REGULARITY FOR GRAPHS WITH BOUNDED ANISOTROPIC MEAN CURVATURE

ANTONIO DE ROSA AND RICCARDO TIONE

Abstract. We prove that m-dimensional Lipschitz graphs with anisotropic mean curvature bounded in L^p, $p > m$, are regular almost everywhere in every dimension and codimension. This provides partial or full answers to multiple open questions arising in the literature. The anisotropic energy is required to satisfy a novel ellipticity condition, which holds for instance in a $C^{1,1}$ neighborhood of the area functional. This condition is proved to imply the atomic condition. In particular we provide the first non-trivial class of examples of anisotropic energies in high codimension satisfying the atomic condition, addressing an open question in the field. As a byproduct, we deduce the rectifiability of varifolds (resp. of the mass of varifolds) with locally bounded anisotropic first variation for a $C^{1,1}$ (resp. C^1) neighborhood of the area functional. In addition to these examples, we also provide a class of anisotropic energies in high codimension, far from the area functional, for which the rectifiability of the mass of varifolds with locally bounded anisotropic first variation holds. To conclude, we show that the atomic condition excludes non-trivial Young measures in the case of anisotropic stationary graphs.

1. Introduction

A celebrated theorem of W. Allard [2] states that, given a rectifiable m-varifold V in $\mathbb{R}^N$ with density greater or equal than 1 and generalized mean curvature bounded in $L^p(\|V\|)$ with $p > m$, then V is regular around $x \in \mathbb{R}^N$ provided x has density ratio sufficiently close to 1. The proof deeply relies on the monotonicity formula of the density ratio, which is strictly related to the special symmetries of the area functional, [3]. For this reason, it is an extremely hard and widely open question whether this result holds for anisotropic energies, [8, Question 1], i.e. assuming an L^p bound on the anisotropic mean curvature, see (2.4) for the definition, with respect to functionals of the form

$$\Sigma_\Psi(V) := \int_\Gamma \Psi(T_x\Gamma)\theta(x)d\mathcal{H}^m(x), \qquad \text{where } V = (\Gamma, \theta) \text{ is a rectifiable } m\text{-varifold.}$$

To the best of our knowledge, the only available result is the regularity for codimension one varifolds with bounded generalized anisotropic mean curvature [1] (further referred to as Ψ-mean curvature), under a *density lower bound assumption* [1, The basic regularity Lemma, Assumption (1)].

The aim of this paper is to provide an affirmative answer to the question above in any dimension and codimension in the case the varifold V is associated to a Lipschitz graph. In particular, we provide an answer to the open question [8, Question 5] for Lipschitz graphs. Namely, we will prove the following main result, see Theorem 4.1 (we refer the reader to Section 2 and Section 3 for notation):

Theorem A. *Let $\Psi \in C^2(\mathbb{G}(N,m),(0,\infty))$ be a functional satisfying (USAC), let $p > m$ and consider an open, bounded set $\Omega \subset \mathbb{R}^m$. Let $u \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ be a map whose graph Γ_u induces a varifold with Ψ-mean curvature H in L^p in $\Omega \times \mathbb{R}^n$. Then there exists $\alpha > 0$ and an open set Ω_0 of full measure in Ω such that*

$$u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^n).$$

Although there have been important results on the regularity of *minimizers* for Σ_Ψ, [4, 14, 17, 20, 30], the regularity for *stationary points* of Σ_Ψ is a completely open question in general codimension. As mentioned above, our proof cannot rely on the monotonicity formula. Hence, we are not able to extend it to general

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rectifiable varifolds. Instead, we introduce a novel ellipticity condition (USAC), which allows us to obtain a Caccioppoli inequality, giving an answer to [8, Question 6].

For the sake of exposition and without loss of generality, in this paper we will treat autonomous integrands as in Theorem A. Nevertheless, we remark that Theorem A can be extended to non autonomous integrands $\Psi \in C^2(\mathbb{R}^N \times \mathbb{G}(N,m),(0,\infty))$ satisfying (USAC) at every $x \in \mathbb{R}^N$, see Remark 4.2. It is enough to observe that the first variation with respect to such integrands carries an additional term, which can be treated as part of the mean curvature term, see [12, Equation (2.5)].

(USAC) can also tackle another open problem in the literature: providing non-trivial examples of anisotropic energies in general codimension satisfying the atomic condition (AC). Indeed, anisotropic energy functionals have attracted an increasing interest since the pioneering works of F. J. Almgren [4, 5]. In particular, the classical Almgren ellipticity (AE), ([5, IV.1(7)] or [4, 1.6(2)]), allowed Almgren to prove regularity for minimizers of anisotropic energies, [4]. Very recently, an ongoing interest on the anisotropic Plateau problem has lead to a series of results, see [9, 11, 13, 15, 19, 22, 26]. In particular, in [12] G. De Philippis, the first author and F. Ghiraldin have introduced (AC) and proved it to be necessary and sufficient for the validity of the rectifiability of varifolds whose anisotropic first variation is a Radon measure. In codimension one and in dimension one, (AC) is proved to be equivalent to the strict convexity of the integrand, [12]. However, in general codimension, characterizing (AC) in terms of more classical conditions (such as (AE), polyconvexity, or others) remains an open problem, [12, Page 2]. The first author and S. Kolasiński have recently obtained one implication, proving that (AC) implies (AE), [16]. However, to date, in general codimension there are no examples of anisotropic energies (besides the area functional) satisfying (AC). We address this question in a result that we can roughly summarize as follows:

Theorem B. *Integrands Ψ in a $C^{1,1}$ neighborhood of the area functional satisfy (USAC); (USAC) implies (AC).*

Hence, the anisotropic energies in a $C^{1,1}$-neighborhood of the area are the first functionals in the literature in general codimension to justify the regularity theory developed in [12]. In particular, we deduce the rectifiability of varifolds with locally bounded anisotropic first variation for a $C^{1,1}$ neighborhood of the area functional.

(AC) can be relaxed to a condition (further denoted (AC1)), which is equivalent to the rectifiability of the mass of varifolds whose anisotropic first variation is a Radon measure, [6]. In codimension one, the convexity of the integrand implies (AC1), compare [6, Section 3.3]. However, in general codimension, there are no non-trivial examples of anisotropic energies satisfying (AC1). We address this problem by proving:

Theorem C. *Integrands Ψ in a C^1 neighborhood of the area functional satisfy (AC1).*

Theorem C implies that, in codimension one, (AC1) is a strictly weaker notion than convexity of the integrand, see Remark 3.9. This shows that the result of [6, Page 656, point (b)] is indeed optimal. We also find a class of examples of functionals satisfying (AC1), which are not C^1-close to the area, see Theorem 6.1.

There are profound connections between anisotropic geometric variational problems and questions arising in the study of polyconvex energies, the latter being roughly speaking a parametric version of the former, see [18, Page 229]. This link was investigated in [8, 23, 34]. In particular, there is a *canonical* way to associate a function $F_\Psi : \mathbb{R}^{n\times m} \to \mathbb{R}$ to an integrand Ψ defined on $\mathbb{G}(N,m)$, in such a way that a Lipschitz graph $[\![\Gamma_u]\!]$ is stationary for Σ_Ψ if and only if u is stationary for the energy

$$\mathbb{E}_{F_\Psi}(u) = \int_\Omega F_\Psi(Du(x))dx.$$

A graph is said to be stationary if and only if it is critical for outer and inner variations. In [34], the second author proves the regularity for 2-dimensional Lipschitz graphs that are stationary with respect to

polyconvex integrands close to the area. This result is close in spirit to Theorem A, but it carries deep differences: in [34], the *closeness* of the two functionals depends on the Lipschitz constant of the stationary graph, while in Theorem A the closeness is more quantitative, depending just on n and m. On the other hand, the proof of [34] yields full regularity for stationary points, and is based on completely different methods coming from the theory of differential inclusions, introduced in [32].

In [8, 23], the second author, together with C. De Lellis, De Philippis, B. Kirchheim and J. Hirsch, investigated the possibility of constructing a nowhere regular stationary graph for Σ_Ψ, exploiting the convex integration techniques introduced by S. Müller & V. Šverák and L. Székelyhidi in [29, 33]. However it is proved that it is impossible to complete this task *using the same strategy* of [29, 33], see [8, 23]. In particular, the authors prove that if the polyconvexity of F_Ψ complies with the stationarity of u, then one can exclude a certain type of Young measures, referred to as T'_N *configurations*, that proved to be the crucial tool in [29, 33]. Here we show a much more systematic result in this direction:

Theorem D. *(AC) excludes non-trivial Young measures in the case of stationary graphs.*

In Subsection 5.1, we will comment further on the importance of the previous result in the context of differential inclusions. Theorem D provides an answer to [8, Question 4]. In [24, Question 1], B. Kirchheim, S. Müller and V. Šverák leave as an open question to find rank-one convex functions whose differential inclusion associated to critical points (for outer variations only) supports only trivial Young measures. This question is largely open, and we provide here an answer in a neighborhood of the area (in arbitrary dimension and codimension), adding the hypothesis of criticality for inner variations. To conclude, we remark that our regularity Theorem A provides partial answers to questions that naturally arose in the context of quasiconvex energies, [25, Page 65], and [24, Question 2].

The paper is organized as follows. In Section 2 we recall some technical preliminaries concerning the theory of varifolds and Young measures. In Section 3 we introduce (SAC1) and (USAC), and we show their connection to (AC1) and (AC), thus proving Theorems B and C. Section 4 is devoted to show the regularity of graphs with L^p-bounded anisotropic mean curvature with respect to a functional satisfying (USAC), i.e. Theorem A. In Section 5 we show the absence of nontrivial Young measures if (AC) holds, i.e. Theorem D. Finally, in Section 6 we give an explicit example of a class of anisotropic energies in high codimension, far from the area functional, satisfying (AC1).

2. Technical preliminaries and notation

In this section, we recall the main definitions and results concerning varifolds and Young measures that we will need in the rest of the paper.

2.1. **Basic notation.** Given $A, B \in \mathbb{R}^{d\times d}$ and $v, w \in \mathbb{R}^d$, we denote the inner products as $\langle A, B\rangle = \sum_{ij=1}^d A_{ij}B_{ij}$ and $(v, w) = \sum_{i=1}^d v_i w_i$. $\|A\|$ and $\|v\|$ will be the norms induced by the previous inner products. A^t will denote the transpose of A.

2.2. **Measures and rectifiable sets.** Given a locally compact separable metric space Y, we denote by $\mathcal{M}(Y)$ the set of positive Radon measure on Y. Given a Radon measure μ we denote by $\mathrm{spt}(\mu)$ its support. For a Borel set E, $\mu \llcorner E$ is the restriction of μ to E, i.e. the measure defined by $[\mu \llcorner E](A) = \mu(E \cap A)$. Eventually, we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure.

A set $K \subset \mathbb{R}^N$ is said to be m-rectifiable if it can be covered, up to an $\mathcal{H}^m$-negligible set, by countably many C^1 m-dimensional submanifolds. In the following we will only consider $\mathcal{H}^m$-measurable sets. Given an m-rectifiable set K, we denote with T_xK the approximate tangent space of K at x, which exists for $\mathcal{H}^m$-almost every point $x \in K$, [31, Chapter 3]. A positive Radon measure $\mu \in \mathcal{M}(\mathbb{R}^N)$ is said to be m-rectifiable if there exists an m-rectifiable set $K \subset \mathbb{R}^N$ such that $\mu = \theta\mathcal{H}^m \llcorner K$ for some Borel function $\theta : \mathbb{R}^N \to (0, \infty)$.

For $\mu \in \mathcal{M}(\mathbb{R}^N)$ we consider its lower and upper m-dimensional densities at x:

$$\theta^m_*(x,\mu) = \liminf_{r\to 0} \frac{\mu(B_r(x))}{\omega_m r^m}, \qquad \theta^{m*}(x,\mu) = \limsup_{r\to 0} \frac{\mu(B_r(x))}{\omega_m r^m},$$

where ω_m is the volume of the m-dimensional unit ball in $\mathbb{R}^m$. In case these two limits are equal, we denote by $\theta^m(x,\mu)$ their common value. Note that if $\mu = \theta\mathcal{H}^m \llcorner K$ with K rectifiable, then $\theta(x) = \theta^m_*(x,\mu) = \theta^{m*}(x,\mu)$ for μ-a.e. x, see [31, Chapter 3].

If $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a Borel map and μ is a Radon measure, we let $\eta_\# \mu = \mu \circ \eta^{-1}$ be the push-forward of μ through η.

2.3. **Varifolds.** We will use $\mathbb{G}(n+m,m)$ to denote the Grassmanian of (un-oriented) m-dimensional linear subspaces in $\mathbb{R}^{n+m}$ (often referred to as m-planes). We will always denote $N = m+n$. Moreover, we identify the spaces

$$\mathbb{G}(N,m) = \{P \in \mathbb{R}^{N\times N} : P = P^t,\ P^2 = P,\ \mathrm{tr}(P) = m\}. \tag{2.1}$$

Definition 2.1. *An m-dimensional varifold V in $\mathbb{R}^N$ is a Radon measure on $\mathbb{R}^N \times \mathbb{G}(N,m)$. The varifold V is said to be rectifiable if there exists an m-rectifiable set Γ and an $\mathcal{H}^m \llcorner \Gamma$-measurable function $\theta : \Gamma \to (0,\infty)$ such that*

$$V(f) = \int_\Gamma f(x, T_x\Gamma)\theta(x) d\mathcal{H}^m(x), \qquad \forall f \in C_c(\mathbb{R}^N \times \mathbb{G}(N,m)).$$

In this case, we denote $V = (\Gamma,\theta)$. If moreover θ takes values in $\mathbb{N}$, V is said to be integer rectifiable. If $\theta = 1$ $\mathcal{H}^m \llcorner \Gamma$-a.e., then we will write $V = [\![\Gamma]\!]$.

We will use $\|V\|$ to denote the projection in $\mathbb{R}^N$ of the measure V, i.e.

$$\|V\|(A) := V(A \times \mathbb{G}(N,m)), \qquad \forall A \subseteq \mathbb{R}^N, A \text{ Borel}.$$

Hence $\|V\| = p_\# V$, where $p : \mathbb{R}^N \times \mathbb{G}(N,m) \to \mathbb{R}^N$ is the projection onto the first factor and the push-forward is intended in the sense of Radon measures.

