

On the Regularity of Optimal Dynamic Blocking Strategies

Alberto Bressan and Maria Teresa Chiri

Department of Mathematics, Penn State University
University Park, Pa. 16802, USA.

e-mails: axb62@psu.edu, mxc6028@psu.edu.

November 25, 2020

Abstract

The paper studies a dynamic blocking problem, motivated by a model of optimal fire confinement. While the fire can expand with unit speed in all directions, barriers are constructed in real time. An optimal strategy is sought, minimizing the total value of the burned region, plus a construction cost. It is well known that optimal barriers exist. In general, they are a countable union of compact, connected, rectifiable sets. The main result of the present paper shows that optimal barriers are nowhere dense. The proof relies on new estimates on the reachable sets and on optimal trajectories for the fire, solving a minimum time problem in the presence of obstacles.

Keywords: Dynamic blocking problem, minimum time problem with obstacles.

Mathematics Subject Classification: 49Q20, 34A60, 49J24, 93B03.

1 Introduction

We consider the dynamic blocking problem introduced in [3], for a model of wildfire propagation [17]. To restrict the spreading of the fire, it is assumed that a barrier can be constructed, in real time. This could be a thin strip of land which is either soaked with water poured down from a helicopter, or cleared from all vegetation using a bulldozer, or sprayed with fire extinguisher by a team of firemen. In all cases, the fire will not cross that particular strip of land. Here the key point is that the barrier is being constructed at the same time as the fire front is advancing.

In this setting, a natural problem is to find the best possible strategy. In other words, we seek the optimal location of the barriers, in order to minimize:

$$[\text{total value of the burned area}] + [\text{total cost for constructing the barriers}] \quad (1.1)$$

among all barriers that can be constructed in real time.

We consider here the simplest situation where the fire initially burns on an open set R_0 , and propagates with unit speed in all directions. We assume

(A1) *The initial set $R_0 \subset \mathbb{R}^2$ is open, bounded, nonempty, connected, with Lipschitz boundary ∂R_0 .*

If barriers are not present, for each $t \geq 0$ the set $R(t)$ reached by the fire is defined as

$$\begin{aligned} R(t) &\doteq \{x(t); x(\cdot) \text{ is 1-Lipschitz, } x(0) \in R_0\} \\ &= \{x \in \mathbb{R}^2; d(x, R_0) < t\}. \end{aligned} \tag{1.2}$$

Here and in the sequel, by 1-Lipschitz we mean a function with Lipschitz constant 1. Moreover, $d(x, \Omega)$ denotes the distance of a point x to the set $\Omega \subset \mathbb{R}^2$, while $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in \mathbb{R}^2 . The closure and the boundary of Ω are denoted by $\bar{\Omega}$ and $\partial\Omega$ respectively. By $B(x, r)$ we denote the open ball centered at x with radius r . More generally, for $\Omega \subset \mathbb{R}^2$, $B(\Omega, r) = \{x; d(x, \Omega) < r\}$ denotes the open neighborhood of radius r around Ω . Finally, m_1, m_2 denote the 1-dimensional and 2-dimensional Hausdorff measure, respectively.

Next, we assume that the spreading of the fire can be controlled by constructing a barrier.

Definition 1.1. *A barrier $\Gamma \subset \mathbb{R}^2$ is a disjoint union of countably many compact connected, rectifiable sets, with finite total length.*

Throughout the following, we write

$$\Gamma = \bigcup_{i \geq 1} \Gamma_i \tag{1.3}$$

to denote a barrier, as a union of its compact, rectifiable, connected components.

Intuitively, we think of a barrier as a family of curves in the plane, which the fire cannot cross. When a barrier Γ is in place, the set reached by the fire is reduced. This leads to the definition of the new reachable set

$$R^\Gamma(t) \doteq \left\{ x(t); x(\cdot) \text{ is 1-Lipschitz, } x(0) \in R_0, x(\tau) \notin \Gamma \text{ for all } \tau \in [0, t] \right\}. \tag{1.4}$$

Clearly, in this case the burned set will be somewhat smaller: $R^\Gamma(t) \subseteq R(t)$ for every $t \geq 0$. Since in our model the barrier is constructed at the same time as the fire propagates, a restriction on its length must be imposed.

Definition 1.2. *Given a construction speed $\sigma > 1$, we say that the barrier Γ is **admissible** if*

$$m_1(\Gamma \cap \overline{R^\Gamma(t)}) \leq \sigma t \quad \text{for all } t \geq 0. \tag{1.5}$$

Remark 1.3. For each $t \geq 0$, the set

$$\gamma(t) \doteq \Gamma \cap \overline{R^\Gamma(t)}$$

appearing in (1.5) is the part of the barrier Γ touched by the fire at time t . This is the portion that actually needs to be put in place within time t , in order to restrain the fire. The remaining portion $\Gamma \setminus \gamma(t)$ can be constructed at a later time. This motivates the above definition. The equivalence between different formulations of the dynamic blocking problem was proved in [8].

Fire propagation can equivalently be described in terms of the minimum time function

$$T^\Gamma(x) \doteq \inf\{t \geq 0; x \in \overline{R^\Gamma(t)}\}. \quad (1.6)$$

From the definition, it follows that T^Γ is lower semicontinuous. We think of $T^\Gamma(x)$ as the minimum time needed for the fire to reach the point x , starting from R_0 and without crossing the barrier. Notice that $T^\Gamma(x) = +\infty$ if the fire never reaches a neighborhood of x . In general, the minimal time function T^Γ can be computed by solving a Hamilton-Jacobi equation with obstacles, namely

$$|\nabla T(x)| = 1 \quad x \notin \Gamma, \quad (1.7)$$

$$T(x) = 0 \quad \text{if } x \in R_0. \quad (1.8)$$

For a precise definition and properties of this solution, see [13]. We recall that T^Γ is locally an SBV function [1]. The set where it has jumps is contained inside Γ . If the function T^Γ is known, we can then recover the region $R^\Gamma(t)$ burned within time t as

$$\overline{R^\Gamma(t)} = \{x \in \mathbb{R}^2; T^\Gamma(x) \leq t\}.$$

Two mathematical problems can now be formulated.

(BP) Blocking Problem. Given a bounded open set R_0 , decide whether there exists an admissible barrier Γ such that the entire region burned by the fire

$$R_\infty^\Gamma \doteq \bigcup_{t>0} R^\Gamma(t) \quad (1.9)$$

is bounded.

(OP) Optimization Problem. Given an initial set R_0 and a constant $c_0 \geq 0$, find an admissible barrier Γ which minimizes the total cost

$$\mathcal{J}(\Gamma) \doteq m_2(R_\infty^\Gamma) + c_0 m_1(\Gamma). \quad (1.10)$$

Remark 1.4. For a given initial domain R_0 , the set R_∞^Γ in (1.9) burned by the fire can be characterized as the union of all connected components of $\mathbb{R}^2 \setminus \Gamma$ which intersect R_0 . For any bounded open set R_0 , it is known [3, 4, 5, 9] that a blocking strategy exists if the construction speed is $\sigma > 2$, while it does not exist if $\sigma \leq 1$. The existence of a blocking strategy for $\sigma \in]1, 2]$ is a challenging open problem. See the review [4] for a more comprehensive discussion.

In a very general setting, the existence of an optimal barrier was proved in [6, 13]. Under the assumption that this optimal barrier is the union of finitely many Lipschitz arcs, various necessary conditions were derived in [3, 10, 18]. Indeed, assuming Lipschitz regularity, one can reformulate the problem in the classical setting of the Calculus of Variations, or within the theory of optimal control [7, 12, 15]. Necessary conditions for optimality are thus obtained in terms of the Euler-Lagrange equations, or by applying the Pontryagin Maximum Principle. For example, when the initial set R_0 is a circle and the construction speed is $\sigma > 2$, among all simple closed curves, it is known that the admissible barrier that encloses the smallest burned area is the union of an arc of circumference and two logarithmic spirals [11].

Unfortunately, the results in [6, 13] only provide the existence of an optimal barrier Γ^* with the minimal regularity properties stated in Definition 1.1. Namely, we only know that Γ^* is the union of countably many compact, connected, rectifiable sets. It remains an outstanding open problem to close this gap, establishing further regularity properties of the optimal barrier, so that necessary conditions for optimality can then be applied. In the present paper we take a step in this direction. Our main goal is to prove

Theorem 1.5. *For the optimization problem (OP), any optimal barrier Γ is nowhere dense.*

This result is motivated by the following considerations. As shown in Fig. 1, left, the optimal barrier can be split as

$$\Gamma = \Gamma^{\text{block}} \cup \Gamma^{\text{delay}}.$$

Here $\Gamma^{\text{block}} = \overline{\partial R_\infty^\Gamma}$ is the portion which actually separates the burned region from the unburned one. On the other hand, Γ^{delay} accounts for the walls whose only purpose is to delay the advancement of the fire front. Eventually, these walls are encircled by the fire on both sides.

We recall that the fire propagates with speed 1, while the barrier is constructed at speed $\sigma > 1$. Building a connected component Γ_1 of the barrier, with length ℓ_1 , thus requires an amount of time $\tau_1 = \ell_1/\sigma$. On the other hand, the fire needs up to time ℓ_1 in order to completely surround Γ_1 . In some cases, it can thus be an advantage to construct some barriers with the sole purpose of slowing down the propagation of the fire.

At an intuitive level, however, building a barrier which contains a large number of very small connected components should be ineffective, because the fire can quickly get around each connected portion. To prove Theorem 1.5, we need to show that a collection of walls which is dense on an open set cannot be optimal. Indeed, some of these walls should be removed, because the time needed to build them is longer than the amount by which they delay the advancement of the fire front.

The heart of the matter is to understand which portions of the barrier can be removed. As shown in Fig. 1, left, the connected component Γ_1 delays the advancement of the fire front. If we remove Γ_1 , then we do not have enough time to construct Γ_2 . Hence, to achieve an admissible barrier $\Gamma' \subseteq \Gamma \setminus \Gamma_1$ satisfying (1.5), we should also remove the component Γ_2 . In turn, this may force us to remove further components $\Gamma_3, \Gamma_4, \dots$. If at the end of this process we need to remove the outer component Γ^{block} as well, then the entire construction fails.

Toward a proof of Theorem 1.5, we shall construct a “flow box” Δ , as shown in Fig. 1, right. Here the lower boundary coincides with the location of the fire front at some time $t_0 > 0$. The two sides are straight lines, consisting of optimal trajectories for the fire which do not intersect any of the barriers. The upper boundary is a curve γ^* , consisting of points having a fixed distance $h > 0$ from γ_0 . These are the points that the fire would reach at time $t^* = t_0 + h$, if no barriers were present. A careful analysis will show that, by removing all the barriers contained inside Δ , the remaining portion $\Gamma^\diamond = \Gamma \setminus \Delta$ is still admissible, and achieves a lower total cost (1.10).

The remainder of the paper is organized as follows. Section 2 is concerned with the minimum time problem for the fire, in the presence of barriers. The first main result, Lemma 2.2, considers a path ξ that crosses some of the connected components Γ_i of the barrier. By

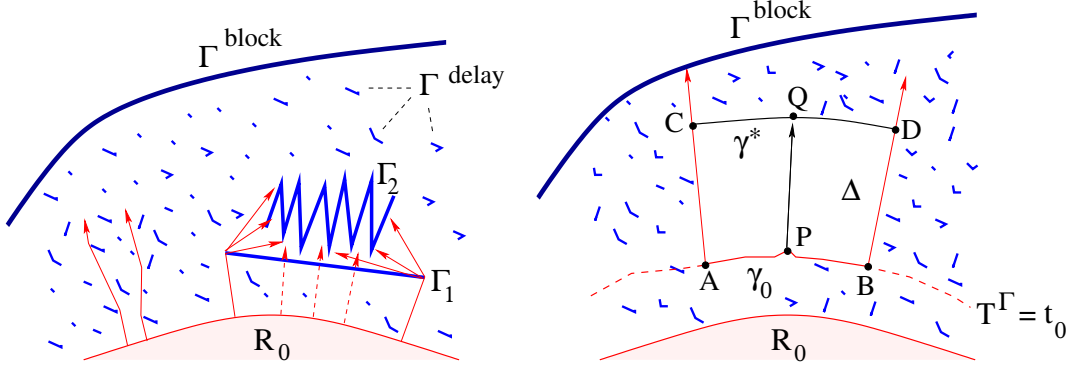


Figure 1: Left: if the connected component Γ_1 of the barrier is removed, then there is not enough time to construct Γ_2 , before the fire reaches it. Right: by removing all barriers inside a carefully chosen “flow box” Δ , the remaining portion $\Gamma^\diamond = \Gamma \setminus \Delta$ still form an admissible barrier, blocking the fire within the same region as before and yielding a smaller total cost.

inserting additional loops, we prove the existence of a modified path $\tilde{\xi}$ which starts and ends at almost the same points as ξ , and does not touch the barrier. Moreover, the difference between the lengths of the two paths is no greater than the total length $\sum_i m_1(\Gamma_i)$ of the components which were crossed. The second main result of this section, Lemma 2.4, shows that the set of times where the fire front touches a component Γ_i is contained in an interval $[a_i, b_i]$ of length $b_i - a_i \leq m_1(\Gamma_i)$. Moreover, when no barrier is touched, the set reached by the fire expands with unit speed in all directions. All these results are intuitively obvious when Γ contains finitely many compact, connected components. However, if Γ is the union of countably many components, possibly everywhere dense, a more careful proof is needed.

In Section 3 we prove some lemmas describing how the minimum time function T^Γ in (1.6) changes when the barrier Γ is perturbed. This analysis is useful, because it allows us to approximate an arbitrary barrier with a polygonal one.

Section 4 continues the study of optimal trajectories for the fire, reaching points $x \in \mathbb{R}^2$ in minimum time without crossing the barrier Γ . The key result in this section (Lemma 4.1) shows that, if the total length of all barriers is small, most of these optimal trajectories for the fire contain long straight segments. This fact can be rigorously stated in terms of an integral inequality. The proof is first achieved in the case of polygonal barriers. The general case follows by an approximation argument.

Section 5 contains another key estimate. Roughly speaking, Lemma 5.5 shows that, if a barrier Γ is “ ε -sparse”, then the additional time needed by the fire to go around it is bounded by $9\varepsilon m_1(\Gamma)$. We observe that the time needed to construct this barrier is $\sigma^{-1}m_1(\Gamma)$, where σ is the construction speed. If $\varepsilon > 0$ is sufficiently small, the time needed to construct this portion of barrier is not compensated by its effectiveness in delaying the advance of the fire front. One can thus conclude that the barrier is not optimal.

The proof of Theorem 1.5 is then completed in Section 6. It consists of two main steps. First, we use Lemma 4.1 to construct a “flow box” Δ , as shown in Fig. 1, whose sides are segments contained in optimal trajectories for the fire which do not cross the barrier Γ . We then use Lemma 5.5 and show that, by removing all the portions of the barrier contained inside Δ , one obtains a new admissible barrier $\Gamma^\diamond = \Gamma \setminus \Delta$, which yields a smaller total cost.

2 Optimal trajectories for the fire

In this section we focus on the optimization problem for the fire. Let $R_0 \subset \mathbb{R}^2$ be a bounded, connected open set, and let $\Gamma = \cup_i \Gamma_i$ be a barrier, consisting of countably many compact, rectifiable, connected components, with finite total length. We seek trajectories that, starting from the closure $\overline{R_0}$, reach points $x \in \mathbb{R}^2$ in minimum time, without crossing Γ . To achieve the existence of these optimal trajectories, referring to Fig. 2 we introduce

Definition 2.1. A trajectory for the fire $t \mapsto x(t)$, $t \in [0, T]$, is **admissible** if there exists a sequence of 1-Lipschitz trajectories $t \mapsto x_n(t)$ such that

$$x_n(0) \in R_0, \quad x_n(t) \notin \Gamma \quad \text{for all } t \in [0, T], \quad (2.1)$$

and moreover $x_n(t) \rightarrow x(t)$ uniformly on $[0, T]$, as $n \rightarrow \infty$.

We say that a trajectory $t \mapsto x(t)$ **does not touch the barrier** Γ if $x(t) \notin \Gamma$ for all times $t \geq 0$. If $x(\cdot)$ is the uniform limit of trajectories $x_n(\cdot)$ that do not touch Γ , we say that $x(\cdot)$ **does not cross the barrier** Γ .

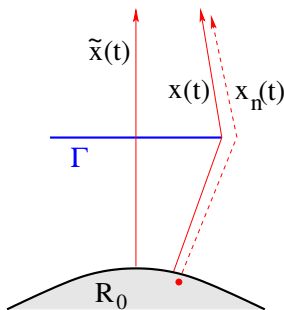


Figure 2: The trajectory $t \mapsto x(t)$ touches the barrier Γ , but does not cross it. Indeed, it can be obtained as a uniform limit of trajectories $x_n(\cdot)$ that do not touch Γ . On the other hand, the trajectory $\tilde{x}(\cdot)$ is not admissible: it goes right across the barrier.

Given a point $\bar{x} \in \mathbb{R}^2$, we seek an admissible trajectory $t \mapsto x(t)$ which starts from a point in the closure $\overline{R_0}$ and reaches \bar{x} in minimum time without crossing Γ . If \bar{x} can be reached in finite time, the existence of such an optimal trajectory is straightforward. Indeed, define

$$T_{\text{inf}} \doteq \lim_{\varepsilon \rightarrow 0} \left[\text{infimum time needed to reach a point in the ball } B(\bar{x}, \varepsilon), \right. \\ \left. \text{starting from } R_0 \text{ and without touching the barrier } \Gamma \right].$$

Let $x_n : [0, T_n] \mapsto \mathbb{R}^2$ be a minimizing sequence of 1-Lipschitz trajectories, satisfying (2.1) together with

$$x_n(T_n) \rightarrow \bar{x}, \quad T_n \rightarrow T_{\text{inf}} \quad \text{as } n \rightarrow \infty.$$

By taking a subsequence we can assume the uniform convergence $x_n \rightarrow x$, for some limit function $x : [0, T_{\text{inf}}] \mapsto \mathbb{R}^2$. According to Definition 2.1, this limit trajectory is admissible. Hence it provides an optimal solution.

Given a trajectory $\xi : [0, \tau] \mapsto \mathbb{R}^2$ that crosses part of the barrier, the next lemma provides the key tool for constructing trajectories that “loop around” each connected component, and reach almost the same endpoint without touching Γ .

