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METRIC HOPF-LAX FORMULA WITH SEMICONTINUOUS DATA

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Abstract. In this paper we study a metric Hopf-Lax formula looking in particular at the Carnot-Carathéodory case. We generalize many properties of the classical euclidean Hopf-Lax formula and we use it in order to get existence results for Hamilton-Jacobi-Cauchy problems satisfying a suitable Hörmander condition.

1. **Introduction.** The aim of this paper is to study some properties of the Hopf-Lax function associated to a generalized distance on $\mathbb{R}^n$ and a semicontinuous datum. We prove, in particular, that for Carnot-Carathéodory metrics which satisfy the Hörmander condition the Hopf-Lax function is a viscosity solution of the Cauchy problem for an Hamilton-Jacobi equation for a state-dependent Hamiltonian, the model being $H(x, Du) = \frac{1}{\alpha}|\sigma(x)Du|^\alpha$, where $\sigma(x)$ is a $m \times n$ matrix satisfying the Hörmander condition.

In Section 2 we define a metric Hopf-Lax formula and we study some of its basic properties. In particular we show that the Hopf-Lax function lower converges to the function g, as $t \to 0^+$, and it is non increasing in t. Moreover it is locally d-Lipschitz in x and locally (euclidean) Lipschitz in t.

In Section 3 we define a minimal-time function and we show that it satisfies a Dynamical Programming Principle. Moreover we prove that, under the Hörmander condition, the Carnot-Carathéodory distance $d(x, y)$ solves in the viscosity sense the horizontal eikonal equation $|\sigma(x)Dd(x, y)| = 1$ in $\mathbb{R}^n\backslash\{y\}$, for any fixed $y \in \mathbb{R}^n$. By the Pansu-Rademacher Theorem, it is an almost everywhere solution, too. In Section 4 we prove an existence result, in the viscosity lower semicontinuous sense, for an Hamilton-Jacobi-Cauchy problem, using the metric Hopf-Lax formula and the eikonal solution built in Sec. 3.

In Section 5 we give some applications for our Hopf-Lax function. First we prove that, if the initial datum is continuous, then the Hopf-Lax function is also. This implies that in this case the Hopf-Lax function is also a viscosity solution following the usual definition of Crandall and Lions. Moreover we pay particular attention to the sub-Riemannian model. Using the Lipschitz regularity results proved in Sec. 2, we remark that, in a such case, the Hopf-Lax solution is an almost everywhere solution for the corresponding Cauchy problem, too. At last, in the particular case

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of the 1-dimensional Heisenberg group, we check that our formula is the same as the Manfredi-Stroffolini formula, proved in [16].

2. **Generalized distances and the metric Hopf-Lax function.** We start by recalling some metric notions.

Definition 1. A *generalized distance on* $\mathbb{R}^n$ is any function $d:\mathbb{R}^n\times\mathbb{R}^n\to[0,+\infty)$ satisfying

$$d(x,y)\geq 0,\ \forall\, x,y\in\mathbb{R}^n,\quad d(y,y)=0,\ \forall\, y\in\mathbb{R}^n, \tag{1}$$

$$d(x,y)\leq d(x,z)+d(z,y),\quad \forall\, x,y,z\in\mathbb{R}^n. \tag{2}$$

In the previous definition we do not require neither the symmetry nor the positive defining property (i.e. $d(x,y)=0$ implies that $x=y$). For example the degenerate distances associated to Finsler metrics are so ([11]).
It is trivial that any finite distance is also a generalized distance, so in particular we will study the case of the Carnot-Carathéodory distances on $\mathbb{R}^n$ satisfying the Hörmander condition. Therefore we recall briefly what are these and some of their properties, which are useful in the study of the associated metric Hopf-Lax function. One can find more informations on Carnot-Carathéodory distances and the Hörmander condition in [5, 17, 21]. Let $X_1(x),...,X_m(x)$ a family of vector fields on $\mathbb{R}^n$ and set $\mathcal{H}(x)=\mathrm{Span}(X_1(x),...,X_m(x))$, then a *distribution* on $\mathbb{R}^n$ is defined as $\mathcal{H}=\{(x,\mathcal{H}(x))\,|\,x\in\mathbb{R}^n\}$.

Definition 2. A *sub-Riemannian metric* in $\mathbb{R}^n$ is a Riemannian metric $\langle\quad\rangle$ defined on the fibers of a distribution $\mathcal{H}$.

Definition 3. An absolutely continuous curve $\gamma:I\to\mathbb{R}^n$ is *admissible* (or also *horizontal*) if $\dot\gamma(t)\in\mathcal{H}(\gamma(t))$, a.e. $t\in I$.

For all the admissible curves, and only for these, it is well defined the following length-functional

$$l(\gamma)=\int_I|\dot\gamma(t)|dt,$$

where $|\dot\gamma(t)|=\langle\dot\gamma(t),\dot\gamma(t)\rangle^{\frac{1}{2}}$.

Definition 4. A *Carnot-Carathéodory distance* is a function defined as

$$d(x,y):=\inf\{l(\gamma)\,|\,\gamma \text{ horizontal curve joining } x \text{ to } y\}. \tag{3}$$

A Carnot-Carathéodory distance, which we call simply C-C distance, is a real distance on all $\mathbb{R}^n$ but sometime it can be infinite for some pair of points. So we introduce the Hörmander condition but first we recall that a bracket between two vector fields X and Y is the vector fields defined, for any smooth real function f, as $[X,Y]f=X(Yf)-Y(Xf)$. Let $\mathcal{L}^0=\{X_1,...,X_m\}$, $\mathcal{L}^1=\{[X_i,X_j]|\,i=1,...,m\}$ and $\mathcal{L}^k=\{[Y_i,Y_j]\,|\,Y_i\in\mathcal{L}^h,Y_j\in\mathcal{L}^l,h,l=0,...,k-1\}\setminus\bigcup_{i=0}^{k-1}\mathcal{L}^i$, then the *Lie algebra* associated to some distribution is the set $\mathcal{L}=\bigcup_{k\in\mathbb{N}}\mathcal{L}^k$.

Definition 5. A C-C distance satisfies the *Hörmander condition* if its distribution is bracket generating, i.e. if the associated Lie algebra spans, in any point, the whole $\mathbb{R}^n$.

The main result for C-C distances satisfying the Hörmander condition is the Chow Theorem (see [5, 17]).

Theorem 1 (Chow Theorem)**.** *Let $\mathcal{H}$ a bracket generating distribution, then there exists a $\mathcal{H}$-horizontal curve joining any two given points.*

A consequence of the Chow Theorem is that the associated C-C distance is finite. The next example shows that the Hörmander condition is not a necessary condition to get a finite C-C distance.

Example 1. We consider on $\mathbb{R}^2$ the following vector fields $X_1((x,y)) = \frac{\partial}{\partial x}$ and $X_2((x,y)) = a(x)\frac{\partial}{\partial y}$, with $a(x) = 1$, if $x \geq 0$ and $a(x) = 0$, if $x < 0$. On the half-plane $x < 0$ we can move only in one direction, then the spanned distribution is not bracket generating. Nevertheless it is easy to write explicitly the associated Carnot-Carathéodory distance, that is

$$d((x,y),(x',y')) = \begin{cases} \sqrt{|x-y|^2 + |x'-y'|^2}, & x_1 \geq 0\ x' \geq 0 \\ |x| + |x'| + |y-y'|, & x < 0\ x' < 0 \\ |x| + \sqrt{|x'|^2 + |y-y'|^2}, & x < 0\ x' \geq 0 \\ |x'| + \sqrt{|x|^2 + |y-y'|^2}, & x \geq 0\ x' < 0 \end{cases}$$

It is immediate to note that d is a finite distance but it is not continuous w.r.t. the euclidean topology. In fact for any $x < 0$

$$\lim_{y \to 0} d((x,0),(x,y)) = 2|x| > 0.$$

The Hörmander condition implies also that the associated distance induces the same topology as the original euclidean topology on $\mathbb{R}^n$, see [17] or Corollary 2.6 in [5]. Moreover the Hörmander condition is also a necessary condition only for analytic vector fields while we look at smooth vector fields and so use it only as sufficient condition.

