THE ANTIFERROMAGNETIC XY MODEL ON THE TRIANGULAR LATTICE: TOPOLOGICAL SINGULARITIES

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Abstract. We study the discrete-to-continuum variational limit of the antiferromagnetic XY model on the two-dimensional triangular lattice in the vortex regime. Within this regime, the spin system cannot overcome the energetic barrier of chirality transitions, hence one of the two chirality phases is prevalent. We find the order parameter that describes the vortex structure of the spin field in the majority chirality phase and we compute explicitly the Γ-limit of the scaled energy, showing that it concentrates on finitely many vortex-like singularities of the spin field.

Contents

1. Introduction

Antiferromagnetic spin systems are magnetic lattice systems in which the exchange interaction between two spins favors anti-alignment. Such systems are said to be geometrically frustrated if, due to the geometry of the lattice, no spin configuration can simultaneously minimize all pairwise interactions. As a consequence of that, ground states of frustrated spin systems may exhibit non-trivial patterns and give rise to unconventional magnetic order, whose understanding has occupied the Statistical Physics and Condensed Matter communities in the last decades [26, 37, 38].

In this paper we are interested in the antiferromagnetic XY spin system on the triangular lattice (AFXY), a system that has attracted the attention of a large scientific community because of its relevance in understanding phase transition properties of frustrated spin models as those governing the physics of Josephson junctions, helimagnets and discotic liquid crystals (see for instance [36] and references therein). Our present contribution is undertaken within the framework of "discrete-to-continuum variational analysis" by means of Γ-convergence (*cf.* [29, 13]). It aims at the first mathematically rigorous derivation of the coarse grained energy of the AFXY system as the lattice spacing vanishes and the energy scaling allows the formation of finitely many spin vortices. This is a further step towards a complete understanding of the AFXY model, whose variational analysis has been initiated in [9] at a different scaling, which leads to interfacial-type energies, as we

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recall below. It is worth mentioning that interfacial energies often result from different frustration mechanisms in the variational analysis of spin systems, *e.g.*, those induced by the competition of ferromagnetic (favoring alignment) and antiferromagnetic interactions [2, 23, 14, 44, 19, 24].

The AFXY is a 2-dimensional nearest-neighbors antiferromagnetic planar spin model on the triangular lattice, *cf.* [26, Chapter 1]. We let $\varepsilon > 0$ be a small parameter and we consider the triangular lattice $\mathcal{L}_\varepsilon$ with spacing ε (see below for the precise definition). To every spin field $u\colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ we associate the energy

$$\sum_{\substack{\varepsilon\sigma,\varepsilon\sigma'\in\mathcal{L}_\varepsilon\\ |\sigma-\sigma'|=1}} u(\varepsilon\sigma)\cdot u(\varepsilon\sigma')\,, \tag{1.1}$$

where $\cdot$ denotes the scalar product. This model is antiferromagnetic since the interaction energy between two neighboring spins is minimized by two opposite vectors. The geometry of the triangular lattice, though, frustrates the system. In fact, already for a single triangular plaquette of the lattice no spin configuration minimizes the energy of all the three interacting pairs, since such a configuration should be made of three pairwise opposite vectors. In order to find the ground states of the system, one can rearrange the indices of the sum in (1.1) to have

$$\sum_T \big(u(\varepsilon i)\cdot u(\varepsilon j) + u(\varepsilon j)\cdot u(\varepsilon k) + u(\varepsilon k)\cdot u(\varepsilon i)\big) = \frac{1}{2}\sum_T \big(|u(\varepsilon i)+u(\varepsilon j)+u(\varepsilon k)|^2 - 3\big)\,, \tag{1.2}$$

where the sum is now running over all triangular plaquettes T with vertices $\varepsilon i, \varepsilon j, \varepsilon k \in \mathcal{L}_\varepsilon$. The formula above shows that in each triangle T the energy is minimized (and is equal to $-\frac{3}{2}$) only when $u(\varepsilon i)+u(\varepsilon j)+u(\varepsilon k) = 0$, *i.e.*, when the vectors of a triple $(u(\varepsilon i), u(\varepsilon j), u(\varepsilon k))$ point at the vertices of an equilateral triangle. The set of all the ground states is then obtained from this configuration thanks to the symmetries of the system, namely the $\mathbb{S}^1$ and the $\mathbb{Z}_2$ symmetry. By the $\mathbb{S}^1$-symmetry, every rotation of a minimizing triple is minimizing, too. By the $\mathbb{Z}_2$-symmetry, triples obtained from a minimizing triple $(u(\varepsilon i), u(\varepsilon j), u(\varepsilon k))$ via a permutation of negative sign as $(u(\varepsilon i), u(\varepsilon k), u(\varepsilon j))$ are also minimizing. The symmetry analysis above shows the existence of two families of ground states that can be distinguished through the *chirality*, a scalar quantity (invariant under rotations) which quantifies the handedness of a certain spin structure. To define the chirality of a spin field u in a triangle T, we need to fix an ordering of its vertices εi, εj, εk. We write the triangular lattice $\mathcal{L}$ as $\mathcal{L} := \{z_1\hat e_1 + z_2\hat e_2 : z_1, z_2 \in \mathbb{Z}\}$ with $\hat e_1 = (1,0)$, and $\hat e_2 = \frac{1}{2}(1,\sqrt{3})$. We introduce also $\hat e_3 := \frac{1}{2}(-1,\sqrt{3})$ as a further unit vector connecting points of $\mathcal{L}$ and define three pairwise disjoint sublattices of $\mathcal{L}$, denoted by $\mathcal{L}^1$, $\mathcal{L}^2$, and $\mathcal{L}^3$, by

$$\mathcal{L}^1 := \{z_1(\hat e_1+\hat e_2) + z_2(\hat e_2+\hat e_3) : z_1, z_2 \in \mathbb{Z}\}\,,\quad \mathcal{L}^2 := \mathcal{L}^1 + \hat e_1\,,\quad \mathcal{L}^3 := \mathcal{L}^1 + \hat e_2\,.$$

We assume that $\varepsilon i \in \varepsilon\mathcal{L}^1$, $\varepsilon j \in \varepsilon\mathcal{L}^2$, $\varepsilon k \in \varepsilon\mathcal{L}^3$ and we set (see (2.1) for the precise definition)

$$\chi(u,T) = \frac{2}{3\sqrt{3}}\big(u(\varepsilon i)\times u(\varepsilon j) + u(\varepsilon j)\times u(\varepsilon k) + u(\varepsilon k)\times u(\varepsilon i)\big) \in [-1,1]\,,$$

where $\times$ is the cross product. We let $\chi(u) \in L^\infty(\mathbb{R}^2)$ denote the function equal to $\chi(u,T)$ on the interior of each triangular plaquette T. By [9, Remark 2.2], the ground states are characterized as those spin configurations u that satisfy either $\chi(u) \equiv 1$ or $\chi(u) \equiv -1$. In order to describe more precisely our framework, let us fix $\Omega \subset \mathbb{R}^2$ open, bounded, and connected (if not we work on each connected component) and let us consider the energy (1.2) restricted to Ω, *i.e.*, computed only on those plaquettes of $\mathcal{L}_\varepsilon$ contained in Ω. We refer the energy to its minimum by removing the energy of the ground states ($-\frac{3}{2}$ for each plaquette) and then divide by the number of lattice points in Ω, which is of order $1/\varepsilon^2$, to obtain the energy per particle. Up to a multiplicative constant the latter reads as

$$E_\varepsilon(u,\Omega) = \sum_{T\subset\Omega} \varepsilon^2 |u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2.$$

In [9] we analyze the energetic regime at which the two families of ground states coexist and the energy of the system concentrates at the interface between the two chiral phases $\{\chi = 1\}$ and $\{\chi = -1\}$. This happens assuming that, as $\varepsilon \to 0$, sequences of spin fields $u_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ can deviate from ground states under the constraint $E_\varepsilon(u_\varepsilon,\Omega) \leq C\varepsilon$. In this case, the chiralities $\chi(u_\varepsilon)$ converge strongly in $L^1(\Omega)$ to some $\chi \in BV(\Omega;\{-1,1\})$. As a result, the continuum limit of $\frac{1}{\varepsilon}E_\varepsilon$ is a function of a partition of Ω into sets of finite perimeter (the phases) where the chirality is either $+1$ or -1. More precisely, it Γ-converges to an anisotropic perimeter of the phase boundary. At this energy scaling, the asymptotic behavior of the AFXY model shares similarities with systems having finitely many phases, such as Ising systems [17, 2, 16] or Potts systems [20].

In this paper we are interested in a much lower energetic regime, which does not allow chirality phase transitions. We turn our attention to sequences of spin fields u_ε that satisfy the bound

$$E_\varepsilon(u_\varepsilon,\Omega) \leq C\varepsilon^2|\log\varepsilon|\,. \tag{1.3}$$

Since $\varepsilon^2|\log\varepsilon| \ll \varepsilon$, within this energy bound the spin system cannot overcome the energetic barrier of the chirality transition (of order ε), hence the chiralities $\chi(u_\varepsilon)$ converge strongly in L^1 to either $\chi \equiv 1$ or $\chi \equiv -1$, see Lemma 4.1. However, within a fixed chirality phase, there is enough energy for the spin field to create finitely many vortices whose complex structure is displayed in Figure 2. Within the framework of "discrete-to-continuum variational analysis", the emergence of vortices in spin systems has been first observed in the ferromagnetic XY model [4], a system which is driven by an energy with neighboring interactions $-u(\varepsilon\sigma)\cdot u(\varepsilon\sigma')$. The latter model has been thoroughly investigated both on the square lattice [39, 4, 5, 6, 15, 21, 22] and on the triangular lattice [18, 25]. Independently of the geometry of the lattice, it has been proved that spin fields that deviate from the ground states by an amount of energy of order $\varepsilon^2|\log\varepsilon|$ may form finitely many vortex-like singularities (topological charges as those arising in the Ginzburg-Landau model [11, 42, 40, 33, 35, 1, 41, 7]).

In order to describe the vortex structure in the AFXY spin system, we assume that the limit chirality is $\chi \equiv 1$. Then, to every spin field $u\colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ we associate the auxiliary field $v\colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ defined by

$$v(\varepsilon i) := u(\varepsilon i)\,, \quad v(\varepsilon j) := R[-\tfrac{2\pi}{3}](u(\varepsilon j))\,, \quad v(\varepsilon k) := R[\tfrac{2\pi}{3}](u(\varepsilon k))\,, \tag{1.4}$$

for $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, $\varepsilon k \in \mathcal{L}^3_\varepsilon$, where $R[\alpha](\cdot)$ denotes the counterclockwise rotation by α. The operation above transforms a ground state with chirality 1 into a set of three parallel vectors, see Figure 1.

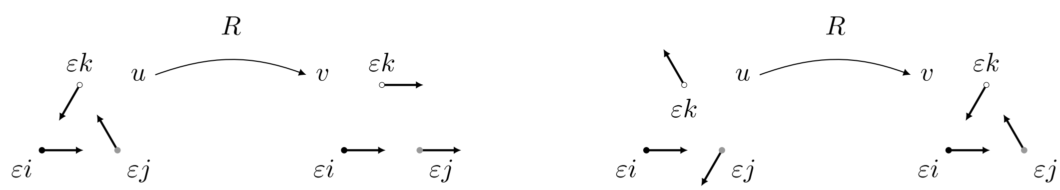

Figure 1. On the left: a ground state u with chirality 1 is transformed into an auxiliary spin field v given by parallel vectors. On the right: a ground state with chirality -1 is transformed in an auxiliary spin field v with a nonzero XY-energy.

The auxiliary variable v introduced above plays a fundamental role in identifying the vortex structure in the AFXY spin system. In the first instance, this is suggested by the asymptotic behavior of E_ε at the bulk scaling, *i.e.*, when assuming that sequences of spin fields u_ε deviate from ground states satisfying the stricter energy constraint $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2$. Under these assumptions we have, *a fortiori*, that the chiralities $\chi(u_\varepsilon)$ converge strongly in L^1 to either $\chi \equiv 1$ or $\chi \equiv -1$. We work in the former case and we associate the auxiliary spin field v_ε to every u_ε as in (1.4). In Theorem 4.2 we prove that, under the previous assumptions, the piecewise affine interpolations of v_ε converge strongly in L^2 to a limit map $v \in H^1(\Omega; \mathbb{S}^1)$ and

$$\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \to \sqrt{3} \int_\Omega |\nabla v|^2 \,\mathrm{d}x\,, \quad \frac{1}{\varepsilon^2} XY_\varepsilon(v_\varepsilon, \Omega) \to \sqrt{3} \int_\Omega |\nabla v|^2 \,\mathrm{d}x\,, \tag{1.5}$$

in the sense of Γ-convergence, where $XY_\varepsilon(v, T) = \frac{1}{2}\varepsilon^2(|v(\varepsilon i) - v(\varepsilon j)|^2 + |v(\varepsilon j) - v(\varepsilon k)|^2 + |v(\varepsilon k) - v(\varepsilon i)|^2)$ is the XY-energy of v in a triangle T. The proof relies on the relation (*cf.* Lemma 2.8 for the precise statement)

$$E_\varepsilon(u, T) \sim XY_\varepsilon(v, T) \quad \text{if} \quad \chi(u, T) \sim 1\,,$$

and the fact that, in the bulk scaling, the regions of Ω where the chirality is far from 1 concentrate around finitely many points, negligible for the limit energy. The asymptotic formula (1.5) and the known results for the XY-energy (see the discussion above) suggest that the limit of $\frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon$ might detect a vortex structure in u by means of the discrete vorticity measure μ_v of the auxiliary variable v (see (3.2) for the precise definition), as in Figure 2. However, the rigorous proof of the latter statement cannot result from a mere comparison between E_ε and XY_ε under assumption (1.3), as the next argument shows.

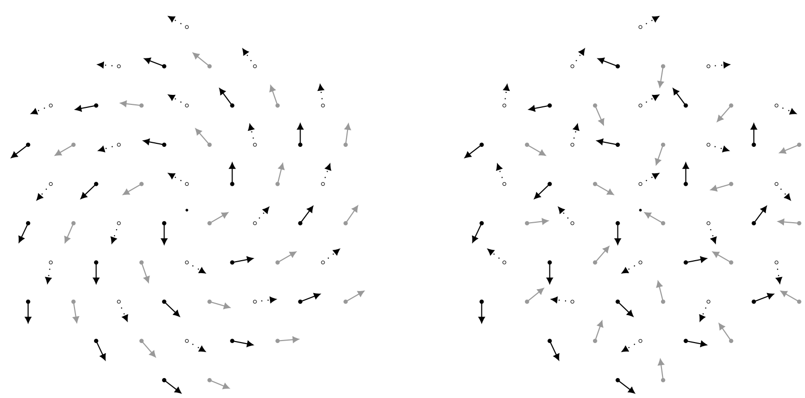

Figure 2. On the left: a vortex for the auxiliary spin field v. On the right: the corresponding spin field u.

In the literature concerning the variational analysis of the XY model [4, 5, 27, 25, 18, 10], the formation of finitely many vortex-like singularities in the limit as $\varepsilon \to 0$ of a sequence of spin fields v_ε is proven if $XY_\varepsilon(v_\varepsilon, \Omega) \leq C\varepsilon^2 |\log \varepsilon|$. However, such a bound does not follow from our working assumption $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log \varepsilon|$ in (1.3) for the corresponding spin field u_ε. This is due to the fact that an inequality $XY_\varepsilon(v, T) \leq CE_\varepsilon(u, T)$ does not hold true. Indeed, if u is a ground state with $\chi(u, T) = -1$, then $E_\varepsilon(u, T) = 0$, but $XY_\varepsilon(v, T) > 0$, see Figure 1. A fine estimate on the measure of the set $\{\chi(u_\varepsilon) \sim -1\}$ allows us in Lemma 6.1 to obtain the sharp bound

$$XY_\varepsilon(v_\varepsilon, \Omega) \leq C\varepsilon^2 |\log \varepsilon|^2. \tag{1.6}$$

This weaker bound for the XY-model is, in general, not sufficient for detecting finitely many vortex-like singularities in the limit as $\varepsilon \to 0$ and usually requires a different type of analysis [34, 41, 42, 5, 31], related to the possible diffusion of the scaled measures $\frac{\mu_{v_\varepsilon}}{|\log \varepsilon|}$. Nevertheless, due to the special structure of μ_{v_ε} in our setting, this phenomenon is ruled out and we are still able to prove that in the limit as $\varepsilon \to 0$ the vorticity measures concentrate on finitely many points (the convergence is made rigorous in the flat topology, see (3.8) for the precise definition). This is contained in the main theorem of the paper stated below.

Theorem 1.1. *Assume that Ω is an open, bounded, and connected set. The following results hold true:*

i) (Compactness) *Let $u_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ be such that $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2 |\log \varepsilon|$. Then, up to a subsequence, either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in $L^1(\Omega)$. Assume that $\chi(u_\varepsilon) \to 1$, let $v_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ be the auxiliary spin field defined as in* (1.4)*. Then there exists $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$ such that, up to a subsequence, $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ for all $\Omega' \subset\subset \Omega$.*

ii) ($\liminf$ inequality) *Let $u_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ be such that $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$, let $v_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ be the auxiliary spin field defined as in* (1.4)*. Let $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$, $x_h \in \Omega$ and assume that $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ for all $\Omega' \subset\subset \Omega$. Then*

$$2\sqrt{3}\pi|\mu|(\Omega) \leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) .$$

iii) ($\limsup$ inequality) *Let $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$. Then there exist $u_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ such that $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega} \to 0$ and*

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) \leq 2\sqrt{3}\pi|\mu|(\Omega) ,$$

where $v_\varepsilon \colon \mathcal{L}_\varepsilon \to \mathbb{S}^1$ is the auxiliary spin field defined as in (1.4).

We now illustrate the main ideas of the proof. To obtain *i)*, from (1.6) we first deduce that $|\mu_{v_\varepsilon}|(\Omega') \leq C|\log \varepsilon|^2$. However, we observe that only $|\log \varepsilon|$ many vortices can occur in the region where $\chi(u_\varepsilon) \sim 1$, which is consistent with the concentration of the energy on finitely many points. Instead, $|\log \varepsilon|^2$ many vortices only appear as ε-close dipoles in the region where $\chi(u_\varepsilon) \sim -1$ (see Figure 3). Those can be shown to be asymptotically irrelevant using a variant of the ball construction [33, 40].

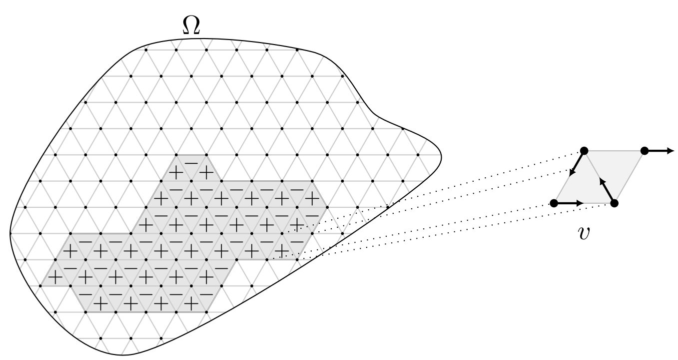

Figure 3. In the grey region the spin field u_ε has chirality -1. There, the auxiliary variable v_ε given by (1.4) may have $|\log \varepsilon|^2$ short dipoles, *e.g.*, as those depicted on the right.

The ball construction is also the main tool for proving the asymptotic lower bound *ii)*, which is the most demanding part of the proof. Here, the choice of a precise expansion rate in the ball construction (see also [27]) allows us to identify finitely many annuli, in which the energy concentrates. The radii of these annuli converge to zero at a much slower scale than ε, thus making possible to estimate the energy by exploiting the bulk scaling limit (1.5) (see also [3] for a similar argument in the context of homogenization). A crucial step in the proof consists in the modification of the spin field in a diverging number of balls, where short dipoles annihilate. It is worth pointing out that in the discrete setting this is an additional source of difficulties, which is solved by proving the extension Lemma 3.5.

2. Notation and preliminary results

Basic notation. We let $\mathrm{d}_{\mathbb{S}^1}(a,b)$ denote the geodesic distance on $\mathbb{S}^1$ between $a,b\in\mathbb{S}^1$ given by $\mathrm{d}_{\mathbb{S}^1}(a,b):=2\arcsin(\frac{1}{2}|a-b|)$. Note that

$$|a-b|\leq \mathrm{d}_{\mathbb{S}^1}(a,b)\leq \frac{\pi}{2}|a-b|\,.$$

For every $0<r<R$ and $x_0\in\mathbb{R}^2$, we define the annulus $A_{r,R}(x_0):=B_R(x_0)\setminus\overline{B}_r(x_0)$. In the case $x_0=0$ we write $A_{r,R}$.

We let $\mathcal{A}(\mathbb{R}^2)$ denote the collection of open sets of $\mathbb{R}^2$. The Lebesgue measure of a measurable set A will be denoted by $|A|$, while $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

Triangular lattice. Here we set the notation for the triangular lattice $\mathcal{L}$. It is given by

$$\mathcal{L}:=\{z_1\hat{e}_1+z_2\hat{e}_2 : z_1,z_2\in\mathbb{Z}\}\,,$$

with $\hat{e}_1=(1,0)$, and $\hat{e}_2=\frac{1}{2}(1,\sqrt{3})$. For later use, we find it convenient here to introduce $\hat{e}_3:=\frac{1}{2}(-1,\sqrt{3})$ and to define three pairwise disjoint sublattices of $\mathcal{L}$, denoted by $\mathcal{L}^1$, $\mathcal{L}^2$, and $\mathcal{L}^3$, by

$$\mathcal{L}^1:=\{z_1(\hat{e}_1+\hat{e}_2)+z_2(\hat{e}_2+\hat{e}_3) : z_1,z_2\in\mathbb{Z}\}\,,\quad \mathcal{L}^2:=\mathcal{L}^1+\hat{e}_1\,,\quad \mathcal{L}^3:=\mathcal{L}^1+\hat{e}_2\,.$$

Eventually, we define the family of triangles subordinated to the lattice $\mathcal{L}$ by setting

$$\mathcal{T}(\mathbb{R}^2):=\big\{T=\mathrm{conv}\{i,j,k\} : i,j,k\in\mathcal{L},\ |i-j|=|j-k|=|k-i|=1\big\}\,,$$

where $\mathrm{conv}\{i,j,k\}$ denotes the closed convex hull of i,j,k.

For $\varepsilon>0$, we consider rescaled versions of $\mathcal{L}$ and $\mathcal{T}(\mathbb{R}^2)$ given by $\mathcal{L}_\varepsilon:=\varepsilon\mathcal{L}$ and $\mathcal{T}_\varepsilon(\mathbb{R}^2):=\varepsilon\mathcal{T}(\mathbb{R}^2)$. With this notation every $T\in\mathcal{T}_\varepsilon(\mathbb{R}^2)$ has vertices $\varepsilon i,\varepsilon j,\varepsilon k\in\mathcal{L}_\varepsilon$. The same notation applies to the sublattices, namely $\mathcal{L}^\alpha_\varepsilon:=\varepsilon\mathcal{L}^\alpha$ for $\alpha\in\{1,2,3\}$. Given a set $A\subset\mathbb{R}^2$ we let $\mathcal{T}_\varepsilon(A):=\{T\in\mathcal{T}_\varepsilon(\mathbb{R}^2)\colon T\subset A\}$ denote the subfamily of triangles contained in A. Eventually, we introduce the set of admissible configurations as the set of all *spin fields*

$$\mathcal{SF}_\varepsilon:=\{u:\mathcal{L}_\varepsilon\to\mathbb{S}^1\}\,.$$

In the case $\varepsilon=1$ we set $\mathcal{SF}:=\mathcal{SF}_1$.

The antiferromagnetic XY model. For every $u\in\mathcal{SF}_\varepsilon$ and $T=\mathrm{conv}\{\varepsilon i,\varepsilon j,\varepsilon k\}\in\mathcal{T}_\varepsilon(\mathbb{R}^2)$ we set

$$E_\varepsilon(u,T):=\varepsilon^2|u(\varepsilon i)+u(\varepsilon j)+u(\varepsilon k)|^2,$$

and we extend the energy to any set $A\subset\mathbb{R}^2$ by setting

$$E_\varepsilon(u,A):=\sum_{T\in\mathcal{T}_\varepsilon(A)}E_\varepsilon(u,T)\,.$$

Chirality. Given $u \in \mathcal{SF}_\varepsilon$ and $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon, \varepsilon j \in \mathcal{L}^2_\varepsilon, \varepsilon k \in \mathcal{L}^3_\varepsilon$ we set

$$\chi(u,T) := \frac{2}{3\sqrt{3}}\big(u(\varepsilon i) \times u(\varepsilon j) + u(\varepsilon j) \times u(\varepsilon k) + u(\varepsilon k) \times u(\varepsilon i)\big)\,. \tag{2.1}$$

Moreover, we define $\chi(u)\colon \Omega \to \mathbb{R}$ almost everywhere by setting $\chi(u)(x) := \chi(u,T)$ if $x \in \mathrm{int}\, T$. Given $u \in \mathcal{SF}_\varepsilon$ it is convenient to rewrite $\chi(u,T)$ in terms of the angular lift of u. More precisely, let $\theta(\varepsilon i), \theta(\varepsilon j), \theta(\varepsilon k)$ be such that $u(x) = \exp(\iota\theta(x))$, $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$. Then

$$\chi(u,T) = \frac{2}{3\sqrt{3}}\big(\sin(\theta(\varepsilon j) - \theta(\varepsilon i)) + \sin(\theta(\varepsilon k) - \theta(\varepsilon j)) + \sin(\theta(\varepsilon i) - \theta(\varepsilon k))\big)\,.$$

Remark 2.1. Let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon, \varepsilon j \in \mathcal{L}^2_\varepsilon, \varepsilon k \in \mathcal{L}^3_\varepsilon$. Given $u \in \mathcal{SF}_\varepsilon$ one can show that if $E_\varepsilon(u,T) = 0$, then $\chi(u,T) \in \{-1,1\}$. Therefore, a continuity argument shows that for every $\eta \in (0,1)$ there exists $C_\eta > 0$ such that for every $u \in \mathcal{SF}_\varepsilon$ the following implication holds:

$$\chi(u,T) \in (-1+\eta, 1-\eta) \implies E_\varepsilon(u,T) \geq \varepsilon^2 C_\eta\,.$$

Given a triangle $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ we introduce the class $\mathcal{N}_\varepsilon(T)$ of its neighboring triangles, namely those triangles in $\mathcal{T}_\varepsilon(\mathbb{R}^2)$ that share a side with T. More precisely, we define

$$\mathcal{N}_\varepsilon(T) := \{T' \in \mathcal{T}_\varepsilon(\mathbb{R}^2) : \mathcal{H}^1(T \cap T') = \varepsilon\}\,.$$

Lemma 2.2. *Let $u \in \mathcal{SF}_\varepsilon$ and let $\eta \in (0,1]$. Let $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ and $T' \in \mathcal{N}_\varepsilon(T)$ and assume that $\chi(u,T) \leq 1-\eta$ and $\chi(u,T') \geq 1-\eta$. Then there exists a constant $C_\eta > 0$ such that $E_\varepsilon(u, T \cup T') \geq \varepsilon^2 C_\eta$.*

Proof. Without loss of generality let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ and $T' = \mathrm{conv}\{\varepsilon i', \varepsilon j, \varepsilon k\}$ with $\varepsilon i, \varepsilon i' \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, and $\varepsilon k \in \mathcal{L}^3_\varepsilon$. Let $u^\eta \in \mathcal{SF}_\varepsilon$ be such that

$$E_\varepsilon(u^\eta, T \cup T') = \min\big\{E_\varepsilon(u, T \cup T')\colon u \in \mathcal{SF}_\varepsilon\,,\ \chi(u,T) \leq 1-\eta\,,\ \chi(u,T') \geq 1-\eta\big\}\,.$$

By a scaling argument u^η is independent of ε. Moreover, since

$$\begin{aligned} E_\varepsilon(u^\eta, T \cup T') &= \varepsilon^2\big(|u^\eta(\varepsilon i) + u^\eta(\varepsilon j) + u^\eta(\varepsilon k)|^2 + |u^\eta(\varepsilon i') + u^\eta(\varepsilon j) + u^\eta(\varepsilon k)|^2\big) \\ &\geq \frac{\varepsilon^2}{2}|u^\eta(\varepsilon i) + u^\eta(\varepsilon j) + u^\eta(\varepsilon k) - (u^\eta(\varepsilon i') + u^\eta(\varepsilon j) + u^\eta(\varepsilon k))|^2 = \frac{\varepsilon^2}{2}|u^\eta(\varepsilon i) - u^\eta(\varepsilon i')|^2, \end{aligned}$$

we have that

$$E_\varepsilon(u^\eta, T \cup T') \geq \frac{\varepsilon^2}{2}\max\Big\{|u^\eta(\varepsilon i) - u^\eta(\varepsilon i')|^2, |u^\eta(\varepsilon i) + u^\eta(\varepsilon j) + u^\eta(\varepsilon k)|^2\Big\} =: \varepsilon^2 C_\eta\,.$$

Now either $u^\eta(\varepsilon i) \neq u^\eta(\varepsilon i')$ in which case it is clear that $C_\eta > 0$. On the other hand, if $u^\eta(\varepsilon i) = u^\eta(\varepsilon i')$, then $\chi(u^\eta, T) = \chi(u^\eta, T') = 1-\eta$. Then, due to Remark 2.1, there exists $C'_\eta > 0$ such that $\varepsilon^2 C_\eta \geq E_\varepsilon(u^\eta, T) \geq \varepsilon^2 C'_\eta$. Thus, $C_\eta > 0$, which concludes the proof. □

Remark 2.3. Let us consider the function

$$\chi(\theta_1, \theta_2) := \frac{2}{3\sqrt{3}}\big(\sin(\theta_1) + \sin(\theta_2 - \theta_1) - \sin(\theta_2)\big)\,.$$

For every $\eta' > 0$ there exists $\eta \in (0,1)$ such that, if $\theta_1, \theta_2 \in [-\pi, \pi]$ satisfy $\chi(\theta_1, \theta_2) > 1-\eta$, then $|\theta_1 - \frac{2\pi}{3}| < \eta'$ and $|\theta_2 + \frac{2\pi}{3}| < \eta'$. This follows from a continuity argument since the global maximum 1 is achieved only at $(\theta_1, \theta_2) = (\frac{2\pi}{3}, -\frac{2\pi}{3})$ in the square $[-\pi, \pi]^2$ (*cf.* [9, Lemma 2.1]). Analogously, if $\theta_1, \theta_2 \in [-\pi, \pi]$ satisfy $\chi(\theta_1, \theta_2) < -1+\eta$, then $|\theta_1 + \frac{2\pi}{3}| < \eta'$ and $|\theta_2 - \frac{2\pi}{3}| < \eta'$.

