OPTIMAL FREE EXPORT/IMPORT REGIONS

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ABSTRACT. We consider the problem of finding two free export/import sets E^+ and E^- that minimize the total cost of some export/import transportation problem (with export/import taxes $g^\pm$), between two densities f^+ and f^-, plus penalization terms on E^+ and E^-. First, we prove existence of such optimal sets under some assumptions on $f^\pm$ and $g^\pm$. Then, we study some properties of these sets such as convexity and regularity. In particular, we show that the optimal free export (resp. import) region E^+ (resp. E^-) has boundary of class C^2 as soon as f^+ (resp. f^-) is continuous and ∂E^+ (resp. ∂E^-) is $C^{2,1}$ provided that f^+ (resp. f^-) is Lipschitz.

1. INTRODUCTION

In this paper we study a shape optimization problem where the functional to be minimized is given by an export/import transportation problem with free export/import zones. Before entering the details of this problem, let us introduce the standard export/import transportation problem. Let f^+ and f^- be two given masses in some bounded region Ω and assume that we want to transport f^+ to f^- paying a transport cost $|x-y|$, for each unit of mass that moves from x to y. But, as the total mass of f^+ can be different than the one for f^-, we are allowed to export or import masses from the boundary $\partial\Omega$ paying two additional costs on the boundary (called *export/import taxes*) $g^+(x)$ and $g^-(y)$, for each unit of mass that comes out/enters at some point of $\partial\Omega$. We note that this problem has already been considered in many papers [8, 3, 4, 5, 2]. In other words, we consider the following problem

$$\min_{\gamma\in\Pi(f^+,f^-)}\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega}g^+(y)\,\mathrm{d}(\Pi_y)_\#\gamma(y)+\int_{\partial\Omega}g^-(x)\,\mathrm{d}(\Pi_x)_\#\gamma(x)\right\}, \tag{1.1}$$

where

$$\Pi(f^+,f^-):=\left\{\gamma\in\mathcal{M}^+(\Omega\times\Omega)\,:\,[(\Pi_x)_\#\gamma]_{|\mathring{\Omega}}=f^+,\,[(\Pi_y)_\#\gamma]_{|\mathring{\Omega}}=f^-\right\}.$$

In [3, 8], the authors proved, using two different approachs, that Problem (1.1) has a dual formulation which is the following

$$\sup\left\{\int_\Omega u(f^+-f^-)\,\mathrm{d}x\,:\,u\in\mathrm{Lip}_1(\Omega),\,-g^-\le u\le g^+\ \text{ on }\ \partial\Omega\right\}.$$

Moreover, this problem has an equivalent minimal flow formulation (see [3, 4]):

$$\min_{\sigma\in\mathcal{M}^d(\Omega),\,\chi\in\mathcal{M}(\partial\Omega)}\left\{\int_\Omega|\sigma|\,\mathrm{d}x+\int_{\partial\Omega}g^+\,\mathrm{d}\chi^++\int_{\partial\Omega}g^-\,\mathrm{d}\chi^-:\nabla\cdot\sigma=f+\chi\ \text{ in }\ \Omega\right\}. \tag{1.2}$$

Now, assume that we have two regions E^+ and E^- inside Ω where the export/import transport is free of charge, i.e. there are no taxes on these special regions. Then, the problem (1.1) becomes

$$W(E^+,E^-)$$

(1.3)
$$= \min_{\gamma\in\Pi(f^+,f^-)} \left\{ \int_{\Omega\times\Omega} |x-y| \, \mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y) \, \mathrm{d}\gamma(x,y) + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x) \, \mathrm{d}\gamma(x,y) \right\}.$$

Then, our shape optimization problem consists of minimizing the total cost $W(E^+,E^-)$ of this export/import transport problem (1.3) plus some penalties on $E^\pm$ (such as paying a cost proportional to the perimeter of $E^\pm$) among all subsets $E^\pm \subset \Omega$, i.e. we minimize

$$\min\left\{W(E^+,E^-) + Per(E^+) + Per(E^-) \ : \ E^\pm \subset \Omega\right\}. \tag{1.4}$$

We note that the authors of [1] have already considered a shape optimization problem slightly similar to Problem (1.4); let us give a brief description of their problem. Let f^+ and f^- be two densities (having the same total mass) in some region Ω and assume that the traffic congestion in $\Omega\setminus E$ is higher than the one in E, so their aim was to find a set E that minimizes

$$\min\left\{\mathcal{J}(E) + Per(E) \ : \ E\subset\Omega\right\}, \tag{1.5}$$

where

$$\mathcal{J}(E) := \min\left\{ \int_E H_1(\sigma)\,\mathrm{d}x + \int_{\Omega\setminus E} H_2(\sigma)\,\mathrm{d}x \ : \ \nabla\cdot\sigma = f^+ - f^- \ \text{in} \ \Omega, \ \sigma\cdot n = 0 \ \text{on} \ \partial\Omega\right\},$$

where H_1 and H_2 are two continuous superlinear convex functions such that $0 \le H_1 \le H_2$. They proved that there exists at least an optimal set E for Problem (1.5). On the contrary, an optimal set E may fail to exist if we replace $Per(E)$ with $|E|$, i.e. the problem

$$\min\left\{\mathcal{J}(E) + |E| \ : \ E\subset\Omega\right\} \tag{1.6}$$

may have no solution. For this reason, they considered instead a relaxed formulation of Problem (1.6) (i.e. with a function $0\le\theta\le 1$ instead of E). More precisely, they showed that the optimal choice for θ is to have $\theta = 0$ on some region E_0 (which represents a high-congestion area), $\theta = 1$ on another region E_1 (a low-congestion area) and $0<\theta<1$ on $\Omega\setminus(E_0\cup E_1)$ (an intermediate congestion area). That is why we consider here a penalization with perimeter, since otherwise it is not clear how to prove existence and even, a solution may not exist.

