Indeterminacy estimates and the size of nodal sets in singular spaces

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Abstract

We obtain the sharp version of the uncertainty principle recently introduced in [53], and improved by [14], relating the size of the zero set of a continuous function having zero mean and the optimal transport cost between the mass of the positive part and the negative one. The result is actually valid for the wide family of metric measure spaces verifying a synthetic lower bound on the Ricci curvature, namely the $\mathsf{MCP}(K, N)$ or $\mathsf{CD}(K, N)$ condition, thus also extending the scope beyond the smooth setting of Riemannian manifolds.

Applying the uncertainty principle to eigenfunctions of the Laplacian in possibly non-smooth spaces, we obtain new lower bounds on the size of their nodal sets in terms of the eigenvalues. Those cases where the Laplacian is possibly non-linear are also covered and applications to linear combinations of eigenfunctions of the Laplacian are derived. To the best of our knowledge, no previous results were known for non-smooth spaces.

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1 Introduction

This paper is motivated by the recent emerging interest on uncertainty estimates and their applications to the behaviour of solutions of certain elliptic equations.

To be more precise: given a continuous function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with zero mean $\int_\Omega f = 0$, with Ω compact, it is natural to interpret f^+, the positive part of f, and f^-, the negative part of f, as two distributions of mass one can compare evaluating their Wasserstein distance W_1 (even though they do not have total mass 1). Then, if it is cheap to transport f^+dx to f^-dx (meaning their Wasserstein distance is small), necessarily most of the mass of f^+ must be close to most of the mass of f^-. Continuity of f implies then that necessarily the nodal set $\{x \in \Omega : f(x) = 0\}$ has to be large. Uncertainty estimates will quantify this relation.

This question was firstly investigated by Steinerberger in dimension 2 [59] and later in any dimension by Sagiv and Steinerberger [53] proving that any continuous function $f : [0,1]^n \to \mathbb{R}$ having zero mean satisfies the following inequality

$$W_1(f^+\,dx, f^-\,dx) \cdot \mathcal{H}^{n-1}\left(\{x \in (0,1)^n : f(x) = 0\}\right) \geq C \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right)^{4-\frac{1}{n}} \|f\|_{L^1}. \tag{1.1}$$

The constant C depends only on n and $\mathcal{H}^{n-1}$ denotes the Hausdorff measure of dimension $n-1$. Subsequently, an improvement on (1.1) has been obtained in [14] by Carroll, Massaneda and Ortega-Cerdà showing the validity of (1.1) with the better exponent $2 - 1/n$ and extending the range of applicability to continuous functions defined on any smooth and compact Riemannian manifold.

It was conjectured [53, 14] however that the sharp exponent for the inequality (1.1) should be 1 in place of $4 - \frac{1}{n}$ or $2 - \frac{1}{n}$. This is indeed one of the first consequences of this note: given $f : [0,1]^n \to \mathbb{R}$ having zero mean, we prove that the following inequality is valid

$$W_1(f^+\,dx, f^-\,dx) \cdot \mathcal{H}^{n-1}\left(\{x \in (0,1)^n : f(x) = 0\}\right) \geq \frac{1}{8} \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right) \|f\|_{L^1}. \tag{1.2}$$

The inequality (1.2) is just a particular case of a much more general *sharp* (in the exponent) uncertainty principle proved in following Theorem 1.1.

The setting for Theorem 1.1 will be real valued continuous (or Sobolev) functions defined over metric measure spaces (m.m.s. for short) $(X, \mathsf{d}, \mathfrak{m})$, meaning a triple with (X, d) a complete and separable metric space and $\mathfrak{m}$ a reference non-negative Radon measure.

Geometric properties of the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ are encoded in a synthetic (meaning not requiring any smoothness assumption on X) lower bound on their Ricci curvature that is called Curvature-Dimension condition and denoted by $\mathsf{CD}(K, N)$. Here K is mimiking the lower bound on the Ricci curvature and N the upper bound on the dimension. All the other terminology and notations will be introduced in Section 2.

Theorem 1.1 (Sharp indeterminacy estimate)**.** *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space verifying $\mathsf{CD}(K, N)$. Let $f \in L^1(X, \mathfrak{m})$ be a continuous function, or alternatively $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, such that $\int_X f\,\mathfrak{m} = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(dx) < \infty$.*

Then the following indeterminacy estimate is valid:

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}\left(\{x \in X : f(x) > 0\}\right) \geq \frac{\|f\|_{L^1(X,\mathfrak{m})}}{\|f\|_{L^\infty(X,\mathfrak{m})}} \frac{\|f\|_{L^1(X,\mathfrak{m})}}{8C_{K,D}}, \tag{1.3}$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,D} := \begin{cases} 1 & K \geq 0, \\ e^{-KD^2/2} & K < 0. \end{cases}$$

The essentially non-branching assumption in Theorem 1.1 is to prevent branch-like behaviour of geodesics and it is trivially satisfied by Riemannian manifolds and verified by the more regular class of $\mathsf{RCD}(K,N)$ spaces (again see Section 2 for the definitions). The notation $\mathsf{Per}(A)$ is used to denote the Perimeter of the set A (see Section 2 for its definition in this abstract setting). In the smooth setting, i.e. an n-dimensional Riemannian manifold endowed with the volume measure, it coincides thanks to De Giorgi's Theorem with the $\mathcal{H}^{n-1}$-measure of the reduced boundary of A (same result has been recently extended to the setting of non-collapsed $\mathsf{RCD}(K,N)$ spaces, see [2] and [13]).

Remark 1.2. For completeness we stress that the inequality (1.2) follows by applying Theorem 1.1 to the m.m.s. $([0,1]^n, \mathcal{L}^n\llcorner_{[0,1]^n}, |\cdot|)$, which is the easiest example of a m.m.s. satisfying $\mathsf{CD}(0,n)$, and using De Giorgi's representation of the Perimeter measure as the Hausdorff measure of dimension $n-1$ restricted to the reduced boundary.

The same result holds true if the unitary cube $[0,1]^n$ is replaced by any other closed and convex set $\Omega \subset \mathbb{R}^n$ with $0 < |\Omega| < \infty$. This will make again the m.m.s. $(\Omega, \mathcal{L}^n\llcorner_\Omega, |\cdot|)$ to satisfy $\mathsf{CD}(0,n)$.

We will now list few detailed comments on Theorem 1.1.
Setting and Sharpness:

- We improve on previous results by including possibly non-smooth spaces, i.e. those spaces verifying the synthetic lower bound on the Ricci curvature, $\mathsf{CD}(K,N)$ condition, see Section 2.1 for the precise definition and for a list of class of spaces falling within this theory. Here we mention that given a complete Riemannian manifold (M,g) one can naturally consider the m.m.s. $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ where where d_g is the geodesic distance and Vol_g the volume measure both induced by the metric g. Then $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ verifies $\mathsf{CD}(K,N)$ if and only if $\mathrm{Ric}_g \geq Kg$ and $\dim(M) \leq N$. In particular *any compact smooth weighted* (meaning with $\mathfrak{m} = e^{-V}\,\mathrm{Vol}_g$ with V smooth) *Riemannian manifold* is included in our setting. Thanks to the well-known stability property of the $\mathsf{CD}(K,N)$ conditions in the measured-Gromov-Hausdorff sense, Theorem 1.1 applies also to any possible limit space of sequences of manifolds having Ric bounded from below uniformly and dimension bounded from above uniformly.

- The estimate does not require the space X to be compact nor the reference measure $\mathfrak{m}$ to be finite, at least when $K \geq 0$. If $K < 0$, then to have a meaningful estimate necessarily the diameter of X has to be finite.

- Inequality (1.3) does not depend on the dimension. In particular the same statement is valid for m.m.s. satisfying $\mathsf{CD}(K,\infty)$ (i.e. no synthetic upper bound on the dimension) for which a localization paradigm is at disposal. In particular if $(X,\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{CD}(K,\infty)$ and the weaker $\mathsf{MCP}(K',N')$ with some other K',N', then (1.3) is still valid. Referring to Theorem 4.1 for the precise statement, here we underline that for any continuous function $f : \mathbb{R}^n \to \mathbb{R}$ having zero mean and satisfying the growth assumptions with respect to $e^{-V}dx$ for some smooth convex function V, then (1.3) holds true.

- As pointed out in [14] by Carroll, Massaneda and Ortega-Cerdà, their version of (1.1) cannot be improved by lowering the exponent $2 - 1/n$ below 1; exponent 1 was known to be reached only in dimension 2. Hence the exponent 1 in (1.3) is *sharp*.

- Theorem 1.1 will be also valid for spaces verifying another synthetic curvature notion called measure-contraction property $\mathsf{MCP}(K,N)$ (see Section 2.1). A long list of subRiemanninan spaces, including the Heinseberg group, verifies this latter condition while failing $\mathsf{CD}(K,N)$. In this framework the constant in the inequality (1.3) will depend on the dimension. See Theorem 4.3 for the precise statement.

Our approach to obtain to Theorem 1.1 will be via a dimensional-reduction argument. In particular, the L^1-optimal transport problem between the positive and the negative part of f gives, as a byproduct, a foliation of the ambient space X into a family of geodesics obtained by considering the integral curves of the gradient of a Kantorovich potential u, i.e. a solution of the dual problem. This non-smooth foliation has few pleasant properties that are summarised in Theorem 2.4 (see Section 2 for all preliminaries). Here we mention that the integral of the function f along almost every geodesics of the foliation is still zero, where the integral is with respect to the corresponding marginal measure. Also the curvature properties of the space are inherited by the one-dimensional "weighted" geodesic in a suitable sense. These two properties permit to reduce the proof of Theorem 1.1 to a one-dimensional analysis (Section 3) and, in turn, to obtain the sharpness.

1.1 Applications: Nodal set of eigenfunctions in singular spaces

Our main application of Theorem 1.1 will be a lower bound on the size of nodal sets for eigenfunctions of the Laplacian (and linear combination of them) in possibly singular spaces verifying synthetic Ricci curvature bounds.

The whole list of topics related to the geometry of Laplace eigenfunctions (for instance the Courant nodal domain theorem or the quasi-symmetry conjecture) goes beyond the scope of this short introduction. However, to put the problem into perspective, we will now recall the long series of contributions to Yau's conjecture and the solution to it.

Yau conjectured in [65] that for any n-dimensional C^∞-smooth closed Riemannian manifold M, hence without boundary and compact, any Laplace eigenfunction

$$-\Delta f_\lambda = \lambda f_\lambda$$

satisfies

$$c\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\{f_\lambda = 0\}) \leq C\sqrt{\lambda},$$

with c, C depend solely on M and not on λ.

First Brüning [12] proved the validity of the lower bound for $n = 2$. Then Donnelly and Fefferman in 1988 [29] established Yau's conjecture in the case of real analytic metrics (for instance spherical harmonics). In the case of smooth manifold Nadirashvili in 1988 [46] proved for $n = 2$ that $\mathcal{H}^1(\{f_\lambda = 0\}) \leq C\lambda \log \lambda$ and later improved [30, 27] to $\mathcal{H}^1(\{f_\lambda = 0\}) \leq C\lambda^{3/4}$. For general $n > 2$, Hardt and Simon [35] obtained the non-polynomial bound $\mathcal{H}^{n-1}(\{f_\lambda = 0\}) \leq C\lambda^{C\sqrt{\lambda}}$.

Few years later the lower bound has been improved to

$$\mathcal{H}^{n-1}(\{f_\lambda = 0\}) \geq c\lambda^{\frac{3-n}{4}}, \tag{1.4}$$

in independent contributions by Colding and Minicozzi [24], Sogge and Zelditch [54, 55] and by Steinerberger [57]. Finally, a breakthrough has been obtained by Logunov in 2018, proving, in the smooth case and for any $n \in \mathbb{N}$, a polynomial upper bound [40] and the lower bound [41] in Yau's conjecture. For an overview on all these result we refer to [42].

To the best of our knowledge there are no results on the size of nodal sets of eigenfunctions of the Laplacian whenever a singularity on the ambient manifold is allowed. Following Steinerberger [59], upper bounds on the W_1-distance between the positive and the negative parts of a Laplace eigenfunctions will yield lower bounds on the size of their nodal sets. This indeed is the content of the following results. The first one will be for spaces verifying $\mathsf{CD}(K,N)$; at this level of generality the Laplacian may not be even a linear operator (see Section 2.4).

Theorem 1.3 (Nodal sets on CD-spaces)**.** *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{CD}(K,N)$ and such that $\mathfrak{m}(X) < \infty$. Let f_λ be an eigenfunction of the Laplacian of eigenvalue $\lambda > 0$ (see Definition 2.17) and assume moreover the existence of $x_0 \in X$ such that $\int_X |f_\lambda(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(dx) < \infty$.*

Then the following estimate on the size of the its nodal set holds true:

$$\mathsf{Per}(\{x \in X : f_\lambda(x) > 0\}) \geq \frac{\sqrt{\lambda}}{8C_{K,D}\sqrt{\mathfrak{m}(X)}} \cdot \frac{\|f_\lambda\|^2_{L^1(X,\mathfrak{m})}}{\|f_\lambda\|_{L^2(X,\mathfrak{m})}\|f_\lambda\|_{L^\infty(X,\mathfrak{m})}},$$

where $D = \mathrm{diam}\,(X)$ and $C_{K,D}$ is the same of Theorem 1.1.