Lemma 2.2. Consider a barrier $\Gamma = \cup_{i \geq 1} \Gamma_i$, written as the union of its connected components. Assume that $\mathbb{R}^2 \setminus \Gamma$ is connected. Let $\xi : [0, \tau] \mapsto \mathbb{R}^2$ be a Lipschitz path, parameterized by arc length, such that

$$\xi(t) \notin \Gamma_i \quad \text{for all } t \in [0, \tau], \quad i \leq \nu. \quad (2.2)$$

Then, for any $\epsilon > 0$, there exists a path $\tilde{\xi} : [0, \tilde{\tau}] \mapsto \mathbb{R}^2$, also parameterized by arc length, such that

$$|\tilde{\xi}(0) - \xi(0)| \leq \epsilon, \quad |\tilde{\xi}(\tilde{\tau}) - \xi(\tau)| \leq \epsilon, \quad (2.3)$$

$$\tilde{\xi}(t) \notin \Gamma \quad \text{for all } t \in [0, \tilde{\tau}], \quad (2.4)$$

and with length

$$\tilde{\tau} \leq \tau + \sum_{i > \nu} m_1(\Gamma_i). \quad (2.5)$$

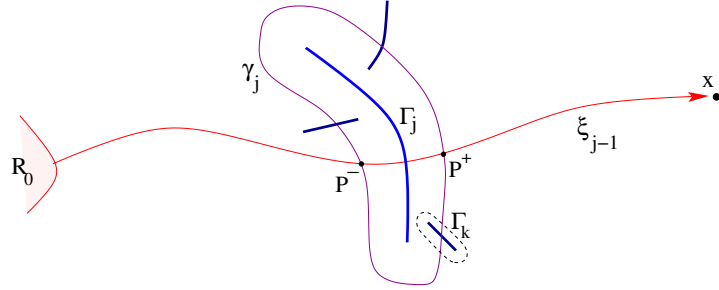


Figure 3: The construction used in the proof of Lemma 3.4. If the trajectory $\xi_{j-1}(\cdot)$ crosses the set Γ_j , we construct a detour γ_j of radius r_j around Γ_j . At a subsequent step, we may be forced to construct a second detour to avoid hitting the component Γ_k . Hence the new path may get closer to Γ_j . In the inductive construction, it is essential to show that all paths keep a uniformly positive distance from Γ_j .

Proof. Let $\epsilon > 0$ be given. The new path $\tilde{\xi}$ will be obtained as limit of a sequence of paths $\xi_j : [0, \tau_j] \mapsto \mathbb{R}^2$, $j \geq 0$, by an inductive procedure. Each inductive step will also determine two auxiliary constants $r_j, \delta_j > 0$.

1. The induction starts by setting $\tau_0 = \tau$, and defining $\xi_0(t) = \xi(t)$ for all $t \geq 0$. Moreover, we choose $\delta_0 > 0$ so that

$$\delta_0 < \frac{\epsilon}{4}, \quad \delta_0 < d(\xi(t), \Gamma_i) \quad \text{for all } t \in [0, \tau], \quad i = 1, \dots, \nu. \quad (2.6)$$

For every $j \geq 1$, the constants $r_j, \delta_j > 0$ and the path $\xi_j : [0, \tau_j] \mapsto \mathbb{R}^2$ will satisfy the following properties.

(i) For every $i \leq j$ and $t \in [0, \tau_j]$ one has

$$d(\xi_j(t), \Gamma_i) \geq (2 - 2^{i-j})\delta_i. \quad (2.7)$$

(ii) For $j \leq \nu$ we simply take $\xi_j = \xi_0$. For $j > \nu$, the length of the path ξ_j satisfies

$$\tau_j < \tau_{j-1} + (1 + \epsilon)m_1(\Gamma_j). \quad (2.8)$$

(iii) The endpoints satisfy

$$|\xi_j(0) - \xi(0)| \leq (1 - 2^{-j})\epsilon, \quad |\xi_j(\tau_j) - \xi(\tau)| \leq (1 - 2^{-j})\epsilon. \quad (2.9)$$

(iv) The constant r_j is chosen so that $r_j < \delta_{j-1}/4$. Moreover, every component Γ_k which intersects $B(\Gamma_j, 2r_j)$ has length $m_1(\Gamma_k) < \delta_{j-1}/4$.

(v) The constant $\delta_j \in]0, r_j/4[$ is chosen so that, for every $k \geq 1$ such that

$$\Gamma_k \cap \left(\overline{B}(\Gamma_j, 2r_j) \setminus B(\Gamma_j, r_j/2) \right) \neq \emptyset, \quad (2.10)$$

one has

$$4\delta_j \leq d(\Gamma_k, \Gamma_j) \doteq \min \left\{ |x - y|; x \in \Gamma_j, y \in \Gamma_k \right\}. \quad (2.11)$$

Note that, even if we choose $\xi_j = \xi_0$ for $j = 1, \dots, \nu$, it is not possible to start the induction procedure at $j = \nu$. Indeed, the initial steps must be performed in order to define suitable constants $r_j, \delta_j, j = 1, \dots, \nu$.

2. Assuming that the induction has been completed up to step $j - 1$, we describe how to accomplish step j .

Consider the path $\xi_{j-1} : [0, \tau_{j-1}]$, and the connected component Γ_j . For a given radius $r > 0$, define

$$\gamma_j \doteq [\text{boundary of the unbounded connected component of } \mathbb{R}^2 \setminus B(\Gamma_j, r)].$$

This is a simple closed curve, that winds around Γ_j , and has length $\approx 2m_1(\Gamma_j)$. We choose $r = r_j > 0$ small enough, so that the following holds:

$$m_1(\gamma_j) < (2 + \epsilon)m_1(\Gamma_j), \quad (2.12)$$

$$r_j < \frac{\delta_{j-1}}{2}, \quad r_j < \frac{1}{4} \min_{1 \leq i < j} d(\Gamma_i, \Gamma_j), \quad (2.13)$$

and moreover

(P_j) Every connected component $\Gamma_k, k \neq j$, that intersects $\overline{B}(\Gamma_j, 2r_j)$ has length $< \delta_{j-1}/4$.

(P'_j) Every disc of radius δ_{j-1} intersects the unbounded connected component of $\mathbb{R}^2 \setminus B(\Gamma_j, r_j)$.

Note that all the above can certainly be achieved, because there are only finitely many components Γ_k whose length is $> \delta_{j-1}/4$. Choosing $r_j > 0$ small enough, γ_j will not intersect any of them. Moreover, the inequality (2.12) follows by well known results in geometric measure theory [1, 14]. Indeed, since Γ_j is rectifiable, the neighborhoods of radius r around Γ_j satisfy

$$\lim_{r \rightarrow 0} m_2 \left(\frac{B(\Gamma_j, r)}{2r} \right) = m_1(\Gamma_j).$$

Hence the co-area formula yields

$$\liminf_{r \rightarrow 0} m_1 \left(\partial B(\Gamma_j, r) \right) \leq 2m_1(\Gamma). \quad (2.14)$$

We then choose a constant δ_j according to (v) above.

Finally, the new path $\xi_j : [0, \tau_j] \mapsto \mathbb{R}^2$ is defined as follows. If $\xi_{j-1}(t) \notin \overline{B}(\Gamma_j, r_j)$ for all $t \in [0, \tau_{j-1}]$, then there is no need to modify the previous path, and we can simply set

$$\tau_j = \tau_{j-1}, \quad \xi_j(t) = \xi_{j-1}(t).$$

Otherwise, we add a detour so that the new path will remain bounded away from the component Γ_j of the barrier. For this purpose, define the times

$$\begin{aligned} t^- &\doteq \inf \{t \in [0, \tau_{j-1}]; \xi_{j-1}(t) \in \overline{B}(\Gamma_j, r_j)\}, \\ t^+ &\doteq \sup \{t \in [0, \tau_{j-1}]; \xi_{j-1}(t) \in \overline{B}(\Gamma_j, r_j)\}, \end{aligned}$$

and the points

$$P^- = \xi_{j-1}(t^-), \quad P^+ = \xi_{j-1}(t^+).$$

Various cases need to be considered (see Fig. 4).

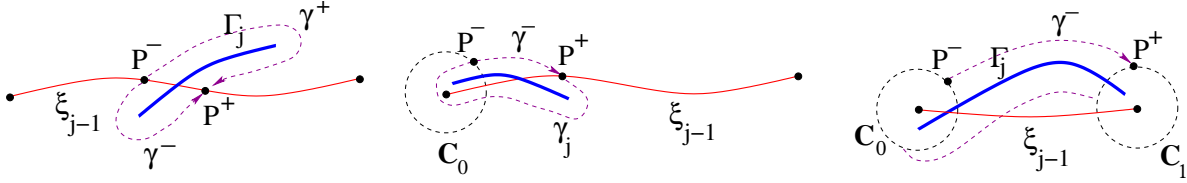


Figure 4: If the path ξ_{j-1} intersects the component Γ_j of the barrier, a detour must be constructed. The figures on the left, center, and right illustrate Cases 1, 2, and 4, respectively.

CASE 1: $0 < t^- \leq t^+ < \tau_{j-1}$, shown in Fig. 4, left.

In this basic case we observe that the two points P^-, P^+ divide the simple closed curve γ_j into two parts, say γ_j^-, γ_j^+ . To fix the ideas, assume

$$s_j \doteq m_1(\gamma_j^-) \leq m_1(\gamma_j^+). \quad (2.15)$$

Let $s \mapsto \gamma_j^-(s)$ be an arc-length parameterization of γ_j^- , with

$$\gamma_j^-(0) = P^-, \quad \gamma_j^-(s_j) = P^+.$$

We then define the new path ξ_j by adding a detour around Γ_j as follows:

$$\xi_j(t) = \begin{cases} \xi_{j-1}(t) & \text{if } t \in [0, t^-], \\ \gamma_j^-(t - t^-) & \text{if } t \in [t^-, t^- + s_j], \\ \xi_{j-1}(t - s_j + t^+ - t^-) & \text{if } t \in [t^- + s_j, \tau_j]. \end{cases} \quad (2.16)$$

Here

$$\tau_j = \tau_{j-1} + s_j - (t^+ - t^-).$$

CASE 2: $0 = t^- \leq t^+ < \tau_{j-1}$, shown in Fig. 4, center.

Call $P^+ = \xi_{j-1}(t^+)$. Observe that, by (2.13) and (\mathbf{P}'_j) , the curve γ_j has non-empty intersection with the circumference

$$\mathbf{C}_0 \doteq \{y \in \mathbb{R}^2; |y - \xi_{j-1}(0)| = \delta_{j-1}\}. \quad (2.17)$$

Therefore, starting at P^+ and moving along the simple closed curve γ_j , we can reach some point P^- on \mathbf{C}_0 in two different ways: clockwise and counterclockwise. We choose $\gamma^- \subset \gamma_j$ to be the shortest among these two paths.

As shown in Fig. 4, center, we parameterize γ^- by arc length, so that

$$\gamma^-(0) = P^-, \quad \gamma^-(s_j) = P^+$$

for some $s_j > 0$. Choosing t^- so that $P^- = \xi_{j-1}(t^-)$, we define the new path $\xi_j(\cdot)$ by setting $\tau_j = \tau_{j-1} - t^- + s_j$ and

$$\xi_j(t) = \begin{cases} \gamma^-(t) & \text{if } t \in [0, s_j], \\ \xi_{j-1}(t + t^- - s_j) & \text{if } t \in [s_j, \tau_{j-1} - t^- + s_j]. \end{cases} \quad (2.18)$$

CASE 3: $0 < t^- \leq t^+ = \tau_{j-1}$.

In this case, the new path $\xi_j(\cdot)$ will connect $\xi_{j-1}(0)$ with a point on the circumference

$$\mathbf{C}_1 \doteq \{y \in \mathbb{R}^2; |y - \xi_{j-1}(\tau_{j-1})| = \delta_{j-1}\}. \quad (2.19)$$

Since this is entirely similar to Case 2, we omit the details.

CASE 4: $0 = t^- < t^+ = \tau_{j-1}$, shown in Fig. 4, right.

In this case, the simple closed curve γ_j intersects both circumferences \mathbf{C}_0 and \mathbf{C}_1 in (2.17), (2.19). We then choose a point $P^- \in \mathbf{C}_0$ and a point $P^+ \in \mathbf{C}_1$ so that the portion $\gamma^- \subset \gamma_j$ connecting P^- with P^+ is as short as possible.

We now parameterize γ^- by arc length, so that

$$\gamma^-(0) = P^-, \quad \gamma^-(s_j) = P^+$$

for some $s_j > 0$. The new path $\xi_j(\cdot)$ is defined simply by setting $\tau_j = s_j$ and

$$\xi_j(t) = \gamma^-(t) \quad \text{for all } t \in [0, s_j]. \quad (2.20)$$

3. Having constructed a sequence of paths $\xi_j : [0, \tau_j] \mapsto \mathbb{R}^2$, $j \geq 0$, by taking the limit as $j \rightarrow \infty$ we will obtain a path $\tilde{\xi}(\cdot)$ which satisfies the properties (2.3)-(2.4), together with

$$\tilde{\tau} \leq \tau + (1 + \epsilon) \sum_{i > \nu} m_1(\Gamma_i). \quad (2.21)$$

For convenience, we extend the definition of each path ξ_j to all of \mathbb{R}_+ by setting

$$\xi_j(t) = \xi_j(\tau_j) \quad \text{for all } t \geq \tau_j.$$

Toward a proof of (2.3), we observe that our construction implies

$$|\xi_j(0) - \xi_{j-1}(0)| \leq \delta_{j-1} < 2^{-j}\epsilon.$$

$$|\xi_j(\tau_j) - \xi_{j-1}(\tau_{j-1})| \leq \delta_{j-1} < 2^{-j}\epsilon.$$

Recalling that $\xi_0 = \xi$ and summing these inequalities from 1 to j , we obtain (2.9).

4. Our construction guarantees that, for every $j \geq 1$, the length τ_j of the new curve $\xi_j(\cdot)$ satisfies (2.8). Notice that, if Case 4 occurs, we have the even sharper bound

$$\tau_j \leq (1 + \epsilon)m_1(\Gamma_j).$$

In addition, since we are assuming that the initial path $\xi(\cdot)$ does not intersect any of the components $\Gamma_1, \dots, \Gamma_\nu$, our choice of $\delta_0 > 0$ in (2.6) guarantees that no modification need to be done in the first ν steps of the algorithm. Hence $\xi_j(\cdot) = \xi_{j-1}(\cdot)$ for all $j = 1, \dots, \nu$. In particular, this implies

$$\tau_j = \tau \quad \text{for all } j = 1, \dots, \nu. \quad (2.22)$$

If one of the Cases 1-2-3 occurs, then our construction yields

$$|\xi_j(t) - \xi_{j-1}(t)| \leq (1 + \epsilon)m_1(\Gamma_j) \quad \text{for all } t \geq 0. \quad (2.23)$$

In Case 4 the above estimate can fail. However, by (2.9), Case 4 in the above construction can occur only finitely many times. Indeed, there are at most finitely many connected components Γ_j of length

$$m_1(\Gamma_j) \geq |\xi(\tau) - \xi(0)| - 4\epsilon.$$

We thus conclude that the sequence $\xi_j(\cdot)$ is Cauchy. As $j \rightarrow \infty$, we have the convergence $\xi_j(t) \rightarrow \xi_\infty(t)$, uniformly for $t \geq 0$.

5. Since all paths ξ_j are Lipschitz continuous with constant 1, the limit path ξ_∞ is 1-Lipschitz as well. We can now parameterize ξ_∞ by arc-length, and obtain a path $\tilde{\xi} : [0, \tilde{\tau}] \mapsto \mathbb{R}^2$, with

$$\tilde{\tau} \leq \liminf_{j \rightarrow \infty} \tau_j \leq \tau + (1 + \epsilon) \sum_{j > \nu} m_1(\Gamma_j).$$

Notice that the above estimates follow from (2.22) and (2.8). This proves (2.21).

The bounds (2.3) are an immediate consequence of (2.9).

6. In this step we show that (2.4) holds. Namely, the path $\tilde{\xi}$ does not touch any of the components Γ_i , $i \geq 1$ of the barrier.

This claim will be proved by showing that, for a fixed $j \geq 1$ and every $k \geq j$ one has

$$d(\xi_k(t), \Gamma_j) \doteq \min\{|\xi_k(t) - y|; y \in \Gamma_j\} \geq \delta_j \quad \text{for all } t \in [0, \tau_k]. \quad (2.24)$$

By construction, it immediately follows that

$$d(\xi_j(t), \Gamma_j) \geq r_j \geq 4\delta_j \quad \text{for all } t \in [0, \tau_j].$$

We now observe that, for any $k > j$, the path $\xi_k(\cdot)$ is obtained from $\xi_j(\cdot)$ by replacing some of its sections by detours $\gamma_i^-(\cdot)$, $i = j + 1, \dots, k$, where

$$d(\gamma_i(s), \Gamma_i) \doteq \min \left\{ |\gamma(s) - y|; y \in \Gamma_i \right\} = r_i \quad \text{for all } s.$$

Three cases must be considered.

CASE 1: (2.10) holds, and hence by construction (2.11) holds as well. In this case we have

$$d(\gamma_k(s), \Gamma_j) \geq d(\Gamma_k, \Gamma_j) - d(\gamma_k(s), \Gamma_k) \geq 4\delta_j - r_k \geq 3\delta_j.$$

CASE 2: $d(\Gamma_k, \Gamma_i) > 2r_j$. In this case, for every s we trivially have

$$d(\gamma_k(s), \Gamma_j) \geq d(\Gamma_k, \Gamma_j) - d(\gamma_k(s), \Gamma_k) \geq 2r_j - r_k > r_j \geq 4\delta_j.$$

CASE 3: $\Gamma_k \subseteq \overline{B}(\Gamma_j, r_j/2)$. If $\xi_k(\cdot) = \xi_{k-1}(\cdot)$, the conclusion (2.24) follows by induction on k . It thus suffices to consider the case where $\xi_k(\cdot)$ is obtained from $\xi_{k-1}(\cdot)$ by inserting some nontrivial portion of a curve $\gamma_k \subseteq \{x; d(x, \Gamma_k) = r_k\}$.

In this case, our algorithm implies that there exists a finite sequence

$$j = i(0) < i(1) < \dots < i(N) = k$$

such that every curve $\gamma_{i(\ell)} \doteq \{x; d(x, \Gamma_k) = r_k\}$ intersects the previous one:

$$\gamma_{i(\ell)} \cap \gamma_{i(\ell-1)} \neq \emptyset \quad \text{for all } \ell = 1, \dots, N.$$

Considering the diameters of the sets $\gamma_{i(\ell)}$, we thus have the bound

$$\begin{aligned} \min_s d(\xi_k(s), \Gamma_j) &\geq \min_s d(\gamma_{i(1)}(s), \Gamma_j) - \sum_{\ell=2}^N \text{diam}(\gamma_{i(\ell)}) \\ &\geq (d(\Gamma_{i(1)}, \Gamma_j) - r_{i(1)}) - \sum_{\ell=2}^N (2r_{i(\ell)} + m_1(\Gamma_{i(\ell)})). \\ &\geq \left(4\delta_j - \frac{\delta_j}{2}\right) - \sum_{\ell=2}^N \left(2 \cdot 2^{j-i(\ell)} \frac{\delta_j}{4} + 2^{j-i(\ell)} \frac{\delta_j}{4}\right) > \delta_j. \end{aligned} \tag{2.25}$$

Indeed, the condition (2.10) applies to $\Gamma_{i(1)}$, hence (2.11) holds. Moreover, using the property (\mathbf{P}_j) with j replaced by $i(1), \dots, i(N)$, we obtain

$$m_1(\Gamma_{i(\ell)}) \leq \frac{\delta_{i(\ell)-1}}{4} \leq 2^{j-i(\ell)} \frac{\delta_j}{4}.$$

Combining the above three cases, we conclude that (2.24) holds. Taking the limit as $k \rightarrow \infty$, we conclude that $d(\tilde{\xi}(t), \Gamma_j) \geq \delta_j$ for all $t \geq 0$ and $j \geq 1$. This establishes (2.4),

7. To obtain the bound (2.5) on the length of the new path, define

$$\hat{\tau} \doteq \min \left\{ \tilde{\tau}, \tau + \sum_{i>\nu} m_1(\Gamma_i) \right\}.$$

By (2.21), we trivially have

$$|\tilde{\tau} - \hat{\tau}| \leq \epsilon \sum_{i>\nu} m_1(\Gamma_i).$$

Therefore, if we replace the path $\tilde{\xi}$ by its restriction to the subinterval $[0, \hat{\tau}]$, the conditions (2.4)-(2.5) are satisfied, while the second inequality in (2.3) will be replaced by

$$|\tilde{\xi}(\hat{\tau}) - \xi(\tau)| \leq \epsilon + \epsilon \sum_{i>\nu} m_1(\Gamma_i) \leq \epsilon(1 + m_1(\Gamma)).$$

Since $\epsilon > 0$ can be chosen arbitrarily small, this completes the proof. \square

The next result is concerned with the length of the portion of the barrier which is touched by the fire at a given time t . We recall that, if Γ is admissible, the linear bound (1.5) must hold.