Now we introduce the Hopf-Lax function associated to the generalized distance d. From now on we indicate by $d(x)$ the generalized distance from the origin to a point x. Let $g : \mathbb{R}^n \to \mathbb{R}$ a lower semicontiuous function such that there exists $C > 0$:

$$g(x) \geq -C(1 + d(x)), \tag{4}$$

and let $\Phi : [0, +\infty) \longrightarrow [0 + \infty)$ continuous, convex, not decreasing, with $\Phi(0) = 0$. The metric Hopf-Lax formula associated to the generalized distance d and the function g is defined by

$$u(x,t) := \inf_{y \in \mathbb{R}^n} \left[g(y) + t\Phi^* \left(\frac{d(x,y)}{t} \right) \right], \tag{5}$$

where Φ^* is the Lagendre transform of Φ, i.e. $\Phi^*(t) = \sup_{s \geq 0}\{ts - \Phi(s)\}$. Note that $\Phi^* : [0, +\infty) \to [0, +\infty]$ is convex, non decreasing and $\Phi^*(0) = 0$.
We begin to study the properties of the metric Hopf-Lax function (5) (for the euclidean case see [1, 2, 13]), remarking that $u(t,x) \leq g(x)$, for any $t > 0$. The next properties are key-points for the viscosity result proved in Sec. 4.

Lemma 1. *Let d a generalized distance inducing the euclidean topology on $\mathbb{R}^n$, then the metric Hopf-Lax function* (5) *lower converges to g, i.e.*

$$\liminf_{(t,x) \to (0^+, \overline{x})} u(t,x) = \inf \left\{ \liminf_{n \to \infty} u(t_n, x_n) \,|\, (t_n, x_n) \to (0^+, \overline{x}) \right\} = g(\overline{x}). \tag{6}$$

Proof. The proof follows the lines of a similar statement in [1]. The main step is to prove, for any $r \geq 0$, the following estimate using the condition (4), the definition of Φ^* and the monotonicity assumption on Φ.

$$u(t,x) \geq \inf_{y\in\mathbb{R}^n} \left[(r-C)\,d(x,y) - C - C\,d(x) - t\Phi(r)\right], \quad \forall\, r \geq 0. \tag{7}$$

By lower semicontinuity of g, for any $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$g(y) \geq g(\overline{x}) - \varepsilon, \quad \forall\, d(\overline{x}, y) < 2\delta. \tag{8}$$

Choosing $\overline{r} > C : \big(\overline{r} - C\big)\delta - C(d(\overline{x}) + \delta) - C \geq g(\overline{x})$ in (7) and $0 < \tau \leq \frac{\varepsilon}{\Phi(\overline{r})}$, we can estimate the Hopf-Lax function outside the ball $B = B^d_{2\delta}(\overline{x})$, i.e.

$$\inf_{y\in\mathbb{R}^n\setminus B}\left[g(y) + t\Phi^*\left(\frac{d(x,y)}{t}\right)\right] \geq g(\overline{x}) - \varepsilon.$$

Inside the ball, using the same inequality with $r = 0$ and $\Phi(0) = 0$, we conclude $g(y) \geq g(\overline{x}) - \varepsilon$. So we have proved that

$$\liminf_{(t,x)\to(0^+,\overline{x})} u(t,x) \geq g(\overline{x}).$$

The opposite inequality follows choosing the sequences of the form $(t_n, \overline{x})$ with $t_n \to 0^+$. In fact, by $\Phi \geq 0$, it immediately that $u(t,\overline{x}) \leq g(\overline{x})$, then

$$\liminf_{(t,x)\to(0^+,\overline{x})} u(t,x) \leq \liminf_{n\to\infty} u(t_n,\overline{x}) \leq g(\overline{x}).$$

□

From (7) with $r = C$, a d-superlinear estimate from below follows.

Lemma 2. *Under assumptions of Lemma 1, set $C' = \max\{C, \Phi(C)\}$, then*

$$u(t,x) \geq -C'(1 + d(x) + t). \tag{9}$$

Lemma 3. *Under assumptions of Lemma 1, the metric Hopf-Lax function* (5) *is lower semicontinuous on* $[0,+\infty) \times \mathbb{R}^n$.

Proof. As for the proof of Lemma (1), we follow the lines of a similar statement in [1]. So we want to show that sublevels of u are closed. Let (t_k, x_k) a sequence such that $u(t_k, x_k) \leq \gamma$, for some $\gamma \in \mathbb{R}$. If $(t_k, x_k) \to (t, x)$, we must prove that $u(t,x) \leq \gamma$. Since $g \in LSC(\mathbb{R}^n)$, we assume $t > 0$. Let $\{y^n_k\}$ a minimizing sequence for (5) in (t_k, x_k), (7) with $r = 1 + C$ gives

$$\begin{aligned}\gamma \geq u(t_k,x_k) &= \liminf_{n\to\infty}\left[g(y^n_k) + t_k\Phi^*\left(\frac{d(x_k,y^n_k)}{t_k}\right)\right]\\ &\geq \liminf_{n\to\infty} d(x_k,y^n_k) - C - C\,d(x_k) - t_k\Phi(C).\end{aligned}$$

By definition of minimum limit with $\varepsilon = 1$ we get $y^n_k \in \overline{B}_k := \overline{B^d_{R(k)}(x_k)}$, with $R(k) = C + Cd(x_k) + t_k\Phi(C) + \gamma + 1$, definitively. Using the convergence of $(t_k, x_k) \to (t, x)$, it is not hard to check that $y_k \in \overline{B^d_R(x)}$ for a suitable constant $R > 0$. We have assumed that d induces the euclidean topology then the relative closed balls are compact. Therefore, as $k \to +\infty$, y_k admits a convergent subsequence to some point $y \in \overline{B^d_R(x)}$. Using the semicontinuity of g, Φ^* and d, we can conclude

$$\gamma \geq u(t_x, x_k) \geq g(y) + t\Phi^*\left(\frac{d(x,y)}{t}\right) \geq u(t,x).$$

□

We show that for any g, bounded and lower semicontinuous, the infimum in formula (5) is a minimum, exactly as in the euclidean case (see [2]).

Lemma 4. *Let d a generalized distance and $g \in BLSC(\mathbb{R}^n)$ (lower semicontinuous and bounded). Then for any $x \in \mathbb{R}^n$ and $t > 0$, the infimum in* (5) *is a minimum.*

Proof. We need only to prove that, for any $x \in \mathbb{R}^n$ and $t > 0$ fixed, there exists a radius $R(t)$ enough large that $g(y) + t\Phi^*\left(\frac{d(x,y)}{t}\right) \geq \|g\|_\infty$, for $y \in \mathbb{R}^n \setminus \overline{B^d_{R(t)}(x)}$. Then, since $u(t,x) \leq g(x) \leq \|g\|_\infty$, then the infimum is attended in $\overline{B^d_{R(t)}(x)}$ and so, by lower semicontinuity of g, it is a minimum.
To prove the previous claim, notice that $\Phi^*(\tau)$ is convex so there exists a supporting line $m\tau + q$. Moreover $\Phi^*(0) = 0$, then $q \leq 0$ and, by the non decreasing property of Φ^*, we can also assume $m > 0$. Then, chose $R(t) = \frac{2\|g\|_\infty - tq}{m}$, for $y \in \mathbb{R}^n \setminus \overline{B^d_{R(t)}(x)}$

$$g(y) + t\Phi^*\left(\frac{d(x,y)}{t}\right) \geq g(y) + md(x,y) + tq \geq -\|g\|_\infty + 2\|g\|_\infty = \|g\|_\infty .$$

