PRESCRIBING GAUSSIAN CURVATURE ON SURFACES WITH CONICAL SINGULARITIES AND GEODESIC BOUNDARY

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Abstract. We study conformal metrics with prescribed Gaussian curvature on surfaces with conical singularities and geodesic boundary in supercritical regimes. Exploiting a variational argument, we derive a general existence result for surfaces with at least two boundary components. This seems to be the first result in this setting. Moreover, we allow to have conical singularities with both positive and negative orders, that is cone angles both less and greater than 2π.

Keywords: Prescribed Gaussian curvature, conformal metrics, conical singularities, geodesic boundary, variational methods.

1. Introduction

The prescribed Gaussian curvature problem on a compact surface M under a conformal change of the metric is a classical problem in geometry dating back to Berger [9], Kazdan-Warner [26] and Chang-Yang [12, 13]. Its singular analog on surfaces with conical singularities has been already considered by Picard [40] and it was later systematically studied by Troyanov [41]. This problem has been studied for several decades and there is by now a huge literature on it, see for example [14, 15, 16, 30, 37] and the more recent results by Malchiodi and his collaborators using PDE methods [3, 4, 6, 11, 33] or by Eremenko, Mondello and Panov using a geometric argument [20, 38, 39]. See also Mazzeo-Zhu [36] for a different approach.

If M has a boundary, it is then natural to prescribe also the geodesic curvature on ∂M. For this problem we still do not have a complete picture and there are fewer results mainly concerning the regular case, see [2, 10, 13, 17, 23, 25] and the recent results by Malchiodi, Ruiz and their group [8, 18, 24, 28].

The higher dimensional analogue is the well-known problem of prescribing the scalar curvature on a manifold and mean curvature on the boundary. In particular, the scalar flat case with constant mean curvature takes the name of Escobar problem which has a deep relation with the classical Yamabe problem, see for example [1, 21, 22, 35] and the references therein.

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We are interested here in the singular flat geodesic case, namely we study conformal metrics with prescribed Gaussian curvature on surfaces with conical singularities and geodesic boundary. It seems that the only result in this direction is the one by Troyanov [41] asserting the existence of such metrics in the subcritical case, see the discussion in the sequel. The goal of this paper is to give a first existence result in the supercritical regime, which holds for a large class of singular surfaces. To state it, we need to introduce some notation.

Let g be a metric on M. A point $p \in M$ is a conical singularity of order $\alpha \in (-1, +\infty)$, or angle $\theta_\alpha = 2\pi(1+\alpha)$, for the metric g if

$$g(z) = \rho(z)|z|^{2\alpha}|dz|^2 \quad \text{locally around } p,$$

for some continuous positive function ρ. We collect the set of conical singularities p_j of orders α_j in the formal sum

$$\boldsymbol{\alpha} = \sum_{j=1}^{N} \alpha_j p_j$$

and denote by $(M, \boldsymbol{\alpha})$ the surface with that set of conical singularities. An important quantity in this study is played by the singular Euler characteristic

$$\chi(M, \boldsymbol{\alpha}) = \chi(M) + \sum_{j=1}^{N} \alpha_j.$$

Here $\chi(M)$ is the Euler characteristic of M, that is $\chi(M) = 2 - \mathfrak{g} - \mathfrak{b}$, where $\mathfrak{g}$ is the genus and $\mathfrak{b}$ is the number of boundary components of M. The critical regime of a singular surface is related to the Moser-Trudinger inequality. Following Troyanov [41] we denote the Trudinger constant of $(M, \boldsymbol{\alpha})$ by

$$\tau(M, \boldsymbol{\alpha}) = 2\left(1 + \min_j \{\alpha_j, 0\}\right)$$

and give the following definition.