Given an m-rectifiable varifold $V = (\Sigma,\theta)$ and $\Psi : \Sigma \to \mathbb{R}^N$ Lipschitz and proper (i.e. $\Psi^{-1}(K)$ is compact for every $K \subset \mathbb{R}^N$ compact), the image varifold of V under ψ is defined by

$$\psi^\# V := (\psi(\Sigma), \tilde\theta), \quad \text{where} \quad \tilde\theta(y) := \sum_{x\in\Sigma\cap\psi^{-1}(y)} \theta(x).$$

Since ψ is proper, we have that $\tilde\theta\mathcal{H}^m \llcorner \psi(\Sigma)$ is locally finite. By the area formula we get

$$\psi^\# V(f) = \int_{\psi(\Sigma)} f(x, T_x\Sigma)\tilde\theta(x) d\mathcal{H}^m(x) = \int_\Sigma f(\psi(x), d_x\psi(T_x\Sigma)) J\psi(x, T_x\Sigma)\theta(x) d\mathcal{H}^m(x),$$

for every $f \in C^0_c(\mathbb{R}^N \times \mathbb{G}(N,m))$. Here $d_x\psi(S)$ is the image of S under the linear map $d_x\psi(x)$ and

$$J\psi(x,S) := \sqrt{\det\left(\left(d_x\psi\big|_S\right)^t \circ d_x\psi\big|_S\right)}$$

denotes the m-Jacobian determinant of the differential $d_x\psi$ restricted to the m-plane S, see [31, Chapter 8]. Note that the image varifold of a varifold V is *not* the same as the push-forward of the Radon measure V through a map ψ defined on $\mathbb{R}^N \times \mathbb{G}(N,m)$ (the latter being denoted with an expressly different notation: $\Psi_\# V$, see Section 2.2).

Given $\Psi \in C^1(\mathbb{G}(N,m))$, we define the anisotropic energy on a rectifiable varifold $V = (\Gamma,\theta)$ as

$$\Sigma_\Psi(V) := \int_\Gamma \Psi(T_x\Gamma)\theta(x) d\mathcal{H}^m(x), \qquad \text{where } V = (\Gamma,\theta). \tag{2.2}$$

We define the first variation of $V = (\Gamma, \theta)$ with respect to Ψ as

$$[\delta_\Psi V](g) := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Sigma_\Psi((\Phi_\varepsilon)^\#(V)), \quad \forall g \in C^1_c(\mathbb{R}^N, \mathbb{R}^N),$$

where $\Phi_\varepsilon := \mathrm{id} + \varepsilon g$ is the flow generated by g. We rely on [12, Lemma A.2] to write the following expression for the variations:

$$[\delta_\Psi V](g) = \int_{\mathbb{R}^N} \langle B_\Psi(T_x\Gamma), Dg(x)\rangle d\|V\|,$$

where $B_\Psi(T)$ is defined through the equality

$$\langle B_\Psi(T), L\rangle := \Psi(T)\langle T, L\rangle + \big\langle D\Psi(T),\, T^\perp L T + (T^\perp L T)^t\big\rangle, \qquad \forall L \in \mathbb{R}^{N\times N}. \tag{2.3}$$

Here, $D\Psi(T)$ denotes the differential of Ψ once extended to a C^1 function defined in a small neighborhood of $\mathbb{G}(N, m)$ in $\mathbb{R}^{N\times N}$. Let $U \subset \mathbb{R}^N$ open, we say that a varifold $V = (\Gamma, \theta)$ has *Ψ-mean curvature in L^p in U* if there exists a map $H \in L^p(\Gamma \cap U, \mathbb{R}^N; \mathcal{H}^m \llcorner \Gamma)$ such that

$$[\delta_\Psi V](g) = -\int_{\mathbb{R}^N} (H(x), g(x)) d\|V\|(x), \quad \forall g \in C^1_c(U, \mathbb{R}^N). \tag{2.4}$$

If H can be chosen to be 0, we say that the varifold V is *stationary*.

The graph Γ_u of a Lipschitz function $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ defines an m-dimensional varifold with multiplicity 1, $\Gamma = [\![\Gamma_u]\!]$. Without loss of generality, we can suppose that the graph is parametrized on the first m coordinates, so that $\Gamma_u := \{(x, y) \in \mathbb{R}^{m+n} : y = u(x)\}$. For notational purposes, we define the following maps:

$$M(X) := \begin{pmatrix} id_m \\ X \end{pmatrix}, \quad \text{and} \quad \mathcal{A}(X) := \sqrt{\det(M(X)^t M(X))}, \quad \forall X \in \mathbb{R}^{n\times m}, \tag{2.5}$$

where $\mathcal{A}(X)$ simply corresponds to the area element of X, and

$$h : \mathbb{R}^{n\times m} \to \mathbb{R}^{N\times N}, \quad h(X) := M(X)(M(X)^t M(X))^{-1} M(X)^t. \tag{2.6}$$

Recalling (2.1), it is easily seen that $h(\mathbb{R}^{n\times m}) \subseteq \mathbb{G}(N, m)$, and that h is injective (it is, in fact, one of the *canonical* charts of $\mathbb{G}(N, m)$). The map h allows us to define the function $F_\Psi : \mathbb{R}^N \times \mathbb{R}^{n\times m} \to (0, +\infty)$ as:

$$F_\Psi(X) := \Psi(h(X))\mathcal{A}(X). \tag{2.7}$$

through the identification (2.1) and (2.6).

2.4. **Young measures.** We refer the reader to [28, Section 3] for the results concerning Young measures that we are going to state without proof. Let $p > 1$ and let $v_j : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^d$ be a sequence of weakly convergent maps in $L^p(\Omega, \mathbb{R}^d)$ to an $L^p(\Omega, \mathbb{R}^d)$ map $v : \Omega \to \mathbb{R}^d$. The fundamental theorem on Young measures states that, up to considering a (non-relabeled) subsequence v_j, there exists a measurable map

$$\Omega \ni x \mapsto \nu_x \in \mathcal{M}(\mathbb{R}^d), \quad \text{with} \quad \nu_x(\mathbb{R}^d) = 1,$$

such that, for every $f \in C(\mathbb{R}^d)$ with

$$|f(\Lambda)| \le C(1 + \|\Lambda\|^s), \qquad \forall \Lambda \in \mathbb{R}^d,$$

for some $s \in [1, p)$, it holds

$$\int_\Omega f(v_j(x))\eta(x)dx \to \int_\Omega \left(\int_{\mathbb{R}^d} f(\Lambda) d\nu_x(\Lambda)\right)\eta(x)dx, \qquad \forall \eta \in C^\infty_c(\Omega).$$

Moreover,

$$v_j \to v \text{ in measure} \quad \Leftrightarrow \quad \nu_x = \delta_{v(x)} \text{ for } \mathcal{H}^m\text{-a.e. } x \in \Omega. \tag{2.8}$$

This in particular implies the strong convergence of v_j to v in $L^q(\Omega, \mathbb{R}^d)$, $\forall q \in [1, p)$.

3. The atomic condition and related ellipticity conditions

Let $\Psi : \mathbb{G}(N,m) \to (0,+\infty)$ be a C^1 function, and let $N = n+m$. As recalled in Subsection 2.3, the formula for the first variation of a varifold $V = (\Gamma, \theta)$ is given by

$$[\delta_\Psi T](g) = \int_\Gamma \langle B_\Psi(T_x\Gamma), Dg(x)\rangle \theta d\mathcal{H}^m(x). \tag{3.1}$$

Recalling (2.3), we can readily compute the following expression of B_Ψ:

$$B_\Psi(T) = \Psi(T)T + T^\perp d\Psi(T)T, \qquad \text{where} \quad d\Psi(T) := D\Psi(T) + D\Psi(T)^t. \tag{3.2}$$

It is crucial to observe that, even if $D\Psi$ is the differential of Ψ as a map defined on $\mathbb{R}^{N\times N}$, i.e. computed after extending Ψ from $\mathbb{G}(N,m)$ to a neighborhood of $\mathbb{G}(N,m)$ in the whole space $\mathbb{R}^{N\times N}$, we have that $B_\Psi(T)$ does not depend on the particular chosen extension. This is true since, as computed in [12, Lemma A.1],

$$T^\perp LT + (T^\perp LT)^t \in \operatorname{Tan}_T \mathbb{G}(N,m), \quad \forall L \in \mathbb{R}^{N\times N},$$

and hence

$$\big\langle D\Psi(T),\, T^\perp LT + (T^\perp LT)^t \big\rangle$$

only depends on the values of Ψ *on* $\mathbb{G}(N,m)$. In particular, $T^\perp d\Psi(T)T$ represents the differential of Ψ on the manifold $\mathbb{G}(N,m)$.

For every Borel probability measure $\mu \in \mathcal{M}(\mathbb{G}(N,m))$, let us define

$$A(\mu) := \int_{\mathbb{G}(N,m)} B_\Psi(T) d\mu(T) \in \mathbb{R}^{N\times N}. \tag{3.3}$$

Definition 3.1. *We say that Ψ satisfies the* atomic condition *(AC) if the following two conditions hold:*
(AC1) $\dim \operatorname{Ker} A(\mu) \le n$ *for every probability measure* $\mu \in \mathcal{M}(\mathbb{G}(N,m))$,
(AC2) if $\dim \operatorname{Ker} A(\mu) = n$, *then* $\mu = \delta_{T_0}$ *for some* $T_0 \in \mathbb{G}(N,m)$.

The aim of this section is to define two classes of integrands, see Definition 3.2 and 3.3. We will show that the first class satisfies (AC1) in Proposition 3.4 and that the second class satisfies (AC) in Proposition 3.5. The interest in these conditions is that they are *open*, see Proposition 3.8 and 3.10. Since the area functional satisfies the atomic condition, we will conclude that the integrands in a C^1 (resp. $C^{1,1}$) neighborhood of the area functional satisfy (AC1) (resp. (AC)), thus providing the first non-trivial class of integrands satisfying (AC1) or (AC) in general dimension and codimension. These results prove Theorems B and C.

Definition 3.2. *We say that $\Psi : \mathbb{G}(N,m) \to \mathbb{R}$ satisfies the* scalar (AC1) condition *(SAC1) if there exists $\delta < \frac{1}{m-1}$ such that*

$$(B_\Psi(T)w, w) \le (1+\delta)\Psi(T)\|w\|^2, \qquad \forall T \in \mathbb{G}(N,m), w \in \mathbb{R}^N. \tag{3.4}$$

Before giving the second definition, let us introduce the following notation. Given a C^1 function $\Psi : \mathbb{G}(N,m) \to \mathbb{R}$, we denote its dual function with $\Psi^* : \mathbb{G}(N,n) \to \mathbb{R}$, namely the function defined as

$$\Psi^*(P) := \Psi(\mathrm{id} - P) = \Psi(P^\perp), \quad \forall P \in \mathbb{G}(N,n).$$

Since one has

$$B_\Psi(T) = \Psi(T)T + T^\perp d\Psi(T)T, \quad d\Psi(T) = D\Psi(T) + D\Psi(T)^t$$

a simple computation shows that

$$d\Psi^*(S^\perp) = -d\Psi(S) \quad \text{and hence} \quad B_{\Psi^*}(S^\perp) = \Psi(S)S^\perp - S d\Psi(S) S^\perp, \quad \forall S \in \mathbb{G}(N,m), \tag{3.5}$$

and hence elementary linear algebra gives us the following useful result:

$$B_\Psi(S)^t B_{\Psi^*}(S^\perp) = 0, \quad \forall S \in \mathbb{G}(N,m). \tag{3.6}$$

Definition 3.3. *We say that Ψ satisfies the* scalar atomic condition *(SAC) if*

$$\langle B_\Psi(T), B_{\Psi^*}(S^\perp)\rangle > 0, \quad \forall T \neq S \in \mathbb{G}(N,m),$$

and it satisfies the uniform scalar atomic condition *(USAC) if there exists a constant $C > 0$ independent of T, S such that*

$$\langle B_\Psi(T), B_{\Psi^*}(S^\perp)\rangle > C\|T-S\|^2, \quad \forall T \neq S \in \mathbb{G}(N,m),$$

We will now show that if Ψ satisfies (SAC1), then Ψ fulfills (AC1).

Proposition 3.4. *If Ψ is a positive integrand satisfying (SAC1), then Ψ satisfies (AC1).*

Proof. First, we observe that for every $T \in \mathbb{G}(N,m)$,

$$\operatorname{tr}(B_\Psi(T)) = m\Psi(T), \tag{3.7}$$

or, in other words, $\operatorname{tr}(T^\perp d\Psi(T)T) = 0$. This can be seen immediately by the properties of $T \in \mathbb{G}(N,m)$. Now, assume by contradiction that there exists a probability measure $\mu \in \mathcal{M}(\mathbb{G}(N,m))$ such that $\dim(\ker(A(\mu))) =: d > n$. Since Ψ is positive, then $\operatorname{tr}(A(\mu)) > 0$ by (3.7), hence $d < N$. Let $\{v_i\}_{i=1}^N$ be an orthonormal basis of $\mathbb{R}^N$ such that $\{v_i\}_{i=1}^d \subset \ker(A(\mu))$. Then, we get the following contradiction

$$\begin{aligned} m\int_{\mathbb{G}(N,m)} \Psi(T)d\mu(T) &\overset{(3.7)}{=} \operatorname{tr}(A(\mu)) = \sum_{i=1}^N (A(\mu)v_i, v_i) = \sum_{i=d+1}^N (A(\mu)v_i, v_i) \\ &= \sum_{i=d+1}^N \int_{\mathbb{G}(N,m)} (B_\Psi(T)v_i, v_i)d\mu(T) \overset{(3.4)}{\leq} \sum_{i=d+1}^N \int_{\mathbb{G}(N,m)} (1+\delta)\Psi(T)d\mu(T) \\ &= (N-d-1)(1+\delta)\int_{\mathbb{G}(N,m)} \Psi(T)d\mu(T) < m\int_{\mathbb{G}(N,m)} \Psi(T)d\mu(T), \end{aligned}$$

the last inequality being true due to the non-negativity of Ψ and the estimate

$$(N-d-1)(1+\delta) < (N-n-1)\frac{m}{m-1} = m.$$

□

Now let us turn to (SAC).

Proposition 3.5. *If Ψ is a positive integrand satisfying (SAC), then Ψ satisfies (AC).*

Proof. Suppose

$$\dim(\operatorname{Ker} A(\mu)) \geq n \tag{3.8}$$

for some probability measure $\mu \in \mathcal{M}(\mathbb{G}(N,m))$. We need to show that in fact $\mu = \delta_{T_0}$ for some $T_0 \in G(N,n)$. Once this is established, it follows that also $\dim(\operatorname{Ker} A(\mu)) = n$. Indeed, in the case $\mu = \delta_{T_0}$, we have

$$0 = A(\mu)w = B_\Psi(T_0)w = \Psi(T_0)T_0w + T_0^\perp d\Psi(T_0)T_0w,$$

and this can happen if and only if $\Psi(T_0)T_0w = 0$ and $T_0^\perp d\Psi(T_0)T_0w = 0$. The sign assumption on Ψ therefore would yield

$$\operatorname{Ker}(B_\Psi(T_0)) = \operatorname{Ker}(T_0),$$

and hence that $B_\Psi(T_0) = A(\mu)$ has n-dimensional kernel.