We conclude remarking that $R(t)$ is not decreasing in $t > 0$. □

Remark 1. When the convex function Φ is a power, i.e. $\Phi(t) = \frac{1}{\alpha}t^\alpha$, with $\alpha \geq 1$, then we can easily show that $\Phi^*(t) = \frac{1}{\beta}t^\beta$ with $\beta = \frac{\alpha}{\alpha-1}$, if $\alpha > 1$, while if $\alpha = 1$, then

$$\Phi^*(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ +\infty, & t > 1 \end{cases}$$

So for convex powers, the Hopf-Lax function is

$$u(t,x) = \inf_{y \in \mathbb{R}^n} \left[g(y) + \frac{1}{\beta}\frac{d(x,y)^\beta}{t^{\beta-1}} \right], \tag{10}$$

if $\alpha > 1$ and

$$u(t,x) = \inf \{ g(y) \mid d(x,y) < t \}, \tag{11}$$

if $\alpha = 1$. In these case, by simple calculations, it is possible to show that previous infimums are attended in the closed d-ball centered in x with radius $R(t) = (2\beta)^{\frac{1}{\beta}} t^{\frac{\beta-1}{\beta}} \|g\|_\infty^{\frac{1}{\beta}}$ and $R(t) = t$, respectively.

The previous lemma is useful to prove the following locally Lipschitz properties. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is d-Lipschitz continuous w.r.t. a non symmetric distance $d(x,y)$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C \max\{d(x,y), d(y,x)\}.$$

Proposition 1. *Let $g \in BLSC(\mathbb{R}^n)$ then, for any generalized distance d, fixed $t > 0$, the metric Hopf-Lax function* (5) *is locally d-Lipschitz continuous in x.*

Proof. By Lemma 4, we can choose $\overline{y}$ such that $u(t,y) = g(\overline{y}) + t\Phi^*\left(\frac{d(y,\overline{y})}{t}\right)$. Φ^* is convex and then locally Lipschitz continuous. Hence, for any $K \subset \mathbb{R}^n$ compact, there exists a constant $C(K) > 0$ such that

$$u(t,x) - u(t,y) \leq C(K)|d(x,\overline{y}) - d(y,\overline{y})|, \quad x, y \in K.$$

By the triangle inequality, $|d(x,\overline{y}) - d(y,\overline{y})| \leq \max\{d(x,y), d(y,x)\}$, so by the previous estimate and swapping x with y we conclude the proof. □

To prove the local Lipschitz continuity in t we need to use the relative geodesics. A *geodesic* is any absolutely continuous horizontal curve which realizes the minimum in the definition (3) of C-C distance. In particular, for C-C distances satisfying the Hörmander condition, $(\mathbb{R}^n, d)$ is a length spaces, that is for any $x, y \in \mathbb{R}^n$ there exists a geodesics γ joining x to y and such that $l(\gamma) = d(x, y)$ (see [17], Theorems I.1.19). As in any length space, we can assume that the geodesic $\gamma : [0, T] \to \mathbb{R}^n$ is parameterized by arc-length, so $l(\gamma) = T$ and $d(\gamma(t), \gamma(s)) = |t-s|$, for $s, t \in [0, l(\gamma)]$ (see [21] Lemma 3.3). To show the local Lipschitz continuity in t, we proceed as in [13], so first we prove a suitable functional identity.

Lemma 5. *Let d be a C-C distance satisfying the Hörmander condition, $g \in BLSC(\mathbb{R}^n)$, then the Hopf-Lax function* (5) *satisfies, for any $0 \le s < t$,*

$$u(t,x) = \inf_{y \in \mathbb{R}^n} \left[u(s,y) + (t-s)\Phi^* \left(\frac{d(x,y)}{t-s} \right) \right]. \tag{12}$$

Proof. By the usual triangle inequality for d and using first the non decreasing property of Φ^* and then its convexity, we get that

$$\Phi^* \left(\frac{d(x,z)}{t} \right) \le \left(1 - \frac{s}{t}\right) \Phi^* \left(\frac{d(x,y)}{t-s} \right) + \frac{s}{t} \Phi^* \left(\frac{d(y,z)}{s} \right),$$

for any $x, y, z \in \mathbb{R}^n$. Fixed x, for any y we choose a minimum point z for $u(s, y)$ (that exists by Lemma 4). Using such point z, we get

$$u(t,x) \le g(z) + t\Phi^* \left(\frac{d(x,z)}{t} \right) \le u(s,y) + (t-s)\Phi^* \left(\frac{d(x,y)}{t-s} \right).$$

Taking the infimum for $y \in \mathbb{R}^n$, we find the following inequality

$$u(t,x) \le \inf_{y \in \mathbb{R}^n} \left[u(s,y) + (t-s)\Phi^* \left(\frac{d(x,y)}{t-s} \right) \right].$$

To prove the inverse inequality, we choose a minimum point w for $u(t, x)$. Put $T = d(x, w)$ there exists $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(T) = w$ and $d(\gamma(s), \gamma(t)) = t - s$, for every $0 \le s \le t \le T$. We define $\overline{y} := \gamma\left(\frac{T(t-s)}{t}\right)$, so that $\frac{d(x,\overline{y})}{t-s} = \frac{d(x,w)}{t} = \frac{d(\overline{y},w)}{s}$, getting the other required inequality

$$\begin{aligned} \inf_{y \in \mathbb{R}^n} \left[u(s,y) + (t-s)\Phi^* \left(\frac{d(x,y)}{t-s} \right) \right] &\le u(s,\overline{y}) + (t-s)\Phi^* \left(\frac{d(x,\overline{y})}{t-s} \right) \le g(w) \\ + s\Phi^* \left(\frac{d(\overline{y},w)}{s} \right) + (t-s)\Phi^* \left(\frac{d(x,\overline{y})}{t-s} \right) &= g(w) + t\Phi^* \left(\frac{d(x,w)}{t} \right) = u(x,t). \end{aligned}$$

□

Remark 2. By choosing $y = x$ in (12), from Lemma 5, , we deduce that the metric Hopf-Lax function (5) is non increasing in t.

Proposition 2. *Let $g \in BLSC(\mathbb{R}^N)$ and assume $t\Phi^*\left(\frac{1}{t}\right)$ convex and decreasing for $t > 0$. Then the Hopf-Lax function associated to a C-C distance satisfying the Hörmander condition, is locally Lipschitz continuous in $t > 0$.*

Proof. Since $u(t, x)$ is non increasing in t, for any $0 \le s \le t$, $u(t, x) - u(s, x) \le 0$. So we need only to check the estimate from below. Choosing a minimum point $\overline{y} = \overline{y}(t)$ for $u(t, x)$, then for any $T_1 \le s < t \le T_2$, we find

$$0 \ge u(t,x) - u(s,x) \ge t\Phi^* \left(\frac{d(x,\overline{y})}{t} \right) - s\Phi^* \left(\frac{d(x,\overline{y})}{s} \right) = I.$$

Fix $t>0$ and let s free. If $d(x,\overline{y})=0$ then $u(t,x)=u(s,x)$ and we have concluded, then we can assume $d(x,\overline{y})\neq 0$ and set $\tau=\frac{t}{d(x,\overline{y})}$ and $\sigma=\frac{s}{d(x,\overline{y})}$. Therefore

$$I=d(x,\overline{y})\left[\tau\Phi^*\left(\frac{1}{\tau}\right)-\sigma\Phi^*\left(\frac{1}{\sigma}\right)\right].$$

Using the local Lipschitz continuity of the convex functions, for any $\widetilde{T}>0$, there exists $C=C(\widetilde{T})>0$ such that

$$I\geq Cd(x,\overline{y})(\tau-\sigma)=C(t-s),\quad \text{for any } \widetilde{T}\leq\sigma\leq\tau.$$

If we choose $\widetilde{T}=\frac{T_1}{R(T_2)}$, by Lemma 4 and the non decreasing property of $R(t)$, we get that for any $s,t\in[T_1,T_2]$, $\sigma,\tau\in[\widetilde{T},+\infty)$ and so we can conclude. ☐

Remark 3. If we look at $\Phi(t)=\frac{1}{\alpha}t^\alpha$ with $\alpha>1$, by Remark 1 it is immediate that $t\Phi^*\left(\frac{1}{t}\right)$ as in Proposition 2. More in general, this property holds whenever Φ^* is strictly convex and there exists $(\Phi^*)''(t)$, in fact $\left(t\Phi^*\left(\frac{1}{t}\right)\right)''=(\Phi^*)''\left(\frac{1}{t}\right)\frac{1}{t^2}\geq 0$ (for example $\Phi(t)=\mathrm{e}^t-1$). Instead in the linear case $\Phi(t)=t$ both of the requirements are not satisfied.