Lemma 2.3. *Given an admissible barrier Γ , consider the function*

$$\varphi(t) = m_1(\overline{R^\Gamma(t)} \cap \Gamma). \quad (2.26)$$

Then

(i) φ is nondecreasing and right continuous.

(ii) The set of times where the constraint is not saturated

$$\mathcal{U} \doteq \left\{ t > 0; \quad m_1(\overline{R^\Gamma(t)} \cap \Gamma) < \sigma t \right\} \quad (2.27)$$

is open.

(iii) The set of times where the constraint is saturated

$$\mathcal{S} \doteq \left\{ t \geq 0; \quad m_1(\overline{R^\Gamma(t)} \cap \Gamma) = \sigma t \right\} \quad (2.28)$$

is closed.

Proof. 1. For any $0 < t_1 < t_2$ we have $R^\Gamma(t_1) \subseteq R^\Gamma(t_2)$. Hence φ is nondecreasing.

2. Next, we claim that φ is right continuous. Indeed, consider a decreasing sequence of times $t_n \downarrow t_0$. Since the fire propagates with unit speed, we have

$$\overline{R^\Gamma(t_n)} \subseteq \overline{B(R^\Gamma(t_0), t_n - t_0)}$$

hence

$$\overline{R^\Gamma(t_0)} \cap \Gamma = \bigcap_{n \geq 1} (\overline{R^\Gamma(t_n)} \cap \Gamma).$$

The right continuity of φ now follows from the dominated convergence theorem.

3. By the previous two steps it follows that φ is upper semicontinuous. Hence the function $t \mapsto \varphi(t) - \sigma t$ is upper semicontinuous as well. We thus conclude that the set \mathcal{U} where $\varphi(t) - \sigma t < 0$ is open. The closure of $\mathcal{S} = \mathbb{R}_+ \setminus \mathcal{U}$ follows immediately. \square

The next lemma will play a key role in the sequel. The intuitive idea is simple: let $t = a_i$ be the first time when the fire front touches the connected component Γ_i . Immediately afterwards, the fire starts going around Γ_i , clockwise as well as counterclockwise, until this connected component is completely surrounded. This will happen at some time b_i with $b_i - a_i \leq m_1(\Gamma_i)$. On the other hand, when the fire front does not touch any of the barriers Γ_j , it expands freely with unit speed in all directions. Therefore, the distance between level sets of the time function T^Γ increases at unit rate.

Lemma 2.4. *Consider a barrier $\Gamma = \cup_{i \geq 1} \Gamma_i$, written as the union of its compact connected components. Assume that $\mathbb{R}^2 \setminus \Gamma$ is connected.*

(i) *For each $i \geq 1$, the set of times when the fire front touches Γ_i*

$$J_i \doteq \left\{ t \geq 0; \partial \overline{R^\Gamma(t)} \cap \Gamma_i \neq \emptyset \right\} \quad (2.29)$$

is contained within an interval $[a_i, b_i]$ of length $b_i - a_i \leq m_1(\Gamma_i)$.

(ii) *For any $0 \leq \tau < \tau'$, one has*

$$B(R^\Gamma(\tau), r) \subseteq \overline{R^\Gamma(\tau')}, \quad \text{with} \quad r = m_1 \left([\tau, \tau'] \setminus \bigcup_{i \geq 1} [a_i, b_i] \right). \quad (2.30)$$

Proof. 1. By the assumptions, each Γ_i is simply connected. Let

$$a_i \doteq \inf \left\{ t \geq 0; \overline{R^\Gamma(t)} \cap \Gamma_i \neq \emptyset \right\} = \min_{x \in \Gamma_i} T^\Gamma(x) \quad (2.31)$$

be the first time when the fire touches Γ_i . By the lower semicontinuity of T^Γ and the compactness of Γ_i , it is clear that a_i is actually a minimum. We will prove part (i) of the lemma by showing that, at any time $\tau > a_i + m_1(\Gamma_i)$, the component Γ_i is entirely contained in the interior of the set $\overline{R^\Gamma(\tau)}$.

2. Toward our goal, we first choose $\varepsilon > 0$ such that

$$4\varepsilon < \tau - a_i - m_1(\Gamma_i), \quad (2.32)$$

then we choose an integer $\nu > i$ so large that

$$\sum_{k > \nu} m_1(\Gamma_k) < \varepsilon. \quad (2.33)$$

Finally, we choose a radius $0 < \rho < \varepsilon$ small enough so that

$$B(\Gamma_i, \rho) \cap \Gamma_j = \emptyset \quad \text{for all } j = 1, \dots, \nu, \quad j \neq i. \quad (2.34)$$

With the above choices, we will show that

$$B(\Gamma_i, \rho) \subset \overline{R^\Gamma(\tau)}. \quad (2.35)$$

3. To prove (2.35), fix any point $x \in B(\Gamma_i, \rho) \setminus \Gamma_i$. For $0 < r < \rho$, consider the open neighborhood $B(\Gamma_i, r)$ of radius r around Γ_i . By a suitable choice of $r > 0$, we claim that the following properties can be achieved.

(i) Calling γ the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus B(\Gamma_i, r)$, we have

$$m_1(\gamma) < 2m_1(\Gamma_i) + \varepsilon. \quad (2.36)$$

(ii) The point x lies in the unbounded connected component of $\mathbb{R}^2 \setminus B(\Gamma_i, r)$.

Indeed, the property (i) follows by the same argument used in (2.14). The property (ii) follows from the fact that Γ_i is compact and simply connected, while $x \notin \Gamma_i$.

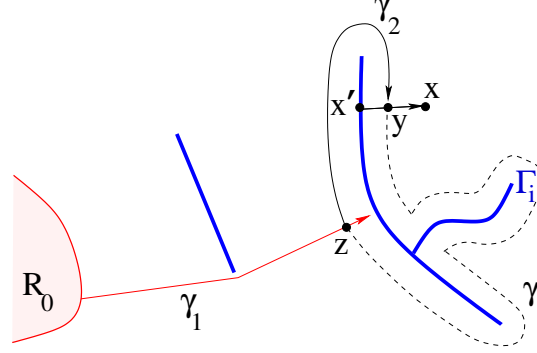


Figure 5: The construction used in the proof of part (i) of Lemma 2.4.

4. As shown in Fig. 5, let $x' \in \Gamma_i$ be one of the points closest to x , so that $|x - x'| < \rho$. By construction, the segment with endpoints x', x intersects the simple closed curve γ at least at one point, say $y \in \gamma$. This implies

$$|x - y| < |x - x'| < \rho \leq \varepsilon. \quad (2.37)$$

Next, since $\overline{R^\Gamma(a_i)} \cap \Gamma \neq \emptyset$, there exists a trajectory for the fire that starts inside R_0 and crosses the curve γ at some point z before time $t = a_i$.

We now consider the path $\xi : [0, \ell] \mapsto \mathbb{R}^2$ obtained by concatenating the following three paths:

- A path γ_1 , starting inside R_0 and reaching $z \in \gamma$ without crossing the barrier Γ . This path has length $\ell_1 < a_i$.
- A path γ_2 contained within γ , starting at z and ending at y . Since we can move along γ both clockwise or counterclockwise, by choosing the shorter path we can assume that γ_2 has length

$$\ell_2 \leq \frac{1}{2}m_1(\gamma) < m_1(\Gamma_i) + \varepsilon.$$

- A path γ_3 consisting of the segment with endpoints y, x . By (2.37), its length is $\ell_3 < \varepsilon$.

The total length of this path $\xi(\cdot)$ is thus

$$\ell = \ell_1 + \ell_2 + \ell_3 < a_i + (m_1(\Gamma_i) + \varepsilon) + \varepsilon < \tau - 2\varepsilon.$$

Notice that, by construction, the path ξ does not cross any of the components $\Gamma_1, \dots, \Gamma_\nu$. Applying Lemma 2.2, for any $\varepsilon' > 0$ we can find a new path $\tilde{\xi} : [0, \tilde{\ell}] \mapsto \mathbb{R}^2 \setminus \Gamma$ such that

$$\tilde{\xi}(0) \in R_0, \quad |\tilde{\xi}(\tilde{\ell}) - x| < \varepsilon',$$

and moreover

$$\tilde{\ell} \leq a_i + m_1(\Gamma_i) + 2\varepsilon + \sum_{k>\nu} m_1(\Gamma_k) < \tau.$$

This implies $x \in \overline{R^\Gamma(\tau)}$, as claimed. Hence part (i) of the lemma is proved.

5. It now remains to prove (ii). Without loss of generality, we can assume that the intervals $[a_i, b_i]$ are labelled according to decreasing length, so that

$$b_1 - a_1 \geq b_2 - a_2 \geq \dots \quad (2.38)$$

Let $0 < \tau < \tau'$ and $\varepsilon > 0$ be given. Choose $\nu > 1$ large enough so that (2.33) holds. We now express the open set

$$]\tau, \tau'[\setminus \left(\bigcup_{1 \leq i \leq \nu} [a_i, b_i] \right) = \bigcup_{k=1}^m]\tau_k, \tau'_k[$$

as the union of finitely many disjoint open intervals.

Next, we choose an integer $\nu' > \nu$ such that

$$\sum_{i>\nu'} m_1(\Gamma_i) < \varepsilon' \doteq \frac{\varepsilon}{m}, \quad (2.39)$$

and define the times

$$t_k \doteq \tau_k + \varepsilon', \quad t'_k \doteq \tau'_k - \varepsilon'.$$

Finally, for $k = 1, \dots, m$, we define the sets of integers

$$I_k \doteq \left\{ i; \nu + 1 \leq i \leq \nu', [a_i, b_i] \cap]\tau_k, \tau'_k[\neq \emptyset \right\}.$$

Notice that, by (2.38), these sets are mutually disjoint.

6. Toward a proof of (2.30) we will show that, for every $k = 1, \dots, m$, one has

$$B(R^\Gamma(t_k), r_k) \subseteq \overline{R^\Gamma(\tau'_k)}, \quad \text{with} \quad r_k = (\tau'_k - \tau_k) - 3\varepsilon' - \sum_{i \in I_k} m_1(\Gamma_i). \quad (2.40)$$

Notice that (2.40) implies

$$B(R^\Gamma(\tau), r) \subseteq \overline{R^\Gamma(\tau')}, \quad (2.41)$$

with

$$\begin{aligned} r &= \sum_{k=1}^m r_k = \sum_{k=1}^m (\tau'_k - \tau_k - \varepsilon') - 3m\varepsilon' - \sum_{k=1}^m \sum_{i \in I_k} m_1(\Gamma_i) \\ &\geq m_1 \left([\tau, \tau'] \setminus \bigcup_{i=1}^{\nu} [a_i, b_i] \right) - 3m\varepsilon' - \sum_{\nu < i \leq \nu'} m_1(\Gamma_i) \\ &\geq m_1 \left([\tau, \tau'] \setminus \bigcup_{i=1}^{+\infty} [a_i, b_i] \right) - \varepsilon - 3m\varepsilon' - \varepsilon \\ &= m_1 \left([\tau, \tau'] \setminus \bigcup_{i=1}^{+\infty} [a_i, b_i] \right) - 5\varepsilon. \end{aligned} \quad (2.42)$$

Since here $\varepsilon > 0$ can be taken arbitrarily small, this yields (2.30).

7. It thus remains to prove (2.40), for each $k \in \{1, \dots, m\}$.

Consider any point $y \in B(R^\Gamma(t_k, r_k))$ with $y \notin \overline{R^\Gamma(t_k)}$, and choose a point $x_0 \in \overline{R^\Gamma(t_k)}$ which minimizes the distance from y . For any given $\rho > 0$, we can then choose a point $y_0 \in R^\Gamma(t_k)$ with $|y_0 - x_0| < \rho$. Notice that we can also assume

$$\lim_{h \rightarrow 0^+} \frac{1}{h^2} m_1(\Gamma \cap B(y_0, h)) = 0, \quad (2.43)$$

because this property holds at a.e. point $x \in \mathbb{R}^2$, w.r.t. Lebesgue measure.

Call γ the segment with endpoints y_0, y , and let $\xi : [0, \ell] \mapsto \mathbb{R}^2$ be an arc-length parameterization of this segment, oriented from y_0 to y . Notice that this implies $\ell < r_k$.

In order to use Lemma 2.2, we claim that, among all the connected components Γ_i , $1 \leq i \leq \nu'$ the only ones that can have a non-empty intersection with γ are the components Γ_i , with $i \in I_k$. Indeed, consider the set of indices

$$I^- \doteq \{i \leq \nu; [a_i, b_i] \subseteq [0, \tau_k]\} \cup I_1 \cup \dots \cup I_{k-1} \quad (2.44)$$

For every $i \in I^-$ we have $b_i \leq \tau_k < t_k$. Hence $\overline{R^\Gamma(t_k)}$ contains a neighborhood of Γ_i . Therefore, since y lies outside $\overline{R^\Gamma(t_k)}$, a segment of minimum length joining y with a point $x_0 \in \overline{R^\Gamma(t_k)}$ cannot intersect Γ_i . The same holds if we choose y_0 sufficiently close to x_0 .

Summarizing the previous discussion, given $y \in B(R^\Gamma(t_k, r_k) \setminus \overline{R^\Gamma(t_k)})$, we can find $y_0 \in R^\Gamma(t_k)$ and a radius $\rho > 0$ small enough such that

- (i) $\ell \doteq |y_0 - y| < r_k$.
- (ii) The segment γ with endpoints y_0, y does not intersect any of the compact connected components Γ_i with $i \in I^-$.
- (iii) The circumference Σ centered at y_0 with radius ρ satisfies

$$\Sigma \doteq \{x \in \mathbb{R}^2; |x - y_0| = \rho\} \subset R^\Gamma(t_k) \setminus \Gamma. \quad (2.45)$$

Notice that the (2.45) is made possible thanks to (2.43).

Next, consider the set of indices

$$I^+ \doteq \{i \leq \nu; a_i \geq \tau'_k\} \cup I_{k+1} \cup \dots \cup I_m.$$

Arguing by contradiction, we show that none of the components Γ_i with $i \in I^+$ can intersect the segment γ . Indeed, if the intersection is nonempty, define

$$\bar{s} \doteq \min\{s \in [0, \ell]; \xi(s) \in \Gamma_i \text{ for some } i \in I^+\}.$$

For every $s < \bar{s}$, an application of Lemma 5.1 would imply the existence of a sequence of paths $\xi_j : [0, \ell_j] \mapsto \mathbb{R}^2 \setminus \Gamma$ such that

$$\xi_j(0) \rightarrow y_0, \quad \xi_j(\ell_j) \rightarrow \xi(s), \quad \text{as } j \rightarrow \infty,$$

and whose length satisfies

$$\limsup_{j \rightarrow \infty} \ell_j \leq s + \sum_{i \in I_k} m_1(\Gamma_i) + \sum_{i > \nu'} m_1(\Gamma_i) < r_k + \sum_{i \in I_k} m_1(\Gamma_i) + \varepsilon'. \quad (2.46)$$

We now observe that, for all j large enough, the path $\xi_j(\cdot)$ crosses the circumference Σ at some point $\xi_j(s_j)$. By taking the restriction of ξ_j to the subinterval $[s_j, \ell_j]$, we obtain a sequence of paths $\tilde{\xi}_j$, of length $\leq \ell_j$, where the initial point lies on $\Sigma \subset R^\Gamma(t_k)$ and the terminal points converge to y . This implies

$$T^\Gamma(\xi(s)) \leq t_k + \liminf_{j \rightarrow \infty} \ell_j \leq t_k + r_k + \sum_{i \in I_k} m_1(\Gamma_i) + \varepsilon'.$$

Since s can be taken arbitrarily close to \bar{s} , recalling (2.40) we conclude that the point $\xi(\bar{s}) \in \Gamma_{i^*}$ lies inside the set $\overline{R^\Gamma(T)}$, with

$$\begin{aligned} T &= t_k + r_k + \sum_{i \in I_k} m_1(\Gamma_i) + \varepsilon' \\ &= (\tau_k + \varepsilon') + \left[(\tau'_k - \varepsilon' - \tau_k) - 3\varepsilon' - \sum_{i \in I_k} m_1(\Gamma_i) \right] + \sum_{i \in I_k} m_1(\Gamma_i) + \varepsilon' \\ &< \tau'_k. \end{aligned}$$

Since $i^* \in I^+$, by definition this implies $a_{i^*} \geq \tau'_k$, reaching a contradiction.

8. In view of the previous step, we can now apply Lemma 5.1 to each segment with endpoints $y_0, \gamma(s)$, for $0 < s < \ell = |y - y_0|$. This yields a sequence of paths $\tilde{\xi}_j : [0, \ell_j] \mapsto \mathbb{R}^2$, joining a point $x_j \in \Sigma \subset R^\Gamma(t_k)$ with a point y_j which becomes arbitrarily close to $\xi(s)$ as $j \rightarrow \infty$. All these paths $\tilde{\xi}_j$ do not cross Γ . Their lengths ℓ_j satisfy the uniform bound

$$\ell_j \leq s + \sum_{i \in I_k^+} m_1(\Gamma_i) + \varepsilon' \leq r_k + \sum_{i \in I_k^+} m_1(\Gamma_i) + \varepsilon' \leq (\tau'_k - t_k) - \varepsilon'.$$

For every $0 \leq s < \ell$, this implies

$$\gamma(s) \in \overline{R^\Gamma(\tau'_k - \varepsilon')} \subseteq \overline{R^\Gamma(\tau'_k)}.$$

Letting $s \rightarrow \ell$, we obtain $\gamma(s) \rightarrow y$, and hence $y \in \overline{R^\Gamma(\tau'_k)}$. This establishes the inclusion (2.40) for every $k = 1, \dots, m$, thus completing the proof. \square

3 Properties of the minimum time function with obstacles

Assume that the initial set R_0 where the fire is burning at $t = 0$ has finite perimeter. Consider a barrier $\Gamma = \cup_i \Gamma_i$, written as the union of its connected components. For every fixed time $\bar{T} > 0$, the truncated function

$$x \mapsto \min\{\bar{T}, T^\Gamma(x)\}$$

has bounded variation. Indeed, as shown in [13], it is an SBV function. By the co-area formula it thus follows

$$\int_0^{\bar{T}} m_1(\partial \overline{R^\Gamma(t)}) dt < \infty. \quad (3.1)$$

As a consequence, for a.e. time $t \in [0, \bar{T}]$, the boundary $\partial \overline{R^\Gamma(t)}$ is a curve with finite length.