In the next lemma we count the number of triangles where the chirality $\chi(u_\varepsilon)$ is far from 1, assuming that $\chi(u_\varepsilon) \to 1$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, and connected and let and let $U \subset\subset \Omega$ with Lipschitz boundary. Let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$. Given $\eta \in (0,1)$ there exists $C_\eta > 0$, depending on U, such that for ε small enough*

$$\#\{T \in \mathcal{T}_\varepsilon(U) : \chi(u_\varepsilon, T) \leq 1 - \eta\} \leq C_\eta \Big(\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega)\Big)^2 .$$

Proof. Without restriction we assume that also U is connected. Let us consider the set

$$N_\varepsilon^\eta := \bigcup \{T \in \mathcal{T}_\varepsilon(\Omega) : \chi(u_\varepsilon, T) \leq 1 - \eta\} .$$

Note that, since $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$,

$$|N_\varepsilon^\eta \cap U| \leq |\{|1 - \chi(u_\varepsilon)| \geq \eta\} \cap U| \to 0 .$$

Thus, by the relative isoperimetric inequality, there exists $C > 0$ depending on U such that

$$|N_\varepsilon^\eta \cap U| = \min\{|U \setminus N_\varepsilon^\eta|, |N_\varepsilon^\eta \cap U|\} \leq C(\mathcal{H}^1(\partial N_\varepsilon^\eta \cap U))^2 \tag{2.2}$$

for ε small enough. We define the collection of triangles

$$\mathcal{T}_\varepsilon^{1-\eta} := \{T \in \mathcal{T}_\varepsilon(\Omega) : \chi(u_\varepsilon, T) \leq 1 - \eta \text{ and } \chi(u_\varepsilon, T') > 1 - \eta \text{ for some } T' \in \mathcal{N}_\varepsilon(T) \cap \mathcal{T}_\varepsilon(\Omega)\}$$

and we remark that $\partial N_\varepsilon^\eta \cap U \subset \partial\big(\bigcup_{T \in \mathcal{T}_\varepsilon^{1-\eta}} T\big)$. From the previous inclusion it follows that

$$\mathcal{H}^1(\partial N_\varepsilon^\eta \cap U) \leq 3\varepsilon \# \mathcal{T}_\varepsilon^{1-\eta}. \tag{2.3}$$

By Lemma 2.2, we obtain that

$$C_\eta \varepsilon^2 \# \mathcal{T}_\varepsilon^{1-\eta} \leq \sum_{T \in \mathcal{T}_\varepsilon^{1-\eta}} \sum_{T' \in \mathcal{N}_\varepsilon(T) \cap \mathcal{T}_\varepsilon(\Omega)} E_\varepsilon(u_\varepsilon, T \cup T') \leq 3 E_\varepsilon(u_\varepsilon, \Omega) .$$

Then (2.2) and (2.3) imply that

$$\frac{\sqrt{3}}{4} \varepsilon^2 \#\{T \in \mathcal{T}_\varepsilon(U) : \chi(u_\varepsilon, T) \leq 1 - \eta\} \leq |N_\varepsilon^\eta \cap U| \leq C(\mathcal{H}^1(\partial N_\varepsilon^\eta \cap U))^2 \leq \frac{C_\eta}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega)^2 .$$

This concludes the proof. □

The ferromagnetic XY model. In this subsection we fix the notation for the ferromagnetic XY model and we recall some properties that relate it to the Ginzburg-Landau functional. For every $v \in \mathcal{SF}_\varepsilon$ and $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ we set

$$XY_\varepsilon(v, T) := \frac{1}{2} \varepsilon^2 \big(|v(\varepsilon i) - v(\varepsilon j)|^2 + |v(\varepsilon j) - v(\varepsilon k)|^2 + |v(\varepsilon k) - v(\varepsilon i)|^2\big)$$

and for any set $A \subset \mathbb{R}^2$

$$XY_\varepsilon(v, A) := \sum_{T \in \mathcal{T}_\varepsilon(A)} XY_\varepsilon(v, T) .$$

Remark 2.5. Given $v \in \mathcal{SF}_\varepsilon$, we let $\hat{v} \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote its piecewise affine interpolation determined by the following conditions: for every $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ the map $\hat{v}$ is affine in T and $\hat{v}(\varepsilon i) = v(\varepsilon i)$, $\hat{v}(\varepsilon j) = v(\varepsilon j)$, $\hat{v}(\varepsilon k) = v(\varepsilon k)$. Then

$$\begin{aligned} XY_\varepsilon(v, T) &= \frac{1}{2} \varepsilon^4 \big(|\nabla \hat{v}\, \hat{e}_1|^2 + |\nabla \hat{v}\, \hat{e}_2|^2 + |\nabla \hat{v}\, \hat{e}_3|^2\big) \\ &= \frac{1}{2} \varepsilon^4 \big(|\partial_1 \hat{v}|^2 + |\tfrac{1}{2} \partial_1 \hat{v} + \tfrac{\sqrt{3}}{2} \partial_2 \hat{v}|^2 + |-\tfrac{1}{2} \partial_1 \hat{v} + \tfrac{\sqrt{3}}{2} \partial_2 \hat{v}|^2\big) \\ &= \frac{3}{4} \varepsilon^4 |\nabla \hat{v}|^2 = \sqrt{3} \varepsilon^2 \int_T |\nabla \hat{v}|^2 \,\mathrm{d}x . \end{aligned}$$

We recall the following key lemma proven in [4, Lemma 2] in the case of the XY-energy on the square lattice. The same proof can be repeated for the XY-energy on the triangular lattice.

Lemma 2.6. *Let $v \in \mathcal{SF}_\varepsilon$ and let $\hat{v}$ be its piecewise affine interpolation. Let $\Omega' \subset\subset \Omega$. Then*

$$\int_{\Omega'} (1 - |\hat{v}|^2)^2 \,\mathrm{d}x \leq C XY_\varepsilon(v, \Omega)\,.$$

Auxiliary spin field. We introduce here an auxiliary variable suited to describe the vortex structure in the AFXY model. Given a vector $u = \exp(\iota\phi) \in \mathbb{S}^1$ with $\phi \in \mathbb{R}$ and an angle $\theta \in \mathbb{R}$, we set

$$R[\theta](u) := \exp\big(\iota(\phi + \theta)\big)\,. \tag{2.4}$$

Let $u \in \mathcal{SF}_\varepsilon$ and let $v \in \mathcal{SF}_\varepsilon$ be defined by

$$v(\varepsilon i) := u(\varepsilon i)\,, \quad v(\varepsilon j) := R[-\tfrac{2\pi}{3}](u(\varepsilon j))\,, \quad v(\varepsilon k) := R[\tfrac{2\pi}{3}](u(\varepsilon k))\,, \tag{2.5}$$

for $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, $\varepsilon k \in \mathcal{L}^3_\varepsilon$. Note that the operation above transforms a ground state with chirality 1 into a set of three parallel vectors.

We can relate the E_ε energy of u to the XY_ε energy of v in a triangle $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$. Letting $\theta(\varepsilon j) \in \mathbb{R}$ be an angle between $u(\varepsilon i)$ and $u(\varepsilon j)$ and letting $\theta(\varepsilon k) \in \mathbb{R}$ be an angle between $u(\varepsilon i)$ and $u(\varepsilon k)$ we get that

$$\begin{aligned}
E_\varepsilon(u, T) &= \varepsilon^2 |u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2 \\
&= \varepsilon^2\big(3 + 2\cos\big(\theta(\varepsilon j)\big) + 2\cos\big(\theta(\varepsilon k) - \theta(\varepsilon j)\big) + 2\cos\big(\theta(\varepsilon k)\big)\big) \\
&= \varepsilon^2\big(3 - \cos\big(\theta(\varepsilon j) - \tfrac{2\pi}{3}\big) - \cos\big(\theta(\varepsilon k) + \tfrac{2\pi}{3} - \theta(\varepsilon j) + \tfrac{2\pi}{3}\big) - \cos\big(\theta(\varepsilon k) + \tfrac{2\pi}{3}\big)\big) \\
&\quad - \sqrt{3}\varepsilon^2\big(\sin(\theta(\varepsilon j) - \tfrac{2\pi}{3}) + \sin(\theta(\varepsilon k) + \tfrac{2\pi}{3} - \theta(\varepsilon j) + \tfrac{2\pi}{3}) - \sin(\theta(\varepsilon k) + \tfrac{2\pi}{3})\big) \\
&= \varepsilon^2\big(3 - v(\varepsilon i)\cdot v(\varepsilon j) - v(\varepsilon j)\cdot v(\varepsilon k) - v(\varepsilon k)\cdot v(\varepsilon i)\big) \\
&\quad + \frac{3}{4}\varepsilon^2\big(2\cos(\theta(\varepsilon j)) + 2\cos(\theta(\varepsilon k) - \theta(\varepsilon j)) + 2\cos(\theta(\varepsilon k))\big) \\
&\quad + \frac{\sqrt{3}}{2}\varepsilon^2\big(\sin(\theta(\varepsilon j)) + \sin(\theta(\varepsilon k) - \theta(\varepsilon j)) + \sin(-\theta(\varepsilon k))\big) \\
&= XY_\varepsilon(v, T) + \frac{3}{4}E_\varepsilon(u, T) - \frac{9}{4}\varepsilon^2(1 - \chi(u, T))\,.
\end{aligned} \tag{2.6}$$

The previous inequality yields

$$E_\varepsilon(u, T) = 4XY_\varepsilon(v, T) - 9\varepsilon^2(1 - \chi(u, T)) \tag{2.7}$$

and, in particular,

$$E_\varepsilon(u, T) \leq 4XY_\varepsilon(v, T)\,. \tag{2.8}$$

Remark 2.7 (Lower bound). In general, a lower bound $E_\varepsilon(u, T) \geq cXY_\varepsilon(v, T)$ does not hold true. For instance, if u is a ground state with negative chirality in T, then $E_\varepsilon(u, T) = 0$ but $XY_\varepsilon(v, T) > 0$. Nonetheless, we show now that this kind of lower bound holds true if u has chirality close to 1 in T.

Lemma 2.8. *Let $u \in \mathcal{SF}_\varepsilon$ and let $v \in \mathcal{SF}_\varepsilon$ be the auxiliary spin field defined by* (2.5)*. Then for every $\lambda \in (0, 1)$ there exists $\eta \in (0, 1)$ such that*

$$(1 - \lambda)XY_\varepsilon(v, T) \leq E_\varepsilon(u, T) \leq (1 + \lambda)XY_\varepsilon(v, T) \tag{2.9}$$

for every $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$, $\varepsilon i \in \mathcal{L}^1_\varepsilon, \varepsilon j \in \mathcal{L}^2_\varepsilon, \varepsilon k \in \mathcal{L}^3_\varepsilon$, such that either $\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon i), v(\varepsilon j)) < \eta$ and $\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon i), v(\varepsilon k)) < \eta$ or $\chi(u, T) > 1 - \eta$.

Proof. Let us fix $\lambda \in (0,1)$ and let $\delta \in (0, \frac{1}{7})$ be such that

$$1-\lambda < \frac{1-7\delta}{1-\delta} \quad \text{and} \quad \frac{1+7\delta}{1+\delta} < 1+\lambda\,. \tag{2.10}$$

On the one hand, we consider the function

$$\chi(\theta_1,\theta_2) := \frac{2}{3\sqrt{3}}\big(\sin(\theta_1) + \sin(\theta_2-\theta_1) - \sin(\theta_2)\big)$$

so that, adopting the notation from the computations in (2.6), $\chi(u,T) = \chi(\theta(\varepsilon j), \theta(\varepsilon k))$. By Taylor expanding χ around the point $(\frac{2\pi}{3}, -\frac{2\pi}{3})$ we obtain that

$$1-\chi(\theta_1,\theta_2) = \frac{1}{3} A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} + \rho_1(\theta_1,\theta_2)\,, \quad A := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3\mathrm{D}^2\chi\big(\tfrac{2\pi}{3}, -\tfrac{2\pi}{3}\big)\,,$$

where $|\rho_1(\theta_1,\theta_2)| \leq C\big(|\theta_1 - \frac{2\pi}{3}|^3 + |\theta_2 + \frac{2\pi}{3}|^3\big)$, the constant C depending only on $\|\mathrm{D}^3\chi\|_{L^\infty}$. There exists $\delta_1 > 0$ (depending only on δ and $\|\mathrm{D}^3\chi\|_{L^\infty}$) such that, if $|\theta_1 - \frac{2\pi}{3}| < \delta_1$ and $|\theta_2 + \frac{2\pi}{3}| < \delta_1$, then

$$\begin{aligned} |\rho_1(\theta_1,\theta_2)| &\leq C\big(|\theta_1 - \tfrac{2\pi}{3}|^3 + |\theta_2 + \tfrac{2\pi}{3}|^3\big) \leq C\delta_1\big(|\theta_1 - \tfrac{2\pi}{3}|^2 + |\theta_2 + \tfrac{2\pi}{3}|^2\big) \\ &\leq \frac{\delta}{3}\big(|\theta_1 - \tfrac{2\pi}{3}|^2 + |\theta_2 + \tfrac{2\pi}{3}|^2\big) \leq \frac{\delta}{3} A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix}. \end{aligned}$$

This implies that

$$\frac{1-\delta}{3} A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \leq 1 - \chi(\theta_1,\theta_2) \leq \frac{1+\delta}{3} A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix}. \tag{2.11}$$

On the other hand, we consider the function

$$f(\theta_1,\theta_2) := 3 - \cos(\theta_1 - \tfrac{2\pi}{3}) - \cos(\theta_2 - \theta_1 + \tfrac{4\pi}{3}) - \cos(\theta_2 + \tfrac{2\pi}{3})\,,$$

so that $XY_\varepsilon(v,T) = \varepsilon^2 f(\theta(\varepsilon j), \theta(\varepsilon k))$. By Taylor expanding f around $(\frac{2\pi}{3}, -\frac{2\pi}{3})$ we obtain that

$$f(\theta_1,\theta_2) = A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} + \rho_2(\theta_1,\theta_2)\,,$$

where $|\rho_2(\theta_1,\theta_2)| \leq C\big(|\theta_1 - \frac{2\pi}{3}|^3 + |\theta_2 + \frac{2\pi}{3}|^3\big)$, the constant C depending only on $\|\mathrm{D}^3 f\|_{L^\infty}$. There exists $\delta_2 > 0$ (depending only on δ and $\|\mathrm{D}^3 f\|_{L^\infty}$) such that, if $|\theta_1 - \frac{2\pi}{3}| < \delta_2$ and $|\theta_2 + \frac{2\pi}{3}| < \delta_2$, then

$$\begin{aligned} |\rho_2(\theta_1,\theta_2)| &\leq C\big(|\theta_1 - \tfrac{2\pi}{3}|^3 + |\theta_2 + \tfrac{2\pi}{3}|^3\big) \leq C\delta_2\big(|\theta_1 - \tfrac{2\pi}{3}|^2 + |\theta_2 + \tfrac{2\pi}{3}|^2\big) \\ &\leq \delta\big(|\theta_1 - \tfrac{2\pi}{3}|^2 + |\theta_2 + \tfrac{2\pi}{3}|^2\big) \leq \delta A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix}. \end{aligned}$$

This implies that

$$(1-\delta) A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \leq f(\theta_1,\theta_2) \leq (1+\delta) A \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 - \frac{2\pi}{3} \\ \theta_2 + \frac{2\pi}{3} \end{pmatrix}. \tag{2.12}$$

Setting $\eta' := \min\{\delta_1, \delta_2\}$, by (2.11) and (2.12) we conclude that

$$3\frac{1-\delta}{1+\delta} f(\theta_1,\theta_2) \leq 9(1-\chi(\theta_1,\theta_2)) \leq 3\frac{1+\delta}{1-\delta} f(\theta_1,\theta_2)\,, \quad \text{for } |\theta_1 - \tfrac{2\pi}{3}| < \eta',\ |\theta_2 + \tfrac{2\pi}{3}| < \eta'\,. \tag{2.13}$$

This and (2.7) imply that (2.9) holds true if $\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon i), v(\varepsilon j)) < \eta'$ and $\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon i), v(\varepsilon k)) < \eta'$.

By Remark 2.3, there exists $\eta'' \in (0,1)$(depending on η') such that, if $\theta_1, \theta_2 \in [-\pi, \pi]$ satisfy $\chi(\theta_1,\theta_2) > 1 - \eta''$, then $|\theta_1 - \frac{2\pi}{3}| < \eta'$, $|\theta_2 + \frac{2\pi}{3}| < \eta'$. By (2.13) we conclude that

$$3\frac{1-\delta}{1+\delta} f(\theta_1,\theta_2) \leq 9(1-\chi(\theta_1,\theta_2)) \leq 3\frac{1+\delta}{1-\delta} f(\theta_1,\theta_2)\,, \quad \text{for } \chi(\theta_1,\theta_2) > 1 - \eta''\,.$$

This, together with (2.7) and (2.10), implies that (2.9) holds true if $\chi(u,T) > 1-\eta''$. Setting $\eta := \min\{\eta',\eta''\}$ concludes the proof. $\square$

3. Topological singularities

In this section we recall the definition of discrete vorticity and its relation with the Jacobian of maps in the continuum. In particular, in Remark 3.2 we introduce an interpolation of discrete spin fields that makes this relation clear.

Discrete vorticity. Let $v \in \mathcal{SF}_\varepsilon$ and let $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$. We define the *discrete vorticity* $\mu_v(T)$ of v in T as follows. Let $\varphi\colon \mathcal{L}_\varepsilon \to \mathbb{R}$ be any function such that $v(x) = \exp(\iota\varphi(x))$. We define the projection on $2\pi\mathbb{Z}$ by

$$P_{2\pi\mathbb{Z}}(t) := \operatorname{argmin}\{|t-s| : s \in 2\pi\mathbb{Z}\}\,,$$

choosing the argmin with minimal modulus when it is not unique. We consider the angle between the vectors $v(x)$ and $v(x')$ given by

$$\mathrm{d}^e\varphi(x,x') := \varphi(x') - \varphi(x) - P_{2\pi\mathbb{Z}}\big(\varphi(x')-\varphi(x)\big)\,. \tag{3.1}$$

We remark that $\mathrm{d}^e\varphi$ does not depend on the choice of φ. Moreover, $\mathrm{d}^e\varphi(x,x') = -\mathrm{d}^e\varphi(x',x)$. Let now (x_1,x_2,x_3) be the vertices of T in counterclockwise order. Then we set

$$\mu_v(T) := \frac{1}{2\pi}\big(\mathrm{d}^e\varphi(x_1,x_2) + \mathrm{d}^e\varphi(x_2,x_3) + \mathrm{d}^e\varphi(x_3,x_1)\big)\,. \tag{3.2}$$

Since $|\mathrm{d}^e\varphi(x,x')| \leq \pi$, we immediately deduce that $\mu_v(T) \in \{-1,0,1\}$. Finally, we define the measure

$$\mu_v := \sum_{T\in\mathcal{T}_\varepsilon(\mathbb{R}^2)} \mu_v(T)\delta_{b(T)}\,,$$

where $b(T) \in \mathbb{R}^2$ denotes the barycenter of the triangle T.

Remark 3.1. There exists a constant $C > 0$ such that for every $v \in \mathcal{SF}_\varepsilon$ and $\Omega' \subset\subset \Omega$ with $\operatorname{dist}(\Omega',\partial\Omega) > \varepsilon$

$$|\mu_v|(\Omega') \leq \frac{C}{\varepsilon^2} XY_\varepsilon(v,\Omega)\,.$$

Indeed, we start by observing that, by the definition of μ_v,

$$|\mu_v|(\Omega') \leq \sum_{T\in\mathcal{T}_\varepsilon(\Omega)} |\mu_v(T)|\,.$$

Thanks to the previous inequality, it is enough to prove that there exists a universal constant $C > 0$ such that $|\mu_v|(T) \leq \frac{C}{\varepsilon^2}XY_\varepsilon(v,T)$ for every $T \in \mathcal{T}_\varepsilon(\Omega)$. Given $T = \operatorname{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\Omega)$, two cases are possible: either $\mu_v(T) = 0$ or $\mu_v(T) \neq 0$. In the former case, we trivially have $|\mu_v|(T) \leq \frac{1}{\varepsilon^2}XY_\varepsilon(v,T)$. In the latter case, let $\varphi(x) \in \mathbb{R}$ be such that $v(x) = \exp(\iota\varphi(x))$ for $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$. Then one value between $|\mathrm{d}^e\varphi(\varepsilon i,\varepsilon j)|$, $|\mathrm{d}^e\varphi(\varepsilon j,\varepsilon k)|$, or $|\mathrm{d}^e\varphi(\varepsilon k,\varepsilon i)|$ is greater than or equal to $\frac{2\pi}{3}$. Since $|v(x)-v(x')|^2 \geq \frac{4}{\pi^2}|\mathrm{d}^e\varphi(x,x')|^2$, we conclude that $\frac{1}{\varepsilon^2}XY_\varepsilon(v,T) \geq \frac{8}{9}|\mu_v(T)|$.

Jacobians and degree. We recall here some definitions and basic results concerning topological singularities. Let $U \subset \mathbb{R}^2$ be an open set and let $v = (v_1,v_2) \in W^{1,1}(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$. We define the *pre-Jacobian* (also known as *current*) of v by

$$j(v) := \frac{1}{2}(v_1\nabla v_2 - v_2\nabla v_1)\,.$$

The *distributional Jacobian* of v is defined by

$$J(v) := \operatorname{curl}(j(v))\,,$$

in the sense of distributions, *i.e.*,

$$\langle J(v), \psi \rangle = -\int_U j(v) \cdot \nabla^\perp \psi \, \mathrm{d}x \quad \text{for every } \psi \in C_c^\infty(U) \, ,$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Note that $J(v)$ is also well-defined when $v \in H^1(U;\mathbb{R}^2)$, and, in that case, it coincides with the L^1 function $\det \nabla v$.

Given $v = (v_1, v_2) \in H^{\frac{1}{2}}(\partial B_\rho(x_0);\mathbb{S}^1)$, its *degree* is defined by

$$\deg(v, \partial B_\rho(x_0)) := \frac{1}{2\pi} \big(\langle \nabla_{\partial B_\rho(x_0)} v_2, v_1 \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} - \langle \nabla_{\partial B_\rho(x_0)} v_1, v_2 \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \big) \, , \tag{3.3}$$

where $\langle \cdot, \cdot \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}$ denotes the duality between $H^{-\frac{1}{2}}(\partial B_\rho(x_0);\mathbb{S}^1)$ and $H^{\frac{1}{2}}(\partial B_\rho(x_0);\mathbb{S}^1)$ and we let $\nabla_{\partial B_\rho(x_0)}$ denote the derivative on $\partial B_\rho(x_0)$ with respect to the unit speed parametrization of $\partial B_\rho(x_0)$. Note that, by definition, the map $v \in H^{\frac{1}{2}}(\partial B_\rho(x_0);\mathbb{S}^1) \mapsto \deg(v, \partial B_\rho(x_0))$ is continuous. We remark that

$$\deg(v, \partial B_\rho(x_0)) = \frac{1}{2\pi} \int_{\partial B_\rho(x_0)} v_1 \nabla_{\partial B_\rho(x_0)} v_2 - v_2 \nabla_{\partial B_\rho(x_0)} v_1 \, \mathrm{d}\mathcal{H}^1 \quad \text{if } v \in H^1(\partial B_\rho(x_0);\mathbb{S}^1)$$

(and thus v is continuous) and this notion coincides with the classical notion of degree.[1] Also when $v \in H^{\frac{1}{2}}(\partial B_\rho(x_0);\mathbb{S}^1)$ is discontinuous, the degree defined in (3.3) inherits from the continuous setting some characterizing properties. In particular, a result due to L. Boutet de Monvel & O. Gabber [12, Theorem A.3] ensures that[2] $\deg(v, \partial B_\rho(x_0)) \in \mathbb{Z}$.

A further fundamental property of the degree is the following. Let $v \in H^1(A_{r,R}(x_0);\mathbb{S}^1)$. By the trace theory, $v|_{\partial B_\rho(x_0)} \in H^{\frac{1}{2}}(\partial B_\rho(x_0);\mathbb{S}^1)$ for every $\rho \in [r, R]$. Then

$$\deg(v, \partial B_\rho(x_0)) = \deg(v, \partial B_{\rho'}(x_0)) \quad \text{for every } \rho, \rho' \in [r, R] \, . \tag{3.4}$$

This follows from the fact that $\deg(v, \partial B_\rho(x_0)) \in \mathbb{Z}$, by the continuity of the degree with respect to the $H^{\frac{1}{2}}$ norm, and by the continuity of the map

$$\rho \in [r, R] \mapsto v(x_0 + \rho \, \cdot)|_{\partial B_1} \in H^{\frac{1}{2}}(\partial B_1;\mathbb{S}^1) \, ,$$

which is a consequence of the trace theory for Sobolev functions.

We conclude this summary about the degree by recalling the following property. Let $v \in H^1(A_{r,R}(x_0);\mathbb{S}^1)$. By the theory of slicing of Sobolev functions (*cf.* [8, Proposition 3.105] with a change of coordinates), for a.e. $\rho \in (r, R)$ the restriction $v|_{\partial B_\rho(x_0)}$ belongs to $H^1(\partial B_\rho(x_0);\mathbb{S}^1)$ and $\nabla_{\partial B_\rho(x_0)}(v|_{\partial B_\rho(x_0)})(y) = \nabla v(y) \tau_{\partial B_\rho(x_0)}(y)$ for $\mathcal{H}^1$-a.e. $y \in \partial B_\rho(x_0)$, where $\tau_{\partial B_\rho(x_0)}(y)$ is the unit tangent vector to $\partial B_\rho(x_0)$ at y. Therefore

$$\deg(v, \partial B_\rho(x_0)) = \frac{1}{\pi} \int_{\partial B_\rho(x_0)} j(v)|_{\partial B_\rho(x_0)} \cdot \tau_{\partial B_\rho(x_0)} \, \mathrm{d}\mathcal{H}^1 \quad \text{for a.e. } \rho \in (r, R) \, , \tag{3.5}$$

which relates the degree to the pre-jacobian and, by Stokes' Theorem, to the distributional Jacobian.

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[1] One can see this by noticing that

$$\frac{1}{2\pi} \int_{\partial B_\rho} v_1 \nabla_{\partial B_\rho} v_2 - v_2 \nabla_{\partial B_\rho} v_1 \, \mathrm{d}\mathcal{H}^1 = \fint_{\partial B_\rho} v^* \omega_{\partial B_\rho} = \deg(v, \partial B_\rho) \fint_{\partial B_\rho} \omega_{\partial B_\rho} = \deg(v, \partial B_\rho) \, ,$$

where $v^* \omega_{\partial B_\rho}$ is the pull-back through v of the volume form $\omega_{\partial B_\rho}$ on ∂B_ρ and the second equality is due to the topological definition of degree.

[2] In [12, Theorem A.3] the degree formula is written in an alternative form, equivalent to (3.3), by interpreting v as a complex-valued function.

An interpolation of discrete spin fields. In the following remark we relate the discrete vorticity of a spin field with the Jacobian of a suitable interpolation.

Remark 3.2 ($\mathbb{S}^1$-interpolation). To every $v \in \mathcal{SF}_\varepsilon$ we associate a map $\widehat{v}\colon \mathbb{R}^2 \to \mathbb{S}^1$ with the following properties:

(1) $\widehat{v} = v$ on the lattice $\mathcal{L}_\varepsilon$;
(2) $\widehat{v} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \operatorname{supp}\mu_v; \mathbb{S}^1)$;
(3) $J(\widehat{v}) = \pi\mu_v$;
(4) $\varepsilon^2 \int_T |\nabla\widehat{v}|^2 \,\mathrm{d}x \le \frac{\pi^2}{4\sqrt{3}} XY_\varepsilon(v, T)$ for every T with $\mu_v(T) = 0$.