Coming back to our shape optimization problem, we can also consider a more general version of Problem (1.4): assume that the export/import transport on $E^\pm$ is not free, but in order to export some mass from E^+, we pay a cost g_0^+ while outside of E^+ we pay a higher cost g_1^+, and to import some mass to E^-, we pay a cost g_0^- while to $\Omega\setminus E^-$ we pay a higher cost g_1^-. In other words, we minimize

$$\min\left\{\tilde{W}(E^+,E^-) + Per(E^+) + Per(E^-) \ : \ E^\pm \subset \Omega\right\},$$

where

$$\tilde{W}(E^+,E^-)$$
$$= \min_{\gamma\in\Pi(f^+,f^-)} \left\{ \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{E^+\times\partial\Omega} g_0^+(y)\,\mathrm{d}\gamma(x,y) + \int_{(\Omega\setminus E^+)\times\partial\Omega} g_1^+(y)\,\mathrm{d}\gamma(x,y) \right.$$
$$\left. + \int_{\partial\Omega\times E^-} g_0^-(x)\,\mathrm{d}\gamma(x,y) \ + \int_{\partial\Omega\times(\Omega\setminus E^-)} g_1^-(x)\,\mathrm{d}\gamma(x,y)\right\}.$$

For simplicity of exposition, we will assume that $g_0^{\pm} = 0$ on $\partial\Omega$. In fact, all the results about the convexity of the optimal free export/import regions hold true in the general case under the assumption that $g_0^{\pm} \leq g_1^{\pm}$ on $\partial\Omega$.

The aim of this paper is to show existence of two optimal export/import sets E^+_{opt} and E^-_{opt} for Problem (1.4) and then, to study some properties of these optimal sets. More precisely, we prove that E^+_{opt} and E^-_{opt} are uniformly convex under some assumptions on $f^{\pm}$ and $g^{\pm}$. Moreover, we will study the regularity of $E^{\pm}_{opt}$; in particular, we will show that $E^{\pm}_{opt}$ is smooth (say $C^{2,1}$) inside $\mathrm{spt}(f^{\pm})$ as soon as $f^{\pm}$ is Lipschitz.

2. Export/Import transport problem with free export/import regions

Let f^+ and f^- be two nonnegative Borel measures on a compact domain $\Omega \subset \mathbb{R}^d$ such that $\mathrm{spt}(f^{\pm}) \subset \mathring{\Omega}$. Let $g^{\pm} : \partial\Omega \mapsto \mathbb{R}^+$ be two given functions. Let $E^{\pm}$ be two subsets of Ω, then we consider the problem

(2.1)
$$\min_{\gamma\in\Pi(f^+,f^-)} \left\{ \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma(x,y) + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x)\,\mathrm{d}\gamma(x,y) \right\}.$$

We have the following:

Proposition 2.1. *Assume that $f^{\pm} \in L^1(\Omega)$ and $g^{\pm} \in C(\partial\Omega)$. Then, the problem (2.1) reaches a minimum.*

Proof. Let $(\gamma_n)_n \subset \Pi(f^+, f^-)$ be a minimizing sequence. Then, it is clear that we can assume that
$$\gamma_n(\partial\Omega \times \partial\Omega) = 0.$$
In this case, we get
$$\begin{aligned} \gamma_n(\Omega\times\Omega) &\leq \gamma_n(\mathring{\Omega}\times\Omega) + \gamma_n(\Omega\times\mathring{\Omega}) \\ &= f^+(\Omega) + f^-(\Omega). \end{aligned}$$
Hence, up to a subsequence, $\gamma_n \rightharpoonup \gamma$ for some $\gamma \in \Pi(f^+, f^-)$. We define the three parts of γ_n as follows
$$\gamma_n^{ii} := {\gamma_n}_{|\mathring{\Omega}\times\mathring{\Omega}},\ \gamma_n^{ib} := {\gamma_n}_{|\mathring{\Omega}\times\partial\Omega},\ \gamma_n^{bi} := {\gamma_n}_{|\partial\Omega\times\mathring{\Omega}}.$$
Yet, we see that $\gamma_n^{ii} \rightharpoonup \gamma_1$, $\gamma_n^{ib} \rightharpoonup \gamma_2$ and $\gamma_n^{bi} \rightharpoonup \gamma_3$ such that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. Moreover, we have $\mathrm{spt}(\gamma_1) \subset \mathring{\Omega}\times\mathring{\Omega}$, $\mathrm{spt}(\gamma_2) \subset \mathring{\Omega}\times\partial\Omega$ and $\mathrm{spt}(\gamma_3) \subset \partial\Omega\times\mathring{\Omega}$. This implies that $\gamma_1 = \gamma^{ii}$, $\gamma_2 = \gamma^{ib}$ and $\gamma_3 = \gamma^{bi}$. On the other hand, let us devide γ_n^{ib} in two parts
$$\gamma_n^{ib} = \gamma_n \cdot 1_{E^+\times\partial\Omega} + \gamma_n \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega}.$$
We see that $\gamma_n \cdot 1_{E^+\times\partial\Omega} \rightharpoonup \gamma_4$ with $\mathrm{spt}(\gamma_4) \subset \overline{E^+}\times\partial\Omega$ and $\gamma_n \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega} \rightharpoonup \gamma_5$ with $\mathrm{spt}(\gamma_5) \subset \overline{(\Omega\setminus E^+)}\times\partial\Omega$ such that
$$\gamma^{ib} = \gamma_4 + \gamma_5.$$
Yet, $f^+ \in L^1(\Omega)$ and $(\Pi_x)_{\#}\gamma^{ib} \leq (\Pi_x)_{\#}(\gamma \cdot 1_{\mathring{\Omega}\times\Omega}) = f^+$, then $\gamma^{ib}(\partial E^+ \times \partial\Omega) = 0$. Hence, we get that
$$\gamma_n \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega} \rightharpoonup \gamma \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega}.$$

Similarly, thanks to the fact that $f^- \in L^1(\Omega)$ and $(\Pi_y)_\#\gamma^{bi} \leq (\Pi_y)_\#(\gamma \cdot 1_{\Omega\times\mathring{\Omega}}) = f^-$, one can prove that

$$\gamma_n \cdot 1_{\partial\Omega\times(\Omega\backslash E^-)} \rightharpoonup \gamma \cdot 1_{\partial\Omega\times(\Omega\backslash E^-)}.$$