We are left to show that (3.8) implies $\mu = \delta_{T_0}$ for some $T_0 \in G(N,n)$. By (3.8) we find an orthonormal system $v_1, \dots, v_n$ inside $\operatorname{Ker} A(\mu)$, and define $P \in \mathbb{G}(N,n)$ to be the orthogonal projection onto it. First, by (3.5), we have $B_{\Psi^*}(P)w = 0$ for every $w \perp \operatorname{span}\{v_1, \dots, v_n\}$. Therefore, since $v_i \in \operatorname{Ker}(A(\mu))$ for every $i = 1, \dots, n$, we see that

$$\int_{\mathbb{G}(N,m)} \langle B_\Psi(T), B_{\Psi^*}(P)\rangle d\mu = \langle A(\mu), B_{\Psi^*}(P)\rangle = \sum_{i=1}^n (A(\mu)v_i, B_{\Psi^*}(P)v_i) = 0.$$

However, by (SAC)

$$0 = \int_{\mathbb{G}(N,m)} \langle B_\Psi(T), B_{\Psi^*}(P)\rangle d\mu \geq 0,$$

with equality if and only if μ is concentrated on $T_0 = P^\perp$, and this finishes the proof. □

Let us comment on the necessity of the sign assumption on Ψ in Proposition 3.5:

Lemma 3.6. *Let $\Psi \in C^1(\mathbb{G}(N,m))$ satisfy (AC). Then Ψ is either nonnegative or nonpositive.*

Proof. The key idea is to restrict Ψ to the space of codimension one planes. Let $T_0 \in \mathbb{G}(N,m)$ be arbitrary but fixed. Suppose T_0 is the projection on $\mathrm{span}\{v_1,\dots,v_m\}$, for an orthonormal system $v_1,\dots,v_m$. We want to show that if $T_1 \in \mathbb{G}(N,m)$ is the projection on $\mathrm{span}\{w,v_2,\dots,v_m\}$, for some $w \in \{v_2,\dots,v_m\}^\perp$ then

$$\Psi(T_0)\Psi(T_1) \geq 0. \tag{3.9}$$

If we show this, then we readily conclude the Lemma by iterating this claim. Hence fix $w \in \{v_2,\dots,v_m\}^\perp$. Without loss of generality, we can assume that $\{e_1,\dots,e_{m+1}\}$ is an orthonormal system spanning $\{w,v_1,v_2,\dots,v_m\}$, where $\{e_1,\dots,e_N\}$ is the canonical orthonormal basis for $\mathbb{R}^N$. We want to define $\Phi \in C^1(\mathbb{G}(m+1,m))$ as a *restriction* of Ψ. To this aim we consider any m-plane π with

$$\pi \subset \mathrm{span}\{e_1,\dots,e_{m+1}\} \sim \mathbb{R}^{m+1},$$

we denote $P \in \mathbb{R}^{(m+1)\times(m+1)}$ to be the projection onto π and simply set $\tilde{P} = MPM^t \in \mathbb{R}^{N\times N}$, where $M = (e_1|\dots|e_{m+1})$. We define

$$\Phi(P) := \Psi(\tilde{P}).$$

Now we show that Φ still fulfills (AC). To this aim, we simply observe that, by definition of Φ, $B_\Phi(P)$ is the $\mathbb{R}^{(m+1)\times(m+1)}$ matrix obtained by

$$B_\Phi(P) = \Phi(P)P + P^\perp d\Phi(P)P, \quad d\Phi(P) = D\Phi(P) + D\Phi(P)^t.$$

By the chain rule we deduce that

$$d\Phi(P) = M^t d\Psi(\tilde{P})M.$$

This in turn implies $B_\Phi(P) = M^t B_\Psi(\tilde{P})M$ and, since Ψ satisfies (AC), we conclude that Φ fulfills (AC) as well. If we prove that Φ has a sign, then (3.9) readily follows.

In the case $m > 1$, we have the following easy contradiction argument. Assume by contradiction that there exists $T_1 \in \mathbb{G}(m+1,m)$ such that $\Phi(T_1) = 0$. By definition of $B_\Phi(T_1)$

$$B_\Phi(T_1) = T_1^\perp d\Phi(T_1)T_1, \quad d\Phi(T_1) = D\Phi(T_1) + D\Phi(T_1)^t. \tag{3.10}$$

Since Φ fulfills (AC), then $\dim(\mathrm{Ker}\, B_\Phi(T_1)) = 1$ and $\dim(\mathrm{Im} B_\Phi(T_1)) = m$. On the other hand, (3.10) implies that $\mathrm{Im} B_\Phi(T_1) \subset \mathrm{Im} T_1^\perp$ and consequently we get the following contradiciton

$$m = \dim(\mathrm{Im} B_\Phi(T_1)) \leq \dim(\mathrm{Im} T_1^\perp) = 1.$$

In the general case, including $m = 1$, we can conclude with the following argument, which borrows ideas from [12, Theorem 1.3]. We define G as the one-homogeneous extension to $\mathbb{R}^{m+1}$ of the following map $\mathbb{S}^m \ni \nu \mapsto \Phi(\mathrm{id} - \nu\otimes\nu)$. Since Φ satisfies (AC), we deduce, as in Step 2 of the proof of [12, Theorem 1.3], that for every $\nu \neq \pm\bar{\nu} \in \mathbb{S}^m$:

$$G(\nu)G(\bar{\nu}) - \langle d_\nu G(\bar{\nu}), \nu\rangle\langle d_\nu G(\nu), \bar{\nu}\rangle \neq 0.$$

Applying [32, Lemma 1] with $K = \mathbb{G}(m+1,m)$, we deduce that either

$$G(\nu)G(\bar{\nu}) - \langle d_\nu G(\bar{\nu}), \nu\rangle\langle d_\nu G(\nu), \bar{\nu}\rangle > 0 \qquad \text{for all } \nu,\bar{\nu} \text{ s.t. } \nu \neq \pm\bar{\nu}, \tag{3.11}$$

or that (3.11) holds with the opposite sign. Without loss of generality we will treat the positive case (3.11). If $G(\bar{\nu}) \neq 0$, we conclude that $G(\nu) > \langle d_\nu G(\bar{\nu}), \nu\rangle$ for every $\nu \neq \pm\bar{\nu} \in \mathbb{S}^m$ as in [12]. Since Φ satisfies (AC), it is easy to see that the zero set of G cannot contain open subsets and, by continuity of G, we conclude that $G(\nu) \geq \langle d_\nu G(\bar{\nu}), \nu\rangle$ for every $\nu,\bar{\nu} \in \mathbb{S}^m$. In other words, G is an even, convex and

one-homogeneous function on $\mathbb{R}^{m+1}$. We claim that this implies G (and hence Φ) is nonnegative. Indeed, assume by contradiction there exists $\nu \in \mathbb{S}^m$ such that $G(\nu) < 0$, then being G even, $G(-\nu) < 0$. Hence, by convexity of G, we deduce $G(0) < 0$, which contradicts the one-homogeneity of G. □

Remark 3.7. *Notwithstanding we focus on positive integrands, in view of Lemma 3.6, it is clear that all the results of this Section hold for negative integrands as well, provided one makes the appropriate change of sign in (SAC) and (USAC).*

Finally, we show that Definitions 3.2 and 3.3 are open conditions.

Proposition 3.8. *Let $\Psi \in C^1(\mathbb{G}(N,m),(0,\infty))$ satisfy (SAC1). Then, there exists $\varepsilon = \varepsilon(\delta, n, m, \min_T \Psi(T)) > 0$ such that if $\Psi' : \mathbb{G}(N,m) \to \mathbb{R}^+$ satisfies*

$$\|\Psi - \Psi'\|_{C^1(\mathbb{G}(N,m))} \leq \varepsilon, \tag{3.12}$$

then Ψ' also satisfies (SAC1). Here $\delta > 0$ is the quantity appearing in Definition 3.2 and depends only on Ψ.

Proof. As observed at the beginning of the section, the term $T^\perp d\Psi(T)T$ is the differential of Ψ on $\mathbb{G}(N,m)$. Thus, we see that

$$\|B_\Psi - B_{\Psi'}\|_{C^0(\mathbb{G}(N,m))} \leq c\|\Psi - \Psi'\|_{C^1(\mathbb{G}(N,m))}, \tag{3.13}$$

where $c > 0$ is a dimensional constant. We assume ε is chosen so small that also Ψ' fulfilling (3.12) is strictly positive and actually we enforce

$$\min_{T\in\mathbb{G}(N,m)} \Psi'(T) \geq \gamma := \frac{1}{2} \min_{T\in\mathbb{G}(N,m)} \Psi(T) > 0. \tag{3.14}$$

The result now easily follows, in fact fix $\delta < \frac{1}{m-1}$ such that

$$(B_\Psi(T)w, w) \leq (1+\delta)\Psi(T), \quad \forall T \in \mathbb{G}(N,m),\ w \in \mathbb{S}^{N-1}.$$

Then, for each $T \in \mathbb{G}(N,m)$ and $w \in \mathbb{S}^{N-1}$

$$\begin{aligned}(B_{\Psi'}(T)w, w) &= ((B_{\Psi'}(T) - B_\Psi(T))w, w) + (B_\Psi(T)w, w) \overset{(3.13)}{\leq} c\|\Psi - \Psi'\|_{C^1(\mathbb{G}(N,m))} + (1+\delta)\Psi(T)\\ &= c\|\Psi - \Psi'\|_{C^1(\mathbb{G}(N,m))} + (1+\delta)(\Psi(T) - \Psi'(T)) + (1+\delta)\Psi'(T)\\ &\overset{(3.14)}{\leq} \frac{(c+1+\delta)}{\gamma}\varepsilon\Psi'(T) + (1+\delta)\Psi'(T).\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, we can still impose $\frac{c+1+\delta}{\gamma}\varepsilon + \delta < \frac{1}{m-1}$ and hence conclude the proof. □

Remark 3.9. *As written in the introduction, Proposition 3.8 yields an interesting immediate corollary: in codimension one, the validity of (AC1) for $\Psi \in \mathbb{G}(m+1,m)$ does not imply (in general) the convexity of the one-homogeneous extension to $\mathbb{R}^{m+1}$ of $\Psi^* \in \mathbb{G}(m+1,1)$. In fact, it is straightforward to see that arbitrarily small C^1 perturbations of convex functions need not to be convex. We recall that (AC1) implies the rectifiability of the mass of a varifold with locally bounded first variation,* [6, Page 656, point (b)]. *This remark shows that the result in* [6, Page 656, point (b)] *is indeed optimal.*

Proposition 3.10. *Let $\Psi \in C^{1,1}(\mathbb{G}(N,m),(0,\infty))$ satisfy (USAC). Then, there exists $\varepsilon = \varepsilon(C, \|\Psi\|_{C^{1,1}}, n, m) > 0$ such that if $\Psi' : \mathbb{G}(N,m) \to \mathbb{R}^+$ satisfies*

$$\|\Psi - \Psi'\|_{C^{1,1}(\mathbb{G}(N,m))} \leq \varepsilon,$$

then Ψ' also satisfies (USAC). Here $C > 0$ is the quantity appearing in Definition 3.3 and only depends on Ψ.

We wish to expressly thank the anonymous referee who suggested the following proof, which is considerably shorter than the original one.

Proof. For any $\Phi \in C^{1,1}(\mathbb{G}(N,m))$, let us denote with $A_\Phi(T) := T^\perp d\Phi(T)T$ and with $\Delta(T) := \Psi(T) - \Psi'(T)$. We start by noticing that for all $\Phi \in C^{1,1}(\mathbb{G}(N,m))$ and all $T, S \in \mathbb{G}(N,m)$,

$$-2\langle B_\Phi(T), B_{\Phi^*}(S^\perp)\rangle = \langle B_\Phi(T) - B_\Phi(S), B_{\Phi^*}(T^\perp) - B_{\Phi^*}(S^\perp)\rangle. \tag{3.15}$$

Indeed, we have

$$\langle B_\Phi(T) - B_\Phi(S), B_{\Phi^*}(T^\perp) - B_{\Phi^*}(S^\perp)\rangle \overset{(3.6)}{=} -(\langle B_\Phi(T), B_{\Phi^*}(S^\perp)\rangle + \langle B_\Phi(S), B_{\Phi^*}(T^\perp)\rangle) \tag{3.16}$$

and

$$\begin{aligned}\langle B_\Phi(S), B_{\Phi^*}(T^\perp)\rangle &= \langle \Phi(S)S + S^\perp A_\Phi(S)S, \Phi(T)T^\perp - TA_\Phi(T)T^\perp\rangle \\ &= \Phi(S)\Phi(T)\langle S, T^\perp\rangle - \langle S^\perp A_\Phi(S)S, TA_\Phi(T)T^\perp\rangle \\ &\quad + \Phi(T)\langle S^\perp A_\Phi(S)S, T^\perp\rangle - \Phi(S)\langle S, TA_\Phi(T)T^\perp\rangle.\end{aligned} \tag{3.17}$$

Since $A_\Phi(T) = A_\Phi(T)^t$ for all $T \in \mathbb{G}(N,m)$, which is immediate by (3.2), we see that the first two addenda in (3.17) remain unchanged if we exchange T and S, while for the last two addenda

$$C(T,S) := \Phi(T)\langle S^\perp A_\Phi(S)S, T^\perp\rangle - \Phi(S)\langle S, TA_\Phi(T)T^\perp\rangle,$$

we have, by the fact that $\mathrm{tr}(S^\perp A_\Phi(S)S) = \mathrm{tr}(TA_\Phi(T)T^\perp) = 0$,

$$\begin{aligned}C(T,S) &= \Phi(T)\langle S^\perp A_\Phi(S)S, T^\perp\rangle - \Phi(S)\langle S, TA_\Phi(T)T^\perp\rangle \\ &= -\Phi(T)\langle S^\perp A_\Phi(S)S, T\rangle + \Phi(S)\langle S^\perp, TA_\Phi(T)T^\perp\rangle \\ &= -\Phi(T)\langle SA_\Phi(S)S^\perp, T\rangle + \Phi(S)\langle S^\perp, T^\perp A_\Phi(T)T\rangle = C(S,T).\end{aligned}$$

From this equality, (3.16) and (3.17), we deduce (3.15). We consider

$$\begin{aligned}D(T,S) &:= \langle B_\Psi(T) - B_\Psi(S), B_{\Psi^*}(T^\perp) - B_{\Psi^*}(S^\perp)\rangle - \langle B_{\Psi'}(T) - B_{\Psi'}(S), B_{(\Psi')^*}(T^\perp) - B_{(\Psi')^*}(S^\perp)\rangle \\ &= \langle B_\Delta(T) - B_\Delta(S), B_{\Psi^*}(T^\perp) - B_{\Psi^*}(S^\perp)\rangle - \langle B_{\Psi'}(T) - B_{\Psi'}(S), B_{\Delta^*}(T^\perp) - B_{\Delta^*}(S^\perp)\rangle.\end{aligned} \tag{3.18}$$

Now, there exists a dimensional constant $k > 0$ such that, for any function $\Phi \in C^{1,1}(\mathbb{G}(N,m))$,

$$|B_\Phi(T) - B_\Phi(S)| \le k\|\Phi\|_{C^{1,1}}|T - S| \text{ and } |B_{\Phi^*}(T) - B_{\Phi^*}(S)| \le k\|\Phi\|_{C^{1,1}}|T - S|.$$