Definition 1.1. *The singular surface* $(M, \boldsymbol{\alpha})$ *is:*

$$\begin{array}{ll} \textit{subcritical} & \textit{if } \chi(M, \boldsymbol{\alpha}) < \tau(M, \boldsymbol{\alpha}) \\ \textit{critical} & \textit{if } \chi(M, \boldsymbol{\alpha}) = \tau(M, \boldsymbol{\alpha}) \\ \textit{supercritical} & \textit{if } \chi(M, \boldsymbol{\alpha}) > \tau(M, \boldsymbol{\alpha}). \end{array}$$

Observe that in the supercritical case we always have $\chi(M, \boldsymbol{\alpha}) > 0$. The existence of conformal metrics with prescribed Gaussian curvature on surfaces with conical singularities and geodesic boundary (possibly with corners) in the subcritical regime has been settled down by Troyanov [41]. Indeed, in this case one is reduced to a minimization problem of a coercive functional. On the contrary, we are not aware of any result concerning the critical/supercritical case. Our main contribution is the following general existence result in the supercritical regime. We will refer to

$$\Gamma_{\boldsymbol{\alpha}} = \left\{ 4\pi n + 8\pi \sum_{j\in J} (1+\alpha_j) \,:\, n \in \mathbb{N}\cup\{0\},\, J \subseteq \{1,\dots,N\} \right\}$$

as the set of critical values. Then, the following holds.

Theorem 1.1. *Let $(M, \boldsymbol{\alpha})$ be a supercritical singular surface with $\alpha_j \geq -\frac{1}{2}$ for $j = 1,\dots,N$ and with at least two boundary components. Let K be a positive Lipschitz function on M. If $4\pi\chi(M,\boldsymbol{\alpha}) \notin \Gamma_{\boldsymbol{\alpha}}$, then there exists a conformal metric with Gaussian curvature K on $(M,\boldsymbol{\alpha})$ and geodesic boundary.*

Remark 1.2. *It is not difficult to see from the proof that we can allow to have geodesic boundary with corners. More precisely, if $\partial M = B_1 \sqcup \cdots \sqcup B_m$ we can treat the case where the boundary components B_l, $l > 1$, have corners at the points $q_j \in B_l$ of angles $\theta_{\beta_j} = 2\pi\left(\frac{1}{2} + \beta_j\right)$ with $\beta_j > 0$, see Lemma 3.7.*

Remark 1.3. *We stress that we allow to have conical singularities with both positive and negative orders, that is cone angles both less and greater than 2π, provided that $\alpha_j \geq -\frac{1}{2}$, see the discussion below.*

The argument is based on Morse theory in the spirit of [3], where the close surface (empty boundary) case is considered, by studying the Liouville PDE (1) and its associated functional J_λ given in (4). More precisely, the desired conformal metric will be realised as a min-max solution of (1), which in turns is produced by the topological changes in the structure of sublevels of J_λ. Indeed, high sublevels have trivial topology while we will show that low sublevels are non-contractible.

By means of improved Moser-Trudinger inequalities, the low sublevels can be described by some formal barycenters of M, that is family of unit measures supported in a finite number of points of $\overline{M}$. Compared to the classical Liouville equation, the difficulties are due both to the presence of conical singularities and the boundary ∂M. Indeed, the unit measures may be supported around a conical singularity or on the boundary, which makes the analysis highly non-trivial. We tackle this problem with the following rough idea: we define the *weight* of a point $p \in \overline{M}$ according to a local Moser-Trudinger inequality around that point, which gives a local volume control in terms of the Dirichlet energy, see Lemma 3.4. We get an *amount* of 8π near regular points, $8\pi\,(1+\min\{\alpha_j, 0\})$ around a singular point and 4π for points lying on the boundary ∂M (this is natural since the volume around a point on the boundary is half the volume of a point in the interior of the surface). The idea is then to retract the barycenters of M onto those of a boundary component of M, where the points have the smallest weight. Here we use the assumptions $\alpha_j \geq -\frac{1}{2}$ and the fact that the surface has at least two boundary components. Indeed, we can prove that the barycenters of a boundary component embed non-trivially into arbitrarily low sublevels of J_λ yielding non-trivial topology of the latter.

