{\partial u^1}{\partial x_1}\right)^2 + \left(\frac{\partial u^2}{\partial x_1}\right)^2 = 1 \\ \left(\frac{\partial u^1}{\partial x_2}\right)^2 + \left(\frac{\partial u^2}{\partial x_2}\right)^2 = 1 \\ \frac{\partial u^1}{\partial x_1}\frac{\partial u^1}{\partial x_2} + \frac{\partial u^2}{\partial x_1}\frac{\partial u^2}{\partial x_2} = 0 \end{cases}$$

and this gives a representation for the *determinant* of the 2×2 matrix Du. In fact, by an algebraic computation, we also find

$$(\det Du)^2 = \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2} - \frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2}\right)^2$$

$$= \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2}\right)^2 + \left(\frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2}\right)^2 - 2\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2}\frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2}.$$

By multiplying side by side the first two equations of (3) we get

$$\left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^1}{\partial x_2}\right)^2 + \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2}\right)^2 + \left(\frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2}\right)^2 + \left(\frac{\partial u^2}{\partial x_1}\frac{\partial u^2}{\partial x_2}\right)^2 = 1$$

and therefore

$$\begin{aligned}(\det Du)^2 &= 1 - \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^1}{\partial x_2}\right)^2 - \left(\frac{\partial u^2}{\partial x_1}\frac{\partial u^2}{\partial x_2}\right)^2 - 2\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2}\frac{\partial u^2}{\partial x_1}\frac{\partial u^1}{\partial x_2} \\ &= 1 - \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^1}{\partial x_2} + \frac{\partial u^2}{\partial x_1}\frac{\partial u^2}{\partial x_2}\right)^2.\end{aligned}$$

Then, by the third equation in (3), we finally get

$$\det Du = \pm 1.$$

The sign of the determinant of the matrix Du gives a *coloration* of the domain Ω, as in Figure 5.

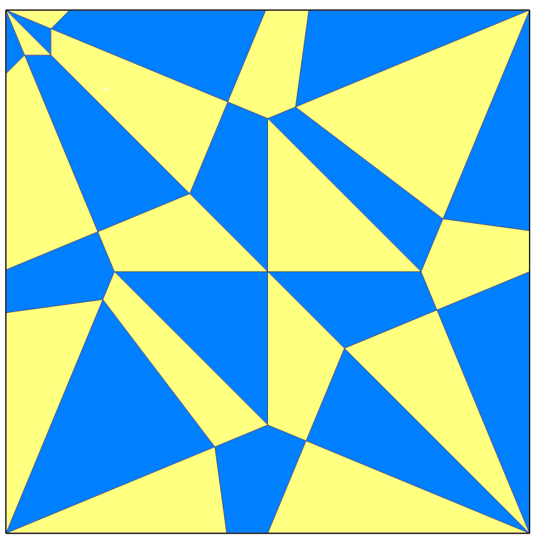

Figure 5: The domain Ω colored by means of the sign of $\det Du = \pm 1$.

4 Boundary value problems and fractal constructions

We have another condition to satisfy: it is a *boundary condition.* That is, we look for maps u with a given value at the boundary $\partial\Omega$ of Ω:

$$u(x) = \varphi(x), \quad x \in \partial\Omega.$$

For instance $u(x) = 0$ for $x \in \partial\Omega$. In order to achieve the boundary datum we must arrive at the boundary with finer and finer subdivisions of the set Ω; i.e., we must have a singular set Σ_u of *fractal form.* This is due to the fact that $\det Du = \pm 1$, in particular $\det Du \neq 0$ and hence, by the implicit function theorem, the map u is locally invertible if it is smooth. This is in contrast with a constant boundary value.

We apply the recovery theorem to a singular set Σ_u of fractal form (at the boundary, with the aim to satisfy a boundary condition), for which the *angle condition* is satisfied. In fact the set Σ_u which we are going to consider has the property that it divides Ω into *two families of colored sets* (see Figures 6):

- grey *rectangles*, where $\det Du = -1$.
- white *convex polygons*, where $\det Du = +1$.