As before, Theorem 1.3 is actually valid also in other frameworks: for spaces verifying $\mathsf{CD}(K,\infty)$ and $\mathsf{MCP}(K',N')$ (Theorem 5.3) or for spaces verifying $\mathsf{MCP}(K,N)$ (Theorem 5.4) with dimension dependent constant.

If in addition we assume the Laplacian to be linear (more precisely the Sobolev space $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ to be an Hilbert space), then more techniques from the classical setting come into play (for instance contraction estimates for the heat flow) permitting to obtain more refined results.

Theorem 1.4 (Nodal sets on RCD spaces I)**.** *Let $K,N\in\mathbb{R}$ with $N>1$. Let $(X,\mathsf{d},\mathfrak{m})$ be a m.m.s. satisfying $\mathsf{RCD}(K,N)$, and such that $\operatorname{diam}(X)=D<\infty$. Let f_λ be an eigenfunction of the Laplacian of eigenvalue $\lambda>2$. Then the following estimate is valid:*

$$\mathsf{Per}\left(\{x\in X:\, f_\lambda(x)>0\}\right)\geq \frac{1}{\bar{C}_{K,D,N}}\sqrt{\frac{\lambda}{\log\lambda}}\cdot\frac{\|f_\lambda\|_{L^1(X,\mathfrak{m})}}{\|f_\lambda\|_{L^\infty(X,\mathfrak{m})}}, \tag{1.5}$$

where $\bar{C}_{K,D,N}$ grows linearly in D if $K\geq 0$ and exponentially if $K<0$ and grows with power $1/2$ in N.

The estimate (1.5) follows directly from the following estimate

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m})\leq C(K,N,D)\sqrt{\frac{\log\lambda}{\lambda}}\|f_\lambda\|_{L^1}, \tag{1.6}$$

(see Proposition 5.6), already obtained in the smooth setting by Steinerberger [58], and recently improved in [14] removing the $\log\lambda$ term but with an approach that seems confined to the smooth setting.

Finally, we notice that using already available L^∞ estimates for Laplace eigenfunctions one can obtain an explicit lower bound on the size of the nodal set of an eigenfunction.

Theorem 1.5 (Nodal sets on RCD spaces II)**.** *Let $K,N\in\mathbb{R}$ with $N>1$. Let $(X,\mathsf{d},\mathfrak{m})$ be a m.m.s. verifying $\mathsf{RCD}(K,N)$, and with $\operatorname{diam}(X)=D<\infty$; finally pose $\mathfrak{m}(X)=1$. Let f_λ be an eigenfunction of the Laplacian of eigenvalue $\lambda>\max\{2,D^{-1}\}$. Then the following estimate is valid:*

$$\mathsf{Per}\left(\{x\in X:\, f_\lambda(x)>0\}\right)\geq \frac{1}{\bar{C}_{K,D,N}}\frac{1}{\sqrt{\log\lambda}}\lambda^{\frac{1-N}{2}}, \tag{1.7}$$

where $\bar{C}_{K,D,N}$ grows linearly in D if $K\geq 0$ and exponentially if $K<0$ and grows with power $1/2$ in N.

Lastly our techniques permits to address also another related problem. Parallel indeed to the study of the zero set of Laplace eigenfunctions, there is the investigation of the zero set of linear combinations of them. That was born with the theory of Sturm [61], [62], [9] and Hurwitz in 1-dimension,[36] and then developed by Donnelly [28], Lebeau and Robbiano [39], and more recently by Decio [26], and Steinerberger. See also [37].

Donnelly in particular proved that the size of the zero set of linear combinations of eigenfunctions is controlled from above by constants depending on the highest frequency appearing in the linear combination. Analogously, a lower bound for the measure of the zero set of a function of the type $\sum_{\lambda\geq\bar{\lambda}} a_\lambda f_\lambda$, in terms of $\bar{\lambda}$ has been proved by Sagiv and Steinerberger [53] and by Steinerberger in [58], [59], giving a multi-dimensional analogue of Sturm-Hurwitz' Theorem.

Our approach to the problem permits to extend the validity of Theorem 1.3 and of Theorem 1.4 also to linear combinations of eigenfunctions. This furnishes therefore a non-smooth analogue of the Sturm-Hurwitz' Theorem mentioned above. Due to the nature of the problem, we necessarily have to assume the Laplacian to be a linear operator making the natural setting for these results the class of RCD spaces. To avoid to overload the introduction, we refer for these results to Section 6.

1.2 Outlook

Consider a compact, smooth, N-dimensional Riemannian manifold M endowed with the geodesic distance d_g and the volume measure Vol_g. For this class of spaces the improved version of (1.6) obtained in [14] holds true. Hence our Theorem 1.1 gives the following inequality

$$\mathcal{H}^{N-1}\left(\{x \in X\colon f_\lambda(x) = 0\}\right) \geq \frac{1}{\bar{C}_{K,D,N}}\sqrt{\lambda}\cdot\frac{\|f_\lambda\|_{L^1(M)}}{\|f_\lambda\|_{L^\infty(M)}},$$

One can then use the inequality $\|f_\lambda\|_\infty \leq \lambda^{\frac{N-1}{4}}\|f_\lambda\|_1$ by Sogge and Zelditch [55] (which is known to be sharp on spherical harmonics) and obtain

$$\mathcal{H}^{N-1}\left(\{x \in X\colon f_\lambda(x) = 0\}\right) \geq \lambda^{\frac{3-N}{4}},$$

reproving the estimate (1.4) by Colding and Minicozzi [24], Sogge and Zelditch [54, 55] and by Steinerberger [57]. It is therefore plausible to expect (1.4) (or its counterpart with the Perimeter) to holds true also for compact RCD-spaces, provided the the validity of the following two inequalities is established

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \leq C(K,N,D)\frac{1}{\sqrt{\lambda}}\|f_\lambda\|_{L^1}, \qquad \|f_\lambda\|_{L^\infty} \leq \lambda^{\frac{N-1}{4}}\|f_\lambda\|_{L^1}.$$

that are left for a future investigation. Similar investigation will be also carried out for the quasi-symmetry property of eigenfunction in the non-smooth setting.

2 Preliminaries

In what follows, $(X, \mathsf{d}, \mathfrak{m})$ will be a complete and separable metric measure space that is (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative Radon measure on X. Also, throughout the note, the various curvature conditions we will assume will imply X to be proper (bounded and closed sets are compact). In various situation this will simplify the presentation (see Section 2.4).

2.1 Synthetic Curvature conditions

We briefly recall the main definitions of curvature bounds for metric measure spaces that we will use throughout the paper referring for more details to the original papers [43, 63, 64].

In the following $\mathcal{P}(X)$ is the space of Borel probability measures on X and, for $p \geq 1$, $\mathcal{P}_p(X)$ is the space of Borel probability measures with finite p-moment.

The *p-Wasserstein distance* W_p on $\mathcal{P}_p(X)$ is defined for any $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ as follows:

$$W_p(\mu_0, \mu_1)^p := \inf_{\pi \in \Pi(\mu_0,\mu_1)} \int_{X\times X} \mathsf{d}^p(x,y)\,\pi(dxdy), \tag{2.1}$$

where

$$\Pi(\mu_0, \mu_1) := \left\{\pi \in \mathcal{P}(X \times X)\colon P^{(1)}_\sharp \pi = \mu_0, P^{(2)}_\sharp \pi = \mu_1\right\}$$

is the set of admissible transport plans between μ_0 and μ_1 and $P^{(i)}$ is the projection on the i-th component, for $i = 1, 2$. We will only considering in this note W_1 and W_2. $\mathrm{Geo}(X)$ denotes the space of constant speed geodesics on X:

$$\mathrm{Geo}(X) := \{\gamma \in C([0,1], X)\colon \mathsf{d}(\gamma(s), \gamma(t)) = |s-t|\mathsf{d}(\gamma(0), \gamma(1)) \text{ for any } s,t \in [0,1]\}.$$

For any $t \in [0,1]$, the evaluation map e_t is defined on $\mathrm{Geo}(X)$ by $\mathrm{e}_t(\gamma) := \gamma(t)$. For any pair of measures μ_0, μ_1 in $\mathcal{P}_2(X)$, the set of dynamical optimal plans is defined by

$$\mathrm{OptGeo}(\mu_0, \mu_1) := \{\nu \in \mathcal{P}(\mathrm{Geo}(X))\colon (\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu \quad \text{realizes the minimum in (2.1)}\}.$$

Definition 2.1 (Essentially non-branching)**.** A subset $G \subset \mathrm{Geo}(X, \mathsf{d})$ of geodesics is called non-branching if for any $\gamma^1, \gamma^2 \in G$ the following holds:

$$\exists\, t \in (0,1) \colon \gamma^1_s = \gamma^2_s \quad \forall\, s \in [0,t] \Longrightarrow \gamma^1_2 = \gamma^2_s \quad \forall\, s \in [0,1].$$

(X, d) is called non-branching if $\mathrm{Geo}(X, \mathsf{d})$ is non-branching. $(X, \mathsf{d}, \mathfrak{m})$ is called essentially non-branching if for any $\mu_0, \mu_1 \ll \mathfrak{m}$ with $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ any $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a Borel non-branching subset $G \subset \mathrm{Geo}(X, \mathsf{d})$.

The above definition was introduced in [51] by Rajala and Sturm. The restriction to essentially non-branching spaces is natural and facilitates avoiding pathological cases. One example is the failure of the local-to-global property for a general $\mathsf{CD}(K,N)$ in [50], property that has been recently verified in [20] under the assumption of essentially non-branching (and finite $\mathfrak{m}$).

Given $K \in \mathbb{R}$ and $\mathcal{N} \in (0, \infty]$, define:

$$D_{K,\mathcal{N}} := \begin{cases} \frac{\pi}{\sqrt{K/\mathcal{N}}} & K > 0\ ,\ \mathcal{N} < \infty \\ +\infty & \text{otherwise} \end{cases}. \tag{2.2}$$

In addition, given $t \in [0,1]$ and $0 < \theta < D_{K,\mathcal{N}}$, define:

$$\sigma^{(t)}_{K,\mathcal{N}}(\theta) := \frac{\sin(t\theta\sqrt{\frac{K}{\mathcal{N}}})}{\sin(\theta\sqrt{\frac{K}{\mathcal{N}}})} = \begin{cases} \frac{\sin(t\theta\sqrt{\frac{K}{\mathcal{N}}})}{\sin(\theta\sqrt{\frac{K}{\mathcal{N}}})} & K > 0\ ,\ \mathcal{N} < \infty \\ t & K = 0 \text{ or } \mathcal{N} = \infty\ , \\ \frac{\sinh(t\theta\sqrt{\frac{-K}{\mathcal{N}}})}{\sinh(\theta\sqrt{\frac{-K}{\mathcal{N}}})} & K < 0\ ,\ \mathcal{N} < \infty \end{cases}$$

and set $\sigma^{(t)}_{K,\mathcal{N}}(0) = t$ and $\sigma^{(t)}_{K,\mathcal{N}}(\theta) = +\infty$ for $\theta \geq D_{K,\mathcal{N}}$. Given $K \in \mathbb{R}$ and $N \in (1, \infty]$, the *distortion coefficients* are defined as:

$$\tau^{(t)}_{K,N}(\theta) := t^{\frac{1}{N}} \sigma^{(t)}_{K,N-1}(\theta)^{1-\frac{1}{N}}.$$

When $N = 1$, set $\tau^{(t)}_{K,1}(\theta) = t$ if $K \leq 0$ and $\tau^{(t)}_{K,1}(\theta) = +\infty$ if $K > 0$.

The *Rényi entropy* functional $\mathcal{E} : \mathcal{P}(X) \to [0, \infty]$ is defined as

$$\mathcal{E}(\mu) := \int_X \rho^{1-\frac{1}{N}}\, \mathfrak{m}, \quad \text{where } \mu = \rho\mathfrak{m} + \mu^s \text{ and } \mu^s \perp \mathfrak{m},$$

and the *Boltzman entropy* $\mathrm{Ent} : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\mathrm{Ent}(\mu) := \int_X \rho \log(\rho)\, \mathfrak{m}, \quad \text{if } \mu = \rho\mathfrak{m}, \text{ and } \mathrm{Ent}(\mu) := \infty \text{ otherwise}$$

Definition 2.2 (CD conditions)**.** $(X, \mathsf{d}, \mathfrak{m})$ verifies the $\mathsf{CD}(K,N)$ (resp. $\mathsf{CD}(K,\infty)$) condition for some $K \in \mathbb{R}$, $N \in (1, \infty)$ if for any pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $\mu_0, \mu_1 \ll \mathfrak{m}$ (and $\mathrm{Ent}(\mu_i) < \infty$, $i = 0,1$), there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ and an optimal plan $\pi \in \Pi(\mu_0, \mu_1)$ such that $\mu_t := (\mathrm{e}_t)_\sharp \nu \ll \mathfrak{m}$ and

$$\mathcal{E}_{N'}(\mu_t) \geq \int \left\{ \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y)) \rho_0^{-\frac{1}{N'}} + \tau^{(t)}_{K,N'}(\mathsf{d}(x,y)) \rho_1^{-\frac{1}{N'}} \right\} \pi(dxdy)$$

for any $N' \geq N$, $t \in [0,1]$ (resp .

$$\mathrm{Ent}(\mu_t) \leq (1-t)\mathrm{Ent}(\mu_0) + t\mathrm{Ent}(\mu_1) - \frac{K}{2} t(1-t) W_2(\mu_0, \mu_1)^2).$$

For our purposes we also need to introduce a weaker variant of CD called Measure-Contraction property, $\mathsf{MCP}(K,N)$ in short, introduced separately by Ohta [47] and Sturm [64] with two definitions that slightly differ in general metric spaces, but that coincide on essentially non-branching spaces.