Thus, after supposing without loss of generality that $\|\Psi - \Psi'\|_{C^{1,1}} \le 1$, we estimate

$$|D(T,S)| \le k'\|\Psi\|_{C^{1,1}}\|\Psi - \Psi'\|_{C^{1,1}}|T - S|^2,$$

for some possibly larger dimensional constant $k' > 0$. In view of (3.15), this concludes the proof. □

Remark 3.11. *The reader may wonder whether the $C^{1,1}$ norm in Proposition 3.10 can be relaxed to the $C^{1,\alpha}$ norm for some $0 \le \alpha < 1$. This is not merely a technical difficulty. Indeed, in the codimension one case it is known by the aforementioned* [12, Theorem 1.3] *that $\Psi \in \mathbb{G}(m+1,m)$ satisfies (AC) if and only if $\Psi^* \in \mathbb{G}(m+1,1)$ is a strictly convex function, once we consider its one-homogeneous extension to $\mathbb{R}^{m+1}$. As in Remark 3.9, we observe that a $C^{1,\alpha}$, $\alpha < 1$, perturbation of the area functional is not, in general, a strictly convex function, hence (AC) cannot have open interior with respect to the $C^{1,\alpha}(\mathbb{G}(N,m))$-topology on integrands.*

Corollary 3.12. *For any $m, n > 0$, (AC1) holds in a C^1 neighborhood of the area functional and (AC) holds in a $C^{1,1}$ neighborhood of the area functional on $\mathbb{G}(N,m)$.*

Proof. Let us denote the area functional as $\Psi \equiv 1$. Therefore,

$$B_\Psi(T) = T.$$

Ψ satisfies (SAC1), since

$$(Tw, w) \le 1, \qquad \forall w \in \mathbb{S}^{m-1}.$$

Moreover, notice that for all $T, S \in \mathbb{G}(N,m)$,

$$\langle T, S^\perp\rangle = m - \langle T, S\rangle = \frac{1}{2}\|T-S\|^2. \tag{3.19}$$

Thus,

$$\langle B_\Psi(T), B_{\Psi*}(S^\perp)\rangle = \langle T, S^\perp\rangle = \frac{1}{2}\|T-S\|^2,$$

and hence the area functional fulfills (USAC) with constant $C = \frac{1}{2}$. Now Propositions 3.4 & 3.8 and 3.5 & 3.10 conclude the proof, once we observe that if we take the C^1 (or $C^{1,1}$) norm sufficiently small, also the perturbed functional will be positive. □

In the next section we will need the following observation, namely that (USAC) implies quasiconvexity:

Proposition 3.13. *Let $\Psi \in C^2(\mathbb{G}(N,m),(0,\infty))$ be a functional that satisfies (USAC) with constant $C > 0$ and F_Ψ as in (2.7). Then F_Ψ is quasiconvex in the following sense. There exists $\alpha(n,m,C,\min\Psi) > 0$ such that, for every open, bounded $\Omega \subset \mathbb{R}^m$ and for every $A \in \mathbb{R}^{n\times m}$ it holds*

$$\int_\Omega F_\Psi(A+D\varphi) - F_\Psi(A)dx \geq \alpha \int_\Omega \mathcal{A}(A+D\varphi) - \mathcal{A}(A)dx, \qquad \forall \varphi \in C^1_c(\Omega,\mathbb{R}^n). \tag{3.20}$$

In particular F_Ψ satisfies the following local uniform Legendre-Hadamard condition, namely for every $R > 0$ there exists $c > 0$ such that

$$D^2F_\Psi(X)[M,M] \geq c\|M\|^2, \qquad \text{for every } X \in B_R \text{ and } M \in \mathbb{R}^{n\times m} \text{ with } \operatorname{rank}(M) = 1. \tag{3.21}$$

Proof. By Proposition 3.10, there exists $\alpha(n,m,C,\min\Psi) > 0$ small enough such that $\Psi' := \Psi - \alpha$ also satisfies (USAC) and is positive. In particular, by Proposition 3.5, we deduce that Ψ' satisfies (AC) and, by [16, Theorem A, Theorem 8.8], Ψ' satisfies the strict Almgren ellipticity condition, [16, Definition 4.5]. In particular we conclude that

$$\int_{\Gamma_{A+\varphi}} \Psi'(T_y\Gamma_{A+\varphi})d\mathcal{H}^m - \int_{\Gamma_A} \Psi'(T_y\Gamma_A)d\mathcal{H}^m > 0,$$

which in turn reads

$$\int_{\Gamma_{A+\varphi}} \Psi(T_y\Gamma_{A+\varphi})d\mathcal{H}^m - \int_{\Gamma_A} \Psi(T_y\Gamma_A)d\mathcal{H}^m > \alpha(\mathcal{H}^m(\Gamma_{A+\varphi}) - \mathcal{H}^m(\Gamma_A)).$$

Hence, by the area formula and (2.7), we conclude (3.20). Moreover, by [28, Lemma 4.3] applied to $F_\Psi - \alpha\mathcal{A}$, we deduce that $F_\Psi - \alpha\mathcal{A}$ is rank-one convex, i.e. $D^2(F_\Psi - \alpha\mathcal{A})(X)[M,M] \geq 0$ for every $X \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}(M) = 1$. One can check that $\mathcal{A}$ satisfies the local uniform Legendre-Hadamard condition, see [34, Lemma 6.5]. [34, Lemma 6.5] is proved for $m = 2$, but it readily extends to every m. Then we deduce the validity of (3.21). □

4. Regularity

In this section, we want to prove the following:

Theorem 4.1. *Let $\Psi \in C^2(\mathbb{G}(N,m),(0,\infty))$ be a functional satisfying (USAC), let $p > m$ and consider an open, bounded set $\Omega \subset \mathbb{R}^m$. Let $u \in \operatorname{Lip}(\Omega,\mathbb{R}^n)$ be a map whose graph Γ_u induces a varifold with Ψ-mean curvature H in L^p in $\Omega \times \mathbb{R}^n$. Then there exists $\alpha > 0$ and an open set Ω_0 of full measure in Ω such that*

$$u \in C^{1,\alpha}(\Omega_0,\mathbb{R}^n).$$

Remark 4.2. *As mentioned in the introduction, without loss of generality, in this paper we treat autonomous integrands as in Theorem 4.1. Nevertheless, we remark that Theorem 4.1 can be extended to non autonomous integrands $\Psi \in C^2(\mathbb{R}^N \times \mathbb{G}(N,m),(0,\infty))$ satisfying (USAC) at every $x \in \mathbb{R}^N$. Requiring (USAC) at every $x \in \mathbb{R}^N$ means that there exists a constant $C > 0$ independent of T, S, x such that for every $x \in \mathbb{R}^N$*

$$\langle B_\Psi(x,T), B_{\Psi^*}(x,S^\perp)\rangle > C\|T-S\|^2, \quad \forall T \neq S \in \mathbb{G}(N,m),$$

where $B_\Psi(x,T)$ is defined in [12, Equation (2.6)]. *Indeed, in this case the first variation for a rectifiable varifold $V=(\Gamma,\theta)$ yields*

$$\int_U \langle d_y\Psi(y,T_y\Gamma),g\rangle + \langle B_\Psi(y,T_y\Gamma),Dg\rangle d\|V\|(y) = -\int_U (H,g)d\|V\|(y). \tag{4.1}$$

We can absorb the term $\langle d_y\Psi(y,T_y\Gamma),g\rangle$ into the right hand side, obtaining an equation similar to (4.3). *We can consequently carry the same regularity analysis.*

The proof goes as follows. We prove, in Proposition 4.3 a Caccioppoli inequality similar to the one obtained by Allard in the case of general varifolds with bounded mean curvature. The latter is the main novelty of our approach, and uses essentially the (USAC) property of Ψ. Subsequently, in Corollary 4.4 we will show how to get a Caccioppoli inequality for u as in Theorem 4.1 from Proposition 4.3. In Proposition 4.5, we will see how this Caccioppoli inequality implies a so-called *decay of the excess*, analogous to [18, Lemma 4.1]. From that point on, the proof becomes rather standard, so we will only sketch how to conclude the proof of Theorem 4.1. The interested reader may consult [18, Section 7].

Proposition 4.3 (Caccioppoli inequality)**.** *Let Ψ as in Theorem 4.1. Let $V=[\![\Gamma,\theta]\!]$ be a rectifiable varifold with Ψ-mean curvature H bounded in L^2 in $U\subset\mathbb{R}^N$. Then, there exists $C_2=C_2(n,m,\|\Psi\|_{C^2},C)>0$, where C is the constant in Definition 3.3 for Ψ, such that:*

$$\fint_{B_r(z)} \|T_y\Gamma - S\|^2 d\|V\|(y) \le \frac{C_2}{r^2}\fint_{B_{2r}(z)} \mathrm{dist}(y-p,\pi)^2 d\|V\|(y) + r^2C_2\fint_{B_{2r}(z)}\|H\|^2 d\|V\|(y), \tag{4.2}$$

for every $p\in\mathbb{R}^N$, $S\in\mathbb{G}(N,m), z\in U, r>0$ such that $\mathrm{dist}(z,\partial U)\ge 4r$, where $\pi=\mathrm{Im}(S)$.

Proof. In this proof we will denote with C_2 a positive constant which may change line by line, but that shall always depend just on $n,m,\|\Psi\|_{C^2},C$. Using the definition of Ψ-mean curvature for varifolds with bounded anisotropic first variation, we have that for every $g\in C_c^\infty(U,\mathbb{R}^N)$,

$$\int_U \langle B_\Psi(T_y\Gamma),Dg\rangle d\|V\|(y) = -\int_U (H,g)d\|V\|(y). \tag{4.3}$$

We fix $S\in\mathbb{G}(N,m)$, and we prove the assertion in the case $r=1, z=0$, the general case being true by scaling and translating. Under these assumptions, choose

$$g(y) := \varphi^2(y)B_{\Psi^*}(S^\perp)(y-p),$$

for a radial $\varphi\in C_c^\infty(\mathbb{R}^N)$, with $\varphi\in[0,1]$, $\varphi\equiv 1$ on $B_1(0)$ and $\varphi\equiv 0$ on $B_2(0)^c$. With this choice of g, (4.3) reads:

$$\begin{aligned}\int_U \varphi^2(y)\langle B_\Psi(T_y\Gamma),B_{\Psi^*}(S^\perp)\rangle d\|V\|(y) + 2\int_U \varphi(y)(B_\Psi(T_y\Gamma)D\varphi(y),B_{\Psi^*}(S^\perp)(y-p))d\|V\|(y)\\ = -\int_U \varphi^2(y)(H,B_{\Psi^*}(S^\perp)(y-p))d\|V\|(y),\end{aligned}$$

that we rewrite as

$$\int_U \varphi^2(y)\langle B_\Psi(T_y\Gamma),B_{\Psi^*}(S^\perp)\rangle d\|V\|(y) \tag{4.4}$$

$$= -\int_U \varphi^2(y)(H,B_{\Psi^*}(S^\perp)(y-p))d\|V\|(y) - 2\int_U \varphi(y)(B_\Psi(T_y\Gamma)D\varphi(y),B_{\Psi^*}(S^\perp)(y-p))d\|V\|(y). \tag{4.5}$$

We bound (4.4) from below using (USAC) for Ψ:

$$C\int_U \varphi^2(y)\|T_y\Gamma - S\|^2 d\|V\|(y) \le \int_U \varphi^2(y)\langle B_\Psi(T_y\Gamma),B_{\Psi^*}(S^\perp)\rangle d\|V\|(y).$$

Now we estimate from above (4.5). The first addendum is estimated simply by Young inequality:

$$-\int_U \varphi^2(y)(H,B_{\Psi^*}(S^\perp)(y-p))d\|V\|(y) \le \frac{1}{2}\int_{B_2}\|H\|^2 d\|V\|(y) + \frac{1}{2}\int_U \varphi^4(y)\|B_{\Psi^*}(S^\perp)(y-p)\|^2 d\|V\|(y).$$

To estimate the second addendum of (4.5), we need to use the algebraic identity (3.6) to rewrite

$$\varphi(y)(B_\Psi(T_y\Gamma)D\varphi(y), B_{\Psi^*}(S^\perp)(y-p)) = \varphi(y)((B_\Psi(T_y\Gamma) - B_\Psi(S))D\varphi(y), B_{\Psi^*}(S^\perp)(y-p))$$

and hence, since $\Psi \in C^2(\mathbb{G}(N,m))$ by assumption, to bound

$$\begin{aligned}|\varphi(y)(B_\Psi(T_y\Gamma)D\varphi(y), B_{\Psi^*}(S^\perp)(y-p))| &\le \varphi(y)\|B_\Psi(T_y\Gamma) - B_\Psi(S)\|\|D\varphi(y)\|\|B_{\Psi^*}(S^\perp)(y-p)\| \\ &\le C_2\varphi(y)\|T_y\Gamma - S\|\|D\varphi(y)\|\|B_{\Psi^*}(S^\perp)(y-p)\| \\ &\le \frac{C}{4}\varphi^2(y)\|T_y\Gamma - S\|^2 + C_2\|D\varphi(y)\|^2\|B_{\Psi^*}(S^\perp)(y-p)\|^2,\end{aligned}$$

where in the third inequality we used again Young's inequality. Combining the previous inequalities, equality (4.4)-(4.5) reads

$$\begin{aligned}\frac{C}{2}\int_U \varphi^2(y)\|T_y\Gamma - S\|^2 d\|V\|(y) \le \frac{1}{2}\int_{B_2}\|H\|^2 d\|V\|(y) + \frac{1}{2}\int_U \varphi^4(y)\|B_{\Psi^*}(S^\perp)(y-p)\|^2 d\|V\|(y) \\ + C_2\int_U \|D\varphi(y)\|^2\|B_{\Psi^*}(S^\perp)(y-p)\|^2 d\|V\|(y).\end{aligned}$$

