Qualitative properties of maximum and average distance minimizers in $\mathbb{R}^n$

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Dedicated to N.N. Uraltseva

Abstract

The paper deals with one-dimensional networks of finite length in $\mathbb{R}^n$ minimizing average distance and maximum distance functionals subject to constraint on the length. We prove that under natural conditions on problem data such minimizers must use maximum available length, cannot contain closed loops (homeomorphic images of a circumference S^1) and have some mild regularity properties.

1 Introduction

Let $A\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a given nondecreasing function and φ be a given finite nonnegative measure with compact nonempty support in $\mathbb{R}^n$. We define then the functional F_φ over all compact connected subsets $\Sigma \subset \mathbb{R}^n$ by the formula

$$F_\varphi(\Sigma) := \int A(\mathrm{dist}\,(x,\Sigma))\, d\varphi(x),$$

where $\mathrm{dist}\,(x,\Sigma)$ stands for the distance between $x \in \mathbb{R}^n$ and Σ, i.e.

$$\mathrm{dist}\,(x,\Sigma) := \inf\{|x-z| \,:\, z \in \Sigma\},$$

$|\cdot|$ standing for the Euclidean norm. One may pose then the following problem.

Problem 1. *Given a number $l > 0$, find a compact connected set $\Sigma_{opt} \subset \mathbb{R}^n$ minimizing the functional F_φ over all compact connected $\Sigma \subset \mathbb{R}^n$ satisfying $\mathcal{H}^1(\Sigma) \le l$ ($\mathcal{H}^1$ standing for the one-dimensional Hausdorff measure).*

This problem will be further referred to as *average distance* problem, while the respective minimizers will be called *average distance minimizers*.

The average distance problem in the above setting stems from many different applications (for an overview the reader may consult [4]). The most easy interpretation is as follows: suppose that φ stands for the density of population in some geographical area, and let the function A represent the cost of movement for each single citizen, so that the cost for covering distance t is given by $A(t)$ (hence the monotonicity condition on A becomes quite natural). If Σ represents some transportation network (i.e. a set of traffic lines), then F_φ gives the cost of reaching the

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network calculated for the whole population (and hence proportional to average cost of reaching the network for the population). Solving Problem 1 amounts then to choosing the "best possible" transportation network of length not exceeding a given $l > 0$ (usually determined by a budget for construction), in the sense that the network has to be chosen so as to minimize the average cost of reaching it for the whole population.

We might as well be interested in another problem, which in a certain sense is similar to Problem 1.

Problem 2. *Given a compact set $M \subset \mathbb{R}^n$, and a number $l > 0$, find a compact connected set $\Sigma_{opt} \subset \mathbb{R}^n$ minimizing the functional F_M defined by the formula*

$$F_M(\Sigma) := \max_{x \in M} \operatorname{dist}(x, \Sigma)$$

over all compact connected $\Sigma \subset \mathbb{R}^n$ satisfying $\mathcal{H}^1(\Sigma) \leq l$.

This problem will be further referred to as *maximum distance* problem, while the respective minimizers will be called *maximum distance minimizers.*

One of the natural motivations to this problem is quite similar to the above discussed one. Namely, suppose that M represent a populated area. One has to construct a transportation network Σ of length not exceeding a given l, so that it be equally accessible to all the people living in M. This means that Σ has to be as near as possible to M in the uniform sense, i.e. it has to minimize F_M.

Minimization problems for average distance and maximum distance functionals are rather usual in economics and in urban planning, with quite similar interpretations. In particular, if, instead of fixing the length as in the above problems, one looks for minimizers satisfying the cardinality constraint $\#\Sigma \leq k$ with given $k \in \mathbb{N}$ ($\#$ standing for the cardinality), then the respective problems are usually referred to as *optimal facility location problems* (for a recent survey the reader may consult [6] or [9]). Average distance minimizing problem with such a cardinality constraint is also known under the name *k-median* (or *multimedian*) problem (see [6]), while maximum distance minimizing problem with this constraint is known as *k-center* (or *multicenter*) problem (see e.g. [10, 11]).

Average distance problems with length constraint have been considered in [2, 3, 4], while maximum distance problems with such a constraint have been considered in [8]. Since existence of minimizers to these problems is an easy exercise, all the above papers were mainly concerned with qualitative properties of minimizers. In what follows we assume, as in [4] the following conditions on the function A in Problem 1:

(α_1) A is Lipschitz continuous over $[0, \operatorname{diam}\operatorname{supp}\varphi]$, i.e. there is a $\Lambda > 0$ such that

$$|A(x) - A(y)| \leq \Lambda |x - y|$$

whenever $\{x, y\} \subset [0, \operatorname{diam}\operatorname{supp}\varphi]$;

(α_2) for every $c > 0$ there is a $\lambda = \lambda(c) > 0$ such that

$$|A(x) - A(y)| \geq \lambda |x - y|$$

whenever $\{x, y\} \subset [c, \operatorname{diam}\operatorname{supp}\varphi]$. In particular, A is injective (i.e. strictly increasing since A is supposed to be nondecreasing) over $[0, \operatorname{diam}\operatorname{supp}\varphi]$.

Note that the above conditions are surely satisfied, say, by the functions $A(t) := t^p$, $p \geq 1$, so that the results we obtain regarding Problem 1 are in particular applicable to the case of functionals

$$F_\varphi(\Sigma) := \int \operatorname{dist}^p(x, \Sigma)\, d\varphi(x), \qquad p \geq 1.$$

In [4] it has been shown, that under such conditions average distance minimizers possess some nontrivial qualitative properties. In particular, if $n = 2$, then they cannot contain closed loops (homeomorphic images of the circumference S^1) and are Ahlfors regular (at least if φ has some extra summability properties). Further, from the results of [3] it follows that when $n = 2$, $A(t) = t$, and φ has some extra summability properties, then the average distance minimizer has only finite number of (topological) endpoints and branching points, while all branching points are triple points and, at least in some reasonably weak sense, are "regular tripods". Some further partial results in this direction have been proven in [4] for the more general case, but they all involved an extra condition on the minimizer, which is non intrinsic in the sense that it is not expressed in terms of only the problem data. For maximum distance minimizers, Ahlfors regularity and absence of closed loops has been proven in [8] for $n = 2$. In this paper we prove, in particular,

- Ahlfors regularity for maximum distance minimizers (Theorem 6.4) and for average distance minimizers (Theorem 6.5). The latter result has been proven under extra summability requirement on φ;
- absence of closed loops for both maximum distance minimizers (Theorem 5.5) and average distance minimizers (Theorem 5.6 and Corollary 5.8). The latter results have been proven under quite general conditions on φ.

All the results are proved for general space dimension n.

Finally, we consider yet another problem in a certain sense "dual" to the maximum distance minimizing problem.

Problem 3. *Given a compact set $M \subset \mathbb{R}^n$ and a number $r > 0$, find a compact connected set $\Sigma_{opt} \subset \mathbb{R}^n$ minimizing the length functional $\mathcal{H}^1(\Sigma)$ over all compact connected $\Sigma \subset \mathbb{R}^n$ satisfying $F_M(\Sigma) \leq r$.*

This problem also admits an easy interpretation. Namely, suppose that we have to provide a gas supply pipeline to every house located in some area M under the condition that the gas supply should reach each house at distance not greater than a given $r > 0$. The company constructing the pipeline will naturally try to minimize its length under the above restriction, which reduces to solving Problem 3.

In [8] it has been shown that Problems 3 and 2 are equivalent when $n = 2$, in the sense that they have the same set of minimizers. Here we prove this result (Theorem 4.2) for the generic space dimension n. Note that this amounts to proving that every minimizer to Problem 2 must have maximum available length l. We also prove the analogous result (Theorem 4.4) for average distance minimizers, namely, we prove that every such minimizer, apart trivial cases, must have maximum available length.

2 Notation and preliminaries

The n-dimensional Lebesgue measure of a set $e \subset \mathbb{R}^n$ will be denoted by $\mathcal{L}^n(e)$ or simply by $|e|$. The measure φ_1 is said to be *absolutely continuous* with respect to the measure φ_2 (written $\varphi_1 \ll \varphi_2$) whenever $\varphi_2(e) = 0$ implies $\varphi_1(e) = 0$.

For a Borel measure φ over $\mathbb{R}^n$, we write $\varphi \in L^p(\mathbb{R}^n)$, if $\varphi = f\mathcal{L}^n$ with $f \in L^p(\mathbb{R}^n)$. We write $\|\varphi\|_p$ for the norm $\|f\|_p$ of f in $L^p(\mathbb{R}^n)$. is absolutely continuous with respect to $\mathcal{L}^n$ (written $\varphi \ll \mathcal{L}^n$), and The restriction $\varphi\llcorner B$ of the measure φ to the Borel set $B \subset \mathbb{R}^n$ is defined by

$$\varphi\llcorner B(e) := \varphi(B \cap e)$$

for all Borel $e \subset \mathbb{R}^n$.

We will need the following elementary lemma (see e.g. [8] for the proof).

Lemma 2.1 (covering)**.** *Let $\Sigma \subset \mathbb{R}^n$ be a bounded set. Then, given $\rho > 0$, there is a finite set of points (called further ρ-lattice of Σ) $\{x_1, \ldots, x_N\} \subset \Sigma$ such that*

$$\bigcup_{j=1}^{N} B_\rho(x_j) \supset \Sigma,$$

while $B_{\rho/2}(x_j)$, $j = 1, \ldots, N$, are pairwise disjoint.

Given an $x \in \Sigma$, a straight line $\Pi \subset \mathbb{R}^n$ such that $x \in \Pi$, and a number $\rho > 0$, we define

$$\beta_{\Sigma,\Pi}(x, \rho) := \sup_{y \in \Sigma \cap B_\rho(x)} \frac{\operatorname{dist}(y, \Pi)}{\rho}.$$

Define then the flatness β_Σ of a set Σ by the formula

$$\beta_\Sigma(x, \rho) = \inf_{\Pi} \beta_{\Sigma,\Pi}(x, \rho)$$

where Π varies among all straight lines of $\mathbb{R}^n$ passing through x. The following auxiliary technical result has been proven in [8].