Definition 2.3 ($\mathsf{MCP}(K,N)$)**.** A m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is said to satisfy $\mathsf{MCP}(K,N)$ if for any $o \in \mathrm{supp}(\mathfrak{m})$ and $\mu_0 \in \mathcal{P}_2(X,\mathsf{d},\mathfrak{m})$ of the form $\mu_0 = \frac{1}{\mathfrak{m}(A)}\mathfrak{m}\llcorner_A$ for some Borel set $A \subset X$ with $0 < \mathfrak{m}(A) < \infty$ (and with $A \subset B(o, \pi\sqrt{(N-1)/K})$ if $K > 0$), there exists $\nu \in \mathrm{OptGeo}(\mu_0, \delta_o)$ such that:

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \geq (\mathrm{e}_t)_\sharp\big(\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))^N \nu(d\gamma)\big) \quad \forall t \in [0,1]. \tag{2.3}$$

If $(X,\mathsf{d},\mathfrak{m})$ is a m.m.s. verifying $\mathsf{MCP}(K,N)$, then $(\mathrm{supp}(\mathfrak{m}),\mathsf{d})$ is Polish, proper and it is a geodesic space. With no loss in generality for our purposes we will assume that $X = \mathrm{supp}(\mathfrak{m})$.

To conclude this part we include a list of notable examples of spaces fitting in the assumptions of our results. The class of essentially non branching $\mathsf{CD}(K,N)$ spaces includes many remarkable family of spaces, among them:

- *Measured Gromov Hausdorff limits of Riemannian N-dimensional manifolds satisfying* $\mathrm{Ric}_g \geq Kg$ *and more generally the class of* $\mathsf{RCD}(K,N)$ *spaces.* Indeed measured Gromov Hausdorff limits of Riemannian N-manifolds satisfying $\mathrm{Ric}_g \geq Kg$ are examples of $\mathsf{RCD}(K,N)$ spaces (see for instance [31] and for the definition of RCD see Section 2.4) and, in particular, are essentially non-branching and $\mathsf{CD}(K,N)$ (see [51]).

- *Alexandrov spaces with curvature $\geq K$.* Petrunin [49] proved that the lower curvature bound in the sense of comparison triangles is compatible with the optimal transport type lower bound on the Ricci curvature given by Lott-Sturm-Villani. Moreover geodesics in Alexandrov spaces with curvature bounded below do not branch. It follows that Alexandrov spaces with curvature bounded from below by K are non-branching $\mathsf{CD}(K(N-1),N)$ spaces.

- *Finsler manifolds where the norm on the tangent spaces is strongly convex, and which satisfy lower Ricci curvature bounds.* More precisely we consider a C^∞-manifold M, endowed with a function $F : TM \to [0,\infty]$ such that $F|_{TM\setminus\{0\}}$ is C^∞ and for each $p \in M$ it holds that $F_p := T_pM \to [0,\infty]$ is a strongly-convex norm, i.e.

  $$g_{ij}^p(v) := \frac{\partial^2(F_p^2)}{\partial v^i \partial v^j}(v) \quad \text{is a positive definite matrix at every } v \in T_pM \setminus \{0\}.$$

  Under these conditions, it is known that one can write the geodesic equations and geodesics do not branch: in other words these spaces are non-branching. We also assume (M,F) to be geodesically complete and endowed with a C^∞ probability measure $\mathfrak{m}$ in a such a way that the associated m.m.s. $(X,F,\mathfrak{m})$ satisfies the $\mathsf{CD}(K,N)$ condition. This class of spaces has been investigated by Ohta [48] who established the equivalence between the Curvature Dimension condition and a Finsler-version of Bakry-Emery N-Ricci tensor bounded from below.

While $\mathsf{CD}(K,N)$ implies the weaker $\mathsf{MCP}(K,N)$, the latter is capable to capture the behaviour of more general family of spaces. In particular, for a complete list of subRiemannian spaces verifying the $\mathsf{MCP}(K,N)$ (and not $\mathsf{CD}(K,N)$), we refer to the recent [44].

2.2 Localization and one-dimensional densities

One of the key tools of our approach to obtain a sharp indeterminacy estimate is the dimensional reduction argument furnished by localization theorem. In its various forms, the following theorem goes back to [10] for the MCP case (with a slightly different presentation), while to [18] for the $\mathsf{CD}(K,N)$ case with $\mathfrak{m}(X) < \infty$ and to [22] for a general Radon measure. We refer to the aforementioned references for all the missing details.

Theorem 2.4. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space with* $\mathrm{supp}(\mathfrak{m}) = X$*. Let $f : X \to \mathbb{R}$ be $\mathfrak{m}$-integrable such that $\int_X f\,\mathfrak{m} = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x,x_0)\,\mathfrak{m}(dx) < \infty$.*

Assume also $(X, \mathsf{d}, \mathfrak{m})$ *verifies* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*) condition for some* $K \in \mathbb{R}$ *and* $N \in [1,\infty)$.

Then the space X *can be written as the disjoint union of two sets* Z *and* $\mathcal{T}$ *with* $\mathcal{T}$ *admitting a partition* $\{X_\alpha\}_{\alpha\in Q}$ *and a corresponding disintegration of* $\mathfrak{m}\llcorner_{\mathcal{T}}$ *such that:*

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ *is a Borel probability measure over* $Q \subset X$ *such that* $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}}) \ll \mathfrak{q}$*, with* $\mathfrak{Q}$ *the quotient map associated to the partition and the map* $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ *satisfying the following properties:*

- *for any* $\mathfrak{m}$*-measurable set* B*, the map* $\alpha \mapsto \mathfrak{m}_\alpha(B)$ *is* $\mathfrak{q}$*-measurable;*
- *for* $\mathfrak{q}$*-a.e.* $\alpha \in Q$*,* $\mathfrak{m}_\alpha$ *is concentrated on* $\mathfrak{Q}^{-1}(\alpha) = X_\alpha$ *(strong consistency);*
- *For* $\mathfrak{q}$*-almost every* $q \in Q$*, it holds* $\int_{X_q} f\, \mathfrak{m}_q = 0$ *and* $f = 0$ $\mathfrak{m}$*-a.e. in* Z.
- *For* $\mathfrak{q}$*-almost every* $q \in Q$*, the set* X_q *is a geodesic (even more a transport ray) and the one dimensional m.m.s.* $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ *verifies* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*).*

Moreover, fixed any $\mathfrak{q}$ *as above such that* $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}}) \ll \mathfrak{q}$*, the disintegration is* $\mathfrak{q}$*-essentially unique.*

Remark 2.5. Via the ray map g associated to the transport set of $f^+\mathfrak{m}$ into $f^-\mathfrak{m}$ (see for instance [10]), we have that for $\mathfrak{q}$-a.e. $q \in Q$

$$\mathfrak{m}_q = g(q,\cdot)_\sharp \left(h_q \cdot \mathcal{L}^1\right),$$

for some function $h_q : \mathrm{Dom}\,(g(q,\cdot)) \subset \mathbb{R} \to [0,\infty)$ where $\mathrm{Dom}\,(g(q,\cdot))$ is an interval $I_q \subset \mathbb{R}$ and $(I_q, |\cdot|, h_q \cdot \mathcal{L}^1)$ is isomorphic to $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$; in particular it verifies $\mathsf{CD}(K,N)$ (resp. $\mathsf{MCP}(K,N)$).

We will therefore spend few lines on one-dimensional m.m.s. verifying curvature bounds.

Definition 2.6 ($\mathsf{CD}(K,N)$ density)**.** Given $K, N \in \mathbb{R}$ and $N \in (1,\infty)$, a non-negative function h defined on an interval $I \subset \mathbb{R}$ is called a $\mathsf{CD}(K,N)$ density on I, if for all $x_0, x_1 \in I$ and $t \in [0,1]$:

$$h(tx_1 + (1-t)x_0)^{\frac{1}{N-1}} \geq \sigma^{(t)}_{K,N-1}(|x_1 - x_0|)h(x_1)^{\frac{1}{N-1}} + \sigma^{(1-t)}_{K,N-1}(|x_1 - x_0|)h(x_0)^{\frac{1}{N-1}},$$

(recalling the coefficients σ from Section 2.1). The case $N = \infty$ request instead

$$\log h(tx_1 + (1-t)x_0) \geq t \log h(x_1) + (1-t)\log h(x_0) + \frac{K}{2} t(1-t)(x_1 - x_0)^2,$$

obtained from the previous one subtracting 1 from both sides, multiplying by $N-1$, and taking the limit as $N \to \infty$. For completeness, we will say that h is a $\mathsf{CD}(K,1)$ density on I iff $K \leq 0$ and h is constant on the interior of I.

Definition 2.7 ($\mathsf{MCP}(K,N)$ density)**.** Given $K, N \in \mathbb{R}$ and $N \in (1,\infty)$, a non-negative function h defined on an interval $I \subset \mathbb{R}$ is called a $\mathsf{MCP}(K,N)$ density on I if for all $x_0, x_1 \in I$ and $t \in [0,1]$:

$$h(tx_1 + (1-t)x_0) \geq \sigma^{(1-t)}_{K,N-1}(|x_1 - x_0|)^{N-1} h(x_0). \tag{2.4}$$

The link between one dimensional m.m.s. with curvature bounds and densities is contained in the next straightforward result.

Theorem 2.8. *If* h *is a* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*) density on an interval* $I \subset \mathbb{R}$ *then the m.m.s.* $(I, |\cdot|, h(t)dt)$ *verifies* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*).*

Conversely, if the m.m.s. $(\mathbb{R}, |\cdot|, \mu)$ *verifies* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*) and* $I = \mathrm{supp}(\mu)$ *is not a point, then* $\mu \ll \mathcal{L}^1$ *and there exists a version of the density* $h = d\mu/d\mathcal{L}^1$ *which is a* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*) density on* I.

The estimate (2.4) implies several known properties that we collect in what follows. To write them in a unified way we define for $\kappa \in \mathbb{R}$ the function $s_\kappa : [0,+\infty) \to \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$)

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta) & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta) & \text{if } \kappa < 0. \end{cases} \tag{2.5}$$

For the moment we confine ourselves to the case $I = (a, b)$ with $a, b \in \mathbb{R}$; hence (2.4) implies

$$\left(\frac{s_{K/(N-1)}(b - x_1)}{s_{K/(N-1)}(b - x_0)} \right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1 - a)}{s_{K/(N-1)}(x_0 - a)} \right)^{N-1}, \tag{2.6}$$

for $x_0 \leq x_1$. Hence denoting with $D = b - a$ the length of I, for any $\varepsilon > 0$ it follows that

$$\sup \left\{ \frac{h(x_1)}{h(x_0)} : x_0, x_1 \in [a + \varepsilon, b - \varepsilon] \right\} \leq C_\varepsilon, \tag{2.7}$$

where C_ε only depends on K, N, provided $2\varepsilon \leq D \leq \frac{1}{\varepsilon}$. In particular, $\mathsf{MCP}(K, N)$ densities will be locally Lipschitz in the interior of their domain and continuous on its closure (see [22] for details).

To conclude we present here a folklore result about localization paradigm in the setting of $\mathsf{CD}(K, \infty)$ spaces. So far Theorem 2.4 is not known for a general $\mathsf{CD}(K, \infty)$ spaces, the missing ingredient being good behaviour of W_2-geodesics. Additionally assuming the space to satisfy $\mathsf{MCP}(K', N')$ for some $K', N' \in \mathbb{R}$ (with possibly K' different from K) excludes all the technical issues and the proof of the following localization result just follows as the one of Theorem 2.4.

Theorem 2.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space with* $\mathrm{supp}(\mathfrak{m}) = X$*. Let $f : X \to \mathbb{R}$ be $\mathfrak{m}$-integrable such that $\int_X f\, \mathfrak{m} = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)|\, \mathsf{d}(x, x_0)\, \mathfrak{m}(dx) < \infty$.*

Assume also $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}(K, \infty)$ and $\mathsf{MCP}(K', N')$ conditions for some $K, K' \in \mathbb{R}$ and $N' \in [1, \infty)$.

Then the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_\alpha\}_{\alpha \in Q}$ and a corresponding disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}}$ such that:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}}) \ll \mathfrak{q}$ and the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ satisfies the following properties:

- *for any $\mathfrak{m}$-measurable set B, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable;*
- *for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha) = X_\alpha$ (strong consistency);*
- *For $\mathfrak{q}$-almost every $q \in Q$, it holds $\int_{X_q} f\, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$-a.e. in Z.*
- *For $\mathfrak{q}$-almost every $q \in Q$, the set X_q is a geodesic (even more a transport ray) and the one dimensional m.m.s. $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{CD}(K, \infty)$.*

Moreover, fixed any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}}) \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$-essentially unique.