We conclude (4.2) observing that

$$\begin{aligned}\|B_{\Psi^*}(S^\perp)(y-p)\| &\overset{(3.5)}{=} \|\Psi(S)S^\perp(y-p) - Sd\Psi(S)S^\perp(y-p)\| \\ &\le \|\Psi(S)S^\perp - Sd\Psi(S)S^\perp\|\,\|S^\perp(y-p)\| \le \|\Psi\|_{C^2} d((y-p),\pi),\end{aligned}$$

where in the last inequality we have used the following elementary identity:

$$\|S^\perp(y-p)\| = d((y-p),\pi). \tag{4.6}$$

□

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, we define

$$(f)_{z,R} := \fint_{B_R(z)} f(y)dy, \qquad (f)_R := \fint_{B_R(0)} f(y)dy.$$

Corollary 4.4 (Caccioppoli inequality for u)**.** *Let Ψ, u be as in Theorem 4.1. Then, for some constant $C_2(n,m,\|\Psi\|_{C^2}, C, \|u\|_{\mathrm{Lip}}) > 0$, where C is the constant in Definition 3.3 for Ψ, and $k = 2(1+\|u\|_{\mathrm{Lip}})$, we have*

$$\fint_{B_r(x_0)}\|Du(x) - A\|^2 dx \le \frac{C_2}{r^2}\fint_{B_{kr}(x_0)}\|u(x) - (u)_{x_0,kr} - A(x-x_0)\|^2 dx + r^2 C_2 \fint_{B_{kr}(x_0)}\|H'\|^2 dx, \tag{4.7}$$

for all $A \in \mathbb{R}^{n\times m}$ with $\|A\| \le 2\|Du\|_\infty$, $x_0 \in \Omega$, and $r > 0$ such that $\mathrm{dist}(x_0, \partial\Omega) > \frac{k}{2}r$. Here,

$$H'(x) := H(x, u(x)) \in L^2(\Omega).$$

Proof. In this proof we will denote with C_2 a positive constant which may change line by line, but that shall always depend just on $n, m, \|\Psi\|_{C^2}, C, \|u\|_{\mathrm{Lip}}$. By (4.2), we know:

$$\fint_{B_R(z)}\|T_y\Gamma_u - S\|^2 d\|V\|(y) \le \frac{C_2}{R^2}\fint_{B_{2R}(z)}\mathrm{dist}(y-p,\pi)^2 d\|V\|(y) + R^2 C_2 \fint_{B_{2R}(z)}\|H\|^2 d\|V\|(y), \tag{4.8}$$

for every $p \in \mathbb{R}^N$ and $S \in \mathbb{G}(N,m), z \in U, R > 0$ such that $\mathrm{dist}(z, \partial U) \ge 4R$, where $\pi = \mathrm{Im}(S)$. Here, $V = [\![\Gamma_u]\!]$. We fix $A \in \mathbb{R}^{n\times m}$, and consequently choose $S = h(A)$, where h is the map defined in (2.6). We choose $p = (x_0, (u)_{x_0,kr})$ and we consider $r > 0$ as in the statement of the theorem fixed. Define also

$$\Gamma_u^r = \{(x, u(x)) : x \in B_r(x_0)\}.$$

Notice that we make a small abuse of notation, denoting with the same symbol balls in $\mathbb{R}^N$ and $\mathbb{R}^m$. Let $L := \|u\|_{\mathrm{Lip}}$. We notice the following:

$$\Gamma_u^{\frac{r}{1+L}} \subset B_r((x_0, u(x_0))) \cap \Gamma_u \subset B_{2r}((x_0, u(x_0))) \cap \Gamma_u \subset \Gamma_u^{2r}. \tag{4.9}$$

From the area formula, we also see that

$$\mathcal{H}^m(B_r((x_0, u(x_0))) \cap \Gamma_u) \leq C_2 r^m, \quad C_2^{-1} r^m \leq \mathcal{H}^m(B_{2r}((x_0, u(x_0))) \cap \Gamma_u) \tag{4.10}$$

Hence, we rewrite (4.8) for $R = r$ and $z = (x_0, u(x_0))$ in the following form:

$$\fint_{B_r(z)\cap\Gamma_u} \|T_y\Gamma_u - S\|^2 d\mathcal{H}^m(y) \leq \frac{C_2}{r^2} \fint_{B_{2r}(z)\cap\Gamma_u} \mathrm{dist}(y-p,\pi)^2 d\mathcal{H}^m + r^2 C_2 \fint_{B_{2r}(z)\cap\Gamma_u} \|H\|^2 d\mathcal{H}^m$$

and then use (4.9),(4.10) to write

$$\fint_{\Gamma_u^{\frac{r}{1+L}}} \|T_y\Gamma_u - S\|^2 d\mathcal{H}^m(y) \leq \frac{C_2}{r^2} \fint_{\Gamma_u^{2r}} \mathrm{dist}(y-p,\pi)^2 d\mathcal{H}^m + r^2 C_2 \fint_{\Gamma_u^{2r}} \|H\|^2 d\mathcal{H}^m. \tag{4.11}$$

Now we use the area formula to rewrite and estimate (4.11), to finally obtain (4.7). Firstly, the area formula and the estimate $\mathcal{H}^m(\Gamma_u^{\frac{r}{1+L}}) \geq C_2^{-1} r^m$ yield

$$\fint_{B_{\frac{r}{1+L}}(x_0)} \|h(Du(x)) - h(A)\|^2 \mathcal{A}(Du(x)) dx \leq C_2 \fint_{\Gamma_u^{\frac{r}{1+L}}} \|T_y\Gamma_u - S\|^2 d\mathcal{H}^m(y). \tag{4.12}$$

Now, $h : \mathbb{R}^{n\times m} \to \mathbb{G}(N,m)$ is invertible on the set $E \subset \mathbb{G}(N,m)$ defined as

$$E = \{T \in \mathbb{G}(N,m) : \det(T') \neq 0\},$$

where T' is the $m \times m$ submatrix obtained by T only considering the first m rows and columns. Moreover, $h^{-1} : E \to \mathbb{R}^{n\times m}$ is locally Lipschitz, hence

$$C_2 \|h(Du(x)) - h(A)\| \geq \|Du(x) - A\|.$$

Since $\mathcal{A}(X) \geq 1$ for every $X \in \mathbb{R}^{n\times m}$, we can finally bound

$$\fint_{B_{\frac{r}{1+L}}(x_0)} \|Du(x) - A\|^2 dx \leq \fint_{B_{\frac{r}{1+L}}(x_0)} \|h(Du(x)) - h(A)\|^2 \mathcal{A}(Du(x)) dx. \tag{4.13}$$

Now we wish to estimate from above the addendum

$$\fint_{B_{2r}(z)\cap\Gamma_u} \mathrm{dist}(y-p,\pi)^2 d\mathcal{H}^m.$$

First of all, as in (4.6), we write

$$\mathrm{dist}(y-p,\pi) = \|S^\perp(y-p)\| = \left\| (\mathrm{id} - h(A)) \begin{pmatrix} x - x_0 \\ u(x) - (u)_{x_0,kr} \end{pmatrix} \right\|. \tag{4.14}$$

Now we claim that

$$\left\| (\mathrm{id} - h(A)) \begin{pmatrix} x - x_0 \\ u(x) - (u)_{x_0,kr} \end{pmatrix} \right\| = \left\| (\mathrm{id} - h(A)) \begin{pmatrix} 0 \\ u(x) - (u)_{x_0,kr} - A(x - x_0) \end{pmatrix} \right\|. \tag{4.15}$$

Indeed

$$(\mathrm{id} - h(A)) \begin{pmatrix} \mathrm{id}_m \\ A \end{pmatrix} \overset{(2.6)}{=} (\mathrm{id} - h(A)) M(A) \overset{(2.6)}{=} (\mathrm{id}_m - M(A)(M(A)^t M(A))^{-1} M(A)^t) M(A) = 0.$$

Combining (4.14) and (4.15), we estimate

$$\mathrm{dist}(y-p,\pi) \leq C_2 \|u(x) - (u)_{x_0,kr} - A(x - x_0)\|. \tag{4.16}$$

In particular, this allows us to write

$$\begin{aligned}\fint_{B_{2r}(z)\cap\Gamma_u} \operatorname{dist}(y-p,\pi)^2 d\mathcal{H}^m &\overset{(4.9)-(4.10)}{\leq} C_2 \fint_{\Gamma_u^{2r}} \operatorname{dist}(y-p,\pi)^2 d\mathcal{H}^m \\ &= C_2 \fint_{B_{2r}(x_0)} \operatorname{dist}((x,u(x))-(x_0,(u)_{x_0,kr}),\pi)^2 \mathcal{A}(Du(x))dx \\ &\overset{(4.16)}{\leq} C_2 \fint_{B_{2r}(x_0)} \|u(x)-(u)_{x_0,kr}-A(x-x_0)\|^2 dx,\end{aligned} \tag{4.17}$$

where in the first equality we used the area formula, and to get the second inequality we used, other than (4.16), also the fact that $u \in \mathrm{Lip}$ to bound $\mathcal{A}(Du(\cdot))$ with a constant depending on L. Finally

$$\fint_{\Gamma_u^{2r}} \|H\|^2 dx = \fint_{B_{2r}(x_0)} \|H'\|^2 \mathcal{A}(Du(x))dx \leq C_2 \fint_{B_{2r}(x_0)} \|H'\|^2 dx, \tag{4.18}$$

that once again exploits the area formula and the fact that $u \in \mathrm{Lip}$. Inequalities (4.11)-(4.12)-(4.13)-(4.17)-(4.18) imply (4.7). □

We will now prove a decay of the following classical quadratic excess:

$$E(x,r) := \fint_{B_r(x)} \|Du(x)-(Du)_{x,r}\|^2 dx.$$

Proposition 4.5 (Excess decay)**.** *Let Ψ, u, H be as in Theorem 4.1. Let moreover $k = 2(1+\|u\|_{\mathrm{Lip}}) > 0$ as in Corollary 4.4. Then, there exists a constant $c > 0$ such that for every $\tau \in \left(0, \frac{1}{4k}\right)$, there exists $\varepsilon = \varepsilon(\tau) > 0$ such that*

$$E(x,r) \leq \varepsilon(\tau) \quad \text{and} \quad r^{1-\frac{m}{p}} \|H'\|_p \leq E(x,r)$$

imply

$$E(x,\tau r) \leq c\tau^2 E(x,r) \tag{4.19}$$

for every $B_r(x) \subset \Omega$. The constant c depends on $n, m, p, \|\Psi\|_{C^2}, C, \|u\|_{\mathrm{Lip}}, k$, where C is as in Definition 3.3.

Proof. The proof is analogous to [18, Lemma 4.1]. We adapt it below to our setting. The key point is a contradiction blow-up argument, that uses the regularity theory for the linearized problem and the Caccioppoli estimate (4.7).

Suppose the thesis were false. Then, for every $c > 0$ there exists $\tau \in \left(0, \frac{1}{4k}\right)$ and a sequence of points $\{x_j\} \subset \Omega$ and radii $r_j > 0$ such that

$$E(x_j, r_j) = \lambda_j^2 \to 0 \quad \text{and} \quad r_j^{1-\frac{m}{p}} \|H'\|_p \leq E(x_j, r_j) = \lambda_j^2 \tag{4.20}$$

but

$$E(x_j, \tau r_j) \geq c\tau^2 E(x_j, r_j) \tag{4.21}$$

We consider blow-ups of u of the following form

$$v_j(z) := \frac{u(x_j + r_j z) - (u)_{x_j,r_j} - r_j A_j z}{\lambda_j r_j}, \qquad \text{with } A_j := (Du)_{x_j,r_j}.$$

For every $j \in \mathbb{N}$ the maps $v_j : B_1(0) \subset \mathbb{R}^m \to \mathbb{R}^n$ enjoy the following properties:

(i) $Dv_j(z) = \frac{Du(x_j+r_j z)-A_j}{\lambda_j}$;
(ii) $(v_j)_1 = 0$, $(Dv_j)_1 = 0$;
(iii) $\fint_{B_1(0)} \|Dv_j\|^2 = 1$;

(iv) $\fint_{B_1(0)} \|v_j\|^2 \leq \gamma$, for some $\gamma > 0$;

(v) $\fint_{B_\tau(0)} \|Dv_j - E_j\|^2 \geq c\tau^2 \fint_{B_1(0)} \|Dv_j\|^2 = c\tau^2, \quad$ where $E_j := (Dv_j)_\tau$.

The first three conditions are easy consequences of the definition of v_j, the fourth is an application of Poincaré's inequality, and the fifth can be seen from (4.21) and the definition of v_j. (iii)-(iv) imply that, up to a non-relabeled subsequence, we can assume

$$v_j \rightharpoonup v \text{ in } W^{1,2}(B_1, \mathbb{R}^n), \qquad v_j \to v \text{ in } L^2(B_1, \mathbb{R}^n)$$

and, since $\{A_j\}_j$ is equibounded, $A_j \to A \in \mathbb{R}^{n\times m}$. Recalling the definition of F_Ψ as in (2.7), we define the sequence of integrands

$$F_j(X) := \frac{1}{\lambda_j^2}(F_\Psi(\lambda_j X + A_j) - F_\Psi(A_j) - \lambda_j \langle DF_\Psi(A_j), X\rangle), \tag{4.22}$$

then, $F_j(X) \to D^2F_\Psi(A)[X, X]$ locally in the C^2 topology. We claim that v is a critical point for the functional

$$G(X) := D^2F_\Psi(A)[X, X],$$

or in other words

$$\int_{B_1} D^2F_\Psi(A)[Dv, Dg]dx = 0, \quad \forall g \in C_c^\infty(B_1, \mathbb{R}^n). \tag{4.23}$$

To see this, fix a test vector-field g and use [8, Proposition 6.8] to find that, since $[\![\Gamma_u]\!]$ has Ψ-mean curvature H bounded in L^p, then we find a constant $C_1 = C_1(\|H\|_{L^p}) > 0$ such that[1] $C_1(0) = 0$, and for which

$$\left|\int_\Omega \langle DF_\Psi(Du(x)), D\eta(x)\rangle dx\right| \leq C_1 \|\eta \mathcal{A}^{\frac{1}{p'}}(Du)\|_{p'}, \qquad \forall \eta \in C_c^\infty(\Omega, \mathbb{R}^n). \tag{4.24}$$

Here we denoted $\frac{1}{p'} + \frac{1}{p} = 1$. Now we plug in the previous inequality $\eta_j(x) = g\left(\frac{x-x_j}{r_j}\right)$. Let us rewrite the right hand side and the left hand side of (4.24) separately. We have

$$\begin{aligned}
\int_\Omega \langle DF_\Psi(Du(x)), D\eta_j(x)\rangle dx &= \frac{1}{r_j}\int_\Omega \left\langle DF_\Psi(Du(x)), Dg\left(\frac{x-x_j}{r_j}\right)\right\rangle dx \\
&= r_j^{m-1}\int_{B_1} \langle DF_\Psi(Du(x_j + r_j y)), Dg(y)\rangle dy \\
&= r_j^{m-1}\int_{B_1} \langle DF_\Psi(Du(x_j + r_j y)) - DF_\Psi(A_j), Dg(y)\rangle dy \\
&= r_j^{m-1}\int_{B_1} \langle DF_\Psi(A_j + \lambda_j Dv_j(y)) - DF_\Psi(A_j), Dg(y)\rangle dy \\
&= \lambda_j r_j^{m-1}\int_{B_1} \langle DF_j(Dv_j(y)), Dg(y)\rangle dy,
\end{aligned}$$

where we used the compactness of the support of g for passing from the second to the third equality and the definitions of v_j and F_j in the rest of the equalities. Now we can turn to the right hand side of (4.24). From now on we will denote with C_2 a positive constant which may change line by line, but that shall always depend just on $n, m, p, \|\Psi\|_{C^2}, C, \|u\|_{\mathrm{Lip}}, k$:

$$\|\eta_j \mathcal{A}^{\frac{1}{p'}}(Du)\|_{p'} \leq C_2\|\eta_j\|_{p'} = C_2\left(\int_\Omega \|\eta_j(x)\|^{p'}dx\right)^{\frac{1}{p'}} = C_2 r_j^{\frac{m}{p'}}\left(\int_{B_1} \|g(y)\|^{p'}dy\right)^{\frac{1}{p'}},$$