We define $\widehat{v}$ in every $T = \operatorname{conv}\{\varepsilon\ell_1, \varepsilon\ell_2, \varepsilon\ell_3\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $(\varepsilon\ell_1, \varepsilon\ell_2, \varepsilon\ell_3)$ ordered counterclockwise by distinguishing two cases: $\mu_v(T) = 0$ or $\mu_v(T) \in \{-1, 1\}$. For every vertex $x \in \{\varepsilon\ell_1, \varepsilon\ell_2, \varepsilon\ell_3\}$ let $\varphi(x) \in \mathbb{R}$ be such that $v(x) = \exp(\iota\varphi(x))$. We set

$$\phi(\varepsilon\ell_1) := \varphi(\varepsilon\ell_1)\,, \qquad \phi(\varepsilon\ell_2) := \phi(\varepsilon\ell_1) + \mathrm{d}^e\varphi(\varepsilon\ell_1, \varepsilon\ell_2)\,, \qquad \phi(\varepsilon\ell_3) := \phi(\varepsilon\ell_2) + \mathrm{d}^e\varphi(\varepsilon\ell_2, \varepsilon\ell_3)\,,$$

where $\mathrm{d}^e\varphi$ is defined in (3.1). In this way,

$$\phi(\varepsilon\ell_3) + \mathrm{d}^e\varphi(\varepsilon\ell_3, \varepsilon\ell_1) = \phi(\varepsilon\ell_1) + 2\pi\mu_v(T)\,. \tag{3.6}$$

If $\mu_v(T) = 0$, we let $\hat{\phi}$ be the function that is affine in T and satisfies $\hat{\phi}(x) = \phi(x)$ for every vertex $x \in \{\varepsilon\ell_1, \varepsilon\ell_2, \varepsilon\ell_3\}$. We set $\widehat{v}(x) := \exp(\iota\hat{\phi}(x))$ for every $x \in T$. Since $\mu_v(T) = 0$, (3.6) implies that $\phi(\varepsilon\ell_1) = \phi(\varepsilon\ell_3) + \mathrm{d}^e\varphi(\varepsilon\ell_3, \varepsilon\ell_1)$. Note that

$$\begin{aligned}
\sqrt{3}\int_T |\nabla\widehat{v}|^2 \,\mathrm{d}x = \sqrt{3}\int_T |\nabla\hat{\phi}|^2 \,\mathrm{d}x &= \frac{1}{2}\varepsilon^2\big(|\nabla\hat{\phi}\cdot\hat{e}_1|^2 + |\nabla\hat{\phi}\cdot\hat{e}_2|^2 + |\nabla\hat{\phi}\cdot\hat{e}_3|^2\big) \\
&= \frac{1}{2}\big(|\phi(\varepsilon\ell_2) - \phi(\varepsilon\ell_1)|^2 + |\phi(\varepsilon\ell_3) - \phi(\varepsilon\ell_2)|^2 + |\phi(\varepsilon\ell_1) - \phi(\varepsilon\ell_3)|^2\big) \\
&= \frac{1}{2}\big(|\mathrm{d}^e\varphi(\varepsilon\ell_1, \varepsilon\ell_2)|^2 + |\mathrm{d}^e\varphi(\varepsilon\ell_2, \varepsilon\ell_3)|^2 + |\mathrm{d}^e\varphi(\varepsilon\ell_3, \varepsilon\ell_1)|^2\big) \\
&= \frac{1}{2}\big(\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon\ell_1), v(\varepsilon\ell_2))^2 + \mathrm{d}_{\mathbb{S}^1}(v(\varepsilon\ell_2), v(\varepsilon\ell_3))^2 + \mathrm{d}_{\mathbb{S}^1}(v(\varepsilon\ell_3), v(\varepsilon\ell_1))^2\big) \\
&\le \frac{\pi^2}{8}\big(|v(\varepsilon\ell_2) - v(\varepsilon\ell_1)|^2 + |v(\varepsilon\ell_3) - v(\varepsilon\ell_2)|^2 + |v(\varepsilon\ell_1) - v(\varepsilon\ell_3)|^2\big) \\
&= \frac{\pi^2}{4\varepsilon^2} XY_\varepsilon(v, T)\,.
\end{aligned} \tag{3.7}$$

We remark that $J(\widehat{v}) \llcorner T = 0$ (using the area formula and noticing that the image of T through the smooth map $\widehat{v}$ is $\mathbb{S}^1$).

If $\mu_v(T) = z$ with $z \in \{-1, 1\}$ we define $\widehat{v}$ in a different way. Namely, on ∂T we define the function $\mathring{\phi}$ by

$$\mathring{\phi}(x) := \begin{cases} \phi(\varepsilon\ell_1) + s\mathrm{d}^e\varphi(\varepsilon\ell_1, \varepsilon\ell_2)\,, & \text{for } x = \varepsilon\ell_1 + s(\varepsilon\ell_2 - \varepsilon\ell_1)\,,\ s \in [0, 1)\,, \\ \phi(\varepsilon\ell_2) + s\mathrm{d}^e\varphi(\varepsilon\ell_2, \varepsilon\ell_3)\,, & \text{for } x = \varepsilon\ell_2 + s(\varepsilon\ell_3 - \varepsilon\ell_2)\,,\ s \in [0, 1)\,, \\ \phi(\varepsilon\ell_3) + s\mathrm{d}^e\varphi(\varepsilon\ell_3, \varepsilon\ell_1)\,, & \text{for } x = \varepsilon\ell_3 + s(\varepsilon\ell_1 - \varepsilon\ell_3)\,,\ s \in [0, 1)\,. \end{cases}$$

Let $b(T) \in T$ be the barycenter of T. We extend $\mathring{\phi}$ to $T \setminus \{b(T)\}$ making it 0-homogeneous with respect to $b(T)$. Notice that $\mathring{\phi}$ is continuous outside the segment $[\varepsilon\ell_1, b(T)]$, where in view of (3.6) it has a jump of $\phi(\varepsilon\ell_3) + \mathrm{d}^e\varphi(\varepsilon\ell_3, \varepsilon\ell_1) - \phi(\varepsilon\ell_1) = 2\pi z$. We define $\widehat{v}(x) := \exp(\iota\mathring{\phi}(x))$ for every $x \in T \setminus \{b(T)\}$, observing that $\widehat{v} \in W^{1,1}(T; \mathbb{S}^1)$ and that $\widehat{v} \in W^{1,\infty}_{\mathrm{loc}}(T \setminus \{b(T)\}; \mathbb{S}^1)$.

Then the Jacobian of $\widehat{v}$ is defined in the sense of distributions. In fact, $J(\widehat{v})$ is a measure and $J(\widehat{v}) \llcorner T = \pi\mu_v \llcorner T$.[3]

The map $\widehat{v}$ is well-defined. Indeed, it satisfies in any case the following property: if $\varphi\colon \mathcal{L}_\varepsilon \to \mathbb{R}$ is such that $v(x) = \exp(\iota\varphi(x))$ for $x \in \mathcal{L}_\varepsilon$, then $\widehat{v}(\varepsilon\ell_1 + s(\varepsilon\ell_2 - \varepsilon\ell_1)) = \exp(\iota\varphi(\varepsilon\ell_1))\exp(\iota s \mathrm{d}^e\varphi(\varepsilon\ell_1,\varepsilon\ell_2))$ for every $\varepsilon\ell_1, \varepsilon\ell_2 \in \mathcal{L}_\varepsilon$ with $|i-j| = 1$. The curve $s \in [0,1] \mapsto \exp(\iota\varphi(\varepsilon\ell_1))\exp(\iota s \mathrm{d}^e\varphi(\varepsilon\ell_1,\varepsilon\ell_2))$ parametrizes a geodesic arc in $\mathbb{S}^1$ that connects $v(\varepsilon\ell_1)$ to $v(\varepsilon\ell_2)$.

Flat convergence. We recall here the notion of convergence relevant for the discrete vorticity of spin fields and for Jacobian of maps. Given a distribution $T \in \mathcal{D}'(U)$, we define its *flat norm*[4] by

$$\|T\|_{\mathrm{flat},U} := \sup\{\langle T, \psi\rangle : \psi \in C^\infty_c(U)\,,\ \|\psi\|_{L^\infty(U)} \leq 1\,,\ \|\nabla\psi\|_{L^\infty(U)} \leq 1\} \tag{3.8}$$

If $\|T\|_{\mathrm{flat},U} < \infty$, then the duality $\langle T,\psi\rangle$ can be extended to Lipschitz functions with compact support $\psi \in C^{0,1}_c(U)$. If T_n is a sequence of distributions such that $\|T_n\|_{\mathrm{flat},U} \to 0$, then $\langle T_n, \psi\rangle \to 0$ for every $\psi \in C^{0,1}_c(U)$.

Lifting of discrete spin fields. In this subsection we discuss the conditions sufficient to define the lifting of a discrete spin field.

Lemma 3.3. *Let $x_0 \in \mathbb{R}^2$, let $v \in \mathcal{SF}_\varepsilon$, and let $\widehat{v}$ be defined as in Remark 3.2. Let $0 < r < R$ and assume that $\mu_v(B_r(x_0)) = 0$ and $|\mu_v|(T) = 0$ for every $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ such that $T \cap A_{r,R}(x_0) \neq 0$. Then there exists $\phi \in W^{1,\infty}(A_{r,R}(x_0))$ such that $\widehat{v}(x) = \exp(\iota\phi(x))$ for all $x \in A_{r,R}(x_0)$ and*

$$|\nabla\phi(x)| = |\nabla\widehat{v}(x)| \quad \textit{for a.e. } x \in A_{r,R}(x_0)\,. \tag{3.9}$$

Proof. We assume, without loss of generality, that $x_0 = 0$ (the arguments in the proof will never use the fact that $0 \in \mathcal{L}_\varepsilon$). From the fact that $|\mu_v|(T) = 0$ for every $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ such that $T \cap A_{r,R} \neq 0$ and by the definition of $\widehat{v}$ in Remark 3.2, we have that $\widehat{v}$ is a continuous function in $A_{r,R}$. We write the annulus $A_{r,R}$ as the union of the two simply connected sets $S^{0,\frac{3\pi}{2}}_{r,R}$ and $S^{\pi,\frac{5\pi}{2}}_{r,R}$ (see Figure 4), where

$$S^{\theta_1,\theta_2}_{r,R} := \{(\rho\cos\theta, \rho\sin\theta) : \rho \in (r,R)\,,\ \theta \in (\theta_1,\theta_2)\}\,.$$

By the simple connectedness of $S^{0,\frac{3\pi}{2}}_{r,R}$ and $S^{\pi,\frac{5\pi}{2}}_{r,R}$, there exist two continuous functions $\phi\colon S^{0,\frac{3\pi}{2}}_{r,R} \to \mathbb{R}$ and $\phi'\colon S^{\pi,\frac{5\pi}{2}}_{r,R} \to \mathbb{R}$ such that $\widehat{v}(x) = \exp(\iota\phi(x))$ for $x \in S^{0,\frac{3\pi}{2}}_{r,R}$ and $\widehat{v}(x) = \exp(\iota\phi'(x))$ for $x \in S^{\pi,\frac{5\pi}{2}}_{r,R}$. We shall prove that ϕ and ϕ' coincide (up to translating ϕ' of an integer multiple of 2π), so that a unique lifting is defined in the annulus $A_{r,R}$.

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[3] The proof of this fact is standard: one can consider for every $\rho \in (0,1)$ the scaled triangle $T^\rho := \rho(T - b(T)) + b(T)$ and define the "conical" approximation $\widehat{v}^\rho\colon T \to \mathbb{R}^2$ given by

$$\widehat{v}^\rho(x) := \begin{cases} \widehat{v}(x)\,, & \text{if } x \in T \setminus T^\rho, \\ \big(1 - \frac{\mathrm{dist}(x,\partial T^\rho)}{\mathrm{dist}(b(T),\partial T^\rho)}\big)\widehat{v}(x)\,, & \text{if } x \in T^\rho. \end{cases}$$

On the one hand, $j(\widehat{v}^\rho) \to j(\widehat{v})$ in $L^1(T;\mathbb{R}^2)$ and thus $J(\widehat{v}^\rho) \rightharpoonup J(\widehat{v})$ in the sense of distributions. On the other hand, $J(\widehat{v}^\rho) = 0$ in $T \setminus T^\rho$ and $\|J(\widehat{v}^\rho)\|_{L^1(T)} = \|J(\widehat{v}^\rho)\|_{L^1(T^\rho)} \leq C$, which implies that $J(\widehat{v}^\rho)$ converges weakly* to a multiple of the Dirac delta at $b(T)$. Moreover $J(\widehat{v}^\rho)(T) \to J(\widehat{v})(T)$ and

$$\begin{aligned}\int_T J(\widehat{v}^\rho)\,\mathrm{d}x &= \int_{T^\rho} J(\widehat{v}^\rho)\,\mathrm{d}x = \int_T J(\widehat{v}^1)\,\mathrm{d}x = \int_T \mathrm{curl}(j(\widehat{v}^1))\,\mathrm{d}x = \int_{\partial T} j(\widehat{v}^1)\cdot\tau\,\mathrm{d}\mathcal{H}^1 = \frac{1}{2}\int_{\partial T} \nabla\mathring{\phi}\cdot\tau\,\mathrm{d}\mathcal{H}^1 \\ &= \frac{1}{2}\Big(\int_{[\varepsilon\ell_1,\varepsilon\ell_2]} \frac{\mathrm{d}^e\varphi(\varepsilon\ell_1,\varepsilon\ell_2)}{\varepsilon}\,\mathrm{d}\mathcal{H}^1 + \int_{[\varepsilon\ell_2,\varepsilon\ell_3]} \frac{\mathrm{d}^e\varphi(\varepsilon\ell_2,\varepsilon\ell_3)}{\varepsilon}\,\mathrm{d}\mathcal{H}^1 + \int_{[\varepsilon\ell_3,\varepsilon\ell_1]} \frac{\mathrm{d}^e\varphi(\varepsilon\ell_3,\varepsilon\ell_1)}{\varepsilon}\,\mathrm{d}\mathcal{H}^1\Big) = \pi z\,.\end{aligned}$$

[4] The name comes from the theory of currents. Interpreting $T \in \mathcal{D}'(U)$ as a 0-current, its flat norm is given by $\mathbb{F}(T) := \inf\{\mathbb{M}(R) + \mathbb{M}(\partial S) : R + \partial S = T\}$, where $\mathbb{M}(\cdot)$ denotes the mass. Then, it holds true that $\mathbb{F}(\cdot) = \|\cdot\|_{\mathrm{flat},U}$ (see [30, 4.1.12]).

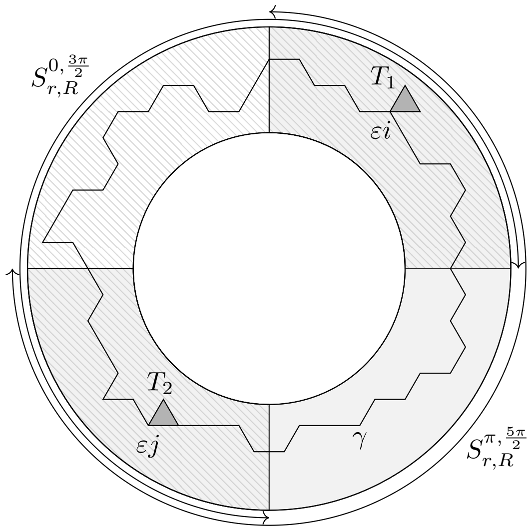

Figure 4. Example of a path γ used in the proof.

We observe that, since $\widehat{v} \in W^{1,\infty}(A_{r,R};\mathbb{S}^1)$, also $\phi \in W^{1,\infty}(S^{0,\frac{3\pi}{2}}_{r,R})$ and $\phi' \in W^{1,\infty}(S^{\pi,\frac{5\pi}{2}}_{r,R})$. Moreover, by the chain rule,

$$2j(\widehat{v})(x) = \nabla\phi(x) \quad \text{for } x \in S^{0,\frac{3\pi}{2}}_{r,R}\,, \quad 2j(\widehat{v})(x) = \nabla\phi'(x) \quad \text{for } x \in S^{\pi,\frac{5\pi}{2}}_{r,R}\,.$$

By the uniqueness of the lifting up to integer multiples of 2π, there exist $z_1 \in \mathbb{Z}$ and $z_2 \in \mathbb{Z}$ such that

$$\phi(x) = \phi'(x) + 2\pi z_1\,, \text{ for every } x \in S^{0,\frac{\pi}{2}}_{r,R}\,, \quad \phi(x) = \phi'(x) + 2\pi z_2\,, \text{ for every } x \in S^{\pi,\frac{3\pi}{2}}_{r,R}\,.$$

Let us prove that $z_1 = z_2$ by exploiting the assumption $\mu_v(B_r) = 0$. Let $T_1 \in \mathcal{T}_\varepsilon(S^{0,\frac{\pi}{2}}_{r,R})$ with a vertex εi and let $T_2 \in \mathcal{T}_\varepsilon(S^{\pi,\frac{3\pi}{2}}_{r,R})$ with a vertex εj. Let $\gamma\colon [0,1] \to A_{r,R}$ be a path such that $\mathrm{supp}(\gamma)$ is the union of edges in the triangular lattice, specifically, $\mathrm{supp}(\gamma) = \bigcup_{h=1}^M [\varepsilon\ell_{h-1}, \varepsilon\ell_h]$ with $|\ell_h - \ell_{h-1}| = 1$. Assume that $\varepsilon\ell_0 = \varepsilon\ell_M = \varepsilon i$, $\varepsilon\ell_N = \varepsilon j$ with $N < M$, $\bigcup_{h=1}^N [\varepsilon\ell_{h-1}, \varepsilon\ell_h] \subset S^{0,\frac{3\pi}{2}}_{r,R}$, and $\bigcup_{h=N+1}^M [\varepsilon\ell_{h-1}, \varepsilon\ell_h] \subset S^{\pi,\frac{5\pi}{2}}_{r,R}$. Moreover, assume that γ is the oriented boundary of an open set $U \subset B_R$ with $0 \in U$ (See Figure 4). Then, by Stokes' Theorem, by the definition of Jacobian, and by Remark 3.2 we infer that

$$\begin{aligned}
2\pi(z_2 - z_1) &= \phi'(\varepsilon i) - \phi(\varepsilon i) + 2\pi z_2 = \phi'(\varepsilon i) - \phi'(\varepsilon j) + \phi(\varepsilon j) - \phi(\varepsilon i)\\
&= \sum_{h=1}^N \phi'(\varepsilon\ell_h) - \phi'(\varepsilon\ell_{h-1}) + \sum_{h=N+1}^M \phi(\varepsilon\ell_h) - \phi(\varepsilon\ell_{h-1})\\
&= \sum_{h=1}^N \int_{[\varepsilon\ell_{h-1},\varepsilon\ell_h]} \nabla\phi'(x)\cdot\tau(x)\,\mathrm{d}\mathcal{H}^1(x) + \sum_{h=N+1}^M \int_{[\varepsilon\ell_{h-1},\varepsilon\ell_h]} \nabla\phi(x)\cdot\tau(x)\,\mathrm{d}\mathcal{H}^1(x)\\
&= 2\sum_{h=1}^M \int_{[\varepsilon\ell_{h-1},\varepsilon\ell_h]} j(\widehat{v})(x)\cdot\tau(x)\,\mathrm{d}\mathcal{H}^1(x) = 2\int_\gamma j(\widehat{v})\cdot\tau\,\mathrm{d}\mathcal{H}^1 = 2\int_U J(\widehat{v})\,\mathrm{d}x = 2\pi\mu(U)\,.
\end{aligned}$$

Since $\mu(U) = 0$, we conclude that $z_1 = z_2$. Therefore, we can extend ϕ to the whole annulus $A_{r,R}$ by setting $\phi(x) := \phi'(x) + 2\pi z_1$ for every $x \in S^{\pi,\frac{5\pi}{2}}_{r,R}$. It satisfies (3.9) by the chain rule. $\square$

Remark 3.4. We point out some properties of the lifting ϕ of $\widehat{v}$ provided by Lemma 3.3.

The first property is the following:

$$\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon i), v(\varepsilon j)) = |\phi(\varepsilon i) - \phi(\varepsilon j)| \,, \quad \text{for every } \varepsilon i, \varepsilon j \in \mathcal{L}_\varepsilon \cap T \text{ with } T \in \mathcal{T}_\varepsilon(A_{r,R}(x_0)) \,. \tag{3.10}$$

Indeed, on the one hand, we have that $\widehat{v}(\varepsilon i + s(\varepsilon j - \varepsilon i)) = \exp(\iota \phi(\varepsilon i + s(\varepsilon j - \varepsilon i)))$ for $s \in [0,1]$. On the other hand, since $\mu_v(T) = 0$, by Remark 3.2, the map $\widehat{v}$ is given by $\widehat{v}(\varepsilon i + s(\varepsilon j - \varepsilon i)) = \exp(\iota \phi(\varepsilon i)) \exp(\iota s \mathrm{d}^e \phi(\varepsilon i, \varepsilon j))$ for $s \in [0,1]$. Hence there exists $z \in \mathbb{Z}$ such that

$$\phi(\varepsilon i + s(\varepsilon j - \varepsilon i)) = \phi(\varepsilon i) + s \,\mathrm{d}^e \phi(\varepsilon i, \varepsilon j) + 2\pi z \,, \quad \text{for every } s \in [0,1] \,.$$

Evaluating the previous formula at $s = 0$, we infer that $z = 0$; evaluating it at $s = 1$, we obtain that $\phi(\varepsilon j) - \phi(\varepsilon i) = \mathrm{d}^e \phi(\varepsilon i, \varepsilon j) \in [-\pi, \pi]$. In particular, $\mathrm{d}_{\mathbb{S}^1}(v(\varepsilon i), v(\varepsilon j)) = |\phi(\varepsilon j) - \phi(\varepsilon i)|$, which concludes the proof of (3.10).

The second property is the following: let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(A_{r,R}(x_0))$. Then $\phi|_T$ is an affine function. Indeed, the previous property yields that $\phi(\varepsilon j) = \phi(\varepsilon i) + \mathrm{d}^e \phi(\varepsilon i, \varepsilon j)$, $\phi(\varepsilon k) = \phi(\varepsilon j) + \mathrm{d}^e \phi(\varepsilon j, \varepsilon k)$. Since $\mu_v(T) = 0$, by the definition of $\widehat{v}$ in Remark 3.2 we have that $\widehat{v}(x) = \exp(\hat{\phi}(x))$ for $x \in T$, where $\hat{\phi}$ is the affine function in T such that $\hat{\phi}(x) = \phi(x)$ for every vertex $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$. Then $\hat{\phi}(x) = \phi(x)$ for every $x \in T$. In particular, from (3.10) and (3.9), we deduce that

$$\frac{1}{\varepsilon^2} XY_\varepsilon(v, T) \leq \sqrt{3} \int_T |\nabla \phi|^2 \,\mathrm{d}x = \sqrt{3} \int_T |\nabla \widehat{v}|^2 \,\mathrm{d}x \leq \frac{\pi^2}{4\varepsilon^2} XY_\varepsilon(v, T) \,, \tag{3.11}$$

where the last inequality follows from Remark 3.2.

Extension of discrete spin fields. We prove now an extension lemma. It is the discrete version of a standard result in the continuum, which states the following: if $v \in H^1(A_{r,R}(x_0); \mathbb{S}^1)$ satisfies $\deg(v, \partial B_\rho(x_0)) = 0$, then it can be extended to a $\overline{v} \in H^1(B_R(x_0; \mathbb{S}^1))$ such that $\int_{B_R(x_0)} |\nabla \overline{v}|^2 \mathrm{d}x \leq C \int_{A_{r,R}(x_0)} |\nabla v|^2 \mathrm{d}x$. In the proof we exploit the interpolation introduced in Remark 3.2.

Lemma 3.5. *There exists a universal constant $C_0 > 0$ such that the following holds true. Let $\varepsilon > 0$, $x_0 \in \mathbb{R}^2$, and $R > r > \varepsilon$, let $C_1 > 1$ and $v_\varepsilon \in \mathcal{SF}_\varepsilon$ with $XY_\varepsilon(v_\varepsilon, A_{r,R}(x_0)) \leq C_1 \varepsilon^2$, $\mu_{v_\varepsilon}(B_r(x_0)) = 0$, and $|\mu_{v_\varepsilon}|(T) = 0$ for every $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ such that $T \cap A_{r,R}(x_0) \neq 0$. Then there exists $\overline{v}_\varepsilon \in \mathcal{SF}_\varepsilon$ such that for $\varepsilon < \frac{R-r}{C_0 C_1} \big(\frac{2\pi}{3}\big)^2$:*

- *$\overline{v}_\varepsilon = v_\varepsilon$ on $\mathcal{L}_\varepsilon \cap (\mathbb{R}^2 \setminus \overline{B}_{\frac{r+R}{2}}(x_0))$;*
- *$|\mu_{\overline{v}_\varepsilon}|(B_R(x_0)) = 0$;*
- *$XY_\varepsilon(\overline{v}_\varepsilon, B_R(x_0)) \leq C(r,R) XY_\varepsilon(v_\varepsilon, A_{r,R}(x_0))$, where $C(r,R) = C_0 \frac{R}{R-r}$.*

Remark 3.6. If there exists $\beta > 1$ such that $R = \beta r$, then the extension constant $C(r,R)$ given in Lemma 3.5 is independent of r. Indeed, $C(r,R) = C_0 \frac{\beta}{\beta - 1} =: C(\beta)$. Moreover, the extension $\overline{v}_\varepsilon$ satisfies the properties in the statement for $\frac{\varepsilon}{r} < \frac{\beta-1}{C_0 C_1} \big(\frac{2\pi}{3}\big)^2$.

The lemma is stated for ε fixed, thus the result can be applied also when $r = r_\varepsilon$ and $R = R_\varepsilon$.

The particular geometry of the triangular lattice does not play a major role in the proof: it is crucial that 3.11 holds true. For instance an analogous result holds true in the case of the square lattice.

Proof of Lemma 3.5. We assume, without loss of generality, that $x_0 = 0$ (the arguments in the proof will never use the fact that $0 \in \mathcal{L}_\varepsilon$) and $2\varepsilon \leq \frac{R-r}{36}$ (Note that this is *a fortiori* satisfied if C_0 is chosen sufficiently large in $\varepsilon < \frac{R-r}{C_0 C_1} \big(\frac{2\pi}{3}\big)^2 < \frac{R-r}{C_0} \big(\frac{2\pi}{3}\big)^2$). Let $\widehat{v}_\varepsilon \in W^{1,\infty}(A_{r,R}; \mathbb{S}^1)$ be defined as in Remark 3.2. By Lemma 3.3 there exists $\phi_\varepsilon \in W^{1,\infty}(A_{r,R})$ such that $\widehat{v}_\varepsilon(x) = \exp(\iota \phi_\varepsilon(x))$ for all $x \in A_{r,R}$ and (3.9) holds true. To define $\overline{v}_\varepsilon$, we start by extending ϕ_ε from the annulus $A_{r,R}$ to a function ϕ'_ε on the ball B_R via a 1-homogeneous extension that starts from a layer of triangles suitably chosen inside $A_{r,R}$. More precisely, we fix $r < r' < R' < \frac{r+R}{2} < R$ such that $R' - r' \geq \frac{R-r}{4}$ and we subdivide $A_{r',R'}$ into the union of annuli

$$A_{r',R'} = \bigcup_{k=1}^{K_\varepsilon} A^k\,, \quad A^k := A_{r_{k-1},r_k}\,, \quad r_k := r' + k\frac{R'-r'}{K_\varepsilon}\,, \quad K_\varepsilon := \lfloor \tfrac{R'-r'}{9\varepsilon} \rfloor\,.$$

Note that $K_\varepsilon \geq \frac{R-r}{36\varepsilon} - 1 \geq \frac{1}{2}\frac{R-r}{36\varepsilon} \geq 1$ and the width of each annulus A^k is $\frac{R'-r'}{K_\varepsilon} \geq 9\varepsilon$. For every ε we find $k_\varepsilon \in \{1,\dots,K_\varepsilon\}$ such that

$$\begin{aligned} C_1 &\geq \frac{1}{\varepsilon^2} XY_\varepsilon(v_\varepsilon, A_{r,R}) \geq C\int_{A_{r',R'}} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x = C\sum_{k=1}^{K_\varepsilon} \int_{A^k} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x \\ &\geq CK_\varepsilon \int_{A^{k_\varepsilon}} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x \geq C\frac{R-r}{\varepsilon^3} XY_\varepsilon(v_\varepsilon, A^{k_\varepsilon})\,, \end{aligned} \tag{3.12}$$

where the second and the last inequality follow from (3.11). Recalling the first property in Remark 3.4, an immediate consequence of the previous inequality is that

$$|\phi_\varepsilon(\varepsilon i) - \phi_\varepsilon(\varepsilon j)|^2 \leq C|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)|^2 \leq \frac{C}{\varepsilon^2} XY_\varepsilon(v_\varepsilon, A^{k_\varepsilon}) \leq \frac{C}{(R-r)\varepsilon} XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq C\frac{C_1\varepsilon}{R-r} \tag{3.13}$$

for every $\varepsilon i, \varepsilon j \in A^{k_\varepsilon}$ with $|i-j| = 1$.