2.3 Perimeters

Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, one can introduce a notion of *perimeter* which extends the classical one on $\mathbb{R}^n$. The following presentation follows [45] and the more recent [3]. We start by recalling the notion of slope (or local Lipschitz constant) of a real-valued function.

Definition 2.10 (Slope)**.** Let (X, d) be a metric space and $u : X \to \mathbb{R}$ be a real valued function. We define the *slope* of f at the point $x \in X$ as

$$|\nabla u|(x) := \begin{cases} \limsup_{y \to x} \frac{|u(x)-u(y)|}{d(x,y)} & \text{if } x \text{ is not isolated} \\ 0 & \text{otherwise.} \end{cases}$$

To fix notations, the space of Lipschitz maps on (X, d) will be denoted by $\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathsf{d})$ while $\mathrm{Lip}_c(X) = \mathrm{Lip}_c(X, \mathsf{d})$ will be the subspace of compactly supported Lipschitz maps. If the function is locally Lipschitz in an open set A, i.e. for every $x \in A$, the function is Lipschitz in a neighborhood of x, then we use the notation $\mathrm{Lip}_{loc}(A)$

Definition 2.11 (Perimeter)**.** Let $E \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the class of Borel sets of (X, d), and let $A \subset X$ be open. We define the perimeter of E relative to A as:

$$\mathsf{Per}(E; A) := \inf \left\{ \liminf_{n \to \infty} \int_A |\nabla u_n| \, \mathfrak{m} \colon \; u_n \in \mathrm{Lip}_{loc}(A), \; u_n \to \chi_E \text{ in } L^1_{loc}(A, \mathfrak{m}) \right\},$$

where $|\nabla u|(x)$ is the slope of u at the point x. If $\mathsf{Per}(E; X) < \infty$, we say that E is a *set of finite perimeter*. We denote $\mathsf{Per}(E; X)$ with $\mathsf{Per}(E)$.

When E is a fixed set of finite perimeter, the map $A \mapsto \mathsf{Per}(E; A)$ is the restriction to open sets of a finite Borel measure on X, defined as

$$\mathsf{Per}(E; B) := \inf \{ \mathsf{Per}(E; A) \colon A \text{ open}, A \supset B \}.$$

For some recent progress on the extension of De Giorgi's rectifiability theorem (relating the perimeter and the Hausdorff measure of codimension 1) to the setting of non-collapsed $\mathsf{RCD}(K, N)$ spaces, we refer to [2] and references therein.

We observe the following fact.

Lemma 2.12. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $E \subseteq X$ be a Borel set. Assume that we are given a strongly consistent disintegration of $\mathfrak{m}$ associated to a zero mean function as given in Theorem 2.9:*

$$\mathfrak{m}_{\llcorner \mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that and $\mathfrak{m}_\alpha \in \mathcal{M}_+(X)$. Then it holds

$$\mathsf{Per}(E) \geq \int_Q \mathsf{Per}_\alpha(E_\alpha) \, \mathfrak{q}(d\alpha),$$

where $E_\alpha = E \cap X_\alpha$ and Per_α is the perimeter functional in the space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$.

Proof. Let $\{f_n\}_n \in \mathrm{Lip}_{\mathrm{loc}}(X)$ be a sequence of functions converging in $L^1(X, \mathfrak{m})$ to χ_E. Then, by disintegration

$$0 = \lim_{n \to +\infty} \int_X |f_n(x) - \chi_E(x)| \, \mathfrak{m}(dx) = \lim_{n \to +\infty} \int_Q \int_{X_\alpha} |f_n(x) - \chi_E(x)| \, \mathfrak{m}_\alpha(dx) \, \mathfrak{q}(d\alpha),$$

so up to extracting a subsequence, that we call again $\{f_n\}$, we have that for $\mathfrak{q}$-a.e. $q \in Q$

$$\lim_{n \to +\infty} \int_{X_\alpha} |f_n(x) - \chi_E(x)| \, \mathfrak{m}_\alpha(dx) = 0.$$

Recalling that each $\mathfrak{m}_\alpha$ is concentrated on X_α and denoting $E_\alpha := E \cap X_\alpha$, we have that $f_{n \llcorner X_\alpha}$ converges on $L^1(X_\alpha, \mathfrak{m}_\alpha)$ to χ_{E_α} for $\mathfrak{q}$-a.e $\alpha \in Q$. We observe in addition that if f_n is Lipschitz then

$f_n \llcorner X_\alpha$ is Lipschitz as well with a smaller local Lipschitz constant. Hence, taken $\{f_n\}_n \in \mathrm{Lip}_{\mathrm{loc}}(X)$ a sequence of functions attaining in the limit $\mathsf{Per}(E)$, we have that

$$\begin{aligned}\mathsf{Per}(E) &= \liminf_{n\to\infty} \int_X |\nabla f_n|\, \mathfrak{m} \geq \liminf_{n\to\infty} \int_Q \int_{X_\alpha} |\nabla f_n|\, \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) \\ &\geq \liminf_{n\to\infty} \int_Q \int_{X_\alpha} |\nabla f_n \llcorner X_\alpha|\, \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) \geq \int_Q \liminf_{n\to\infty} \int_{X_\alpha} |\nabla f_n \llcorner X_\alpha|\, \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) \\ &\geq \int_Q \mathsf{Per}_\alpha(E_\alpha)\, \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha),\end{aligned}$$

and the claim follows. □

We also include the following easy fact about the perimeter in the weighted one dimensional case. For a proof we refer to [21, Proposition 3.1] where there is an analogous statement for $\mathsf{CD}(K,N)$ densities.

Lemma 2.13. *Let $\mathfrak{m} = h\mathcal{L}^1$ be a non-negative measure on $\mathbb{R}$, with h a $\mathsf{CD}(K,\infty)$ density on its support, which in particular is an interval. Let E be an open set in $\mathrm{supp}(\mathfrak{m})$. Let $C_1,\dots,C_n$ be its connected components, with n possibly $+\infty$. We consider the set $\cup_{k=0}^n \bar{C}_k$. We observe that $\cup_{k=0}^n \bar{C}_k = \cup_{k=0}^m [a_k,b_k]$, with $[a_k,b_k]$ disjoint, with m possibly $+\infty$, $a_k,b_k \in \mathbb{R}\cup\{\pm\infty\}$. Then, setting $B(E) := \cup_{k=0}^m \{a_k,b_k\} \setminus \{\inf(\mathrm{supp}(\mathfrak{m})), \sup(\mathrm{supp}(\mathfrak{m}))\}$, it holds*

$$\mathsf{Per}_h(E) = \sum_{x\in B(E)} h(x) = \sum_{k=1}^m h(a_k) + h(b_k),$$

where Per_h is the Perimeter functional in the space $(\mathrm{supp}(\mathfrak{m}), |\cdot|, h\mathcal{L}^1)$.

2.4 Laplacian, Heat Flow and RCD

The main references for this part are [23, 4, 5, 32, 31, 7, 17] or [1] for a survey on the subject.

We recall the definition of the Cheeger energy of an L^2 function, which will be used to define Sobolev spaces on metric measure spaces. We will be only concerned with the case $p=2$.

Let $f \in L^p(X,\mathfrak{m})$, the *Cheeger energy* of f is defined as

$$\mathsf{Ch}(f) := \inf\left\{\liminf_{n\to\infty} \frac{1}{2}\int |\nabla f_n|^2\, \mathfrak{m} \colon f_n \in \mathrm{Lip}(X)\cap L^2(X,\mathfrak{m}),\ \|f_n - f\|_{L^2} \to 0\right\}, \tag{2.8}$$

where $|\nabla f_n|(x)$ is the slope of f_n at the point x. Then $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is defined as the space of functions $f \in L^2(X,\mathfrak{m})$ with finite Cheeger energy, endowed with the norm

$$\|f\|_{W^{1,2}(X,\mathsf{d},\mathfrak{m})} := \left\{\|f\|_{L^2(X,\mathfrak{m})} + \mathsf{Ch}(f)^{\frac{1}{2}}\right\}$$

which makes $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ a Banach space. For any $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, one can single out a distinguished object $|\nabla f|_w \in L^2(X,\mathfrak{m})$, which plays the role of the modulus of the gradient and provides the integral representation

$$\mathsf{Ch}(f) = \frac{1}{2}\int |\nabla f|_w^2\, \mathfrak{m};$$

this function is called the *minimal weak upper gradient* (after its identification with the minimal relaxed gradient). For Lipschitz functions $\mathrm{Lip}(f)$ is a weak upper gradient for f.

Next we review the definition of Laplacian. Throughout the note, the various curvature conditions we will assume will always imply X to be proper (bounded and closed sets are compact) thus simplifying the presentation.

We recall that a *Radon functional over* X is a linear functional $T : \mathrm{LIP}_c(X) \to \mathbb{R}$ such that for every compact subset $W \subset X$ there exists a constant $C_W \geq 0$ so that

$$|T(f)| \leq C_W \max_W |u|, \quad \text{for all } u \in \mathrm{LIP}_c(\Omega) \text{ with } \mathrm{supp}(u) \subset X.$$

Finally for any f, u locally in the Sobolev space (see [32]), define the functions $D^{\pm} f(\nabla u) : X \to \mathbb{R}$ by

$$D^+ f(\nabla u) := \inf_{\varepsilon > 0} \frac{|D(u + \varepsilon f)|_w^2 - |Du|_w^2}{2\varepsilon},$$

while $D^- f(\nabla u)$ is obtained replacing $\inf_{\varepsilon>0}$ with $\sup_{\varepsilon<0}$.

Definition 2.14 (General definition)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and $f : X \to \mathbb{R}$ be a Borel function. The function $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is in the domain of the Laplacian in X, $f \in D(\mathbf{\Delta})$, provided there exists a Radon functional $T : \mathrm{Lip}_c(X) \to \mathbb{R}$ such that

$$\int D^- u(\nabla f)\, \mathfrak{m} \leq -T(u) \leq \int D^+ u(\nabla f)\, \mathfrak{m},$$

for each $u \in \mathrm{Lip}_c(X)$. In this case we write $T \in \mathbf{\Delta}(f)$.

Definition 2.15 (Eigenfunction)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and f be in $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$. The function f is an eigenfunction for $-\mathbf{\Delta}$ if there exists $\lambda > 0$ such that

$$-\lambda f \mathfrak{m} \in \mathbf{\Delta} f;$$

i.e.

$$\int D^- u(\nabla f)\, \mathfrak{m} \leq \lambda \int f u\, \mathfrak{m} \leq \int D^+ u(\nabla f)\, \mathfrak{m},$$

for each $u \in \mathrm{Lip}_c(X)$.

Remark 2.16. It is straightforward to check that any eigenfunction has zero mean, provided $\mathfrak{m}(X) < \infty$. Here we only sketch the argument when X is proper. Consider any sequence (χ_n) of 1-Lipschitz functions with bounded support and values in $[0, 1]$ such that $\chi_n \equiv 1$ in $B_n(\bar{x})$, for some fixed $\bar{x} \in X$. Since we are assuming X to be proper, $\chi_n \in \mathrm{Lip}_c(X)$ and therefore

$$\int D^- \chi_n(\nabla f)\, \mathfrak{m} \leq \lambda \int \chi_n f\, \mathfrak{m} \leq \int D^+ \chi_n(\nabla f)\, \mathfrak{m};$$

for both quantities,

$$\left| \int D^{\pm} \chi_n(\nabla f)\, \mathfrak{m} \right| \leq \int_{X \setminus B_n(\bar{x})} |\nabla f|_w\, \mathfrak{m}$$

that are both converging to zero, provided $\mathfrak{m}(X) < \infty$, giving $\int f\, \mathfrak{m} = 0$ by dominated convergence theorem.

From the lower semi-continuity and convexity of $\mathsf{Ch} : D(\mathsf{Ch}) \subset L^2(X, \mathfrak{m}) \to [0, \infty)$, it is natural to consider an alternative definition of Laplacian related to the sub-differential $\partial^- \mathsf{Ch}$ of convex analysis. We recall that the sub-differential $\partial^- \mathsf{Ch}$ is the multivalued operator in $L^2(X, \mathfrak{m})$ defined at all $f \in D(\mathsf{Ch})$ by the family of inequalities

$$g \in \partial^- \mathsf{Ch}(f) \iff \int_X g(h - f)\, \mathfrak{m} \leq \mathsf{Ch}(h) - \mathsf{Ch}(f), \quad \forall\, h \in L^2(X, \mathfrak{m}).$$

Definition 2.17 (L^2-Laplacian)**.** The Laplacian $-\Delta f \in L^2(X, \mathfrak{m})$ of a function $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is the element of minimal $L^2(X, \mathfrak{m})$-norm in the sub-differential $\partial^- \mathsf{Ch}(f)$, provided the latter is non-empty. Accordingly, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is an eigenfunction provided $-\Delta f = \lambda f$, for some $\lambda > 0$.

It will be clear from the proof of our results that the minimality requirement in the previous definition does not play any role: our main results will be valid for any element of the sub-differential.

We also mention that at this level of generality, the two notions of Laplacian do not coincide, however ([32, Proposition 4.9]) if $f, g \in L^2(X, \mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$ and $-g \in \partial^- \mathsf{Ch}(f)$, then $g \in D(\boldsymbol{\Delta})$ and $g\, \mathfrak{m} \in \boldsymbol{\Delta} f$.