[1] In fact, as written at the end of the proof of [8, Proposition 6.8], C_1 can be taken to be exactly $\|H'\|_{L^p}$.

where we bounded $\mathcal{A}^{\frac{1}{p'}}(Du) \leq C_2$ using the fact that u is Lipschitz. These computations allows us to rewrite (4.24) as

$$\lambda_j r_j^{m-1} \int_{B_1} \langle DF_j(Dv_j(y)), Dg(y)\rangle dy \leq C_1 C_2 r_j^{\frac{m}{p'}} \left(\int_{B_1} \|g(y)\|^{p'} dy \right)^{\frac{1}{p'}}.$$

Dividing by $\lambda_j r_j^{m-1}$, we see that the right hand side becomes

$$C_1 C_2 \frac{r_j^{1-\frac{m}{p}}}{\lambda_j} \left(\int_{B_1} \|g(y)\|^{p'} dy \right)^{\frac{1}{p'}} \overset{(4.20)}{\leq} \lambda_j \frac{C_1 C_2}{\|H'\|_p} \left(\int_{B_1} \|g(y)\|^{p'} dy \right)^{\frac{1}{p'}},$$

that converges to 0 since $\lambda_j^2 \to 0$. Of course, if $\|H'\|_{L^p} = 0$, the previous computation cannot be performed, but in that case we see that the right-hand side of (4.24) is identically 0, since the constant appearing in (4.24) satisfies $C_1(0) = 0$. To finish the proof, we need to show that

$$\int_{B_1} \langle DF_j(Dv_j), Dg\rangle dx \to \int_{B_1} D^2 F_\Psi(A)[Dv, Dg] dx = 0.$$

To do so, using the definition of F_j we rewrite:

$$DF_j(Dv_j) = \frac{DF_\Psi(A_j + \lambda_j Dv_j) - DF_\Psi(A_j)}{\lambda_j} = \int_0^1 D^2 F_\Psi(A_j + s\lambda_j Dv_j(x)) Dv_j(x) ds.$$

By assumption Dv_j converges weakly to Dv, hence it suffices to prove that

$$d_j(x) := \int_0^1 D^2 F_\Psi(A_j + s\lambda_j Dv_j(x)) ds \to D^2 F_\Psi(A)$$

strongly in L^2 to conclude. First, by the definition of v_j, we infer that $\{\lambda_j v_j\}_j$ is an equilipschitz sequence, and hence that $\{A_j + s\lambda_j Dv_j\}_j$ is a sequence equibounded in L^∞. Furthermore, as $\{v_j\}_j$ is equibounded in $W^{1,2}$, $\{\lambda_j v_j\}_j$ converges to 0 strongly in L^2, and we may assume (up to non-relabeled subsequences) that $\lambda_j Dv_j \to 0$ pointwise a.e.. Then dominated convergence implies the convergence of d_j in L^p for every $p \in [1, +\infty)$, and hence we find that our claim (4.23) holds.

Since v is a weak solution of a linear elliptic systems with constant coefficients, see (4.23), then classical elliptic regularity theory, [27], yields:

$$\sup_{B_{\frac{1}{2}}(0)} \|D^2 v\|^2 \leq C_2 \fint_{B_1} \|Dv\|^2 dx \overset{(iii)}{\leq} C_2. \tag{4.25}$$

Furthermore, from (4.7) and Hölder inequality we get:

$$\begin{aligned} \fint_{B_r(x_0)} \|Du(x) - A\|^2 dx &\leq \frac{C_2}{r^2} \fint_{B_{kr}(x_0)} \|u(x) - (u)_{x_0,kr} - A(x - x_0)\|^2 dx + r^2 C_2 \fint_{B_{kr}(x_0)} \|H'\|^2 dx \\ &\leq \frac{C_2}{r^2} \fint_{B_{kr}(x_0)} \|u(x) - (u)_{x_0,kr} - A(x - x_0)\|^2 dx + C_2 \left[r^{1-\frac{m}{p}} \left(\int_{B_{kr}(x_0)} \|H'\|^p dx \right)^{\frac{1}{p}} \right]^2. \end{aligned} \tag{4.26}$$

We rewrite (4.26) in terms of v_j choosing $r = \tau r_j$, $A = (Du)_{x_j, r_j k\tau}$, simply by rescaling, translating, dividing by λ_j^2 and estimating H' with its L^p-norm:

$$\fint_{B_\tau} \|Dv_j - B_j\|^2 dx \leq \frac{C_2}{\tau^2} \fint_{B_{k\tau}} \|v_j - b_j - B_j x\|^2 dx + \frac{C_2}{\lambda_j^2} \left(r_j^{1-\frac{m}{p}} \|H'\|_p \right)^2, \tag{4.27}$$

where $b_j := (\lambda_j r_j)^{-1}((u)_{x_j,k\tau r_j} - (u)_{x_j,r_j}) = (v_j)_{k\tau}$ and $B_j = (Dv_j)_{k\tau}$. By (4.20), the last addendum converges to 0 as $j \to \infty$. It is well-known that

$$\fint_{B_\tau} \|Dv_j - E_j\|^2 dx \le \fint_{B_\tau} \|Dv_j - B_j\|^2 dx, \tag{4.28}$$

and hence, denoting $B = (Dv)_{k\tau}$, we obtain:

$$\begin{aligned} c\tau^2 &\overset{(v)}{\le} \limsup_j \fint_{B_\tau} \|Dv_j - E_j\|^2 dx \overset{(4.28)}{\le} \limsup_j \fint_{B_\tau} \|Dv_j - B_j\|^2 dx \\ &\overset{(4.27)}{\le} \frac{C_2}{\tau^2} \limsup_j \left[\fint_{B_{k\tau}} \|v_j - b_j - B_j x\|^2 dx + \frac{1}{\lambda_j^2} \left(r_j^{1-\frac{m}{p}} \|H'\|_p \right)^2 \right] \\ &\overset{(4.20)}{=} \frac{C_2}{\tau^2} \fint_{B_{k\tau}} \|v - b - Bx\|^2 dx \le C_2 \fint_{B_{k\tau}} \|Dv - B\|^2 dx \overset{(4.25)}{\le} C_2 \tau^2, \end{aligned}$$

where the last line is obtained by Poincaré inequality, using that $(v_j)_{k\tau} = b_j$ implies $(v)_{k\tau} = b$. Choosing $c > C_2$, we obtain the desired contradiction. □

Now we can finally give the proof of Theorem 4.1:

Proof of Theorem 4.1: We define

$$\Omega_0 := \{x \in \Omega : \lim_{r\to 0} E(x,r) = 0\}.$$

This set is of full measure in Ω as it contains all Lebesgue points of Du. We want to show that Ω_0 is open and $Du_{|\Omega_0} \in C^\alpha$ for some $\alpha \in (0,1)$. In the ongoing proof, we fix $\tau \in (0,(4k)^{-1})$, where $k = 2(1 + \|u\|_{\mathrm{Lip}})$, satisfying $c\tau^{2-\beta} < 1$ and $\tau^\beta < \frac{1}{8}$, where c is the constant found in Proposition 4.5 and $\beta := 1 - \frac{m}{p}$. Further, let $x_0 \in \Omega_0$, and define the auxiliary excess:

$$F(s) := E(x_0, s) + \Lambda s^\beta \|H'\|_p, \quad \Lambda := \frac{8}{\tau^m}.$$

We choose $r > 0$ such that

$$E(x_0, r) \le F(r) < \varepsilon(\tau), \tag{4.29}$$

where ε is given by Proposition 4.5. From now on r and τ are fixed. If $r^\beta \|H'\|_p \le E(x_0,r)$, by Proposition 4.5 we also find

$$F(\tau r) = E(x_0, \tau r) + \Lambda \tau^\beta r^\beta \|H'\|_p \overset{(4.19)}{\le} c\tau^2 E(x_0,r) + \Lambda \tau^\beta r^\beta \|H'\|_p \overset{c\tau^{2-\beta}<1}{\le} \tau^\beta F(r).$$

On the other hand, if $r^\beta \|H'\|_p > E(x_0,r)$, we have

$$\begin{aligned} F(\tau r) = E(x_0,\tau r) + \Lambda \tau^\beta r^\beta \|H'\|_p &\le \frac{1}{\tau^m} E(x_0,r) + \Lambda \tau^\beta r^\beta \|H'\|_p < (\tau^{-m} r^\beta + \Lambda \tau^\beta r^\beta)\|H'\|_p \\ &= (\tau^{-m}\Lambda^{-1} + \tau^\beta)\Lambda r^\beta \|H'\|_p \overset{\Lambda = 8\tau^{-m}}{\le} (8^{-1} + \tau^\beta)\Lambda r^\beta \|H'\|_p \\ &\overset{\tau^\beta < 8^{-1}}{\le} \frac{1}{4}\Lambda r^\beta \|H'\|_p \le \frac{1}{4} F(r). \end{aligned}$$

In particular, if r satisfies (4.29), we always have $F(\tau r) \le \frac{1}{4} F(r)$. This inequality allows us to iterate the reasoning with τr instead of r (notice in fact that τr still satisfies (4.29)). Hence we find, for every $\ell \in \mathbb{N}$,

$$F(\tau^\ell r) \le 4^{-\ell} F(r), \quad \forall \ell \in \mathbb{N}.$$

From this, one easily find the existence of $\alpha \in (0,1)$ such that

$$F(R) \le c_1 R^{2\alpha}, \quad \forall R \in (0,r),$$

for some constant $c_1 > 0$ depending only on r and τ. Now the key observation is that, fixed $r > 0$, for points x sufficiently close to x_0, one still has

$$E(x,r) + \Lambda r^\beta \|H'\|_p < \varepsilon(\tau),$$

that is an easy consequence of the continuity of $x \mapsto E(x,r)$. Therefore, we find that there exists $\rho > 0$ such that

$$E(x,R) + \Lambda R^\beta \|H'\|_p < c_1 R^{2\alpha}, \quad \forall x \in B_\rho(x_0), R \in (0,r).$$

In particular, we infer

$$E(x,R) < c_2 R^{2\alpha}, \quad \forall x \in B_\rho(x_0), R \in (0,r).$$

This shows that $Du : B_\rho(x_0) \to \mathbb{R}^n$ is in a Campanato space, and it is well-know that this yields Hölder regularity for Du, see for instance [7, Proof of Theorem 3.2], and this concludes the proof. □

5. Compactness

Aim of this section is to prove Theorem D. This will be obtained combining the following Theorems 5.1 and 5.2. In order to precisely state Theorem D, we can use the notion of differential inclusions. In this way, Theorem D is equivalent to say that the only Young measures generated by *div-curl* inclusions supported in

$$K_{F_\Psi} := \left\{ A \in \mathbb{R}^{(2n+m)\times m} : A = \begin{pmatrix} X \\ DF_\Psi(X) \\ X^T DF_\Psi(X) - F_\Psi(X)\,\mathrm{id}_m \end{pmatrix} \right\}$$

are trivial in the case Ψ satisfies (AC). In particular, this answers [8, Question 9] for Ψ satisfying (AC). We will not enter in the details of the theory of differential inclusions and we refer the reader to [8, Section 2] for a thorough explanation of the terminology. Nonetheless, we will try to give an informal explanation of the results of this section in Subsection 5.1.

Theorem 5.1. *Let $\Psi \in C^2(\mathbb{G}(N,m),(0,\infty))$ and F_Ψ as in (2.7). Let $\Omega \subset \mathbb{R}^m$ be open and bounded. Consider sequences $u_j : \Omega \to \mathbb{R}^n$, $A_j : \Omega \to \mathbb{R}^{n\times m}$, $B_j : \Omega \to \mathbb{R}^{m\times m}$ such that u_j is equibounded in $W^{1,\infty}$ and A_j, B_j are equibounded in L^∞. Suppose further that $\operatorname{div} A_j$ and $\operatorname{div} B_j$ are equibounded in L^1. Define*

$$W_j := \begin{pmatrix} Du_j \\ A_j \\ B_j \end{pmatrix}$$

and suppose $\operatorname{dist}(W_j, K_{F_\Psi}) \to 0$ pointwise a.e. as $j \to \infty$. Suppose further that $u_j \rightharpoonup u$ in $W^{1,2}(\Omega,\mathbb{R}^n)$. Then, the associated varifolds $V_j := [\![\Gamma_{u_j}]\!]$ converge in the sense of varifolds (i.e. weakly* as measures on $\Omega \times \mathbb{R}^n \times \mathbb{G}(N,m)$) to $V = [\![\Gamma_u]\!]$.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^m$ be open and bounded and $p > m$. Let $u_j : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a sequence of maps such that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^n)$ and $Du_j \overset{*}{\rightharpoonup} (\nu_x)_x$ in the sense of Young measures. Suppose the graphs $[\![\Gamma_{u_j}]\!]$ converge in the sense of varifolds to $[\![\Gamma_u]\!]$. Then u_j converges to u in the strong $W^{1,q}(\Omega,\mathbb{R}^n)$-topology for every $1 \le q < p$. Conversely, if $u_j : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^n$ satisfies $u_j \to u$ strongly in $W^{1,p}(\Omega,\mathbb{R}^n)$, then $[\![\Gamma_{u_j}]\!]$ converges in the sense of varifolds to $[\![\Gamma_u]\!]$.*

Proof of Theorem 5.1. Since u_j is equibounded in $W^{1,\infty}$, then $u_j \to u$ uniformly by Ascoli-Arzelà compactness criterion and V_j is an equibounded sequence of measures. Therefore, up to extracting a subsequence, $V_j \overset{*}{\rightharpoonup} V$. We need to prove that $V = [\![\Gamma_u]\!]$. First, we claim that

$$\operatorname{spt}(\|V\|) \subseteq \Gamma_u. \tag{5.1}$$

To see (5.1), fix $y \in \mathbb{R}^N \setminus \Gamma_u$ and $r > 0$ such that $B_{2r}(y) \cap \Gamma_u = \emptyset$. Then, by the uniform convergence of u_j to u, up to taking j large enough $\|V_j\|(B_r(y)) = 0$. Since $\|V_j\| \overset{*}{\rightharpoonup} \|V\|$, by lower semicontinuity we deduce that $\|V\|(B_r(y)) = 0$. In particular $y \in \mathbb{R}^N \setminus \operatorname{spt}(\|V\|)$ and we deduce (5.1).

We want to prove that the limit varifold V is integer rectifiable. We wish to apply [15, Theorem 4.1].