Consequently,

$$\liminf_n \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma_n + \int_{(\Omega\backslash E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma_n(x,y) + \int_{\partial\Omega\times(\Omega\backslash E^-)} g^-(x)\,\mathrm{d}\gamma_n(x,y)$$
$$= \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\backslash E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma(x,y) + \int_{\partial\Omega\times(\Omega\backslash E^-)} g^-(x)\,\mathrm{d}\gamma(x,y).$$

This implies that the transport plan $\gamma \in \Pi(f^+, f^-)$ is in fact a minimizer for the problem (2.1). $\square$

Let γ_{opt} be an optimal transport plan for Problem (2.1). We devide γ_{opt} into three parts γ^{ii}_{opt}, γ^{ib}_{opt} and γ^{bi}_{opt}, where

$$\gamma^{ii}_{opt} = \gamma_{opt} \cdot 1_{\mathring{\Omega}\times\mathring{\Omega}},\ \gamma^{ib}_{opt} = \gamma_{opt} \cdot 1_{\mathring{\Omega}\times\partial\Omega} \ \text{ and } \ \gamma^{bi}_{opt} = \gamma_{opt} \cdot 1_{\partial\Omega\times\mathring{\Omega}}.$$

Set

$$\nu^+_{opt} := (\Pi_x)_\#(\gamma^{ib}_{opt}) \quad \text{and} \quad \nu^-_{opt} := (\Pi_y)_\#(\gamma^{bi}_{opt}).$$

We consider the three following problems:

$$\text{(P1)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma \,:\, \gamma \in \Pi(f^+ - \nu^+_{opt}, f^- - \nu^-_{opt})\right\},$$

$$\text{(P2)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma \,+\, \int_{(\Omega\backslash E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma \,:\, (\Pi_x)_\#\gamma = \nu^+_{opt},\, \mathrm{spt}((\Pi_y)_\#\gamma) \subset \partial\Omega\right\},$$

$$\text{(P3)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma \,+\, \int_{\partial\Omega\times(\Omega\backslash E^-)} g^-(x)\,\mathrm{d}\gamma : (\Pi_y)_\#\gamma = \nu^-_{opt},\, \mathrm{spt}((\Pi_x)_\#\gamma) \subset \partial\Omega\right\}.$$

Then, we have the following

Proposition 2.2. *The plans γ^{ii}_{opt}, γ^{ib}_{opt} and γ^{bi}_{opt} solve (P1), (P2) and (P3), respectively. In addition, $\gamma^{ib}_{opt} = (Id, T_{E^+})_\#\nu^+_{opt}$ where T_{E^+} is the Borel selector function of the multivalued map*

$$T_{E^+}(x) = \begin{cases} P(x) := \mathit{argmin}\{|x-y| \,:\, y \in \partial\Omega\}, & \text{if } x \in E^+, \\ T^+(x) := \mathit{argmin}\{|x-y| + g^+(y) \,:\, y \in \partial\Omega\}, & \text{if } x \in \Omega\backslash E^+. \end{cases}$$

On the other hand, $\gamma^{bi}_{opt} = (T_{E^-}, Id)_\#\nu^-_{opt}$ where T_{E^-} is the following Borel selector function

$$T_{E^-}(y) = \begin{cases} P(y) := \mathit{argmin}\{|x-y| \,:\, x \in \partial\Omega\}, & \text{if } y \in E^-, \\ T^-(y) := \mathit{argmin}\{|x-y| + g^-(x) \,:\, x \in \partial\Omega\}, & \text{if } y \in \Omega\backslash E^-. \end{cases}$$

Proof. This follows immediately from the fact that γ^{ii}_{opt}, γ^{ib}_{opt} and γ^{bi}_{opt} are admissible in (P1), (P2) and (P3), respectively. Moreover, if γ_1, γ_2 and γ_3 minimize (P1), (P2) and (P3), respectively, then $\gamma_1 + \gamma_2 + \gamma_3$ minimizes Problem (2.1) and so, γ^{ii}_{opt}, γ^{ib}_{opt} and γ^{bi}_{opt} minimize (P1), (P2) and

(P3), respectively. On the other hand, if $\gamma \neq (Id, T_{E^+})_\# \nu^+_{opt}$ is admissible in (P2), then we have

$$\int_{\Omega\times\partial\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma = \int_{E^+\times\partial\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} \left[|x-y| + g^+(y)\right]\,\mathrm{d}\gamma$$

$$> \int_{E^+\times\partial\Omega} |x - T_{E^+}(x)|\,\mathrm{d}\gamma \; + \int_{(\Omega\setminus E^+)\times\partial\Omega} \left[|x - T_{E^+}(x)| + g^+(T_{E^+}(x))\right]\,\mathrm{d}\gamma.$$

This shows that $\gamma^{ib}_{opt} = (Id, T_{E^+})_\# \nu^+_{opt}$. Similarly, we also get that $\gamma^{bi}_{opt} = (T_{E^-}, Id)_\# \nu^-_{opt}$. □

We conclude this section by the following

Proposition 2.3. *Assume $f^\pm \in L^1(\Omega)$. Let γ_{opt} be an optimal transport plan for (2.1) and consider the transport plans γ^{ii}_{opt}, γ^{ib}_{opt} and γ^{bi}_{opt}. Let $\nu^+_{opt} := (\Pi_x)_\#\gamma^{ib}_{opt}$ and $\nu^-_{opt} := (\Pi_y)_\#\gamma^{bi}_{opt}$. Then, there are two sets $E^\pm \subset \mathrm{spt}(f^\pm)$ such that $\nu^\pm_{opt} = f^\pm \cdot 1_{E^\pm}$.*

Proof. Assume that this is not the case for ν^+_{opt} (of course, the proof will be the same for ν^-_{opt}), i.e. there is some set A such that $0 < \nu^+_{opt} < f^+$ on A. This means that on A we split the mass f^+ in two parts: one is going to $f^- - \nu^-_{opt}$ and the second one is exported to $\partial\Omega$. But, this is a contradiction since $|A| > 0$ while it is well known that the set of double points (that is the set of points whose belong to different transport rays) is negligible for the Lebesgue measure (see [9]). □