We stress once more that this idea allows us to consider conical singularities with both positive and negative orders with a simple unified approach. Up to now, positive and negative singularities have been studied separately with different arguments, see for example [3, 4, 33] for the positive case and [11] for the negative one.

The paper is organized as follows. In section 2 we introduce the Liouville PDE and in section 3 we prove the main result.

2. The Liouville equation

In this section we set up the PDE approach. Let g_0 be a smooth metric representing any given conformal structure on M and consider a new metric $g = e^u g_0$ with conical singularities at the points p_j. The curvatures then transform according to the following law:

$$\begin{cases} -\Delta u + 2K_0 = 2Ke^u & \text{in } M \setminus \{p_1, \dots, p_N\}, \\ \dfrac{\partial u}{\partial \nu} + 2h_0 = 2he^{\frac{u}{2}} & \text{on } \partial M, \end{cases} \tag{1}$$

where $\Delta = \Delta_{g_0}$ stands for the Laplace-Beltrami operator associated to the metric g_0 and ν is the outward normal vector to ∂M. Here K_0, K are the Gaussian curvatures and h_0, h are the geodesic curvature of the boundary with respect to metrics g_0 and g, respectively. The conical singularities are encoded by the behavior

$$u(z) = 2\alpha_j \log|z| + O(1) \quad \text{locally around } p_j.$$

First, we desingularize the behavior of u around the conical points by writing

$$v = u + 4\pi \sum_{j=1}^{N} \alpha_j G_{p_j},$$

where G_{p_j} is the fundamental solution of the Laplace equation on M with pole at p_j, i.e. the unique solution to

$$\begin{cases} -\Delta G_{p_j} = \delta_{p_j} - \dfrac{1}{|M|} & \text{in } M, \\ \dfrac{\partial G_{p_j}}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

with $\int_M G_{p_j} = 0$, where δ_{p_j} is the Dirac delta with pole p_j, $|M|$ is the area of M. Here and in the rest of the paper the volume form is induced by the metric g_0. Next, we can always assume that our initial metric has constant Gaussian curvature and that ∂M is geodesic, see for example Proposition 3.1 in [28]. Since we are interested in target metrics inducing geodesic boundary, we are led to consider the following problem:

$$\begin{cases} -\Delta v + 2K_0 + \dfrac{4\pi}{|M|}\displaystyle\sum_{j=1}^N \alpha_j = K_{\boldsymbol{\alpha}} e^v & \text{in } M, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{2}$$

where $K_0 \equiv const.$ and $K_{\boldsymbol{\alpha}} = 2Ke^{-4\pi\sum_{j=1}^N \alpha_j G_{p_j}}$. Observe that we have the singular behavior

$$K_{\boldsymbol{\alpha}}(z) = C|z|^{2\alpha_j}(1+O(1)) \quad \text{locally around } p_j.$$

Now, by the Gauss-Bonnet formula (recall that we have flat geodesic) one necessarily has

$$\int_M K_{\boldsymbol{\alpha}} e^v = \int_M 2K_0 + 4\pi\sum_{j=1}^N \alpha_j = 4\pi\chi(M) + 4\pi\sum_{j=1}^N \alpha_j = 4\pi\chi(M,\boldsymbol{\alpha}).$$

We can thus rewrite the problem (2) as

$$\begin{cases} -\Delta v + 2K_0 + \dfrac{4\pi}{|M|}\displaystyle\sum_{j=1}^N \alpha_j = \lambda \dfrac{K_{\boldsymbol{\alpha}} e^v}{\int_M K_{\boldsymbol{\alpha}} e^v} & \text{in } M, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{3}$$

with $\lambda = 4\pi\chi(M,\boldsymbol{\alpha})$. Therefore, the main existence result in Theorem 1.1 will follow once we prove the following result.