To conclude the study of the properties of the metric Hopf-Lax function, we point out a link with a problem in the calculus of variation, see [13], Section 3.3.1, for the euclidean case. We look at the minimization problem

$$v(t,x)=\inf\left\{\int_0^t\Phi^*(|\dot{\gamma}(s)|)ds+g(\gamma(t))\,\middle|\,\gamma\text{ a.c., horizontal, with }\gamma(0)=x\right\}\tag{13}$$

Proposition 3. *Let $g\in LSC(\mathbb{R}^n)$ and d a C-C distance satisfying the Hörmander condition, then the infimum* (13) *coincides with the metric Hopf-Lax function* (5)*.*

Proof. From the Jensen inequality it follows immediately that

$$\Phi^*\left(\frac{1}{t}\int_0^t|\dot{\gamma}(s)|ds\right)\leq\frac{1}{t}\int_0^t\Phi^*(|\dot{\gamma}(s)|)ds.$$

So for $t>0$ and all the a.c. horizontal curve $\gamma:[0,t]\to\mathbb{R}^n$, joining x to a point $y\in\mathbb{R}^n$, it holds

$$g(y)+t\Phi^*\left(\frac{d(x,y)}{t}\right)\leq g(y)+t\Phi^*\left(\frac{l(\gamma)}{t}\right)\leq g(y)+\int_0^t\Phi^*(|\dot{\gamma}(s)|)ds.$$

Taking the infimum, we get $u(t,x)\leq v(t,x)$. To prove the reverse inequality we must use the length structure. Fix $t>0$ and $y\in\mathbb{R}^n$, there exists γ geodesic parameterized by arc-length and joining x to y. Set $T=d(x,y)$ and define $\widetilde{\gamma}(s):=\gamma(\frac{Ts}{t})$, then $|\dot{\widetilde{\gamma}}(s)|=\frac{T}{t}|\dot{\gamma}(\overline{s})|=\frac{T}{t}$, so

$$\int_0^t\Phi^*(|\dot{\widetilde{\gamma}}(s)|)ds=\int_0^t\Phi^*\left(\frac{T}{t}\right)ds=t\Phi^*\left(\frac{d(x,y)}{t}\right).$$

Adding $g(y)$, we get

$$v(t,x)\leq g(y)+t\Phi^*\left(\frac{d(x,y)}{t}\right)$$

and so we conclude taking the infimum in $y\in\mathbb{R}^n$. ☐

Remark 4. By Lemma 4 and Proposition 3, for a C-C distance satisfying the Hörmander condition and $g \in BLSC(\mathbb{R}^n)$, it is possible to express the value function of the calculus of variation problem (13) as the solution of a minimization problem in $\mathbb{R}^n$.

If $\Phi(t) = \frac{1}{2}t^2$, then the metric Hopf-Lax function (5) coincides with the metric inf-convolution w.r.t. the distance d. One can find information about euclidean inf-convolutions in [7, 2]. For a study of metric inf-convolutions in the Carnot-Carathéodory case see the work in progress [12].

3. **The generalized eikonal equation.** In this section we study a generalized eikonal equation under a suitable Hörmander-type condition on the Hamiltonian. Generalized solutions and in particular viscosity solutions of eikonal equations are studied by P.L. Lions in [15] for convex geometrical Hamiltonians and then by A. Siconolfi in [20] in the non convex case. Therefore we look at

$$H_0(x, Du(x)) = 1, \tag{14}$$

where H_0 is a *geometrical Hamiltonian*, i.e. $H_0 : \mathbb{R}^{2n} \longrightarrow [0, +\infty)$ is continuous in both variables, convex and positively homogeneous of degree 1 with respect to p. Moreover, we assume that there exists a $m \times n$-matrix $\sigma(x)$, with C^∞ coefficients, such that the distribution, spanned by its lines, satisfies the Hörmander condition and

$$\sigma^t(x)\overline{B_1(0)} \subset \partial H_0(x, 0), \quad \text{for any } x \in \mathbb{R}^n \tag{15}$$

where $\sigma^t(x)$ is the transpose matrix of $\sigma(x)$ and $\partial H_0(x, 0)$ is the subgradient of the convex function $p \longmapsto H_0(x, p)$, in the point $(x, 0)$, and $\overline{B_1(0)}$ is the closed ball in $\mathbb{R}^m$, with $m \leq n$.

Under assumption (15), for any fixed point $y \in \mathbb{R}^n$ we build a generalized distance which is a viscosity solution of the vanishing Dirichlet eikonal problem in $\mathbb{R}^n \backslash \{y\}$. At this purpose we look at the differential inclusion

$$\dot{X}(t) \in \partial H_0(X(t), 0), \quad t \in (0, +\infty). \tag{16}$$

A solution of (16) is an absolutely continuous function $X : (0, +\infty) \to \mathbb{R}^n$ satisfying (16) almost everywhere.
Let $F_{x,y}$ the set of all solutions $X(\cdot)$ of (16), joining x to y in some finite time (i.e. $X(0) = x$ and $X(T) = y$ for some $0 \leq T = T(X(\cdot)) < +\infty$).
First note that (15) implies $F_{x,y} \neq \emptyset$, for any pair of points $x, y \in \mathbb{R}^n$. In fact the set of solutions of (16) includes the solutions of the control systems

$$\begin{cases} \dot{X}(t) = \sigma^t(X(t))\alpha(t), & t \in (0, +\infty) \\ X(0) = x, \end{cases} \tag{17}$$

where the control $\alpha : [0, +\infty) \to \mathbb{R}^m$ is a measurable function, with $|\alpha(t)| \leq 1$ a.e. $t > 0$.
So, by Chow's Theorem 1, there exists a solution of (17), joining x to y in a finite time, and then $F_{x,y} \neq \emptyset$. Therefore we define

$$d(x, y) := \inf_{X(\cdot) \in F_{x,y}} T(X(\cdot)). \tag{18}$$

We show now that the minimal-time function is a generalized distance.