3. Properties of the optimal free export/import regions

In this section, we prove existence of optimal free export/import sets E^+_{opt} and E^-_{opt} for Problem (1.4). Then, we introduce some regularity results on $E^\pm_{opt}$. Fix $\lambda^\pm > 0$, then we consider the following shape optimization problem

$$\min\left\{W(E^+,E^-) + \lambda^+ Per(E^+) + \lambda^- Per(E^-)\,:\, E^\pm \subset \Omega\right\}, \tag{3.1}$$

where

$$W(E^+,E^-)$$
$$= \min_{\gamma\in\Pi(f^+,f^-)} \left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma(x,y) + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x)\,\mathrm{d}\gamma(x,y)\right\}.$$

First, we have the following

Proposition 3.1. *Assume that $f^\pm \in L^p(\Omega)$, for some $p > 1$, and $g^\pm \in C(\partial\Omega)$. Then, there exist two optimal free export/import regions E^+_{opt} and E^-_{opt} for Problem (3.1).*

Proof. Let $(E^+_k, E^-_k)_k$ be a minimizing sequence in Problem (3.1). For each k, let γ_k be an optimal transport plan for $W(E^+_k, E^-_k)$ (see Proposition 2.1). Recall that $\gamma_k \in \Pi(f^+, f^-)$ and one can assume that γ_k gives zero mass to $\partial\Omega\times\partial\Omega$, so there is some constant C such that $\gamma_k(\Omega\times\Omega) \le C$, for every k. As $g^\pm \in C(\partial\Omega)$, so we have $W(E^+_k, E^-_k) \ge -C||g||_\infty$, for all k. Then, we get that

$$Per(E^\pm_k) \le C, \;\text{ for every } k.$$

Yet, $E^\pm_k \subset \Omega$, for every k, and Ω is bounded. Hence, up to a subsequence, $(E^+_k, E^-_k) \to (E^+, E^-)$ in L^1, where E^+ and E^- are two subsets of Ω. In particular, we have

$$Per(E^\pm) \le \liminf_k Per(E^\pm_k).$$

Moreover, we have

$$W(E_k^+,E_k^-)=\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma_k+\int_{(\Omega\setminus E_k^+)\times\partial\Omega}g^+(y)\,\mathrm{d}\gamma_k(x,y)+\int_{\partial\Omega\times(\Omega\setminus E_k^-)}g^-(x)\,\mathrm{d}\gamma_k(x,y).$$

Set

$$\nu_k^+ := (\Pi_x)_\#(\gamma_k\cdot 1_{\Omega\times\partial\Omega}) \quad \text{and} \quad \nu_k^- := (\Pi_y)_\#(\gamma_k\cdot 1_{\partial\Omega\times\Omega}).$$

From Proposition 2.2, we know that

$$<\gamma_k\cdot 1_{\Omega\times\partial\Omega},\phi>=\int_{E_k^+}\phi(x,P(x))\,\mathrm{d}\nu_k^+ \;+\int_{\Omega\setminus E_k^+}\phi(x,T^+(x))\,\mathrm{d}\nu_k^+,\;\;\text{for all}\;\;\phi\in C(\Omega\times\Omega),$$

and

$$<\gamma_k\cdot 1_{\partial\Omega\times\Omega},\psi>=\int_{E_k^-}\psi(P(y),y)\,\mathrm{d}\nu_k^- \;+\int_{\Omega\setminus E_k^-}\psi(T^-(y),y)\,\mathrm{d}\nu_k^-,\;\;\text{for all}\;\;\psi\in C(\Omega\times\Omega).$$

Yet, up to a subsequence, we see that $\gamma_k \rightharpoonup \gamma$ (in the sense of measures) for some $\gamma\in\Pi(f^+,f^-)$ and $\nu_k^\pm \rightharpoonup \nu^\pm$ in L^p, where

$$<\gamma\cdot 1_{\Omega\times\partial\Omega},\phi>=\int_{E^+}\phi(x,P(x))\,\mathrm{d}\nu^+ \;+\int_{\Omega\setminus E^+}\phi(x,T^+(x))\,\mathrm{d}\nu^+,\;\;\text{for all}\;\;\phi\in C(\Omega\times\Omega)$$

and

$$<\gamma\cdot 1_{\partial\Omega\times\Omega},\psi>=\int_{E^-}\psi(P(y),y)\,\mathrm{d}\nu^- \;+\int_{\Omega\setminus E^-}\psi(T^-(y),y)\,\mathrm{d}\nu^-,\;\;\text{for all}\;\;\psi\in C(\Omega\times\Omega).$$

Consequently, we have

$$\begin{aligned}W(E^+,E^-) &\leq \int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{(\Omega\setminus E^+)\times\partial\Omega}g^+(y)\,\mathrm{d}\gamma(x,y)+\int_{\partial\Omega\times(\Omega\setminus E^-)}g^-(x)\,\mathrm{d}\gamma(x,y)\\ &= \liminf_k W(E_k^+,E_k^-).\end{aligned}$$

Finally, we infer that

$$W(E^+,E^-)+\lambda^+Per(E^+)+\lambda^-Per(E^-)\leq\liminf_k\left[W(E_k^+,E_k^-)+\lambda^+Per(E_k^+)+\lambda^-Per(E_k^-)\right],$$

which means that the pair (E^+,E^-) minimizes Problem (3.1). □

On the other hand, we have the following monotonicity property.