Again from the lower semi-continuity and convexity of Ch, invoking the classical theory of gradient flows of convex functionals in Hilbert spaces, it follows that: for any $f \in L^2(X, \mathfrak{m})$ there exists a unique continuous curve $(f_t)_{t \geq 0}$ in $L^2(X, \mathfrak{m})$ locally absolutely continuous in $[0, \infty)$ with $f_0 = f$, such that $\frac{d}{dt} f_t \in \partial^- \mathsf{Ch}(f_t)$ for a.e. $t > 0$.

The existence of the flow for all $f \in L^2(X, \mathfrak{m})$ stems from the density of $D(\mathsf{Ch})$ in $L^2(X, \mathfrak{m})$. This gives rise to a semigroup $(H_t)_{t \geq 0}$ on $L^2(X, \mathfrak{m})$ defined by $H_t f = f_t$, where f_t is the unique L^2-gradient flow of Ch.

It follows that $f_t \in D(\Delta)$ and

$$\frac{d^+}{dt} f_t = \Delta f_t, \qquad \forall\, t \in (0, \infty),$$

according to Definition 2.17.

On the other hand, one can study the metric gradient flow of the Boltzmann entropy Ent in $\mathcal{P}_2(X, \mathsf{d})$. If $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}(K, \infty)$, it has been proven in [4] that for any $\mu \in D(\mathrm{Ent})$ there exists a unique gradient flow of Ent starting from μ (for details we refer to [4]). This gives rise to a semigroup $(\mathfrak{H}_t)_{t \geq 0}$ on $P_2(X, \mathsf{d})$ defined by $\mathfrak{H}_t \mu = \mu_t$ where μ_t is the unique gradient flow of Ent starting from μ.

One of the main result of [4] is the identification of the two gradient flows: if $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}(K, \infty)$ space and $f \in L^2(X, \mathfrak{m})$ such that $f \mathfrak{m} = \mu \in \mathcal{P}_2(X, \mathsf{d})$, then

$$\mathfrak{H}_t \mu = (H_t f)\, \mathfrak{m}, \qquad \forall\, t \geq 0. \tag{2.9}$$

In particular we will only use the notation H_t for both semi-groups.

Definition 2.18 (RCD condition)**.** We say that $(X, \mathsf{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if the Cheeger energy Ch_2 defined in (2.8) is a quadratic form on $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Finally $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{RCD}(K, N)$ condition if it satisfies the $\mathsf{CD}(K, N)$ condition and it is infinitesimally Hilbertian.

Under the RCD condition, powerful contraction estimates for the heat flow are at disposal.

Theorem 2.19 (Theorem 3 of [31])**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space verifying $\mathsf{RCD}(K, N)$, then for any $\mu, \nu \in \mathcal{P}_2(X)$ and $s, t > 0$*

$$W_2(H_t \mu, H_s \nu)^2 \leq e^{-K\tau(s,t)} W_2(\mu, \nu)^2 + 2N \frac{1 - e^{-K\tau(s,t)}}{K\tau(s,t)} (\sqrt{t} - \sqrt{s})^2, \tag{2.10}$$

where $\tau(s, t) = 2(t + s + \sqrt{ts})/3$.

Also, if X is infinitesimally Hilbertian then the two notions of Laplacian coincide and the previous implication can be reversed. Indeed if $f, g \in L^2(X, \mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$ and $f \in D(\boldsymbol{\Delta})$ with $g\mathfrak{m} \in \boldsymbol{\Delta} f$ then $-g \in \partial^- \mathsf{Ch}(f)$.

Finally if $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is an Hilbert space (hence in the RCD case) and $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is an eigenfunction for the Laplacian in the sense of Definition 2.15, then $H_t f = e^{-\lambda t} f$. It is indeed enough to check that $-\lambda e^{-\lambda t} f \in \partial^- \mathsf{Ch}(e^{-\lambda t} f)$ that is equivalent to $-\lambda f \in \partial^- \mathsf{Ch}(f)$ and this follows from [32, Proposition 4.12].

3 One dimensional indeterminacy estimates

In this section we will obtain the one-dimensional version of the uncertainty principle we will then integrate via Disintegration Theorem. A slightly different version of the following Proposition 3.1 was already present in the literature [60, Theorem 4].

Throughout this section we will tacitly assume the Wasserstein distance to be defined on any couple of non-negative Borel measures having the same finite mass (not necessarily coinciding with 1).

We fix some notations that will be useful in this section and in the following one.
Given a function $f : I \to \mathbb{R}$ with zero mean, with I real closed interval, possibly of infinite length, satisfying the hypotheses of Theorem 2.4 we say that the trasport of f goes along a unique trasport ray if applying Theorem 2.4 one has that the partition in $\{X_\alpha\}_\alpha$ is made of only one element.
We underline in addition that, recalling the notations of Lemma 2.13, in the case of f being a continuous function $B(\{x \mid f(x) > 0\})$ is a subset of the zero set of f.

Proposition 3.1. *Let $f \colon [0,1] \to \mathbb{R}$ be a continuous function having zero mean w.r.t Lebesgue measure, i.e.*

$$\int_{(0,1)} f^+(x)\,dx = \int_{(0,1)} f^-(x)\,dx,$$

and assume that the transport of f goes along a unique transport ray, (see notations above). Then it holds:

$$W_1(f^+\mathcal{L}^1, f^-\mathcal{L}^1)\,\mathcal{H}^0\left(B(\{x \mid f(x) > 0\})\right) \geq \frac{\|f^+\|^2_{L^1(0,1)}}{2\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}. \tag{3.1}$$

Proof. **Step 1.** We claim that given two non-negative functions $h, g \in L^\infty(0,1)$ such that $\int_{[0,1]} h(x)\,dx = \int_{[0,1]} g(x)\,dx$ and which satisfy the following condition on the supports: there exists $\bar{x} \in (0,1)$ such that

$$\operatorname{supp}\{h\} \subseteq [0,\bar{x}], \quad \operatorname{supp}\{g\} \subseteq [\bar{x},1], \tag{3.2}$$

then one has

$$W_1(h,g) \geq \frac{1}{2}\frac{\|h\|^2_{L^1}}{\min\{\|h\|_{L^\infty}, \|g\|_{L^\infty}\}}. \tag{3.3}$$

Indeed we can consider the two following rearrangement of the masses

$$r_h\mathcal{L}^1 := \|h\|_{L^\infty}\,\chi_{(\bar{x}-\tau_h,\bar{x})}\mathcal{L}^1, \quad r_g\mathcal{L}^1 := \|g\|_{L^\infty}\,\chi_{(\bar{x},\bar{x}+\tau_g)}\mathcal{L}^1,$$

with τ_h and τ_g chosen so that the total mass of $r_h\mathcal{L}^1$ is the same total mass of $h\mathcal{L}^1$, and the same for $r_g\mathcal{L}^1$ and $g\mathcal{L}^1$. We notice that by direct calculation it holds

$$W_1(r_h\mathcal{L}^1, r_g\mathcal{L}^1) = \frac{1}{2}\left(\frac{\|h\|^2_{L^1(0,1)}}{\|h\|_{L^\infty(0,1)}} + \frac{\|g\|^2_{L^1(0,1)}}{\|g\|_{L^\infty(0,1)}}\right), \tag{3.4}$$

and then we observe that

$$W_1(h\mathcal{L}^1, g\mathcal{L}^1) \geq W_1(r_h\mathcal{L}^1, r_g\mathcal{L}^1). \tag{3.5}$$

Indeed for any π optimal transport plan between $h\mathcal{L}^1$ and $g\mathcal{L}^1$, one has

$$\begin{aligned} W_1(h\mathcal{L}^1, g\mathcal{L}^1) &= \int |x-y|\pi(dxdy) = \int_{(\bar{x},1)} (y-\bar{x})g(y)\,dy + \int_{(0,\bar{x})} (\bar{x}-x)h(x)\,dx \\ &= \int_{(\bar{x},1)} yg(y)\,dy + \int_{(0,\bar{x})} -xh(x)\,dx \geq \int_{(\bar{x},1)} yr_g(y)\,dy + \int_{(0,\bar{x})} -xr_h(x)\,dx \end{aligned}$$

where the last inequality follow from the two following observations:

- $g - r_g \le 0$ in $(\bar{x}, \bar{x} + \tau_g)$ and $g - r_g \ge 0$ in $(\bar{x} + \tau_g, 1)$,
- $h - r_h \le 0$ in $(0, \bar{x} - \tau_h)$ and $h - r_h \ge 0$ in $(\bar{x} - \tau_h, \bar{x})$,

and the fact that for a function $\psi : (0, +\infty) \to \mathbb{R}$ with zero mean and such that $\psi \le 0$ in $(0, a)$ and $\psi \ge 0$ in $(a, +\infty)$ it holds that $\int_{(0,+\infty)} x\psi(x)\, dx \ge 0$. So finally putting (3.4) and (3.5) togheter we obtain

$$W_1(h\mathcal{L}^1, g\mathcal{L}^1) \ge \frac{\|h\|^2_{L^1(0,1)}}{2\min\{\|h\|_{L^\infty(0,1)}, \|g\|_{L^\infty(0,1)}\}}.$$

Step 2. Let $f : [0, 1] \to \mathbb{R}$ be as in the hypotheses.
We define the sets D_k as follows: let $C_1, \dots, C_n$ be the connected components of $\{x \in [0, 1] \mid f(x) > 0\}$, with n possibly $+\infty$, then the sets $\{D_k\}_k$ are the closed disjoint intervals such that

$$\cup_{k=0}^{n} \bar{C}_k = \cup_{k=0}^{m} D_k.$$

We observe that if $m = +\infty$ then $\mathcal{H}^0(B(\{x \mid f(x) > 0\})) = +\infty$ and the statement is trivially true. So we assume that $m < +\infty$.

Let $T : [0, 1] \to \mathbb{R}$ be an optimal transport map for the problem.
We prove the following **claim:**

$$W_1(f^+ \llcorner_{D_k} \mathcal{L}^1, T_\sharp(f^+ \llcorner_{D_k} \mathcal{L}^1)) \ge \frac{1}{2} \frac{\|f^+ \llcorner_{D_k}\|^2_{L^1(0,1)}}{\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}, \quad \forall\, 1 \le k \le m.$$

Proof of the claim.
We observe that $T_\#(f^+ \llcorner_{D_k} \mathcal{L}^1) \le f^- \mathcal{L}^1$ (actually equality holds), for any k, so in particular it is absolutely continuous with respect to the Lebesgue measure. We consider

$$h := f^+ \llcorner_{D_k}, \quad g := \frac{dT_\sharp(f^+ \llcorner_{D_k} \mathcal{L}^1)}{d\mathcal{L}^1}, \tag{3.6}$$

and we notice that they satisfy the hypotheses of the previous step. Indeed

$$\mathrm{supp}(h) \subseteq D_k, \quad \mathrm{supp}(g) \subseteq T(D_k) \text{ and } T(x) \ge x \quad \forall\, x \in D_k \text{ or } T(x) \le x \quad \forall\, x \in D_k.$$

To see this observe that, since the transport of f goes along a unique transport ray, we have that either $u(x) := -x$ or $u(x) := x$ is a Kantorovich potential for the problem. Assuming without loss of generality $u(x) = -x$ and using the definition of Kantorovich potential we have that each couple $(x, T(x))$ with $x \in \mathrm{supp} f^+$ satisfies $u(x) - u(T(x)) = |x - T(x)|$ so in particular $T(x) = x + |x - T(x)|$ and $T(x) \ge x$. The claim follows by applying the result of the previous step to h and g and observing that $\|g\|_{L^\infty} \le \|f^-\|_{L^\infty}$.

Once that we proved the claim, we notice that being the sets C_k disjoint, we can sum over k the inequalities (3.6), and we get

$$W_1(f^+\mathcal{L}^1, f^-\mathcal{L}^1) = \sum_{k=1}^{m} W_1(f^+ \llcorner_{D_k} \mathcal{L}^1, T_\sharp(f^+ \llcorner_{D_k} \mathcal{L}^1))) \ge \frac{1}{2} \sum_{k=1}^{m} \frac{\|f^+ \llcorner_{D_k}\|^2_{L^1(0,1)}}{\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}.$$

Applying Cauchy-Schwartz inequality the result follows:

$$\begin{aligned} W_1(f^+\mathcal{L}^1, f^-\mathcal{L}^1) &\ge \frac{1}{2\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}} \frac{1}{m} \left(\sum_{k=1}^{m} \|f^+ \llcorner_{D_k}\|_{L^1(0,1)} \right)^2 \\ &= \frac{1}{2m} \frac{\|f^+\|^2_{L^1(0,1)}}{\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}. \end{aligned}$$

□

Remark 3.2. We observe that in the preceeding proposition the fact that the interval in which we are working in is exactly $[0, 1]$ plays no role, so it analogously holds for an interval $[a, b]$ o in general for intervals of infinite length provided that the function f is in L^1.

3.1 One dimensional densities with curvature bounds

We now obtain the one-dimensional estimate also for a reference measure other than the Lebesgue one.

As before, we will first consider the case of functions defined on a compact interval $[0, D]$ and then we will discuss the non-compact case in the following Remark 3.4.