Proposition 2.2 (existence of tangent lines)**.** *Let $\Sigma \subset \mathbb{R}^n$ be a closed connected set such that $\mathcal{H}^1(\Sigma) < +\infty$. Then in $\mathcal{H}^1$-a.e. $x \in \Sigma$ there exists a "tangent" line Π to Σ at x in the sense that $x \in \Pi$ and*

$$\lim_{\rho \to 0^+} \beta_{\Sigma,\Pi}(x, \rho) = 0.$$

Finally, given a set $X \subset \mathbb{R}^n$ we denote by $(X)_\rho := \{y \in \mathbb{R}^n : \operatorname{dist}(y, X) < \rho\}$ the ρ-neighborhood of X.

3 Auxiliary lemmata

We start with the following technical assertion.

Lemma 3.1. *Let $\Sigma \subset \mathbb{R}^n$ be a compact connected set with $\mathcal{H}^1(\Sigma) < +\infty$ and let ψ be a finite Borel measure on Σ. Then for all $\varepsilon \in (0, \operatorname{diam} \Sigma/4)$ there exists an $x = x(\varepsilon) \in \Sigma$ such that*

$$\psi(B_\varepsilon(x)) \geq \frac{\varepsilon \psi(\Sigma)}{2\mathcal{H}^1(\Sigma)}.$$

Proof. Let $x_1, \ldots, x_N$ be an ε-lattice for Σ as given by Lemma 2.1. Since Σ is connected and $\varepsilon/2 < \operatorname{diam} \Sigma/2$, we have $\Sigma \cap \partial B_{\varepsilon/2}(x_i)) \neq \emptyset$ and hence $\mathcal{H}^1(\Sigma \cap B_{\varepsilon/2}(x_i) \geq \varepsilon/2$ for all $i = 1, \ldots, N$. Thus

$$\mathcal{H}^1(\Sigma) \geq \sum_{i=1}^{N} \mathcal{H}^1(\Sigma \cap B_{\varepsilon/2}(x_i)) \geq N\frac{\varepsilon}{2}. \tag{1}$$

Choose now an $j \in 1, \ldots, N$ such that

$$\psi(B_\varepsilon(x_j)) = \max\{\psi(B_\varepsilon(x_i)) : i = 1, \ldots, N\}.$$

Therefore,

$$\psi(\Sigma) \leq \sum_{i=1}^{N} \psi(B_\varepsilon(x_i)) \leq N\psi(B_\varepsilon(x_j))$$

which together with (1), gives

$$\psi(B_\varepsilon(x_j)) \geq \frac{\psi(\Sigma)}{N} \geq \frac{\varepsilon \psi(\Sigma)}{2\mathcal{H}^1(\Sigma)}.$$

It is enough to set now $x := x_j$ to conclude the proof. ☐

REMARK. Note that for every Borel set $\Sigma \subset \mathbb{R}^n$ and for every positive Radon measure ψ in $\mathbb{R}^n$ one has that

$$\Theta_1^*(\psi, x) := \limsup_{\rho \to 0^+} \frac{\psi(B_\rho(x))}{2\rho} \geq \frac{\psi(\Sigma)}{4\mathcal{H}^1(\Sigma)} \tag{2}$$

for a set of $x \in \Sigma$ of positive $\mathcal{H}^1$ measure. In fact, otherwise, for $\mathcal{H}^1$-a.e. $x \in \Sigma$ one would have $\Theta_1^*(\psi, x) \leq \psi(\Sigma)/4\mathcal{H}^1(\Sigma)$, and hence, theorem 2.56 from [1] gives the contradiction $\psi(\Sigma) \leq \psi(\Sigma)/2$. The estimate (2) implies that for a set of $x \in \Sigma$ of positive $\mathcal{H}^1$ measure one has

$$\psi(B_\varepsilon(x)) \geq \frac{\varepsilon\psi(\Sigma)}{2\mathcal{H}^1(\Sigma)}$$

for a sequence of $\varepsilon := \varepsilon_\nu \to 0^+$ as $\nu \to \infty$.

The following lemma will be used frequently in the sequel to modify a given set so as to decrease the distance to this set from sufficiently distant points.

Lemma 3.2. *Given a $\rho > 0$, consider the set $\mathbb{K}_\rho \subset \mathbb{R}^n$ defined as*

$$\mathbb{K}_\rho = \bigcup_{k=1}^{n} \{te_k \,:\, t \in [-\rho, \rho]\}$$

where $\{e_k\}_{k=1}^n$ is the standard orthonormal basis of $\mathbb{R}^n$.

Then

(i) $\mathbb{K}_\rho$ is connected and $0 \in \mathbb{K}_\rho$,

(ii) $\mathcal{H}^1(\mathbb{K}_\rho) = 2n\rho$, and

(iii) if $y \in \mathbb{R}^n$ is given with $|y| \geq \sqrt{n}\rho$, then

$$\operatorname{dist}(y, \mathbb{K}_\rho) \leq |y| - \frac{\rho}{2\sqrt{n}}$$

Proof. Only the claim (iii) is nontrivial. Scaling in ρ, we may assume without loss of generality that $\rho = 1$, $y = (y_1, \ldots, y_n)$, $y_k \geq 0$ for all $k = 1, \ldots, n$, and $y_1 = \max\{y_1, \ldots, y_n\}$. Then

$$\begin{aligned} d(y, \mathbb{K}_1) &\leq \sqrt{(y_1 - 1)^2 + y_2^2 + \ldots + y_n^2} \\ &= \sqrt{|y|^2 + 1 - 2y_1} \leq \sqrt{|y|^2 - \frac{2}{\sqrt{n}}|y| + 1}. \end{aligned}$$

By Lemma 3.3 below with $\alpha = 2/\sqrt{n}$ and $\beta = 1$ we conclude that if $|y| \geq \sqrt{n}$, then $\operatorname{dist}(y, \mathbb{K}_1) \leq |y| - 1/(2\sqrt{n})$. □

Given $x \in \mathbb{R}^n$, we denote

$$\mathbb{K}_\rho(x) := x + \mathbb{K}_\rho,$$

where $\mathbb{K}_\rho$ is defined by Lemma 3.2.

The elementary lemma below is used both in the proof of Lemma 3.2 and in that of Proposition 3.4.

Lemma 3.3. *Let $\alpha, \beta > 0$ such that $\alpha^2 \leq 4\beta$. Then $x \geq 2\beta/\alpha$ implies*

$$\sqrt{x^2 - \alpha x + \beta} \leq x - \frac{\alpha}{4}.$$

Proof. Simply note that under conditions of the statement being proven the function $f\colon \mathbb{R}\to\mathbb{R}$ defined by

$$f(x) := x - \sqrt{x^2 - \alpha x + \beta}$$

is nondecreasing, and that

$$\begin{aligned} f(2\beta/\alpha) &= 2\beta/\alpha - \sqrt{(2\beta/\alpha)^2 - \beta} \\ &= 2\beta/\alpha - \sqrt{(2\beta/\alpha - \alpha/4)^2 - \alpha^2/16} \\ &\geq 2\beta/\alpha - (2\beta/\alpha - \alpha/4) \\ &= \alpha/4. \end{aligned}$$

□

Another auxiliary result to be used in the sequel gives the possibility to decrease the distance to a given set once the latter is sufficiently "flat" (the flatter is the set, the less additional length will be used).

Proposition 3.4. *Let $\rho > 0$, $\beta \in [0,1]$, $r \geq 9n\rho$ and $a, b \in [0,\rho]$, be given. Let $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^{n-1}$ and consider the cylinder $R := [-a,b]\times\bar{B}_{\beta\rho}$, where $\bar{B}_{\beta\rho}\subset\mathbb{R}^{n-1}$ stands for the closed ball of radius $\beta\rho$ centered at zero. Then there exists a set $X := X^-\cup X^+$ such that*

(i) $(-a,0)\in X^-$ *and* $(b,0)\in X^+$,

(ii) $X^\pm$ *are compact and connected sets,*

(iii) for every $y\in\mathbb{R}^n$ *satisfying* $|y|\geq 2r$ *one has*

$$\mathrm{dist}\,(y, X) \leq d(y,R) - \frac{\beta\rho}{2} - \frac{3\rho^2}{2r},$$

(iv) $\mathcal{H}^1(X^\pm) \leq 8n\sqrt{n}(\beta + \frac{\rho}{r})\rho$,

(v) $X \subset B_{r/\sqrt{n}}(0)$.

Proof. Define

$$\lambda := 4\sqrt{n}(\beta+\rho/r)\rho, \quad X^- := \mathbb{K}_\lambda(-a,0), \quad X^+ := \mathbb{K}_\lambda(b,0)$$

Assertions (i) and (ii) are clearly valid, while (iv) holds in view of Lemma 3.2. Since $X\subset\bar{B}_{\rho+\lambda}(0)$, the assertion (v) is a consequence of the estimate

$$\rho+\lambda < \frac{r}{9n} + 8\sqrt{n}\rho \leq \frac{r}{9\sqrt{n}} + \frac{8r}{9\sqrt{n}} \leq \frac{r}{\sqrt{n}}.$$

To prove (iii) let $y := (y_1, y')\in\mathbb{R}\times\mathbb{R}^{n-1}$ satisfy the condition $|y|\geq 2r$.

Case A. Suppose that $y_1\in[0,\rho]$ (the case $y_1\in[-\rho,0]$ is completely symmetric). Then

$$\begin{aligned} \mathrm{dist}\,(y, \mathbb{K}_\lambda(b,0)) &\leq \sqrt{|y_1-b|^2 + \mathrm{dist}^2(y',\mathbb{K}_\lambda)} \\ &\leq \sqrt{\rho^2 + \mathrm{dist}^2(y',\mathbb{K}_\lambda)}. \end{aligned}$$

Since $|y|\geq 2r\geq 18n\rho$, while $|y_1|\leq\rho$, then $|y'|\geq 17n\rho\geq\sqrt{n}\lambda$, and hence, by Lemma 3.2

$$\begin{aligned} \mathrm{dist}\,(y, \mathbb{K}_\lambda(b,0)) &\leq \sqrt{\rho^2 + \left(|y'| - \frac{\lambda}{2\sqrt{n}}\right)^2} \\ &= \sqrt{\rho^2 + (|y'| - 2\beta\rho - 2\rho^2/r)^2}. \end{aligned}$$

Let $t := |y'| - \beta\rho - \rho^2/r$ so that

$$\begin{aligned}\operatorname{dist}(y, \mathbb{K}_\lambda(b,0)) &\le \sqrt{\rho^2 + (t-\beta\rho-\rho^2/r)^2} \\ &= \sqrt{t^2 - 2(\beta\rho+\rho^2/r)t + \rho^2 + (\beta\rho+\rho^2/r)^2}.\end{aligned}$$