Using that $\phi_\varepsilon \in W^{1,\infty}(A_{r,R})$,[5] we find $\rho_\varepsilon \in (r_{k_\varepsilon - 1} + 4\varepsilon, r_{k_\varepsilon} - 4\varepsilon)$ such that the restriction $y \in \partial B_{\rho_\varepsilon} \mapsto \phi_\varepsilon|_{\partial B_{\rho_\varepsilon}}(y)$ belongs to $W^{1,\infty}(\partial B_{\rho_\varepsilon})$, $\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})(y) = \nabla\phi_\varepsilon(y)\cdot\tau_{\partial B_{\rho_\varepsilon}}(y)$ for $\mathcal{H}^1$-a.e. $y \in \partial B_{\rho_\varepsilon}$, and

$$\begin{aligned} \int_{A^{k_\varepsilon}} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x = \int_{A^{k_\varepsilon}} |\nabla \phi_\varepsilon|^2 \,\mathrm{d}x &\geq \int_{r_{k_\varepsilon - 1}+4\varepsilon}^{r_{k_\varepsilon}-4\varepsilon} \int_{\partial B_\rho} |\nabla\phi_\varepsilon(y)\cdot\tau_{\partial B_\rho}(y)|^2 \,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\rho \\ &\geq \Big(\frac{R'-r'}{K_\varepsilon} - 8\varepsilon\Big) \int_{\partial B_{\rho_\varepsilon}} |\nabla\phi_\varepsilon(y)\cdot\tau_{\partial B_{\rho_\varepsilon}}(y)|^2 \,\mathrm{d}\mathcal{H}^1(y) \\ &\geq \varepsilon \int_{\partial B_{\rho_\varepsilon}} \big|\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})(y)\big|^2 \,\mathrm{d}\mathcal{H}^1(y)\,. \end{aligned}$$

In particular, by the previous formula and by (3.12), we infer that

$$\int_{\partial B_{\rho_\varepsilon}} \big|\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})\big|^2 \,\mathrm{d}\mathcal{H}^1 \leq \frac{1}{\varepsilon^3 K_\varepsilon} XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq \frac{C}{(R-r)\varepsilon^2} XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq C\frac{C_1}{R-r}\,. \tag{3.14}$$

Setting $a_\varepsilon := \fint_{\partial B_{\rho_\varepsilon}} \phi_\varepsilon \,\mathrm{d}\mathcal{H}^1$, by Poincare's Inequality on $\partial B_{\rho_\varepsilon}$ we have that

$$\int_{\partial B_{\rho_\varepsilon}} \big|\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}} - a_\varepsilon\big|^2 \,\mathrm{d}\mathcal{H}^1 \leq C\rho_\varepsilon^2 \int_{\partial B_{\rho_\varepsilon}} \big|\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})\big|^2 \,\mathrm{d}\mathcal{H}^1, \tag{3.15}$$

[5] Notice that $\phi_\varepsilon|_{\partial B_\rho} \in W^{1,\infty}(\partial B_\rho)$ for every $\rho \in (r', R')$: the function ϕ_ε is piecewise affine by Remark 3.2 and thus it is C^1 outside a finite union of segments, which intersect ∂B_ρ only in a finite number of points (depending on ε).

for a scale-independent constant C. By (3.14) and the Sobolev Embedding Theorem in one dimension, we have that ϕ_ε is $\frac{1}{2}$-Hölder continuous and

$$\begin{aligned} \|\phi_\varepsilon - a_\varepsilon\|^2_{L^\infty(\partial B_{\rho_\varepsilon})} &\leq \frac{C\rho_\varepsilon}{(R-r)\varepsilon^2} XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq C\frac{C_1}{R-r}\rho_\varepsilon\,, \\ \sup_{\substack{x,y\in\partial B_{\rho_\varepsilon}\\ x\neq y}} \frac{|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^2}{|x-y|} &\leq \frac{C}{(R-r)\varepsilon^2} XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq C\frac{C_1}{R-r}\,. \end{aligned} \tag{3.16}$$

The function $\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}}$ is also Lipschitz continuous, but its Lipschitz constant might depend on ε. We define the auxiliary function $\phi'_\varepsilon \in W^{1,\infty}(B_R)$ via the 1-homogeneous extension

$$\phi'_\varepsilon(x) := \begin{cases} \phi_\varepsilon(x)\,, & \text{if } x \in A_{\rho_\varepsilon,R}\,, \\ a_\varepsilon + \frac{|x|}{\rho_\varepsilon}\big(\phi_\varepsilon\big(\frac{x}{|x|}\rho_\varepsilon\big) - a_\varepsilon\big)\,, & \text{if } x \in \overline{B}_{\rho_\varepsilon}\,, \end{cases} \quad v'_\varepsilon(x) := \exp(\iota\phi'_\varepsilon(x)) \quad \text{for } x\in B_R\,.$$

Note that $v'_\varepsilon = \widehat{v}_\varepsilon$ in $A_{\rho_\varepsilon,R}$. To define the spin field $\overline{v}_\varepsilon$, we suitably sample v'_ε. Applying Lemma 3.7 below, we find $\overline{x}_\varepsilon \in \mathbb{R}^2$ with $|\overline{x}_\varepsilon| \leq \varepsilon$ such that

$$\frac{1}{\varepsilon^2} XY_\varepsilon(v'_\varepsilon(\,\cdot\,+\overline{x}_\varepsilon), B_{\rho_\varepsilon}) \leq C\int_{B_{R'}} |\nabla v'_\varepsilon|^2 \mathrm{d}x \tag{3.17}$$

and for $\varepsilon i \in \mathcal{L}_\varepsilon \cap B_R$ we set

$$\overline{\phi}_\varepsilon(\varepsilon i) := \begin{cases} \phi'_\varepsilon(\varepsilon i)\,, & \text{if } \varepsilon i \in A_{\rho_\varepsilon,R}\,, \\ \phi'_\varepsilon(\varepsilon i + \overline{x}_\varepsilon)\,, & \text{if } \varepsilon i \in \overline{B}_{\rho_\varepsilon}\,, \end{cases} \quad \overline{v}_\varepsilon(\varepsilon i) := \exp(\iota\overline{\phi}_\varepsilon(\varepsilon i)) \quad \text{for } \varepsilon i \in B_R\,.$$

We extend $\overline{v}_\varepsilon$ outside B_R by setting $\overline{v}_\varepsilon(\varepsilon i) := v_\varepsilon(\varepsilon i)$ for $\varepsilon i \in \mathbb{R}^2 \setminus B_R$. By construction we have that $\overline{v}_\varepsilon = v_\varepsilon$ on $\mathcal{L}_\varepsilon \cap (\mathbb{R}^2 \setminus \overline{B}_{\rho_\varepsilon})$ and thus on $\mathcal{L}_\varepsilon \cap (\mathbb{R}^2 \setminus \overline{B}_{\frac{r+R}{2}})$.

Let us prove that $|\mu_{\overline{v}_\varepsilon}|(B_R) = 0$. Let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ be such that $T \cap B_R \neq \emptyset$. If $T \subset \mathbb{R}^2 \setminus \overline{B}_{\rho_\varepsilon}$, there is nothing to prove, as $\overline{v}_\varepsilon = v_\varepsilon$ on $\mathcal{L}_\varepsilon \cap T$ and $|\mu_{v_\varepsilon}|(T) = 0$ (since $T \cap A_{r,R} \neq \emptyset$). Let us thus assume that $T \cap \overline{B}_{\rho_\varepsilon} \neq \emptyset$ and let us prove that $|\overline{\phi}_\varepsilon(\varepsilon i) - \overline{\phi}_\varepsilon(\varepsilon j)|, |\overline{\phi}_\varepsilon(\varepsilon j) - \overline{\phi}_\varepsilon(\varepsilon k)|, |\overline{\phi}_\varepsilon(\varepsilon k) - \overline{\phi}_\varepsilon(\varepsilon i)| < \frac{2\pi}{3}$, which implies $|\mu_{\overline{v}_\varepsilon}|(T) = 0$. We only prove it for $|\overline{\phi}_\varepsilon(\varepsilon i) - \overline{\phi}_\varepsilon(\varepsilon j)|$, the other inequalities being analogous. We start by observing that

$$|\overline{\phi}_\varepsilon(\varepsilon i) - \overline{\phi}_\varepsilon(\varepsilon j)|^2 \leq 3|\phi'_\varepsilon(\varepsilon i + \overline{x}_\varepsilon) - \phi'_\varepsilon(\varepsilon i)|^2 + 3|\phi'_\varepsilon(\varepsilon j) - \phi'_\varepsilon(\varepsilon i)|^2 + 3|\phi'_\varepsilon(\varepsilon j + \overline{x}_\varepsilon) - \phi'_\varepsilon(\varepsilon j)|^2 \tag{3.18}$$

and $\varepsilon i, \varepsilon i + \overline{x}_\varepsilon, \varepsilon j, \varepsilon j + \overline{x}_\varepsilon \in \overline{B}_{\rho_\varepsilon + 2\varepsilon}$. Therefore, to conclude it is enough to prove that for all $x, y \in \overline{B}_{\rho_\varepsilon+2\varepsilon}$ such that $|x-y| \leq \varepsilon$ we have

$$|\phi'_\varepsilon(x) - \phi'_\varepsilon(y)|^2 \leq \frac{C}{(R-r)\varepsilon} XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq C\frac{C_1\varepsilon}{R-r} \tag{3.19}$$

from which it follows that

$$|\overline{\phi}_\varepsilon(\varepsilon i) - \overline{\phi}_\varepsilon(\varepsilon j)|^2 \leq 9C\frac{C_1\varepsilon}{R-r} < \Big(\frac{2\pi}{3}\Big)^2 \tag{3.20}$$

for $\varepsilon < \big(\frac{2\pi}{3}\big)^2 \frac{R-r}{C_0 C_1}$ and $C_0 > 9C$. To prove (3.19) we distinguish three cases.

Case 1: $x, y \in \overline{B}_{\rho_\varepsilon+2\varepsilon} \setminus B_{\rho_\varepsilon}$. Since $B_{\rho_\varepsilon+4\varepsilon} \setminus \overline{B}_{\rho_\varepsilon-4\varepsilon} \subset A^{k_\varepsilon}$, we find $T', T'' \in \mathcal{T}_\varepsilon(A^{k_\varepsilon})$ such that $x \in T'$, $y \in T''$, and $T' \cap T'' \neq \emptyset$. Let $z \in T' \cap T''$. Since $\phi_\varepsilon|_{T'}$ and $\phi_\varepsilon|_{T''}$ are affine and using (3.13) we obtain that

$$|\phi'_\varepsilon(x) - \phi'_\varepsilon(y)|^2 = |\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2 \leq 2|\phi_\varepsilon(x) - \phi_\varepsilon(z)|^2 + 2|\phi_\varepsilon(z) - \phi_\varepsilon(y)|^2 \leq \frac{C}{(R-r)\varepsilon} XY_\varepsilon(v_\varepsilon, A_{r,R})\,. \tag{3.21}$$

Case 2: $x, y \in \overline{B}_{\rho_\varepsilon}$. By (3.16) we get that

$$\begin{aligned}|\phi'_\varepsilon(x) - \phi'_\varepsilon(y)|^2 &= \big|\tfrac{|x|}{\rho_\varepsilon}\big(\phi_\varepsilon\big(\tfrac{x}{|x|}\rho_\varepsilon\big) - a_\varepsilon\big) - \tfrac{|y|}{\rho_\varepsilon}\big(\phi_\varepsilon\big(\tfrac{y}{|y|}\rho_\varepsilon\big) - a_\varepsilon\big)\big|^2 \\ &\leq 2\big|\tfrac{|x|-|y|}{\rho_\varepsilon}\big|^2\|\phi_\varepsilon - a_\varepsilon\|^2_{L^\infty(\partial B_{\rho_\varepsilon})} + 2\tfrac{|x|^2}{\rho_\varepsilon^2}\big|\phi_\varepsilon\big(\tfrac{x}{|x|}\rho_\varepsilon\big) - \phi_\varepsilon\big(\tfrac{y}{|y|}\rho_\varepsilon\big)\big|^2 \\ &\leq \Big(\frac{\varepsilon^2}{\rho_\varepsilon} + \varepsilon\Big)\frac{C}{(R-r)\varepsilon^2}XY_\varepsilon(v_\varepsilon, A_{r,R}) \leq \frac{C}{(R-r)\varepsilon}XY_\varepsilon(v_\varepsilon, A_{r,R})\,.\end{aligned}$$

Case 3: $x \in \overline{B}_{\rho_\varepsilon}$, $y \in A_{\rho_\varepsilon,R}$. We find $z \in \partial B_{\rho_\varepsilon}$ such that $|x-z| + |y-z| = |x-y|$. Using Case 1 for x, z and Case 2 for z, y we obtain

$$|\phi'_\varepsilon(x) - \phi'_\varepsilon(y)|^2 \leq 2|\phi'_\varepsilon(x) - \phi'_\varepsilon(z)|^2 + 2|\phi'_\varepsilon(z) - \phi'_\varepsilon(y)|^2 \leq \frac{C}{(R-r)\varepsilon}XY_\varepsilon(v_\varepsilon, A_{r,R})\,. \tag{3.22}$$

This concludes the proof of the fact that $|\mu_{\overline{v}_\varepsilon}|(T) = 0$. For the next estimates it is worth to mention that, as a byproduct of (3.18) and (3.21)–(3.22), we also obtain that

$$\frac{1}{\varepsilon^2}XY_\varepsilon(\overline{v}_\varepsilon, T) \leq \frac{C}{(R-r)\varepsilon}XY_\varepsilon(v_\varepsilon, A_{r,R}) \quad \text{for every } T \in \mathcal{T}_\varepsilon(B_R) \text{ such that } T \cap \partial B_{\rho_\varepsilon} \neq \emptyset\,. \tag{3.23}$$

It remains to prove that $XY_\varepsilon(\overline{v}_\varepsilon, B_R) \leq C(r,R)XY_\varepsilon(v_\varepsilon, A_{r,R})$. First of all, we observe that (3.17) and the definition of v'_ε imply

$$\frac{1}{\varepsilon^2}XY_\varepsilon(\overline{v}_\varepsilon, B_{\rho_\varepsilon}) \leq C\int_{B_{\rho_\varepsilon}}|\nabla v'_\varepsilon|^2\mathrm{d}x + C\int_{A_{\rho_\varepsilon,R'}}|\nabla \widehat{v}_\varepsilon|^2\mathrm{d}x \leq C\int_{B_{\rho_\varepsilon}}|\nabla v'_\varepsilon|^2\mathrm{d}x + \frac{C}{\varepsilon^2}XY_\varepsilon(v_\varepsilon, A_{r,R})\,.$$

Let us estimate $\int_{B_{\rho_\varepsilon}}|\nabla v'_\varepsilon|^2\mathrm{d}x$. Using that $\nabla\phi'_\varepsilon(x) = \frac{1}{\rho_\varepsilon}\frac{x}{|x|}\big(\phi_\varepsilon\big(\frac{x}{|x|}\rho_\varepsilon\big) - a_\varepsilon\big) + \nabla\phi_\varepsilon\big(\frac{x}{|x|}\rho_\varepsilon\big)\frac{x^\perp}{|x|} \otimes \frac{x^\perp}{|x|}$, by Fubini's Theorem, by (3.15), and by (3.14), we obtain that

$$\begin{aligned}\int_{B_{\rho_\varepsilon}}|\nabla v'_\varepsilon|^2\,\mathrm{d}x &= \int_{B_{\rho_\varepsilon}}|\nabla\phi'_\varepsilon|^2\,\mathrm{d}x = \int_{B_{\rho_\varepsilon}}\big|\nabla\phi'_\varepsilon(x)\cdot\tfrac{x}{|x|}\big|^2 + \big|\nabla\phi'_\varepsilon(x)\cdot\tfrac{x^\perp}{|x|}\big|^2\,\mathrm{d}x \\ &= \int_{B_{\rho_\varepsilon}}\tfrac{1}{\rho_\varepsilon^2}\big|\phi_\varepsilon\big(\tfrac{x}{|x|}\rho_\varepsilon\big) - a_\varepsilon\big|^2 + \big|\nabla\phi_\varepsilon\big(\tfrac{x}{|x|}\rho_\varepsilon\big)\cdot\tfrac{x^\perp}{|x|}\big|^2\,\mathrm{d}x \\ &= \int_0^{\rho_\varepsilon}\int_{\partial B_{\rho_\varepsilon}}\Big(\tfrac{1}{\rho_\varepsilon^2}\big|\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}}(y) - a_\varepsilon\big|^2 + \big|\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})(y)\big|^2\Big)\tfrac{\rho}{\rho_\varepsilon}\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\rho \\ &\leq CR\int_{\partial B_{\rho_\varepsilon}}\Big(\tfrac{1}{\rho_\varepsilon^2}\big|\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}}(y) - a_\varepsilon\big|^2 + \big|\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})(y)\big|^2\Big)\,\mathrm{d}\mathcal{H}^1(y) \\ &\leq CR\int_{\partial B_{\rho_\varepsilon}}\big|\nabla_{\partial B_{\rho_\varepsilon}}(\phi_\varepsilon|_{\partial B_{\rho_\varepsilon}})(y)\big|^2\,\mathrm{d}\mathcal{H}^1(y) \leq \frac{CR}{(R-r)\varepsilon^2}XY_\varepsilon(v_\varepsilon, A_{r,R})\,.\end{aligned}$$

To conclude, let us estimate the energy on triangles that are not contained in B_{ρ_ε}. Let us fix $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(B_R)$. If $T \subset A_{\rho_\varepsilon,R}$, then $\overline{v}_\varepsilon = v_\varepsilon$ on $\mathcal{L}_\varepsilon \cap T$ and thus $XY_\varepsilon(\overline{v}_\varepsilon, T) = XY_\varepsilon(v_\varepsilon, T)$. Finally, since $\#\{T \in \mathcal{T}_\varepsilon(B_r) : T \cap \partial B_{\rho_\varepsilon} \neq \emptyset\} \leq \frac{C\rho_\varepsilon}{\varepsilon} \leq \frac{CR}{\varepsilon}$, inequality (3.23) yields

$$\sum_{\substack{T \in \mathcal{T}_\varepsilon(B_R)\\ T\cap\partial B_{\rho_\varepsilon}\neq\emptyset}}\frac{1}{\varepsilon^2}XY_\varepsilon(\overline{v}_\varepsilon, T) \leq \frac{CR}{(R-r)\varepsilon^2}XY_\varepsilon(v_\varepsilon, A_{r,R})\,.$$

Putting together the previous estimates yields $XY_\varepsilon(\overline{v}_\varepsilon, B_R) \leq \frac{CR}{(R-r)}XY_\varepsilon(v_\varepsilon, A_{r,R})$. Choosing $C_0 \geq C$ such that (3.20) is satisfied yields the statement of the lemma. □

We prove a lemma concerning sampling of H^1 functions used in the previous proof.

Lemma 3.7. *Let $T_0 := \mathrm{conv}\{0, \varepsilon\hat{e}_1, \varepsilon\hat{e}_2\}$. There exists a universal constant $C > 0$ such that the following holds true: given $U \subset \mathbb{R}^2$, $v \in H^1(U;\mathbb{S}^1)$, $U' \subset\subset U$ and $\varepsilon > 0$ with $\mathrm{dist}(U', \partial U) > \varepsilon$, there exists a point $\overline{x} \in T_0$ (possibly depending on U') such that*

$$\frac{1}{\varepsilon^2} XY_\varepsilon(v(\,\cdot\, + \overline{x}), U') \leq C \int_U |\nabla v|^2 \mathrm{d}x\,.$$

Proof. Let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(U')$. Note that, if $T' \cap T \neq \emptyset$, then $T' \subset U$. Moreover, by the theory of slicing of Sobolev functions (*cf.* [8, Proposition 3.105]), for a.e. $x \in T_0$ (actually, for $\mathcal{H}^1$-a.e. $x \in T_0$) we have that $v|_{[x+\varepsilon i, x+\varepsilon j]} \in H^1([x+\varepsilon i, x+\varepsilon j]; \mathbb{S}^1)$ and $\frac{\mathrm{d}}{\mathrm{d}t} v\big(x + \varepsilon i + t(\varepsilon j - \varepsilon i)\big) = \nabla v\big(x + \varepsilon i + t(\varepsilon j - \varepsilon i)\big)(\varepsilon j - \varepsilon i)$ for $t \in (0,1)$. Thus, by Jensen's Inequality and Fubini's Theorem,

$$\begin{aligned}
\int_{T_0} |v(\varepsilon i + x) - v(\varepsilon j + x)|^2 \,\mathrm{d}x &= \int_{T_0} \Big| \int_0^1 \nabla v\big(x + \varepsilon i + t(\varepsilon j - \varepsilon i)\big)(\varepsilon j - \varepsilon i)\,\mathrm{d}t \Big|^2 \,\mathrm{d}x \\
&\leq \int_{T_0} \int_0^1 \varepsilon^2 \big|\nabla v\big(x + \varepsilon i + t(\varepsilon j - \varepsilon i)\big)\big|^2 \,\mathrm{d}t\,\mathrm{d}x \\
&\leq \varepsilon^2 \int_{T + T_0} \big|\nabla v(y)\big|^2 \,\mathrm{d}y \leq \varepsilon^2 \sum_{T' \cap T \neq \emptyset} \int_{T'} \big|\nabla v(y)\big|^2 \,\mathrm{d}y\,.
\end{aligned}$$

Arguing analogously with the other vertices of the triangle, we get that for every $T \in \mathcal{T}_\varepsilon(U')$

$$\int_{T_0} \frac{1}{\varepsilon^2} XY_\varepsilon(v(\,\cdot\, + x), T)\,\mathrm{d}x \leq 3\varepsilon^2 \sum_{T' \cap T \neq \emptyset} \int_{T'} \big|\nabla v(y)\big|^2 \,\mathrm{d}y\,.$$

Hence, there exists $\overline{x} \in T_0$ such that

$$\begin{aligned}
XY_\varepsilon(v(\,\cdot\, + \overline{x}), U') &\leq \int_{T_0} \frac{4}{\sqrt{3}\varepsilon^2} XY_\varepsilon(v(\,\cdot\, + x), U')\,\mathrm{d}x = \sum_{T \in \mathcal{T}_\varepsilon(U')} \frac{4}{\sqrt{3}} \int_{T_0} \frac{1}{\varepsilon^2} XY_\varepsilon(v(\,\cdot\, + x), T)\,\mathrm{d}x \\
&\leq \sum_{T \in \mathcal{T}_\varepsilon(U')} 4\sqrt{3}\varepsilon^2 \sum_{T' \cap T \neq \emptyset} \int_{T'} \big|\nabla v(y)\big|^2 \,\mathrm{d}y \leq 52\sqrt{3}\varepsilon^2 \int_U \big|\nabla v(y)\big|^2 \,\mathrm{d}y\,,
\end{aligned}$$

where we used that each triangle T' is counted at most 13 times. This concludes the proof. □

Relations between chirality and vorticity. We describe here the relations between the chirality of $u \in \mathcal{SF}_\varepsilon$ and the vorticity of the auxiliary spin field $v \in \mathcal{SF}_\varepsilon$ defined as in (2.5).

Remark 3.8. If the chirality of u is close enough to 1 in a triangle of the lattice, there the auxiliary spin field v cannot have vorticity. More precisely, there exists $\eta \in (0,1)$ such that $\mu_v(T) = 0$ for every $u \in \mathcal{SF}_\varepsilon$ and $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $\chi(u,T) > 1 - \eta$. Indeed, by Remark 2.3 there exists $\eta \in (0,1)$ such that $|\theta_1 - \frac{2\pi}{3}| < \frac{\pi}{2}$ and $|\theta_2 + \frac{2\pi}{3}| < \frac{\pi}{2}$ for every $\theta_1, \theta_2 \in [-\pi, \pi]$ with $\chi(\theta_1, \theta_2) > 1 - \eta$. Let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, $\varepsilon k \in \mathcal{L}^3_\varepsilon$ and let $u(x) = \exp(\iota\theta(x))$ for $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$ be such that $\theta(\varepsilon j) - \theta(\varepsilon i) \in [-\pi, \pi]$ and $\theta(\varepsilon k) - \theta(\varepsilon i) \in [-\pi, \pi]$. If $\chi(u,T) = \chi(\theta(\varepsilon j) - \theta(\varepsilon i), \theta(\varepsilon k) - \theta(\varepsilon i)) > 1 - \eta$, then $|\theta(\varepsilon j) - \theta(\varepsilon i) - \frac{2\pi}{3}| < \frac{\pi}{2}$ and $|\theta(\varepsilon k) - \theta(\varepsilon i) + \frac{2\pi}{3}| < \frac{\pi}{2}$. Let

$$\varphi(\varepsilon i) := \theta(\varepsilon i)\,, \quad \varphi(\varepsilon j) := \theta(\varepsilon j) - \frac{2\pi}{3}\,, \quad \varphi(\varepsilon k) := \theta(\varepsilon k) + \frac{2\pi}{3}\,,$$

so that $v(x) = \exp(\iota\varphi(x))$ for $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$. In particular, $|\varphi(\varepsilon j) - \varphi(\varepsilon i)| < \frac{\pi}{2}$, $|\varphi(\varepsilon k) - \varphi(\varepsilon i)| < \frac{\pi}{2}$, and $|\varphi(\varepsilon k) - \varphi(\varepsilon j)| < \pi$. The latter conditions imply that

$$\mathrm{d}^e\varphi(\varepsilon i, \varepsilon j) + \mathrm{d}^e\varphi(\varepsilon j, \varepsilon k) + \mathrm{d}^e\varphi(\varepsilon k, \varepsilon i) = 0 = \mathrm{d}^e\varphi(\varepsilon i, \varepsilon k) + \mathrm{d}^e\varphi(\varepsilon k, \varepsilon j) + \mathrm{d}^e\varphi(\varepsilon j, \varepsilon i)$$

and thus $\mu_v(T) = 0$, independent of the ordering of the vertices $\varepsilon i, \varepsilon j, \varepsilon k$.

Remark 3.9. Conversely to Remark 3.8, if the vorticity of the auxiliary spin field v is 0 in a triangle of the lattice, there the chirality of u cannot be close to -1. More precisely, there exists $\eta' \in (0,1)$ such that $\chi(u,T) \geq -1+\eta'$ for every $u \in \mathcal{SF}_\varepsilon$ and $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $\mu_v(T) = 0$. Indeed, as in Remark 2.3 there exists $\eta' \in (0,1)$ such that $|\theta_1 + \frac{2\pi}{3}| < \frac{\pi}{6}$ and $|\theta_2 - \frac{2\pi}{3}| < \frac{\pi}{6}$ for every $\theta_1, \theta_2 \in [-\pi,\pi]$ with $\chi(\theta_1,\theta_2) < -1+\eta'$. Let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, $\varepsilon k \in \mathcal{L}^3_\varepsilon$, let $u(x) = \exp(\iota\theta(x))$ for $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$. If $\chi(u,T) < -1+\eta'$, then $|\theta(\varepsilon j) - \theta(\varepsilon i) + \frac{2\pi}{3}| < \frac{\pi}{6}$ and $|\theta(\varepsilon k) - \theta(\varepsilon i) - \frac{2\pi}{3}| < \frac{\pi}{6}$. Let now $\varphi(\varepsilon i) := \theta(\varepsilon i)$, $\varphi(\varepsilon j) := \theta(\varepsilon j) - \frac{2\pi}{3}$, $\varphi(\varepsilon k) := \theta(\varepsilon k) + \frac{2\pi}{3}$, so that $v(x) = \exp(\iota\varphi(x))$ for $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$. Then

$$|\varphi(\varepsilon j) - \varphi(\varepsilon i) + \tfrac{4\pi}{3}| < \tfrac{\pi}{6}\,, \quad |\varphi(\varepsilon i) - \varphi(\varepsilon k) + \tfrac{4\pi}{3}| < \tfrac{\pi}{6}\,, \quad |\varphi(\varepsilon k) - \varphi(\varepsilon j) - \tfrac{8\pi}{3}| < \tfrac{\pi}{3}\,,$$

which yields $\mathrm{d}^e\varphi(\varepsilon i,\varepsilon j) = \varphi(\varepsilon j) - \varphi(\varepsilon i) + 2\pi$, $\mathrm{d}^e\varphi(\varepsilon j,\varepsilon k) = \varphi(\varepsilon k) - \varphi(\varepsilon j) - 2\pi$, $\mathrm{d}^e\varphi(\varepsilon k,\varepsilon i) = \varphi(\varepsilon i) - \varphi(\varepsilon k) + 2\pi$, whence $\mu_v(T) = 1$, assuming that the counterclockwise order of the vertices is $(\varepsilon i, \varepsilon j, \varepsilon k)$. If instead the counterclockwise order is $(\varepsilon i, \varepsilon k, \varepsilon j)$, then the antisymmetry condition $\mathrm{d}^e\varphi(x,x') = -\mathrm{d}^e\varphi(x',x)$ implies that $\mu_v(T) = -1$.

Although a control $XY_\varepsilon \leq CE_\varepsilon$ does not hold true, we show that it is feasible in the regions where the spin field has no vorticity.

Lemma 3.10. *There exists a constant $C > 0$ such that the following holds true. Let $u \in \mathcal{SF}_\varepsilon$, let $v \in \mathcal{SF}_\varepsilon$ the auxiliary spin field defined according to* (2.5)*, and let $T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$. Then*

$$\mu_v(T) = 0 \quad \Longrightarrow \quad XY_\varepsilon(v,T) \leq CE_\varepsilon(u,T)\,. \tag{3.24}$$

Proof. Let us fix $\lambda = \frac{1}{2} \in (0,1)$ and let $\eta > 0$ be the corresponding number given by Lemma 2.8. If $\chi(u,T) > 1-\eta$, by Lemma 2.8 we have that $\frac{1}{2}XY_\varepsilon(v,T) \leq E_\varepsilon(u,T)$. Therefore, we are left to prove the bound when $\chi(u,T) \leq 1-\eta$. Let now $\eta' > 0$ be as in Remark 3.9. From the condition $\mu_v(T) = 0$, we infer that $\chi(u,T) \geq -1+\eta'$. Let $\eta'' := \min\{\eta, \eta'\}$. Since $-1+\eta'' \leq \chi(u,T) \leq 1-\eta''$, by Remark 2.1 we deduce the existence of a constant $C_{\eta''} > 0$ such that $E_\varepsilon(u,T) \geq C_{\eta''}\varepsilon^2$. On the other hand $XY_\varepsilon(v,T) \leq 6\varepsilon^2 \leq \frac{6}{C_{\eta''}}E_\varepsilon(u,T)$. This concludes the proof. □

Remark 3.11. The constant C found in (3.24) is not optimal.

4. Γ-limit in the bulk scaling

In this section we compute the Γ-limit of the AFXY-energy in the bulk scaling. We start with a lemma concerning the spin fields that cannot overcome the energetic barrier of the chirality transition (of order ε).

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, and connected set and let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $E_\varepsilon(u_\varepsilon,\Omega) \leq C\delta_\varepsilon$ with $\delta_\varepsilon \ll \varepsilon$. Then, up to a subsequence, either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in $L^1(\Omega)$.*

Proof. Let

$$\hat\chi_\varepsilon := \begin{cases} 1\,, & \text{if } \chi(u_\varepsilon) > 0\,, \\ -1\,, & \text{if } \chi(u_\varepsilon) \leq 0\,. \end{cases}$$

For every $\Omega' \subset\subset \Omega$ connected set such that $\mathrm{dist}(\Omega', \partial\Omega) > \sqrt{3}\varepsilon$, due to Lemma 2.2 applied with $\eta = 1$, there exists $C > 0$ such that

$$\mathcal{H}^1(\partial\{\hat\chi_\varepsilon = 1\} \cap \Omega') \leq \frac{C}{\varepsilon}E_\varepsilon(u_\varepsilon,\Omega) \leq C\frac{\delta_\varepsilon}{\varepsilon}\,,$$

for ε small enough. Since Ω' is connected, this implies that, up to a subsequence, either $\hat{\chi}_\varepsilon \to 1$ or $\hat{\chi}_\varepsilon \to -1$ strongly in $L^1(\Omega)$. Moreover, due to Remark 2.1, for every $\eta \in (0,1)$ and every $\Omega' \subset\subset \Omega$ with $\mathrm{dist}(\Omega', \partial\Omega) > 3\varepsilon$ we have that

$$\lim_{\varepsilon \to 0} |\{|\hat{\chi}_\varepsilon - \chi(u_\varepsilon)| > \eta\} \cap \Omega'| = 0\,,$$

which implies that either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in measure and therefore also strongly in $L^1(\Omega)$. □

We compute the Γ-limit in the bulk scaling in the case where $\chi(u_\varepsilon) \sim 1$. An analogous statement holds true if $\chi(u_\varepsilon) \sim -1$ (in that case, the auxiliary variable has to be redefined accordingly).

We state the next theorem for Ω connected. In case Ω is not connected, the result holds true in every connected component of Ω.