We now consider a sequence of barriers $\Gamma^{(n)}$, $n \geq 1$, converging to a barrier Γ , and study the behavior of the corresponding minimum time functions $T^{\Gamma^{(n)}}$. Two cases will be studied. The first lemma deals with the case where each barrier has a finite number of connected components. The second lemma is concerned with barriers having countably many components. For the definition and properties of the Hausdorff distance between compact sets we refer to [2, 7].

Lemma 3.1. *Let a bounded open set $R_0 \subset \mathbb{R}^2$ be given. Consider a barrier $\Gamma = \cup_{i=1}^N \Gamma_i$ which is the union of finitely many compact, simply connected, rectifiable components. Let $\Gamma^{(n)} = \cup_{i=1}^N \Gamma_i^{(n)}$, with $n \geq 1$, be an approximating sequence of barriers. Assume the convergence w.r.t. the Hausdorff distance:*

$$\lim_{n \rightarrow \infty} d_H(\Gamma_i^{(n)}, \Gamma_i) = 0 \quad \text{for each } i = 1, \dots, N. \quad (3.2)$$

(i) For every $x \in \mathbb{R}^2 \setminus \Gamma$, one has

$$T^\Gamma(x) = \lim_{n \rightarrow \infty} T^{\Gamma^{(n)}}(x). \quad (3.3)$$

(ii) For each $n \geq 1$, let $\xi_n : [0, \tau_n] \mapsto \mathbb{R}^2$ be an optimal trajectory reaching a point $x \in \mathbb{R}^2 \setminus \Gamma$ in minimum time without crossing $\Gamma^{(n)}$. If $\tau_n \rightarrow \bar{\tau}$ and $\xi_n(\cdot) \rightarrow \xi(\cdot)$ uniformly on every compact subset of $[0, \bar{\tau}[$, then $\xi(\cdot)$ is an optimal trajectory reaching x in minimum time without crossing Γ .

Proof. 1. Toward a proof of (i), consider a minimizing sequence of 1-Lipschitz paths $\xi_\nu : [0, \tau_\nu] \mapsto \mathbb{R}^2$, satisfying

$$\begin{aligned} \xi_\nu(0) &\in R_0, & \xi_\nu(t) &\notin \Gamma \quad \text{for all } t \in [0, \tau_\nu], \\ |\xi(\tau_\nu) - x| &< \frac{1}{\nu}, & \tau_\nu &\rightarrow T^\Gamma(x) \quad \text{as } \nu \rightarrow \infty. \end{aligned}$$

Since Γ is compact and by assumption $x \notin \Gamma$, we conclude that, for all ν large enough, the segment joining $\xi(\tau_\nu)$ with x will not touch Γ . By adding this segment to the path ξ_ν we obtain another sequence of 1-Lipschitz paths $\tilde{\xi}_\nu : [0, \tilde{T}_\nu] \mapsto \mathbb{R}^2 \setminus \Gamma$, with

$$\tilde{\xi}_\nu(\tilde{T}_\nu) = x \quad \text{for all } \nu, \quad \lim_{\nu \rightarrow \infty} \tilde{T}_\nu = T^\Gamma(x). \quad (3.4)$$

By the compactness of the barriers $\Gamma^{(n)}$, for each $\nu \geq 1$ there exist an integer n_ν large enough such that $\tilde{\xi}_\nu(t) \notin \Gamma^{(n)}$ for all $n \geq n_\nu$ and all $t \in [0, \tilde{T}_\nu]$. This immediately implies

$$\tilde{T}_\nu \geq \limsup_{n \rightarrow \infty} T^{\Gamma^{(n)}}(x).$$

Together with (3.4), this yields

$$T^\Gamma(x) \geq \limsup_{n \rightarrow \infty} T^{\Gamma^{(n)}}(x). \quad (3.5)$$

2. To prove (ii) we observe that, by (3.5), $\bar{\tau} \leq T^\Gamma(x)$. It thus only remains to prove that the limit path $\xi(\cdot)$ is admissible. Since $x \notin \Gamma$, there exists $\rho > 0$ such that $B(x, \rho) \cap \Gamma = \emptyset$. In the following, to simplify notation, we still denote by ξ the set of points $\{\xi(t); t \in [0, \bar{\tau}]\} \subset \mathbb{R}^2$. For a given $0 < r \ll \rho$ sufficiently small, consider the neighborhood $B(\xi, r)$, and let Σ be the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus B(\xi, r)$. This is a simple closed curve, which we can parameterize by arc-length, oriented counterclockwise.

As shown in Fig. 6, within Σ we distinguish an arc Σ_1 connecting a point in $P_1 \in R_0$ with a point $Q_1 \in B(x, \rho)$, and an arc Σ_2 connecting a point $Q_2 \in B(x, \rho)$ with a point $P_2 \in R_0$, moving counterclockwise. we call $\widehat{\Omega}$ the open set bounded by Σ . Moreover, we consider the open subset

$$\Omega \doteq \widehat{\Omega} \setminus (\overline{R_0} \cup \overline{B}(x, \rho))$$

with its open subsets $\Omega_1 \subset \Omega$, bounded between Σ_1 and ξ , and $\Omega_2 \subset \Omega$ bounded between ξ and Σ_2 .

3. In the following, for simplicity we consider the case where $N = 1$, so that Γ and all the approximating barriers $\Gamma^{(n)}$ contain only one component. Since all components are compact and have a positive distance from each other, the general case follows by the same arguments.

Fix a point $z \in \Gamma$, outside the region enclosed by Σ . We claim that, for every $y \in \Omega_1 \cap \Gamma$, there exists a path $\gamma^y \subseteq \Gamma$ connecting y with z , and touching Σ_1 without entering Ω_2 . More precisely, we claim that there is a map $\gamma^y : [0, \bar{s}] \mapsto \Gamma$ such that

$$\begin{aligned} \gamma^y(0) &= y, & \gamma^y(\bar{s}) &= z, \\ \gamma^y(s^*) &\in \Sigma_1, & \text{where } s^* &\doteq \inf \left\{ s \in [0, \bar{s}]; \gamma^y(s) \notin \Omega \right\}. \end{aligned}$$

To prove this claim, we observe that there exist sequences $y_n, z_n \in \Gamma^{(n)}$, with $y_n \rightarrow y$ and $z_n \rightarrow z$. Since $\Gamma^{(n)}$ is connected, for each $n \geq 1$ there is a path γ_n joining y_n with z_n , and remaining inside $\Gamma^{(n)}$. By possibly selecting a subsequence and relabeling, we obtain a limit path $\gamma : [0, \bar{s}] \mapsto \Gamma$, joining y with z . If $\gamma(s) \in \Omega_2$ for some $0 < s < s^*$, then γ crosses the path ξ . By uniform convergence $\gamma_n \rightarrow \gamma$ and $\xi_n \rightarrow \xi$, this would imply that every ξ_n , with n suitably large, crosses $\Gamma^{(n)}$, a contradiction.

We observe that, after reaching the boundary Σ_1 , for $s \in [s^*, \bar{s}]$ the path γ^y can re-enter inside Ω . However, it cannot cross ξ . Namely, if it enters through Σ_1 , it must eventually leave through Σ_1 . If it enters through Σ_2 , it must leave through Σ_2 . Otherwise, being a limit of paths γ_n contained in the approximating barriers $\Gamma^{(n)}$, these paths would cross the corresponding paths ξ_n .

4. Within the compact curve ξ , for $k = 1, 2$ we define the subset

$$V_k \doteq \left\{ y \in \xi \cap \Gamma; \text{ there is a path inside } \Gamma \text{ joining } y \text{ with } z, \text{ exiting through } \Sigma_k \right\}. \quad (3.6)$$

Since Γ is rectifiable and compact, it follows that V_1, V_2 are both compact. We claim that they are disjoint. Indeed, assume that $y = \xi(s) \in V_1 \cap V_2$. Then, inside $\Omega \cap \Gamma$, we can find a path joining y with a point $y_1 \in \Omega_1$, and another path joining y with a point $y_2 \in \Omega_2$. In turn, by the previous step, there is a path γ^{y_1} joining y_1 to z , and a path γ^{y_2} joining y_2 with

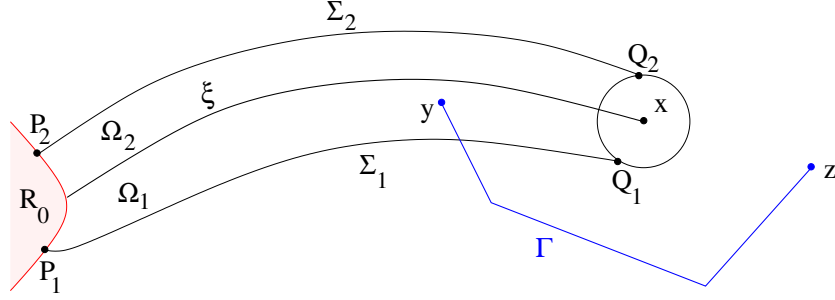


Figure 6: Showing that the limit path ξ is admissible.

z . The union of these paths is a multiply connected rectifiable subset of Γ . This yields a contradiction.

5. By the previous step, we can cover the disjoint compact sets $V_1, V_2 \subset [0, \bar{\tau}]$ with finitely many disjoint intervals, say $[a_j, b_j]$ and $[c_j, d_j]$, so that

$$V_1 \subseteq \bigcup_{j=1}^m [a_j, b_j], \quad V_2 \subseteq \bigcup_{j=1}^m [c_j, d_j].$$

Define the corresponding portions of curve

$$\gamma_1^{(j)} \doteq \{\gamma(s); s \in [a_j, b_j]\}, \quad \gamma_2^{(j)} \doteq \{\gamma(s); s \in [c_j, d_j]\}.$$

We claim that, by choosing a radius $\delta > 0$ small enough, for every $j = 1, \dots, m$ one has

$$\Gamma \cap B(\gamma_1^{(j)}, \delta) \cap \Omega_2 = \emptyset, \quad \Gamma \cap B(\gamma_2^{(j)}, \delta) \cap \Omega_1 = \emptyset. \quad (3.7)$$

Indeed, if no such radius $\delta > 0$ exists, we could find a point $y \in V_1$ and a sequence of points $y_n \rightarrow y$ with $y_n \in \Gamma \cap \Omega_2$ for all $n \geq 1$. By step 3, for each y_n there exists a path joining y_n to z , remaining inside $\Gamma \setminus \Omega_1$. By taking a limit, we obtain a path joining y with z , remaining inside $\Gamma \setminus \Omega_1$. This would yield $y \in V_2$, reaching a contradiction because in step 2 we proved that $V_1 \cap V_2 = \emptyset$.

6. We now describe how to make a small modification of the path ξ , so that it does not touch the barrier Γ . Fix $\varepsilon > 0$ and consider the finitely many circumferences with radius ε , centered at the points

$$A_j = \xi(a_j), \quad B_j = \xi(b_j), \quad C_j = \xi(c_j), \quad D_j = \xi(d_j).$$

In addition, for $0 < \varepsilon' \ll \varepsilon$, call Σ' the simple closed curve obtained by taking the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus B(\xi, \varepsilon')$. As in step 2, we distinguish a lower and an upper portion of this boundary, which we call Σ'_1, Σ'_2 , respectively.

As shown in Fig. 7, for each $j = 1, \dots, m$, the portion of the path $\{\xi(s); s \in [a_j, b_j]\}$ between A_j and B_j , is replaced by two arcs of circumferences centered at A_j, B_j together with a portion of the curve Σ'_2 . Similarly, the portion of the path $\{\xi(s); s \in [c_j, d_j]\}$ between C_j and D_j , is replaced by two arcs of circumferences centered at C_j, D_j together with a portion of the curve Σ'_1 . By the previous analysis, for all $\varepsilon, \varepsilon' > 0$ sufficiently small, this new curve does

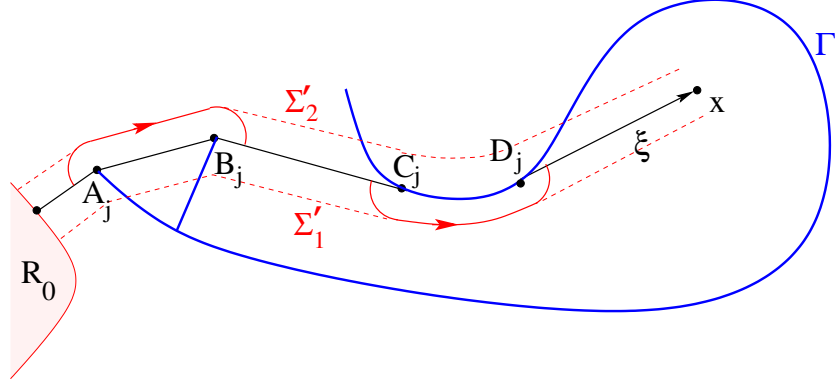


Figure 7: By modifying the trajectory $\xi(\cdot)$ along the arcs where it touches the barrier Γ , one can show that ξ is admissible.

not intersect Γ . Moreover, letting $\varepsilon, \varepsilon' \rightarrow 0$, we recover the original path ξ in the limit. This shows that $\xi(\cdot)$ is admissible, proving (ii).

8. By (ii) it now follows

$$T^\Gamma(x) \leq \liminf_{n \rightarrow \infty} T^{\Gamma^{(n)}}(x).$$

Together with (3.5), this yields (3.3), completing the proof. \square

Remark 3.2. In the above lemma, the assumptions that each Γ_i is simply connected and that $x \notin \Gamma$ play an essential role. In Figure 8 shows two cases where these assumptions are not satisfied, and the conclusions fail.

Remark 3.3. In (3.6), one can think of V_1 is the set of points where the barrier Γ touches the optimal trajectory ξ on the right, while V_2 is the set of points where Γ touches ξ on the left. Calling $\dot{\xi}(t) = (\cos \theta, \sin \theta) \in \mathbb{R}^2$ the tangent vector, by construction the map $t \mapsto \theta(t)$ is non-increasing along each interval $[a_j, b_j]$, non-decreasing along each interval $[c_j, d_j]$, and constant everywhere else.

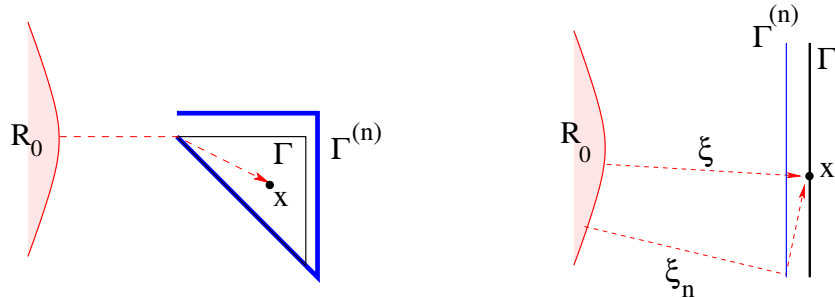


Figure 8: Left: an example showing that (3.3) can fail, if the barrier Γ is not simply connected. For each $n \geq 1$, the barrier $\Gamma^{(n)}$ is the union of three segments, and $\mathbb{R}^2 \setminus \Gamma^{(n)}$ is connected. However, the limit barrier Γ is the boundary of a triangle, which is not simply connected. None of the points x in the interior of this triangle can be reached from R_0 , without crossing Γ . Right: an example showing that, if $x \in \Gamma$, the inequality (3.3) can fail. Indeed, here $T^\Gamma(x) < \lim_{n \rightarrow \infty} T^{\Gamma^{(n)}}(x)$.

Lemma 3.4. *Let a bounded open set $R_0 \subset \mathbb{R}^2$ be given. Consider a barrier $\Gamma = \cup_{i=1}^{\infty} \Gamma_i$ and assume that $\mathbb{R}^2 \setminus \Gamma$ is connected. For each $\nu \geq 1$, consider the finite union $\Gamma_\nu = \cup_{i=1}^{\nu} \Gamma_i$. Call T^Γ, T^{Γ_ν} the corresponding minimum time functions.*

(i) *For every $x \in \mathbb{R}^2$ one has*

$$T^\Gamma(x) = \lim_{\nu \rightarrow \infty} T^{\Gamma_\nu}(x). \quad (3.8)$$

(ii) *For each $\nu \geq 1$, let $\xi_\nu : [0, \tau_\nu] \mapsto \mathbb{R}^2$ be an optimal trajectory reaching x in minimum time without crossing Γ_ν . If $\tau_\nu \rightarrow \bar{\tau}$ and $\xi_\nu(\cdot) \rightarrow \xi(\cdot)$ uniformly on every compact subset of $[0, \bar{\tau}[$, then $\xi(\cdot)$ is an optimal trajectory reaching x in minimum time without crossing Γ .*

Proof. 1. To prove (3.8), fix $x \in \mathbb{R}^2$ and, for every $\nu \geq 1$, call $\tau_\nu \doteq T^{\Gamma_\nu}(x)$. Denote by $\zeta_\nu : [0, \tau_\nu] \rightarrow \mathbb{R}^2$ an optimal trajectory reaching the point x without crossing Γ_ν . According to Definition 2.1, there exists a second path $\xi_\nu : [0, \tau_\nu] \rightarrow \mathbb{R}^2 \setminus \Gamma_\nu$ such that

$$\xi_\nu(0) \in R_0, \quad |\xi(t) - \zeta(t)| < \frac{1}{\nu} \quad \text{for all } t \in [0, \tau_\nu].$$

Applying Lemma 2.2, we obtain a further path $\tilde{\xi}_\nu : [0, \tilde{\tau}_\nu] \rightarrow \mathbb{R}^2$, also parameterized by arc length, such that

$$\begin{aligned} \tilde{\xi}(0) \in R_0, \quad & |\tilde{\xi}_\nu(\tilde{\tau}_\nu) - x| \leq \frac{2}{\nu}, \\ \tilde{\xi}_\nu(t) \notin \Gamma \quad & \text{for all } t \in [0, \tilde{\tau}_\nu], \end{aligned}$$

and with length

$$\tilde{\tau}_\nu < \tau_\nu + \sum_{i>\nu} m_1(\Gamma_i).$$

Therefore (3.8) follows from

$$\limsup_{\nu \rightarrow \infty} \tau_\nu \leq T^\Gamma(x) \leq \liminf_{\nu \rightarrow \infty} \tilde{\tau}_\nu = \liminf_{\nu \rightarrow \infty} \left[\tau_\nu + \sum_{i>\nu} m_1(\Gamma_i) \right] = \liminf_{\nu \rightarrow \infty} \tau_\nu.$$

2. To prove part (ii), as usual we assume that all the optimal trajectories ξ_ν are parameterized by arc length. By the previous step one has

$$T^\Gamma(x) = \lim_{\nu \rightarrow \infty} T^{\Gamma_\nu}(x) = \lim_{\nu \rightarrow \infty} \tau_\nu = \bar{\tau}.$$

To achieve the proof it thus suffices to check that the limit trajectory $\xi(\cdot)$ is admissible.