Theorem 2.1. *Let M be a surface with at least two boundary components. Let $\alpha_j \geq -\frac{1}{2}$ for $j = 1,\dots,N$ and let K be a positive Lipschitz function on M. If $\lambda \notin \Gamma_{\boldsymbol{\alpha}}$, then there exists a solution to* (3).

The proof will be presented in the next section and is based on the variational structure of the problem. Since the equations (3) are invariant up to an additive constant, we shall restrict ourselves to the subspace of functions with zero average

$$\overline{H}^1(M) = \left\{ v \in H^1(M) \,:\, \int_M v = 0 \right\}$$

and look for critical points of the Euler-Lagrange functional

$$J_\lambda(v) = \frac{1}{2}\int_M |\nabla v|^2 - \lambda \log \int_M K_{\boldsymbol{\alpha}} e^v, \quad v \in \overline{H}^1(M). \tag{4}$$

3. The proof of the main result

By the discussion in the previous section, the existence of a conformal metric as described in Theorem 1.1 will follow by solving the PDE (3). Therefore, we prove here Theorem 2.1 by looking at the functional J_λ given in (4). We are inspired here by the argument proposed in [3], where the closed surface case is considered. The aim will be to detect a change of topology between its sublevels

$$J_\lambda^a = \left\{ v \in \overline{H}^1(M) \,:\, J_\lambda(v) \le a \right\}.$$

We start with the following general blow-up picture, referring to [6] for what concerns the case of positive singularities, to [5] for negative singularities and to [7, 42] for boundary blow-up.

Proposition 3.1. ([5, 6, 7, 42]) *Let $(v_n)_n$ be a sequence of solutions to* (3) *with $\lambda_n \to \lambda$. Then, up to a subsequence, one of the following alternatives holds:*

1. (Compactness): *v_n are uniformly bounded.*
2. (Blow-up): $\max_{\overline{M}} v_n \to +\infty$ *and there exists a finite blow-up set $S = \{q_1, \dots, q_m\}$ such that*
$$\lambda_n \frac{K_{\boldsymbol{\alpha}} e^{v_n}}{\int_M K_{\boldsymbol{\alpha}} e^{v_n}} \rightharpoonup \sum_{j=1}^m \sigma_j \delta_{q_j}$$
in the sense of measures, where
$$\begin{cases} \sigma_j = 8\pi(1+\alpha_j) & \text{if } q_j = p_j, \\ \sigma_j = 8\pi & \text{if } q_j \in M \setminus \{p_1, \dots, p_N\}, \\ \sigma_j = 4\pi & \text{if } q_j \in \partial M. \end{cases}$$

In particular, if $\lambda \notin \Gamma_{\boldsymbol{\alpha}}$ then v_n are uniformly bounded.

The latter compactness property is needed to bypass the Palais-Smale condition, as it was shown in [29], where a deformation lemma is used to derive the following crucial result.

Lemma 3.2. *Suppose $\lambda \notin \Gamma_{\boldsymbol{\alpha}}$ and that J_λ has no critical levels inside $[a, b]$. Then, J_λ^a is a deformation retract of J_λ^b.*

Next, by Proposition 3.1, J_λ has no critical points above some high level $b \gg 0$. Therefore, the deformation Lemma 3.2 can be applied to obtain the following topological property (see also Corollary 2.8 in [31]).

Proposition 3.3. *Suppose $\lambda \notin \Gamma_{\boldsymbol{\alpha}}$. Then, there exists $b \gg 0$ such that J_λ^b is a deformation retract of $\overline{H}^1(M)$ and it is thus contractible.*

We are left with showing that the low sublevels of J_λ are non-contractible. To this end we will need improved versions of the Moser-Trudinger (Troyanov) inequality. This is done by means of a localized version of the Moser-Trudinger inequality, which is based on cut-off functions and spectral decomposition, see for example the approach in Proposition 2.3 in [34], where the Toda system is considered. This idea has its origin in [14] and has been then adapted by many authors, see for example [11] and [19] for the singular and regular case, respectively. Observe that the inequality depends on whether we are localizing it around a regular point, a conical point or at the boundary.