Applying now Lemma 3.3, we get that

$$\operatorname{dist}(y, \mathbb{K}_\lambda(b,0)) \le t - \frac{\beta\rho+\rho^2/r}{2} = |y'| - \frac{3}{2}(\beta\rho+\rho^2/r), \tag{3}$$

whenever t satisfies

$$t \ge \frac{\rho^2 + (\beta\rho+\rho^2/r)^2}{\beta\rho+\rho^2/r}. \tag{4}$$

Since $|y| \ge 2r$, we know that

$$\begin{aligned}t = |y'| - \beta\rho - \rho^2/r &\ge |y| - \rho - \beta\rho - \rho^2/r \ge 2r - 3\rho \ge r + 9n\rho - 3\rho \\ &\ge 6n\rho + r \ge 2\rho + r.\end{aligned}$$

But

$$\begin{aligned}\frac{\rho^2 + (\beta\rho+\rho^2/r)^2}{\beta\rho+\rho^2/r} &= \beta\rho + \rho^2/r + \frac{\rho^2}{\beta\rho+\rho^2/r} \\ &= \beta\rho + \rho^2/r + \frac{r}{\beta r/\rho + 1} \le 2\rho + r,\end{aligned}$$

which provides that the condition (4) is satisfied. Since $|y'| = \operatorname{dist}(y, R) + \beta\rho$, then by (3) we obtain

$$\operatorname{dist}(y, \mathbb{K}_\lambda(b,0)) \le |y'| - \frac{3}{2}(\beta\rho+\rho^2/r) \le \operatorname{dist}(y,R) - \frac{1}{2}\beta\rho - \frac{3}{2}\rho^2/r.$$

Case B. Consider now the case $y_1 \ge \rho$ (the case $y_1 \le -\rho$ is symmetric). Since $|y| \ge \rho$, we have

$$\begin{aligned}|y-(b,0)| &\ge |y| - \rho \ge r - \rho \ge 8n\rho \\ &\ge 4n(\beta\rho+\rho^2/r) = \sqrt{n}\lambda,\end{aligned}$$

so that by Lemma 3.2

$$\begin{aligned}\operatorname{dist}(y, \mathbb{K}_\lambda(b,0)) &\le |y-(b,0)| - \frac{\lambda}{2\sqrt{n}} \le \operatorname{dist}(y,R) + \beta\rho - \frac{\lambda}{2\sqrt{n}} \\ &= \operatorname{dist}(y,R) + \beta\rho - 2\beta\rho - 2\rho^2/r \le \operatorname{dist}(y,R) - \frac{1}{2}\beta\rho - \frac{3}{2}\rho^2/r.\end{aligned}$$

Summing up, we get the validity of the assertion (iii), and hence conclude the proof. □

The following result shows how to decrease the functional $F_\varphi(\Sigma)$ by a quantity of order ε^2 using an additional length of order ε (for small $\varepsilon > 0$). This result is similar to Lemma 3.4 in [4] where the order of the gain is estimated by $\varepsilon^{(n+1)/2}$. Hence the result of [4] is better when $n = 2$.

Lemma 3.5. *Assume (α_2) holds. Let $l > 0$ be given and let H and K be two Borel subsets of $\mathbb{R}^n$ such that $\varphi(K) > 0$ and*

$$r := \inf\{\operatorname{dist}(y, H) \,:\, y \in K\} > 0.$$

Then given any compact connected set $\Sigma \subset H$ *with* $\mathcal{H}^1(\Sigma) \leq l$*, one has for all* $\varepsilon < r/\sqrt{n}$ *the existence of a compact connected set* $\Sigma' \supset \Sigma$ *such that*

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) + 2n\varepsilon,$$
$$F_\varphi(\Sigma') \leq F_\varphi(\Sigma) - C_1\varepsilon^2,$$

where

$$C_1 := \frac{\lambda(r)\varphi(K)}{32nl}.$$

and $\lambda(r)$ *is the constant defined in* (α_2)*.*

Proof. Let $k : \mathbb{R}^n \to \Sigma$ stand for a Borel projection map on Σ i.e. a Borel function such that $|y - k(y)| = d(y, \Sigma)$ and let $\psi := k_\# \varphi \llcorner K$. Given $x \in \Sigma$ and $\varepsilon \in (0, r/\sqrt{n})$ consider the set

$$T(x, \varepsilon) := k^{-1}(B_{\varepsilon/(4\sqrt{n})}(x)) \cap K.$$

and notice that $\varphi(T(x, \varepsilon)) = \psi(B_{\varepsilon/(4\sqrt{n})}(x))$.

By Lemma 3.1 there exists an $x \in \Sigma$ such that

$$\varphi(T(x, \varepsilon)) = \psi(B_{\varepsilon/(4\sqrt{n})}(x)) \geq \frac{\varepsilon\psi(\Sigma)}{8\sqrt{n}\mathcal{H}^1(\Sigma)} = \frac{\varepsilon\varphi(K)}{8\sqrt{n}\mathcal{H}^1(\Sigma)}.$$

Define

$$\Sigma' := \Sigma \cup \mathbb{K}_\varepsilon(x).$$

Given $y \in T(x, \varepsilon)$ recall that $|y - x| \geq r \geq \sqrt{n}\varepsilon$ and $|k(y) - x| \leq \varepsilon/(4\sqrt{n})$. Therefore, by Lemma 3.2 we have

$$\begin{aligned} \operatorname{dist}(y, \Sigma') &\leq \operatorname{dist}(y, \mathbb{K}_\varepsilon(x)) \leq |y - x| - \frac{\varepsilon}{2\sqrt{n}} \\ &\leq |y - k(y)| + |k(y) - x| - \frac{\varepsilon}{2\sqrt{n}} \leq \operatorname{dist}(y, \Sigma) - \frac{\varepsilon}{4\sqrt{n}}. \end{aligned}$$

As a consequence, using (α_2) we get

$$\begin{aligned} F_\varphi(\Sigma) - F_\varphi(\Sigma') &\geq \int_{T(x,\varepsilon)} [A(\operatorname{dist}(y, \Sigma)) - A(\operatorname{dist}(y, \Sigma'))] \, d\varphi(y) \\ &\geq \int_{T(x,\varepsilon)} \lambda(r)[\operatorname{dist}(y, \Sigma) - \operatorname{dist}(y, \Sigma')] \, d\varphi(y) \\ &\geq \frac{\lambda(r)\varepsilon}{4\sqrt{n}}\varphi(T(x, \varepsilon)) \geq \frac{\lambda(r)\varphi(K)}{32n\mathcal{H}^1(\Sigma)}\varepsilon^2 \geq \frac{\lambda(r)\varphi(K)}{32nl}\varepsilon^2. \end{aligned}$$

□

4 Basic properties

In this section we show that solutions to Problems 1 and 2 under natural conditions on problem data have to have maximum available length l. We further show the equivalence of Problems 2 and 3.

The principal technical tool in this section is given by the proposition below in which a set Σ is modified with the addition of a piece of small length to obtain a new set such that all points which were not too close to Σ become closer to the modified set. The idea of the proof is an adaptation to the case of arbitrary space dimension n of the core of the proof of Theorem 3.7 from [8].

Proposition 4.1. *Let $\Sigma \subset \mathbb{R}^n$ be a compact connected set with $\mathcal{H}^1(\Sigma) < \infty$ and $r > 0$ be some given number. Then for each $c > 0$ there exists a compact connected $\Sigma' \subset \mathbb{R}^n$, $\Sigma \subset \Sigma'$, such that $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) + c$, such that for every $y \in \mathbb{R}^n$ satisfying* $\mathrm{dist}\,(y, \Sigma) \geq 3r/4$ *one has*

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma) - C,$$

where $C > 0$ is some constant depending on Σ, r, c and the space dimension n and independent of y.

Proof. In view of Proposition 2.2 one has $\lim_{k\to\infty} \beta_\Sigma(x, 1/k) = 0$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$. Consider an $\varepsilon > 0$ to be chosen later. By Egorov Theorem there exists a set $\Sigma_\varepsilon \subset \Sigma$ such that $\mathcal{H}^1(\Sigma_\varepsilon) \leq \varepsilon$ and

$$\lim_{k\to\infty} \sup_{x\in\Sigma\setminus\Sigma_\varepsilon} \beta_\Sigma(x, 1/k) = 0.$$

Let $r' := r/4$,

$$\beta := \min\left\{1, \frac{c}{18n\sqrt{n}\cdot 8\mathcal{H}^1(\Sigma)}\right\}, \tag{5}$$

and choose a $k \in \mathbb{N}$ such that for $\rho := 1/k > 0$ one has

$$\rho \leq \min\left\{\beta r', \frac{r'}{9n}\right\}, \qquad \sup_{x\in\Sigma\setminus\Sigma_\varepsilon} \beta_\Sigma(x, \rho) \leq \beta \text{ and } \rho < \mathrm{diam}\,\Sigma. \tag{6}$$

Consider now a ρ-lattice $\{x_1, \ldots, x_N\}$ of $\Sigma \setminus \Sigma_\varepsilon$ as provided by Lemma 2.1, so that the balls of radius $\rho/2$ centered in these points are all disjoint, while the balls of radius ρ centered in these points cover the whole set $\Sigma \setminus \Sigma_\varepsilon$. Note that since Σ is connected and $\rho/2 < \mathrm{diam}\,\Sigma/2$ by (6), then we have $\mathcal{H}^1(\Sigma \cap B_{\rho/2}(x_i)) \geq \rho/2$ and hence

$$\frac{N\rho}{2} \leq \sum_{i=1}^{N} \mathcal{H}^1\big(\Sigma \cap B_{\rho/2}(x_i)\big) \leq \mathcal{H}^1(\Sigma) \tag{7}$$

i.e.

$$\rho \leq 2\mathcal{H}^1(\Sigma)/N. \tag{8}$$

Let now $i \in \{1, \ldots, N\}$ be fixed and consider the line Π through x_i such that $\mathrm{dist}\,(x, \Pi) \leq \rho\beta_\Sigma(x_i, \rho) \leq \beta\rho$ for all $x \in \Sigma \cap B_\rho(x_i)$. Consider now an orthonormal system of coordinates in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ such that $x_i = (0, 0)$ and such that the line Π coincides with the first coordinate axis.