Proposition 3.2. *For $\lambda_1^\pm$, $\lambda_2^\pm>0$, let $(E^+_{\lambda_1^+},E^-_{\lambda_1^-})$ and $(E^+_{\lambda_2^+},E^-_{\lambda_2^-})$ be the corresponding optimal free export/import regions for Problem (1.4) with $\lambda^\pm=\lambda_1^\pm$ and $\lambda^\pm=\lambda_2^\pm$, respectively. Let γ_{λ_1} and γ_{λ_2} be two optimal transport plans for $W(E^+_{\lambda_1^+},E^-_{\lambda_1^-})$ and $W(E^+_{\lambda_2^+},E^-_{\lambda_2^-})$, respectively. Set $\nu^+_{\lambda_1}=(\Pi_x)_\#(\gamma_{\lambda_1}\cdot 1_{\Omega\times\partial\Omega})$, $\nu^+_{\lambda_2}=(\Pi_x)_\#(\gamma_{\lambda_2}\cdot 1_{\Omega\times\partial\Omega})$, $\nu^-_{\lambda_1}=(\Pi_y)_\#(\gamma_{\lambda_1}\cdot 1_{\partial\Omega\times\Omega})$ and $\nu^-_{\lambda_2}=(\Pi_y)_\#(\gamma_{\lambda_2}\cdot 1_{\partial\Omega\times\Omega})$. Assume that $\frac{\lambda_2^\pm}{\lambda_1^\pm}<\frac{\min g^\pm}{\max g^\pm}$ and $\operatorname{spt}(\nu^\pm_{\lambda_1})\subset\operatorname{spt}(\nu^\pm_{\lambda_2})$, then we have $E^\pm_{\lambda_1^\pm}\subset E^\pm_{\lambda_2^\pm}$.*

Proof. From the optimality of $(E^+_{\lambda_1^+}, E^-_{\lambda_1^-})$ and $(E^+_{\lambda_2^+}, E^-_{\lambda_2^-})$ in Problem (1.4) with $\lambda^\pm = \lambda_1^\pm$ and $\lambda^\pm = \lambda_2^\pm$, respectively, we have the following

$$W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-}) + \lambda_1^+ Per(E^+_{\lambda_1^+}) + \lambda_1^- Per(E^-_{\lambda_1^-})$$
$$\leq W(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}, E^-_{\lambda_1^-}) + \lambda_1^+ Per(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}) + \lambda_1^- Per(E^-_{\lambda_1^-})$$

and

$$W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) + \lambda_2^+ Per(E^+_{\lambda_2^+}) + \lambda_2^- Per(E^-_{\lambda_2^-})$$
$$\leq W(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) + \lambda_2^+ Per(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}) + \lambda_2^- Per(E^-_{\lambda_2^-}).$$

But,

$$Per(E \cup E') + Per(E \cap E') \leq Per(E) + Per(E').$$

Then, we get

$$\frac{1}{\lambda_1^+}(W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-}) - W(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}, E^-_{\lambda_1^-})) \leq \frac{1}{\lambda_2^+}(W(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) - W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-})).$$

Yet, we have

$$W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-}) - W(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}, E^-_{\lambda_1^-}) \geq -\int_{E^+_{\lambda_1^+} \setminus E^+_{\lambda_2^+}} g^+(P(x))\, \mathrm{d}\nu_{\lambda_1^+}(x)$$

and

$$W(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) - W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) \leq -\int_{E^+_{\lambda_1^+} \setminus E^+_{\lambda_2^+}} g^+(T^+(x))\, \mathrm{d}\nu_{\lambda_2^+}(x).$$

We infer that

$$\int_{E^+_{\lambda_1^+} \setminus E^+_{\lambda_2^+}} \left(\frac{g^+(T^+(x))\, \nu_{\lambda_2^+}(x)}{\lambda_2^+} - \frac{g^+(P(x))\, \nu_{\lambda_1^+}(x)}{\lambda_1^+} \right) \mathrm{d}x \leq 0.$$

Similarly, we prove that

$$\int_{E^-_{\lambda_1^-} \setminus E^-_{\lambda_2^-}} \left(\frac{g^-(T^-(x))\, \nu_{\lambda_2^-}(x)}{\lambda_2^-} - \frac{g^-(P(x))\, \nu_{\lambda_1^-}(x)}{\lambda_1^-} \right) \mathrm{d}x \leq 0.$$

This concludes the proof. $\square$

Remark 3.1. *Assume that $f^- = 0$. For $\lambda_1, \lambda_2 > 0$, let E_{λ_1} and E_{λ_2} be the corresponding optimal free export regions for Problem (1.4) with $\lambda^+ = \lambda_1$ and $\lambda^+ = \lambda_2$, respectively. Assume that $\frac{\lambda_2}{\lambda_1} < \frac{\min g^+}{\max g^+}$. Then, we have $E_{\lambda_1} \subset E_{\lambda_2}$.*

The aim now is to study some properties of the optimal free export/import regions E^+_{opt} and E^-_{opt}. First, let us start by introducing the following

Definition 3.1. *For a subset $E \subset \Omega$, we define the $\Omega-$convex hull of E as $\bigcup_{k \in \mathbb{N}} P_k$, where $P_k \subset \Omega$ is a polygone with $k-$vertices on ∂E.*

Definition 3.2. *Let E be a subset of Ω. We say that E is $\Omega-$convex if the $\Omega-$convex hull of E is the set E itself.*

Remark 3.2. *We note that the $\Omega-$convex hull of a subset $E \subset \Omega$ is not necessarily convex, unless Ω is convex or the convex hull of E is contained in Ω.*

Proposition 3.3. *Assume $d = 2$. Let E^+_{opt} (resp. E^-_{opt}) be the optimal free export (resp. import) region. Then, E^+_{opt} (resp. E^-_{opt}) is $\Omega-$convex.*

Proof. Let us prove that the optimal free export region E^+_{opt} is $\Omega-$convex. Suppose that this is not the case and let E^{++}_{opt} be the $\Omega-$convex hull of E^+_{opt}. It is not difficult to see that

$$Per(E^{++}_{opt}) < Per(E^+_{opt}).$$

Moreover, let γ_{opt} be an optimal transport plan for $W(E^+_{opt}, E^-_{opt})$. Then, the following holds

$$\begin{aligned} W(E^{++}_{opt}, E^-_{opt}) &\leq \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma_{opt} + \int_{(\Omega\setminus E^{++}_{opt})\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma_{opt} + \int_{\partial\Omega\times(\Omega\setminus E^-_{opt})} g^-(x)\,\mathrm{d}\gamma_{opt} \\ &\leq \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma_{opt} + \int_{(\Omega\setminus E^{+}_{opt})\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma_{opt} + \int_{\partial\Omega\times(\Omega\setminus E^-_{opt})} g^-(x)\,\mathrm{d}\gamma_{opt} \\ &= W(E^+_{opt}, E^-_{opt}). \end{aligned}$$