Lemma 3.4. ([11, 14, 19]) *Let $\delta > 0$ and let $\Omega \subset \widetilde{\Omega} \subset \overline{M}$ be such that $d\left(\Omega, \partial\widetilde{\Omega}\right) > \delta$.*

1. (Regular case): *if $d\left(\widetilde{\Omega}, p_j\right) > \delta$ for all j's and $d\left(\widetilde{\Omega}, \partial M\right) > \delta$, then there exists $C_{\varepsilon,\delta} > 0$ such that for all $v \in \overline{H}^1(M)$*
$$8\pi \log \int_\Omega K_{\boldsymbol{\alpha}} e^v \le \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla v|^2 + \varepsilon \int_M |\nabla v|^2 + C.$$
2. (Singular case): *if $p_j \in \Omega$ for some j, $d\left(\widetilde{\Omega}, p_l\right) > \delta$ for all $l \neq j$ and $d(\Omega, \partial M) > \delta$, then there exists $C_{\varepsilon,\delta} > 0$ such that for all $v \in \overline{H}^1(M)$*
$$8\pi\left(1 + \min\{\alpha_j, 0\}\right) \log \int_\Omega K_{\boldsymbol{\alpha}} e^v \le \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla v|^2 + \varepsilon \int_M |\nabla v|^2 + C.$$
3. (Boundary case): *if $d\left(\widetilde{\Omega}, p_j\right) > \delta$ for all j's, then there exists $C_{\varepsilon,\delta} > 0$ such that for all $v \in \overline{H}^1(M)$*
$$4\pi \log \int_\Omega K_{\boldsymbol{\alpha}} e^v \le \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla v|^2 + \varepsilon \int_M |\nabla v|^2 + C.$$

The above different scenarios make the Morse approach for Liouville equations in supercritical regimes quite challenging, especially in the case where both positive and negative singularities are present. Here we avoid such complexity by using the following idea: we define the *weight* of a point $p \in \overline{M}$ according to the constant in the above local Moser-Trudinger inequalities around that point, which indicates the local volume control in terms of the Dirichlet energy. Therefore, regular points have weight 8π, singular points $8\pi\left(1 + \min\{\alpha_j, 0\}\right)$ and boundary points 4π. The idea is to focus on points with the smallest weight, that is points on the boundary (recall $\alpha_j \ge -\frac{1}{2}$), through a suitable retraction, see the discussion later on. We will indeed show that the boundary generates non-trivial topology of the low sublevels of J_λ.

To this end, we start by observing that, in Lemma 3.4, a concentration of the conformal volume $K_{\boldsymbol{\alpha}} e^v$ in Ω, in the sense

$$\int_\Omega \frac{K_{\boldsymbol{\alpha}} e^v}{\int_M K_{\boldsymbol{\alpha}} e^v} \geq \gamma, \quad \text{for some } \gamma > 0,$$

would give a global volume control in terms of the Dirichlet energy. We conclude that whenever $K_{\boldsymbol{\alpha}} e^v$ is concentrated in different regions of $\overline{M}$ an improved Moser-Trudinger inequality holds just by summing up the local inequalities. Improved inequalities in turn give improved lower bounds on the functional J_λ. Therefore, in the low sublevels, $K_{\boldsymbol{\alpha}} e^v$ cannot be concentrated in too many different regions, i.e. we have the following property. Here we have just to observe that, since $\alpha_j \geq -\dfrac{1}{2}$, any local Moser-Trudinger inequality in Lemma 3.4 gives a volume control of at least 4π.