We have $\Sigma \cap B_\rho(x_i) \subset [-\rho, \rho] \times \bar{B}_{\beta\rho}$, where $\bar{B}_{\beta\rho} \subset \mathbb{R}^{n-1}$ stands for the closed ball of radius $\beta\rho$ centered at zero. Thus we can define $a_i^+, a_i^- \in [0, \rho]$ to be the minimum numbers such that for $R_i := [-a_i^-, a_i^+] \times \bar{B}_{\beta\rho}$ one has $\Sigma \cap B_\rho(x_i) \subset R_i$. Then we clearly have that both disks $\{\pm a_i^\pm\} \times \bar{B}_{\beta\rho}$ intersect Σ.

Let X_i be the set constructed in Proposition 3.4 (applied with $a := a_i^-$, $b := a_i^+$, r' instead of r and hence $R = R_i$). Let

$$X_i' := X_i \cup S_i^+ \cup S_i^-,$$

where $S_i^\pm$ is the segment joining the point $(\pm a_i^\pm, 0)$ with a point in $(\{\pm a_i^\pm\} \times \bar{B}_{\beta\rho}) \cap$

Σ. Hence, $\Sigma \cup X_i'$ is connected. We calculate now

$$\begin{aligned}
\mathcal{H}^1(X_i') &\le \mathcal{H}^1(X_i) + \mathcal{H}^1(S_i^+) + \mathcal{H}^1(S_i^-) \\
&\le 16n\sqrt{n}(\beta\rho + \rho^2/r') + 2\beta\rho \\
&\le 18n\sqrt{n}\beta\rho + 16n\sqrt{n}\rho^2/r' \\
&\le 18n\sqrt{n}\beta\rho + 16n\sqrt{n}\beta\rho \qquad \text{by (6)} \\
&\le 36n\sqrt{n}\beta\rho \\
&\le \frac{c\rho}{4\mathcal{H}^1(\Sigma)} \qquad \text{by (5)} \\
&\le \frac{c}{2N} \qquad \text{by (8).}
\end{aligned} \tag{9}$$

Denote by $\tilde{x}_i$ the center of the cylinder R_i. We know from Proposition 3.4 that if $|y - \tilde{x}_i| \ge 2r' = r/2$, then

$$\operatorname{dist}(y, X_i) \le \operatorname{dist}(y, R_i) - \frac{1}{2}\beta\rho - \frac{3}{2}\rho^2/r' \le \operatorname{dist}(y, R_i) - \beta\rho/2.$$

Let now

$$R_i' := \{x \in \mathbb{R}^n \,:\, \operatorname{dist}(x, R_i) < \beta\rho/4\}$$

stand for the open $\beta\rho/4$-neighborhood of R_i. Since

$$\bigcup_{i=1}^N B_\rho(x_i) \supset \Sigma \setminus \Sigma_\varepsilon \text{ and } \Sigma \cap B_\rho(x_i) \subset R_i \subset R_i',$$

then one has

$$\bigcup_{i=1}^N R_i' \supset \bigcup_{i=1}^N (\Sigma \cap B_\rho(x_i)) \supset \Sigma \setminus \Sigma_\varepsilon.$$

Further, if $|y - \tilde{x}_i| \ge r/2$ we conclude that $\operatorname{dist}(y, X_i) \le \operatorname{dist}(y, R_i') - \beta\rho/4$.

Consider the set

$$Z := \Sigma \setminus \bigcup_{i=1}^N R_i' \subset \Sigma_\varepsilon.$$

Since all R_i' are open sets and Σ is compact, then Z is a compact set.

Choose

$$\delta := \frac{3r}{4(2n+1)}. \tag{10}$$

Since the spherical Hausdorff measure of the rectifiable set is equal to the usual Hausdorff measure, then there exists an at most countable number of balls $B_{\delta_i}(z_i)$ with $z_i \in Z$ and $\delta_i < \delta$ such that

$$\bigcup_i B_{\delta_i}(z_i) \supset Z \text{ and } \sum_i 2\delta_i \le \mathcal{H}^1(Z) \le \mathcal{H}^1(\Sigma_\varepsilon) \le \varepsilon \tag{11}$$

The compactness of Z permits us to assume that there is only a finite number M of such balls.

Consider now the sets $Y_i := \mathbb{K}_{2\sqrt{n}\delta_i}(z_i)$ as defined in Lemma 3.2. We finally define

$$\Sigma' := \Sigma \cup \bigcup_{i=1}^N X_i \cup \bigcup_{i=1}^M Y_i.$$

By the properties of X_i and Y_i we know that Σ' is compact and connected. Further, minding that $\mathcal{H}^1(Y_i) \leq 4n\sqrt{n}\delta_i$ in view of Lemma 3.2, we get using (9) and (11) the estimate

$$\mathcal{H}^1(\Sigma') - \mathcal{H}^1(\Sigma) \leq \sum_{i=1}^{N} \mathcal{H}^1(X_i) + \sum_{i=1}^{M} \mathcal{H}^1(Y_i) \leq \frac{c}{2} + \sum_{i=1}^{M} 4n\sqrt{n}\delta_i \leq \frac{c}{2} + 2n\sqrt{n}\varepsilon.$$

Choosing $\varepsilon := c/4n\sqrt{n}$ we get $\mathcal{H}^1(\Sigma') - \mathcal{H}^1(\Sigma) \leq c$.

Suppose that $y \in \mathbb{R}^n$ is such that $\mathrm{dist}\,(y, \Sigma) \geq 3r/4$. Consider a point $x \in \Sigma$ such that $|x - y| = \mathrm{dist}\,(y, \Sigma)$. Only two cases may happen: either $x \in R'_i$ for some $i \in \{1, \ldots, N\}$ or $x \in B_{\delta_i}(z_i)$ for some $i \in \{1, \ldots, M\}$. We consider them separately.

Case A: $x \in R'_i$. Minding (6), we get the estimate

$$\begin{aligned} |y - \tilde{x}_i| &\geq |y - x| - |x - \tilde{x}_i| \\ &\geq 3r/4 - \sqrt{(\beta\rho)^2 + \rho^2} - \beta\rho/4 \\ &= 3r/4 - (\sqrt{\beta^2 + 1} + \beta/4)\rho \geq r/2, \end{aligned}$$

because $\beta \leq 1$ by (5) (hence $\sqrt{\beta^2 + 1} + \beta/4 \leq 8$) and $\rho \leq r'/9 \leq r'/8 = r/32$ by (6).

Therefore,

$$\begin{aligned} \mathrm{dist}\,(y, \Sigma') \quad \leq \quad \mathrm{dist}\,(y, X_i) \quad &\leq \quad \mathrm{dist}\,(y, R'_i) - \beta\rho/4 \leq |y - x| - \beta\rho/4 \\ &= \quad \mathrm{dist}\,(y, \Sigma) - \beta\rho/4. \end{aligned} \tag{12}$$

Case B: $x \in B_{\delta_i}(z_i)$.

We know that $y \notin B_{2n\delta_i}(z_i)$, since by (10) one has

$$|y - x_i| \geq |y - x| - |x - z_i| \geq 3r/4 - \delta = 2n\delta \geq 2n\delta_i.$$

Thus, by Lemma 3.2,

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, Y_i) \leq |y - x| - \delta_i \leq r - \delta', \tag{13}$$

where $\delta' := \min\{\delta_i \,:\, i = 1, \ldots, M\}$.

Combining (12) and (13), and setting

$$C := \min\{\beta\rho/4, \delta'\},$$

we conclude the proof. □

Basic properties of maximum distance minimizers

At this point we are able to prove that minimizers of maximum distance functional must always have maximum available length.

Theorem 4.2. *If Σ_{opt} solves Problem 2, and $\Sigma_{opt} \not\supset M$ (hence $F_M(\Sigma_{opt}) > 0$), then $\mathcal{H}^1(\Sigma_{opt}) = l$.*

Remark. The assumption $\Sigma_{opt} \not\supset M$ is necessarily satisfied, for instance, if $\mathcal{H}^1(M) = +\infty$.

Proof. Suppose the contrary, i.e. that $\mathcal{H}^1(\Sigma_{opt}) = l - c$ for some $c > 0$. Let $r := F_M(\Sigma_{opt})$ and consider a Σ' provided by Proposition 4.1 (applied with $\Sigma := \Sigma_{opt}$). We have that $\mathcal{H}^1(\Sigma') \leq l$. Besides, for every $y \in \mathbb{R}^n$ one has

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma_{opt})$$

since $\Sigma_{opt} \subset \Sigma'$. Hence, if $y \in M$ and $\mathrm{dist}\,(y, \Sigma_{opt}) \leq 3r/4$, we get that $\mathrm{dist}\,(y, \Sigma') \leq 3r/4$. On the other hand, if $\mathrm{dist}\,(y, \Sigma_{opt}) \geq 3r/4$, then Proposition 4.1 implies

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma_{opt}) - C \leq F_M(\Sigma_{opt}) - C$$

for some $C > 0$ independent of y. This means $F_M(\Sigma') \leq F_M(\Sigma_{opt}) - C$, which contradicts the optimality of Σ_{opt}. □

Corollary 4.3. *If Σ_{opt} solves Problem 2, and $\Sigma_{opt} \not\supset M$ (hence $F_M(\Sigma_{opt}) > 0$), then it solves also Problem 3 with $r := F_M(\Sigma_{opt})$.*

Proof. If it is not so, then there is a Σ' solving Problem 3 (i.e. such that $F_M(\Sigma') \leq r = F_M(\Sigma_{opt})$, but with $\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma_{opt}) \leq l$. Hence Σ' still solves Problem 2, but $\mathcal{H}^1(\Sigma') < l$ contrary to Theorem 4.2. □

We remark that the following "dual" result to the above corollary has been proven in [8]: if Σ_{opt} solves Problem 3, then it also solves Problem 2 with $l := \mathcal{H}^1(\Sigma_{opt})$. Together with the above corollary this can be interpreted as the "equivalence" of Problems 2 and 3.

Maximal length for average distance minimizers

The statement below is analogous to Theorem 4.2, but applies to average distance minimizers instead of maximum distance ones.