Hence, we get

$$W(E^{++}_{opt}, E^-_{opt}) + \lambda^+ Per(E^{++}_{opt}) + \lambda^- Per(E^-_{opt}) < W(E^+_{opt}, E^-_{opt}) + \lambda^+ Per(E^+_{opt}) + \lambda^- Per(E^-_{opt}),$$

which is a contradiction as (E^+_{opt}, E^-_{opt}) minimizes Problem (3.1). The proof of $\Omega-$convexity for E^-_{opt} is eventually the same. □

More precisely, we have the following

Proposition 3.4. *Assume $d = 2$. Let $E^\pm_{opt}$ be the optimal free export/import region and let $\nu^\pm_{opt}$ the mass to be exported/imported. Then, $E^\pm_{opt}$ is convex in the interior of $\mathrm{spt}(\nu^\pm_{opt})$. Moreover, the part of $E^\pm_{opt}$ which is outside $\mathrm{spt}(\nu^\pm_{opt})$ is a part of the $\Omega-$convex hull of $\mathrm{spt}(\nu^\pm_{opt})$.*

Proof. The convexity of $E^\pm_{opt}$ in the interior of $\mathrm{spt}(\nu^\pm_{opt})$ follows immediately from Proposition 3.3. Let $\mathcal{C}$ be a part of $\partial E^+_{opt}\setminus \mathrm{spt}(\nu^+_{opt})$ and x, y be the endpoints of $\mathcal{C}$ such that $[x, y] \subset \Omega\setminus \mathrm{spt}(\nu^+_{opt})$. Then, it is clear that the segment $[x, y]$ is better than $\mathcal{C}$ in E^+_{opt}. □

Corollary 3.5. *Assume that $d = 2$ and $\mathrm{spt}(\nu^\pm_{opt})$ is $\Omega-$convex. Then, $E^\pm_{opt}$ is contained in $\mathrm{spt}(\nu^\pm_{opt})$.*

Anyway, we also have the following

Corollary 3.6. *Assume that $d = 2$ and $\mathrm{spt}(f^\pm)$ is $\Omega-$convex. Then, the optimal free export/import region $E^\pm_{opt}$ is contained in $\mathrm{spt}(f^\pm)$.*

Moreover, we have the following

Proposition 3.7. *Assume that $d = 2$ and the convex hull of $\mathrm{spt}(\nu^\pm_{opt})$ is contained in Ω. Then, the optimal free export/import region $E^\pm_{opt}$ is convex.*

Proof. This follows immediately from Proposition 3.4. □

Remark 3.3. *Notice that it is not obvious when $d > 2$ if the optimal free export/import regions E^+_{opt} and E^-_{opt} are $\Omega-$convex or not, since it is not true in dimensions 3 or greater that the perimeter of the $\Omega-$convex hull of a set is less than the perimeter of the set itself. However,* *it is possible to prove convexity of $E^{\pm}_{opt}$ under the assumption that $E^{\pm}_{opt}$ are smooth. In fact, from the optimality conditions on E^{+}_{opt} in the problem*

$$\min_E \Big\{ \lambda^+ Per(E) - \int_E \big[|x - T^+(x)| + g^+(T^+(x)) - |x - P(x)| \big] \, \mathrm{d}\nu^+_{opt}(x) \Big\},$$

one can prove (see, for instance, [7]) that for any smooth vector field V such that $V \cdot n = 0$ on $\partial\Omega$, one has

$$\int_{\partial E^+_{opt}} \big[\lambda^+ \mathcal{K}(x) - \big[|x - T^+(x)| + g^+(T^+(x)) - |x - P(x)| \big] \, \nu^+_{opt}(x) \big] \; V \cdot n = 0,$$

where $\mathcal{K}$ denotes the mean curvature of ∂E^+_{opt} and n is the exterior normal vector to ∂E^+_{opt}. Since V is arbitrary, we get

$$\mathcal{K} = \frac{1}{\lambda^+} \, \big[|x - T^+(x)| + g^+(T^+(x)) - |x - P(x)| \big] \, \nu^+_{opt}(x) \geq 0 \quad on \quad \partial E^+_{opt}.$$

Proposition 3.8. *Assume that Ω is convex. Then, the optimal free export/import set $E^{\pm}_{opt}$ intersects the boundary of $\operatorname{spt}(\nu^{\pm}_{opt})$, unless $g^{\pm} = c$ on some arc of $\partial\Omega$.*

Proof. Assume that $E^+_{opt} \cap \partial[\operatorname{spt}(\nu^+_{opt})] = \emptyset$. Let R be the union of all the transport rays between $\nu^+_{opt} \cdot 1_{E^+_{opt}}$ and its projection on the boundary $P_{\#}[\nu^+_{opt} \cdot 1_{E^+_{opt}}]$. Set $E := (R \cap \operatorname{spt}(\nu^+_{opt})) \backslash E^+_{opt}$ and $\mathcal{C} := P(R)$. As the transport rays cannot intersect at their interiors, then we see that this set E is also exported (with a tax g^+) to the same arc $\mathcal{C}$ with $T^+(x) = P(x) \in \mathcal{C}$, for all $x \in E$. Yet, this implies that there is some constant c such that $g^+ = c$ on $\mathcal{C}$. □

Now, we will study the regularity of $E^{\pm}_{opt}$.