Lemma 3.5. *Suppose $\lambda < 4(k+1)\pi$. Then, for any $\varepsilon, r > 0$, there exists $L = L(\varepsilon, r) > 0$ such that for any $v \in J_\lambda^{-L}$ there exist at most k points $\{q_1, \dots, q_k\} \subset \overline{M}$ such that*

$$\int_{\bigcup_{i=1}^k B_r(q_i)} \frac{K_{\boldsymbol{\alpha}} e^v}{\int_M K_{\boldsymbol{\alpha}} e^v} \geq 1 - \varepsilon. \tag{5}$$

Proof. Assume (5) does not hold. Then, by a standard covering lemma (see for instance [31], Lemma 3.3), there exists $\delta > 0$ and $\Omega_1, \dots, \Omega_{k+1} \subset \overline{M}$ such that

$$d(\Omega_i, \Omega_j) \geq 2\delta, \ \forall i \neq j \qquad \int_{\Omega_j} \frac{K_{\boldsymbol{\alpha}} e^v}{\int_M K_{\boldsymbol{\alpha}} e^v} \geq \delta.$$

For any $j = 1, \dots, k+1$ we apply Lemma 3.4 with $\Omega = \Omega_j, \widetilde{\Omega} = B_\delta(\Omega_j)$. In the *boundary case*, we get

$$\begin{aligned} 4\pi \log \int_M K_{\boldsymbol{\alpha}} e^v &\leq 4\pi \log \int_{\Omega_j} K_{\boldsymbol{\alpha}} e^v + 4\pi \log \frac{1}{\delta} \\ &\leq \frac{1}{2} \int_{B_\delta(\Omega_j)} |\nabla v|^2 + \varepsilon \int_M |\nabla v|^2 + C. \end{aligned}$$

In the *regular case*, since Jensen's inequality gives $\log \int_M K_{\boldsymbol{\alpha}} e^v \geq -C$, then

$$\begin{aligned} 4\pi \log \int_M K_{\boldsymbol{\alpha}} e^v &\leq 8\pi \log \int_M K_{\boldsymbol{\alpha}} e^v + C \\ &\leq 8\pi \log \int_{\Omega_j} K_{\boldsymbol{\alpha}} e^v + C \\ &\leq \frac{1}{2} \int_{B_\delta(\Omega_j)} |\nabla v|^2 + \varepsilon \int_M |\nabla v|^2 + C. \end{aligned}$$

The same computation holds true in the *singular case*, since $8\pi\,(1 + \min\{\alpha_j, 0\}) \geq 4\pi$; therefore, summing on all j's and taking account that $B_\delta(\Omega_i) \cap B_\delta(\Omega_j) =$

$\emptyset$ for $i \neq j$, we get

$$4(k+1)\pi \log \int_M K_{\boldsymbol{\alpha}} e^v \leq \left(\frac{1}{2} + k\varepsilon\right) \int_M |\nabla v|^2 + C.$$

It follows that

$$J_\lambda(v) \geq \frac{1}{2}\left(1 - \frac{\lambda}{4(k+1)\pi}(1+2k\varepsilon)\right) \int_M |\nabla v|^2 - C.$$

Since $\lambda < 4(k+1)\pi$, we can choose $\varepsilon > 0$ such that $4(k+1)\pi = \lambda(1+2k\varepsilon)$ and get $J_\lambda(v) \geq -L$ for some $L > 0$, which concludes the proof. $\square$

This naturally leads us to describe the low sublevels by unit measures supported in (at most) k points of $\overline{M}$, known as the formal barycenters of M of order k:

$$M_k = \left\{ \sum_{i=1}^k t_i \delta_{q_i} \,:\, \sum_{i=1}^k t_i = 1, t_i \geq 0, q_i \in \overline{M}, \forall\, i = 1, \dots, k \right\}. \tag{6}$$

Indeed, one can project the measure $\dfrac{K_{\boldsymbol{\alpha}} e^v}{\int_M K_{\boldsymbol{\alpha}} e^v}$ on the closest element in M_k, see Lemma 4.9 in [19].