Theorem 4.4. *Suppose that in Problem 1 one has $\mathcal{H}^1(\mathrm{supp}\,\varphi) \geq l$ and A is strictly increasing. The for each Σ_{opt} solving Problem 1 one has $\mathcal{H}^1(\Sigma_{opt}) = l$.*

Proof. Suppose the contrary, i.e. that $\mathcal{H}^1(\Sigma_{opt}) = l - c$ for some $c > 0$. We choose an $r > 0$ such that

$$\varphi(D_r) > 0, \text{ where } D_r := \{y \in \mathbb{R}^n : \mathrm{dist}\,(y, \Sigma_{opt}) > 3r/4\}.$$

Such an $r > 0$ exists since otherwise φ would be concentrated over Σ_{opt}, i.e. $\mathrm{supp}\,\varphi \subset \Sigma_{opt}$ contradicting the assumption on φ. Consider a Σ' provided by Proposition 4.1 (applied with $\Sigma := \Sigma_{opt}$). We have that $\mathcal{H}^1(\Sigma') \leq l$. Besides, for every $y \in \mathbb{R}^n$ one has

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma_{opt})$$

since $\Sigma_{opt} \subset \Sigma'$, while

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma_{opt}) - C$$

for some $C > 0$ (independent of y) whenever $y \in D_r$. Hence, minding the strict monotonicity of A, we get

$$\begin{aligned} F_\varphi(\Sigma') &= \int_{\mathbb{R}^n \setminus D_r} A(\mathrm{dist}\,(y, \Sigma'))\, d\varphi(y) + \int_{D_r} A(\mathrm{dist}\,(y, \Sigma'))\, d\varphi(y) \\ &\leq \int_{\mathbb{R}^n \setminus D_r} A(\mathrm{dist}\,(y, \Sigma_{opt}))\, d\varphi(y) + \int_{D_r} A(\mathrm{dist}\,(y, \Sigma_{opt}) - C)\, d\varphi(y) \\ &< \int_{\mathbb{R}^n \setminus D_r} A(\mathrm{dist}\,(y, \Sigma_{opt}))\, d\varphi(y) + \int_{D_r} A(\mathrm{dist}\,(y, \Sigma_{opt}))\, d\varphi(y) \\ &= F_\varphi(\Sigma_{opt}), \end{aligned}$$

contradicting the optimality of Σ_{opt}. □

5 Absence of loops

In this section we prove that both minimizers of average distance functionals and those of maximum distance functional under certain natural conditions do not contain simple closed curves (homeomorphic images of S^1).

We now recall briefly the following topological notions which will be used in the sequel.

Definition 5.1. *Let Σ be a connected space. Then $x \in \Sigma$ is called noncut point of Σ, if $\Sigma \setminus \{x\}$ is connected. Otherwise, x is called cut point of Σ.*

Let us recall that according to the Moore theorem (theorem IV.5 from [7, § 47]), every continuum (i.e. compact connected space) has at least two noncut points.

We also need the following statement.

Lemma 5.2. *Let $\Sigma \subset \mathbb{R}^n$ be a closed connected set satisfying $\mathcal{H}^1(\Sigma) < \infty$ which contains a simple closed curve S. Then $\mathcal{H}^1$-a.e. point $x \in S$ is a noncut point for Σ.*

Proof. If $x \in S$ is a cut point for Σ then there is a continuum L_x such that $L_x \cap S = \{x\}$ and $L_x \setminus \{x\} \neq \emptyset$. One has then $L_x \cap L_y = \emptyset$ whenever $x \neq y$. But then $\mathcal{H}^1(L_x) > 0$ for only at most a countable number of points x, otherwise one would have $\mathcal{H}^1(\Sigma) = +\infty$. □

The following general result states that every noncut point possesses a connected neighborhood which can be cut leaving the set connected.

Lemma 5.3. *Let Σ be a locally connected metric continuum consisting of more than one point and $x \in \Sigma$ be a noncut point of Σ. Then there is a sequence of open sets $D_\nu \subset \Sigma$ satisfying*

(i) $x \in D_\nu$ *for all sufficiently large* ν*;*

(ii) $\Sigma \setminus D_\nu$ *are connected for all* ν*;*

(iii) $\operatorname{diam} D_\nu \searrow 0$ *as* $\nu \to \infty$*;*

(iv) D_ν *are connected for all* ν*.*

Proof. Let $z \in \Sigma$, $z \neq x$. For every couple of points $\{y_1, y_2\} \subset \Sigma$ we will say that y_1 is connected to y_2 through a closed connected set $\Gamma \subset \Sigma$, if $\{y_1, y_2\} \subset \Gamma$. Define now

$$\begin{aligned} X_\nu &:= \{y \in \Sigma : y \text{ connected to } z \text{ through a } \Gamma \subset \Sigma \setminus B_{1/\nu}(x)\}, \\ O_\nu &:= \Sigma \setminus X_\nu. \end{aligned}$$

We show first that O_ν are open for all ν, while

(i') $x \in O_\nu$ for all sufficiently large ν;

(ii') $\Sigma \setminus O_\nu$ are connected for all ν;

(iii') $\operatorname{diam} O_\nu \searrow 0$ as $\nu \to \infty$.

We first observe, that all X_ν are closed, hence O_ν are open. In fact, if $y_k \in X_\nu$, $y_k \to y$ as $k \to \infty$, then each y_k is connected to z through a set Γ_k satisfying $\Gamma_k \subset \Sigma \setminus B_{1/\nu}(x)$. Then, up to a subsequence (not relabeled) $\Gamma_k \to \Gamma$ in the sense of Hausdorff convergence, while Γ is a closed connected set which connects y to z, and $\Gamma \subset \Sigma \setminus B_{1/\nu}(x)$. In other words, $y \in X_\nu$.

Now, clearly, (i') holds since otherwise one would have $z = x$. Moreover, (ii') is also immediate from the definition of X_ν. To prove (iii'), suppose the contrary, namely, that there is an $r > 0$ such that for all ν outside of $B_r(x)$ there are points $y_\nu \in O_\nu \setminus B_r(x)$. It follows that for every set Γ_ν connecting y_ν with z one has

$$\Gamma_\nu \cap \bar{B}_{1/\nu}(x) \neq \emptyset. \tag{14}$$

Consider an arbitrary accumulation point y of a sequence $\{y_\nu\}$. The local connectedness of Σ implies (see theorem I.2 from [7, § 49]) that for each sufficiently large ν there is a closed connected set $C_\nu \subset \Sigma$ satisfying $C_\nu \cap B_{r/2}(x) = \emptyset$ which connects y and y_ν. Therefore, for every set $\Gamma \subset \Sigma$ connecting y to z one has $x \in \Gamma$. In fact, otherwise minding the compactness of Γ we would have $(\Gamma \cup C_\nu) \cap B_\varepsilon(x) = \emptyset$ for some $\varepsilon > 0$ independent of ν, although $\Gamma \cup C_\nu$ connects y_ν to z, in contradiction with (14).

Mind now that $y \notin B_r(x)$ and hence $y \in \Sigma \setminus \{x\}$. But the latter space is locally connected (as an open subset of a locally connected space Σ) and, being completely metrizable (again, as an open subset of Σ), is therefore also locally arcwise connected. Since it is supposed to be connected (because x is a noncut point of Σ), then it is arcwise connected. In particular, this means that there is an arc $\gamma \subset \Sigma \setminus \{x\}$ connecting y to z. But according to what has just been proven, every arc in Σ connecting y to z must pass through x. This contradiction thus proves (iii').

Let now D_ν stand for the connected component of O_ν which contains x. Then D_ν is relatively (with respect to O_ν) open in view of local connectedness of Σ and hence is open in Σ. One immediately observes that (i) and (iii) hold in view of (i') and (iii') respectively. Since (iv) automatically follows from the construction, it remains to verify (ii). The latter, in view of (ii') will be shown once we prove that for every $y \in O_\nu \setminus D_\nu$ there is a set Γ connecting y to z such that $\Gamma \cap D_\nu = \emptyset$.

To prove the latter claim, suppose the contrary, i.e. that for some $y \in O_\nu \setminus D_\nu$ and for every Γ connecting y to z one has $\Gamma \cap D_\nu \neq \emptyset$. Let γ be an arbitrary arc connecting y to z (so that $\gamma(0) = y$ and $\gamma(1) = z$), and set

$$\bar{t} := \sup\{t \in [0,1] \, : \, \gamma(s) \notin D_\nu \text{ for all } s \in [0,t)\}, \qquad \tilde{x} := \gamma(\bar{t}).$$

Consider now the arc $[y, \tilde{x}] \subset \gamma$ which connects y to $\tilde{x}$, and let $[y, \tilde{x}) := [y, \tilde{x}] \setminus \{\tilde{x}\}$. We claim

$$[y, \tilde{x}) \subset O_\nu. \tag{15}$$

In fact, if $v \in [y, \tilde{x})$, then for every set Γ' connecting v with z there is an $x' \in \Gamma' \cap D_\nu$, since otherwise, minding that for $[y, v] \subset \gamma$ one has $[y, v] \cap D_\nu = \emptyset$ and hence, contrary to our assumption, $([y, v] \cup \Gamma') \cap D_\nu = \emptyset$ despite the fact that $[y, v] \cup \Gamma'$ connects y to z. Since $x' \in O_\nu$ then necessarily $\Gamma' \cap \bar{B}_{1/\nu}(x) \neq \emptyset$. Since the latter holds for an arbitrary closed connected Γ' connecting v to z, then by construction $v \in O_\nu$ which implies (15) in view of arbitrarity of v.

We prove now $\tilde{x} \in O_\nu$. In fact, if this is not the case, then there is a set $\tilde{\Gamma}$ connecting $\tilde{x}$ to z such that $\tilde{\Gamma} \subset X \setminus B_{1/\nu}(x)$, hence, $\tilde{\Gamma} \cap O_\nu = \emptyset$ and in particular $\tilde{\Gamma} \cap D_\nu = \emptyset$. But then $([y, \tilde{x}] \cup \tilde{\Gamma}) \cap D_\nu = \emptyset$ contrary to our assumption since $[y, \tilde{x}] \cup \tilde{\Gamma}$ connects y to z.

Combining the latter observation with (15), we get

$$[y, \tilde{x}] = \gamma([0, \bar{t}]) \subset O_\nu.$$

But since O_ν is open then there is a $t' > \bar{t}$ such that $\gamma([0, t']) \subset O_\nu$, which means that $\gamma([0, t'])$ belongs to the same connected component of O_ν. Since by definition of $\bar{t}$ one has $\gamma([0, t']) \cap D_\nu \neq \emptyset$, then this component is D_ν, and in particular, $y \in D_\nu$. This however contradicts the choice of y and thus proves the claim. □

Absence of loops for maximum distance minimizers

We are able to prove that minimizers of maximum distance functional never contain simple closed curves. Our construction is based on the following lemma which states that a set Σ which contains a loop can be modified in a set Σ' which has less length and is such that the set of points of $\mathbb{R}^n$ which increase their distance from the set, is contained in a small ball.