Theorem 4.2. *Assume that Ω is an open, bounded, and connected set with Lipschitz boundary. The following results hold:*

i) (Compactness) *Let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $E_\varepsilon(u_\varepsilon, \Omega) \le C\varepsilon^2$. Then, up to a subsequence, either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in $L^1(\Omega)$. Assume that $\chi(u_\varepsilon) \to 1$, let $v_\varepsilon \in \mathcal{SF}_\varepsilon$ be the auxiliary spin field defined as in* (2.5)*, and let $\hat{v}_\varepsilon$ be its piecewise affine interpolation. Then there exists a subsequence (not relabeled) and $v \in H^1(\Omega;\mathbb{S}^1)$ such that $\hat{v}_\varepsilon \to v$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and $\hat{v}_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$.*

ii) ($\liminf$ inequality) *Let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$, let $v_\varepsilon \in \mathcal{SF}_\varepsilon$ be the auxiliary spin field defined as in* (2.5)*, and let $\hat{v}_\varepsilon$ be its piecewise affine interpolation. Let $v \in H^1(\Omega;\mathbb{S}^1)$ and assume that $\hat{v}_\varepsilon \to v$ strongly in $L^2(\Omega;\mathbb{R}^2)$. Then*

$$\sqrt{3}\int_\Omega |\nabla v|^2 \,\mathrm{d}x \le \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega)\,.$$

iii) ($\limsup$ inequality) *Let $v \in H^1(\Omega;\mathbb{S}^1)$. Then there exist $u_\varepsilon \in \mathcal{SF}_\varepsilon$ such that $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$ and $\hat{v}_\varepsilon \to v$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and*

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \le \sqrt{3}\int_\Omega |\nabla v|^2 \,\mathrm{d}x\,,$$

where $v_\varepsilon \in \mathcal{SF}_\varepsilon$ is the auxiliary spin field defined as in (2.5) *and $\hat{v}_\varepsilon$ is its piecewise affine interpolation.*

Proof. Let us prove *i)*. The fact that either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in $L^1(\Omega)$ (up to a subsequence) follows from Lemma 4.1. In the following, we assume that $\chi(u_\varepsilon) \to 1$.

Let us fix $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be such that both Ω' and Ω'' have Lipschitz boundary. For ε small enough, $\sqrt{3}\varepsilon < \mathrm{dist}(\Omega'', \partial\Omega)$ holds true. We fix $\lambda \in (0,1)$ and we consider the corresponding $\eta \in (0,1)$ provided by Lemma 2.8. By Lemma 2.4 we get that there exists $C_\eta > 0$ depending also on Ω'' such that

$$\#\{T \in \mathcal{T}_\varepsilon(\Omega'') : \chi(u_\varepsilon, T) \le 1 - \eta\} \le C_\eta \Big(\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega)\Big)^2 \le C_\eta.$$

Therefore, up to a subsequence, we can assume that $\#\{T \in \mathcal{T}_\varepsilon(\Omega'') : \chi(u_\varepsilon, T) \le 1 - \eta\} = \overline{M}$, the number $\overline{M}$ possibly depending on η and Ω''. This yields

$$\sum_{\substack{T \in \mathcal{T}_\varepsilon(\Omega'')\\ \chi(u_\varepsilon, T) \le 1-\eta}} \frac{1}{\varepsilon^2} XY_\varepsilon(v_\varepsilon, T) \le 6\#\{T \in \mathcal{T}_\varepsilon(\Omega'') : \chi(u_\varepsilon, T) \le 1 - \eta\} \le 6\overline{M}\,.$$

We apply Lemma 2.8 to obtain that

$$
\begin{aligned}
C \geq \frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon,\Omega) &\geq \sum_{\substack{T\in\mathcal{T}_\varepsilon(\Omega'')\\ \chi(u_\varepsilon,T)>1-\eta}} \frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon,T) \geq (1-\lambda)\sum_{\substack{T\in\mathcal{T}_\varepsilon(\Omega'')\\ \chi(u_\varepsilon,T)>1-\eta}} \frac{1}{\varepsilon^2}XY_\varepsilon(v_\varepsilon,T)\\
&\geq (1-\lambda)\frac{1}{\varepsilon^2}XY_\varepsilon(v_\varepsilon,\Omega'') - (1-\lambda)6\overline{M}\,.
\end{aligned}
$$

In conclusion, applying Remark 2.5,

$$
C \geq \frac{1}{\varepsilon^2}XY_\varepsilon(v_\varepsilon,\Omega'') \geq \sqrt{3}\int_{\Omega'}|\nabla \hat{v}_\varepsilon|^2\,\mathrm{d}x\,.
$$

From this we deduce that, up to a subsequence, $\hat{v}_\varepsilon \rightharpoonup v$ in $H^1(\Omega';\mathbb{R}^2)$ and $\hat{v}_\varepsilon \to v$ a.e. in Ω', with $v\in H^1(\Omega';\mathbb{R}^2)$. To prove that $|v|=1$ we apply Lemma 2.6 to infer that

$$
\int_{\Omega'}(1-|\hat{v}_\varepsilon|^2)^2\mathrm{d}x \leq CXY_\varepsilon(v_\varepsilon,\Omega) \leq C\varepsilon^2 \to 0
$$

to obtain, due to Fatou's Lemma, that $|v|=1$ a.e. in Ω''.

By a diagonal argument we find a $v\in H^1(\Omega;\mathbb{S}^1)$ and we extract a subsequence such that for every $\Omega'\subset\subset\Omega$ we have $\hat{v}_\varepsilon\rightharpoonup v$ in $H^1(\Omega';\mathbb{R}^2)$. Finally, $\hat{v}_\varepsilon\to v$ in $L^2(\Omega;\mathbb{R}^2)$ by the Dominated Convergence Theorem.

Let us prove *ii)*. We let $\Omega'\subset\subset\Omega''\subset\subset\Omega$ and $\lambda\in(0,1)$, $\eta\in(0,1)$ as in the proof of *i)*. We prove in the same way that $\#\{T\in\mathcal{T}_\varepsilon(\Omega''):\chi(u_\varepsilon,T)\leq 1-\eta\}=\overline{M}$ with $\overline{M}$ possibly depending on η and Ω''. Let $b^1_\varepsilon,\dots,b^{\overline{M}}_\varepsilon$ be the barycenters of the triangles in $\{T\in\mathcal{T}_\varepsilon(\Omega''):\chi(u_\varepsilon,T)\leq 1-\eta\}$. There exist $b^1,\dots,b^M\in\Omega$ with $M\leq\overline{M}$ such that, up to a subsequence, each of the points $b^1_\varepsilon,\dots,b^{\overline{M}}_\varepsilon$ converges to one of the points in $\{b^1,\dots,b^M\}$. Let us fix $\rho>0$. For ε small enough, every triangle $T\in\mathcal{T}_\varepsilon\big(\Omega''\setminus\bigcup_{h=1}^M B_\rho(b^h)\big)$ satisfies $\chi(u_\varepsilon,T)>1-\eta$. In particular, Lemma 2.8 and Remark 2.5 yield

$$
\begin{aligned}
\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon,\Omega) &\geq \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}E_\varepsilon\Big(u_\varepsilon,\Omega''\setminus\bigcup_{h=1}^M B_\rho(b^h)\Big)\\
&\geq (1-\lambda)\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}XY_\varepsilon\Big(v_\varepsilon,\Omega''\setminus\bigcup_{h=1}^M B_\rho(b^h)\Big)\\
&\geq (1-\lambda)\liminf_{\varepsilon\to0}\sqrt{3}\int_{\Omega'\setminus\bigcup_{h=1}^M B_{2\rho}(b^h)}|\nabla\hat{v}_\varepsilon|^2\,\mathrm{d}x\\
&\geq (1-\lambda)\sqrt{3}\int_{\Omega'\setminus\bigcup_{h=1}^M B_{2\rho}(b^h)}|\nabla v|^2\,\mathrm{d}x\,,
\end{aligned}
$$

where we used that $\hat{v}_\varepsilon\rightharpoonup v$ in $H^1(\Omega';\mathbb{R}^2)$. The claim is proven by letting, in the order, $\rho\to0$, $\lambda\to0$, and $\Omega'\nearrow\Omega$.

Let us prove *iii)*. Let $v\in H^1(\Omega;\mathbb{S}^1)$. Thanks to the regularity of the boundary we find $\widetilde{\Omega}\supset\supset\Omega$ open, bounded set with Lipschitz boundary and we extend v to a map in $H^1(\widetilde{\Omega};\mathbb{S}^1)$, which we still denote, with a slight abuse of notation, by v. This can be achieved via a reflection argument in an open neighborhood of $\partial\Omega$. More details can be found, *e.g.*, in [20, Step 2 in proof of Proposition 4.3]. For the moment, let us assume that $v\in C^\infty(\widetilde{\Omega};\mathbb{S}^1)\cap H^1(\widetilde{\Omega};\mathbb{S}^1)$. Later we will prove the result for a generic $v\in H^1(\widetilde{\Omega};\mathbb{S}^1)$ with a regularization argument. We define the discrete spin field $u_\varepsilon\in\mathcal{SF}_\varepsilon$

as follows:

$$v_\varepsilon(\varepsilon i) := v(\varepsilon i)\,, \quad v_\varepsilon(\varepsilon j) := v(\varepsilon j)\,, \quad v_\varepsilon(\varepsilon k) := v(\varepsilon k)\,,$$
$$u_\varepsilon(\varepsilon i) := v_\varepsilon(\varepsilon i)\,, \quad u_\varepsilon(\varepsilon j) := R[\tfrac{2\pi}{3}](v_\varepsilon(\varepsilon j))\,, \quad u_\varepsilon(\varepsilon k) := R[-\tfrac{2\pi}{3}](v_\varepsilon(\varepsilon k))\,,$$

for $\varepsilon i \in \mathcal{L}^1_\varepsilon \cap \widetilde{\Omega}$, $\varepsilon j \in \mathcal{L}^2_\varepsilon \cap \widetilde{\Omega}$, $\varepsilon k \in \mathcal{L}^3_\varepsilon \cap \widetilde{\Omega}$. For points of $\mathcal{L}_\varepsilon$ outside $\widetilde{\Omega}$, we define u_ε arbitrarily.

Let $\hat{v}_\varepsilon$ be the affine interpolation of v_ε and let us prove that $\hat{v}_\varepsilon \to v$ in $L^2(\Omega;\mathbb{R}^2)$ and

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon,\Omega) \le \sqrt{3}\int_\Omega |\nabla v|^2 \,\mathrm{d}x\,. \tag{4.1}$$

Let $\Omega \subset\subset U \subset\subset \widetilde{\Omega}$ and let $T \in \mathcal{T}_\varepsilon(U)$. Then, for $x \in T$ we have that $|\hat{v}_\varepsilon(x) - v(x)| \le 3\|\nabla v\|_{L^\infty(U)}\varepsilon$. This yields $\|\hat{v}_\varepsilon - v\|_{L^2(\Omega)} \to 0$.

Let now $\alpha \in \{1,2,3\}$ and let $\varepsilon i, \varepsilon j$ be two vertices of $T \in \mathcal{T}_\varepsilon(\Omega)$ (not necessarily $\varepsilon i \in \mathcal{L}^1_\varepsilon$ and $\varepsilon j \in \mathcal{L}^2_\varepsilon$) with $j - i = \hat{e}_\alpha$. By a Taylor expansion there exists ξ belonging to the segment $[\varepsilon i, \varepsilon j]$ such that

$$\begin{aligned}
|\nabla \hat{v}_\varepsilon(x)\hat{e}_\alpha - \nabla v(x)\hat{e}_\alpha| &= \Big|\frac{v(\varepsilon j) - v(\varepsilon i)}{\varepsilon} - \nabla v(x)\hat{e}_\alpha\Big| \\
&= \Big|\nabla v(\varepsilon i)\hat{e}_\alpha + \frac{1}{2}\mathrm{D}^2 v(\xi)(\varepsilon j - \varepsilon i)\cdot(j-i) - \nabla v(x)\hat{e}_\alpha\Big| \\
&\le \Big|\nabla v(\varepsilon i)\hat{e}_\alpha - \nabla v(x)\hat{e}_\alpha\Big| + \frac{1}{2}\|\mathrm{D}^2 v\|_{L^\infty(U)}\varepsilon \\
&\le C\|\mathrm{D}^2 v\|_{L^\infty(U)}\varepsilon
\end{aligned}$$

for every $x \in T$, which yields $\|\nabla\hat{v}_\varepsilon - \nabla v\|_{L^2(\Omega)} \to 0$. Let us fix $\lambda \in (0,1)$ and let $\eta \in (0,1)$ be as in Lemma 2.8. Let $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\Omega)$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, $\varepsilon k \in \mathcal{L}^3_\varepsilon$. For ε small enough we have that

$$\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i), v_\varepsilon(\varepsilon j)) \le \frac{\pi}{2}|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)| \le \frac{\pi}{2}\|\nabla v\|_{L^\infty(\Omega)}\varepsilon < \min\big\{\eta, \tfrac{\pi}{3}\big\}\,,$$
$$\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i), v_\varepsilon(\varepsilon k)) \le \frac{\pi}{2}|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon k)| \le \frac{\pi}{2}\|\nabla v\|_{L^\infty(\Omega)}\varepsilon < \min\big\{\eta, \tfrac{\pi}{3}\big\}\,.$$

In particular, this implies that $\chi(u_\varepsilon) > 0$, as $(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j), u_\varepsilon(\varepsilon k))$ are in a counterclockwise order (see [9, Remark 2.3]). By Lemma 2.8, by Remark 2.5, and since $\|\nabla\hat{v}_\varepsilon - \nabla v\|_{L^2(\Omega)} \to 0$,

$$\begin{aligned}
\limsup_{\varepsilon\to0}\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon,\Omega) &\le (1+\lambda)\limsup_{\varepsilon\to0}\frac{1}{\varepsilon^2}XY_\varepsilon(v_\varepsilon,\Omega) \le (1+\lambda)\limsup_{\varepsilon\to0}\sqrt{3}\int_\Omega|\nabla\hat{v}_\varepsilon|^2\,\mathrm{d}x \\
&\le (1+\lambda)\sqrt{3}\int_\Omega |\nabla v|^2\,\mathrm{d}x\,.
\end{aligned}$$

Letting $\lambda \to 0$ we conclude the proof of (4.1). Let us prove that $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$. From (4.1) we get that $\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon,\Omega) \le C$. Using Lemma 4.1 and using the fact that $\chi(u_\varepsilon) > 0$ (independent of the subsequence), we conclude that $\chi(u_\varepsilon) \to 1$ in $L^1(\Omega)$.

We assume now that $v \in H^1(\widetilde{\Omega};\mathbb{S}^1)$ and we regularize it. By Schoen-Uhlenbeck's approximation theorem for Sobolev maps between manifolds [43, Section 4], there exists a sequence $v_n \in C^\infty(\widetilde{\Omega};\mathbb{S}^1) \cap H^1(\widetilde{\Omega};\mathbb{S}^1)$ such that $\|v_n - v\|_{H^1(\widetilde{\Omega};\mathbb{R}^2)} \le \frac{1}{n}$ (see also [32, 5.1, Theorem 3 and Remark 1]). Then we conclude the proof of the limsup inequality by a standard diagonal argument. $\square$

A consequence of the Γ-limit result in the bulk scaling is the following lower bound for the energy under a degree constraint on the spin field. To properly set the constraint, we define the set of admissible spin fields with degree d in an annulus $A_{r,R}$ by

$$\mathrm{Adm}^{\varepsilon}_{r,R}(d) := \Big\{u \in \mathcal{SF}_\varepsilon : \mu_v(T) = 0 \text{ for every } T \in \mathcal{T}_\varepsilon(\mathbb{R}^2) \text{ with } T \cap A_{r,R} \neq \emptyset\,,\ \mu_v(B_r) = d\Big\}\,,$$

where $v \in \mathcal{SF}_\varepsilon$ is the auxiliary spin field associated to u defined as in (2.5).

Proposition 4.3. *For every $0 < r < R$, we have*

$$\liminf_{\varepsilon\to 0}\ \inf\Big\{\frac{1}{\varepsilon^2}E_\varepsilon(u, A_{r,R}) : u \in \mathrm{Adm}^{\varepsilon}_{r,R}(d)\Big\} \geq 2\sqrt{3}\pi|d|^2 \log\frac{R}{r}\,.$$

Proof. For every $\varepsilon > 0$ let $u_\varepsilon \in \mathrm{Adm}^{\varepsilon}_{r,R}(d)$ be such that

$$\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon, A_{r,R}) \leq \inf\Big\{\frac{1}{\varepsilon^2}E_\varepsilon(u, A_{r,R}) : u \in \mathrm{Adm}^{\varepsilon}_{r,R}(d)\Big\} + \varepsilon\,. \tag{4.2}$$

Without loss of generality, we assume that $\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon, A_{r,R})$ is equibounded. By Remark 3.9, there exists $\eta' \in (0,1)$ such that $\chi(u_\varepsilon) > -1+\eta'$ in $A_{r,R}$. Let $r < r' < R' < R$. By Theorem 4.2-*i)*, up to a subsequence, either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in $L^1(A_{r',R'})$. Since $\chi(u_\varepsilon) > -1+\eta'$, the latter possibility is ruled out. Via a diagonal argument, we obtain that $\chi(u_\varepsilon) \to 1$ in $L^1(A_{r',R'})$ for every $r < r' < R' < R$. By Theorem 4.2-*i)* and via a diagonal argument we find $v \in H^1(A_{r,R};\mathbb{S}^1)$ such that $\hat v_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(A_{r,R};\mathbb{R}^2)$, up to a subsequence that we do not relabel.

Let us prove that $\deg(v, \partial B_\rho) = d$ for every $\rho \in [r,R]$. By (4.2), by Theorem 4.2–*ii)*, since $2|j(v)| = |\nabla v|$, and (3.5), this yields that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\ \inf\Big\{\frac{1}{\varepsilon^2}E_\varepsilon(u, A_{r,R}) : u \in \mathrm{Adm}^{\varepsilon}_{r,R}(d)\Big\} \geq \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}E_\varepsilon(u_\varepsilon, A_{r,R}) \geq \sqrt{3}\int_{A_{r,R}}|\nabla v|^2\,\mathrm{d}x\\
&\geq 4\sqrt{3}\int_r^R\int_{\partial B_\rho}\big|j(v)|_{\partial B_\rho}\cdot\tau_{\partial B_\rho}\big|^2\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}\rho \geq \sqrt{3}\int_r^R\frac{2}{\pi\rho}\Big|\int_{\partial B_\rho} j(v)|_{\partial B_\rho}\cdot\tau_{\partial B_\rho}\,\mathrm{d}\mathcal{H}^1\Big|^2\,\mathrm{d}\rho\\
&= 2\sqrt{3}\pi\int_r^R\frac{1}{\rho}|d|^2\,\mathrm{d}\rho = 2\sqrt{3}\pi|d|^2\log\frac{R}{r}\,.
\end{aligned}$$

To prove that $\deg(v, \partial B_\rho) = d$ for every $\rho \in [r,R]$, by (3.4) it is enough to show that $\deg(v, \partial B_r) = d$. Let $\widehat v_\varepsilon \colon \mathbb{R}^2 \to \mathbb{S}^1$ be the map associated to v_ε given by Remark 3.2. Let $r < r' < R' < R$. We claim that $j(\widehat v_\varepsilon) - j(\hat v_\varepsilon) \to 0$ in $L^1(A_{r',R'};\mathbb{R}^2)$. Indeed, given $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(A_{r,R})$, since $\mu_{v_\varepsilon}(T) = 0$, we get $\widehat v_\varepsilon = \exp(\iota\hat\phi_\varepsilon)$, where $\hat\phi_\varepsilon$ is the affine function in T with $v_\varepsilon(x) = \exp(\iota\hat\phi_\varepsilon(x))$ for $x \in \{\varepsilon i, \varepsilon j, \varepsilon k\}$, see Remark 3.2. Let $\alpha \in \{1,2,3\}$ and let $j - i = \hat e_\alpha$. For every $x \in T$ we have that

$$\begin{aligned}
|\nabla\widehat v_\varepsilon(x)\hat e_\alpha - \nabla\hat v_\varepsilon(x)\hat e_\alpha| &= |\exp(\iota\hat\phi_\varepsilon(x))^\perp\nabla\hat\phi_\varepsilon(x)\cdot\hat e_\alpha - \nabla\hat v_\varepsilon(x)\hat e_\alpha|\\
&= \Big|\exp(\iota\hat\phi_\varepsilon(x))^\perp\frac{\phi_\varepsilon(\varepsilon j) - \phi_\varepsilon(\varepsilon i)}{\varepsilon} - \frac{\exp(\iota\phi_\varepsilon(\varepsilon j)) - \exp(\iota\phi_\varepsilon(\varepsilon i))}{\varepsilon}\Big|\\
&= \Big|\exp(\iota\hat\phi_\varepsilon(x))^\perp\frac{\phi_\varepsilon(\varepsilon j) - \phi_\varepsilon(\varepsilon i)}{\varepsilon} - \exp(\iota\xi_{i,j})^\perp\frac{\phi_\varepsilon(\varepsilon j) - \phi_\varepsilon(\varepsilon i)}{\varepsilon}\Big|\\
&= |\exp(\iota\hat\phi_\varepsilon(x))^\perp - \exp(\iota\xi_{i,j})^\perp||\nabla\hat\phi_\varepsilon(x)\cdot\hat e_\alpha|\\
&\leq |\hat\phi_\varepsilon(x) - \xi_{i,j}||\nabla\widehat v_\varepsilon(x)\hat e_\alpha| \leq \varepsilon|\nabla\hat\phi_\varepsilon(x)||\nabla\widehat v_\varepsilon(x)\hat e_\alpha| \leq \varepsilon|\nabla\widehat v_\varepsilon(x)|^2,
\end{aligned}$$

where $\xi_{i,j}$ belongs to the segment $[\phi_\varepsilon(\varepsilon i), \phi_\varepsilon(\varepsilon j)]$. Moreover, by a straightforward computation[6] one shows that for every $x \in T$

$$|\widehat v_\varepsilon(x) - \hat v_\varepsilon(x)| \leq C\varepsilon|\nabla\hat\phi_\varepsilon(x)| = C\varepsilon|\nabla\widehat v_\varepsilon(x)|\,.$$

[6] *e.g.*, in the case $j - i = \hat e_1$ and $k - i = \hat e_2$, one writes $x = \varepsilon i + s\varepsilon\hat e_1 + t\varepsilon\hat e_2$ with $s,t \in [0,1]$, $\widehat v_\varepsilon(x) = \exp(\iota\hat\phi_\varepsilon(x)) = \exp(\iota\phi_\varepsilon(\varepsilon i))\exp\big(\iota s(\phi_\varepsilon(\varepsilon j) - \phi_\varepsilon(\varepsilon i))\big)\exp\big(\iota t(\phi_\varepsilon(\varepsilon k) - \phi_\varepsilon(\varepsilon i))\big)$,2 and $\hat v_\varepsilon(x) = \exp(\iota\phi_\varepsilon(\varepsilon i)) + s\big(\exp(\iota\phi_\varepsilon(\varepsilon j)) - \exp(\iota\phi_\varepsilon(\varepsilon i))\big) + t\big(\exp(\iota\phi_\varepsilon(\varepsilon k)) - \exp(\iota\phi_\varepsilon(\varepsilon i))\big)$.

The previous inequalities and (3.7) yield

$$\begin{aligned}\int_T |j(\widehat{v}_\varepsilon) - j(\hat{v}_\varepsilon)|\,\mathrm{d}x &\leq C\int_T |\widehat{v}_\varepsilon - \hat{v}_\varepsilon||\nabla\widehat{v}_\varepsilon| + |\hat{v}_\varepsilon||\nabla\widehat{v}_\varepsilon - \nabla\hat{v}_\varepsilon|\,\mathrm{d}x\\ &\leq C\int_T |\widehat{v}_\varepsilon - \hat{v}_\varepsilon||\nabla\widehat{v}_\varepsilon| + |\nabla\widehat{v}_\varepsilon - \nabla\hat{v}_\varepsilon|\,\mathrm{d}x \leq C\varepsilon\int_T |\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x\\ &\leq \frac{C}{\varepsilon}XY_\varepsilon(v_\varepsilon,T) \leq \frac{C}{\varepsilon}E_\varepsilon(u_\varepsilon,T)\,,\end{aligned}$$

where in the last inequality we used the fact that $\mu_{v_\varepsilon}(T) = 0$ and we applied Lemma 3.10. Summing over all triangles $T \in \mathcal{T}_\varepsilon(A_{r,R})$ that intersect $A_{r',R'}$ we conclude that

$$\int_{A_{r',R'}} |j(\widehat{v}_\varepsilon) - j(\hat{v}_\varepsilon)|\,\mathrm{d}x \leq \frac{C}{\varepsilon}E_\varepsilon(u_\varepsilon, A_{r,R}) \leq C\varepsilon \to 0\,,$$

which in turn implies that $j(\widehat{v}_\varepsilon) \to j(v)$ in $L^1(A_{r',R'};\mathbb{R}^2)$. We are now in a position to prove that $\deg(v,\partial B_r) = d$. Let $\psi(x) := 1 - \min\{\frac{1}{R'-r'}\operatorname{dist}(x,B_{r'}),1\}$. Using the convergence of $j(\widehat{v}_\varepsilon)$ together with the fact that $v \in H^1(A_{r,R};\mathbb{S}^1)$, by (3.5) and (3.4) we have that

$$\begin{aligned}\pi d = \pi\mu_{v_\varepsilon}(B_r) = \int_{B_R} J(\widehat{v}_\varepsilon)\psi\,\mathrm{d}x = -\int_{A_{r',R'}} j(\widehat{v}_\varepsilon)\cdot\nabla^\perp\psi\,\mathrm{d}x \to\\ \to -\int_{A_{r',R'}} j(v)\cdot\nabla^\perp\psi\,\mathrm{d}x = \frac{1}{R'-r'}\int_{r'}^{R'}\int_{\partial B_\rho} j(v)|_{\partial B_\rho}\cdot\tau_{\partial B_\rho}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}\rho = \pi\deg(v,\partial B_r)\,,\end{aligned}$$

which concludes the proof. □

5. Ball Construction

In this section we prove a variant of the well-known ball construction [40, 33] suited for our arguments.

Let $\mathcal{B} = \{B_{r_i}(x_i)\}_{i=1}^N$ be a finite family of open balls such that $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset$ for every $i,j \in \{1,\dots,N\}$, $i \neq j$. Let $\mu = \sum_{i=1}^N d_i\delta_{x_i}$, $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \mathbb{R}^2$, and let $\mathcal{E}(\mathcal{B},\mu,\cdot)\colon \mathcal{A}(\mathbb{R}^2) \to [0,+\infty]$ be an increasing set function satisfying the following properties:

- (B1) $\mathcal{E}(\mathcal{B},\mu,U\cup V) \geq \mathcal{E}(\mathcal{B},\mu,U) + \mathcal{E}(\mathcal{B},\mu,V)$ for every $U,V \in \mathcal{A}(\mathbb{R}^2)$ such that $U \cap V = \emptyset$.
- (B2) for every annulus $A_{r,R}(x) = B_R(x)\setminus\overline{B}_r(x)$, $0 < r < R$ with $A_{r,R}(x) \cap \bigcup_{i=1}^N B_{r_i}(x_i) = \emptyset$, it holds
$$\mathcal{E}(\mathcal{B},\mu,A_{r,R}(x)) \geq c_0|\mu(B_r(x))|\log\frac{R}{r}\,, \tag{5.1}$$
for some constant $c_0 > 0$.

Given a ball B, we let $r(B)$ denote its radius. For a family of balls $\mathcal{B}$, we let $\mathcal{R}(\mathcal{B}) := \sum_{B\in\mathcal{B}} r(B)$.

Lemma 5.1 (Ball construction). *Let $\mathcal{B}$, μ, and $\mathcal{E}$ be as above. Let $\sigma > 0$. Then there exists a one-parameter family $\{\mathcal{B}(t)\}_{t\geq 0}$ of balls such that*

1. *the following inclusions hold:*
$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}(t_1)} B \subset \bigcup_{B\in\mathcal{B}(t_2)} B\,,\quad \textit{for every } 0 \leq t_1 \leq t_2\,;$$
2. $\overline{B}\cap\overline{B}' = \emptyset$ *for every* $B, B' \in \mathcal{B}(t)$, $B \neq B'$, *and* $t \geq 0$*;*

(3) *for every* $0 \le t_1 \le t_2$ *and every* $U \in \mathcal{A}(\mathbb{R}^2)$ *we have that*

$$\mathcal{E}\Big(\mathcal{B},\mu,U\cap\Big(\bigcup_{B\in\mathcal{B}(t_2)}B\setminus\bigcup_{B\in\mathcal{B}(t_1)}\overline{B}\Big)\Big)\ge c_0\sum_{\substack{B\in\mathcal{B}(t_2)\\ B\subset U}}|\mu(B)|\log\frac{1+t_2}{1+t_1}\,;$$

(4) $|\mu|(B_{r+\sigma}(x)\setminus B_{r-\sigma}(x))=0$ *for every* $B=B_r(x)\in\mathcal{B}(t)$ *and for every* $t\ge 0$*;*
(5) *for every* $t\ge 0$ *we have that* $\mathcal{R}(\mathcal{B}(t))\le(1+t)\,(\mathcal{R}(\mathcal{B})+N\sigma)$*;*
(6) *for every* $t\ge 0$*,* $B\in\mathcal{B}$*, and* $B'\in\mathcal{B}(t)$ *with* $B\subset B'$ *we have that* $r(B')\ge(1+t)r(B)$*.*

Proof. In order to construct the family $\mathcal{B}(t)$, we closely follow the strategy of the ball construction due to Sandier [40] and Jerrard [33]. We adapt the argument in order to be sure that condition (4) holds true, *i.e.*, that the measure μ is supported far from the boundaries of the balls of the constructed family.

The ball construction consists in letting the balls alternatively expand and merge into each other. We let $T_0 := 0$ and we define the family $\mathcal{B}(T_0)$ by distinguishing the following two cases: If $\overline{B}_{r_i+\sigma}(x_i)\cap\overline{B}_{r_j+\sigma}(x_j)\neq\emptyset$ for some of the starting balls with $i,j\in\{1,\dots,N\}$, $i\neq j$, then the construction starts with a merging phase and $T_0=0$ is the first merging time. This phase consists in identifying a suitable partition $\{S^0_j\}_{j=1,\dots,N_0}$ of the family $\{B_{r_i+\sigma}(x_i)\}_{i=1}^N$ which satisfies the following: for each $j\in\{1,\dots,N_0\}$ there exists a ball $B_{r^0_j}(x^0_j)$ which contains all the balls in S^0_j and such that

i) $\overline{B}_{r^0_j}(x^0_j)\cap\overline{B}_{r^0_\ell}(x^0_\ell)=\emptyset$ for every $j,\ell\in\{1,\dots,N_0\}$, $j\neq\ell$,
ii) $r^0_j\le\sum_{B\in S^0_j}r(B)$.