Proposition 3.3. *Let $h\colon [0, D] \to [0, +\infty)$ be a $\mathsf{CD}(K, \infty)$-density (recall Definition 2.6). Let $f\colon [0, D] \to \mathbb{R}$ be a continuous function having zero mean w.r.t the measure $h\mathcal{L}^1$: $\int_{(0,D)} f(x)h(x)\,dx = 0$. Assume also that the transport of fh goes along a unique transport ray. Then it holds*

$$W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1)\left(\sum_{x\in B(\{f>0\})} h(x)\right) \geq \frac{\|fh\|^2_{L^1(0,D)}}{8C_{K,D}\|f\|_{L^\infty(0,D)}}, \tag{3.7}$$

(see Lemma 2.13 for the definition of $B(\{f > 0\})$) where

$$C_{K,D} := \begin{cases} 1 & K \geq 0, \\ e^{-KD^2/2} & K < 0. \end{cases} \tag{3.8}$$

Proof. **Step 1.** We make the following preliminary observation. From $\mathsf{CD}(K, \infty)$ assumption it follows that the map

$$[0, D] \ni x \mapsto \log h(x) + K\frac{(x - \bar{x})^2}{2},$$

is concave. In particular, for each $\bar{x} \in (0, D)$ either is increasing in $[0, \bar{x}]$ or is decreasing in $[\bar{x}, D]$. Hence in the first case

$$\log h(x) + K\frac{(x - \bar{x})^2}{2} \leq \log h(\bar{x}), \qquad \forall\, x \in [0, \bar{x}];$$

while in the second case:

$$\log h(x) + K\frac{(x - \bar{x})^2}{2} \leq \log h(\bar{x}), \qquad \forall\, x \in [\bar{x}, D];$$

The combination of the two previous inequalities yields

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \leq h(\bar{x})C_{K,D}. \tag{3.9}$$

where $C_{K,D}$ is the defined in (3.8).
Similarly to Step 1 of the previous proof we make a base estimate that we will use in the next step: we take two non negative bounded functions $f, g : [0, D] \to \mathbb{R}$ such that $\int_{[0,D]} fh\,dx = \int_{[0,D]} gh\,dx$, satisfying

$$\mathrm{supp}\{f\} \subseteq [0, \bar{x}], \quad \mathrm{supp}\{g\} \subseteq [\bar{x}, D]. \tag{3.10}$$

We can now apply (3.3) to fh, gh (recalling that h is positive) and

$$\begin{aligned} W_1(fh\mathcal{L}^1, gh\mathcal{L}^1) &\geq \frac{\|fh\|^2_{L^1(0,D)}}{2\min\{\|fh\|_{L^\infty(0,D)}, \|gh\|_{L^\infty(0,D)}\}} \\ &\geq \frac{\|fh\|^2_{L^1(0,D)}}{2C_{K,D}\max\{\|f\|_{L^\infty(0,D)}, \|g\|_{L^\infty(0,D)}\}h(\bar{x})}, \end{aligned} \tag{3.11}$$

where the second inequality follows from (3.9).

Step 2. Consider $C_1, \ldots, C_n$ the connected components of $\{f > 0\}$ with n possibly $+\infty$. As in Lemma 2.13 we consider the set $\cup_{k=0}^n \bar{C}_k$. We observe that it is the union of disjoint closed intervals:

$\cup_{k=0}^{n}\bar{C}_k = \cup_{k=0}^{m}[a_k, b_k]$, with m possibly $+\infty$. We will proceed as in the proof of Proposition 3.1: we consider an optimal trasport map T and we obtain

$$W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1) = \sum_{k=0}^{m} W_1(f^+h\mathcal{L}^1\llcorner_{(a_k,b_k)}, T_\sharp(f^+h\mathcal{L}^1\llcorner_{(a_k,b_k)})).$$

Then we can apply (3.11) (as in the previous proof using the fact that the transport of f^+h into f^-h goes along a unique transport ray) to obtain:

$$\begin{aligned} W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1) = & \sum_{k=0}^{m} W_1(f^+h\mathcal{L}^1\llcorner_{(a_k,b_k)}, T_\sharp(f^+h\mathcal{L}^1\llcorner_{(a_k,b_k)})) \\ \geq & \sum_{k=0}^{m} \frac{\|f^+h\|^2_{L^1(a_k,b_k)}}{2C_{K,D}\,\|f\|_{L^\infty(a_k,b_k)}\,(h(a_k)+h(b_k))} \\ \geq & \frac{1}{2C_{K,D}\,\|f\|_{L^\infty(0,D)}} \sum_{k=0}^{m} \frac{\|f^+h\|^2_{L^1(a_k,b_k)}}{(h(a_k)+h(b_k))} \\ \geq & \frac{\|fh\|^2_{L^1(0,D)}}{8C_{K,D}\,\|f\|_{L^\infty(0,D)}\sum_{k=0}^{m}(h(a_k)+h(b_k))}, \end{aligned}$$

with the convention that if $a_k = 0$ (resp. $b_k = D$) the term $h(a_k)$ (resp. $h(b_k)$) does not appear. From this we get

$$8C_{K,D}W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1)\left(\sum_{k=0}^{m}(h(a_k)+h(b_k))\right) \geq \frac{\|fh\|^2_{L^1(0,D)}}{\|f\|_{L^\infty(0,D)}},$$

with the same convention on $h(0)$, $h(D)$ as above, from which the conclusion follows. □

Remark 3.4. The case of non-compact intervals of definition holds without modifications. The only relevant case is $K \geq 0$ and $D = \infty$ indeed for $K < 0$ and $D = \infty$, the claim becomes empty. Notice that D plays a role only in (3.9) where, in the relevant cases, it becomes independent on D.

3.2 The case of $\mathsf{MCP}(K,N)$ densities

We now address the case of an $\mathsf{MCP}(K,N)$-density. As it is clear from the proof of Proposition 3.3, the only place where the $\mathsf{CD}(K,\infty)$ assumption has been used is to ensure $h > 0$ over $(0,D)$ and to derive (3.9). A similar estimate, with suitable variations, can be obtained also for $\mathsf{MCP}(K,N)$-densities.

Lemma 3.5. *Let $h : [0,D] \to [0,\infty]$ be an $\mathsf{MCP}(K,N)$-density for some real parameters K, N with $N \geq 1$. Then for any $\bar{x} \in [0,D]$ the following estimates holds true:*

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \leq h(\bar{x})C_{K,N,D}, \tag{3.12}$$

where

$$C_{K,N,D} := \begin{cases} 2^{N-1} & K \geq 0 \\ 2^{N-1}e^{\sqrt{-K(N-1)}\frac{D}{2}} & K < 0. \end{cases} \tag{3.13}$$

Proof. The claim will follow from simple manipulations of (2.6). For clarity we recall it: for all $0 \leq x_0 \leq x_1 \leq D$

$$\left(\frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)}\right)^{N-1};$$

indeed for $x \in [0,\bar{x}]$

$$h(x) \leq \left(\frac{s_{K/(N-1)}(D-x)}{s_{K/(N-1)}(D-\bar{x})}\right)^{N-1} h(\bar{x}) \leq \frac{h(\bar{x})}{s_{K/(N-1)}(D-\bar{x})^{N-1}} \sup_{0\leq x\leq \bar{x}} s_{K/(N-1)}(D-x)^{N-1}$$

and for $x \in [\bar{x}, D]$

$$h(x) \le \left(\frac{s_{K/(N-1)}(x)}{s_{K/(N-1)}(\bar{x})}\right)^{N-1} h(\bar{x}) \le \frac{h(\bar{x})}{s_{K/(N-1)}(\bar{x})^{N-1}} \sup_{\bar{x}\le x\le D} s_{K/(N-1)}(x)^{N-1}.$$

Then if $K \ge 0$, in particular h will be $\mathsf{MCP}(0, N)$ giving

$$\sup_{0\le x\le \bar{x}} h(x) \le h(\bar{x}) \left(\frac{D}{D-\bar{x}}\right)^{N-1}, \qquad \sup_{\bar{x}\le x\le D} h(x) \le h(\bar{x}) \left(\frac{D}{\bar{x}}\right)^{N-1},$$

and therefore

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \le h(\bar{x}) D^{N-1} \min\{1/(D-\bar{x}), 1/\bar{x}\}^{N-1} \le 2^{N-1} h(\bar{x}),$$

proving the inequality if $K \ge 0$. If $K < 0$, arguing analogously one gets

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \le h(\bar{x}) 2^{N-1} e^{\sqrt{-K(N-1)}\frac{D}{2}},$$

concluding the proof. □

Putting together the proof of Proposition 3.3 and Lemma 3.5 we straightforwardly obtain the next

Proposition 3.6. *Let $h\colon [0, D] \to [0, +\infty)$ be an $\mathsf{MCP}(K, N)$-density. Let $f\colon [0, D] \to \mathbb{R}$ be a continuous function having zero mean w.r.t the measure with density h: $\int_{(0,D)} f(x)h(x)\,dx = 0$. Assume also that the transport of fh goes along a unique transport ray: $\int_{(0,s)} f(x)h(x)\,dx \ge 0$ for all $s \in [0, D]$. Then it holds*

$$W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1) \left(\sum_{\{x\in B(\{f>0\})\}} h(x)\right) \ge \frac{\|fh\|^2_{L^1(0,D)}}{8C_{K,N,D}\|f\|_{L^\infty(0,D)}}, \tag{3.14}$$

where $C_{K,N,D}$ is given by (3.13).

Remark 3.7. The case of non-compact intervals of definition holds again without modifications. The only relevant case here will be $K = 0$ and $D = \infty$; if $K > 0$, then MCP implies that $D < D_{K,\mathcal{N}}$ (see (2.2)) while if $K < 0$ and $D = \infty$, the claim becomes empty. Notice that D plays a role only in (3.9) that is the content of Lemma 3.5.

4 Indeterminacy estimates for metric measure spaces

We now use the one-dimensional estimates of the previous section to deduce the following sharp indeterminacy estimates.

Theorem 4.1. *Let $K, K', N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. satisfying either $\mathsf{CD}(K, N)$ or $\mathsf{MCP}(K', N)$ and $\mathsf{CD}(K, \infty)$. Let $f \in L^1(X, \mathfrak{m})$ a continuous function or, alternatively, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ be such that $\int_X f\,\mathfrak{m} = 0$. Assume also the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(dx) < \infty$. Then the following indeterminacy estimate is valid:*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}(\{x \in X : f(x) > 0\}) \ge \frac{\|f\|^2_{L^1(X,\mathfrak{m})}}{8C_{K,D}\|f\|_{L^\infty(X,\mathfrak{m})}}, \tag{4.1}$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,D} := \begin{cases} 1 & K \ge 0, \\ e^{-KD^2/2} & K < 0. \end{cases}$$

Remark 4.2. Notice that curvature assumptions $\mathsf{CD}(K,N)$ and $\mathsf{MCP}(K,N)$ imply $D<\infty$ only in the range $K>0$ and $N\in(1,\infty)$. Hence under the second set of assumptions ($\mathsf{MCP}(K',N)$ and $\mathsf{CD}(K,\infty)$), the result (4.1) for $K\geq 0$ gives a non-trivial bound also in the non-compact case $D=\infty$.

Proof. Given f as in the assumptions, we can invoke localization paradigm (Theorem 2.4 and Theorem 2.9) yielding a decomposition of the space X as $X=Z\cup\mathcal{T}$, where f is zero $\mathfrak{m}$-a.e. in Z and $\mathcal{T}$ can be partitioned into $\{X_\alpha\}_\alpha$ with α in a Borel set $Q\subset X$, and a disintegration of $\mathfrak{m}$,

$$\mathfrak{m}\llcorner_{\mathcal{T}}=\int_Q \mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha),$$

with $\mathfrak{q}$ Borel probability measure with $\mathfrak{q}(Q)=1$ and $Q\ni\alpha\mapsto\mathfrak{m}_\alpha\in\mathcal{M}_+(X)$ satisfying the properties of Theorem 2.4; in particular, $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha)$ is a $\mathsf{CD}(K,N)$ space (or $\mathsf{CD}(K,\infty)$ see Theorem 2.9), $\int_{X_\alpha} f\,\mathfrak{m}_\alpha=0$ and every X_α is a transport ray associated to the L^1-optimal trasport of $f^+\mathfrak{m}$ into $f^-\mathfrak{m}$.

Step 1. As proven in [22, Proposition 4.4] for the case of signed distance functions, $\mathfrak{q}$ can be identified with a test plan, see [4, Definition 5.1]; hence, if $f\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, by the identification between different definitions of Sobolev spaces [4, Theorem 6.2], for $\mathfrak{q}$-a.e. α the function f restricted to the geodesic X_α is Sobolev and therefore continuous.

As said in Remark 2.5, we have an isomorphism between each space $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha)$ and spaces $(I_\alpha,|\cdot|,h_\alpha\cdot\mathcal{L}^1)$, with I_α a real interval (of possible infinite length) satisfying the same $\mathsf{CD}(K,N)$ (or $\mathsf{CD}(K,\infty)$) condition, $\int_{I_\alpha} f_\alpha(x)h_\alpha(x)\,dx=0$ being f_α the corresponding of $f\llcorner_{X_\alpha}$ through the isomorphism and I_α transport ray for f_α. Whenever possible, for simplicity of notation, we will use $f=f_\alpha$.