Lemma 5.4. *Suppose that $\Sigma \subset \mathbb{R}^n$ be a compact connected set satisfying $\mathcal{H}^1(\Sigma) < +\infty$ and containing a simple closed curve S. Given a $\beta \in (0,1]$, for $\mathcal{H}^1$-a.e. $\bar{x} \in S$, and for every $r > 0$, there exists a $\rho \in (0,r)$ and a closed connected set $\Sigma' \subset \mathbb{R}^n$ such that*

$$
\begin{aligned}
&\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \rho/2 + C_2\beta\rho, \\
&\Sigma \setminus \Sigma' \subset B_\rho(\bar{x}), \\
&\Sigma' \setminus \Sigma \subset B_{32n\rho}(\bar{x}), \\
&\operatorname{dist}(y, \Sigma') \leq \operatorname{dist}(y, \Sigma), && \text{for all } y \notin B_{64n\sqrt{n}\rho}(\bar{x}), \\
&\operatorname{dist}(y, \Sigma') \leq \operatorname{dist}(y, \Sigma) + \rho, && \text{for all } y \in B_{64n\sqrt{n}\rho}(\bar{x}),
\end{aligned}
$$

where $C_2 > 0$ is a constant depending only on n.

Proof. Let $\gamma : [0,1] \to \Sigma$ be a Lipschitz parameterization of the simple closed curve S in Σ. Let $\bar{x} := \gamma(\bar{t})$ be such that $\bar{t} \in (0,1)$, γ is differentiable in $\bar{t}$, $\lim_{\rho\to 0^+} \beta_\Sigma(\bar{x}, \rho) = 0$ and $\bar{x}$ is a noncut point of Σ. In view of Proposition 2.2 and of Lemma 5.2 this is true for $\mathcal{H}^1$-a.e. $\bar{x} \in S$. Take a system of orthonormal coordinates in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ such that $\bar{x} = (0,0)$, $\gamma'(\bar{t}) = (|\gamma'(\bar{t})|, 0)$.

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Choose a $\rho_0 \in (0,r)$ such that

$$\beta(\bar{x}, \rho) \leq \beta \qquad \text{for all } \rho \leq \rho_0.$$

Let D_ν be a neighborhood of $\bar{x}$ in Σ as given by Lemma 5.3 such that $\operatorname{diam} D_\nu \leq \rho_0$. Let $\rho > 0$ be such that $\bar{B}_\rho(\bar{x})$ is the smallest ball containing D_ν. Hence $D_\nu \subset \Sigma \cap B_\rho(\bar{x}) \subset [-\rho, \rho] \times \bar{B}_{\beta\rho}$, where $\bar{B}_{\beta\rho} \subset \mathbb{R}^{n-1}$ stands for the closed ball of radius $\beta\rho$ centered at zero.

Let $a, b \in [0, \rho]$ be the smallest numbers such that $D_\nu \subset [-a, b] \times \bar{B}_{\beta\rho}$. Thus there exist $x_1 \in \Sigma \cap B_{\beta\rho}(-a, 0)$ and $x_2 \in \Sigma \cap B_{\beta\rho}(b, 0)$.

Choose $r' := 32n\sqrt{n}\rho$. Then Proposition 3.4 applied with r' instead of r gives us the set $X = X^+ \cup X^-$. Let S^- be the segment joining x_1 to $(-a, 0)$ and S^+ the segment joining x_2 to $(b, 0)$. Define

$$\Sigma' := \Sigma \setminus D_\nu \cup X \cup S^+ \cup S^-.$$

We observe that Σ' is connected since so is $\Sigma \setminus D_\nu$. Further, $\Sigma \setminus \Sigma' \subset D_\nu \subset B_\rho(\bar{x})$ by construction. On the other hand, $\Sigma' \setminus \Sigma \subset B_{r'/\sqrt{n}}(\bar{x}) = B_{32n\rho}(\bar{x})$, by Proposition 3.4, while $S^\pm \subset \bar{B}_{\rho+\beta\rho}(\bar{x})$.

Note that $\mathcal{H}^1(S^\pm) \leq \beta\rho$ and $\mathcal{H}^1(X) = 16n\sqrt{n}(\beta\rho + \rho^2/r')$, while $\mathcal{H}^1(D_\nu) \geq \operatorname{diam} D_\nu \geq \rho$. Clearly we have then

$$
\begin{aligned}
\mathcal{H}^1(\Sigma') &\leq \mathcal{H}^1(\Sigma) - \rho + 2\beta\rho + 16n\sqrt{n}\beta\rho + 16n\sqrt{n}\rho^2/r' \\
&\leq \mathcal{H}^1(\Sigma) - \rho + 2\beta\rho + 16n\sqrt{n}\beta\rho + \rho/2 \\
&\leq \mathcal{H}^1(\Sigma) - \rho/2 + C_2\beta\rho,
\end{aligned}
$$

where $C_2 := 16n\sqrt{n} + 2$.

Finally, the first estimate on $\operatorname{dist}(y, \Sigma')$ in the statement of the lemma being proven follows immediately from Proposition 3.4, while the second one follows from the fact that $\Sigma \setminus \Sigma' \subset B_\rho(\bar{x})$. □

We are able now to claim that maximum distance minimizers, apart from trivial cases, never contain closed loops. Again we recall the assumption $F_M(\Sigma_{opt}) > 0$ is satisfied if, for instance, $H(M) = +\infty$.

Theorem 5.5. *Let Σ_{opt} solve Problem 3 and $F_M(\Sigma_{opt}) > 0$. Then Σ_{opt} contains no simple closed curve (a homeomorphic image of S^1).*

Proof. Supposing the contrary and choosing $r := F_M(\Sigma_{opt})/65n\sqrt{n}$, $\beta := 1/(4C_2)$ (where $C_2 > 0$ is the constant defined in the statement of Lemma 5.4), we consider then an $\bar{x} \in \Sigma_{opt}$, a $\rho < r$ and a Σ' as provided by Lemma 5.4 (with Σ_{opt} in place of Σ).

By Lemma 5.4 we have $\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma_{opt})$, while if

$$|y - \bar{x}| \geq 64n\sqrt{n}\rho,$$

then $\operatorname{dist}(y, \Sigma') \leq \operatorname{dist}(y, \Sigma_{opt}) \leq F_M(\Sigma_{opt})$. Otherwise, if $|y - \bar{x}| < 64n\sqrt{n}\rho$, then

$$\operatorname{dist}(y, \Sigma') \leq \operatorname{dist}(y, \Sigma_{opt}) + \rho \leq 64n\sqrt{n}\rho + \rho \leq 65n\sqrt{n}r \leq F_M(\Sigma_{opt}).$$

Summing up, we get

$$F_M(\Sigma') \leq F_M(\Sigma_{opt})$$

which is in contradiction with the minimality of Σ_{opt}. □

Absence of loops for average distance minimizers

We may now announce the result regarding the absence of loops for minimizers of average distance problems analogous to that of Theorem 5.5.

Theorem 5.6. *Let Σ_{opt} solve Problem 1, the function A satisfy conditions (α_1) and (α_2), and suppose that $\varphi(\Sigma_{opt}) = 0$. Then Σ_{opt} contains no simple closed curve (a homeomorphic image of S^1).*

Proof. Define $L := \mathcal{H}^1(\Sigma_{opt})$. We may suppose $L > 0$ (otherwise there is nothing to prove).

Since $\varphi(\Sigma_{opt}) = 0$, there exists a compact set K, disjoint from Σ_{opt} and such that $\varphi(K) > 0$. Let

$$R := \frac{1}{2} \min\{\operatorname{dist}(y, \Sigma_{opt}) \,:\, y \in K\} > 0$$

and let $H := (\Sigma_{opt})_R$ be the R-neighborhood of Σ_{opt}. Suppose by contradiction that there exists a simple closed curve $S \subset \Sigma_{opt}$. Let $\beta := (4C_2)^{-1}$, where C_2 is the constant defined in Lemma 5.4.

Given an $x \in \Sigma_{opt}$, define

$$\omega(x, \rho) := \frac{\varphi(B_\rho(x))}{\rho}.$$

Since $\varphi(S) = 0$, from the direct consequence of theorem 2.56 from [1] for $\mathcal{H}^1$-a.e. $x \in S$ one has

$$\lim_{\rho \to 0^+} \omega(x, \rho) = 0.$$

Let $r > 0$ to be chosen later. We apply therefore Lemma 5.4 to find a point $\bar{x} \in S$ with $\omega(x, t) \to 0$ as $t \to 0^+$, a $\rho \in (0, r)$ and a connected set Σ' such that

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{opt}) - \rho/2 + C_2\beta\rho = \mathcal{H}^1(\Sigma_{opt}) - \rho/4, \tag{16}$$

while

$$
\begin{aligned}
F_\varphi(\Sigma') \leq\ & F_\varphi(\Sigma_{opt}) \\
&+ \int_{B_{64n\sqrt{n}\rho}(\bar{x})} (A(\mathrm{dist}\,(y, \Sigma_{opt}) + \rho) - A(\mathrm{dist}\,(y, \Sigma_{opt}))\ d\varphi(y) \\
&\leq F_\varphi(\Sigma_{opt}) + 64n\sqrt{n}\Lambda\rho^2\omega(\bar{x}, 64n\sqrt{n}\rho).
\end{aligned}
\tag{17}
$$

Apply now Lemma 3.5 with $\varepsilon := \rho/8n$, Σ' in place of Σ, R in place of r, and L in place of l (mind that $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{opt}) = L$) to find a connected compact set $\Sigma'' \supset \Sigma'$ such that, by (16)

$$
\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2n\varepsilon \leq \mathcal{H}^1(\Sigma_{opt}) - \rho/4 + 2n\varepsilon = \mathcal{H}^1(\Sigma_{opt}),
$$

while, by (17),

$$
F_\varphi(\Sigma'') \leq F_\varphi(\Sigma') - C_1\varepsilon^2 \tag{18}
$$

$$
\leq F_\varphi(\Sigma) + 64n\sqrt{n}\Lambda\rho^2\omega(\bar{x}, 64n\sqrt{n}\rho) - \frac{C_1}{16n^2}\rho^2, \tag{19}
$$

where C_1 is the constant introduced in Lemma 3.5. Hence, choosing an $r > 0$ such that

$$
64n\sqrt{n}\Lambda\omega(\bar{x}, 64n\sqrt{n}\rho) < \frac{C_1}{16n^2}
$$

for all $\rho < r$, we get from (18) that $F_\varphi(\Sigma'') < F_\varphi(\Sigma_{opt})$ contradicting the optimality of Σ_{opt}. □

We recall the following well-known notion.