Proposition 3.6. *Suppose $\lambda < 4(k+1)\pi$. Then, there exists a projection $\Psi : J_\lambda^{-L} \to M_k$, for some $L \gg 0$.*

Now, to restrict our target on barycenters supported only on the boundary, we need the following result. We recall that M is assumed to have at least two boundary components and we write

$$\partial M = B_1 \sqcup \cdots \sqcup B_m,$$

with $m > 1$, where $B_i \simeq \mathbb{S}^1$.

Lemma 3.7. *Suppose $\lambda < 4(k+1)\pi$. Then, there exists a map $\Psi_\Pi : J_\lambda^{-L} \to (B_1)_k$, for some $L \gg 0$.*

Proof. We start by defining a global retraction $\Pi : M \to B_1$. To this end, consider the space $\mathbb{R}^3 \ni (x, y, z)$ and the projection $P : \mathbb{R}^3 \to \{z = 0\}$. We point out that any two compact surfaces with the same genus and same number of boundary components are homeomorphic. Therefore, we can assume without loss of generality that M is embedded in $\mathbb{R}^3$ such that in the holes B_1 and B_2 passes the same line parallel to the z-axis such that $P(M)$ is a disk with at least one hole H with $\partial H = P(B_1)$, see Figure 1. Therefore, there exists a retraction $R : P(M) \to P(B_1)$ which induces a retraction $\Pi : M \to B_1$.

Now, by Proposition 3.6 there exists a projection $\Psi : J_\lambda^{-L} \to M_k$, for some $L \gg 0$. The desired map $\Psi_\Pi : J_\lambda^{-L} \to (B_1)_k$ is then defined through the composition

$$J_\lambda^{-L} \xrightarrow{\Psi} M_k \xrightarrow{\Pi_*} (B_1)_k,$$

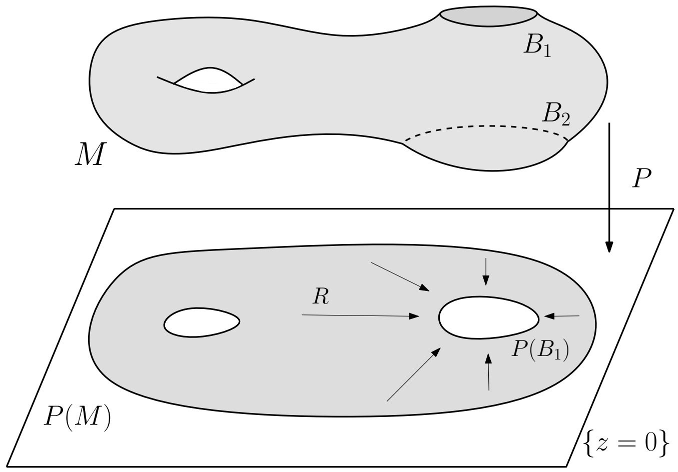

FIGURE 1. The construction of the retraction $\Pi : M \to B_1$.

where Π_* denotes here the push-forward of measures induced by the above retraction. □

To gain some topological properties of the low sublevels we construct now a reverse map $\Phi : (B_1)_k \to J_\lambda^{-L}$. To this end, we consider a family of regular bubbles centered at the boundary component B_1 and we set, for $\Lambda > 0$, $\Phi : (B_1)_k \to \overline{H}^1(M)$ as

$$\Phi : \sigma = \sum_{i=1}^{k} t_i \delta_{q_i} \quad \mapsto \quad \varphi_{\Lambda,\sigma} - \overline{\varphi}_{\Lambda,\sigma}, \tag{7}$$

where

$$\varphi_{\Lambda,\sigma}(y) = \log \sum_{i=1}^{k} t_i \left(\frac{\Lambda}{1 + \Lambda^2 d(y, q_i)^2} \right)^2,$$

and $\overline{\varphi}_{\Lambda,\sigma}$ is the average of $\varphi_{\Lambda,\sigma}$. Then, the following estimates hold true.