We then define

$$\mathcal{B}(T_0):=\{B_{r^0_j}(x^0_j)\colon j=1\,,\dots\,,N_0\}\,. \tag{5.2}$$

If, instead, $\overline{B}_{r_i+\sigma}(x_i)\cap\overline{B}_{r_j+\sigma}(x_j)=\emptyset$ for every $i,j\in\{1,\dots,N\}$, $i\neq j$, then we let $N_0:=N$, $B_{r^0_j}(x^0_j):=B_{r_j+\sigma}(x_j)$ for $j=1,\dots,N$ in (5.2), and we start with an expansion phase. During this first expansion phase, we let the balls expand without changing their centres, in such a way that the new radius $r^0_j(t)$ of the ball centred in x^0_j satisfies

$$\frac{r^0_j(t)}{r^0_j}=\frac{1+t}{1+T_0}=1+t\,,$$

for every $t\ge T_0=0$ and every $j\in\{1,\dots,N_0\}$. We continue the first expansion phase as a long as

$$\overline{B}_{r^0_j(t)}(x_j)\cap\overline{B}_{r^0_\ell(t)}(x_\ell)=\emptyset\ \text{ for every } j,\ell\in\{1,\dots,N_0\}\,,\ j\neq\ell\,, \tag{5.3}$$

and we let T_1 denote the smallest $t\ge T_0=0$ such that (5.3) is violated. (Note that $T_1>0$.) At time T_1, following the same procedure described above, a merging phase starting from the balls $\{B_{r^0_j(T_1)}(x^0_j)\}_{j=1}^{N_0}$ begins, that defines a new family of balls $\{B_{r^1_j}(x^1_j)\}_{j=1}^{N_1}$.

We iterate this procedure by alternating merging and expansion phases to obtain the following: a discrete set of times $\{T_0,\dots,T_K\}$, $K\le N$; for each $k\in\{1,\dots,K\}$, a partition $\{S^k_j\}_{j=1}^{N_k}$ of $\{B_{r^{k-1}_j(T_k)}(x^{k-1}_j)\}_{j=1}^{N_{k-1}}$; for each subclass S^k_j, a ball $B_{r^k_j}(x^k_j)$, which contains the balls in S^k_j and such that the following properties are satisfied:

i) $\overline{B}_{r^k_j}(x^k_j)\cap\overline{B}_{r^k_\ell}(x^k_\ell)=\emptyset$ for every $j,\ell\in\{1,\dots,N_k\}$, $j\neq\ell$,
ii) $r^k_j\le\sum_{B\in S^k_j}r(B)$.

For $t\ge 0$, the family $\mathcal{B}(t)$ is given by $\{B_{r^k_j(t)}(x^k_j)\}_{j=1}^{N_k}$ for $t\in[T_k,T_{k+1})$ and $k=0,\dots,K$, where we set $T_{K+1}:=+\infty$ (in other words, it consists of a single expanding ball for $t\ge T_K$). For every

$t \in [T_k, T_{k+1})$ and for $j = 1, \dots, N_k$, the radii satisfy

$$\frac{r_j^k(t)}{r_j^k} = \frac{1+t}{1+T_k}\,. \tag{5.4}$$

Note that

$$\mathcal{R}(\mathcal{B}(T_0)) = \sum_{j=1}^{N_0} r_j^0 \leq \mathcal{R}(\mathcal{B}) + N\sigma\,. \tag{5.5}$$

It remains to check that conditions (1)–(5) hold true. By construction, it is clear that (1) and (2) are satisfied.

Let us prove (3). We note that, by (1),

$$\sum_{\substack{B \in \mathcal{B}(\tau_1)\\ B \subset U}} |\mu(B)| \geq \sum_{\substack{B \in \mathcal{B}(\tau_2)\\ B \subset U}} |\mu(B)| \quad \text{for every } 0 < \tau_1 < \tau_2\,. \tag{5.6}$$

Let $t_1 < \bar{t} < t_2$. In view of (5.6), since $\mathcal{E}$ is an increasing sub-additive set-function, if we show that (3) holds true for the pairs $(t_1, \bar{t})$ and $(\bar{t}, t_2)$, then (3) also follows for t_1 and t_2. Therefore we can assume, without loss of generality, that $T_k \notin (t_1, t_2)$ for every $k = 1, \dots, K$. Let $t_1 < \tau < t_2$ and let $B \in \mathcal{B}(\tau)$. Then, there exists a unique ball $B' \in \mathcal{B}(t_1)$ such that $B' \subset B$. By construction $\mu(B) = \mu(B')$ and, by (5.1), we have that

$$\mathcal{E}(\mathcal{B}, \mu, B \setminus B') \geq c_0 |\mu(B')| \log \frac{1+\tau}{1+t_1} = c_0 |\mu(B)| \log \frac{1+\tau}{1+t_1}\,.$$

Summing up over all $B \in \mathcal{B}(\tau)$ with $B \subset U$ and using (5.6) yields

$$\mathcal{E}\Big(\mathcal{B}, \mu, U \cap \Big(\bigcup_{B \in \mathcal{B}(t_2)} B \setminus \bigcup_{B \in \mathcal{B}(t_1)} B \Big)\Big) \geq c_0 \sum_{\substack{B \in \mathcal{B}(\tau)\\ B \subset U}} |\mu(B)| \log \frac{1+\tau}{1+t_1} \geq c_0 \sum_{\substack{B \in \mathcal{B}(t_2)\\ B \subset U}} |\mu(B)| \log \frac{1+\tau}{1+t_1}\,.$$

Property (3) follows by letting $\tau \to t_2$.

Let us prove (4). Let $t \geq 0$ and let $B = B_r(x) \in \mathcal{B}(t)$. Let us fix an initial ball $B_{r_i}(x_i)$. By construction, $B_{r_i+\sigma}(x_i)$ is contained in some ball $B_{r'}(y) \in \mathcal{B}(t)$, *i.e.*, $B_{r_i}(x_i) \subset B_{r'-\sigma}(y)$. Then $B_{r_i}(x_i) \cap B_{r+\sigma}(x) \subset B_{r-\sigma}(x)$, since condition (2) implies that $\overline{B}_{r'-\sigma}(y) \cap \overline{B}_{r+\sigma}(x) = \emptyset$ whenever $y \neq x$. This yields

$$B_{r+\sigma}(x) \cap \bigcup_{i=1}^N B_{r_i}(x_i) \subset B_{r-\sigma}(x) \quad \Longrightarrow \quad B_{r+\sigma}(x) \setminus B_{r-\sigma}(x) \subset B_{r+\sigma}(x) \setminus \bigcup_{i=1}^N B_{r_i}(x_i)\,.$$

Therefore

$$|\mu|(B_{r+\sigma}(x) \setminus B_{r-\sigma}(x)) \leq |\mu|\Big(\mathbb{R}^2 \setminus \bigcup_{i=1}^N B_{r_i}(x_i)\Big) = 0\,,$$

where we used the fact that μ is supported on $\{x_1, \dots, x_N\}$. This proves (4).

To prove (5), we start by observing that, by (5.4),

$$\mathcal{R}(\mathcal{B}(t)) = \sum_{j=1}^{N_k} r_j^k(t) = \sum_{j=1}^{N_k} \frac{1+t}{1+T_k} r_j^k = \frac{1+t}{1+T_k} \mathcal{R}(\mathcal{B}(T_k))$$

for every $t \in [T_k, T_{k+1})$ and every $k \in \{0, \dots, K\}$. It thus suffices to show that $\mathcal{R}(\mathcal{B}(T_k)) \leq (1+T_k)(\mathcal{R}(\mathcal{B}) + N\sigma)$ for every $k \in \{0, \dots, K\}$. For $k = 0$ this is a direct consequence of (5.5).

For $k \geq 1$, it follows inductively by applying (5) for $t \in [T_{k-1}, T_k)$ and observing that

$$\mathcal{R}(\mathcal{B}(T_k)) = \sum_{j=1}^{N_k} r_j^k \leq \sum_{j=1}^{N_k} \sum_{B \in S_j^k} r(B) = \sum_{j=1}^{N_{k-1}} r_j^{k-1}(T_k) = \limsup_{t \nearrow T_k} \mathcal{R}(\mathcal{B}(t)) \leq (1 + T_k)\left(\mathcal{R}(\mathcal{B}) + N\sigma\right) ,$$

which follows from *ii*).

Finally, property (6) holds true by construction. □

6. Proof of Theorem 1.1-*i)* and *ii)*

In this section we prove Theorem 1.1-*i)* and *ii)*. We start by proving a first estimate on the XY-energy of the auxiliary spin field, from which, however, the compactness statement does not follow straightforwardly.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, and connected let $\Omega' \subset\subset \Omega$ with Lipschitz boundary. Let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log\varepsilon|$ and $\chi(u_\varepsilon) \to 1$, and let $v_\varepsilon \in \mathcal{SF}_\varepsilon$ be the auxiliary spin field defined as in* (2.5)*. There exists a constant $C > 0$ depending on Ω' such that*

$$XY_\varepsilon(v_\varepsilon, \Omega') \leq C\varepsilon^2|\log\varepsilon|^2,$$

for ε sufficiently small.

Proof. Let $\Omega' \subset\subset \Omega$ with Lipschitz boundary and assume $\mathrm{dist}(\Omega', \partial\Omega) > \sqrt{3}\varepsilon$. Fix $\lambda \in (0, 1)$ and let $\eta \in (0, 1)$ be given by Lemma 2.8. For every $T \in \mathcal{T}_\varepsilon(\Omega')$ with $\chi(u_\varepsilon, T) > 1 - \eta$, by Lemma 2.8 we have that

$$(1 - \lambda)XY_\varepsilon(v_\varepsilon, T) \leq E_\varepsilon(u_\varepsilon, T)\,. \tag{6.1}$$

For $T \in \mathcal{T}_\varepsilon(\Omega')$ with $\chi(u_\varepsilon, T) \leq 1 - \eta$ we estimate

$$XY_\varepsilon(v_\varepsilon, T) \leq 6\varepsilon^2.$$

and we count the number of such triangles. By Lemma 2.4 we obtain that there exists $C > 0$ depending on η and Ω' such that

$$\#\{T \in \mathcal{T}_\varepsilon(\Omega') : \chi(u_\varepsilon, T) \leq 1 - \eta\} \leq \frac{C_\eta}{\varepsilon^4} E_\varepsilon(u_\varepsilon, \Omega)^2 \leq C|\log\varepsilon|^2 \tag{6.2}$$

for ε sufficiently small. Putting together (6.1)–(6.2), we infer that

$$\begin{aligned} XY_\varepsilon(v_\varepsilon, \Omega') &\leq \sum_{\substack{T \in \mathcal{T}_\varepsilon(\Omega') \\ \chi(u_\varepsilon, T) > 1-\eta}} XY_\varepsilon(v_\varepsilon, T) \quad + \sum_{\substack{T \in \mathcal{T}_\varepsilon(\Omega') \\ \chi(u_\varepsilon, T) \leq 1-\eta}} XY_\varepsilon(v_\varepsilon, T) \\ &\leq CE_\varepsilon(u_\varepsilon, \Omega) + C\varepsilon^2|\log\varepsilon|^2 \leq C\varepsilon^2|\log\varepsilon|^2, \end{aligned}$$

thus concluding the proof of the lemma. □

We are now in a position to prove the compactness statement Theorem 1.1-*i)*. Let Ω be an open, bounded set. Let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log\varepsilon|$. The fact that either $\chi(u_\varepsilon) \to 1$ or $\chi(u_\varepsilon) \to -1$ in $L^1(\Omega)$ (up to a subsequence) follows from Lemma 4.1. In the following, we assume that $\chi(u_\varepsilon) \to 1$ and we let $v_\varepsilon \in \mathcal{SF}_\varepsilon$ be the auxiliary spin field defined as in (2.5). We plan to apply the Ball Construction of Lemma 5.1 to the measures μ_{v_ε}.

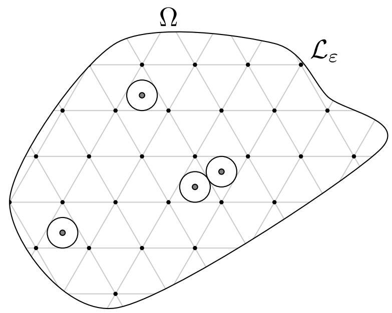

Figure 5. Example of a possible family of balls $\mathcal{B}_\varepsilon$, from which the ball construction starts.

Let us fix $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ with Lipschitz boundary. By Remark 3.1 and by Lemma 6.1, for ε sufficiently small we have that there exists a constant $C > 0$ depending on Ω'' such that

$$\#\mathrm{supp}(\mu_{v_\varepsilon}) \cap \Omega' \leq |\mu_{v_\varepsilon}|(\Omega') \leq \frac{C}{\varepsilon^2} XY_\varepsilon(v_\varepsilon, \Omega'') \leq C|\log \varepsilon|^2. \tag{6.3}$$

We consider the family of balls

$$\mathcal{B}_\varepsilon := \{B_{\frac{\varepsilon}{2\sqrt{3}}}(x) : x \in \mathrm{supp}(\mu_{v_\varepsilon}) \cap \Omega'\}\,. \tag{6.4}$$

Notice that each of these balls is fully contained in a triangle of the lattice, see Figure 5. For every $0 < r < R$ and for every $x \in \mathbb{R}^2$ such that $A_{r,R}(x) \cap \bigcup_{B \in \mathcal{B}_\varepsilon} B = \emptyset$ we set

$$\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{v_\varepsilon}, A_{r,R}(x)) := |\mu_{v_\varepsilon}(B_r(x))| \log \frac{R}{r}\,,$$

and we extend $\mathcal{E}$ to every $A \in \mathcal{A}(\mathbb{R}^2)$ by

$$\begin{aligned}\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{v_\varepsilon}, A) := \sup \Big\{ \sum_{j=1}^N \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{v_\varepsilon}, A^j) : N \in \mathbb{N}\,,\ A^j = A_{r_j,R_j}(x_j)\,,\ A^j \cap \bigcup_{B \in \mathcal{B}_\varepsilon} B = \emptyset\,, \\ A^j \cap A^k = \emptyset \text{ for } j \neq k\,,\ A^j \subset A \text{ for all } j \Big\}\,.\end{aligned} \tag{6.5}$$

We apply Lemma 5.1 with $\sigma = 3\varepsilon$ to $\mathcal{B} = \mathcal{B}_\varepsilon$, $\mu = \mu_{v_\varepsilon}$, and $\mathcal{E}$ defined in (6.5), which satisfy the assumptions (B1) and (B2) with $c_0 = 1$. Hence, there exists a family of balls $\{\mathcal{B}_\varepsilon(t)\}_{t \geq 0}$ satisfying (1)–(6) of Lemma 5.1. Due to (6.3) and (6.4), we have that

$$\mathcal{R}(\mathcal{B}_\varepsilon) \leq C\varepsilon |\log \varepsilon|^2. \tag{6.6}$$

Moreover, by property (6) in Lemma 5.1,

$$r(B) \geq (1+t)\frac{\varepsilon}{2\sqrt{3}} \quad \text{for every } B \in \mathcal{B}_\varepsilon(t)\,. \tag{6.7}$$

In the next lemma we deduce an upper bound for the set function $\mathcal{E}$.

Lemma 6.2. *Let $\mathcal{E}$, $\mathcal{B}_\varepsilon$, and μ_{v_ε} be as above. Let $U_\varepsilon(t) := \bigcup_{B \in \mathcal{B}_\varepsilon(t)} B$ for all $\varepsilon > 0$ and $t \geq 0$. Then we have the following inequalities*

$$\mathcal{E}(\mathcal{B}_\varepsilon, \mu_{v_\varepsilon}, \Omega'' \setminus \overline{U}_\varepsilon(0)) \leq C \int_{\Omega'' \setminus \overline{U}_\varepsilon(0)} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x \leq \frac{C}{\varepsilon^2} E_\varepsilon(u_\varepsilon, \Omega) \leq C|\log \varepsilon|\,. \tag{6.8}$$

Proof. We set $U_\varepsilon := \bigcup_{B\in\mathcal{B}_\varepsilon} B$ and we let $0<r<R$ and $x_0\in\mathbb{R}^2$ be such that $A_{r,R}(x_0)\cap U_\varepsilon=\emptyset$. Since $J(\widehat{v}_\varepsilon)=\pi\mu_{v_\varepsilon}$ and by Stokes' theorem (see also (3.5)), we have that

$$\pi\mu_{v_\varepsilon}(B_s(x_0)) = J(\widehat{v}_\varepsilon)(B_s(x_0)) = \int_{B_s(x_0)} \mathrm{curl}(j(\widehat{v}_\varepsilon))\,\mathrm{d}x = \int_{\partial B_s(x_0)} j(\widehat{v}_\varepsilon)\cdot\tau_{\partial B_s(x_0)}\,\mathrm{d}\mathcal{H}^1$$

for a.e. $s\in(r,R)$.[7] Note that, since $|\widehat{v}_\varepsilon|=1$, we have $2|j(\widehat{v}_\varepsilon)|=|\nabla\widehat{v}_\varepsilon|$. Therefore, by Jensen's inequality,

$$\Big|\int_{\partial B_s(x_0)} j(\widehat{v}_\varepsilon)\cdot\tau_{\partial B_s(x_0)}\,\mathrm{d}\mathcal{H}^1\Big|^2 \le \frac{1}{4}\Big(\int_{\partial B_s(x_0)}|\nabla\widehat{v}_\varepsilon|\,\mathrm{d}\mathcal{H}^1\Big)^2 \le \frac{\pi}{2}s\int_{\partial B_s(x_0)}|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}\mathcal{H}^1.$$

Since $A_{r,R}\cap U_\varepsilon=\emptyset$, $\mu_{v_\varepsilon}(B_s(x_0))=\mu_{v_\varepsilon}(B_r(x_0))\in\mathbb{Z}$ for every $s\in(r,R)$. Thus, the two previous inequalities imply that

$$\frac{2\pi}{s}|\mu_{v_\varepsilon}(B_r(x_0))| \le \frac{2\pi}{s}|\mu_{v_\varepsilon}(B_r(x_0))|^2 \le \int_{\partial B_s(x_0)}|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}\mathcal{H}^1.$$

Integrating in s from r to R, by the coarea formula we obtain

$$\mathcal{E}(\mathcal{B}_\varepsilon,\mu_{v_\varepsilon},A_{r,R}(x_0)) \le C\int_{A_{r,R}(x_0)}|\nabla\widehat{v}_\varepsilon|^2\mathrm{d}x\,. \tag{6.9}$$

Let now $A\in\mathcal{A}(\mathbb{R}^2)$. For all A^j admissible in (6.5), we have $A^j\subset A\setminus\overline{U}_\varepsilon$ and $A^j\cap A^k=\emptyset$ for $j\neq k$. Therefore, using (6.9), we get

$$\sum_j \mathcal{E}(\mathcal{B}_\varepsilon,\mu_{v_\varepsilon},A^j) \le \sum_j C\int_{A^j}|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x \le C\int_{A\setminus\overline{U}_\varepsilon}|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x\,.$$

Taking the supremum over all admissible A^j, we infer that

$$\mathcal{E}(\mathcal{B}_\varepsilon,\mu_{v_\varepsilon},A) \le C\int_{A\setminus\overline{U}_\varepsilon}|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x\,.$$

We are now in a position to prove (6.8). Note that Lemma 5.1 gives

$$\Omega''\setminus\overline{U}_\varepsilon(0) \subset \bigcup_{\substack{T\in\mathcal{T}_\varepsilon(\Omega)\\ |\mu_{v_\varepsilon}|(T)=0}} T$$

thanks to the choice $\sigma=3\varepsilon$. Hence, by Lemma 3.10 and the properties of $\widehat{v}_\varepsilon$, we obtain

$$\begin{aligned}\mathcal{E}(\mathcal{B}_\varepsilon,\mu_{v_\varepsilon},\Omega''\setminus\overline{U}_\varepsilon(0)) &\le C\int_{\Omega''\setminus\overline{U}_\varepsilon(0)}|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x \le C\sum_{\substack{T\in\mathcal{T}_\varepsilon(\Omega)\\ |\mu_{v_\varepsilon}|(T)=0}}\int_T|\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x\\ &\le \frac{C}{\varepsilon^2}\sum_{\substack{T\in\mathcal{T}_\varepsilon(\Omega)\\ |\mu_{v_\varepsilon}|(T)=0}} XY_\varepsilon(v_\varepsilon,T) \le \frac{C}{\varepsilon^2}\sum_{\substack{T\in\mathcal{T}_\varepsilon(\Omega)\\ |\mu_{v_\varepsilon}|(T)=0}} E_\varepsilon(u_\varepsilon,T) \le \frac{C}{\varepsilon^2}E_\varepsilon(u_\varepsilon,\Omega) \le C|\log\varepsilon|\,.\end{aligned}$$

This concludes the proof of (6.8). $\square$

In the next lemma we estimate the number of merging times in the ball construction and show that the trivial estimate of order $|\log\varepsilon|^2$ can be improved to become of order $|\log\varepsilon|$. By inspecting the proof, we get a better insight on the structure of the vorticity measure μ_{v_ε}: the possible $|\log\varepsilon|^2$ short dipoles in the region $\chi(u_\varepsilon)\sim -1$ are annihilated at the first step of the ball construction.

[7] In fact, $\widehat{v}_\varepsilon|_{\partial B_s(x_0)}\in H^1(\partial B_s(x_0);\mathbb{S}^1)$ for every $s\in(r,R)$. See also Footnote 5.

Lemma 6.3. *Let $\mathcal{B}_\varepsilon(t)$ be as above and let*

$$\mathrm{T}_\varepsilon^{\mathrm{merg}} := \{t \in [0,+\infty) : \#\mathcal{B}_\varepsilon(t^+) < \#\mathcal{B}_\varepsilon(t^-)\}$$

denote the set of merging times. Then there exists $M > 0$ such that

$$\#\mathrm{T}_\varepsilon^{\mathrm{merg}} \leq M|\log \varepsilon|\,. \tag{6.10}$$

Proof. We start by proving that there exists $c > 0$ such that

$$E_\varepsilon(u_\varepsilon, B) \geq c\varepsilon^2 \quad \text{for every } B \in \mathcal{B}_\varepsilon(0)\,. \tag{6.11}$$

Given $B = B_r(x) \in \mathcal{B}_\varepsilon(0)$, there exists $T_1 \in \mathcal{T}_\varepsilon(B)$ such that $|\mu_{v_\varepsilon}|(T_1) = 1$. Letting $\eta \in (0,1)$ be given by Remark 3.8, we have that $\chi(u_\varepsilon, T_1) \leq 1 - \eta$ (otherwise, the vorticity of v_ε would be zero in T_1). If additionally $-1 + \eta \leq \chi(u_\varepsilon, T_1) \leq 1 - \eta$, then, by Remark 2.1, $E_\varepsilon(u_\varepsilon, T_1) \geq C_\eta \varepsilon^2$ for some constant $C_\eta > 0$ and thus (6.11) holds true. If, instead, $\chi(u_\varepsilon, T_1) < -1 + \eta$, then we argue as follows. Thanks to the choice $\sigma = 3\varepsilon$, there exists $T' \in \mathcal{T}_\varepsilon(B_r(x) \setminus \overline{B}_{r-\sigma}(x))$. Property (4) in Lemma 5.1 implies that $|\mu_{v_\varepsilon}|(T') = 0$. Letting $\eta' \in (0,1)$ be given by Remark 3.9, we have that $-1 + \eta' \leq \chi(u_\varepsilon, T')$. If $-1 + \eta' \leq \chi(u_\varepsilon, T') \leq 1 - \eta'$, then $E_\varepsilon(u_\varepsilon, T') \geq C_{\eta'}\varepsilon^2$ for some constant $C_{\eta'} > 0$ and thus (6.11) holds true. Then we assume $1 - \eta' < \chi(u_\varepsilon, T')$. We find now a chain of triangles $\{T_1, \dots, T_L = T'\} \subset \mathcal{T}_\varepsilon(B)$ with $T_{\ell+1} \in \mathcal{N}_\varepsilon(T_\ell)$ for all $\ell = 1, \dots, L-1$, see Figure 6. Since $\chi(u_\varepsilon, T_1) < -1 + \eta$ and $1 - \eta' < \chi(u_\varepsilon, T_N)$, there exists $\ell \in \{1, \dots, L-1\}$ such that $\chi(u_\varepsilon, T_\ell) < 0$ and $\chi(u_\varepsilon, T_{\ell+1}) \geq 0$. Then (6.11) follows from Lemma 2.2.

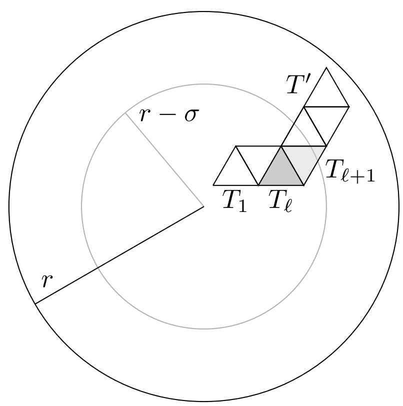

Figure 6. Choice of a chain of triangles $\{T_1, \dots, T_L = T'\} \subset \mathcal{T}_\varepsilon(B)$.

Estimate (6.10) is a consequence of (6.11) since

$$c\varepsilon^2 \#\mathrm{T}_\varepsilon^{\mathrm{merg}} \leq c\varepsilon^2 \#\mathcal{B}_\varepsilon(0) \leq \sum_{B \in \mathcal{B}_\varepsilon(0)} E_\varepsilon(u_\varepsilon, B) \leq E_\varepsilon(u_\varepsilon, \Omega) \leq C\varepsilon^2|\log\varepsilon|\,,$$

hence (6.10) follows. □

Let us fix $p \in (0,1)$. (At the very end of the proof we will let $p \to 1$.) We construct an auxiliary sequence of measures $\mu_{\varepsilon,p}$ such that $\mu_{\varepsilon,p}$ have equibounded mass and $\mu_{\varepsilon,p}$ are close to μ_{v_ε} in the flat norm. For $k = 0, \dots, \lfloor 2M|\log\varepsilon|\rfloor$ we set[8]

$$\beta_p := \exp\Big(\frac{\sqrt{p}(1-\sqrt{p})}{2M}\Big)\,, \quad t_{\varepsilon,p}^k := (\beta_p)^k \varepsilon^{\sqrt{p}-1} - 1\,, \tag{6.12}$$

[8]The choice of these particular expansion times will become clearer later when we deduce (6.22). Similar arguments can be found, *e.g.*, in [27, 3].

and

$$\mathcal{K}_\varepsilon := \Big\{k \in \{1,\dots,\lfloor 2M|\log\varepsilon|\rfloor\} : (t^{k-1}_{\varepsilon,p}, t^k_{\varepsilon,p}] \cap \mathrm{T}^{\mathrm{merg}}_\varepsilon = \emptyset\Big\}. \tag{6.13}$$

By (6.10), we have that $\#\mathcal{K}_\varepsilon \geq M|\log\varepsilon|$. We choose $k_\varepsilon \in \mathcal{K}_\varepsilon$ (depending also on p) such that

$$\int_{\Omega''\cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\setminus \overline{U}_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x \leq \frac{1}{\#\mathcal{K}_\varepsilon} \sum_{k\in\mathcal{K}_\varepsilon} \int_{\Omega''\cap U_\varepsilon(t^{k}_{\varepsilon,p})\setminus \overline{U}_\varepsilon(t^{k-1}_{\varepsilon,p})} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x\,.$$

By conditions (1)–(2) in Lemma 5.1 and by (6.8) we have that

$$\sum_{k\in\mathcal{K}_\varepsilon} \int_{\Omega''\cap U_\varepsilon(t^{k}_{\varepsilon,p})\setminus \overline{U}_\varepsilon(t^{k-1}_{\varepsilon,p})} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x \leq \int_{\Omega''\setminus\overline{U}_\varepsilon(0)} |\nabla \widehat{v}_\varepsilon|^2\,\mathrm{d}x \leq C|\log\varepsilon|\,,$$

whence

$$\int_{\Omega''\cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\setminus \overline{U}_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})} |\nabla \widehat{v}_\varepsilon|^2 \,\mathrm{d}x \leq C\frac{|\log\varepsilon|}{\#\mathcal{K}_\varepsilon} \leq C_1\,. \tag{6.14}$$

We define

$$\mu_{\varepsilon,p} := \sum_{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})} \mu_{v_\varepsilon}(B)\delta_{x_B}\,, \tag{6.15}$$

where we let x_B denote the center of the ball B.

Lemma 6.4. *Let μ_{v_ε} be as above and let $\mu_{\varepsilon,p}$ be the measure defined in* (6.15)*. Then*

$$|\mu_{\varepsilon,p}|(\Omega') \leq \frac{C}{1-\sqrt{p}} =: C_p \quad \textit{and} \quad \|\mu_{v_\varepsilon} - \mu_{\varepsilon,p}\|_{\mathrm{flat},\Omega'} \to 0\,. \tag{6.16}$$

Proof. We start by estimating the radii of the balls in the family $\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})$ used in the definition of $\mu_{\varepsilon,p}$. Recalling that $\sigma = 3\varepsilon$ and that the number of balls at the start of the ball construction is $N \leq C|\log\varepsilon|^2$, by condition (5) in Lemma 5.1 and by (6.6), we infer that

$$\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})) \leq (1+t^{k_\varepsilon}_{\varepsilon,p})(\mathcal{R}(\mathcal{B}_\varepsilon) + C\varepsilon|\log\varepsilon|^2) \leq C(\beta_p)^{2M|\log\varepsilon|}\varepsilon^{\sqrt{p}-1}\varepsilon|\log\varepsilon|^2 = C\varepsilon^p|\log\varepsilon|^2, \tag{6.17}$$

where C depends on Ω''. In particular, the balls of the family $\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})$ have infinitesimal radius as $\varepsilon\to 0$. Hence, by (6.8) and by property (3) in Lemma 5.1, for ε small enough

$$\begin{aligned} C|\log\varepsilon| &\geq \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{v_\varepsilon}, \Omega''\setminus\overline{U}_\varepsilon(0)) \geq \mathcal{E}(\mathcal{B}_\varepsilon, \mu_{v_\varepsilon}, \Omega''\cap U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\setminus\overline{U}_\varepsilon(0)) \\ &\geq \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\in\Omega'}} |\mu_{v_\varepsilon}(B)|\log(1+t^{k_\varepsilon}_{\varepsilon,p}) \geq |\mu_{\varepsilon,p}|(\Omega')\,|\log(1+t^0_{\varepsilon,p})| \geq |\mu_{\varepsilon,p}|(\Omega')(1-\sqrt{p})|\log\varepsilon|\,, \end{aligned}$$

which yields the estimate in (6.16).