Lemma 6. *Let H_0 a geometrical Hamiltonian satisfying* (15)*, then* (18) *is a generalized distance, inducing on $\mathbb{R}^n$ the euclidean topology.*

Proof. As previously observed, by the Hörmander condition (15), d is finite. Also property (1) is trivial. We must check (2). Let $E_{x,y} := \{T = T(X(\cdot)) \,|\, X(\cdot) \in F_{x,y}\}$, for any $T_1 \in E_{x,y}$ and $T_2 \in E_{y,z}$ we put $X_i(\cdot)$ the trajectory with respect to T_i, for $i = 1, 2$, and consider the path

$$X(t) := \begin{cases} X_1(t), & 0 \leq t \leq T_1, \\ X_2(t - T_1), & T_1 \leq t \leq T_1 + T_2. \end{cases}$$

It is trivial to check that $X(\cdot)$ satisfies the differential inclusion (16). Moreover $X(0) = x$ and $X(T_1 + T_2) = z$. So $X(\cdot) \in F_{x,z}$, i.e. $T_1 + T_2 \in E_{x,z}$. Then $d(x, z) \leq T_1 + T_2$ and taking the infimum in $E_{x,y}$ and $E_{y,z}$ respectively, we can conclude that $d(x, z) \leq d(x, y) + d(y, z)$. Note that d is non symmetric in general; indeed, for $X(\cdot) \in F_{x,y}$, the inverse path $\widetilde{X}(t) := \widetilde{X}(T - t)$ may not satisfy (16). Finally, by assumption (15), d induces on $\mathbb{R}^n$ the euclidean topology (see proof of Theorem 2.3 in [17]) and this concludes the proof. □

To prove that $d(x, y)$ is a viscosity solution of the eikonal equation (14) in $\mathbb{R}^n \backslash \{y\}$, we proceed as in [2], using a Dynamical Programming Principle.

Lemma 7 (DPP)**.** *Under the assumptions of Lemma 6, for any $x, y \in \mathbb{R}^n$*

$$d(x, y) = \inf_{X(\cdot) \in F_{x,y}} [t + d(X(t), y)], \quad \forall\, 0 \leq t \leq d(x, y). \tag{19}$$

Proof. First we prove that

$$d(x, y) \leq \inf_{X(\cdot) \in F_{x,y}} [t + d(X(t), y)]. \tag{20}$$

Let $y \in \mathbb{R}^n$ and $X(\cdot) \in F_{x,y}$, we set $z = X(t)$. Since $d(x, y) = \inf_{X(\cdot) \in F_{z,y}} T(X(\cdot))$, for any $\varepsilon > 0$ there exists $\tilde{X}(\cdot) \in F_{z,y}$ such that $d(z, y) > T(\tilde{X}(\cdot)) - \varepsilon$.
We define

$$\overline{X}(s) := \begin{cases} X(s), & 0 < s \leq t, \\ \tilde{X}(s - t), & t < s. \end{cases}$$

It is trivial to check that $\overline{X}(\cdot) \in F_{x,y}$, so, for $\varepsilon > 0$, $d(x, y) \leq T(\overline{X}(\cdot)) = t + T(\tilde{X}(\cdot)) < t + d(z, y) + \varepsilon$ and then $d(x, y) < t + d(z, y) + \varepsilon$. Passing to the limit as $\varepsilon \to 0^+$ we get $d(x, y) \leq t + d(z, y) = t + d(X(t), y)$. At this point, (20) follows by taking the infimum over $X(\cdot) \in F_{x,y}$.
To prove the reverse inequality, fix $y \in \mathbb{R}^n$ and remark that, for $X(\cdot) \in F_{x,y}$ and $0 \leq t \leq d(x, y) \leq T(X(\cdot))$, $T(X(\cdot)) \geq t + d(X(t), y)$. Taking the infimum over $X(\cdot) \in F_{x,y}$ we get the last inequality. □

Using DPP we are able to solve, in the viscosity sense, the horizontal eikonal problem

$$\begin{cases} |\sigma(x) Du(x)| = 1, & \text{in } \mathbb{R}^n \backslash \{y\}, \\ u(y) = 0, \end{cases} \tag{21}$$

for any $y \in \mathbb{R}^n$.
For analogous results using a such technique, one can see [2, 13, 15]. Moreover, there exists a result where there is proved (by a DPP) that the minimal-time function (18) is a viscosity solution, starting from a generic Lipschitz multifunction in place of $\partial H_0(x, 0)$, [6].

The next lemma shows that for the model $H_0(x,Du)=|\sigma(x)Du|$ the minimal-time distance $d(x,y)$, defined by (18), coincides with the Carnot-Carathèodory distance $d_\sigma(x,y)$ associated to the Hörmander-matrix $\sigma(x)$ ([5]). We indicate by $|\cdot|_n$ the euclidean norm in $\mathbb{R}^n$ and by $\langle\,\cdot\,,\,\cdot\,\rangle_n$ the inner product in $\mathbb{R}^n$.

Lemma 8. *Let $H_0(x,p)=|\sigma(x)p|_m$, where $\sigma(x)$ is a $m\times n$ Hörmander-matrix with rank equal to $m\le n$ and C^∞ coefficients, then*

$$\sigma^t(x)\overline{B_1(0)}=\partial H_0(x,0),$$

for any $x\in\mathbb{R}^n$. Therefore $d(x,y)=d_\sigma(x,y)$ in whole $\mathbb{R}^n\times\mathbb{R}^n$.

Proof. Let $p\in\sigma^t(x)\overline{B_1(0)}$, then there exists $\alpha\in\overline{B_1(0)}$ such that $p=\sigma^t(x)\alpha$. By the Cauchy-Schwartz inequality, for any $q\in\mathbb{R}^n$, we get

$$\langle p,q\rangle_n=\langle\sigma^t(x)\alpha,q\rangle_n=\langle\alpha,\sigma(x)q\rangle_m\le|\alpha|_m\,|\sigma(x)q|_m\le|\sigma(x)q|_m=H_0(x,q).$$

Hence $p\in\partial H(x,0)$ and so we can conclude that $\sigma^t(x)\overline{B_1(0)}\subset\partial H(x,0)$.
In order to prove the reverse inequality, we fix x and omit to write the dependence on it. Since the $\mathrm{Rank}(\sigma)=m$, we can write $\mathbb{R}^n=\mathrm{Ker}(\sigma)\oplus\mathrm{Im}(\sigma^t)$. If $v\in\partial H_0(x,0)$, then

$$\langle v,p\rangle_n\le H_0(x,0)=|\sigma\,p|_m,\quad\forall\,p\in\mathbb{R}^n.$$

Choosing $p\in\mathrm{Ker}(\sigma)$, we get $\langle v,p\rangle_n\le 0$, that implies $v\in\mathrm{Im}(\sigma^t)$. So there exists $w\in\mathbb{R}^m$ such that $v=\sigma^t w$. Hence

$$\langle\sigma^t w,p\rangle_n=\langle w,\sigma\,p\rangle_m\le|\sigma\,p|_m,\quad\forall\,p\in\mathbb{R}^n.$$

Since $\mathrm{Rank}(\sigma)=m$, there exists a $\overline{p}\in\mathbb{R}^n$ such that $\sigma\,p=w$, so we find

$$\langle w,w\rangle_m=|w|_m^2\le|w|_m,$$

that implies $w\in\overline{B_1(0)}\subset\mathbb{R}^m$.
Therefore $v\in\sigma^t(x)\overline{B_1(0)}$, so that $\partial H_0(x,0)\subset\sigma^t(x)\overline{B_1(0)}$. □

Theorem 2. *Let $\sigma(x)$ Hörmander-matrix as in Lemma 8, then the associated Carnot-Carathéodory distance $d_\sigma(x,y)$ is a viscosity solution of the eikonal problem* (21).