Lemma 3.8. *Let $\varphi_{\Lambda,\sigma}$, $\sigma \in (B_1)_k$ be the functions defined above. Then, for $\Lambda \to +\infty$ we have*

$$\begin{aligned} \frac{1}{2} \int_M |\nabla \varphi_{\Lambda,\sigma}|^2 &= 8k\pi(1 + o(1)) \log \Lambda, \\ \log \int_M K_{\boldsymbol{\alpha}} e^{\varphi_{\Lambda,\sigma} - \overline{\varphi}_{\Lambda,\sigma}} &= 2(1 + o(1)) \log \Lambda. \end{aligned}$$

Moreover,

$$\frac{K_{\boldsymbol{\alpha}} e^{\varphi_{\Lambda,\sigma}}}{\int_M K_{\boldsymbol{\alpha}} e^{\varphi_{\Lambda,\sigma}}} \rightharpoonup \sigma \in (B_1)_k, \tag{8}$$

in the sense of measures.

The latter estimates are by now standard and we refer for example to Proposition 4.2 in [32] where the regular close surface case is considered. The presence of the boundary can be handled with obvious modifications. Indeed, the only difference is the main contribution of the Dirichlet energy (8) comes from half-balls around the centers of the bubbles and it is thus divided by a factor 2. Observe that we can neglect the effect of the singularities since we are considering bubbles centered on the boundary component B_1 which does not have conical points.

By plugging the above estimates into the functional J_λ it is then easy to conclude the following.

Proposition 3.9. *Suppose $\lambda > 4k\pi$ and let Φ be given as in* (7)*. Then, for any $L > 0$ there exists $\Lambda \gg 0$ such that $\Phi : (B_1)_k \to J_\lambda^{-L}$.*

Proof. Indeed, we have

$$J_\lambda(\Phi(\sigma)) \leq (8k\pi - 2\lambda + o(1)) \log \Lambda$$

and since $\lambda > 4k\pi$ the thesis follows by choosing suitably $\Lambda \gg 0$. □

We can now prove the main result.

Proof of Theorem 2.1. Take $\lambda \in (4k\pi, 4(k+1)\pi) \setminus \Gamma_{\boldsymbol{\alpha}}$. By Lemma 3.7 there exists a map $\Psi_\Pi : J_\lambda^{-L} \to (B_1)_k$, for some $L \gg 0$. On the other hand, by Proposition 3.9 we have a map $\Phi : (B_1)_k \to J_\lambda^{-L}$. Such maps are natural in the sense that the composition

$$\begin{array}{ccccc} (B_1)_k & \xrightarrow{\Phi} & J_\lambda^{-L} & \xrightarrow{\Psi_\Pi} & (B_1)_k \\ \sigma & \mapsto & \left(\varphi_{\lambda,\sigma} - \overline{\varphi}_{\lambda,\sigma}\right) & \mapsto & \dfrac{K_{\boldsymbol{\alpha}} e^{\varphi_{\lambda,\sigma}}}{\int_M K_{\boldsymbol{\alpha}} e^{\varphi_{\lambda,\sigma}}} \simeq \sigma \end{array}$$

is homotopic to the identity on $(B_1)_k$. Here we recall (8) holds true. We refer to Proposition 4.4 in [32] for more details on this point. Passing to the induced maps Φ^*, Ψ_Π^* between homological groups H_* we derive $\Psi_\Pi^* \circ \Phi^* = \mathrm{Id}^*_{(B_1)_k}$. In particular,

$$H_*((B_1)_k) \hookrightarrow H_*\left(J_\lambda^{-L}\right)$$

injectively. Since $(B_1)_k \simeq \left(\mathbb{S}^1\right)_k \simeq \mathbb{S}^{2k-1}$, see for example Proposition 3.2 in [4], we deduce that J_λ^{-L} is not contractible. But J_λ^b is contractible for $b \gg 0$ by Proposition 3.3. The existence of a solution to (3) follows by Lemma 3.2. □

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