Proposition 3.9. *Assume $d = 2$ and $f^{\pm} \in L^p(\Omega)$ with $p > 2$. Then, the optimal free export/import region $E^{\pm}_{opt}$ is C^1 in the interior of Ω. Moreover, $E^{\pm}_{opt}$ is C^1 provided that $\partial\Omega$ is C^1.*

Proof. Assume that this is not the case at some point $x \in \partial E^+_{opt}$. After a rotation and translation of axes, we can assume that $x = (0,0)$ and the x_1-axis is below the two tangent lines to ∂E^+_{opt} at x. Let α_1 and α_2 be the parameterizations of the two parts of ∂E^+_{opt} around x. Take $\varepsilon > 0$ small enough and let $\delta > 0$ be such that $\alpha_1(\varepsilon) = \alpha_2(-\delta)$. Now, let $\mathcal{C}$ be the part of ∂E^+_{opt} between $(\varepsilon, \alpha_1(\varepsilon))$ and $(-\delta, \alpha_2(-\delta))$ and let $\hat{\mathcal{C}}$ be the segment joining these two points. Let E^{++}_{opt} be such that $\partial E^{++}_{opt} = (\partial E^+_{opt} \backslash \mathcal{C}) \cup \hat{\mathcal{C}}$. Then, we have

$$Per(E^{++}_{opt}) - Per(E^+_{opt}) = \varepsilon + \delta - \int_0^{\varepsilon} \sqrt{1 + \alpha_1'(s)^2} \, \mathrm{d}s - \int_{-\delta}^0 \sqrt{1 + \alpha_2'(s)^2} \, \mathrm{d}s.$$

Thanks to the convexity of E^+_{opt} (see Proposition 3.4), we have that $|\alpha_1'(s)|$, $|\alpha_2'(s)| \geq c > 0$, for s small enough. Hence, we get that

$$Per(E^{++}_{opt}) - Per(E^+_{opt}) \leq (1 - \sqrt{1 + c^2})(\varepsilon + \delta).$$

On the other hand, let γ_{opt} be an optimal transport plan for $W(E^+_{opt}, E^-_{opt})$. Then, we have

$$W(E^{++}_{opt}, E^-_{opt}) - W(E^+_{opt}, E^-_{opt}) \leq \int_{(E^+_{opt} \backslash E^{++}_{opt}) \times \partial\Omega} g^+(y) \, \mathrm{d}\gamma_{opt} = \int_{(E^+_{opt} \backslash E^{++}_{opt})} g^+(P(x)) \, \mathrm{d}\nu^+_{opt}(x)$$
$$\leq ||g^+||_{L^\infty} ||\nu^+_{opt}||_{L^p} |E^+_{opt} \backslash E^{++}_{opt}|^{\frac{1}{q}}.$$

Yet,

$$|E^+_{opt} \backslash E^{++}_{opt}| = (\varepsilon+\delta)\alpha_1(\varepsilon) - \int_0^\varepsilon \alpha_1(s)\,\mathrm{d}s - \int_{-\delta}^0 \alpha_2(s)\,\mathrm{d}s \le C(\varepsilon+\delta)^2.$$

Consequently, we get

$$W(E^{++}_{opt}, E^-_{opt}) + \lambda^+ Per(E^{++}_{opt}) - W(E^+_{opt}, E^-_{opt}) - \lambda^+ Per(E^+_{opt})$$

$$\le \Big[\lambda^+(1-\sqrt{1+c^2}) + ||g^+||_{L^\infty}||\nu^+_{opt}||_{L^p} C(\varepsilon+\delta)^{\frac{2}{q}-1}\Big](\varepsilon+\delta),$$

which is a contradiction for ε, δ small enough, as (E^+_{opt}, E^-_{opt}) is a minimizer for Problem (2.1) and $p > 2$. □

Proposition 3.10. *Assume $d = 2$, $g^\pm \ge c > 0$ and $f^\pm$ is continuous on* $\mathrm{spt}(f^\pm)$. *Let $\nu^\pm_{opt}$ the mass to be exported/imported and $E^\pm_{opt}$ be the optimal free export/import region. Then, $E^\pm_{opt}$ is strictly convex in $\{\nu^\pm_{opt} > 0\}$. In particular, if $f^\pm$ is bounded from below, then $E^\pm_{opt}$ is uniformly convex inside* $\mathrm{spt}(\nu^\pm_{opt})$.

Proof. Let γ_{opt} be an optimal transport plan for $W(E^+_{opt}, E^-_{opt})$ such that $\nu^+_{opt} = (\Pi_x)_\#(\gamma_{opt} \cdot 1_{\Omega\times\partial\Omega})$. It is easy to see that the set E^+_{opt} minimizes

$$\min_E \Big\{ \min\Big\{ \int_{\Omega\times\Omega} |x-y|\mathrm{d}\gamma + \int_{(\Omega\backslash E)\times\partial\Omega} g^+(y)\mathrm{d}\gamma \,:\, (\Pi_x)_\#\gamma = \nu^+_{opt},\, (\Pi_y)_\#\gamma \subset \partial\Omega \Big\} + \lambda^+ Per(E) \Big\}.$$

On the other hand, from Proposition 2.2, the transport plan $\gamma^{ib}_{opt} := (Id, T_E)_\#\nu^+_{opt}$, where the map T_E is defined as follows

$$T_E(x) = \begin{cases} P(x) := \mathrm{argmin}\{|x-y| \,:\, y \in \partial\Omega\}, & \text{if } x \in E,\\ T^+(x) := \mathrm{argmin}\{|x-y| + g^+(y) \,:\, y \in \partial\Omega\}, & \text{if } x \in \Omega\backslash E,\end{cases}$$

is a minimizer for the problem

$$\min\Big\{ \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\backslash E)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma \,:\, (\Pi_x)_\#\gamma = \nu^+_{opt},\, \mathrm{spt}((\Pi_y)_\#\gamma) \subset \partial\Omega \Big\}.$$

So, we infer that E^+_{opt} minimizes

$$\min_E \Big\{ \int_\Omega |x - T_E(x)|\,\mathrm{d}\nu^+_{opt}(x) + \int_{\Omega\backslash E} g^+(T_E(x))\,\mathrm{d}\nu^+_{opt}(x) + \lambda^+ Per(E) \Big\}.$$