Proof. In order to prove the theorem, we use the expression of d_σ as the minimal-time function (18) and the corresponding DPP.
First we prove that $u(x)=d(x,y)$ is a viscosity subsolution in $\mathbb{R}^n\backslash\{y\}$. At this purpose, let $x\neq y$ and $\varphi\in C^1(\mathbb{R}^n)$ such that $u-\varphi$ has a local maximum at x, i.e. $\exists\,R>0:$

$$\varphi(x)-\varphi(z)\le d(x,y)-d(z,y),\quad\forall\,z\in B_R(x).$$

Let $\alpha\in\overline{B_1(0)}$ and $X_\alpha(\cdot)$ a solution of the control system with constant control α,

$$\begin{cases}\dot X_\alpha(t)=\sigma^t(X_\alpha(t))\alpha\\ X_\alpha(0)=x\end{cases}$$

Note that, since $\sigma\in C^\infty$, $X_\alpha(\cdot)\in C^\infty$ and so in particular is $\dot X_\alpha(0)=\sigma^t(x)\alpha$.
Remark that for enough small time t, $X_\alpha(t)\in B_R(x)$. Therefore,

$$\varphi(x)-\varphi(X_\alpha(t))\le d(x,y)-d(X_\alpha(t),y)\le t+d(X_\alpha(t),y)-d(X_\alpha(t),y)=t,$$

so that

$$\frac{\varphi(x)-\varphi(X_\alpha(t))}{t}\le 1.$$

Since $X_\alpha(\cdot)$ is smooth, we can pass to the limit, as $t \to 0^+$, getting

$$-\left\langle D\varphi(x), \dot{X}_\alpha(0)\right\rangle_n = -\left\langle D\varphi(x), \sigma^t(x)\alpha\right\rangle_n = \left\langle \sigma(x)D\varphi(x), -\alpha\right\rangle_m \leq 1, \tag{22}$$

for any $|\alpha|_m \leq 1$. Taking the infimum among all $\alpha \in \overline{B_1(0)}$ we find $|\sigma(x)D\varphi(x)|_m \leq 1$.

It remains to prove that u is also a viscosity supersolution. So fix $x \neq y$, by DPP we know that, for any $\varepsilon > 0$, there exists $\overline{X}_\varepsilon(\cdot) \in F_{x,y}$ such that

$$d(x,y) > d(\overline{X}_\varepsilon(t), y) + t - \varepsilon t. \tag{23}$$

Let $\varphi \in C^1(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x, i.e. $\exists\, R > 0:$

$$d(x,y) - d(z,y) \leq \varphi(x) - \varphi(z), \quad \forall\, z \in B_R(x).$$

$\overline{X}_\varepsilon$ is absolutely continuous, so for enough small t, $\overline{X}_\varepsilon(t) \in B_R(x)$. Therefore,

$$d(x,y) - d(\overline{X}_\varepsilon(t), y) \leq \varphi(x) - \varphi(\overline{X}_\varepsilon(t)). \tag{24}$$

Using (23) in (24), we get

$$\frac{\varphi(x) - \varphi(\overline{X}_\varepsilon(t))}{t} \geq 1 - \varepsilon. \tag{25}$$

In general $\overline{X}_\varepsilon$ is not differentiable so we cannot pass directly to the limit, as $t \to 0^+$, in order to conclude.
Nevertheless, by the absolutely continuity, we have

$$\begin{aligned}\frac{\varphi(x) - \varphi(\overline{X}_\varepsilon(t))}{t} &= -\frac{1}{t}\int_0^t \left\langle D\varphi\big(\overline{X}_\varepsilon(s)\big), \dot{\overline{X}}_\varepsilon(s)\right\rangle_n ds \\ &= -\frac{1}{t}\int_0^t \left\langle D\varphi\big(\overline{X}_\varepsilon(s)\big), \sigma^t(\overline{X}_\varepsilon(s))\overline{\alpha}(s)\right\rangle_n ds\end{aligned}$$

Now we can add and subtract $\pm\left\langle D\varphi\big(\overline{X}_\varepsilon(s)\big), \sigma^t(x)\overline{\alpha}(s)\right\rangle_n$ and $\pm\left\langle D\varphi(x), \sigma^t(x)\overline{\alpha}(s)\right\rangle_n$ inside the previous integral.
Since the coefficients of σ is smooth and $\phi \in C^1$, by the absolutely continuity of $\overline{X}_\varepsilon(s)$, it is easy to show that for $0 < t << 1$, we have

$$\begin{aligned}-\frac{1}{t}\int_0^t \left\langle D\varphi\big(\overline{X}_\varepsilon(s)\big), \sigma^t(\overline{X}_\varepsilon(s))\overline{\alpha}(s)\right\rangle_n ds &\leq -\frac{1}{t}\int_0^t \left\langle D\varphi(x), \sigma^t(x)\overline{\alpha}(s)\right\rangle_n ds + o(1) \\ = -\frac{1}{t}\int_0^t \left\langle \sigma(x)D\varphi(x), \overline{\alpha}(s)\right\rangle_m ds + o(1) &\leq |\sigma(x)D\varphi(x)|_m + o(1),\end{aligned} \tag{26}$$

since $|\overline{\alpha}(s)|_m = 1$ a.e. s. From (25) and (26), it follows that

$$1 - \varepsilon \leq |\sigma(x)D\varphi(x)|_m + o(1)$$

Passing to the limit, as $t \to 0^+$, we find $\sigma(x)D\varphi(x)|_m \geq 1 - \varepsilon$. Hence, passing to the limit, as $\varepsilon \to 0^+$, we can conclude that $|\sigma(x)D\varphi(x)|_m \geq 1$. □

Remark 5. Note that $\sigma(x)Du$ is exactly the horizontal gradient Xu, made w.r.t. the sub-Riemannian geometry induced by the lines of $\sigma(x)$. Hence, by the Pansu-Rademacher Theorem ([17]), we get that, for any fixed y,

$$|Xd_\sigma(x,y)| = 1, \quad \text{a.e. } x \in \mathbb{R}^n.$$

So we find the result proved in [18] (Theorem 3.1).

4. **Hopf-Lax solution for the Cauchy problem.** In this section we consider the Cauchy problem

$$\begin{cases} u_t + H(x, Du) = 0, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u = g, & \text{in } \mathbb{R}^n \times \{0\}. \end{cases} \tag{27}$$

with $g \in LSC(\mathbb{R}^n)$ and H of the form

$$H(x, p) = \Phi(H_0(x, p))$$

where H_0 is a geometrical Hamiltonian and Φ is a convex function.
More precisely, we assume that

- **(H1):** $H_0 : \mathbb{R}^{2n} \longrightarrow [0, +\infty)$ continuous in both variables, convex and positively homogeneous of degree 1 with respect to p,
- **(H2):** $\Phi : [0, +\infty) \longrightarrow [0, +\infty)$, differentiable, convex, not decreasing with $\Phi(0) = 0$ and $\lim_{t \to 0^+} \Phi'(t) = 0$.

The model example is

$$\Phi(H_0(x, p)) = \frac{1}{\alpha} |\sigma(x) p|^{\alpha}, \tag{28}$$

with $\alpha > 1$ and $\sigma(x)$ an Hörmander-matrix .

The Hopf-Lax solution for the above Cauchy problem has been studied in [2, 13, 15] for continuous initial data g and H_0 independent of x, in [1, 3] for H_0 independent of x and semicontinuous g and in [8, 9] in the more general setting of the present paper. Let us recall the following definition from [1, 4].