Definition 5.7. *The Hausdorff dimension* $\dim\varphi$ *of the Borel measure* φ *over* $\mathbb{R}^n$ *is defined as*

$$
\dim\,\varphi = \sup\{k\,:\,\varphi \ll \mathcal{H}^k\},
$$

where $\mathcal{H}^k$ *stands for the* k*-dimensional Hausdorff measure.*

For instance, if $\varphi \ll \mathcal{L}^n$, then $\dim\,\varphi = n$.

The above notion allows to formulate the following immediate corollary of the above Theorem 5.6.

Corollary 5.8. *Let* Σ_{opt} *solve Problem 1, the function* A *satisfy conditions* (α_1) *and* (α_2)*, and suppose that* $\dim\varphi > 1$*. Then* Σ_{opt} *contains no simple closed curve (a homeomorphic image of* S^1*). In particular, this assertion holds true when* $\varphi \ll \mathcal{L}^n$*.*

6 Ahlfors regularity

We show now that both the minimizers of Problem 1 and those of Problem 3 (hence also those of Problem 2 in view of Corollary 4.3) possess some mild regularity properties, namely that they are is Ahlfors regular. Recall that a set $\Sigma \subset \mathbb{R}^n$ is called Ahlfors regular, if there exist two constants $c > 0$ and $C > 0$ such that for every positive $\rho < \mathrm{diam}\,\Sigma$ and for every $x \in \Sigma$ one has

$$
c\rho \leq \mathcal{H}^1(\Sigma \cap B_\rho(x)) \leq C\rho \tag{20}
$$

(while a singleton is considered to be Ahlfors regular by definition). It is worth mentioning that Ahlfors regularity of a closed connected set Σ implies the so-called *uniform rectifiability* on Σ, which, as it has been shown in [5], provides several nice analytical properties of Σ. This condition can be considered a kind of "quantitative rectifiability" which is somewhat stronger than the classical rectifiability used in geometric measure theory.

We remark that to verify Ahlfors regularity of $\Sigma \subset \mathbb{R}^n$, it is enough to show the existence of some $r_0 \in (0, \operatorname{diam}\Sigma)$ (independent of $x \in \Sigma$) such that (20) holds just for $r \le r_0$. In fact, in this case, when $r > r_0$ and $r < \operatorname{diam}\Sigma$, we have

$$\begin{aligned}\frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r} &= \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r_0}\frac{r_0}{r}\\ &\ge \frac{\mathcal{H}^1(\Sigma \cap B_{r_0}(x))}{r_0}\frac{r_0}{\operatorname{diam}\Sigma} \ge c\frac{r_0}{\operatorname{diam}\Sigma},\end{aligned}$$

and on the other hand,

$$\frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r} \le \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r_0} \le \frac{\mathcal{H}^1(\Sigma)}{r_0}.$$

We further remark that if Σ is closed and connected, then the lower estimate in (20) is trivial: in fact, for all $\rho < r_0 := \operatorname{diam}\Sigma/2$ one has $\Sigma \cap \partial B_\rho(x) \ne \emptyset$, and hence

$$\mathcal{H}^1(\Sigma \cap \partial B_\rho(x)) \ge \rho$$

when $x \in \Sigma$. Hence, for such Σ the proof of Ahlfors regularity reduces to verifying that for every $x \in \Sigma$ and for all $\rho > 0$ sufficiently small (independently of x) one has

$$\lambda(x, \rho) \le C, \tag{21}$$

where

$$\lambda(x, \rho) := \frac{\mathcal{H}^1(\Sigma \cap B_\rho(x))}{\rho}. \tag{22}$$

We will need the following auxiliary assertion which estimates the length of the minimal connection of k given points.

Lemma 6.1. *Given k points $\{z_i\}_{i=1}^k \subset \bar{B}_\rho(\bar{x}) \subset \mathbb{R}^n$, there is a compact connected set $\Sigma_0 \subset \bar{B}_{\sqrt{n}\rho}(\bar{x})$ containing all these points, such that*

$$\mathcal{H}^1(\Sigma_0) \le C_3 k^{1-1/n}\rho$$

for some $C_3 = C_3(n) > 0$.

Proof. Up to a rescaling we may suppose that $\rho = 1/2$ and up to a translation we suppose that all the points $z_1, \ldots, z_k$ are contained in the unit cube $I^n = [0,1]^n$.

Let Γ^j be the set of all $(x_1, \ldots, x_n) \in I^n$ such that $jx_i \in \mathbb{N}$ for all $i = 1, \ldots, n$ except at most one (i.e. Γ^j is a uniform one-dimensional grid of step $1/j$ in I^n). Then clearly $\Gamma^j \subset I^n$ is connected and

$$\mathcal{H}^1(\Gamma^j) \le n(j+1)^{n-1} \text{ and } \operatorname{dist}(y, \Gamma^j) \le \sqrt{n}/2j$$

for all $y \in I^n$. For each $i = 1, \ldots, k$ denote by z_i^j an arbitrary projection of z_i to Γ^j, and let $[z_i, z_i^j]$ stand for the segment connecting these two points. Then the set

$$\tilde{\Gamma}^j := \Gamma^j \cup \bigcup_{i=1}^k [z_i, z_i^j]$$

connects all the points z_i, and hence

$$\mathcal{H}^1(\tilde{\Gamma}^j) \le \mathcal{H}^1(\Gamma^j) + \sum_{i=1}^k \operatorname{dist}(z_i, \Gamma^j) = n(j+1)^{n-1} + \frac{k\sqrt{n}}{2j}.$$

Taking in the above relationship $\Sigma_0 := \tilde{\Gamma}^j \subset I^n \subset \bar{B}_{\sqrt{n}\rho}(\bar{x})$ with $j := \lfloor k^{1/n} \rfloor$, where $\lfloor\cdot\rfloor$ stands for the integer part, we get the desired conclusion. □

Ahlfors regularity for maximum distance minimizers

We are able now to prove Ahlfors regularity of solutions to problem 3.

The following lemma will be necessary in our proof.

Lemma 6.2. *Suppose that $\Sigma \subset \mathbb{R}^n$ be a compact connected set satisfying $\mathcal{H}^1(\Sigma) < +\infty$, and $x \in \Sigma$. Then there exists a constant $C_4 > 0$ depending only on n, such that for every $\rho > 0$ and for every $x \in \Sigma$ there exists a closed connected set $\Sigma' \subset \mathbb{R}^n$ such that*

$$\begin{aligned} &\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \lambda(x,\rho)\rho + C_4(\lambda(x,2\rho)^\alpha + 1)\rho, \\ &\Sigma \setminus \Sigma' \subset B_{2\rho}(x), \\ &\Sigma' \setminus \Sigma \subset B_{8\sqrt{n}\rho}(x), \\ &\operatorname{dist}(y,\Sigma') < \operatorname{dist}(y,\Sigma), \qquad \text{for all } y \notin B_{4n\rho}(x), \\ &\operatorname{dist}(y,\Sigma') \leq \operatorname{dist}(y,\Sigma) + 2\rho, \qquad \text{for all } y \in B_{4n\rho}(x) \end{aligned}$$

where $\alpha = 1 - 1/n$ and $\lambda(x,\rho)$ is defined by (22).

Proof. Let $x \in \Sigma$ be an arbitrary point and set

$$k(x,\rho) := \#\Sigma \cap \partial B_\rho(x).$$

By the coarea formula one has

$$2\rho\lambda(x,2\rho) \geq \int_0^{2\rho} k(x,t)\,dt \geq \int_\rho^{2\rho} k(x,t)\,dt$$

and hence there exists a $t \in [\rho, 2\rho]$ such that

$$k(x,t) \leq 2\lambda(x,2\rho). \tag{23}$$

Let $\Sigma_0(t)$ be the set constructed in Lemma 6.1 with respect to the $k(x,t)$ points $\Sigma \cap \partial B_t(x)$ so that $\mathcal{H}^1(\Sigma_0(t)) \leq C_3 k^{1-1/n}(x,t)t$. ¿From now on C will stand for a positive constant depending only on n which may be different from line to line. Lemma 3.2 defines the set $\Sigma_1(t) := \mathbb{K}_{2\sqrt{n}t}(x)$ such that $x \in \Sigma_1(t)$, $\mathcal{H}^1(\Sigma_1(t)) = 4n\sqrt{n}t$ and given any $y \in \mathbb{R}^n$ with $|y - x| \geq 2nt$ one has

$$\operatorname{dist}(y, \Sigma_1(t)) \leq |y - x| - t < \operatorname{dist}(y,\Sigma). \tag{24}$$

With a possible rotation of $\Sigma_1(t)$ around x we may suppose $\Sigma_0(t) \cap \Sigma_1(t) \neq \emptyset$. Then the set

$$\Sigma' := \Sigma \setminus B_t(x) \cup \Sigma_0(t) \cup \Sigma_1(t)$$

is connected and closed. Also, clearly one has $\Sigma \setminus \Sigma' \subset B_{2\rho}(x)$, which in its turn implies that

$$\operatorname{dist}(y,\Sigma') \leq \operatorname{dist}(y,\Sigma) + 2\rho$$

for all $y \in \mathbb{R}^n$. At the same time, for $y \notin B_{2nt}(x)$, the estimate (24) implies $\operatorname{dist}(y,\Sigma') < \operatorname{dist}(y,\Sigma)$.

To prove the estimate on $\mathcal{H}^1(\Sigma)$, we observe that

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B_t(x)) + \mathcal{H}^1(\Sigma_0(t)) + \mathcal{H}^1(\Sigma_1(t)).$$

By definition one has $\mathcal{H}^1(\Sigma \cap B_t(x)) = t\lambda(x,t) \geq \rho\lambda(x,\rho)$. By Lemma 6.1 and (23) one has

$$\mathcal{H}^1(\Sigma_0(t)) \leq C_3 k^\alpha(x,t)t \leq 2C_3\lambda^\alpha(x,2\rho)t$$

where $\alpha = 1 - 1/n$. Also $\mathcal{H}^1(\Sigma_1(t)) = 4n\sqrt{n}t$ by construction. Summing up, we get

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \lambda(x,\rho)\rho + C_4(\lambda^\alpha(x,2\rho)+1)\rho$$

with $C_4 = \max\{8n\sqrt{n}, 4C_3\}$.