To deduce the convergence in (6.16), we estimate the flat distance between $\mu_{\varepsilon,p}$ and μ_{v_ε}. The argument to do this is standard (see, *e.g.*, [28, Lemma 2.2]). One lets $\psi\in C^{0,1}_c(\Omega')$ be such that $\|\psi\|_{L^\infty(\Omega')}\leq 1$, $\|\nabla\psi\|_{L^\infty(\Omega')}\leq 1$. Since the balls in $\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})$ are pairwise disjoint,

$$\begin{aligned} \langle \mu_{v_\varepsilon} - \mu_{\varepsilon,p}, \psi\rangle &= \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\in\Omega'}} \int_B \psi\,\mathrm{d}(\mu_{v_\varepsilon} - \mu_{\varepsilon,p}) + \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\notin\Omega'}} \int_{B\cap\mathrm{supp}(\psi)} \psi\,\mathrm{d}\mu_{v_\varepsilon} \\ &\leq \sum_{\substack{B\in\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\\ x_B\in\Omega'}} \mathrm{osc}_B(\psi)\big(|\mu_{v_\varepsilon}| + |\mu_{\varepsilon,p}|\big)(\Omega') + \mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}))|\mu_{v_\varepsilon}|(\Omega') \\ &\leq 2\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}))\big(|\mu_{v_\varepsilon}| + |\mu_{\varepsilon,p}|\big)(\Omega')\,. \end{aligned}$$

Taking the supremum over ψ in the previous inequality, by (6.17), (6.3), and the uniform bound in (6.16), we get that

$$\|\mu_{v_\varepsilon} - \mu_{\varepsilon,p}\|_{\mathrm{flat},\Omega'} \leq C\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}))\big(|\mu_{v_\varepsilon}| + |\mu_{\varepsilon,p}|\big)(\Omega') \leq C\varepsilon^p|\log\varepsilon|^4 \to 0\,,$$

hence the convergence in (6.16) is proved. □

Thanks to the previous lemma, we conclude the proof of the compactness statement Theorem 1.1-*i)*. Indeed, by (6.16) the measures $\mu_{\varepsilon,p}\llcorner\Omega'$ converge weakly* to some measure μ in Ω', up to a subsequence. Moreover, μ is a finite sum of Dirac deltas with centers in Ω' and with integer weights, because of the structure of $\mu_{\varepsilon,p}$ in (6.15) and the uniform bound on the mass (6.16). Finally, we have that[9] $\|\mu_{\varepsilon,p} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ and thus, by (6.16), $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$. We argue for every $\Omega' \subset\subset \Omega$ and by a diagonal argument to obtain that $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ for every $\Omega' \subset\subset \Omega$. The finiteness of $|\mu|(\Omega)$ will follow from Theorem 1.1-*ii)*.

Let us now prove Theorem 1.1-*ii)*. Let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ and assume that $\chi(u_\varepsilon) \to 1$. We let $v_\varepsilon \in \mathcal{SF}_\varepsilon$ be the auxiliary spin field defined as in (2.5). Let $\mu = \sum_{h=1}^N d_h\delta_{x_h}$ with $d_h \in \mathbb{Z}$, $x_h \in \Omega$ and assume that $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ for all $\Omega' \subset\subset \Omega$. Let us prove that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) \geq 2\sqrt{3}\pi|\mu|(\Omega)\,. \tag{6.18}$$

We can assume, without loss of generality, that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega) < +\infty\,.$$

Let us fix $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ with Lipschitz boundary. We assume that $0 \in \Omega'$ and $\mu\llcorner\Omega' = d\delta_0$ for some $d \in \mathbb{Z}\setminus\{0\}$, hence $\|\mu_{v_\varepsilon} - d\delta_0\|_{\mathrm{flat},\Omega'} \to 0$. (The fact that μ is supported in 0 is not relevant for the discussion.) Thanks to the superadditivity of the lim inf and the non-negativity of the energy, it will be enough to prove the claim in Ω'.

We apply the ball construction and we define $\mu_{\varepsilon,p}$ as done above for the compactness result. By Lemma 6.4 and the assumptions made above, we have that

$$\mu_{\varepsilon,p}\llcorner\Omega' \overset{*}{\rightharpoonup} d\delta_0\,. \tag{6.19}$$

We classify the balls of the family $\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})$ into two subclasses

$$\begin{aligned} \mathcal{B}^{=0}_\varepsilon &:= \{B \in \mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) : \mu_{v_\varepsilon}(B) = 0\,,\ x_B \in \Omega'\}\,,\\ \mathcal{B}^{\neq 0}_\varepsilon &:= \{B \in \mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p}) : \mu_{v_\varepsilon}(B) \neq 0\}\,. \end{aligned} \tag{6.20}$$

We modify the spin field u_ε in such a way that we can assume $\mathcal{B}^{=0}_\varepsilon = \emptyset$ without loss of generality. Then we will work only with balls in the family $\mathcal{B}^{\neq 0}_\varepsilon$, which are relevant from the energetic point of view.

Lemma 6.5. *Let u_ε be as above, let $\mathcal{B}^{=0}_\varepsilon$ be as in (6.20), and let $c_p := \frac{\beta_p+1}{2\beta_p} \in (0,1)$. Then there exists $\overline{u}_\varepsilon \in \mathcal{SF}_\varepsilon$ such that $\overline{u}_\varepsilon = u_\varepsilon$ on $\Omega' \setminus \bigcup_{B_R(x)\in\mathcal{B}^{=0}_\varepsilon} B_{c_pR}(x)$, $|\mu_{\overline{v}_\varepsilon}|(B) = 0$ for all $B \in \mathcal{B}^{=0}_\varepsilon$, and*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(\overline{u}_\varepsilon, \Omega') \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega')\,.$$

[9] This is due to the fact that the flat norm metrizes the weak convergence of measures with equibounded mass.

Proof. Let $B_{R_\varepsilon}(x_\varepsilon) \in \mathcal{B}^{=0}_\varepsilon$. Since $k_\varepsilon \in \mathcal{K}_\varepsilon$, by (6.13) no merging occurs in the interval $(t^{k_\varepsilon-1}_{\varepsilon,p}, t^{k_\varepsilon}_{\varepsilon,p}]$ and therefore there exists $B_{r_\varepsilon}(x_\varepsilon) \in \mathcal{B}_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})$ (*i.e.*, a ball with the same center). Note that, by (6.7),

$$\frac{\varepsilon}{r_\varepsilon} \leq \frac{C}{1+t^{k_\varepsilon-1,p}_{\varepsilon}} = \frac{C\varepsilon^{1-\sqrt{p}}}{(\beta_p)^{k_\varepsilon-1}} \leq C\varepsilon^{1-\sqrt{p}} \to 0\,. \tag{6.21}$$

Let r'_ε be the radius of the ball centred in x_ε at the last merging time $T \leq t^{k_\varepsilon-1}_{\varepsilon,p}$ (in the case no merging occurred before $t^{k_\varepsilon-1}_{\varepsilon,p}$, let $T=0$). By construction, recalling (6.12), we have that

$$\frac{r_\varepsilon}{r'_\varepsilon} = \frac{1+t^{k_\varepsilon-1}_{\varepsilon,p}}{1+T}\,, \quad \frac{R_\varepsilon}{r'_\varepsilon} = \frac{1+t^{k_\varepsilon}_{\varepsilon,p}}{1+T} \implies \frac{R_\varepsilon}{r_\varepsilon} = \beta_p\,.$$

Note that $\mu_{v_\varepsilon}(B_{r_\varepsilon}(x_\varepsilon)) = 0$ and, by property (4) in Lemma 5.1 and due to the choice $\sigma = 3\varepsilon$, $|\mu_{v_\varepsilon}|(A_{r_\varepsilon-3\varepsilon,R_\varepsilon+3\varepsilon}(x_\varepsilon)) = 0$. Furthermore, due to (3.11) and to (6.14), we have that

$$\frac{1}{\varepsilon^2}XY_\varepsilon(v_\varepsilon, A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) \leq \sqrt{3}\int_{A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)} |\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x \leq C_1\,. \tag{6.22}$$

Therefore, we are in a position to apply Lemma 3.5, see also Remark 3.6. We obtain $\overline{v}_\varepsilon \in \mathcal{SF}_\varepsilon$ such that $\overline{v}_\varepsilon = v_\varepsilon$ on $A_{c_pR_\varepsilon,R_\varepsilon}(x_\varepsilon)$ (observe that $\frac{r_\varepsilon+R_\varepsilon}{2} = c_pR_\varepsilon$), $|\mu_{\overline{v}_\varepsilon}|(B_{R_\varepsilon}(x_\varepsilon)) = 0$, and

$$XY_\varepsilon(\overline{v}_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \leq C(\beta_p)XY_\varepsilon(v_\varepsilon, A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) \tag{6.23}$$

for ε small enough (*i.e.*, such that $\frac{\varepsilon}{r_\varepsilon} < \frac{\beta_p-1}{C_0C_1}(\frac{2\pi}{3})^2$, *cf.* (6.21), where C_0 is given by Lemma 3.5). We set

$$\overline{u}_\varepsilon(\varepsilon i) := \overline{v}_\varepsilon(\varepsilon i)\,, \quad \overline{u}_\varepsilon(\varepsilon j) := R\big[\tfrac{2\pi}{3}\big](\overline{v}_\varepsilon(\varepsilon j))\,, \quad \overline{u}_\varepsilon(\varepsilon k) := R\big[-\tfrac{2\pi}{3}\big](\overline{v}_\varepsilon(\varepsilon k))$$

for $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, $\varepsilon k \in \mathcal{L}^3_\varepsilon$ in accordance with (2.5). By (2.8), (6.23), and (6.22), we get

$$\frac{1}{\varepsilon^2}E_\varepsilon(\overline{u}_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \leq \frac{C}{\varepsilon^2}XY_\varepsilon(\overline{v}_\varepsilon, B_{R_\varepsilon}(x_\varepsilon)) \leq \frac{C}{\varepsilon^2}XY_\varepsilon(v_\varepsilon, A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)) \leq C\int_{A_{r_\varepsilon,R_\varepsilon}(x_\varepsilon)} |\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x\,. \tag{6.24}$$

We apply this construction for all $B \in \mathcal{B}^{=0}_\varepsilon$ in order to obtain $\overline{u}_\varepsilon \in \mathcal{SF}_\varepsilon$ such that $\overline{u}_\varepsilon = u_\varepsilon$ on $\Omega' \setminus \bigcup_{B_R(x)\in\mathcal{B}^{=0}_\varepsilon} B_{c_pR}(x)$, $|\mu_{\overline{v}_\varepsilon}|(B) = 0$ for all $B \in \mathcal{B}^{=0}_\varepsilon$, and

$$\frac{1}{\varepsilon^2}E_\varepsilon\Big(\overline{u}_\varepsilon, \bigcup_{B\in\mathcal{B}^{=0}_\varepsilon} B\Big) \leq C\int_{U_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})\setminus\overline{U}_\varepsilon(t^{k_\varepsilon-1}_{\varepsilon,p})} |\nabla\widehat{v}_\varepsilon|^2\,\mathrm{d}x \leq C\,, \tag{6.25}$$

where we exploited (6.24) and (6.14). Using (6.25), we therefore obtain

$$\begin{aligned}
\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(\overline{u}_\varepsilon,\Omega') &\leq \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}\Big(E_\varepsilon(u_\varepsilon,\Omega') + E_\varepsilon\Big(\overline{u}_\varepsilon, \bigcup_{B\in\mathcal{B}^{=0}_\varepsilon} B\Big)\Big)\\
&\leq \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega') + \limsup_{\varepsilon\to0}\frac{C}{|\log\varepsilon|}\\
&= \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega')\,.
\end{aligned}$$

This concludes the proof. □

Thanks to Lemma 6.5, we replace u_ε by $\overline{u}_\varepsilon$ and thus we can assume hereafter that the collection $\mathcal{B}^{=0}_\varepsilon$ is empty without loss of generality. Hence, it remains to prove the lower bound for the sequence u_ε using only the family of balls $\mathcal{B}^{\neq0}_\varepsilon$. Before going further with the proof, we obtain the lower bound in a simpler framework. Afterwards, we shall reduce to this setting. We recall that

$$\mathrm{Adm}^\varepsilon_{r,R}(d) := \Big\{u \in \mathcal{SF}_\varepsilon : \mu_v(T) = 0 \text{ for every } T \in \mathcal{T}_\varepsilon(\mathbb{R}^2)\,,\ T\cap A_{r,R} \neq \emptyset\,,\ \mu_v(B_r) = d\Big\}\,,$$

where $v \in \mathcal{SF}_\varepsilon$ is the auxiliary spin field associated to u defined as in (2.5).

Lemma 6.6. *Let $d \in \mathbb{Z} \setminus \{0\}$ and let $0 < q_1 < q_2 < 1$. Then*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} \inf\{E_\varepsilon(u, A_{\varepsilon^{q_2},\varepsilon^{q_1}}) : u \in \mathrm{Adm}^{\varepsilon}_{\varepsilon^{q_2},\varepsilon^{q_1}}(d)\} \geq (q_2-q_1)2\sqrt{3}\pi|d|^2.$$

Proof. For every ε let $u'_\varepsilon \in \mathrm{Adm}^{\varepsilon}_{\varepsilon^{q_2},\varepsilon^{q_1}}(d)$ be such that

$$E_\varepsilon(u'_\varepsilon, A_{\varepsilon^{q_2},\varepsilon^{q_1}}) \leq \inf\{E_\varepsilon(u, A_{\varepsilon^{q_2},\varepsilon^{q_1}}) : u \in \mathrm{Adm}^{\varepsilon}_{\varepsilon^{q_2},\varepsilon^{q_1}}(d)\} + \varepsilon^2 .$$

We fix $R > 1$, we set $M_{\varepsilon,R} := \lfloor (q_2-q_1)\frac{|\log\varepsilon|}{\log R}\rfloor$ and $A^{m,\varepsilon} := A_{R^{m-1}\varepsilon^{q_2},R^m\varepsilon^{q_2}}$. We remark that $\bigcup_{m=1}^{M_{\varepsilon,R}} A^{m,\varepsilon} \subset A_{\varepsilon^{q_2},\varepsilon^{q_1}}$. Let $\overline{m} = \overline{m}_{\varepsilon,R}$ be such that

$$E_\varepsilon(u'_\varepsilon, A^{\overline{m},\varepsilon}) \leq E_\varepsilon(u'_\varepsilon, A^{m,\varepsilon}) , \quad \text{for } m = 1, \dots, M_{\varepsilon,R} .$$

We let $\eta_\varepsilon := \varepsilon/R^{\overline{m}-1}\varepsilon^{q_2}$ and we define $u'_{\eta_\varepsilon}(\eta_\varepsilon i) := u'_\varepsilon(\varepsilon i)$ for every $i \in \mathcal{L}$. Then we have

$$\frac{1}{\varepsilon^2}E_\varepsilon(u'_\varepsilon, A_{\varepsilon^{q_2},\varepsilon^{q_1}}) \geq \sum_{m=1}^{M_{\varepsilon,R}} \frac{1}{\varepsilon^2}E_\varepsilon(u'_\varepsilon, A^{m,\varepsilon}) \geq \frac{M_{\varepsilon,R}}{\varepsilon^2}E_\varepsilon(u'_\varepsilon, A^{\overline{m},\varepsilon}) = \frac{M_{\varepsilon,R}}{\eta_\varepsilon^2}E_{\eta_\varepsilon}(u'_{\eta_\varepsilon}, A_{1,R}) .$$

Since $M_{\varepsilon,R} \geq (q_2-q_1)\frac{|\log\varepsilon|}{\log R} - 1$, from the previous inequalities, and by Proposition 4.3 it follows that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} &\inf\{E_\varepsilon(u, A_{\varepsilon^{q_2},\varepsilon^{q_1}}) : u \in \mathrm{Adm}^{\varepsilon}_{\varepsilon^{q_2},\varepsilon^{q_1}}(d)\} \geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u'_\varepsilon, A_{\varepsilon^{q_2},\varepsilon^{q_1}}) \\
&\geq \liminf_{\varepsilon\to 0} \frac{M_{\varepsilon,R}}{|\log\varepsilon|}\frac{1}{\eta_\varepsilon^2}E_{\eta_\varepsilon}(u'_{\eta_\varepsilon}, A_{1,R}) \geq \liminf_{\varepsilon\to 0} \Big[\Big(\frac{q_2-q_1}{\log R} - \frac{1}{|\log\varepsilon|}\Big)\frac{1}{\eta_\varepsilon^2}E_{\eta_\varepsilon}(u'_{\eta_\varepsilon}, A_{1,R})\Big] \\
&\geq \frac{q_2-q_1}{\log R} \liminf_{\eta\to 0} \ \inf\Big\{\frac{1}{\eta^2}E_\eta(u, A_{1,R}) : u \in \mathrm{Adm}^{\eta}_{1,R}(d)\Big\} \geq (q_2-q_1)2\sqrt{3}\pi|d|^2.
\end{aligned}$$

This concludes the proof. □

In view of (6.16), we have that $\#\mathcal{B}^{\neq 0}_\varepsilon \leq C_p$ and therefore we can assume that (up to a subsequence) $\#\mathcal{B}^{\neq 0}_\varepsilon = L$ for all $\varepsilon > 0$ for some $L \in \mathbb{N}$. Let $\mathcal{B}^{\neq 0}_\varepsilon = \{B_{r^\ell_\varepsilon}(x^\ell_\varepsilon)\}_{\ell=1}^L$. By definition (6.15), we have that $\{x^1_\varepsilon, \dots, x^L_\varepsilon\}$ is the support of the measure $\mu_{\varepsilon,p}$. The points x^ℓ_ε converge (up to a subsequence) to a finite set of points $\{0 = \xi^1, \dots, \xi^{L'}\}$ contained in $\overline{\Omega}$ with $L' \leq L$. Fix $\rho > 0$ such that $B_\rho \subset\subset \Omega'$ and $B_\rho(\xi^h) \cap B_\rho = \emptyset$ for all $h = 2, \dots, L'$. For $\varepsilon > 0$ small enough we have that either $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \cap B_\rho = \emptyset$ or $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset\subset B_\rho$. Furthermore, by (6.15), (6.19), and the fact that $|\mu|(\partial B_\rho) = 0$, we have that

$$\sum_{x^\ell_\varepsilon \in B_\rho} \mu_{v_\varepsilon}(B_{r^\ell_\varepsilon}(x^\ell_\varepsilon)) = d . \tag{6.26}$$

We prove that

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon, B_\rho) \geq 2\sqrt{3}\pi|d| .$$

Since our estimate is local, we can assume that $|\mu_{v_\varepsilon}|(\mathbb{R}^2 \setminus B_\rho) = 0$, which implies that $x^\ell_\varepsilon \in B_\rho$, *i.e.*, $B_{r^\ell_\varepsilon}(x^\ell_\varepsilon) \subset B_\rho$, for $\ell = 1, \dots, L$ and ε small enough. To reduce to the setting in Lemma 6.6 we follow an argument introduced, *e.g.*, in [27] or [3]. It is aimed at separating the scales of the radii of the balls charged by μ_{v_ε}.

Fix $0 < p' < p'' < p$ such that $\mathcal{R}(\mathcal{B}_\varepsilon(t^{k_\varepsilon}_{\varepsilon,p})) \leq \varepsilon^{p''}$ (this is possible due to (6.17)). We consider the function $g_\varepsilon \colon [p', p''] \to \{1, \dots, L\}$ such that $g_\varepsilon(q)$ gives the number of connected components of

$\bigcup_{\ell=1}^{L} B_{\varepsilon^q}(x_\varepsilon^\ell)$. For each $\varepsilon > 0$, the function g_ε is monotonically non-decreasing so that it can have at most $\hat{L} \leq L-1$ discontinuity points. We let $\{q_1^\varepsilon, \dots, q_{\hat{L}}^\varepsilon\}$ denote these discontinuity points with

$$p' \leq q_1^\varepsilon < \ldots < q_{\hat{L}}^\varepsilon \leq p''.$$

There exists a finite set $\mathfrak{D} = \{q_0, \dots, q_{\widetilde{L}+1}\}$ with $q_h < q_{h+1}$ such that, up to a subsequence, $(q_j^\varepsilon)_\varepsilon$ converges to some point in $\mathfrak{D}$ as $\varepsilon \to 0$, for $j = 1, \dots, \hat{L}$. We set $q_0 = p'$, $q_{\widetilde{L}+1} = p''$, and thus $\widetilde{L} \leq \hat{L}$. Let us fix $\lambda > 0$ with $2\lambda < \min_h(q_{h+1} - q_h)$. For $\varepsilon > 0$ small enough (that is, such that for $h' = 1, \dots, \hat{L}$ one has $|q_{h'}^\varepsilon - q_h| < \lambda/2$ for some $q_h \in \mathfrak{D}$) the function g_ε is constant in the interval $[q_h + \lambda/2, q_{h+1} - \lambda/2]$ with constant value M_h^ε, where $M_h^\varepsilon \leq L$. Up to extracting a subsequence, we assume that $M_h^\varepsilon = M_h$. We now construct a family of annuli $\{A_\varepsilon^{h,m}\}_{m=1}^{M_h}$ where we can apply Lemma 6.6.

Lemma 6.7. *In the assumptions above, for ε sufficiently small, for every $h = 0, \dots, \widetilde{L}$ there exists a family of pairwise disjoint annuli $\{A_\varepsilon^{h,m}\}_{m=1}^{M_h}$ with $A_\varepsilon^{h,m} := B_{\varepsilon^{q_h+\lambda}}(z_\varepsilon^{h,m}) \setminus \overline{B}_{\varepsilon^{q_{h+1}-\lambda}}(z_\varepsilon^{h,m})$ such that the sets in the family $\{\bigcup_{m=1}^{M_h} A_\varepsilon^{h,m}\}_{h=1}^{\widetilde{L}}$ are pairwise disjoint and*

$$\bigcup_{\ell=1}^{L} B_{r_\varepsilon^\ell}(x_\varepsilon^\ell) \subset \bigcup_{m=1}^{M_h} B_{\varepsilon^{q_{h+1}-\lambda}}(z_\varepsilon^{h,m}) \tag{6.27}$$

for $h = 0, \dots, \widetilde{L}$. Moreover, the points $z_\varepsilon^{h,m}$ can be chosen in $\mathcal{L}_\varepsilon \cap \bigcup_{\ell=1}^{L} B_\varepsilon(x_\varepsilon^\ell)$.

Proof. Let $h \in \{0, \dots, \widetilde{L}\}$. Since $g_\varepsilon \equiv M_h$ on $[q_h + \frac{\lambda}{2}, q_{h+1} - \frac{\lambda}{2}]$, we find a partition $\{\mathcal{I}_\varepsilon^{h,m}\}_{m=1}^{M_h}$ of $\{1, \dots, L\}$ such that $\{\bigcup_{\ell \in \mathcal{I}_\varepsilon^{h,m}} B_{\varepsilon^q}(x_\varepsilon^\ell)\}_{m=1}^{M_h}$ are the M_h connected components of $\bigcup_{\ell=1}^{L} B_{\varepsilon^q}(x_\varepsilon^\ell)$ for $q \in [q_h + \frac{\lambda}{2}, q_{h+1} - \frac{\lambda}{2}]$. For $m = 1, \dots, M_h$ we choose arbitrarily $\ell(m) \in \mathcal{I}_\varepsilon^{h,m}$ and $z_\varepsilon^{h,m} \in \mathcal{L}_\varepsilon \cap B_\varepsilon(x_\varepsilon^{\ell(m)})$. For ε small enough the balls in $\{B_{\varepsilon^q}(z_\varepsilon^{h,m})\}_{m=1}^{M_h}$ are pairwise disjoint for $q \in [q_h + \lambda, q_{h+1} - \lambda]$, since $B_{\varepsilon^q}(z_\varepsilon^{h,m}) \subset B_{\varepsilon^q+\varepsilon}(x_\varepsilon^{\ell(m)}) \subset B_{\varepsilon^{q_h+\frac{\lambda}{2}}}(x_\varepsilon^{\ell(m)})$, thus each of the balls is contained in a different connected component. Moreover, (6.27) holds true by construction. Indeed, let $x \in B_{r_\varepsilon^\ell}(x_\varepsilon^\ell) \subset B_{\varepsilon^{p''}}(x_\varepsilon^\ell)$ for some $\ell \in \{1, \dots, L\}$, and let $m_\ell \in \{1, \dots, M_h\}$ with $\ell \in \mathcal{I}_\varepsilon^{h,m_\ell}$. Then

$$|x - z_\varepsilon^{h,m}| \leq |x - x_\varepsilon^\ell| + |x_\varepsilon^\ell - z_\varepsilon^{h,m_\ell}| \leq \varepsilon^{p''} + \varepsilon + (\#\mathcal{I}_\varepsilon^{h,m_\ell} - 1)\varepsilon^{q_{h+1}-\frac{\lambda}{2}} \leq \varepsilon^{p''} + \varepsilon + L\varepsilon^{q_{h+1}-\frac{\lambda}{2}} \ll \varepsilon^{q_{h+1}-\lambda},$$

for ε sufficiently small (depending on λ), which gives (6.27). Let us finally prove that

$$\bigcup_{m=1}^{M_h} B_{\varepsilon^{q_h+\lambda}}(z_\varepsilon^{h,m}) \subset \bigcup_{n=1}^{M_{h-1}} B_{\varepsilon^{q_h-\lambda}}(z_\varepsilon^{h-1,n}) \quad \text{for } h = 1, \dots, \widetilde{L}, \tag{6.28}$$

which implies that $\bigcup_{m=1}^{M_h} A_\varepsilon^{h,m}$ and $\bigcup_{n=1}^{M_{h-1}} A_\varepsilon^{h-1,n}$ are disjoint. To prove (6.28), let $m \in \{1, \dots, M_h\}$ and let $x \in B_{\varepsilon^{q_h+\lambda}}(z_\varepsilon^{h,m})$ with $z_\varepsilon^{h,m} \in \mathcal{L}_\varepsilon \cap B_\varepsilon(x_\varepsilon^{\ell(m)})$. Moreover, let $n_m \in \{1, \dots, M_{h-1}\}$ with $\ell(m) \in \mathcal{I}_\varepsilon^{h-1,n_m}$. Then a similar argument as above shows that

$$|x - z_\varepsilon^{h-1,n_m}| \leq |x - z_\varepsilon^{h,m}| + |z_\varepsilon^{h,m} - z_\varepsilon^{h-1,n_m}| \leq \varepsilon^{q_h+\lambda} + 2\varepsilon + L\varepsilon^{q_h-\frac{\lambda}{2}} \ll \varepsilon^{q_h-\lambda},$$

for ε sufficiently small. $\square$

Finally, we conclude by exploiting the annuli $A_\varepsilon^{h,m}$ to prove the lower bound. Note that, for ε small enough $A_\varepsilon^{h,m} \subset\subset \Omega'$ for $h = 0, \dots, \widetilde{L}$ and $m = 1, \dots, M_h$. Moreover, in view of (6.15) and (6.16), we have that $|\mu_{v_\varepsilon}(B_{\varepsilon^{q_{h+1}-\lambda}}(z_\varepsilon^{h,m}))| \leq C$ for $h = 0, \dots, \widetilde{L}$ and $m = 1, \dots, M_h$. Therefore, up to extracting a further subsequence, $\mu_{v_\varepsilon}(B_{\varepsilon^{q_{h+1}-\lambda}}(z_\varepsilon^{h,m})) = d_{h,m} \in \mathbb{Z}\setminus\{0\}$ with M_h and $d_{h,m}$ independent of ε. Finally, by (6.26), we have

$$\sum_{m=1}^{M_h} d_{h,m} = d\,. \tag{6.29}$$

As $u_\varepsilon(\,\cdot\, - z_\varepsilon^{h,m}) \in \mathrm{Adm}^\varepsilon_{\varepsilon^{q_{h+1}-\lambda},\varepsilon^{q_h+\lambda}}(d_{h,m})$, since $\mathcal{B}_\varepsilon^{=0} = \emptyset$, by property (4) in Lemma 5.1 (recalling that $\sigma = 3\varepsilon$), by (6.27), and by Lemma 6.6, for every h and m we get

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, A_\varepsilon^{h,m}) \geq (q_{h+1}-q_h-2\lambda)2\sqrt{3}\pi|d_{h,m}|^2 \geq (q_{h+1}-q_h-2\lambda)2\sqrt{3}\pi|d_{h,m}|\,,$$

which, summing over h and m and using (6.29), yields

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon, \Omega') &\geq \sum_{h=0}^{\widetilde{L}}\sum_{m=1}^{M_h}(q_{h+1}-q_h-2\lambda)2\sqrt{3}\pi|d_{h,m}| \geq \sum_{h=0}^{\widetilde{L}}(q_{h+1}-q_h-2\lambda)2\sqrt{3}\pi|d| \\ &= (p''-p'-2(\widetilde{L}+1)\lambda)2\sqrt{3}\pi|d| = (p''-p'-2(\widetilde{L}+1)\lambda)2\sqrt{3}\pi|\mu|(\Omega')\,.\end{aligned}$$

The claim follows letting $\lambda\to 0$, $p'\to 0$, $p''\to p$, and $p\to 1$ in the previous inequality. Thanks to the arbitrariness of Ω', we have proven (6.18).