Toward this goal, the key tool is again provided by Lemma 2.2. For each $\nu \geq 1$, using the lemma we obtain a path $\tilde{\xi}_\nu : [0, \tilde{\tau}_\nu] \rightarrow \mathbb{R}^2$ such that

$$\begin{aligned} \tilde{\xi}_\nu(0) \in R_0, \quad & \tilde{\xi}_\nu(t) \notin \Gamma \quad \text{for all } t \in [0, \tilde{\tau}_\nu], \\ |\tilde{\xi}_\nu(\tilde{\tau}_\nu) - \xi_\nu(\tau_\nu)| & \leq \frac{1}{\nu}, \end{aligned}$$

and with length

$$\tilde{\tau}_\nu \leq \tau_\nu + \sum_{i>\nu} m_1(\Gamma_i).$$

Recalling (2.23) in the proof of Lemma 2.2, w.l.o.g. we can assume that

$$|\tilde{\xi}_\nu(t) - \xi_\nu(t)| \leq \left(1 + \frac{1}{\nu}\right) \sum_{i>\nu} m_1(\Gamma_i), \quad (3.9)$$

for all $t \geq 0$. It is understood that here ξ_ν and $\tilde{\xi}_\nu$ are extended as constant functions, for $t \geq \tau_\nu$ and $t \geq \tilde{\tau}_\nu$, respectively.

By (3.9), as $\nu \rightarrow \infty$ the sequence of paths $\tilde{\xi}_\nu(\cdot)$ converges uniformly to $\xi(\cdot)$. Hence ξ is admissible.

4 A regularity property of optimal trajectories

Aim of this section is to study a property of the optimal trajectories for the fire, in the presence of barriers. We begin with a few observations.

- If no barriers are present, all optimal trajectories are straight lines, and the minimum time function is trivially $T(x) = d(x, R_0)$.
- Next, assume that R_0 has a C^2 boundary. For each point $x \notin R_0$, consider the shortest segment connecting x with a point $y \in \overline{R_0}$. If the total length of all barriers $m_1(\Gamma) = \sum_i m_1(\Gamma_i) < \varepsilon$ is sufficiently small, then most of these segments will not cross Γ . Hence, as shown in Fig. 9, center, they will yield optimal trajectories for the fire also when barriers are present. Notice that this remains true even if the set Γ of all barriers is dense in \mathbb{R}^2 .
- For a general set R_0 , however, even if the total length of all barriers is very small, it can happen that most of the optimal trajectories touch one of the barriers. As shown in Fig. 9, this is the case when R_0 has cusps, and some of the barriers are placed very close to these cusps.

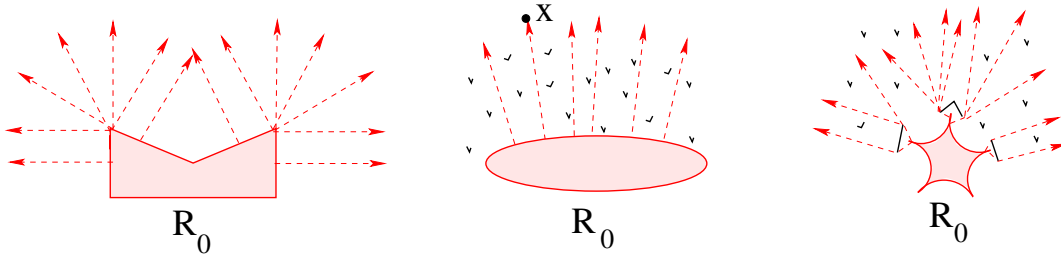


Figure 9: Left: if no barriers are present, all optimal trajectories are straight lines. Center: if the initial set R_0 has smooth boundary and the total length of all barriers is small, then most of the optimal trajectories do not hit the barrier, and hence are still straight lines. Right: for a general set R_0 whose boundary contains cusps, even if the total length of the barriers is small, most of the optimal trajectories may touch one of the barriers.

Since in general it is not true that most optimal trajectories are straight lines, in this section we prove a somewhat weaker property. Namely: most optimal trajectories contain long straight segments. This property will play a key role in the proof of Theorem 1.5.

Consider again the optimization problem for the fire, in the presence of a barrier Γ . Given an initial set R_0 , let

$$R_1 \doteq \{x \in \mathbb{R}^2; d(x, R_0) < 1\} \quad (4.1)$$

be the neighborhood of radius 1 around R_0 . For any $x \in \mathbb{R}^2$, call $T^\Gamma(x)$ the minimum time needed to reach x from R_0 without crossing Γ . Moreover, given an admissible trajectory $t \mapsto \xi^x(t)$ reaching x in minimum time, we denote by $\rho(x)$ the length of the last portion of this trajectory which is a straight line. More precisely,

$$\rho(x) \doteq \sup \left\{ \tau \geq 0; \text{ there exists a trajectory } t \mapsto \xi^x(t) \text{ reaching } x \text{ in minimum time} \right. \\ \left. \text{without crossing } \Gamma, \text{ and the velocity } \dot{\xi}^x \text{ is constant on } [T^\Gamma(x) - \tau, T^\Gamma(x)] \right\}. \quad (4.2)$$

Lemma 4.1. *Let R_0 be a bounded, open set, and call R_1 the set in (4.1). Then, for any barrier Γ one has*

$$\int_{R_1} \rho(x) dx \geq \int_{R_1} d(x, R_0) dx - \frac{\widehat{T}^2 + \widehat{T}}{2} \cdot m_1(\Gamma), \quad (4.3)$$

where

$$\widehat{T} \doteq \sup_{x \in R_1} T^\Gamma(x).$$

Remark 4.2. In the case where no barriers are present, one has $\rho(x) = d(x, R_0)$ and the bound (4.3) is obvious. We observe that a lower bound on the left hand side of (4.3) cannot be achieved by the trivial estimate

$$\rho(x) \geq \inf_{y \in \Gamma} |y - x|, \quad (4.4)$$

because the set $\Gamma = \cup_i \Gamma_i$ can be everywhere dense. In this case the right hand side of (4.4) is identically zero.

Remark 4.3. Assuming that $\mathbb{R}^2 \setminus \Gamma$ is connected, so that all barriers are only delaying the fire, by Lemma 2.2 it follows that

$$\widehat{T} \leq 1 + m_1(\Gamma). \quad (4.5)$$

4.1 Polygonal barriers.

We shall give a proof of Lemma 4.1 first in a special case where explicit computations can be performed. The general case will then be handled by an approximation argument. In this section, we assume

(A2) *The initial set R_0 is the union of finitely many open discs, while the barrier Γ is the union of finitely many (not necessarily disjoint) closed segments.*

Notice that this special setting implies

- (i) Every optimal trajectory for the fire, reaching a point $x \in \mathbb{R}^2$ in minimum time without crossing the barriers, is a polygonal, say with vertices P_0, P_1, \dots, P_N . Here $P_0 \in \overline{R_0}$, $P_N = x$, while $P_i \in \Gamma$ for all $i \in \{1, \dots, N-1\}$. Indeed, each P_i will be an edge of one of the segments forming the barrier Γ .
- (ii) For every $t > 0$, the boundary of the reachable set $\partial \overline{R^\Gamma(t)}$ is the union of finitely many arcs of circumferences.
- (iii) The set of points which can be reached in minimum time by two distinct trajectories is the union of finitely many segments, or arcs of hyperbolas.

To prove the estimate (4.3) we shall study a family of problems, parameterized by time. Call

$$\Gamma(t) = \Gamma \cap \overline{R^\Gamma(t)}$$

the portion of the walls which are touched by the fire within time t . We obviously have

$$\Gamma(s) \subseteq \Gamma(t) \quad \text{for } s < t.$$

For every $t \geq 0$, call $\rho(t, x)$ the function defined at (4.2), but with Γ replaced by the smaller set $\Gamma(t)$.

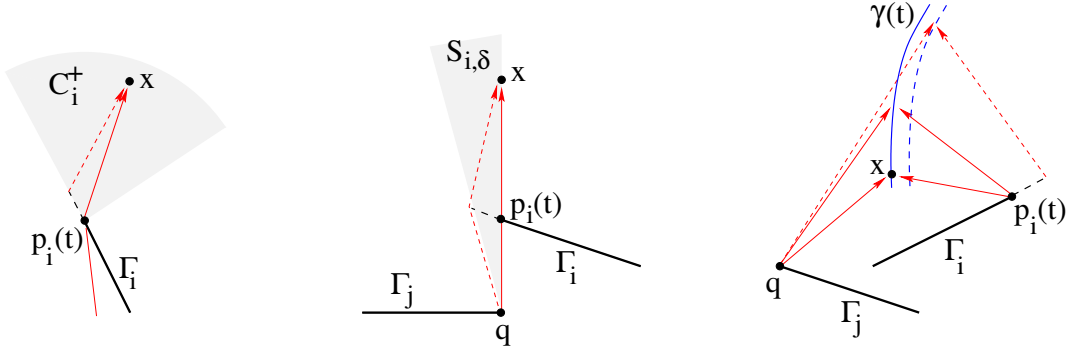


Figure 10: Left: as the segment Γ_i becomes longer, the value $\rho(x)$ can decrease at all points x in the shaded region. Center: at time increases, the value $\rho(x)$ jumps downward from $|x - q|$ to $|x - p_i(t)|$. Right: the curve $\gamma(t)$ denotes the set of points x reached in minimum time by two distinct trajectories. As time increases, this curve changes in time. At points x on this curve, the value of ρ jumps upward from $|x - p_i(t)|$ to $|x - q|$.

The estimate (4.3) will be achieved by showing that, for a.e. $t \in [0, 1]$,

$$-\frac{d}{dt} \int_{R_1} \rho(t, x) dx \leq \frac{\widehat{T}^2 + \widehat{T}}{2} \cdot \frac{d}{dt} m_1(\Gamma(t)). \quad (4.6)$$

To fix the ideas, let $\Gamma_i(t) \subseteq \Gamma(t)$ be one of the segments of the barrier, with an endpoint $p_i(t) \in \partial \overline{R^\Gamma(t)}$ moving along the edge of the advancing fire. For a fixed time t , referring to the optimization problem with barrier $\Gamma(t)$, three cases must be considered.

1. Points x reached in minimum time by a trajectory which touches the point $p_i(t)$.

The set of all these points, that we shall call Ω_i , is contained within a half disc C_i , with center at $p_i(t)$ and radius $\widehat{T} - t$. As time increases, for all $x \in \Omega_i$ we have

$$\frac{d}{dt}\rho(t, x) = \frac{d}{dt}|p_i(t) - x| = \left\langle \dot{p}_i(t), \frac{p_i(t) - x}{|p_i(t) - x|} \right\rangle \geq -|\dot{p}_i(t)|. \quad (4.7)$$

Observe that the quantity in (4.7) can be negative only on the quarter disc

$$C_i^+ \doteq \{x \in C_i; \langle \dot{p}_i(t), p_i(t) - x \rangle < 0\},$$

corresponding to the shaded region in Fig. 10, left. Using (4.7) we compute

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_i} \rho(t, x) dx &\geq \int_{C_i^+} \left\langle \dot{p}_i(t), \frac{p_i(t) - x}{|p_i(t) - x|} \right\rangle dx \\ &\geq -|\dot{p}_i(t)| \cdot \int_0^{\widehat{T}-t} \int_0^{\pi/2} \cos \theta d\theta r dr \geq -\frac{\widehat{T}^2}{2} |\dot{p}_i(t)|. \end{aligned} \quad (4.8)$$

2. Next, we consider points x reached in minimum time by a trajectory whose last portion is a segment with endpoints q and x , and such that $p_i(t)$ is a point inside this segment (see Fig. 10, center).

The set of all these points, which we will call D_i , is contained on a half line starting at q and passing through $p_i(t)$, so that

$$|x - p_i(t)| \leq |x - q| < 1 \quad \text{for all } x \in D_i.$$

As time increases, the value of ρ along D_i jumps downward from $|x - q|$ to $|x - p_i(t)|$. To compute the rate of decrease in the integral $\int \rho(t, x) dx$ due to such points, fix $\delta > 0$ small and consider the region $D_{i,\delta}$ of all points x such that

$$\rho(t, x) = |x - q|, \quad \rho(t + \delta, x) = |x - p_i(t + \delta)|.$$

By the triangle inequality one obtains

$$\int_{D_{i,\delta}} (\rho(t, x) - \rho(t + \delta, x)) dx \leq m_2(D_{i,\delta}) \cdot |p_i(t + \delta) - q|. \quad (4.9)$$

Observe that $D_{i,\delta}$ is contained in a circular sector $S_{i,\delta}$ with radius \widehat{T} (as the one shaded in Fig. 10, center) whose area can be computed using the vector product

$$m_2(S_{i,\delta}) = \frac{\widehat{T}}{2} \frac{|(p_i(t + \delta) - p_i(t)) \times (p_i(t) - q)|}{|p_i(t) - q|} + o(\delta). \quad (4.10)$$

Combining (4.9) with (4.10), we conclude

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{D_{i,\delta}} (\rho(t, x) - \rho(t + \delta, x)) dx \leq \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} m_2(S_{i,\delta}) \cdot |p_i(t + \delta) - q| \leq \frac{\widehat{T}}{2} |\dot{p}_i(t)|. \quad (4.11)$$

3. Points x on a curve $\gamma(t)$ reached in minimum time by two distinct optimal trajectories.

As shown in Fig. 10, right, we can assume that one of these touches $p_i(t) \in \Gamma_i$, while the other touches some other point $q \in \Gamma_j$. These two trajectories have the same length, therefore

$$|x - q| + T^\Gamma(q) = |x - p_i(t)| + T^\Gamma(p_i(t)) = |x - p_i(t)| + t. \quad (4.12)$$

Since $T^\Gamma(q) \leq t$, this implies $|x - q| \geq |x - p_i(t)|$ for all $x \in \gamma(t)$.

Notice that $\gamma(t)$ is a branch of hyperbola. For a.e. $y \in \gamma(t)$ we can choose a neighborhood V of y such that, for every $x \in V$, one has

$$T^{\Gamma(t)}(x) = \min \left\{ |x - q| + T^\Gamma(q), \quad |x - p_i(t)| + t \right\}. \quad (4.13)$$

Assume that, when the barrier is $\Gamma(t)$, a point $x \in V$ is reached in minimum time by a trajectory passing through q , namely

$$|x - q| + T^\Gamma(q) < |x - p_i(t)| + t.$$

As time increases from t to $t + \delta$ and the point $p_i(t)$ is replaced by $p_i(t + \delta)$, by (4.13) we have

$$\begin{aligned} |x - p_i(t + \delta)| + T^{\Gamma(t + \delta)}(p_i(t + \delta)) &\geq |x - p_i(t + \delta)| + T^{\Gamma(t)}(p_i(t + \delta)) \\ &\geq |x - p_i(t)| + t > |x - q| + T^\Gamma(q). \end{aligned} \quad (4.14)$$

According to (4.14), when the barrier increases from $\Gamma(t)$ to $\Gamma(t + \delta)$, the point x is still reached in minimum time by a trajectory passing through q . We conclude that, for $x \in V$,

$$\rho(t, x) = |x - q| \quad \Longrightarrow \quad \rho(t + \delta, x) = |x - q|.$$

In other words, the function $\rho(\cdot, x)$ cannot have a downward jump. However, it may well jump upward, from $|x - p_i(t)|$ to $|x - q|$.

4. Combining the previous steps **1-2-3**, for a.e. time $t > 0$ we obtain

$$\frac{d}{dt} \int_{R_1} \rho(t, x) dx \geq -\frac{\widehat{T}^2 + \widehat{T}}{2} \cdot \sum_i |\dot{p}_i(t)| = -\frac{\widehat{T}^2 + \widehat{T}}{2} \cdot \frac{d}{dt} m_1(\Gamma(t)).$$

This proves (4.6).

To achieve the estimate (4.3), we now observe that the integral in (4.6) depends continuously on time, except at finitely many times τ_k where the topology of $\Gamma(t)$ changes. To understand what happens at these exceptional times, as shown in Fig. 11, left, assume that the barrier $\Gamma(t)$ contains two segments Γ_1 and Γ_2 with moving endpoints $p_1(t)$, $p_2(t)$. Assume that, at time $t = \tau$, the two segments join together: $p_1(\tau) = p_2(\tau)$ as in Fig. 11, right.

Let $x \in R_1$ and assume that, for $t = \tau - \delta$ with $\delta > 0$ small enough, the point x is reached in minimum time by a trajectory passing through $p_1(t)$. On the other hand, for $t = \tau$, assume that $\rho(\tau, x) = |x - q|$, for some point q along a different optimal trajectory which reaches x without crossing $\Gamma(\tau)$. For $t < \tau$ we now have

$$\rho(t, x) = |x - p_1(t)| = T^{\Gamma(t)}(x) - t, \quad (4.15)$$

while at time τ

$$\rho(\tau, x) = |x - q| = T^{\Gamma(\tau)}(x) - T^{\Gamma}(q). \quad (4.16)$$

Observing that

$$T^{\Gamma(\tau)}(x) \geq \lim_{t \rightarrow \tau^-} T^{\Gamma(t)}(x), \quad T^{\Gamma}(q) \leq \tau,$$

by (4.15)-(4.16) we conclude

$$\rho(\tau, x) \geq \lim_{t \rightarrow \tau^-} \rho(t, x). \quad (4.17)$$

This shows that, at a time τ where the topology of the barrier $\Gamma(\cdot)$ changes, the function ρ can only have upward jumps.

It remains to observe that, when $t = 0$, one trivially has $\Gamma(0) = \emptyset$ and

$$\int_{R_1} \rho(0, x) dx = \int_{R_1} d(x, R_0) dx.$$

Hence from (4.6) we conclude (4.3).

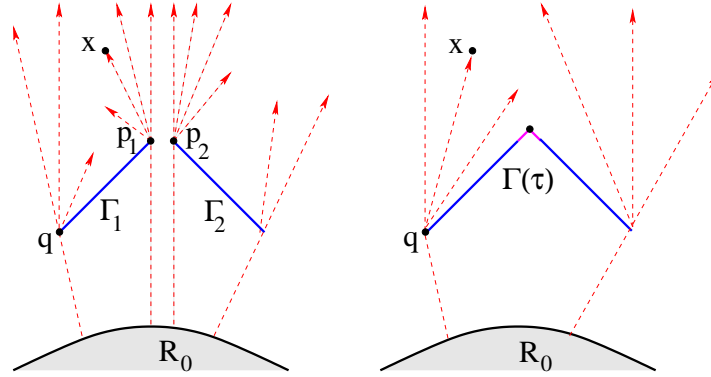


Figure 11: As time t reaches a critical value τ when two portions of the barrier join together, the topology of $\Gamma(t)$ changes. Both the minimum time $T^{\Gamma(t)}(x)$ and the value $\rho(t, x)$ jump upward.

5. The previous analysis has established the estimate (4.3) in the case where the boundary of R_0 is a finite union of circular arcs, and the barrier Γ is the union of finitely many segments. By an approximation argument, we shall extend the result to a general initial domain R_0 and a general barrier Γ .

As an intermediate step, we show that the estimate (4.3) holds for a general initial set R_0 , assuming that Γ has finitely many connected components: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_N$.