So now we can apply Proposition 3.3 and we have that $\mathfrak{q}$-a.e. $\alpha\in Q$ it holds

$$W_1(f_\alpha^+h_\alpha\mathcal{L}^1,f_\alpha^-h_\alpha\mathcal{L}^1)\left(\sum_{x\in B(\{f_\alpha>0\})}h_\alpha(x)\right)\geq\frac{\|f\|^2_{L^1(X_\alpha,\mathfrak{m}_\alpha)}}{8C_{K,D}\|f\|_{L^\infty(X_\alpha,\mathfrak{m}_\alpha)}}. \tag{4.2}$$

By Lemma 2.13 $\sum_{x\in B(\{f_\alpha>0\})}h_\alpha(x)=\mathsf{Per}_{h_\alpha}(\{x\in I_\alpha:\ f_\alpha(x)>0\})$, hence using the isomorphisms of metric measure spaces, we have

$$W_1(f^+\mathfrak{m}_\alpha,f^-\mathfrak{m}_\alpha)\,\mathsf{Per}_\alpha\left(\{x\in X_\alpha:\ f(x)>0\}\right)\geq\frac{\|f\|^2_{L^1(X_\alpha,\mathfrak{m}_\alpha)}}{8C_{K,D}\|f\|_{L^\infty(X_\alpha,\mathfrak{m}_\alpha)}},$$

where Per_α is the perimeter in $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha)$ and Per_{h_α} in $(I_\alpha,|\cdot|,h_\alpha\cdot\mathcal{L}^1)$. In the previous factor we have tacitly used that $C_{K,D}\geq C_{K,D_\alpha}$, where D_α is the length of X_α. Integrating the square root of the inequality with respect to the measure $\mathfrak{q}$ on Q and applying Holder inequality, we get

$$\begin{aligned}
&\left(\int_Q W_1(f^+\mathfrak{m}_\alpha,f^-\mathfrak{m}_\alpha)\,\mathfrak{q}(d\alpha)\right)^{\frac12}\left(\int_Q\mathsf{Per}_\alpha(\{x\in X_\alpha:\ f(x)>0\})\,\mathfrak{q}(d\alpha)\right)^{\frac12}\\
&\geq\int_Q\left(W_1(f^+\mathfrak{m}_\alpha,f^-\mathfrak{m}_\alpha)\cdot\mathsf{Per}_\alpha(\{x\in X_\alpha:\ f(x)>0\})\right)^{\frac12}\,\mathfrak{q}(d\alpha)\\
&\geq\int_Q\frac{\|f\|_{L^1(X_\alpha,\mathfrak{m}_\alpha)}}{2\sqrt{2C_{K,D}}\|f\|^{\frac12}_{L^\infty(X_\alpha,\mathfrak{m}_\alpha)}}\,\mathfrak{q}(d\alpha)\\
&\geq\frac{1}{2\sqrt{2C_{K,D}}\|f\|^{\frac12}_{L^\infty(X,\mathfrak{m})}}\int_Q\int_{X_\alpha}|f(x)|\mathfrak{m}_\alpha(dx)\,\mathfrak{q}(d\alpha)\\
&=\frac{\|f\|_{L^1(X,\mathfrak{m})}}{2\sqrt{2C_{K,D}}\|f\|^{\frac12}_{L^\infty(X,\mathfrak{m})}}.
\end{aligned}$$

Clearly $\int_Q W_1(f^+\mathfrak{m}_\alpha, f^-\mathfrak{m}_\alpha)\,\mathfrak{q}(d\alpha) = W_1(f^+\mathfrak{m}, f^-\mathfrak{m})$; therefore

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m})^{\frac{1}{2}} \left(\int_Q \mathsf{Per}_\alpha(\{x \in X_\alpha\colon\, f(x) > 0\})\,\mathfrak{q}(d\alpha) \right)^{\frac{1}{2}} \geq \frac{\|f\|_{L^1(X,\mathfrak{m})}}{2\sqrt{2C_{K,D}}\|f\|^{\frac{1}{2}}_{L^\infty(X,\mathfrak{m})}}.$$

The conclusion follows using Lemma 2.12. □

Repeating the same argument of the previous proof and using Proposition 3.6, we also obtain the analogous estimate for spaces verifying the weaker $\mathsf{MCP}(K,N)$; as expected, weaker curvature assumptions yields a dependence on the dimension of the estimate.

Theorem 4.3. *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$.*

Let $f \in L^1(X,\mathfrak{m})$ a continuous function or, alternatively, $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ be such that $\int_X f\,\mathfrak{m} = 0$. Assume also the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x,x_0)\,\mathfrak{m}(dx) < \infty$. Then the following indeterminacy estimate is valid:

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}\left(\{x \in X\colon\, f(x) > 0\}\right) \geq \frac{\|f\|^2_{L^1(X,\mathfrak{m})}}{8C_{K,N,D}\|f\|_{L^\infty(X,\mathfrak{m})}}, \tag{4.3}$$

where $\mathrm{diam}\,(X) = D$ *and*

$$C_{K,N,D} := \begin{cases} 2^{N-1} & K \geq 0, \\ 2^{N-1} e^{\sqrt{-K(N-1)}\frac{D}{2}} & K < 0. \end{cases}$$

5 Nodal Sets of Eigenfunctions

The plan for this section is to obtain lower bounds on the nodal set of eigenfunctions under curvature assumptions. Building on the previous Theorem 4.1 and Theorem 4.3, this will reduce to find an upper bound on the W_1 distance between the positive and the negative part of the eigenfunction.

5.1 Nodal set under MCP and CD

Here, as throughout the paper, the W_1 distance is understood to be tacitly extended between any finite non-negative measure with the same total mass.

Lemma 5.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. verifying $\mathsf{MCP}(K,N)$ and with finite total mass, $\mathfrak{m}(X) < \infty$. Let f be an eigenfunction of the Laplacian with eigenvalue $\lambda \neq 0$ accordingly to Definition 2.17 and assume moreover the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x,x_0)\,\mathfrak{m}(dx) < \infty$.*

Then

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \leq \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}}\|f\|_{L^2(X,\mathfrak{m})}$$

Proof. First from Remark 2.16, $\int f\,\mathfrak{m} = 0$ and, by definition, $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$. By assumption Kantorovich duality has a solution and therefore exists a 1-Lipschitz Kantorovich Potential $u : X \to \mathbb{R}$ such that

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) = \int_X (f^+(x) - f^-(x))u(x)\,\mathfrak{m}(dx) = \int_X f(x)u(x)\,\mathfrak{m}(dx). \tag{5.1}$$

Since f is a eigenfunction in the sense of Definition 2.17, then the following integration by-parts formula

$$\int_X D^- g(\nabla f)\,\mathfrak{m} \leq \lambda \int_X g f\,\mathfrak{m} \leq \int_X D^+ g(\nabla f)\,\mathfrak{m},$$

is valid for any $g \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ (see for instance the proof of [32, Proposition 4.9]).

From $\mathfrak{m}(X) < \infty$ it follows that $u \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, hence together with (5.1) gives

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \leq \frac{1}{\lambda} \int_X D^+u(\nabla f)\,\mathfrak{m} \leq \frac{1}{\lambda} \int_X |Du|_w\,|Df|_w\,\mathfrak{m} \leq \frac{\mathrm{Lip}(u)}{\lambda} \int_X |Df|_w\,\mathfrak{m},$$

where we used the fact that $|D^{\pm}u(\nabla f)| \leq |Du|_w\,|Df|_w$ and that $\mathrm{Lip}(u)$ is a weak upper gradient for u. Then by Holder inequality we have

$$\int_X |Df|_w\,\mathfrak{m} \leq \mathfrak{m}(X)^{\frac{1}{2}} \left(\int_X |Df|_w^2\,\mathfrak{m} \right)^{\frac{1}{2}} = \mathfrak{m}(X)^{\frac{1}{2}} \left(\int_X D^-f(\nabla f)\,\mathfrak{m} \right)^{\frac{1}{2}} \leq \mathfrak{m}(X)^{\frac{1}{2}} \sqrt{\lambda} \left(\int_X f^2\,\mathfrak{m} \right)^{\frac{1}{2}}$$

noticing that $Df^+(\nabla f) = |Df|_w^2$ (see [32, (3.6)]) and f itself as test-function. □

Remark 5.2. The same claim can be obtained assuming f to be an eigenfunction for the more general notion of Laplacian of Definition 2.14, provided one additionally knows f to be Lipschitz regular, yielding integration by-parts formula against any Sobolev functions (and in particular yielding f to be an eigenfunction for the Laplacian of Definition 2.17.)

Putting together Lemma 5.1 and the previous results we obtain the next

Theorem 5.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying either $\mathsf{CD}(K,N)$ or $\mathsf{MCP}(K', N')$ and $\mathsf{CD}(K,\infty)$ and such that $\mathfrak{m}(X) < \infty$.*

Let f be an eigenfunction of the Laplacian of eigenvalue $\lambda > 0$ accordingly to to Definition 2.17 and assume moreover the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(dx) < \infty$. Then the following estimate on the size of the its nodal set holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \geq \frac{\sqrt{\lambda}}{8C_{K,D}\sqrt{\mathfrak{m}(X)}} \cdot \frac{\|f\|_{L^1(X,\mathfrak{m})}^2}{\|f\|_{L^2(X,\mathfrak{m})}\|f\|_{L^\infty(X,\mathfrak{m})}},$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,D} := \begin{cases} 1 & K \geq 0, \\ e^{-KD^2/2} & K < 0. \end{cases}$$

Proof. Theorem 4.1 and Lemma 5.1 imply the claim. □

Using Theorem 4.3, we obtain the following analogous statement for spaces verifying the weaker $\mathsf{MCP}(K,N)$ condition with dimension-dependent constant appearing. The proof, being completely the same is omitted.

Theorem 5.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$ and such that $\mathfrak{m}(X) < \infty$.*

Let f be an eigenfunction of the Laplacian of eigenvalue $\lambda > 0$ accordingly to to Definition 2.17 and assume moreover the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(dx) < \infty$.

Then the following estimate on the size of the its nodal set holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \geq \frac{\sqrt{\lambda}}{8C_{K,N,D}\sqrt{\mathfrak{m}(X)}} \cdot \frac{\|f\|_{L^1(X,\mathfrak{m})}^2}{\|f\|_{L^2(X,\mathfrak{m})}\|f\|_{L^\infty(X,\mathfrak{m})}},$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,N,D} := \begin{cases} 2^{N-1} & K \geq 0, \\ 2^{N-1} e^{\sqrt{-K(N-1)}\frac{D}{2}} & K < 0. \end{cases}$$

5.2 The infinitesimally Hilbertian case

Assuming the heat flow to be linear yields more sophisticated argument and sharper estimtes. We start with the following folklore result whose proof is included as no proof as been found in the literature.

Lemma 5.5. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s. with* $\operatorname{diam}(X) < D$*. Let* $f, g : X \to [0, \infty)$ *be functions with* $\|f\|_{L^1(X,\mathfrak{m})} = \|g\|_{L^1(X,\mathfrak{m})}$*. Then*

$$W_1(f\,\mathfrak{m}, g\,\mathfrak{m}) \le D\|f - g\|_{L^1(X,\mathfrak{m})}.$$

Proof. Construct an admissible plan $\bar{\pi} \in \Pi(f\,\mathfrak{m}, g\,\mathfrak{m})$, with $\bar{\pi} = \pi_1 + \pi_2$ by defining

$$\pi_1 := (Id, Id)_\sharp \left(g\,\mathfrak{m}\llcorner_{\{g \le f\}}\right) + (Id, Id)_\sharp \left(f\,\mathfrak{m}\llcorner_{\{g > f\}}\right)$$

and considering any $\pi_2 \in \Pi((f - g)^+\mathfrak{m}, (f - g)^-\mathfrak{m})$. Then it is straightforward to check that

$$W_1(f\,\mathfrak{m}, g\,\mathfrak{m}) \le \int_{X\times X} \mathsf{d}(x, y)\,\pi_2(dxdy) \le D\,\pi_2(X \times X) = D\int_X (f - g)^+\,\mathfrak{m}(dx),$$

proving the claim. □

Proposition 5.6. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s. verifying* $\mathsf{RCD}(K, N)$ *and such that* $\operatorname{diam}(X) = D < \infty$*. Let* f *be an eigenfunction of eigenvalue* $\lambda > 2$*. Then*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \le C(K, N, D)\sqrt{\frac{\log\lambda}{\lambda}}\|f\|_{L^1(X,\mathfrak{m})},$$

with $C(K, N, D)$ *growing linearly in* D *and as square root in* N*.*

Proof. We define

$$\mu_0^\pm := f^\pm\,\mathfrak{m}, \qquad \mu_t^\pm := H_t\mu_0^\pm,$$

where H_t is the heat flow (see Section 2.4) and by triangular inequality

$$W_1(\mu_0^+, \mu_0^-) \le W_1(\mu_0^+, \mu_t^+) + W_1(\mu_t^+, \mu_t^-) + W_1(\mu_t^-, \mu_0^-),$$

notice indeed that $\mu_t^+(X) = \mu_0^+(X) = \mu_0^-(X) = \mu_t^-(X)$. Then by Theorem 2.19 we deduce that

$$\begin{aligned}
W_1(\mu_t^\pm, \mu_0^\pm) &= \left(\int_X f^+\,\mathfrak{m}\right) W_1(\mu_t^\pm/\mu_t^\pm(X), \mu_0^\pm/\mu_0^\pm(X)) \\
&\le \|f\|_{L^1(X,\mathfrak{m})} W_2(\mu_t^\pm/\mu_t^\pm(X), \mu_0^\pm/\mu_0^\pm(X)) \\
&\le \sqrt{t}\|f\|_{L^1(X,\mathfrak{m})} C(t, K, N),
\end{aligned}$$

where $C(t, K, N) := \left(2N\frac{1-e^{-K2t/3}}{K2t/3}\right)^{1/2}$, (with $C(t, K, N) \le \sqrt{2N}$ if $K \ge 0$).