Remark 6.8. It is possible to obtain a non-sharp lower bound on $E_\varepsilon(u,\Omega)$ in terms of another auxiliary variable – the spin field u^1 obtained by restricting u to the sublattice $\mathcal{L}^1_\varepsilon$. Let $\hat{T}$ be a plaquette in the sublattice $\mathcal{L}^1_\varepsilon$, namely $\hat{T} = \mathrm{conv}\{\varepsilon i, \varepsilon i', \varepsilon i''\}$, $\varepsilon i, \varepsilon i', \varepsilon i'' \in \mathcal{L}^1_\varepsilon$, $|\varepsilon i - \varepsilon i'| = |\varepsilon i - \varepsilon i''| = |\varepsilon i' - \varepsilon i''| = \sqrt{3}\varepsilon$. We define

$$\begin{aligned}XY_{\sqrt{3}\varepsilon}(u,\hat{T}) &= \frac{3}{2}\varepsilon^2(|u(\varepsilon i)-u(\varepsilon i')|^2 + |u(\varepsilon i)-u(\varepsilon i'')|^2 + |u(\varepsilon i')-u(\varepsilon i'')|^2) \\ &= (\sqrt{3}\varepsilon)^2\sqrt{3}\int_{\hat{T}}|\nabla\hat{u}^1|^2\mathrm{d}x\,,\end{aligned} \tag{6.30}$$

where $\hat{u}^1$ is the affine interpolation in $\hat{T}$ of the spin field u^1. Let H be the hexagon composed of the 6 triangles in $\mathcal{T}_\varepsilon(\mathbb{R}^2)$ that intersect the interior of $\hat{T}$. By convexity of $x\mapsto|x|^2$ we get

$$E_\varepsilon(u,H) \geq \frac{1}{2}\varepsilon^2(|u(\varepsilon i)-u(\varepsilon i')|^2 + |u(\varepsilon i)-u(\varepsilon i'')|^2 + |u(\varepsilon i')-u(\varepsilon i'')|^2) = \frac{1}{3}XY_{\sqrt{3}\varepsilon}(u^1,\hat{T})\,.$$

Summing over all triangles $\hat{T}$ of the sublattice $\mathcal{L}^1_\varepsilon$ and noticing that the energy of every hexagon H is counted twice, we obtain

$$2E_\varepsilon(u,\Omega) \geq \frac{1}{3}XY_{\sqrt{3}\varepsilon}(u^1,\Omega')\,,$$

for all $\Omega' \subset\subset \Omega$ such that $\mathrm{dist}(\Omega',\partial\Omega) > \varepsilon$. We therefore obtain the following non-sharp lower bound (*cf.* [25] and (6.30)): If $u_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfies $\|\mu_{u^1_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ for all $\Omega' \subset\subset \Omega$, then

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u,\Omega) \geq \liminf_{\varepsilon\to 0}\frac{1}{2}\frac{1}{(\sqrt{3}\varepsilon)^2|\log(\sqrt{3}\varepsilon)|}XY_{\sqrt{3}\varepsilon}(u^1_\varepsilon,\Omega') \geq \sqrt{3}\pi|\mu|(\Omega')\,.$$

7. Proof of Theorem 1.1-*iii)*

In this section we prove Theorem 1.1-*iii)*. We start with an upper bound of the XY-energy of the prototypical function with a vortex-like singularity.

Lemma 7.1. *For $2\varepsilon \leq r \leq R$ one has that*

$$XY_\varepsilon\Big(\Big(\frac{x}{|x|}\Big)^d, A_{r,R}\Big) \leq 2\sqrt{3}\pi|d|^2\varepsilon^2\log\Big(\frac{R}{r}\Big) + C\varepsilon^2\,. \tag{7.1}$$

Proof. The computation is standard, but we present it for the sake of completeness. Set $v(x) := \big(\frac{x}{|x|}\big)^d$ for $x \in \mathbb{R}^2 \setminus \{0\}$ and $v_\varepsilon(x) := \big(\frac{x}{|x|}\big)^d$ for $x \in \mathcal{L}_\varepsilon \setminus \{0\}$ and $v_\varepsilon(0) := e_1$. Let $\alpha \in \{1,2,3\}$ and let $\varepsilon i, \varepsilon j \in T$ with $j-i$ parallel to $\hat e_\alpha$. For every $x \in T$, we have that

$$\begin{aligned}|\nabla \hat v_\varepsilon(x)\hat e_\alpha|^2 = \frac{|v(\varepsilon j)-v(\varepsilon i)|^2}{\varepsilon^2} &= \frac{1}{\varepsilon^2}\Big|\int_0^1 \nabla v(\varepsilon i + t(\varepsilon j - \varepsilon i))(\varepsilon j - \varepsilon i)\,\mathrm{d}t\Big|^2\\ &\le \int_0^1 |\nabla v(\varepsilon i + t(\varepsilon j-\varepsilon i))\hat e_\alpha|^2\,\mathrm{d}t\\ &\le |\nabla v(x)\hat e_\alpha|^2 + \int_0^1 \Big(|\nabla v(\varepsilon i + t(\varepsilon j - \varepsilon i))\hat e_\alpha|^2 - |\nabla v(x)\hat e_\alpha|^2\Big)\,\mathrm{d}t\,.\end{aligned}$$

We let $z := \varepsilon i + t(\varepsilon j - \varepsilon i)$ and we find ξ in the segment $[x,z]$ such that

$$\begin{aligned}|\nabla v(z)\hat e_\alpha|^2 - |\nabla v(x)\hat e_\alpha|^2 &\le |\nabla v(z) - \nabla v(x)|(|\nabla v(z)| + |\nabla v(x)|) \le \varepsilon|\nabla^2 v(\xi)|(|\nabla v(z)| + |\nabla v(x)|)\\ &\le \varepsilon\frac{C(d)}{|\xi|^2}\Big(\frac{|d|}{|z|} + \frac{|d|}{|x|}\Big) \le \varepsilon\frac{C}{(|x|-\varepsilon)^3}\,.\end{aligned}$$

where we used the fact that $|\nabla v(x)| = \frac{|d|}{|x|}$, $|\nabla^2 v(\xi)| \le \frac{C(d)}{|\xi|^2}$,[10] and $\min\{|x|,|z|,|\xi|\} \ge |x| - \varepsilon$. We conclude that for every $x \in T$

$$|\nabla \hat v_\varepsilon(x)|^2 \le |\nabla v(x)|^2 + \varepsilon\frac{C}{(|x|-\varepsilon)^3}\,.$$

Therefore, by Remark 2.5

$$\begin{aligned}XY_\varepsilon(v_\varepsilon, A_{r,R}) &\le \sqrt{3}\varepsilon^2\int_{A_{r,R}} |\nabla \hat v_\varepsilon(x)|^2\,\mathrm{d}x \le \sqrt{3}\varepsilon^2\int_{A_{r,R}}|\nabla v(x)|^2\,\mathrm{d}x + \int_{A_{r,R}}\frac{C\varepsilon^3}{(|x|-\varepsilon)^3}\,\mathrm{d}x\\ &= \sqrt{3}\varepsilon^2\int_r^R\int_0^{2\pi}\frac{|d|^2}{\rho}\,\mathrm{d}\theta\,\mathrm{d}\rho + C\varepsilon^3\int_r^R\int_0^{2\pi}\frac{\rho}{(\rho-\varepsilon)^3}\,\mathrm{d}\theta\,\mathrm{d}\rho\\ &\le 2\sqrt{3}\pi|d|^2\varepsilon^2\log\Big(\frac{R}{r}\Big) + C\varepsilon^2\Big(\frac{\varepsilon}{r-\varepsilon} - \frac{\varepsilon}{R-\varepsilon} + \frac{\varepsilon^2}{2(r-\varepsilon)^2} - \frac{\varepsilon^2}{2(R-\varepsilon)^2}\Big)\,,\end{aligned}$$

whence (7.1). □

Let us prove Theorem 1.1-*iii)*. Let $\mu = \sum_{h=1}^N d_h\delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$. Let us prove that there exist $u_\varepsilon \in \mathcal{SF}_\varepsilon$ such that $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega} \to 0$, where v_ε is as in (2.5), and

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega) \le 2\sqrt{3}\pi|\mu|(\Omega)\,. \tag{7.2}$$

Step 1. (The case $\mu = \pm\delta_{x_1}$) Let $x_1 \in \Omega$ and $\mu = \pm\delta_{x_1}$. It is not restrictive to assume that $x_1 = 0 \in \Omega$ and $\mu = \delta_0$. We define $v_\varepsilon \in \mathcal{SF}_\varepsilon$ by setting $v_\varepsilon(x) := \frac{x}{|x|}$ for every $x \in \mathcal{L}_\varepsilon \setminus \{0\}$, $v_\varepsilon(0) := e_1$ and we set

$$u_\varepsilon(\varepsilon i) := v_\varepsilon(\varepsilon i)\,, \quad u_\varepsilon(\varepsilon j) := R[\tfrac{2\pi}{3}](v_\varepsilon(\varepsilon j))\,, \quad u_\varepsilon(\varepsilon k) := R[-\tfrac{2\pi}{3}](v_\varepsilon(\varepsilon k))\,, \tag{7.3}$$

for $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, and $\varepsilon k \in \mathcal{L}^3_\varepsilon$, where $R[\,\cdot\,]$ is as in (2.4). We now estimate $E_\varepsilon(u_\varepsilon,\Omega)$ in terms of $XY_\varepsilon(v_\varepsilon,\Omega)$, then we can conclude using (7.1). To this end, let us fix $\lambda \in (0,1)$ and let $\eta \in (0,1)$ be

[10] This follows, *e.g.*, , by a computation in polar coordinates which shows that, for $h = 1,2$,

$$\nabla^2 v^h(x) = \frac{1}{\rho^2}\begin{pmatrix} 2\partial_\theta v^h\sin\theta\cos\theta + \partial^2_\theta v^h\sin^2\theta & -\partial_\theta v^h\cos(2\theta) - \partial^2_\theta v^h\sin\theta\cos\theta\\ -\partial_\theta v^h\cos(2\theta) - \partial^2_\theta v^h\sin\theta\cos\theta & -2\partial_\theta v^h\sin\theta\cos\theta + \partial^2_\theta v^h\cos^2\theta\end{pmatrix}\,.$$

as in Lemma 2.8. We observe that for every $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon(\mathbb{R}^2)$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, and $\varepsilon k \in \mathcal{L}^3_\varepsilon$ we have

$$\frac{2}{\pi}\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i), v_\varepsilon(\varepsilon j)) \leq |v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)| = \Big|\frac{i}{|i|} - \frac{j}{|j|}\Big| \leq \Big|\frac{i}{|i|} - \frac{j}{|i|}\Big| + \Big|\frac{j}{|i|} - \frac{j}{|j|}\Big| \leq \frac{2\varepsilon}{|\varepsilon i|}\,. \tag{7.4}$$

Since the same reasoning holds for $\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i), v_\varepsilon(\varepsilon k))$, we find $K \in \mathbb{N}$ (depending on η) such that

$$\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i), v_\varepsilon(\varepsilon j)) < \min\left\{\eta, \tfrac{\pi}{2}\right\} \quad \text{and} \quad \mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i), v_\varepsilon(\varepsilon k)) < \min\left\{\eta, \tfrac{\pi}{2}\right\}, \tag{7.5}$$

whenever $T \cap (\mathbb{R}^2 \setminus B_{K\varepsilon}) \neq \emptyset$. Thanks to Lemma 2.8 this allows us to estimate $E_\varepsilon(u_\varepsilon, \Omega)$ via

$$E_\varepsilon(u_\varepsilon,\Omega) \leq E_\varepsilon(u_\varepsilon, B_{(K+2)\varepsilon}) + (1+\lambda) XY_\varepsilon(v_\varepsilon, \Omega \setminus \overline{B}_{K\varepsilon}) \leq E_\varepsilon(u_\varepsilon, B_{(K+2)\varepsilon}) + (1+\lambda) XY_\varepsilon(v_\varepsilon, A_{K\varepsilon,R})\,, \tag{7.6}$$

where $R > 0$ is chosen large enough such that $\Omega \subset\subset B_R$. Moreover, we have

$$E_\varepsilon(u_\varepsilon, B_{(K+2)\varepsilon}) \leq 3\varepsilon^2 \# \mathcal{T}_\varepsilon(B_{(K+2)\varepsilon}) \leq C(K+2)^2\varepsilon^2.$$

Thus, from (7.6) together with Remark 7.1 we infer

$$\frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,\Omega) \leq \frac{1}{\varepsilon^2|\log\varepsilon|}E_\varepsilon(u_\varepsilon,B_R) \leq \frac{C(K+2)^2}{|\log\varepsilon|} + (1+\lambda)2\sqrt{3}\pi\frac{1}{|\log\varepsilon|}\log\frac{R}{K\varepsilon}\,, \tag{7.7}$$

from which we deduce (7.2) by letting $\varepsilon \to 0$ and then $\lambda \to 0$. To conclude the proof it thus remains to show that $\|\mu_{v_\varepsilon} - \delta_0\|_{\mathrm{flat},\Omega} \to 0$. First of all, due to Theorem 1.1-*i)*, we have that there exists $\mu = \sum_{h=1}^N d_h\delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$ such that, up to a subsequence $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega'} \to 0$ for all $\Omega' \subset\subset \Omega$. Note that, thanks to (7.5), we have $\mu_{v_\varepsilon} = 0$ on $\mathbb{R}^2 \setminus B_{K\varepsilon}$, which in turn implies that $\|\mu_{v_\varepsilon} - d\delta_0\|_{\mathrm{flat},\Omega} \to 0$ for some $d \in \mathbb{Z}$. We claim that $d = 1$. Indeed, let $\widehat{v}_\varepsilon$ be the interpolation defined as in Remark 3.2. Note that $\widehat{v}_\varepsilon \in W^{1,\infty}(A_{1,2};\mathbb{S}^1)$, since $\mu_{v_\varepsilon} = 0$ on $\mathbb{R}^2 \setminus B_{K\varepsilon}$. Let $\zeta\colon [0,3] \to \mathbb{R}$ be the piecewise affine function satisfying $\zeta \equiv 1$ on $[0,1]$, $\zeta \equiv 0$ on $[2,3]$, and ζ affine on $[1,2]$ and set $\psi(x) := \zeta(|x|)$. Then,

$$\langle d\delta_0, \psi\rangle = \lim_{\varepsilon\to 0}\langle \mu_{v_\varepsilon}, \psi\rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0}\langle J(\widehat{v}_\varepsilon), \psi\rangle = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{A_{1,2}} j(\widehat{v}_\varepsilon)\cdot\nabla^\perp\psi\,\mathrm{d}x = -\frac{1}{\pi}\int_{A_{1,2}} j\big(\tfrac{x}{|x|}\big)\cdot\nabla^\perp\psi\,\mathrm{d}x\,,$$

where in the last step we used that $\widehat{v}_\varepsilon \rightharpoonup \frac{x}{|x|}$ weakly in $H^1(A_{1,2};\mathbb{R}^2)$. Moreover, $\nabla^\perp\psi(x) = -\frac{x^\perp}{|x|}$ on $A_{1,2}$, thus a direct computation shows that

$$\langle d\delta_0, \psi\rangle = \frac{1}{2\pi}\int_{A_{1,2}}\frac{1}{|x|}\,\mathrm{d}x = 1\,,$$

consequently $d = 1$ and the whole sequence converges.

Step 2. (The case $\mu = \sum_{h=1}^N \pm\delta_{x_h}$) We first construct a recovery sequence when $\mu = \delta_{x_1} + \delta_{x_2}$ with $x_1, x_2 \in \Omega$ and $x_1 \neq x_2$. To simplify the exposition and the notation we assume that $x_1 = 0$ and we set $\overline{x} := x_2$. Then, to define a recovery sequence u_ε for $\mu = \delta_0 + \delta_{\overline{x}}$, we choose $\overline{x}_\varepsilon \in \mathcal{L}_\varepsilon \cap B_{2\varepsilon}(\overline{x})$ and we set $w_\varepsilon(x) := \frac{x}{|x|}$ for $x \in \mathcal{L}_\varepsilon \setminus \{0\}$, $\overline{w}_\varepsilon(x) := \frac{x - \overline{x}_\varepsilon}{|x - \overline{x}_\varepsilon|}$ for $x \in \mathcal{L}_\varepsilon \setminus \{\overline{x}_\varepsilon\}$ and $w_\varepsilon(0) = \overline{w}_\varepsilon(\overline{x}_\varepsilon) := e_1$. Eventually, we define $v_\varepsilon \in \mathcal{SF}_\varepsilon$ by setting $v_\varepsilon(x) := w_\varepsilon(x) \odot \overline{w}_\varepsilon(x)$ for every $x \in \mathcal{L}_\varepsilon$, where $\odot$ denotes the complex product, and we define u_ε according to (7.3). Suppose now that $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ with $\varepsilon i \in \mathcal{L}^1_\varepsilon$, $\varepsilon j \in \mathcal{L}^2_\varepsilon$, and $\varepsilon k \in \mathcal{L}^3_\varepsilon$. Then

$$\begin{aligned} |v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)| &= \Big|\Big(\frac{\varepsilon i}{|\varepsilon i|} - \frac{\varepsilon j}{|\varepsilon j|}\Big)\odot\frac{\varepsilon i - \overline{x}_\varepsilon}{|\varepsilon i - \overline{x}_\varepsilon|} + \frac{\varepsilon j}{|\varepsilon j|}\odot\Big(\frac{\varepsilon i - \overline{x}_\varepsilon}{|\varepsilon i - \overline{x}_\varepsilon|} - \frac{\varepsilon j - \overline{x}_\varepsilon}{|\varepsilon j - \overline{x}_\varepsilon|}\Big)\Big| \\ &\leq |w_\varepsilon(\varepsilon i) - w_\varepsilon(\varepsilon j)| + |\overline{w}_\varepsilon(\varepsilon i) - \overline{w}_\varepsilon(\varepsilon j)|\,. \end{aligned} \tag{7.8}$$

Taking and expanding the square in (7.8) yields

$$|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)|^2 \leq |w_\varepsilon(\varepsilon i) - w_\varepsilon(\varepsilon j)|^2 + |\overline{w}_\varepsilon(\varepsilon i) - \overline{w}_\varepsilon(\varepsilon j)|^2 + \frac{2}{\varepsilon^2}XY_\varepsilon(w_\varepsilon,T)^{\frac12}XY_\varepsilon(\overline{w}_\varepsilon,T)^{\frac12}. \tag{7.9}$$

The same estimates hold true when either εi or εj is replaced by εk. Thus, in view of (7.8) and (7.9) we can estimate $E_\varepsilon(u_\varepsilon,\Omega)$ as follows: Letting $\lambda\in(0,1)$ and $\eta\in(0,1)$ be as in Step 1, from (7.8) we deduce the existence of $K\in\mathbb{N}$ such that $\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i),v_\varepsilon(\varepsilon j))<\min\left\{\eta,\frac{\pi}{2}\right\}$ and $\mathrm{d}_{\mathbb{S}^1}(v_\varepsilon(\varepsilon i),v_\varepsilon(\varepsilon k))<\min\left\{\eta,\frac{\pi}{2}\right\}$, whenever $T\cap(\mathbb{R}^2\setminus(B_{K\varepsilon}\cup B_{K\varepsilon}(\overline{x}_\varepsilon)))\neq\emptyset$. Then, thanks to Lemma 2.8 and (7.9) we get

$$E_\varepsilon(u_\varepsilon,\Omega)\le C(K+2)^2\varepsilon^2+(1+\lambda)\Big(XY_\varepsilon\big(w_\varepsilon,\Omega\setminus\overline{B}_{K\varepsilon}\big)+XY_\varepsilon\big(\overline{w}_\varepsilon,\Omega\setminus\overline{B}_{K\varepsilon}(\overline{x}_\varepsilon)\big)+6I_\varepsilon\Big)\,,$$

where the remainder I_ε is given by

$$I_\varepsilon:=\sum_{T\in\mathcal{T}_\varepsilon(\Omega\setminus(\overline{B}_{K\varepsilon}\cup\overline{B}_{K\varepsilon}(\overline{x}_\varepsilon))}XY_\varepsilon(w_\varepsilon,T)^{\frac12}XY_\varepsilon(\overline{w}_\varepsilon,T)^{\frac12}\,.$$

To conclude as in (7.7), it is enough to show that $I_\varepsilon\le C\varepsilon^2$. We split the sum in the definition of I_ε. We fix $r>K\varepsilon$ such that $B_{r+2\varepsilon}\cap B_{r+2\varepsilon}(\overline{x}_\varepsilon)=\emptyset$ and $B_{r+2\varepsilon}\cup B_{r+2\varepsilon}(\overline{x}_\varepsilon)\subset\subset\Omega$. We also fix $R>r+2\varepsilon$ such that $\Omega\subset\subset B_R\cap B_R(\overline{x}_\varepsilon)$. Then, by the Cauchy-Schwarz Inequality and Lemma 7.1,

$$\begin{aligned}\sum_{T\in\mathcal{T}_\varepsilon(\Omega\setminus(\overline{B}_r\cup\overline{B}_r(\overline{x}_\varepsilon))}XY_\varepsilon(w_\varepsilon,T)^{\frac12}XY_\varepsilon(\overline{w}_\varepsilon,T)^{\frac12}&\le\big(XY_\varepsilon(w_\varepsilon,A_{r,R})\big)^{\frac12}\big(XY_\varepsilon(\overline{w}_\varepsilon,A_{r,R}(\overline{x}_\varepsilon))\big)^{\frac12}\\&\le2\sqrt{3}\pi|d|^2\varepsilon^2\log\Big(\frac{R}{r}\Big)+C\varepsilon^2\le C\varepsilon^2.\end{aligned}\tag{7.10}$$

Let $T\in\mathcal{T}_\varepsilon(\mathbb{R}^2\setminus\overline{B}_{K\varepsilon})$. Estimate (7.4) implies that $\frac{1}{\varepsilon^2}XY_\varepsilon(w_\varepsilon,T)\le\frac{12\varepsilon^2}{\mathrm{dist}(T,0)^2}$. In particular,

$$XY_\varepsilon(w_\varepsilon,T)^{\frac12}\le\frac{\sqrt{12}\varepsilon^2}{\mathrm{dist}(T,0)}=\fint_T\frac{\sqrt{12}\varepsilon^2}{\mathrm{dist}(T,0)}\,\mathrm{d}x\le\fint_T\frac{\sqrt{12}\varepsilon^2}{|x|-\varepsilon}\,\mathrm{d}x\,.$$

Moreover, if $T\subset\mathbb{R}^2\setminus\overline{B}_r$, then $XY_\varepsilon(w_\varepsilon,T)^{\frac12}\le2\frac{\sqrt{12}}{r}\varepsilon^2$. Analogously, if $T\subset\mathbb{R}^2\setminus\overline{B}_r(\overline{x}_\varepsilon)$, then $XY_\varepsilon(\overline{w}_\varepsilon,T)^{\frac12}\le2\frac{\sqrt{12}}{r}\varepsilon^2$. From the previous inequalities it follows that

$$\begin{aligned}\sum_{T\in\mathcal{T}_\varepsilon(B_{r+2\varepsilon}\setminus\overline{B}_{K\varepsilon})}XY_\varepsilon(w_\varepsilon,T)^{\frac12}XY_\varepsilon(\overline{w}_\varepsilon,T)^{\frac12}&\le\sum_{T\in\mathcal{T}_\varepsilon(B_{r+2\varepsilon}\setminus\overline{B}_{K\varepsilon})}XY_\varepsilon(w_\varepsilon,T)^{\frac12}C\varepsilon^2\\&\le\sum_{T\in\mathcal{T}_\varepsilon(B_{r+2\varepsilon}\setminus\overline{B}_{K\varepsilon})}\fint_T\frac{C\varepsilon^4}{|x|-\varepsilon}\,\mathrm{d}x\le\int_{B_{r+2\varepsilon}\setminus\overline{B}_{K\varepsilon})}\frac{C\varepsilon^2}{|x|-\varepsilon}\,\mathrm{d}x\le C\varepsilon^2\,.\end{aligned}$$

Analogously,

$$\sum_{T\in\mathcal{T}_\varepsilon(B_{r+2\varepsilon}\setminus\overline{B}_{K\varepsilon}(\overline{x}_\varepsilon))}XY_\varepsilon(w_\varepsilon,T)^{\frac12}XY_\varepsilon(\overline{w}_\varepsilon,T)^{\frac12}\le C\varepsilon^2\,.\tag{7.11}$$

Summing (7.10)–(7.11) we obtain that $I_\varepsilon\le C\varepsilon^2$.

It remains to show that $\|\mu_{v_\varepsilon}-\mu\|_{\mathrm{flat},\Omega}\to0$. By the same reasoning as in Step 1 we first obtain that, up to a subsequence, $\|\mu_{v_\varepsilon}-(d\delta_0+\overline{d}\delta_{\overline{x}})\|_{\mathrm{flat},\Omega}\to0$, where we have used that $\overline{x}_\varepsilon\to x$ as $\varepsilon\to0$. We are then left to show that $d=\overline{d}=1$. This will be done by localising the argument in Step 1. Namely, letting $\zeta\colon[0,3]\to\mathbb{R}$ be as in Step 1, we choose $r>0$ sufficiently small such that $B_{3r}\cap B_{3r}(\overline{x})=\emptyset$ and we set $\psi(x):=\zeta\big(\frac{|x|}{r}\big)$, $\overline{\psi}(x):=\zeta\big(\frac{|x-\overline{x}|}{r}\big)$ for every $x\in\mathbb{R}^2$. We let $\widehat{v}_\varepsilon$ denote the interpolation of v_ε as in Remark 3.2 and we set $v(x):=\frac{x}{|x|}\odot\frac{x-\overline{x}}{|x-\overline{x}|}=:w(x)\odot\overline{w}(x)$ for every $x\in\mathbb{R}^2\setminus\{0,\overline{x}\}$. Thanks to the choice of r and the fact that $\overline{x}_\varepsilon\to\overline{x}$, we have that $\widehat{v}_\varepsilon\rightharpoonup v$ in $H^1(A_{r,2r}\cup A_{r,2r}(\overline{x});\mathbb{R}^2)$. In particular, as in Step 1 we deduce that

$$\langle d\delta_0,\psi\rangle=\lim_{\varepsilon\to0}\langle\mu_{v_\varepsilon},\psi\rangle=\frac{1}{\pi}\lim_{\varepsilon\to0}\langle J(\widehat{v}_\varepsilon),\psi\rangle=-\frac{1}{\pi}\lim_{\varepsilon\to0}\int_{A_{r,2r}}j(\widehat{v}_\varepsilon)\cdot\nabla^\perp\psi\,\mathrm{d}x=-\frac{1}{\pi}\int_{A_{r,2r}}j(v)\cdot\nabla^\perp\psi\,\mathrm{d}x\,.$$

Moreover, a direct computation yields $j(v) = j(w) + j(\overline{w})$ and $\nabla^\perp\psi(x) = -\frac{x^\perp}{r|x|}$, hence

$$\langle d\delta_0, \psi\rangle = \frac{1}{2\pi}\int_{A_{r,2r}} \frac{1}{r|x|}\,\mathrm{d}x - \frac{1}{\pi}\int_{A_{r,2r}} j(\overline{w})\cdot\nabla^\perp\psi\,\mathrm{d}x = 1 - \frac{1}{\pi}\int_{A_{r,2r}} j(\overline{w})\cdot\nabla^\perp\psi\,\mathrm{d}x\,. \tag{7.12}$$

Eventually, the choice of $r > 0$ ensures that $\overline{w} \in H^1(A_{r,2r,};\mathbb{S}^1)$ with $\deg(\overline{w}, \partial B_\rho) = 0$ for every $\rho \in [r, 2r]$. Since in addition $\nabla^\perp\psi = -\frac{1}{r}\tau_{\partial B_\rho}$ for every $\rho \in [r, 2r]$, applying the coarea formula and (3.5) yields

$$-\frac{1}{\pi}\int_{A_{r,2r}} j(\overline{w})\cdot\nabla^\perp\psi\,\mathrm{d}x = \int_r^{2r}\frac{1}{r\pi}\int_{\partial B_\rho} j(\overline{w})|_{\partial B_\rho}\cdot\tau_{\partial B_\rho}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}\rho = \int_r^{2r}\frac{1}{r}\deg(\overline{w},\partial B_\rho)\,\mathrm{d}\rho = 0\,.$$

Thus, from (7.12) we deduce that $d = 1$. By repeating the argument in (7.12) with $\langle d\delta_0, \psi\rangle$ replaced by $\langle \overline{d}\delta_{\overline{x}}, \overline{\psi}\rangle$ and exchanging the roles of w and $\overline{w}$ we obtain $\overline{d} = 1$, hence $\|\mu_{v_\varepsilon} - \mu\|_{\mathrm{flat},\Omega} \to 0$, which concludes the proof of the limsup inequality.

Since the case $\mu = \pm\delta_{x_1} \pm \delta_{x_2}$ can be treated similarly, the case $\mu = \sum_{h=1}^N \pm\delta_{x_h}$ now follows by an iterative construction.

Step 3. (The general case) The general case follows from Step 2 via a diagonal argument. More in detail, given $\mu = \sum_{h=1}^N d_h\delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$ we approximate μ with a sequence of measures μ_n which are admissible for Step 2 as follows: For every $n \in \mathbb{N}$ and every $h \in \{1, \dots, N\}$ we choose $|d_h|$ points $x^1_{h,n}, \dots, x^{|d_h|}_{h,n} \in B_{\frac{1}{n}}(x_h)$ and we set

$$\mu_n := \sum_{h=1}^N \sum_{m=1}^{|d_h|} \mathrm{sign}(d_h)\delta_{x^m_{h,n}}\,.$$

By construction, $|\mu_n|(\Omega) = \sum_h |d_h| = |\mu|(\Omega)$. Thus, for every $n \in \mathbb{N}$ there exist $u_{\varepsilon,n} \in \mathcal{SF}_\varepsilon$ and corresponding spin fields $v_{\varepsilon,n} \in \mathcal{SF}_\varepsilon$ such that $\|\mu_{v_{\varepsilon,n}} - \mu_n\|_{\mathrm{flat},\Omega} \to 0$ as $\varepsilon \to 0$ and

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_{\varepsilon,n}, \Omega) \le 2\sqrt{3}\pi|\mu_n|(\Omega) = 2\sqrt{3}\pi|\mu|(\Omega)\,.$$

Thus, since $\|\mu_n - \mu\|_{\mathrm{flat},\Omega} \to 0$ as $n \to +\infty$, a diagonal argument provides us with a sequence $(u_{\varepsilon,n(\varepsilon)})$ such that $\|\mu_{v_{\varepsilon,n(\varepsilon)}} - \mu\|_{\mathrm{flat},\Omega} \to 0$ as $\varepsilon \to 0$ and (7.2) holds true. This concludes the proof in the general case.

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