Definition 6. A function $u \in LSC\big([0, +\infty) \times \mathbb{R}^n\big)$, is a *lower semicontinuous viscosity solution* of the Hamilton-Jacobi equation

$$u_t + H(x, Du) = 0, \tag{29}$$

if, for any test function $\varphi \in C^1(\Omega)$ such that $u - \varphi$ admits a local minimum in (t_0, x_0), it holds

$$\frac{\partial}{\partial t} \varphi(t_0, x_0) + H(x_0, D\varphi(t_0, x_0)) = 0. \tag{30}$$

Theorem 3. *Assume that $d(x, y)$ is a generalized distance inducing on $\mathbb{R}^n$ the euclidean topology. Assume also that, for any fixed y, $x \to d(x, y)$ is a viscosity solution of the eikonal equation $H_0(x, Du(x)) = 1$ in $\mathbb{R}^n \backslash \{y\}$. Let $g \in LSC(\mathbb{R}^n)$ such that* (4) *holds, then the Hopf-Lax function* (5) *is a lower semicontinuous viscosity solution of the Cauchy problem* (27) *and moreover the estimate* (9) *holds.*

Proof. By Lemma 3 and Lemma 1 we know that $u(t, x)$ is lower semicontinuous in $[0, +\infty) \times \mathbb{R}^n$ and assumes the initial data g in the lower semicontinuity sense. Moreover, by Lemma 2 the estimate (2) holds. To prove the theorem, it remains only to check that u satisfies Definition 6.
We show that u is a solution because infimum of solutions of equation (29). So we must prove that, for fixed $y \in \mathbb{R}^n$, the following function

$$v^y(t, x) := g(y) + t \Phi^* \left(\frac{d(x, y)}{t} \right) \tag{31}$$

is a lower semicontinuous viscosity solution of (29).
We introduce a strictly convex approximation of Φ, set $\Phi_\delta(s) := \Phi(s) + \frac{\delta}{2} s^2$, with

$\delta > 0$. Then we prove that v_δ^y, defined replacing Φ_δ^* to Φ^* in (31) is a lower semicontinuous viscosity solution of the Hamilton-Jacobi equation

$$v_t(t,x) + \Phi_\delta\big(H_0(x, Dv(t,x))\big) = 0, \text{ in } (0,+\infty)\times\mathbb{R}^n. \tag{32}$$

It is trivial that v_δ^y is lower semicontinuous. Let $\varphi \in C^1$ such that $v_\delta^y - \varphi$ has local minimum (of 0) at (t_0, x_0), i.e. there exists $r > 0$ and $0 < \overline{t} < t_0$ such that, for any $x \in B_r(x_0)$ and $t_0 - \overline{t} < t < t_0 + \overline{t}$, it holds

$$v_\delta^y(t,x) - \varphi(t,x) \geq v_\delta^y(t_0,x_0) - \varphi(t_0,x_0) = 0. \tag{33}$$

Writing (33) in $x = x_0$, we get that $T(t) := v_\delta^y(t,x_0) - \varphi(t,x_0)$ has a local minimum at $t = t_0$. Moreover $T \in C^1$, then $\dot{T}(t_0) = 0$, i.e.

$$\varphi_t(t_0,x_0) = \big(v_\delta^y(t,x)\big)_t(t_0,x_0) = \Phi_\delta^*\left(\frac{d(x_0,y)}{t_0}\right) - t_0\big(\Phi_\delta^*\big)'\left(\frac{d(x_0,y)}{t_0}\right)\frac{d(x_0,y)}{t_0^2}. \tag{34}$$

Since Φ_δ is strictly convex, the duality formula $\Phi\big((\Phi^*)'(\tau)\big) + \Phi^*(\tau) = \tau\big(\Phi^*\big)'(\tau)$ holds. Writing it in the point $s = \frac{d(x_0,y)}{t_0}$, (34) becomes

$$\varphi_t(t_0,x_0) = -\Phi_\delta\big(\Phi_\delta^*\big)'\left(\frac{d(x_0,y)}{t_0}\right).$$

So we need only to check that

$$\big(\Phi_\delta^*\big)'\left(\frac{d(x_0,y)}{t_0}\right) = H_0(x_0, D\varphi(x_0,t_0)). \tag{35}$$

If $x = y$, (35) is trivial. In fact, by the assumptions on Φ, the left-side is 0. Moreover (adding a suitable quadratic perturbation and a constant) by (33), it is not difficult to show that $\varphi(t_0, x)$ attends a local maximum at x_0, so that $D\varphi(t_0,x_0) = 0$. Hence the left-side of (35) is 0, too.

If $x \neq y$ we use the fact that d is a viscosity solution of the associated eikonal equation (14).

So fix $t = t_0 > 0$, by (33) and adding a suitable constant to the test-function, we have that for any $x \in B_r(x_0)$

$$\Phi_\delta^*\left(\frac{d(x,y)}{t_0}\right) - \frac{1}{t_0}\varphi(t_0,x) \geq \Phi_\delta^*\left(\frac{d(x_0,y)}{t_0}\right) - \frac{1}{t_0}\varphi(t_0,x_0) = 0. \tag{36}$$

By the assumptions on Φ, in particular we have that Φ_δ is strictly convex and $\lim_{t\to 0^+}\Phi_\delta'(t) = 0$, then it is not difficult to check that Φ_δ^* is strictly increasing. Then Φ_δ^* is invertible in $[0,+\infty)$ and moreover its inverse function is non decreasing. Therefore by (36) we get $(\Phi_\delta^*)^{-1}\left(\Phi_\delta^*\left(\frac{d(x,y)}{t_0}\right)\right) \geq (\Phi_\delta^*)^{-1}\left(\frac{\varphi(t_0,x)}{t_0}\right)$, or, equivalently,

$$d(x,y) - t_0\big(\Phi_\delta^*\big)^{-1}\left(\frac{\varphi(t_0,x)}{t_0}\right) \geq 0, \tag{37}$$

where we have put Φ_δ^{-1} equal to zero for any negative numbers.

If we set $k(x) := d(x,y) - t_0\big(\Phi_\delta^*\big)^{-1}\big(\frac{\varphi(t_0,x)}{t_0}\big)$, (37) implies that k has a local minimum at x_0. Now we use $\psi(x) := t_0\big(\Phi_\delta^*\big)^{-1}\big(\frac{\varphi(t_0,x)}{t_0}\big)$ as test-function for the eikonal viscosity solution d in the point x_0 (in fact $\psi \in C^1$). So

$$H_0(x_0, D\psi(x_0)) = 1. \tag{38}$$

Since $x_0 \neq y$ then $\varphi(t_0,x) > 0$ near x_0, so that $(\Phi^*_\delta)^{-1}$ is strictly positive and

$$D\psi(x_0) = t_0 D\left[\left(\Phi^*_\delta\right)^{-1}\left(\frac{\varphi(t_0,x)}{t_0}\right)\right]\Big|_{x=x_0} = \left[\left(\Phi^*_\delta\right)'\left(\frac{d(x_0,y)}{t_0}\right)\right]^{-1} D\varphi(t_0,x_0). \tag{39}$$

Put (39) in (38) we get

$$H_0\left(x_0, \left[\left(\Phi^*_\delta\right)'\left(\frac{d(x_0,y)}{t_0}\right)\right]^{-1} D\varphi(t_0,x_0)\right) = 1.$$

Since $H_0(x,p)$ is positively homogeneous with respect to p, we get (35). So we can conclude that v^y_δ is a lower semicontinuous viscosity solution of (32).
Remark that v^y is lower semicontinuous and pointwise-limit of lower semicontinuous viscosity solutions of (32), in fact $v^y_\delta(t,x) \to v^y(t,x)$, as $\delta \to 0^+$, for any $(t,x) \in (0,+\infty)\times\mathbb{R}^n$. Set $H_\delta = \Phi_\delta \circ H_0$, it is immediate that $H_\delta \to H$, as $\delta \to 0^+$. Therefore v^y is a lower semicontinuous viscosity solution of (29).
Recall that the metric Hopf-Lax function (5) is lower semicontinuous (see Lemma 3) and moreover it is the infimum of lower semicontinuous viscosity solutions of (29). Since the lower semicontinuous viscosity solutions are stable with respect to the infimum operation, then the Hopf-Lax function is a lower semicontinuous viscosity solution of the Hamilton-Jacobi equation (29). □

Example 2. Some positive-convex functions satisfying our assumptions are $\Phi(t) = \frac{1}{\alpha}t^\alpha$ with $\alpha > 1$ and $\Phi(t) = \mathrm{e}^t - t - 1$. While the functions $\Phi(t) = t$ and $\Phi(t) = \mathrm{e}^t - 1$ don't satisfy all them, since $\lim_{t\to 0^+}\Phi'(t) = 1$ in both these cases.

Using in Theorem 3 the eikonal solution built in Sec.3, Theorem 2, we conclude with the following existence result.