To this aim, we show that $[\delta_\Psi V]$ is a Radon measure. This is an easy consequence of [8, Lemma 7.3], that yields for every $g = (g^1, \dots, g^N) \in C^1_c(\Omega \times \mathbb{R}^n)$:

$$[\delta_\Psi([\![\Gamma_{u_j}]\!])](g) = \int_\Omega \langle B(Du_j(x)), D(g_1(x, u_j(x)))\rangle dx + \int_\Omega \langle A(Du_j(x)), D(g_2(x, u_j(x)))\rangle dx, \tag{5.2}$$

where $g_1(x,y) := (g^1(x,y), \dots, g^m(x,y))$, $g_2(x,y) := (g^{m+1}(x,y), \dots, g^{m+n}(x,y))$ and $A(X)$ and $B(X)$ are defined by $A(X) = DF_\Psi(X), B(X) = X^T DF_\Psi(X) - F_\Psi(X)\mathrm{id}_m$. Since $W \mapsto [\delta_\Psi(W)]$ is continuous with respect to the weak-$*$ convergence of varifolds, the left hand side of (5.2) converges to $[\delta_\Psi(V)](g)$ as $j \to \infty$, while the right hand side can be rewritten as:

$$\begin{aligned}\int_\Omega \langle B(Du_j(x)) - B_j(x), D(g_1(x, u_j(x)))\rangle dx + \int_\Omega \langle A(Du_j(x)) - A_j(x), D(g_2(x, u_j(x)))\rangle dx \\ + \int_\Omega \langle B_j(x), D(g_1(x, u_j(x)))\rangle dx + \int_\Omega \langle A_j(x), D(g_2(x, u_j(x)))\rangle dx.\end{aligned} \tag{5.3}$$

We may assume, up to passing to a non-relabeled subsequence, that $\operatorname{div} B_j$ and $\operatorname{div} A_j$ weakly-$*$ converge as Radon measures to μ and ν, respectively. Therefore, taking the limit as $j \to \infty$, since $\operatorname{dist}(W_j, K_{F_\Psi}) \to 0$ pointwise a.e. and $g_1(x, u_j(x)) - g_1(x, u(x))$, $g_2(x, u_j(x)) - g_2(x, u(x))$ converge uniformly to 0, then (5.3) converges to

$$-\int_\Omega g_1(x, u(x)) d\mu(x) - \int_\Omega g_2(x, u(x)) d\nu(x).$$

Hence, $[\delta_\Psi V]$ is a Radon measure. Secondly, in order to apply [15, Theorem 4.1], we need to prove that $\theta^m_*(y, \|V\|) > 0$, for $\|V\|$-a.e. $y \in \Omega \times \mathbb{R}^n$. Let $\pi : \Omega \times \mathbb{R}^n \to \Omega$ be the projection map on the first factor and denote $L := \|u\|_{\mathrm{Lip}}$. Fix a point $z = \pi(y) \in \Omega$. As $\pi_\# \|V_j\| \overset{*}{\rightharpoonup} \pi_\# \|V\|$, for every $r \in (0, \operatorname{dist}(z, \partial\Omega))$ we compute

$$\begin{aligned}\frac{r^m}{2^m}\omega_m &\le \limsup_j \pi_\# \|V_j\|(\overline{B_{\frac{r}{2}}(z)}) \le \pi_\# \|V\|(\overline{B_{\frac{r}{2}}(z)}) \le \pi_\# \|V\|(B_r(z)) \\ &\le \|V\|\left(\Gamma_u \cap \pi^{-1}(B_r(z))\right) \le \|V\|(\Gamma_u \cap B_{r\sqrt{1+L^2}}(y)) = \|V\|(B_{r\sqrt{1+L^2}}(y)).\end{aligned} \tag{5.4}$$

which implies $\theta^m_*(y, \|V\|) > 0$, as desired. We apply [15, Theorem 4.1] to deduce that V is an integer rectifiable varifold. Moreover (5.4) implies that $\Gamma_u \subseteq \operatorname{spt}(\|V\|)$, which combined with (5.1) gives $\Gamma_u = \operatorname{spt}(\|V\|)$. We deduce that $V = (\Gamma_u, \theta)$. Exploiting the graphicality of $\{V_j\}$, we have the equality

$$[\![\Omega \times \{0\}^n]\!] = \pi^\# V_j \overset{*}{\rightharpoonup} \pi^\# V = (\Omega \times \{0\}^n, \theta \circ (\pi_{|\Gamma_u})^{-1}).$$

This implies that $\pi^\# V = [\![\Omega \times \{0\}^n]\!]$, and in particular that $\theta(y) = 1$ for $\mathcal{H}^m$-a.e. $y \in \operatorname{spt}(\|V\|)$. □

Proof of Theorem 5.2. In order to prove that u_j converges to u in the strong $W^{1,q}(\Omega, \mathbb{R}^n)$-topology for every $1 \le q < p$, by (2.8), it is sufficient to prove that $\nu_x = \delta_{Du(x)}$ for $\mathcal{H}^m$-a.e. $x \in \Omega$. To do so, we consider the function h defined in (2.6). Note that

$$\|h(\Lambda)\|_{\mathrm{op}} \le 1, \qquad \forall \Lambda \in \mathbb{R}^{n\times m},$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm of a linear operator. This implies that

$$\|h(\Lambda)\mathcal{A}(\Lambda)\|_{\mathrm{op}} \le \mathcal{A}(\Lambda) \le C(1 + \|\Lambda\|^m), \qquad \forall \Lambda \in \mathbb{R}^{n\times m}.$$

By the definition of Young measures, we have, for any $\eta \in C^\infty_c(\Omega)$,

$$\int_\Omega h(Du_j)(x)\mathcal{A}(Du_j)(x)\eta(x)dx \to \int_\Omega \left(\int_{\mathbb{R}^{n\times m}} h(\Lambda)\mathcal{A}(\Lambda) d\nu_x(\Lambda)\right)\eta(x)dx. \tag{5.5}$$

We choose the following function $f : \Omega \times \mathbb{R}^n \times \mathbb{G}(n+m, m) \to \mathbb{R}^{N\times N}$:

$$f((x,y), T) := \eta(x)\psi(y)T,$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ enjoys the following properties:

$$\psi \equiv 1 \text{ on } B_{2M}(0), \quad \text{and} \quad \psi \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{3M+1}(0), \qquad \text{where } M = \max_{\mathrm{spt}(\eta)} \|u\|.$$

With a slight abuse of notation, we will anyway write $V_j(f)$, even though f is matrix-valued. This can be easily corrected considering the composition between f and the function $a_{ij} : \mathbb{R}^{n\times m} \to \mathbb{R}$ that gives the (i,j) component of the matrix. Since $V_j := [\![\Gamma_{u_j}]\!] \rightharpoonup V := [\![\Gamma_u]\!]$ as varifolds, we deduce

$$\int_{\Gamma_{u_j}} T_x\Gamma_{u_j}\eta(x)\psi(y)d\mathcal{H}^m(x,y) = V_j(f) \to V(f) = \int_{\Gamma_u} T_x\Gamma_u\eta(x)\psi(y)d\mathcal{H}^m(x,y),$$

which, by the area formula [8, Proposition 6.4], reads

$$\int_\Omega h(Du_j(x))\mathcal{A}(Du_j(x))\psi(u_j(x))\eta(x)dx \to \int_\Omega h(Du(x))\mathcal{A}(Du(x))\psi(u(x))\eta(x)dx.$$

By our choice of ψ and the uniform convergence $u_j \to u$, we rewrite the previous limit as

$$\int_\Omega h(Du_j(x))\mathcal{A}(Du_j(x))\eta(x)dx \to \int_\Omega h(Du(x))\mathcal{A}(Du(x))\eta(x)dx. \tag{5.6}$$

Combining (5.6) with (5.5), we deduce that for $\mathcal{H}^m$-a.e. $x \in \Omega$

$$h(Du(x))\mathcal{A}(Du(x)) = \int_{\mathbb{R}^{n\times m}} h(\Lambda)\mathcal{A}(\Lambda)d\nu_x(\Lambda).$$

Taking the inner product with $h(Du(x))^\perp$, we obtain

$$\begin{aligned} 0 = \langle h(Du(x))^\perp, h(Du(x))\rangle\mathcal{A}(Du(x)) &= \int_{\mathbb{R}^{n\times m}} \langle h(Du(x))^\perp, h(\Lambda)\rangle\mathcal{A}(\Lambda)d\nu_x(\Lambda) \\ &\overset{(3.19)}{=} \frac{1}{2}\int_{\mathbb{R}^{n\times m}} \|h(Du(x)) - h(\Lambda)\|^2\mathcal{A}(\Lambda)d\nu_x(\Lambda). \end{aligned}$$

Since, $\mathcal{A}(\Lambda) > 0$ for every $\Lambda \in \mathbb{R}^{n\times m}$, we deduce that

$$\nu_x = \delta_{Du(x)}, \qquad \text{for } \mathcal{H}^m\text{-a.e. } x \in \Omega \tag{5.7}$$

as claimed.

Conversely, we notice that it is sufficient to show that for every subsequence $[\![\Gamma_{u_{j_k}}]\!]$, there exists a further subsequence $[\![\Gamma_{u_{j_{k_\ell}}}]\!]$ converging to $[\![\Gamma_u]\!]$. In particular, avoiding relabeling, since we have the strong convergence in $W^{1,p}$ of u_j to u, we can further suppose that u_j and Du_j converge pointwise a.e. to u and Du respectively. By the area formula [8, Proposition 6.4], for every $f \in C_c(\mathbb{R}^N \times \mathbb{G}(N,m))$,

$$\begin{aligned} [\![\Gamma_{u_j}]\!](f) &= \int_{\Gamma_{u_j}} f(y, T_y\Gamma_{u_j})d\mathcal{H}^m(x) \\ &= \int_\Omega f(x, u_j(x), h(Du_j(x)))\mathcal{A}(Du_j(x))dx \to \int_\Omega f(x, u(x), h(Du(x)))\mathcal{A}(Du(x))dx \\ &= \int_{\Gamma_u} f(y, T_y\Gamma_u)d\mathcal{H}^m(x) = [\![\Gamma_u]\!](f). \end{aligned}$$

Let us justify the passage to the limit in the second line. By [8, Appendix B], we know that

$$|D\mathcal{A}(Z)| \le C(1 + |Z|^{\min\{m,n\}-1}), \quad \forall Z \in \mathbb{R}^{n\times m}.$$

Thus, for all $X, Y \in \mathbb{R}^{n\times m}$,

$$|\mathcal{A}(X) - \mathcal{A}(Y)| \le C(1 + \max\{|X|, |Y|\}^{\min\{m,n\}-1})|X - Y| \tag{5.8}$$

From (5.8), it readily follows by Hölder's inequality that if $Du_j \to Du$ strongly in $L^p(\Omega)$, for $p > m$, then $\mathcal{A}(Du_j) \to \mathcal{A}(Du)$ strongly in $L^q(\Omega)$ for some $q > 1$. As f is continuous and compactly supported, $x \mapsto f(x, u_j(x), h(Du_j(x)))$ is uniformly bounded in $L^\infty(\Omega)$. Therefore,

$$g_j(x) := f(x, u_j(x), h(Du_j(x)))\mathcal{A}(Du_j(x))$$

is a sequence of maps equibounded in $L^q(\Omega)$ for some $q > 1$ and pointwise converging to

$$g(x) := f(x, u(x), h(Du(x)))\mathcal{A}(Du).$$

This shows the strong convergence in $L^1(\Omega)$ of g_j to g and concludes the proof. □

5.1. **Additional comments.** We wish to add a few remarks to put Theorems 5.1-5.2 into context. Consider a compact set $K \subset \mathbb{R}^k$, a linear operator with constant coefficients $\mathbb{A}$ acting on $C_c^\infty(\mathbb{R}^m, \mathbb{R}^k)$ and an open set $\Omega \subset \mathbb{R}^m$. Studying the differential inclusion associated to $\mathbb{A}$ and K means to study properties of solutions $z \in L^\infty(\Omega, \mathbb{R}^k)$, $\Omega \subset \mathbb{R}^m$, to the following system:

$$\mathbb{A}(z) = 0 \text{ in the sense of distributions}, \quad z(x) \in K \text{ a.e. in } \Omega. \tag{5.9}$$

Differential inclusions provide a unified framework for the study of PDEs of completely different nature, for instance the Euler equations [10] and the aforementioned critical points of quasiconvex/polyconvex functionals, [29, 33]. In the case under consideration, $K = K_{F_\Psi}$ and $\mathbb{A}$ is a mixed *div-curl* operator. Depending on the set K, one may need to study different properties of solutions to (5.9). For instance, in the case of Euler equations, one wishes to construct non-conservative weak solutions, while in the case of [29, 33] one wishes to construct irregular solutions to (5.9). In both cases, the following question is of crucial importance:

Compactness: given a sequence $\{z_j\}_j$ weakly-* convergent in L^∞ to z, which satisfies $\mathrm{dist}(z_j, K) \to 0$ in L^1 and $\mathbb{A}(z_j) = 0$ for all $j \in \mathbb{N}$, is it true that the sequence converges strongly in L^1 to z?

Let us only analyze the case of [29, 33]. The strategy in [29, 33] consists first in finding a T_N-configuration $Z = \{z_1, \dots, z_N\} \subset K$. T_N-configurations are of great relevance, as they support non-trivial Young measures. In particular the previous question has negative answer by (2.8). Roughly speaking, one would like to turn an approximate solution as in the description of the Compactness property into an exact one. In order to do so, the second step consists in proving that, near Z, one can find a manifold of T_N-configurations inside K. This is enough in [29, 33] to produce an exact, very oscillatory and thus irregular, solution to (5.9). To summarize, the underlying idea of [29, 33] is to use approximate solutions and iterative procedures in order to eliminate errors, i.e. it is crucial to find (many) non-trivial Young measures in K. These features are present in [10] too, even though in [10] plane waves substitute T_N-configurations. In fact the authors do not know an example of a set K for which the Compactness property is true and which still allows for rough exact solutions constructed via convex integration. The second author and collaborators have already proved in [8, 23] results which hinted at a certain rigidity of the system (5.9) in the case considered in this section. In particular, it is known that it is not possible to find a T_N-configuration inside K_{F_Ψ}, under certain general assumptions on Ψ. Theorems 5.1-5.2 are much more systematic answers to the issue studied in [8, 23], in that they exclude *any* non-trivial Young measure inside K_{F_Ψ} as long as Ψ satisfies (AC). Thus, although we are not able at present to prove a partial regularity result under the sole (AC) assumption, Theorems 5.1-5.2 prevent the use of convex integration techniques to construct irregular solutions to (5.9).

6. Example: l^p norms

In this final section, we provide explicit examples of integrands $\Psi : \mathbb{G}(4, 2) \to (0, \infty)$ satisfying (AC1). In order to do so, we use another well-know representation of $\mathbb{G}(4, 2)$ in terms of the simple 2-vectors[2]. Recall that $\Lambda_2(\mathbb{R}^4)$ is the space of 2-vectors of $\mathbb{R}^4$, i.e. the vector space given by finite linear combinations of elements of the form $v_1 \wedge v_2$, with $v_i \in \mathbb{R}^4$. We also let $\Lambda_2^s(\mathbb{R}^4) \subset \Lambda_2(\mathbb{R}^4)$ be the space of simple 2-vectors,

[2]More generally, one can identify $\mathbb{G}(N, m)$ with the space of simple m-vectors of $\mathbb{R}^N$, see [21, Section 2.1].

i.e. all elements $\tau \in \Lambda_2(\mathbb{R}^4)$ such that $\tau = v_1 \wedge v_2$. Let $\{e_1, e_2, e_3, e_4\}$ be a canonical basis of $\mathbb{R}^4$, then $\{e_i \wedge e_j : 1 \leq i < j \leq 4\}$ is a *canonical* basis of $\Lambda_2(\mathbb{R}^4)$. The vector space $\Lambda_2(\mathbb{R}^4)$ can be endowed with a scalar product that is defined on simple vectors as

$$\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle := \det(X),$$

where $X \in \mathbb{R}^{2\times 2}$ is defined as $X_{ij} = (v_i, w_j)$. Consequently, we define $\|\tau\| := \sqrt{\langle \tau, \tau \rangle}$.