Indeed, consider a sequence of open sets $(R_{0,n})_{n \geq 1}$ such that:

- (i) The boundary of each $R_{0,n}$ is a finite union of circular arcs.
- (ii) As $n \rightarrow \infty$ the closures of these sets converge in the Hausdorff distance [2], namely $d_H(\overline{R_{0,n}}, \overline{R_0}) \rightarrow 0$

Moreover, for each $k = 1, \dots, N$, let $(\Gamma_{k,n})_{n \geq 1}$ be a sequence of compact connected sets such that:

(iii) Each $\Gamma_{k,n}$ is the union of finitely many segments.

(iv) $m_1(\Gamma_{k,n}) \leq m_1(\Gamma_k)$.

(v) As $n \rightarrow \infty$ we have the convergence in the Hausdorff distance: $d_H(\Gamma_{k,n}, \Gamma_k) \rightarrow 0$.

For $x \in R_1$, let $\gamma_{x,n}$ be a polygonal line reaching x in minimum time. More precisely, $\gamma_{x,n}$ minimizes $m_1(\gamma_{x,n})$ among all polygonal lines connecting x to some point $y \in \partial R_0$ without crossing the barrier $\Gamma_n = \cup_{k=1}^N \Gamma_{k,n}$.

We now observe that, for a.e. point $x \in R_1$, the function ρ defined at (4.2) satisfies

$$\rho(x) \geq \limsup_{n \rightarrow \infty} \rho_n(x). \quad (4.18)$$

Indeed, we can parameterize every curve $\gamma_{x,n}$ by arc-length, say $s \mapsto \gamma_{x,n}(s)$, with

$$\gamma_{x,n}(0) = x, \quad \gamma_{x,n}(m_1(\gamma_{x,n})) \in \partial R_{0,n}.$$

By taking a subsequence, we can assume the uniform convergence $\gamma_{x,n} \rightarrow \gamma_x$ on every subinterval $[0, \ell]$ with $\ell < m_1(\gamma_x)$. If now the derivatives $\dot{\gamma}_{x,n}$ are constant over some initial interval $[0, \bar{s}]$, the same is true of the derivative $\dot{\gamma}_x$ of the limit function γ_x . This proves (4.18).

From the inequality (3.3) in Lemma 3.1 it follows

$$\widehat{T} \doteq \sup_{x \in R_1} T^\Gamma(x) \geq \limsup_{n \rightarrow \infty} \widehat{T}^{(n)} \doteq \limsup_{n \rightarrow \infty} \sup_{x \in R_1} T^{\Gamma^{(n)}}(x).$$

In turn, since all functions ρ_n are uniformly bounded, we have

$$\begin{aligned} \int_{R_1} \rho(x) dx &\geq \limsup_{n \rightarrow \infty} \int_{R_1} \rho_n(x) dx \geq \limsup_{n \rightarrow \infty} \int_{R_1} d(x, R_{0,n}) dx - \frac{\widehat{T}_n + \widehat{T}_n^2}{2} m_1(\Gamma_n) \\ &\geq \int_{R_1} d(x, R_0) dx - \frac{\widehat{T} + \widehat{T}^2}{2} m_1(\Gamma). \end{aligned} \quad (4.19)$$

6. Finally, we consider the general case where $\Gamma = \cup_{k \geq 1} \Gamma_k$ is the union of countably many compact, connected components. We call $\rho_\nu(\cdot)$ the map in (4.2), replacing Γ with a finite union $\Gamma_\nu \doteq \cup_{k=1}^\nu \Gamma_k$.

Thanks to Lemma 3.4, the same argument used to prove (4.18) now yields

$$\rho(x) \geq \limsup_{\nu \rightarrow \infty} \rho_\nu(x) \quad \text{for a.e. } x \in R_1. \quad (4.20)$$

Moreover, since $\Gamma_\nu \subset \Gamma$ for every $\nu \geq 1$, we trivially have

$$\widehat{T} \doteq \sup_{x \in R_1} T^\Gamma(x) \geq \sup_{x \in R_1} T^{\Gamma_\nu}(x) \doteq \widehat{T}_\nu.$$

By the previous steps, we already know that the estimate (4.3) holds for every ρ_ν . Taking the limit as $\nu \rightarrow \infty$ and using Lemma 3.4 we conclude

$$\begin{aligned} \int_{R_1} \rho(x) dx &\geq \limsup_{\nu \rightarrow \infty} \int_{R_1} \rho_\nu(x) dx \geq \int_{R_1} d(x, R_0) dx - \lim_{\nu \rightarrow \infty} \frac{\widehat{T}_\nu^2 + \widehat{T}_\nu}{2} m_1(\Gamma) \\ &\geq \int_{R_1} d(x, R_0) dx - \lim_{\nu \rightarrow \infty} \frac{\widehat{T}^2 + \widehat{T}}{2} m_1(\Gamma). \end{aligned}$$

This completes the proof of Lemma 4.1. \square

5 Avoiding barriers more efficiently

As before, we assume that $\mathbb{R}^2 \setminus \Gamma$ is connected. By the analysis in Lemma 2.2, if $p, q \notin \Gamma$, then for any $\varepsilon > 0$ we can connect these two points with a path that does not cross Γ and has length $\leq |p - q| + (1 + \varepsilon)m_1(\Gamma)$. Indeed, one can start with the segment having p, q as endpoints, and then insert detours to avoid crossing each connected component of Γ .

In this section we prove a sharper result. Namely, if the barrier is sufficiently sparse, we can connect the two points p, q with a path that avoids Γ and has length just slightly larger than $|p - q|$. We begin by studying the case where Γ is the union of finitely many (possibly intersecting) closed segments, then generalize.

Lemma 5.1. *In the t - x plane, consider a barrier Γ consisting of finitely many (possibly intersecting) segments, none of which is parallel to the x -axis. Assume that, for every $t > 0$, the total length of the portion of Γ contained in the strip $[0, t] \times \mathbb{R}$ satisfies*

$$\psi(t) \doteq m_1\left(\Gamma \cap ([0, t] \times \mathbb{R})\right) \leq \sqrt{2}\varepsilon t, \quad t \in [0, T], \quad (5.1)$$

for some $0 < \varepsilon < 1$. Then there exists a continuous map $\xi : [0, T] \mapsto \mathbb{R}$ with Lipschitz constant ε , which satisfies $\xi(0) = 0$ and whose graph does not cross Γ .

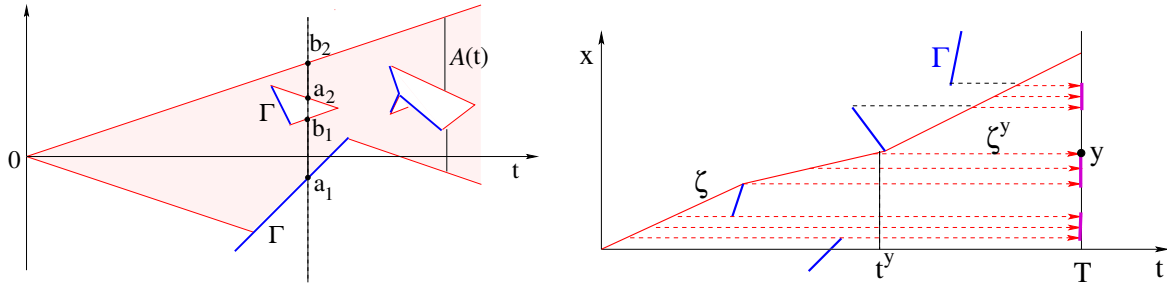


Figure 12: Left: for each $t > 0$, the set $\mathcal{A}(t)$ in (5.2) is the union of finitely many segments $[a_k(t), b_k(t)]$. Right: the functions ζ and ζ^y constructed in the proof of Lemma 5.3.

Proof. 1. For every $t > 0$, consider the set $\mathcal{A}(t) \subset \mathbb{R}$ of all values that can be attained by ε -Lipschitz functions, which are zero at the origin and whose graph does not cross Γ . Namely, as shown in Fig. 12), left,

$$\begin{aligned} \mathcal{A}(t) \doteq \left\{ \xi(t); \quad \xi \text{ is absolutely continuous, } \xi(0) = 0, \right. \\ \left. \|\dot{\xi}\|_{\mathbf{L}^\infty} \leq \varepsilon, \quad (s, \xi(s)) \notin \Gamma \text{ for all } s \in [0, t] \right\}. \end{aligned} \quad (5.2)$$

Since Γ is the union of finitely many segment, we observe that each $\mathcal{A}(t)$ is the union of finitely many intervals, say

$$\mathcal{A}(t) = \bigcup_k]a_k(t), b_k(t)[.$$

At any given time t , we denote by $\mathcal{B}(t)$ the set of the endpoints a_k, b_k which lie along the barrier Γ , and by $\mathcal{F}(t)$ the set of the endpoints which are free, i.e. they do not lie on Γ . The total length of the attainable set $\mathcal{A}(t)$ changes at the rate

$$\begin{aligned} \frac{d}{dt}(\text{meas}(\mathcal{A}(t))) &= \sum_k (\dot{b}_k(t) - \dot{a}_k(t)) \\ &\geq \sum_{b_k(t) \in \mathcal{F}(t)} \dot{b}_k(t) - \sum_{a_k(t) \in \mathcal{F}(t)} \dot{a}_k(t) - \sum_{b_k(t) \in \mathcal{B}(t)} |\dot{b}_k(t)| - \sum_{a_k(t) \in \mathcal{B}(t)} |\dot{a}_k(t)|. \end{aligned} \quad (5.3)$$

On the other hand, from the definition of ψ at (5.1), it follows

$$\begin{aligned} \dot{\psi}(t) &\geq \sum_{a_k(t) \in \mathcal{B}(t)} \sqrt{1 + \dot{a}_k^2(t)} + \sum_{b_k(t) \in \mathcal{B}(t)} \sqrt{1 + \dot{b}_k^2(t)} \\ &\geq \sum_{a_k(t) \in \mathcal{B}(t)} \frac{1 + |\dot{a}_k(t)|}{\sqrt{2}} + \sum_{b_k(t) \in \mathcal{B}(t)} \frac{1 + |\dot{b}_k(t)|}{\sqrt{2}}. \end{aligned} \quad (5.4)$$

2. To estimate the right hand side of (5.3), consider the function

$$f(t) \doteq \text{meas}(\mathcal{A}(t)) - \sqrt{2}(\varepsilon t - \psi(t)). \quad (5.5)$$

By (5.1) it follows

$$\text{meas}(\mathcal{A}(t)) = f(t) + \sqrt{2}(\varepsilon t - \psi(t)) \geq f(t).$$

Therefore, as long as $f(t) > 0$, we have $\mathcal{A}(t) \neq \emptyset$. In the remainder of the proof we will show that f is positive and nondecreasing.

To begin, we observe that, for $t > 0$ small, no barriers are present. Hence

$$f(t) = \text{meas}(\mathcal{A}(t)) - \sqrt{2}\varepsilon t = 2\varepsilon t - \sqrt{2}\varepsilon t > 0.$$

Next, using (5.3) and (5.4), from (5.5) we obtain

$$\begin{aligned} \frac{d}{dt}f(t) &= \frac{d}{dt}\text{meas}(\mathcal{A}(t)) - \sqrt{2}\varepsilon + \sqrt{2} \cdot \dot{\psi}(t) \\ &\geq \left(\#\mathcal{F}(t) \cdot \varepsilon - \sum_{b_k(t) \in \mathcal{B}(t)} |\dot{b}_k(t)| - \sum_{a_k(t) \in \mathcal{B}(t)} |\dot{a}_k(t)| \right) - \sqrt{2}\varepsilon \\ &\quad + \sqrt{2} \cdot \left(\sum_{a_k(t) \in \mathcal{B}(t)} \frac{1 + |\dot{a}_k(t)|}{\sqrt{2}} + \sum_{b_k(t) \in \mathcal{B}(t)} \frac{1 + |\dot{b}_k(t)|}{\sqrt{2}} \right) \\ &\geq (\#\mathcal{F}(t) + \#\mathcal{B}(t)) \cdot \varepsilon - \sqrt{2}\varepsilon \geq 2\varepsilon - \sqrt{2}\varepsilon > 0. \end{aligned} \quad (5.6)$$

Here $\#\mathcal{F}$ and $\#\mathcal{B}$ denote the cardinality of the sets of free and constrained endpoints, respectively. We observe that, as long as $\mathcal{A}(t)$ does not vanish, its boundary contains at least two points. This yields the last inequality in (5.6). \square

Next, instead of (5.2), we consider the sets

$$\begin{aligned} \tilde{\mathcal{A}}(t) \doteq \left\{ \xi(t); \xi \text{ is absolutely continuous, } \dot{\xi}(s) \in [\varepsilon, 3\varepsilon] \text{ for a.e. } s \in [0, t], \right. \\ \left. \xi(0) = 0, \quad (s, \xi(s)) \notin \Gamma \text{ for all } s \in [0, t] \right\}. \end{aligned} \quad (5.7)$$

The same argument used to prove Lemma 5.1 yields

Corollary 5.2. *In the same setting as Lemma 5.1, let (5.1) be replaced by*

$$\psi(t) \doteq m_1\left(\Gamma \cap ([0, t] \times \mathbb{R})\right) \leq \frac{\sqrt{2}\varepsilon}{1+2\varepsilon} t, \quad t \in [0, T]. \quad (5.8)$$

Then, for every $t \in [0, T]$ the set $\tilde{\mathcal{A}}(t)$ in (5.7) is non-empty.

Proof. Given a barrier $\Gamma \subset \mathbb{R}^2$ satisfying (5.8), consider the shifted barrier

$$\Gamma^{2\varepsilon} = \{(t, x); (t, x + 2\varepsilon t) \in \Gamma\}.$$

In view of (5.8), this set satisfies the inequality

$$\psi(t) \doteq m_1\left(\Gamma^{2\varepsilon} \cap ([0, t] \times \mathbb{R})\right) \leq (1+2\varepsilon)m_1\left(\Gamma \cap ([0, t] \times \mathbb{R})\right) \leq \sqrt{2}\varepsilon t, \quad t \in [0, T]. \quad (5.9)$$

Applying Lemma 5.1 we obtain an ε -Lipschitz function $t \mapsto \xi(t)$ such that $\xi(0) = 0$ and $(t, \xi(t)) \notin \Gamma^{2\varepsilon}$ for all $t \in [0, T]$.

Introducing the function $\tilde{\xi}(t) \doteq \xi(t) + 2\varepsilon t$, we obtain $\tilde{\xi}(t) \in \tilde{\mathcal{A}}(t)$ for all $t \in [0, T]$. Hence $\tilde{\mathcal{A}}(t)$ is nonempty. \square

In the next lemma, instead of (5.2), for $t \in [0, T]$ we consider the attainable sets

$$\begin{aligned} \mathcal{A}_3(t) \doteq \left\{ \xi(t); \xi \text{ is absolutely continuous, } \|\dot{\xi}(s)\|_{\mathbf{L}^\infty} \leq 3\varepsilon, \right. \\ \left. \xi(0) = 0, \quad (s, \xi(s)) \notin \Gamma \text{ for all } s \in [0, t] \right\}. \end{aligned} \quad (5.10)$$

Lemma 5.3. *In the t - x plane, consider a barrier Γ consisting of finitely many (possibly intersecting) segments, none of which is parallel to the x -axis. Assume that, for some $0 < \varepsilon < 1$,*

$$\psi(t) \doteq m_1\left(\Gamma \cap ([0, t] \times \mathbb{R})\right) \leq \frac{\varepsilon}{2} t, \quad \text{for all } t \in [0, T]. \quad (5.11)$$

Moreover assume that the total length of the barrier satisfies

$$h \doteq m_1(\Gamma) \leq \frac{\varepsilon T}{3}. \quad (5.12)$$

Then

$$m_1(\mathcal{A}_3(T) \cap [0, 3h]) \geq 2h. \quad (5.13)$$

Proof. 1. Applying Corollary 5.2, we obtain an absolutely continuous map $t \mapsto \zeta(t)$, with $\zeta(0) = 0$, $\dot{\zeta}(t) \in [\varepsilon, 3\varepsilon]$, and whose graph does not intersect Γ .

2. Call

$$V \doteq \{x \in \mathbb{R}; (t, x) \in \Gamma \text{ for some } t \in [0, T]\}$$

the perpendicular projection of Γ on the x -axis. By (5.12) it follows $m_1(V) \leq h$. Hence

$$m_1([0, 3h] \setminus V) \geq 2h. \quad (5.14)$$

3. Since $\dot{\zeta}(t) \geq \varepsilon$, by (5.12) for every $y \in [0, 3h] \setminus V$, there exists a unique time $t^y \in [0, T]$ such that $\zeta(t^y) = y$. As shown in Fig. 12, right, consider the map

$$t \mapsto \zeta^y(t) = \begin{cases} \zeta(t) & \text{if } t \in [0, t^y], \\ y & \text{if } t \in [t^y, T]. \end{cases} \quad (5.15)$$

Our construction implies $(t, \zeta^y(t)) \notin \Gamma$ for all $t \in [0, T]$. Hence $\zeta^y(T) = y \in \mathcal{A}_3(T)$. We thus conclude

$$m_1(\mathcal{A}_3(T) \cap [0, 3h]) \geq m_1([0, 3h] \setminus V) \geq 2h. \quad \square$$

Remark 5.4. Let $z = \zeta^y(\cdot)$ be one of the functions considered at (5.15). The length of its graph is computed by

$$\begin{aligned} \ell &= \int_0^T \sqrt{1 + \dot{z}^2(t)} dt \leq \int_0^T \sqrt{1 + 3\varepsilon \dot{z}(t)} dt \leq \int_0^T \left(1 + \frac{3\varepsilon}{2} \dot{z}(t)\right) dt \\ &\leq T + \frac{3\varepsilon}{2} z(T) \leq T + \frac{3\varepsilon}{2} \cdot 3h = T + \frac{9\varepsilon}{2} \cdot m_1(\Gamma). \end{aligned} \quad (5.16)$$

This is a crucial bound, because it shows that the presence of a very sparse barrier can lengthen the trajectories of the fire only by an amount $\mathcal{O}(\varepsilon) \cdot m_1(\Gamma)$. As a consequence, the time $\sigma^{-1} \cdot m_1(\Gamma)$ spent for constructing these walls is not compensated by the additional time needed for the fire to go around them.

The final result proved in this section extends the previous lemmas to a general barrier $\Gamma = \cup_{i \geq 1} \Gamma_i$, which is the union of countably many compact, connected, rectifiable sets. As in Definition 2.1, we say that a path $t \mapsto \gamma(t) \in \mathbb{R}^2$, $t \in [0, \ell]$, is **admissible** if there exists a sequence of 1-Lipschitz paths $t \mapsto \gamma_n(t)$ such that $\gamma_n(t) \notin \Gamma$ for all $t \geq 0$, and moreover $\lim_{n \rightarrow \infty} \gamma_n(t) = \gamma(t)$, uniformly for $t \in [0, \ell]$.

Lemma 5.5. *In the t - x plane, consider the points $P = (-\kappa, 0)$, $Q = (\kappa, 0)$. Let $\Gamma \subset \mathbb{R}^2$ be a barrier such that, for every $r > 0$,*

$$m_1\left(\Gamma \cap ([-\kappa, -\kappa + r] \times \mathbb{R})\right) < \frac{\varepsilon}{3} r, \quad m_1\left(\Gamma \cap ([\kappa - r, \kappa] \times \mathbb{R})\right) < \frac{\varepsilon}{3} r. \quad (5.17)$$

Moreover, assume

$$h \doteq m_1(\Gamma) < \frac{2\kappa\varepsilon}{3}. \quad (5.18)$$

Then there exists an admissible path $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ such that

$$\gamma(0) = P, \quad \gamma(\ell) = Q, \quad (5.19)$$

and with length

$$\ell \leq 2\kappa + 9\varepsilon m_1(\Gamma). \quad (5.20)$$

Proof. 1. We begin by studying the case where $\Gamma = \cup_{i=1}^{\nu} \Gamma_i$ is the union of finitely many compact, connected component. Then we will extend the result to the general case.