Theorem 4. *Let $H(x,p) = \Phi(|\sigma(x)p|)$ with Φ satisfying assumptions $(H2)$ and $\sigma(x)$ $m\times n$ Hörmander-matrix with C^∞ coefficients. If $g \in LSC(\mathbb{R}^n)$ satisfies* (4)*, then the Carnot-Carathéodory Hopf-Lax function*

$$u(x,t) := \inf_{y\in\mathbb{R}^n}\left[g(y) + t\Phi^*\left(\frac{d_\sigma(x,y)}{t}\right)\right], \tag{40}$$

is a lower semicontinuous viscosity solution of the Hamilton-Jacobi-Cauchy problem (27) *and moreover estimate* (9) *holds.*

About the uniqueness for the v. solutions of Cauchy problems (27), let us mention that [4] contains a uniqueness result which covers our model case $\frac{1}{\alpha}|\sigma(x)Du|^\alpha$ with $\alpha = 1$ and σ bounded.
For the model case with $\alpha > 1$, comparison and uniqueness results for continuous solutions (i.e. starting from continuous initial data) have been proved recently in [10].

5. **Examples and applications.**

5.1. **Various remarks.** We show that if the initial data g is continuous then the metric Hopf-Lax function is so.

Proposition 4. *If $g \in C(\mathbb{R}^n)$ then the Hopf-Lax function* (5)*, is continuous in $[0,+\infty)\times\mathbb{R}^n$.*

Proof. By Lemma (3) we know that the Hopf funtion u is lower semicontinuous in $[0,+\infty)\times\mathbb{R}^n$. So we only need to show that u is also upper semicontinuous, i.e. we want to prove that its upperlevel sets are closed.
Fixed $\gamma\in\mathbb{R}$, and let (t_x,x_k) be a sequence in the γ-upperlevel. We must check that, if $(t_k,x_k)\to(t,x)$, as $k\to+\infty$, then $u(t,x)\geq\gamma$.
As in proof of Lemma 3 we can assume $t>0$. From definition (5) it follows that

$$u(t_k,x_k)\leq g(y)+t_k\Phi^*\left(\frac{d(x_k,y)}{t_k}\right). \tag{41}$$

for every $y\in\mathbb{R}^n$. The right-hand side of (41) is continuous, so if we pass to the upper limit we obtain

$$\limsup_{k\to+\infty}u(t_k,x_k)\leq g(y)+t\Phi^*\left(\frac{d(x,y)}{t}\right). \tag{42}$$

Taking the infimum in (42) for $y\in\mathbb{R}^n$, we conclude that

$$\gamma\leq\limsup_{k\to+\infty}u(t_k,x_k)\leq u(t,x).$$

□

Since H is convex in p, when the initial data g is continuous, the metric Hopf-Lax function is also a viscosity solution, following the usual definition of Crandall and Lions (see [4] and note that a such proof holds also in our case).

In the case when the Hamiltonian depends only on the gradient-variable, we find the well-known Hopf-Lax formula for the solution of the Cauchy problem, that is

$$u(x,y)=\inf_{y\in\mathbb{R}^n}\left[g(y)+H^*\left(\frac{|x-y|}{t}\right)\right],$$

where H^* is the Legendre-Fenchel transform of H (see [1, 2, 13]).
In fact, if we consider the Hamilton-Jacobi equation $u_t+H(|Du|)=0$, the associated eikonal equation is $|Du|=1$ whose viscosity solution is the euclidean distance $d(x,y)=|x-y|$, see for example [2]. Therefore in this case we can remark that formula (5) reduces obviously to the classical one.

5.2. **The Carnot-Carathéodory case.** Now we want look at the sub-Riemannian model $H(x,p)=\frac{1}{\alpha}|\sigma(x)p|^\alpha$, with $\alpha>1$.
Whenever g is bounded and lower semicontinuous, by Propositions 1 and 2 we can deduce that u_t and $Xu=\sigma(x)Du$ exist for almost every $t>0$ and $x\in\mathbb{R}^n$. In fact, the time derivative exists almost everywhere thanks to the classic Rademacher Theorem, while the almost everywhere existence of the horizontal gradient Xu is insured by the sub-Riemannian generalization of the Rademacher Theorem (see [17, 18, 19, 22] for more details about this point). Therefore from the locally Lipschitz properties proved in Sec. 2, we conclude that the Hopf-Lax formula (40) satisfies the equation almost everywhere, exactly as in the eikonal case.

Finally, we look at the model problem (28) with $\alpha>1$, in a particular sub-Riemannian case, the 1-dimensional Heisenberg group, and show that the Hopf-Lax function (10) coincides with the formula proved by Manfredi and Stroffolini in [16]. Set $p=(x,y,z)$ and $q=(x',y',z')$, we recall that the 1-dimensional Heisenberg group $\mathbb{H}^1$ is the sub-Riemannian geometry generated by the vector fields $X_1(p)=\frac{\partial}{\partial x}-\frac{y}{2}\frac{\partial}{\partial z}$ and $X_2(p)=\frac{\partial}{\partial y}+\frac{x}{2}\frac{\partial}{\partial z}$.

Then $X_3(p) = \frac{\partial}{\partial z} = [X_1(p), X_2(p)]$ and $\sigma(x,y)Du = (u_x - \frac{y}{2}u_z, u_y + \frac{x}{2}u_z, u_z)$ is the horizontal gradient of $\mathbb{H}^1$. So in this case the explicit expression of the Cauchy problem (27) is

$$\begin{cases} u_t + \dfrac{1}{\alpha}\left(\left(u_x - \dfrac{y}{2}u_z\right)^2 + \left(u_y - \dfrac{x}{2}u_z\right)^2\right)^{\frac{\alpha}{2}} = 0, \text{ in } \mathbb{R}^n \times (0,+\infty) \\ u = g, \quad \text{in } \mathbb{R}^n \times \{0\} \end{cases}$$

We recall also that the group operation in $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ is given by $p \cdot q = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$ and the intrinsic dilatation δ_λ defined inside $\mathbb{H}^1$ is $\delta_\lambda(x,y,z) = (\lambda x, \lambda y, \lambda^2 z)$ for $\lambda > 0$ (see [14]). Therefore

$$\frac{1}{t} d_{\mathbb{H}^1}(p,p) = d_{\mathbb{H}^1}\left(\delta_{\frac{1}{t}}(p), \delta_{\frac{1}{t}}(q)\right) = d_{\mathbb{H}^1}\left(\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t^2}\right), \left(\frac{x'}{t}, \frac{y'}{t}, \frac{z'}{t^2}\right)\right), \tag{43}$$

where by $d_{\mathbb{H}^1}$ is the Carnot-Carathéodory distance in $\mathcal{H}^1$ defined by (3).
We define the Heisenberg gauge as $|p|_{\mathbb{H}^1} = d(0,p)$ so, since $d_{\mathbb{H}^1}$ is by definition a left invariant distance, then $d_{\mathbb{H}^1}(p,q) = |p^{-1} \cdot q|_{\mathbb{H}^1}$, where the inverse element is given by $p^{-1} = (-x, -y, -z)$. Hence, (43) gives

$$\frac{1}{t} d_{\mathbb{H}^1}(p,q) = \left|\left(\frac{x'-x}{t}, \frac{y'-y}{t}, \frac{z'-z+\frac{1}{2}(x'y - xy')}{t^2}\right)\right|_{\mathbb{H}^1}.$$

Using formula (10) we can write the metric Hopf-Lax function in the Heisenberg group as

$$u(q,t) = \inf_{p \in \mathbb{R}^3}\left[g(p) + \frac{t}{\beta}\left(\left|\left(\frac{x'-x}{t}, \frac{y'-y}{t}, \frac{z'-z+\frac{1}{2}(x'y - xy')}{t^2}\right)\right|_{\mathbb{H}^1}\right)^\beta\right]$$

which is the same formula found in [16] .

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