Non-zero simple vectors are in natural surjection with $\mathbb{G}(4,2)$, as for every non-zero $\tau = v_1 \wedge v_2 \in \Lambda_2^s(\mathbb{R}^4)$, one can associate the projection on $\mathrm{span}\{v_1, v_2\}$. We will denote such a projection matrix onto $\mathrm{span}\{v_1, v_2\}$ as $T(v_1 \wedge v_2)$. For any even and 1-homogeneous function $\mathcal{G} : \mathbb{R}^6 \to \mathbb{R}$, we can define $\Phi : \Lambda_2(\mathbb{R}^4) \to \mathbb{R}$ as:

$$\Phi(\tau) = \mathcal{G}(\langle v_1 \wedge v_2, e_1 \wedge e_2 \rangle, \langle v_1 \wedge v_2, e_1 \wedge e_3 \rangle, \langle v_1 \wedge v_2, e_1 \wedge e_4 \rangle, \langle v_1 \wedge v_2, e_2 \wedge e_3 \rangle, \langle v_1 \wedge v_2, e_2 \wedge e_4 \rangle, \langle v_1 \wedge v_2, e_3 \wedge e_4 \rangle). \tag{6.1}$$

for $\tau = v_1 \wedge v_2$. For ease of notation, denote $v_{ij} := \langle v_1 \wedge v_2, e_i \wedge e_j \rangle$, for every $1 \leq i, j \leq 4$. Notice that $v_{ij} = -v_{ji}$. Every such function Φ provides a well defined function $\Psi_{\mathcal{G}} : \mathbb{G}(4,2) \to \mathbb{R}$, simply by setting

$$\Psi_{\mathcal{G}}(T) := \Phi\left(\frac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|}\right), \quad \text{if } T = T(v_1 \wedge v_2).$$

In this case, we will denote $T = T(v_1 \wedge v_2)$. Notice that the evenness and 1-homogeneity of $\mathcal{G}$ imply that Ψ is well-defined.

The main result of this section is the following:

Theorem 6.1. *Let $p \in (1, +\infty)$, and let $\mathcal{G} = \|\cdot\|_{\ell^p}$ in $\mathbb{R}^6$. Then, $\Psi_{\mathcal{G}}$ satisfies (AC1).*

Notice that in the case $p = 2$, we recover the area functional. To prove the previous theorem, first we need to understand how the (AC) condition (3.1) reads with respect to these coordinates. We have:

Lemma 6.2. *Let $v_1, v_2 \in \mathbb{R}^4$ be linearly independent. Then*

$$B_{\Psi_{\mathcal{G}}}(T(v_1 \wedge v_2)) = B_{\mathcal{G}}\left(\frac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|}\right),$$

where, for all $1 \leq a, b \leq 4$, denoting with $\partial_{(ab)}\mathcal{G}$ the partial derivative of $\mathcal{G}$ in the component (ab),

$$(B_{\mathcal{G}})_{ab}(v_1 \wedge v_2) := \sum_{a<j} \partial_{(aj)}\mathcal{G}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) v_{bj} + \sum_{a>j} \partial_{(ja)}\mathcal{G}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) v_{jb}. \tag{6.2}$$

Proof. Consider a varifold $V = (\Gamma, \theta)$. We let $w_1(x), w_2(x)$ be an orthonormal basis for $T_x\Gamma$ at $\mathcal{H}^m \llcorner \Gamma$-a.e. x, i.e. $\|w_1(x)\| = \|w_2(x)\| = 1$ at $\mathcal{H}^m \llcorner \Gamma$-a.e. x, $\mathrm{span}\{w_1(x), w_2(x)\}$ is the approximate tangent space to Γ at $\mathcal{H}^m \llcorner \Gamma$-a.e. x, and $\|w_1(x) \wedge w_2(x)\| = 1$ at $\mathcal{H}^m \llcorner \Gamma$-a.e. x. Moreover, for the integrand Φ defined as in (6.1), we denote

$$\nabla\Phi(v_1 \wedge v_2) = \sum_{i<j} \partial_{(ij)}\mathcal{G}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) e_i \wedge e_j.$$

By [23, Lemma A.1], one can compute the variation of a varifold[3] $V = (\Gamma, \theta)$:

$$[\delta_\Phi V](g) = \int_\Gamma \langle \nabla\Phi(w_1 \wedge w_2(x)), Dg(x)w_1(x) \wedge w_2(x) + w_1(x) \wedge Dg(x)w_2(x) \rangle \theta(x) d\mathcal{H}^2(x). \tag{6.3}$$

We define $B_{\mathcal{G}}(\tau)$ by requiring that for every $L \in \mathbb{R}^{4\times 4}$:

$$\langle \nabla\Phi(v_1 \wedge v_2), (Lv_1) \wedge v_2 + v_1 \wedge (Lv_2) \rangle = \langle B_{\mathcal{G}}(v_1 \wedge v_2), L \rangle. \tag{6.4}$$

[3]The computation in [23] is actually carried out for currents, but the evenness of $\mathcal{G}$ allows us to immediately extend it to varifolds.

Some simple but lengthy computations yield that for $1 \le a, b \le 4$, $B_{\mathcal{G}}(v_1 \wedge v_2)$ has components given by (6.2). Through (6.4), we write (6.3) as

$$[\delta_\Phi V](g) = \int_\Gamma \langle B_{\mathcal{G}}(w_1 \wedge w_2), Dg(x)\rangle \theta(x) d\mathcal{H}^2(x).$$

Since $\Psi_{\mathcal{G}}(T(v_1 \wedge v_2)) = \Phi\left(\frac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|}\right)$, it is simple to see that

$$[\delta_{\Psi_{\mathcal{G}}} V](g) = [\delta_\Phi V](g) = \int_\Gamma \langle B_{\mathcal{G}}(w_1 \wedge w_2), Dg(x)\rangle \theta(x) d\mathcal{H}^2(x), \qquad \forall g \in C^1_c(\mathbb{R}^N, \mathbb{R}^N), \quad \forall V = (\Gamma, \theta).$$

By (3.1) and the arbitrariness of V and g, this yields $B_{\Psi_{\mathcal{G}}}(T(v_1 \wedge v_2)) = B_{\mathcal{G}}(v_1 \wedge v_2)$, and hence proves the Lemma. □

Finite measures on $\mathbb{G}(4,2)$ and even measures on the space

$$G := \{v_1 \wedge v_2 \in \Lambda^s_2(\mathbb{R}^4) : \|v_1 \wedge v_2\| = 1\}$$

are in bijective correspondence, through the map $v_1 \wedge v_2 \mapsto T(v_1 \wedge v_2)$. This consideration and the previous Lemma imply the following:

Lemma 6.3. *The map $\Psi_{\mathcal{G}} : \mathbb{G}(4,2) \to \mathbb{R}$ satisfies (AC1) if and only for every even probability measure μ on G,*

$$\dim \operatorname{Ker} \int_G B_{\mathcal{G}}(v_1 \wedge v_2) d\mu(v_1 \wedge v_2) \le 2.$$

The map $\Psi_{\mathcal{G}} : \mathbb{G}(4,2) \to \mathbb{R}$ satisfies (AC2) if and only if for every even probability measure μ on G,

$$\dim \operatorname{Ker} \int_G B_{\mathcal{G}}(v_1 \wedge v_2) d\mu(v_1 \wedge v_2) = 2 \qquad \Longleftrightarrow \qquad \mu = \frac{\delta_{\tau_0} + \delta_{-\tau_0}}{2} \quad \text{for some } \tau_0 = v_1^0 \wedge v_2^0 \in G.$$

Proof of Theorem 6.1. Fix $p \in (1, +\infty)$. We use the characterization given by Lemma 6.3: given an even probability measure μ on G, we denote

$$A(\mu) = \int B_{\mathcal{G}}(v_1 \wedge v_2) d\mu(v_1 \wedge v_2).$$

We need to show that $\operatorname{rank}(A(\mu)) \ge 2$. First of all, by formula (6.2), we have

$$(B_{\mathcal{G}})_{ab}(v_1 \wedge v_2) = \frac{\sum_{a<j} \operatorname{sign}(v_{aj})|v_{aj}|^{p-1} v_{bj} + \sum_{a>j} \operatorname{sign}(v_{ja})|v_{ja}|^{p-1} v_{jb}}{\mathcal{G}^{p-1}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34})} = \frac{\sum_j \operatorname{sign}(v_{aj})|v_{aj}|^{p-1} v_{bj}}{\mathcal{G}^{p-1}(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34})}.$$

Motivated by this expression, we define the even finite measure μ' on G:

$$\mu'(f) = \int_G \frac{f(v_1 \wedge v_2)}{\Phi^{p-1}(v_1 \wedge v_2)} d\mu(v_1 \wedge v_2), \qquad \text{for all } f \in C(G).$$

The goal now is therefore to show that

$$A(\mu) = \sigma(\mu') := \int B_{\mathcal{G}}(v_1 \wedge v_2) \Phi^{p-1}(v_1 \wedge v_2) d\mu'$$

cannot have rank smaller than 2. For ease of notation, denote

$$a_{ij} := \int |v_{ij}|^p d\mu',$$

for all $i, j \in \{1, \dots, 4\}$. We observe that $a_{ij} = a_{ji}$ if $i > j$ and $a_{ii} = 0$. The principal 2×2 subminor S_{ij} of $\sigma(\mu')$, i.e. the submatrix of $\sigma(\mu')$ obtained by considering only the i-th and j-th rows and columns, are:

$$S_{ij} = \begin{pmatrix} a_{1i} + a_{2i} + a_{3i} + a_{4i} & \sum_{s=1}^4 \int \operatorname{sign}(v_{is})|v_{is}|^{p-1} v_{js} d\mu' \\ \sum_{s=1}^4 \int \operatorname{sign}(v_{js})|v_{js}|^{p-1} v_{is} d\mu' & a_{1j} + a_{2j} + a_{3j} + a_{4j} \end{pmatrix}.$$

It suffices to prove that at least one of the S_{ij} must be invertible to conclude the proof. We first use Hölder inequality to estimate

$$|(S_{ij})_{21}| = \left|\sum_{s=1}^{4}\int \operatorname{sign}(v_{js})|v_{js}|^{p-1}v_{is}d\mu'\right| = \left|\sum_{s\neq i, s\neq j}\int \operatorname{sign}(v_{js})|v_{js}|^{p-1}v_{is}d\mu'\right|$$
$$\leq \sum_{s\neq i,s\neq j}\left(\int |v_{js}|^p d\mu'\right)^{\frac{p-1}{p}}\left(\int |v_{is}|^p d\mu'\right)^{\frac{1}{p}} = \sum_{s\neq i,s\neq j} a_{js}^{\frac{p-1}{p}} a_{is}^{\frac{1}{p}}$$

and analogously,

$$|(S_{ij})_{12}| = \left|\sum_{s=1}^{4}\int \operatorname{sign}(v_{is})|v_{is}|^{p-1}v_{js}d\mu'\right| \leq \sum_{s\neq i,s\neq j} a_{is}^{\frac{p-1}{p}} a_{js}^{\frac{1}{p}}.$$

Thus, we can estimate:

$$\begin{aligned}\det(S_{ij}) &= (S_{ij})_{11}(S_{ij})_{22} - (S_{ij})_{12}(S_{ij})_{21} \geq (S_{ij})_{11}(S_{ij})_{22} - |(S_{ij})_{21}(S_{ij})_{12}| \\ &\geq \sum_{s\neq i} a_{is}\sum_{s\neq j} a_{js} - \sum_{s\neq i,s\neq j} a_{is}^{\frac{p-1}{p}} a_{js}^{\frac{1}{p}} \sum_{s\neq i,s\neq j} a_{js}^{\frac{p-1}{p}} a_{is}^{\frac{1}{p}}. \end{aligned} \tag{6.5}$$

We use again Hölder inequality to write

$$\sum_{s\neq i,s\neq j} a_{is}^{\frac{p-1}{p}} a_{js}^{\frac{1}{p}} \leq \left(\sum_{s\neq i,s\neq j} a_{is}\right)^{\frac{p-1}{p}}\left(\sum_{s\neq i,s\neq j} a_{js}\right)^{\frac{1}{p}} \leq \left(\sum_{s\neq i} a_{is}\right)^{\frac{p-1}{p}}\left(\sum_{s\neq j} a_{js}\right)^{\frac{1}{p}} \tag{6.6}$$

and analogously

$$\sum_{s\neq i,s\neq j} a_{is}^{\frac{1}{p}} a_{js}^{\frac{p-1}{p}} \leq \left(\sum_{s\neq i} a_{is}\right)^{\frac{1}{p}}\left(\sum_{s\neq j} a_{js}\right)^{\frac{p-1}{p}}. \tag{6.7}$$

Plugging (6.6)-(6.7) in (6.5), we get $\det(S_{ij}) \geq 0$. If $\det(S_{ij}) = 0$, then the second inequality of (6.6) would hold with an equality, implying $a_{ij} = 0$. If we had $\det(S_{ij}) = 0$ for every i, j, then this would mean

$$a_{ij} = \int |v_{ij}|^p d\mu' = 0. \tag{6.8}$$

As μ' is supported on G, we have $\|\tau\|^2 = \|v_1 \wedge v_2\|^2 = \sum_{i<j}|v_{ij}|^2 = 1$ μ'-a.e.. Since μ is a nonnegative measure, if follows that μ' is a nonnegative measure. Therefore, (6.8) implies that μ' is the zero measure. In turn, this implies that μ is the zero measure, which is in contradiction with the fact that μ is a probability measure. This reasoning shows that at least one of the S_{ij} must be invertible, as desired. □

Remark 6.4. *Let us remark that the considerations at the beginning of the section and Lemmas 6.2-6.3 hold in $\mathbb{G}(N, m)$, but we have chosen here to use only $\mathbb{G}(4, 2)$ for ease of notation. We do not know whether Theorem 6.1 holds for higher dimensions or codimensions. Moreover, numerical simulations indicated that for $\mathcal{G}(\cdot) = \|\cdot\|_{\ell_p}, p \in (1, \infty)$, $\Psi_{\mathcal{G}}$ fulfills (SAC), hence actually satisfies (AC), by Proposition 3.5. Unfortunately, we were not able to prove it analytically.*

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Antonio De Rosa
Department of Mathematics, University of Maryland, 4176 Campus Dr, College Park, Maryland 20742, United States
Email address: anderosa@umd.edu

Riccardo Tione
Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany
Email address: riccardo.tione@mis.mpg.de