To bound $W_1(\mu_t^+, \mu_t^-)$ we use Lemma 5.5. Call g_t the evolution of a function g through the heat flow ($g_t = H_t g$), by the identification (2.9), it follows that (recall that $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ by definition)

$$\mu_t^\pm = (H_t f^\pm)\,\mathfrak{m} = f_t^\pm\,\mathfrak{m}.$$

Notice that by infinitesimal Hilbertianity

$$f_t^+ - f_t^- = H_t(f^+ - f^-) = H_t(f) = e^{-\lambda t} f,$$

where the last identity is a consequence of f being an eigenfunction (see Section 2.4). Then we have that

$$W_1(\mu_t^+, \mu_t^-) \le D\|f_t^+ - f_t^-\|_{L^1(X,\mathfrak{m})} = D\|f_t\|_{L^1(X,\mathfrak{m})} = De^{-\lambda t}\|f\|_{L^1(X,\mathfrak{m})}.$$

So finally

$$W_1(\mu_0^+, \mu_0^-) \leq \left(\sqrt{t}C(t,K,N) + De^{-\lambda t}\right) \|f\|_{L^1(X,\mathfrak{m})}.$$

Choosing $t = \frac{1}{\lambda}\log(\lambda)$ we obtain

$$W_1(f^+\,\mathfrak{m}, f^-\,\mathfrak{m}) \leq C(K,D,N)\sqrt{\frac{\log\lambda}{\lambda}}\|f\|_{L^1(X,\mathfrak{m})},$$

with $C(K,N,D)$ growing linearly in D and as square root in N. □

Hence we can state one of the main results of this note.

Theorem 5.7 (Nodal set RCD-spaces)**.** *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. satisfying $\mathsf{RCD}(K,N)$. Assume moreover* $\operatorname{diam}(X) = D < \infty$*. Let f be an eigenfunction of the Laplacian of eigenvalue $\lambda > 2$. Then the following estimate is valid:*

$$\mathsf{Per}\left(\{x \in X : f(x) > 0\}\right) \geq \sqrt{\frac{\lambda}{\log\lambda}} \cdot \frac{\|f\|_{L^1(X,\mathfrak{m})}}{\bar{C}_{K,D,N}\|f\|_{L^\infty(X,\mathfrak{m})}}, \tag{5.2}$$

where $\bar{C}_{K,D,N}$ grows linearly in D if $K \geq 0$ and exponentially if $K < 0$ and grows with power $1/2$ in N.

Proof. Since $\operatorname{diam}(X) < \infty$, it follows that $\mathfrak{m}(X) < \infty$ and therefore $f \in L^1(X,\mathfrak{m})$, it has zero mean and satisfies the growth conditions and regularity needed to invoke Theorem 4.1. Hence Theorem 4.1 implies that

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}\left(\{x \in X : f(x) > 0\}\right) \geq \frac{\|f\|^2_{L^1(X,\mathfrak{m})}}{8C_{K,D}\|f\|_{L^\infty(X,\mathfrak{m})}},$$

that together with Proposition 5.6 implies that

$$\mathsf{Per}\left(\{x \in X : f(x) > 0\}\right) \geq \sqrt{\frac{\lambda}{\log\lambda}}\frac{\|f\|_{L^1(X,\mathfrak{m})}}{C(K,N,D)C_{K,D}\|f\|_{L^\infty(X,\mathfrak{m})}},$$

giving therefore the claim. □

We are now in position of obtaining the explicit lower bound on the size of the nodal set of an eigenfunction stated in Theorem 1.5.

Proof of Theorem 1.5. It is a straightforward consequence of Theorem 5.7 and of the following observation: given an eigenfunction f of eigenvalue λ, there exists a constant $C = C(K,N,D)$ such that

$$\|f\|_{L^\infty(X,\mathfrak{m})} \leq C\lambda^{\frac{N}{2}}\|f\|_{L^1(X,\mathfrak{m})},$$

provided $\lambda \geq D^{-2}$. Indeed from [8, Proposition 7.1] and assuming $\mathfrak{m}(X) = 1$, one has that

$$\|f\|_{L^\infty(X,\mathfrak{m})} \leq C\lambda^{\frac{N}{4}}\|f\|_{L^2(X,\mathfrak{m})} \leq C\lambda^{\frac{N}{4}}\|f\|^{\frac{1}{2}}_{L^\infty(X,\mathfrak{m})}\|f\|^{\frac{1}{2}}_{L^1(X,\mathfrak{m})},$$

from which the claim follows dividing by the L^∞ norm and squaring both sides. □

6 Linear combination of eigenfunctions

We now consider functions obtained as linear combination of eigenfunctions. As expected, for the following results it will be necessary to assume the linearity of the Laplacian, i.e. infinitesimal Hilbertianity.

We will however present two different upper bounds for the W_1 distance between the positive and the negative part of the function, one following the lines of Proposition 5.6 valid for RCD spaces and one following Lemma 5.1 valid for MCP spaces.

Proposition 6.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying* $\mathsf{MCP}(K, N)$ *with* $\operatorname{diam}(X) = D < \infty$*; assume moreover $(X, \mathsf{d}, \mathfrak{m})$ to be infinitesimally Hilbertian.*

Let f be a continuous function or, alternatively, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, such that it satisfies in L^2 sense $f = \sum_{\lambda_k \geq \lambda} a_k f_{\lambda_k}$, $k \in \mathbb{N}$, where each f_{λ_k} is an eigenfunction with eigenvalue λ_k.

Then the following estimate on the size of the nodal set of f holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \geq \frac{\sqrt{\lambda}}{\sqrt{\mathfrak{m}(X)} C_{K,N,D}} \cdot \frac{\|f\|^2_{L^1}}{\|f\|_{L^2} \|f\|_{L^\infty}},$$

where $C_{K,N,D}$ is given by Theorem 4.3.

Proof. From $\operatorname{diam}(X) < \infty$, it follows that $\mathfrak{m}(X) < \infty$ and therefore $f \in L^1(X, \mathfrak{m})$, it has zero mean and satisfies the growth conditions needed to apply Theorem 4.3. To prove the claim it will be therefore sufficient to obtain an upper bound for $W_1(f^+ \mathfrak{m}, f^- \mathfrak{m})$.

Using the Kantorovich formulation, there exists a 1-Lipschitz function such that

$$\begin{aligned} W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) &= \int_X f u \, \mathfrak{m} = \sum_{\lambda_k \geq \lambda} a_k \int_X f_{\lambda_k} u \, \mathfrak{m} \leq \sum_{\lambda_k \geq \lambda} \frac{a_k}{\lambda_k} \| |\nabla f_{\lambda_k}|_w \|_{L^2} \| |\nabla u| \|_{L^2} \\ &\leq \sqrt{\mathfrak{m}(X)} \sum_{\lambda_k \geq \lambda} \frac{1}{\sqrt{\lambda_k}} \|a_k f_{\lambda_k}\|_{L^2} \leq \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}} \sum_{\lambda_k \geq \lambda} \|a_k f_{\lambda_k}\|_{L^2} = \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}} \|f\|_{L^2}, \end{aligned}$$

where we used in the third identity $\|\frac{1}{\lambda_k} |\nabla f_{\lambda_k}|_w\|^2_{L^2} = \frac{1}{\lambda_k} \|f_{\lambda_k}\|^2_{L^2}$, and in the last one the orthogonality of $\{f_{\lambda_k}\}_{k \in \mathbb{N}}$ given by infinitesimally Hilbertianity. □

Lemma 6.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. verifying* $\mathsf{RCD}(K, N)$ *and such that* $\operatorname{diam}(X) = D < \infty$ *and $K \geq 0$. Let $f : X \to \mathbb{R}$ be a continuous or, alternatively, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, such that*

$$f = \sum_{\lambda_k \geq \lambda} \langle f, f_{\lambda_k} \rangle f_{\lambda_k}, \qquad \{\lambda_k\}_{k \in \mathbb{N}},$$

where $\{f_{\lambda_k}\}_{k \in \mathbb{N}}$ are eigenfunctions of the Laplacian of unitary L^2-norm with eigenvalue λ_k, $\langle f, f_{\lambda_k} \rangle$ is the scalar product of $L^2(X, \mathfrak{m})$, $\langle f_{\lambda_j}, f_{\lambda_k} \rangle = \delta_{j,k}$ and convergence of the series is in $L^2(X, \mathfrak{m})$. Then

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \leq C(K, N, D, \mathfrak{m}(X)) \left(\frac{1}{\lambda} \log \left(\lambda \frac{\|f\|_{L^2}}{\|f\|_{L^1}} \right) \right)^{\frac{1}{2}} \|f\|_{L^1},$$

with $C(K, N, D, \mathfrak{m}(X))$ an explicit constant, provided that $\lambda \geq 2\sqrt{\mathfrak{m}(X)}$.

Proof. Following the approach and the same notation of the proof of Proposition 5.6 we have

$$W_1(\mu_0^+, \mu_0^-) \leq W_1(\mu_0^+, \mu_t^+) + W_1(\mu_t^+, \mu_t^-) + W_1(\mu_t^-, \mu_0^-),$$

and deduce from Theorem 2.19 that

$$W_1(\mu_t^\pm, \mu_0^\pm) \leq \sqrt{t} \|f\|_{L^1(X,\mathfrak{m})} C(t, K, N),$$

where $C(t, K, N) := \left(2N \frac{1 - e^{-K2t/3}}{K2t/3} \right)^{1/2}$. Then to bound $W_1(\mu_t^+, \mu_t^-)$, again using Lemma 5.5, by orthonormality of $\{f_{\lambda_k}\}_k$ it follows that

$$\begin{aligned} \|f_t\|^2_{L^1(X,\mathfrak{m})} &= \left\| \sum_{\lambda_k \geq \lambda} e^{-\lambda_k t} \langle f, f_{\lambda_k} \rangle f_{\lambda_k} \right\|^2_{L^1(X,\mathfrak{m})} \leq \mathfrak{m}(X) \left\| \sum_{\lambda_k \geq \lambda} e^{-\lambda_k t} \langle f, f_{\lambda_k} \rangle f_{\lambda_k} \right\|^2_{L^2(X,\mathfrak{m})} \\ &= \mathfrak{m}(X) \sum_{\lambda_k \geq \lambda} e^{-2\lambda_k t} |\langle f, f_{\lambda_k} \rangle|^2 \leq \mathfrak{m}(X) e^{-2\lambda t} \|f\|^2_{L^2(X,\mathfrak{m})}. \end{aligned} \tag{6.1}$$

So finally

$$W_1(\mu_0^+,\mu_0^-) \le \sqrt{t}\|f\|_{L^1(X,\mathfrak{m})}C(t,K,N) + D\sqrt{\mathfrak{m}(X)}e^{-\lambda t}\|f\|_{L^2(X,\mathfrak{m})}.$$

Using that $K \ge 0$ (so $C(t,K,N) \le \sqrt{2N}$) and choosing $t = \frac{1}{\lambda}\log\left(\frac{\lambda\|f\|_{L^2(X,\mathfrak{m})}}{\|f\|_{L^1(X,\mathfrak{m})}}\right)$, it holds

$$W_1(\mu_0^+,\mu_0^-) \le C(K,N,D,\mathfrak{m}(X))\left(\frac{1}{\lambda}\log\left(\lambda\frac{\|f\|_{L^2}}{\|f\|_{L^1}}\right)\right)^{\frac{1}{2}}\|f\|_{L^1},$$

proving the claim. □

The following result is then a straightforward consequence

Corollary 6.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be a m.m.s. verifying $\mathsf{RCD}(K,N)$ and such that* diam $(X) = D < \infty$.
Let $f : X \to \mathbb{R}$ be a continuous or, alternatively, $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ such that

$$f = \sum_{\lambda_k \ge \lambda} \langle f, f_{\lambda_k}\rangle f_{\lambda_k}, \qquad \{\lambda_k\}_{k\in\mathbb{N}},\ \lambda > 0,$$

where $\{f_{\lambda_k}\}_{k\in\mathbb{N}}$ are eigenfunctions of the Laplacian of unitary L^2-norm with eigenvalue λ_k, $\langle f, f_{\lambda_k}\rangle$ is the scalar product of $L^2(X,\mathfrak{m})$, $\langle f_{\lambda_j}, f_{\lambda_k}\rangle = \delta_{j,k}$ and convergence of the series is in $L^2(X,\mathfrak{m})$.
Then the following estimate on the size of the nodal set of f holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \ge \frac{\sqrt{\lambda}}{C(K,N,D,\mathfrak{m}(X))}\log\left(\lambda\frac{\|f\|_{L^2}}{\|f\|_{L^1}}\right)^{-1/2}\cdot\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}},$$

with $C(K,N,D,\mathfrak{m}(X))$ the same constant of Lemma 6.2, provided that $\lambda \ge 2\sqrt{\mathfrak{m}(X)}$.

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