For any $\delta > 0$, we can approximate each component Γ_i with another connected set Γ'_i , which is the union of finitely many closed segments, so that their Hausdorff distance satisfies

$$d_H(\Gamma_i, \Gamma'_i) < \delta \quad \text{for all } i = 1, \dots, \nu. \quad (5.21)$$

Moreover, we can assume that (5.17) still holds, with Γ replaced by $\Gamma' = \cup_{i=1}^{\nu} \Gamma'_i$.

An application of Lemma 5.3, with $[0, T]$ replaced by $[-\kappa, 0]$ yields the existence of a set $\mathcal{A}^- \subseteq [0, 3h]$ with the following properties.

$$m_1(\mathcal{A}^-) \geq 2h, \quad (5.22)$$

For every $y \in \mathcal{A}^-$, there exists a Lipschitz function $t \mapsto \zeta^y(t) \in [0, y]$ such that

$$\begin{aligned} \dot{\zeta}^y(t) &\in [0, 3\varepsilon] \quad \text{for a.e. } t \in [-\kappa, 0], \\ \zeta^y(-\kappa) &= 0, \quad \zeta^y(0) = y, \quad (t, \zeta^y(t)) \notin \Gamma' \quad \text{for all } t \in [-\kappa, 0]. \end{aligned}$$

Repeating the same argument on the interval $[0, \kappa]$, we obtain the existence of a set $\mathcal{A}^+ \subseteq [0, 3h]$ such that

$$m_1(\mathcal{A}^+) \geq 2h, \quad (5.23)$$

For every $y \in \mathcal{A}^+$, there exists a Lipschitz function $t \mapsto \zeta^y(t) \in [0, y]$ such that

$$\begin{aligned} \dot{\zeta}^y(t) &\in [-3\varepsilon, 0] \quad \text{for a.e. } t \in [0, \kappa], \\ \zeta^y(0) &= y, \quad \zeta^y(\kappa) = 0, \quad (t, \zeta^y(t)) \notin \Gamma' \quad \text{for all } t \in [0, \kappa]. \end{aligned}$$

By (5.22) and (5.23), we can choose $y \in \mathcal{A}^- \cap \mathcal{A}^+$. Combining the two previous constructions on $[-\kappa, 0]$ and on $[0, \kappa]$, we obtain a Lipschitz function $\zeta : [-\kappa, \kappa] \mapsto [0, 3h]$ such that $|\dot{\zeta}(t)| \leq 3\varepsilon$ for a.e. t , and moreover

$$\zeta(0) = y, \quad \zeta(-\kappa) = \zeta(\kappa) = 0, \quad (t, \zeta(t)) \notin \Gamma' \quad \text{for all } t \in [-\kappa, \kappa]. \quad (5.24)$$

By the same argument used in Remark 5.4, the length of the graph of ζ is bounded by

$$\ell = \int_{-\kappa}^{\kappa} \sqrt{1 + \dot{\zeta}^2(t)} dt \leq 2\kappa + 9\varepsilon \cdot m_1(\Gamma). \quad (5.25)$$

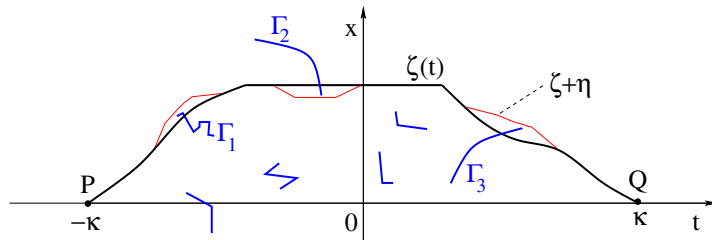


Figure 13: Replacing the function ζ with $\zeta + \eta$, we obtain a new function whose graph which does not intersect any of the connected components $\Gamma_1, \dots, \Gamma_\nu$. In this figure we have $1, 3 \in I^-$ while $2 \in I^+$.

2. The graph of the function ζ constructed in the previous step does not touch the components Γ'_i , $1 \leq i \leq \nu$ of the approximated barrier. However, it may well cross some components Γ_i of the original barrier. In this step (see Fig. 13) we perform a small modification and construct a new map $z : [-\kappa, \kappa] \mapsto \mathbb{R}$ whose graph will not cross $\Gamma_1 \cup \dots \cup \Gamma_\nu$.

We begin by splitting

$$\{1, \dots, \nu\} = I^+ \cup I^-,$$

where I^+ labels the components Γ'_i lying above the graph of ζ , while I^- labels the components Γ'_i lying below the graph of ζ . We then set

$$\Gamma = \Gamma^- \cup \Gamma^+, \quad \Gamma^- \doteq \bigcup_{i \in I^-} \Gamma_i, \quad \Gamma^+ \doteq \bigcup_{i \in I^+} \Gamma_i.$$

Consider the functions

$$\zeta^+(t) \doteq \zeta(t) + 2\delta, \quad \zeta^-(t) \doteq \zeta(t) - 2\delta.$$

For any $\delta > 0$ sufficiently small compared with ε , by (5.21) the graph of z^+ does not intersect Γ^- , while the graph of z^- does not intersect Γ^+ . Call

$$\delta_\nu \doteq \min \left\{ |p - q|; p \in \Gamma_i, q \in \Gamma_j, 1 \leq i < j \leq \nu \right\} > 0, \quad (5.26)$$

and notice that

$$\min \left\{ |p - q|; p \in \Gamma^-, q \in \Gamma^+ \right\} \geq \delta_\nu. \quad (5.27)$$

Consider the sets of times

$$\mathcal{T}^- \doteq \left\{ t \in [-\kappa, \kappa]; (t, x) \in \Gamma^+ \text{ for some } x < \zeta(t) \right\},$$

$$\mathcal{T}^+ \doteq \left\{ t \in [-\kappa, \kappa]; (t, x) \in \Gamma^- \text{ for some } x > \zeta(t) \right\}.$$

By (5.27), there exists $\delta_\nu^* > 0$ independent of δ , such that

$$\inf \left\{ |t - t'|; t \in \mathcal{T}^+, t' \in \mathcal{T}^- \right\} \geq \delta_\nu^*. \quad (5.28)$$

We now construct a Lipschitz function

$$\eta : [-\kappa, \kappa] \mapsto [-2\delta, 2\delta]$$

such that

$$\eta(-\kappa) = \eta(\kappa) = 0, \quad \eta(t) = \begin{cases} 2\delta & \text{if } t \in \mathcal{T}^+, \\ -2\delta & \text{if } t \in \mathcal{T}^-. \end{cases}$$

By choosing $\delta > 0$ sufficiently small, the Lipschitz constant of η can be rendered as small as we like. In particular, we can assume

$$\|\dot{\eta}\|_{\mathbf{L}^\infty} \leq 2^{-\nu}.$$

The new function

$$z(t) \doteq \zeta(t) + \eta(t), \quad t \in [-\kappa, \kappa],$$

has Lipschitz constant $\text{Lip}(z) \leq 3\varepsilon + 2^{-\nu}$. Moreover, its graph does not intersect any of the components $\Gamma_1, \dots, \Gamma_\nu$. Recalling (5.25), the length of the graph can be bounded as

$$\begin{aligned} \ell &= \int_{-\kappa}^{\kappa} \sqrt{1 + (\dot{\zeta}(t) + \dot{\eta}(t))^2} dt \leq \int_{-\kappa}^{\kappa} \sqrt{1 + \dot{\zeta}^2(t)} dt + \int_{-\kappa}^{\kappa} (\dot{\zeta}(t) + \dot{\eta}(t)) \dot{\eta}(t) dt \\ &\leq 2\kappa + 9\varepsilon m_1(\Gamma) + 2\kappa(3\varepsilon + 2^{-\nu})2^{-\nu} \leq 2\kappa + 9\varepsilon m_1(\Gamma) + 8\kappa 2^{-\nu}. \end{aligned} \quad (5.29)$$

3. Next, consider the path $s \mapsto \gamma(s)$, $s \in [0, \ell]$, obtained by parameterizing the graph of z by arc-length. This is a 1-Lipschitz path that connects P with Q , without touching any of the connected components $\Gamma_1, \dots, \Gamma_\nu$. However, it may well cross many of the remaining components Γ_i , for $i > \nu$.

To cope with this issue, we now use Lemma 2.2 choosing $\varepsilon = 2^{-\nu}$, and obtain a new path

$$\tilde{\gamma} : [0, \tilde{\ell}] \mapsto \mathbb{R}^2,$$

such that

$$\begin{aligned} |\tilde{\gamma}(0) - P| &\leq 2^{-\nu}, & |\tilde{\gamma}(\tilde{\ell}) - Q| &\leq 2^{-\nu}, \\ \tilde{\gamma}(s) &\notin \Gamma & \text{for all } s &\in [0, \tilde{\ell}]. \end{aligned}$$

The length of this new path is bounded by

$$\tilde{\ell} \leq \ell + \sum_{i>\nu} m_1(\Gamma_i) \leq 2\kappa + 9\varepsilon m_1(\Gamma) + 8\kappa 2^{-\nu} + \sum_{i>\nu} m_1(\Gamma_i).$$

4. By the previous steps, for every $\nu \geq 1$ there exists a 1-Lipschitz path

$$\gamma_\nu : [0, \ell_\nu] \mapsto \mathbb{R}^2 \setminus \Gamma$$

such that

$$|\tilde{\gamma}_\nu(0) - P| \leq 2^{-\nu}, \quad |\tilde{\gamma}_\nu(\tilde{\ell}) - Q| \leq 2^{-\nu}.$$

Moreover, its length satisfies

$$\ell_\nu \leq 2\kappa + 9\varepsilon m_1(\Gamma) + 8\kappa 2^{-\nu} + \sum_{i>\nu} m_1(\Gamma_i).$$

By Ascoli's compactness theorem, taking a subsequence we achieve the convergence $\gamma_\nu \rightarrow \gamma$, where $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ is a 1-Lipschitz path joining P with Q , with length

$$\ell \leq 2\kappa + 9\varepsilon m_1(\Gamma).$$

By construction, this is an admissible path, satisfying the conclusion of the lemma. \square

Remark 5.6. By the above construction, it follows that each path γ_ν differs by an amount $\mathcal{O}(1) \cdot \sum_{i>\nu} m_1(\Gamma_i)$ from the graph of a continuous function with Lipschitz constant $3\varepsilon + 2^{-\nu}$. Taking the limit, we thus obtain an admissible path $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ which is the graph of a Lipschitz function $x = z(t)$, $t \in [-\kappa, \kappa]$ with Lipschitz constant 3ε .

Remark 5.7. For simplicity, in the statements of Lemmas 5.1 and 5.3 we assumed a bound on the intersection of Γ with the vertical strip $[0, T] \times \mathbb{R}$. Looking at the proofs, it is clear that we only needed a bound on the intersection of Γ with the cone $\{(t, x) ; t \in [0, T], |x| \leq 3\varepsilon t\}$. In particular, the conclusion of Lemma 5.3 remains valid if (5.11) is replaced by

$$m_1\left(\Gamma \cap \{(t, x) ; t \in [0, T], |x| \leq 3\varepsilon t\}\right) \leq \frac{\varepsilon}{2}t, \quad \text{for all } t \in [0, T]. \quad (5.30)$$

The same remark applies to Lemma 5.5. Namely, all steps in the proof remain valid if, for $-\kappa < t < \kappa$, the assumption (5.17) is replaced by

$$\begin{aligned} m_1\left(\Gamma \cap \{(t, x) ; |x| \leq 4\varepsilon(t + \kappa)\}\right) &< \frac{\varepsilon}{3}(t + \kappa), \\ m_1\left(\Gamma \cap \{(t, x) ; |x| \leq 4\varepsilon(\kappa - t)\}\right) &< \frac{\varepsilon}{3}(\kappa - t). \end{aligned} \quad (5.31)$$

Thanks to the previous remarks, from Lemma 5.5 we deduce

Corollary 5.8. *Given $\theta_0, \varepsilon_0 > 0$, there exists $\varepsilon > 0$ small enough so that the following holds. Consider a triangle Δ_0 with vertices*

$$P = (-\kappa, 0), \quad Q = (\kappa, 0), \quad Z = (0, \theta_0\kappa).$$

Let $\Gamma \subset \mathbb{R}^2$ be a barrier such that, for every $r > 0$,

$$m_1\left(\Gamma \cap \Delta_0 \cap B(P, r)\right) \leq \varepsilon r, \quad m_1\left(\Gamma \cap \Delta_0 \cap B(Q, r)\right) \leq \varepsilon r. \quad (5.32)$$

Then there exists a path $\xi : [0, \ell] \mapsto \Delta_0$, joining P with Q without crossing the barrier Γ , with length bounded by

$$\ell \leq |P - Q| + \varepsilon_0 m_1(\Gamma \cap \Delta_0). \quad (5.33)$$

6 Proof of Theorem 1.5

Let Γ be an optimal barrier for the optimization problem **(OP)**, and let $\Omega \subset \mathbb{R}^2$ be any open set. We need to prove that the closure $\overline{\Gamma}$ does not contain all of Ω .

Without loss of generality, we can assume $\Omega \subset \overline{R_\infty^\Gamma}$. Otherwise, we can remove all barriers contained in the set $\Omega \setminus \overline{R_\infty^\Gamma}$, i.e., all portions of the wall which are never touched by the fire, and get a strictly smaller barrier. This yields a blocking strategy with a strictly lower cost.

As shown in Fig. 1, right, the proof will be achieved by constructing a quadrilateral domain $\Delta \subset \Omega$ with the following properties:

- (i) The lower boundary γ_0 is the portion of a level set $\{x \in \mathbb{R}^2 ; T^\Gamma(x) = t_0\}$, between the two points A and B .

(ii) For a suitable $h > 0$, the upper boundary γ^* is the portion of the curve

$$\{x \in \mathbb{R}^2; d(x, \gamma_0) = h\} \quad (6.1)$$

between the points C and D .

(iii) The two sides AC and BD are segments which do not cross Γ , and are part of optimal trajectories for the fire. Their lengths satisfy

$$|C - A| = |D - B| = d(C, \gamma_0) = d(D, \gamma_0) = h. \quad (6.2)$$

(iv) The total amount of barriers contained in Δ is small. Namely, for some $\varepsilon > 0$ suitably small, one has

$$m_1(\Gamma \cap \Delta) \leq \varepsilon, \quad (6.3)$$

where σ is the construction speed. Moreover, for every $s > 0$ one has

$$\begin{aligned} m_1(\{y \in \Gamma \cap \Delta; d(y, \gamma_0) < s\}) &\leq 6\varepsilon s, \\ m_1(\{y \in \Gamma \cap \Delta; d(y, \gamma^*) < s\}) &\leq 12\varepsilon s. \end{aligned} \quad (6.4)$$

The first part of the proof, based on Lemma 4.1, works out a construction of the “flow box” Δ . In the second part of the proof, using Lemma 5.5, we show that the reduced barrier

$$\Gamma^\diamond = \Gamma \setminus \Delta \quad (6.5)$$

is still admissible, and yields a strictly lower cost. We split the argument in several steps.

1. Let $\varepsilon > 0$ be given. Since the minimum time function T^Γ is in SBV, it is differentiable at a.e. point \bar{x} . Moreover, the limit

$$\lim_{r \rightarrow 0^+} \frac{m_1(B(\bar{x}, r) \cap \Gamma)}{r^2} = 0 \quad (6.6)$$

also holds at a.e. point $\bar{x} \in \Omega$. We thus choose such a point \bar{x} , and consider a system of coordinates with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_2 = \nabla T^\Gamma(\bar{x})$. Call $\bar{t} = T^\Gamma(\bar{x})$.

We now perform an affine transformation of time and space coordinates, so that (\bar{t}, \bar{x}) becomes the new origin of coordinates:

$$\Lambda(t', s_1, s_2) = (\bar{t} + rt', \bar{x} + s_1 r \mathbf{e}_1 + s_2 r \mathbf{e}_2). \quad (6.7)$$

Since we are only interested in the local behavior of optimal trajectories for the fire in a neighborhood of \bar{x} , we consider a new problem where the initial open set burned by the fire is

$$R_0 \doteq \text{int}(\overline{R^\Gamma(\bar{t})}).$$

Working in the (t', s_1, s_2) coordinates, after renaming the variables and choosing a rescaling factor $r > 0$ sufficiently small, we are led to study the following situation.

- The total length of all barriers contained in the square $Q_2 = [-2, 2] \times [-2, 2]$ satisfies

$$m_1(\Gamma \cap Q_2) \leq \varepsilon. \quad (6.8)$$

- The initial set R_0 satisfies

$$\{(x_1, x_2) \in Q_2; x_2 < -\varepsilon|x_1|\} \subseteq R_0 \cap Q_2 \subseteq \{(x_1, x_2) \in Q_2; x_2 < \varepsilon|x_1|\}. \quad (6.9)$$

2. Call

$$\Gamma_{1/3} \doteq \Gamma \cap Q_2 \cap \overline{R^\Gamma(1/3)} \quad (6.10)$$

the portion of the barrier contained in the square Q_2 and touched by the fire within time $t = 1/3$. By (6.8) we trivially have $m_1(\Gamma_{1/3}) \leq \varepsilon$.

Our next goal is to apply Lemma 4.1 in this particular situation. As in (4.1), let R_1 be the neighborhood of radius 1 around R_0 . For $x \in R_1$, define $\rho(x)$ as in (4.2), with Γ replaced by $\Gamma_{1/3}$. By (4.4) and (4.5) it now follows

$$\begin{aligned} \int_{R_1} \rho(x) dx &\geq \int_{R_1} d(x, R_0) dx - \frac{\widehat{T}^2 + \widehat{T}}{2} \cdot m_1(\Gamma_{1/3}) \\ &\geq \int_{R_1} d(x, R_0) dx - \frac{(1+\varepsilon)^2 + (1+\varepsilon)}{2} \varepsilon \geq \int_{R_1} d(x, R_0) dx - 2\varepsilon, \end{aligned} \quad (6.11)$$

provided that $\varepsilon > 0$ is small enough.