Finally, we note that $\Sigma_1(t) \in \bar{B}_{2\sqrt{n}t}(x)$ by construction, while clearly $\Sigma_0(t) \in \bar{B}_t(x)$, which in particular implies

$$\Sigma' \setminus \Sigma \subset B_{4\sqrt{n}t}(x) \subset B_{8\sqrt{n}\rho}(x).$$

concluding the proof. □

Lemma 6.3. *Let $\Sigma \subset \mathbb{R}^n$ be bounded and suppose that there exists an $r > 0$ such that for all $x \in \Sigma$ and for all $\rho \in (0, r]$ one has*

$$\lambda(x,\rho) \leq a\lambda^\alpha(x,2\rho) + b$$

for some constants $a > 0$, $b \geq 0$ and $\alpha \in (0,1)$. Then there exists a constant $C > 0$ depending only on $a, b, r, \mathcal{H}^1(\Sigma)$ such that for all $x \in \Sigma$ and all $\rho \in (0, r]$ one has

$$\lambda(x,\rho) \leq C.$$

Proof. Consider an arbitrary $x \in \Sigma$. Let for all $\nu \in \mathbb{N}$

$$\lambda_\nu(x) := \lambda(x, r/2^\nu).$$

Let $\bar{\lambda}$ stand for the unique positive solution of the equation

$$\lambda = a\lambda^\alpha + b.$$

Then, for every $\lambda > \bar{\lambda}$ one has

$$a\lambda^\alpha + b > \lambda.$$

Then for each $\nu \in \mathbb{N}$ either $\lambda_\nu(x) \leq \bar{\lambda}$, or else $\lambda_{\nu+1}(x) \leq \lambda_\nu(x)$. Summing up, we have that

$$\lambda_\nu(x) \leq \max\{\bar{\lambda}, \lambda_0(x)\}.$$

But

$$\lambda_0(x) = \lambda(x,r) = \frac{\mathcal{H}^1(\Sigma \cap B_r(x))}{r} \leq \frac{\mathcal{H}^1(\Sigma)}{r}.$$

Hence,

$$\lambda_\nu(x) \leq C' := \max\{\bar{\lambda}, \mathcal{H}^1(\Sigma)/r\}.$$

If $\rho \leq r$ is an arbitrary positive number, then there is a $\nu \in \mathbb{N}$ such that $\rho \in [r/2^{\nu+1}, r/2^\nu]$. Therefore,

$$\rho\lambda(x,\rho) \leq \frac{r}{2^\nu}\lambda(x, r/2^\nu) = \frac{r}{2^\nu}\lambda_\nu(x) \leq 2\rho C',$$

which implies $\lambda(x,\rho) \leq C := 2C'$. □

Theorem 6.4. *Let Σ_{opt} solve Problem 3. Then Σ_{opt} is Ahlfors regular. The constants C and c in (20) depend only on M and on the space dimension n.*

Proof. Define $r := F_M(\Sigma)/6n$, and for arbitrary $\rho \in (0, r]$ and $x \in \Sigma_{opt}$ consider a set Σ' as provided by Lemma 6.2 (applied with Σ_{opt} in place of Σ). If $y \notin B_{4n\rho}(x)$, then

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma_{opt}) \leq F_M(\Sigma_{opt})$$

by Lemma 6.2. Otherwise, if $y \notin B_{4n\rho}(x)$, then

$$\mathrm{dist}\,(y, \Sigma') \leq \mathrm{dist}\,(y, \Sigma_{opt}) + 2\rho \leq 4n\rho + 2\rho \leq 6n\rho \leq 6nr = F_M(\Sigma_{opt}).$$

Summing up, we get

$$F_M(\Sigma') \leq F_M(\Sigma_{opt}),$$

and hence the optimality of Σ_{opt} yields

$$\mathcal{H}^1(\Sigma) \leq \mathcal{H}^1(\Sigma').$$

By Lemma 6.2 the latter relationship implies

$$\lambda(x,\rho) - C\,(\lambda^\alpha(x, 2\rho) + 1) \leq 0.$$

In view of Lemma 6.3, we have therefore $\lambda(x,\rho) \leq C$ for some $C > 0$ and for all $\rho \leq r$, where C depends only on the problem data. □

Ahlfors regularity for average distance minimizers

We also prove Ahlfors regularity of solutions to Problem 1.

Theorem 6.5. *Let Σ_{opt} solve Problem 1, the function A satisfy conditions (α_1) and (α_2), and $\varphi \in L^p(\mathbb{R}^n)$, $p \geq n/(n-1)$, $n > 1$. Then Σ_{opt} is Ahlfors regular.*

Remark. A better result for $n = 2$ has been proven in [4] (it required only $\varphi \in L^p(\mathbb{R}^n)$, $p \geq 4/3$).

Proof. We may assume $l > 0$ (otherwise Σ_{opt} is a singleton and hence there is nothing to prove). Set $r := \mathrm{diam}\,\Sigma_{opt}$, $x \in \Sigma_{opt}$. There is a $c \in (0, r)$ such that for $K := \mathbb{R}^n \setminus (\Sigma_{opt})_{2c}$ one has $\varphi(K) > 0$ (since $\varphi(\Sigma_{opt}) = 0$ because $\mathcal{L}^n(\Sigma_{opt}) = 0$). Choose now an arbitrary $\rho < c/8\sqrt{n}$.

Let Σ' be the set defined by Lemma 6.2 applied with Σ_{opt} in place of Σ. Minding (α_1), we have then

$$F_\varphi(\Sigma') \leq F_\varphi(\Sigma_{opt}) + 2\Lambda\rho\varphi(B_{4n\rho}(x)).$$

By Hölder inequality, one has

$$\varphi(B_{4n\rho}(x)) \leq \|\varphi\|_p \cdot |B_{4n\rho}(x)|^{1/p'} \leq C\rho^{n/p'}, \qquad 1/p' + 1/p = 1,$$

for some $C > 0$ depending only on n and $\|\varphi\|_p$. Hence, we have

$$F_\varphi(\Sigma') \leq F_\varphi(\Sigma_{opt}) + C\rho^{n/p'+1}, \tag{25}$$

where $C > 0$ depends neither on ρ nor on x (from now on it may be different from line to line).

On the other hand, we note that

$$\mathcal{H}^1(\Sigma') - \mathcal{H}^1(\Sigma_{opt}) \geq \rho\lambda(x,\rho) - \rho C\,(\lambda^\alpha(x, 2\rho) + 1)\,.$$

If the right-hand side of the above relationship is negative, then

$$\rho\lambda(x,\rho) - \rho C\,(\lambda^\alpha(x, 2\rho) + 1) \leq 0,$$

and hence by Lemma 6.3 one has $\lambda(x,\rho) \leq C$ (where $C > 0$ depends only on the problem data). Otherwise, note that, setting $H := (\Sigma_{opt})_c$, we have that $\Sigma' \subset H$ since $\rho < c/8\sqrt{n}$ by our choice and $\Sigma' \setminus \Sigma \subset B_{8\sqrt{n}\rho}(x)$ by Lemma 6.2. We use then Lemma 3.5 with

$$\varepsilon := \rho\lambda(x,\rho) - \rho C\,(\lambda^\alpha(x, 2\rho) + 1)\,,$$

and Σ' in place of Σ to get a new set Σ'' such that

$$F_\varphi(\Sigma'') \leq F_\varphi(\Sigma') - C_1 \left(\rho\lambda(x,\rho) - \rho C \left(\lambda^\alpha(x,2\rho)+1\right)\right)^2, \tag{26}$$

where C_1 is the constant defined in Lemma 3.5. Putting together (26) and (25) and minding that $F_\varphi(\Sigma'') \geq F_\varphi(\Sigma_{opt})$ by the optimality assumption on Σ_{opt}, we get

$$\left(\rho\lambda(x,\rho) - \rho C \left(\lambda^\alpha(x,2\rho)+1\right)\right)^2 \leq C'\rho^{n/p'+1},$$

where the constant $C' > 0$ depends only on $\|\varphi\|_p$, n, c, $\varphi(K)$, $\mathcal{H}^1(\Sigma_{opt})$ (hence, since c and K depend on Σ_{opt}, then C' depends on the problem data and on Σ_{opt}). This is only possible when

$$\lambda(x,\rho) - C\left(\lambda^\alpha(x,2\rho)+1\right) \leq C'\rho^\delta,$$

where $\delta := n/2p' - 1/2$. Minding the assumption on p, we have that $\delta \geq 0$, and hence $\rho^\delta \leq\leq (\operatorname{diam}\Sigma_{opt})^\delta$. Applying again Lemma 6.3 one has $\lambda(x,\rho) \leq C$, where $C > 0$ depends only on problem data and on Σ_{opt}. □

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford mathematical monographs. Oxford University Press, Oxford, 2000.

[2] G. Buttazzo, E. Oudet, and E. Stepanov. Optimal transportation problems with free Dirichlet regions. In *Progress in Nonlinear Diff. Equations and their Applications*, volume 51, pages 41–65. Birkhäuser, 2002.

[3] G. Buttazzo and E. Stepanov. Optimal transportation networks as free Dirichlet regions in the Monge-Kantorovich problem. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, II(4):631–678, 2003.

[4] G. Buttazzo and E. Stepanov. Minimization problems for average distance functionals, 2004. To appear in: *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi*, D. Pallara (ed.), Quaderni di Matematica, Series edited by Dipartimento di Matematica, Seconda Università di Napoli.

[5] G. David and S. Semmes. *Analysis of and on uniformly rectifiable sets*, volume 38 of *Math. Surveys Monographs.* Amer. Math. Soc., Providence, RI, 1993.

[6] Z. Drezner, editor. *Facility location: a survey of applications and methods.* Springer series in operations research. Springer Verlag, 1995.

[7] C. Kuratowski. *Topologie*, volume 1. Państwowe Wydawnictwo Naukowe, Warszawa, 1958. in French.

[8] M. Miranda, E. Paolini, and E. Stepanov. On one-dimensional continua uniformly approximating planar sets, 2004. `http://cvgmt.sns.it/papers/mirpaoste04`.

[9] F. Morgan and R. Bolton. Hexagonal economic regions solve the location problem. *Amer. Math. Monthly*, 109(2):165–172, 2001.

[10] A. Suzuki and Z. Drezner. The *p*-center location. *Location science*, 4(1-2):69–82, 1996.

[11] A. Suzuki and A. Okabe. Using Voronoi diagrams. In Z. Drezner, editor, *Facility location: a survey of applications and methods*, Springer series in operations research, pages 103–118. Springer Verlag, 1995.