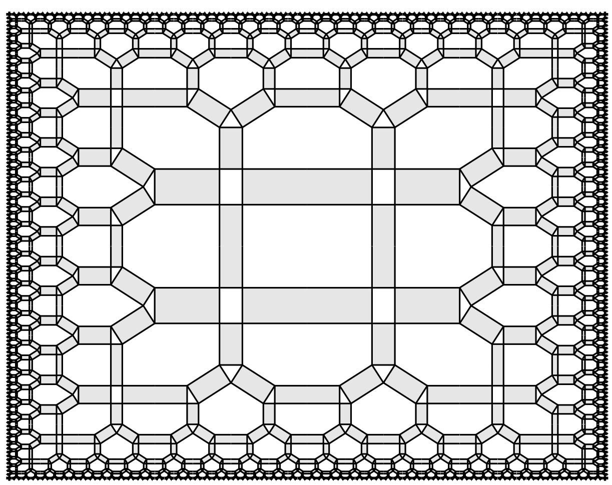

Figure 6: Escher-type not-periodic picture satisfying the *angle condition* at every vertex, with *fractal structure* at the boundary which allows to fix a *boundary value*

Each vertex of the singular set Σ_u is shared by two rectangles, hence the *angle condition* holds. We see the shape of the sets that we consider: it is an *Escher-type not-periodic picture.*

There exists a piecewise C^1 rigid map $u : \bar{\Omega} \to \mathbb{R}^2$ (flat origami), with singular set Σ_u as in Figure 6, such that $u = \varphi$ on $\partial\Omega$. Thus u satisfies the *Dirichlet problem*

$$\begin{cases} Du \in O(2), & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases}$$

for some given boundary values φ (see [5]).

From the *"scalar"* picture (see Figure 6) we can also *"read"* the boundary value of the *vectorial* map u.

We end by giving a picture with a $3-$*dimensional flat origami*. It is a *mathematical origami*, being a rigid application from $\mathbb{R}^3 \to \mathbb{R}^3$.

Theorem 2 (3D Dirichlet Problem) *On the cube $\Omega = [0,1]^3$ it is possible to define a piecewise C^1 rigid map $u : \Omega \to \mathbb{R}^3$ such that $u = 0$ on the boundary. The singular set Σ_u is represented in Figure 7.*

This result was first obtained by Cellina and Perrotta [3] and extended in [6] to general n-dimensional origami.

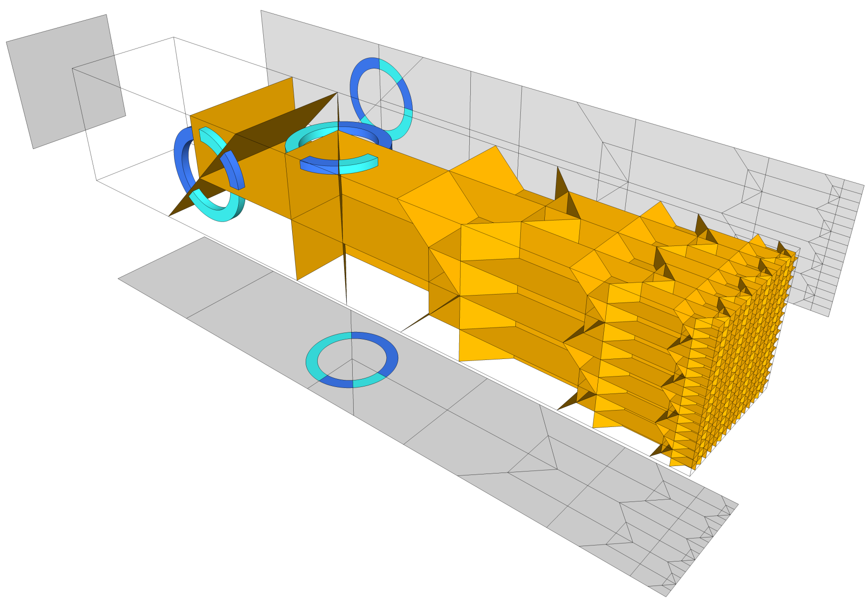

Figure 7: The singular set which defines a 3-dimensional origami. The angle condition is satisfied on every edge (the rings highlight the measures of the alternating angles)

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