From (6.11) we wish to conclude that, within the square $Q_1 = [-1, 1] \times [-1, 1]$, most of the optimal trajectories for the fire contain long straight segments. Since $\Gamma_{1/3} \subset B(0, 3)$, by the triangle inequality we have

$$\rho(x) = d(x, R_0) \quad \text{for all } x \in R_1 \setminus B(0, 4). \quad (6.12)$$

By Lemma 2.2 it follows

$$\rho(x) \leq T^{\Gamma_{1/3}}(x) \leq d(x, R_0) + \varepsilon,$$

and hence

$$\int_{R_1 \cap (B(0,4) \setminus Q_1)} \rho(x) dx \leq \int_{R_1 \cap (B(0,4) \setminus Q_1)} d(x, R_0) dx + \varepsilon m_2(R_1 \cap (B(0,4) \setminus Q_1)). \quad (6.13)$$

From (6.11), using (6.12) and then (6.13) we deduce

$$\begin{aligned} \left(\int_{R_1 \cap Q_1} + \int_{R_1 \cap (B(0,4) \setminus Q_1)} \right) \rho(x) dx &\geq \left(\int_{R_1 \cap Q_1} + \int_{R_1 \cap (B(0,4) \setminus Q_1)} \right) d(x, R_0) dx - 2\varepsilon, \\ \int_{R_1 \cap Q_1} \rho(x) dx &\geq \int_{R_1 \cap Q_1} d(x, R_0) dx - \varepsilon m_2(B(0,4) \setminus Q_1) - 2\varepsilon \\ &\geq \int_{R_1 \cap Q_1} d(x, R_0) dx - (16\pi + 2)\varepsilon. \end{aligned} \quad (6.14)$$

3. As shown in Fig. 14, consider in Q_1 the four rectangles

$$\begin{aligned} \Omega_1 &= [-1, -1/2] \times [1, 1/6], & \Omega_2 &= [1/2, 1] \times [1, 1/6], \\ \Omega_3 &= [-1, -1/2] \times [5/6, 1], & \Omega_4 &= [1/2, 1] \times [5/6, 1]. \end{aligned}$$

Consider the lower side of Ω_3 . This is the horizontal segment U with endpoints $P = (-1, 5/6)$ and $P' = (-1/2, 5/6)$. By (6.14), if $\varepsilon > 0$ is small enough, there exists a 1-dimensional subset $\tilde{U} \subseteq U$ such that $\rho(x) > 3/4$ for all $x \in \tilde{U}$. By choosing $\varepsilon > 0$ small, we can make the size of \tilde{U} as close to $1/2$ as we like. Say,

$$m_1(\tilde{U}) > 1/3.$$

Given two distinct points $x, x' \in \tilde{U}$, let y, y' be the points where the optimal trajectories reaching x, x' cross the boundary $\partial R^\Gamma(1/3) = \{z \in \mathbb{R}^2; T^\Gamma(z) = 1/3\}$. Call S, S' the segments with endpoints x, y and x', y' , respectively. Since these optimal trajectories are straight lines and cannot cross each other within their last segment of length $3/4$, we can find a constant $\lambda > 0$ independent of ε such that

$$B(S, \lambda|x - x'|) \cap B(S', \lambda|x - x'|) = \emptyset. \quad (6.15)$$

In other words, optimal trajectories reaching distinct points $x \in \tilde{U}$ remain bounded away from each other. A measure-theoretic argument now implies that, if $m(\Gamma \cap Q_1) \leq \varepsilon$ with $\varepsilon > 0$ small enough, we can find at least one segment S with endpoints x, y as above, which does not intersect Γ .

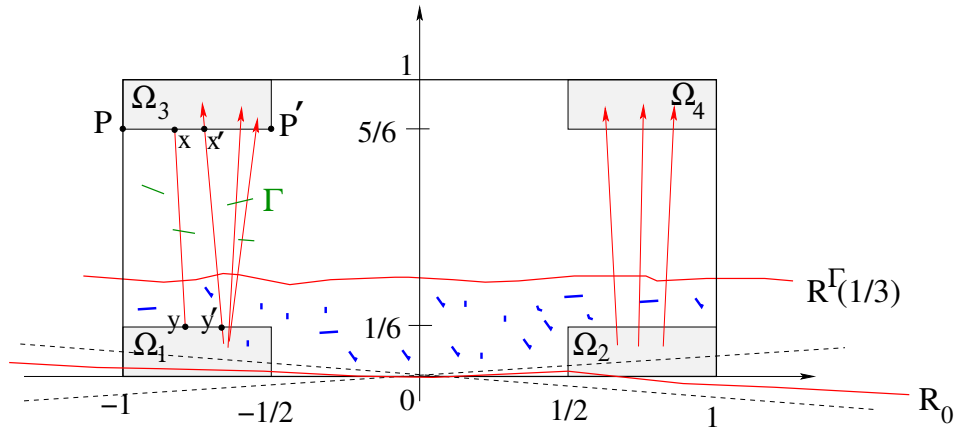


Figure 14: Since the total length of all barriers is $\mathcal{O}(1) \cdot \varepsilon$, by Lemma 4.1 there exists many optimal trajectories that contain long straight segments, with one endpoint in Ω_1 and the other in Ω_3 . Since optimal trajectories do not cross each other, a barrier $\Gamma \subset [-1, 1] \times [1/4, 5/6]$, whose total length is sufficiently small, cannot cross all of these segments.

4. We are now ready to construct the quadrilateral domain Δ satisfying the conditions (i)–(iv), as shown in Fig. 15.

By the previous step, we can find four points

$$A' \in \Omega_1, \quad B' \in \Omega_2, \quad C' \in \Omega_3, \quad D' \in \Omega_4,$$

with the following properties:

- (i) When the barrier is taken to be $\Gamma_{1/3} \doteq \Gamma \cap \overline{R^\Gamma(1/3)}$, the segment $A'C'$ is the last portion of a trajectory reaching C' in minimum time. Similarly, the segment $B'D'$ is the last portion of a trajectory reaching D' in minimum time.

(ii) The segments $A'C'$ and $B'D'$ do not cross Γ .

As a consequence, for any $\tau \in [1/3, 3/4]$, the segments $A'C'$ and $B'D'$ are still part of an optimal trajectory for the fire, in case the barrier Γ is replaced by $\Gamma_\tau \doteq \Gamma \cap \overline{R^\Gamma(\tau)}$.

At this stage, it would be tempting to choose Δ as the quadrilateral having $\overline{\partial R^\Gamma(1/3)}$ as lower boundary, the segments $A'C'$ and $B'D'$ as sides, and the curve

$$\gamma^* = \{x \in \mathbb{R}^2; d(x, \overline{R^\Gamma(1/3)}) = 1/3\}$$

as upper boundary. However, with this choice there is no guarantee that the bounds (6.4) will be satisfied.

To cope with this difficulty, the lower boundary will be chosen to be $\gamma_0 = \overline{\partial R^\Gamma(t_0)}$, for some $t_0 \in [1/3, 1/2]$, while the upper boundary γ^* will be the set of points having distance h from the lower boundary, for some $h \in [1/4, 1/3]$. The values of t_0, h must be carefully chosen, in order to satisfy (6.4).

Consider the nondecreasing function

$$\varphi(t) = m_1(\Gamma \cap Q_2 \cap \overline{R^\Gamma(t)} \setminus R_0).$$

By (6.8),

$$\varphi(0) \geq 0, \quad \varphi(1) \leq \varepsilon.$$

Using Riesz' sunrise lemma (see for example [16], p.319) we can find $t_0 \in [1/3, 1/2]$ such that

$$\varphi(t_0 + s) - \varphi(t_0) \leq 6\varepsilon s \leq \frac{s}{2} \quad \text{for all } s \in [t_0, 1]. \quad (6.16)$$

As in Lemma 2.4, call $[a_i, b_i]$ the intervals during which the fire front touches the components Γ_i . By (2.30) we have

$$\begin{aligned} \overline{R^\Gamma(t_0 + s)} &\supseteq B\left(R^\Gamma(t_0), m_1\left([t_0, t_0 + s] \setminus \bigcup_{i \geq 1} [a_i, b_i]\right)\right) \\ &\supseteq B(R^\Gamma(t_0), s - \varphi(t_0 + s) + \varphi(t_0)) \supseteq B(R^\Gamma(t_0), s/2). \end{aligned}$$

In turn, by (6.16) this yields

$$\begin{aligned} m_1\left(\{y \in \Gamma \cap \Delta; d(y, \gamma_0) < s\}\right) &\leq m_1\left(\Gamma \cap Q_1 \cap \overline{R^\Gamma(t_0 + 2s)} \setminus \overline{R^\Gamma(t_0)}\right) \\ &\leq \varphi(t_0 + 2s) - \varphi(t_0) \leq 12\varepsilon s. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough, this yields the first inequality in (6.4).

In a similar way, we now choose h so that the second inequality in (6.4) is satisfied as well. Consider the nondecreasing function

$$\psi(t) \doteq m_1\left(\Gamma \cap Q_1 \cap \{x; d(x, \overline{R^\Gamma(t_0)}) \leq t - t_0\}\right)$$

By (6.8),

$$\psi(t_0) \geq 0, \quad \psi(1) \leq \varepsilon.$$

Using Riesz' sunrise lemma we can find $h \in [1/4, 1/3]$ such that

$$\psi(t_0 + h) - \psi(t) \leq 12\varepsilon(t_0 + h - t) \quad \text{for all } t \in [t_0, t_0 + h]. \quad (6.17)$$

Define the set

$$\gamma^* \doteq \{x \in Q_1; d(x, \overline{R^\Gamma(t_0)}) = h\}.$$

By (6.17) it now follows

$$\begin{aligned} & m_1\left(\Gamma \cap Q_1 \cap \{x; d(x, \gamma^*) \leq s\} \cap \{x; d(x, \overline{R^\Gamma(t_0)}) \leq h\}\right) \\ & \leq m_1\left(\Gamma \cap Q_1 \cap \{x; d(x, \overline{R^\Gamma(t_0)}) \in [h-s, h]\}\right) \\ & \leq \psi(t_0+h) - \psi(t_0+h-s) \leq 12\varepsilon s. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we thus obtain the second inequality in (6.4).

As shown in Fig. 15, the quadrilateral domain Δ is now defined to be the set of all points $x \in Q_1$ such that

$$0 < d(x, \overline{R^\Gamma(t_0)}) < h,$$

bounded between the two segments $A'C'$ and $B'D'$.

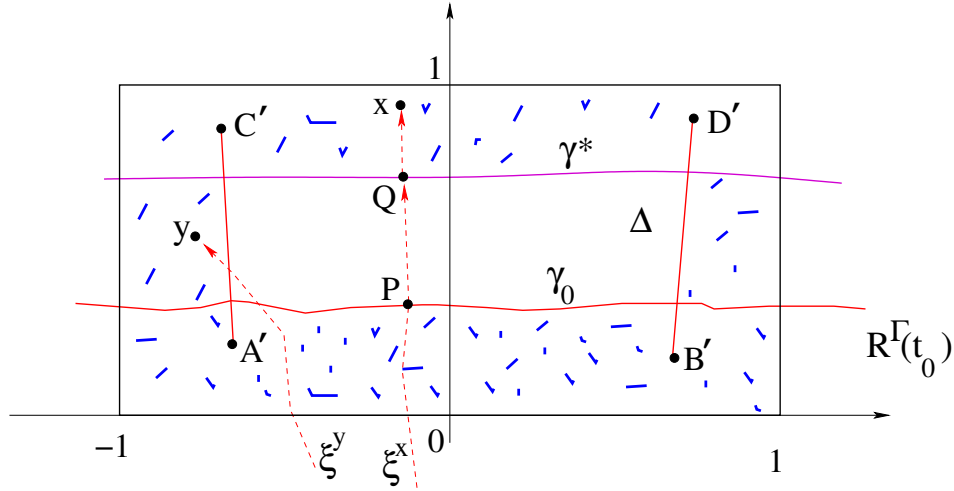


Figure 15: If the domain Δ satisfies all properties (i)–(iv), optimal trajectories for the fire cannot exit from Δ through the lateral boundaries $A'C'$ or $B'D'$. In particular, ξ^y is not optimal.

5. Having constructed the domain Δ , we now define the reduced barrier Γ^\diamond as in (6.5), by removing all portions inside Δ . Using the fact that Γ is admissible, we will show that Γ^\diamond is admissible as well. By (1.5), this means

$$m_1(\Gamma^\diamond \cap \overline{R^{\Gamma^\diamond}(t)}) \leq \sigma t \quad \text{for all } t \geq 0. \quad (6.18)$$

For $t \leq t_0$ we trivially have

$$m_1(\Gamma^\diamond \cap \overline{R^{\Gamma^\diamond}(t)}) = m_1(\Gamma \cap \overline{R^\Gamma(t)}) \leq \sigma t.$$

For $t_0 < t < t_0 + h$, we claim that

$$m_1(\Gamma^\diamond \cap \overline{R^{\Gamma^\diamond}(t)}) \leq m_1(\Gamma \cap \overline{R^\Gamma(t)}) \leq \sigma t. \quad (6.19)$$

To prove (6.19) we show that, for every $y \notin \Delta$, one has the implication

$$T^\Gamma(y) < t_0 + h \quad \implies \quad T^{\Gamma^\diamond}(y) = T^\Gamma(y). \quad (6.20)$$

Indeed, let $t \mapsto \xi^y(t)$ be an optimal trajectory for the fire, reaching y in minimum time without crossing the barrier Γ^\diamond . If $T^{\Gamma^\diamond}(y) < T^\Gamma(y)$, then ξ^y must cross some barrier contained in $\Gamma \cap \Delta$. As shown in Fig. 15, this trajectory must be partly inside Δ , then exit through one of the sides, either $A'C'$ or $B'D'$. But this is impossible, because our construction implies that both of these segments are part of optimal trajectories for the fire, and two optimal trajectories cannot cross each other. For $t < t_0 + h$, the inequality (6.19) is an immediate consequence of (6.20).

To achieve a bound valid for $t \geq t_0 + h$, we claim that

$$\sup_{x \in \gamma^*} T^\Gamma(x) \leq t_0 + h + \frac{1}{2\sigma} m_1(\Gamma \cap \Delta). \quad (6.21)$$

Indeed, consider any point $Q \in \gamma^*$, and let $P \in \gamma$ be a point such that $d(P, Q) = d(P, \gamma) = h$. Using Corollary 5.8, if $\varepsilon > 0$ was chosen sufficiently small, we can find a path $\xi : [0, \ell] \mapsto \Delta$, joining P with Q without crossing the original barrier Γ , whose length satisfies

$$\ell \leq h + \frac{1}{2\sigma} m_1(\Gamma \cap \Delta).$$

This yields (6.21). In turn, for every $x \in \overline{R_\infty^\Gamma}$ with $T^\Gamma(x) \geq t_0 + h$, the inequality (6.21) implies

$$T^\Gamma(x) \leq T^{\Gamma^\diamond}(x) + \frac{1}{2\sigma} m_1(\Gamma \cap \Delta). \quad (6.22)$$

Therefore

$$\overline{R^{\Gamma^\diamond}(t)} \subseteq \overline{R^\Gamma\left(t + \frac{1}{2\sigma} m_1(\Gamma \cap \Delta)\right)}.$$

For any $t \geq t_0 + h$, the admissibility of Γ now implies

$$\begin{aligned} m_1\left(\Gamma^\diamond \cap \overline{R^{\Gamma^\diamond}(t)}\right) &\leq m_1\left(\Gamma \cap \overline{R^\Gamma\left(t + \frac{1}{2\sigma} m_1(\Gamma \cap \Delta)\right)}\right) - m_1(\Gamma \cap \Delta) \\ &\leq \sigma \cdot \left(t + \frac{1}{2\sigma} m_1(\Gamma \cap \Delta)\right) - m_1(\Gamma \cap \Delta) \leq \sigma t - \frac{1}{2} m_1(\Gamma \cap \Delta), \end{aligned} \quad (6.23)$$

showing that the reduced barrier Γ^\diamond is admissible as well.

6. Since $\overline{R_\infty^{\Gamma^\diamond}} = \overline{R_\infty^\Gamma}$, but $m_1(\Gamma^\diamond) < m_1(\Gamma)$, if $c_0 > 0$ we immediately conclude that the total cost of the strategy Γ^\diamond is strictly smaller:

$$m_2(R_\infty^{\Gamma^\diamond}) + c_0 m_1(\Gamma^\diamond) < m_2(R_\infty^\Gamma) + c_0 m_1(\Gamma).$$

This contradicts the optimality of Γ .

In the case $c_0 = 0$ we observe that, by (6.23), having removed all barriers contained inside Δ , we are left with a little extra time: $(2\sigma)^{-1} m_1(\Gamma \cap \Delta)$. We can use this time to construct a circumference that forever shields a small disc from the fire. More precisely, let D_0 be an open disc with radius r_0 , so that the length of its boundary $\Gamma_0 = \partial D_0$ satisfies

$$m_1(\Gamma_0) = 2\pi r_0 \leq \frac{1}{2} m_1(\Gamma \cap \Delta).$$

We choose $D_0 \subset \overline{R_\infty^\Gamma}$ so that

$$\overline{D_0} \cap \overline{R^{\Gamma^\diamond}(t_0 + h)} = \emptyset.$$

In this way, the barrier $\Gamma^* = \Gamma^\diamond \cup \Gamma_0$ is still admissible. The corresponding burned set satisfies

$$\overline{R_\infty^{\Gamma^*}} \subseteq \overline{R_\infty^\Gamma} \setminus D_0,$$

which has a strictly smaller area. Again, this yields a contradiction with the optimality of Γ , proving the theorem. \square

References

- [1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press, 2000.
- [2] J. P. Aubin and A. Cellina, *Differential inclusions. Set-Valued Maps and Viability Theory*. Springer-Verlag, Berlin, 1984.
- [3] A. Bressan, Differential inclusions and the control of forest fires, *J. Differential Equations* (special volume in honor of A. Cellina and J. Yorke), **243** (2007), 179–207.
- [4] A. Bressan, Dynamic blocking problems for a model of fire propagation. In *Advances in Applied Mathematics, Modeling, and Computational Science*, pp. 11–40. R. Melnik and I. Kotsireas editors. Fields Institute Communications, Springer, New York, 2013.
- [5] A. Bressan, M. Burago, A. Friend, and J. Jou, Blocking strategies for a fire control problem, *Analysis and Applications* **6** (2008), 229–246.
- [6] A. Bressan and C. De Lellis, Existence of optimal strategies for a fire confinement problem, *Comm. Pure Appl. Math.* **62** (2009), 789–830.
- [7] A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control*, AIMS Series in Applied Mathematics, Springfield Mo. 2007.
- [8] A. Bressan and T. Wang, Equivalent formulation and numerical analysis of a fire confinement problem, *ESAIM; Control Optim. Calc. Var.* **16** (2010), 974–1001.
- [9] A. Bressan and T. Wang, The minimum speed for a blocking problem on the half plane, *J. Math. Anal. Appl.*, **356** (2009), 133–144.
- [10] A. Bressan and T. Wang, Global necessary conditions for a dynamic blocking problem, *ESAIM; Control Optim. Calc. Var.* **18** (2012), 124–156.
- [11] A. Bressan and T. Wang, On the optimal strategy for an isotropic blocking problem. *Calculus of Variations & PDE* **45** (2012), 125–145.
- [12] L. Cesari, *Optimization - Theory and Applications*, Springer-Verlag, New York, 1983.
- [13] C. De Lellis and R. Robyr, Hamilton-Jacobi equations with obstacles, *Arch. Rational Mech. Anal.* **200** (2011), 1051–1073.
- [14] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1996.

- [15] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.
- [16] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*. Dover, 1975.
- [17] A. L. Sullivan, Wildland surface fire spread modeling, 1990-2007. *Internat. J. Wildland Fire*, **18** (2009), 349–403.
- [18] T. Wang, Optimality conditions for a blocking strategy involving delaying arcs, *J. Optim. Theory Appl.* **152** (2012), 307–333.