Fix $x_0 \in \partial E^+_{opt}$. Let $\mathcal{C}$ be a part of ∂E^+_{opt} around x_0 and $x_1 := (s_1, t_1)$, $x_2 := (s_2, t_2)$ be the endpoints of $\mathcal{C}$. Assume that $\mathcal{C}$ is the graph of a function α_{opt}. Then, we see that α_{opt} minimizes the following problem

$$\min\Big\{ \int_{s_1}^{s_2}\int_0^{\alpha(s)} u(s,t)\,\nu^+_{opt}(s,t)\,\mathrm{d}t\,\mathrm{d}s + \lambda^+ \int_{s_1}^{s_2} \sqrt{1+\alpha'(s)^2}\,\mathrm{d}s \,:\, \alpha(s_1) = t_1,\, \alpha(s_2) = t_2 \Big\},$$

where

$$u(s,t) := |(s,t) - T^+(s,t)| + g^+(T^+(s,t)) - |(s,t) - P(s,t)|.$$

From the optimality conditions on α_{opt} and thanks to the continuity of ν^+_{opt} on $\mathrm{spt}(\nu^+_{opt})$ (see Proposition 2.3), we get that

$$\left[\frac{\alpha'_{opt}(s)}{\sqrt{1+\alpha'_{opt}(s)^2}}\right]' = \frac{1}{\lambda^+}\,u(s,\alpha_{opt}(s))\,\nu^+_{opt}(s,\alpha_{opt}(s)).$$

This implies that

$$\kappa(s_0) = \frac{1}{\lambda^+}u(s_0,\alpha_{opt}(s_0))\,\nu^+_{opt}(s_0,\alpha_{opt}(s_0)),$$

where $\kappa(s_0)$ denotes the curvature of ∂E^+_{opt} at $x_0 := (s_0,\alpha_{opt}(s_0))$. Yet, we have $u(s,t) \geq g^+(T(s,t)) \geq c > 0$. This concludes the proof. □

Corollary 3.11. *Assume $d = 2$ and $f^\pm$ is continuous on $\mathrm{spt}(f^\pm)$. Then, the optimal free export/import region $E^\pm_{opt}$ is C^2 in the interior of $\mathrm{spt}(\nu^\pm_{opt})$ and $C^{1,1}$ in the interior of Ω. Moreover, $E^\pm_{opt}$ is $C^{1,1}$ provided that $\partial\Omega$ is $C^{1,1}$.*

Proof. This follows from Propositions 3.4 & 3.9, the continuity of $\nu^\pm_{opt}$ (see Proposition 2.3) and u, and the fact that the curvature κ of ∂E^+_{opt} in the interior of Ω satisfies (see Proposition 3.10)

$$\kappa(s) = \frac{1}{\lambda^+}u(s,\alpha_{opt}(s))\,\nu^+_{opt}(s,\alpha_{opt}(s)).$$

Now, assume that $x_0 \in \partial E^+_{opt} \cap \partial\Omega$. Similarly to Proposition 3.10, we denote by $\mathcal{C}$ the arc of ∂E^+_{opt} around x_0 and by $x_1 := (s_1,t_1)$, $x_2 := (s_2,t_2)$ the endpoints of $\mathcal{C}$. Assume that $\mathcal{C}$ (resp. $\partial\Omega$) is the graph of a function α_{opt} (resp. ψ). Then, we have that α_{opt} solves

$$\min\left\{\int_{s_1}^{s_2}\int_0^{\alpha(s)} u(s,t)\,\nu^+_{opt}(s,t)\,\mathrm{d}t\mathrm{d}s + \lambda^+\int_{s_1}^{s_2}\sqrt{1+\alpha'(s)^2}\mathrm{d}s : \alpha \geq \psi,\ \alpha(s_1) = t_1,\ \alpha(s_2) = t_2\right\}.$$

From the optimality conditions on α_{opt}, we get

$$\kappa(s) \leq \frac{1}{\lambda^+}u(s,\alpha_{opt}(s))\,\nu^+_{opt}(s,\alpha_{opt}(s)).$$

As $\partial\Omega$ is $C^{1,1}$, then it is clear that κ is bounded from below as well. This concludes the proof. □

Corollary 3.12. *Assume that $d = 2$ and $f^\pm$ is $C^{0,\alpha}$ on $\mathrm{spt}(f^\pm)$, with $\alpha \in (0,1)$. Then, the optimal free export/import region $E^\pm_{opt}$ is $C^{2,\alpha}$ in the interior of $\mathrm{spt}(\nu^\pm_{opt})$. In particular, $E^\pm_{opt}$ is $C^{2,1}$ inside $\mathrm{spt}(\nu^\pm_{opt})$ as soon as $f^\pm$ is Lipschitz on $\mathrm{spt}(f^\pm)$.*

Remark 3.4. *We see that if $f^\pm$ is $C^{k,\alpha}$ on $\mathrm{spt}(f^\pm)$ and $g^\pm = c^\pm > 0$, then the optimal free export/import set $E^\pm_{opt}$ is $C^{k+2,\alpha}$ in the interior of $\mathrm{spt}(\nu^\pm_{opt})$.*

Remark 3.5. *It is not clear how to adapt Proposition 3.9 to the case $d > 2$. On the other hand, we know that any part $E \subset \partial E^+_{opt}$ is a solution of the variational problem*

$$\min\left\{\int_{B(\varepsilon)}\int_0^{\alpha(\bar{y})} u(\bar{y},y_d)\nu^+_{opt}(\bar{y},y_d)\mathrm{d}y_d\,\mathrm{d}\bar{y} + \lambda^+\int_{B(\varepsilon)}\sqrt{1+|\nabla\alpha(\bar{y})|^2}\mathrm{d}\bar{y} : Id\times\alpha_{|\partial B(\varepsilon)} = \partial E\right\}.$$

From the optimality conditions on α_{opt}, we get that α_{opt} satisfies the following equation

$$\nabla\cdot\left[\frac{\nabla\alpha_{opt}(\bar{y})}{\sqrt{1+|\nabla\alpha_{opt}(\bar{y})|^2}}\right] = \frac{1}{\lambda^+}\,u(\bar{y},\alpha_{opt}(\bar{y}))\,\nu^+_{opt}(\bar{y},\alpha_{opt}(\bar{y}))$$

or equivalently,

$$\sum_{i,j} a_{i,j}\ \partial^2_{x_i x_j}\alpha_{opt} = \frac{1}{\lambda^+}\, u(\bar{y},\alpha_{opt}(\bar{y}))\, \nu^+_{opt}(\bar{y},\alpha_{opt}(\bar{y})),$$

where

$$a_{i,j} = \frac{\delta_{i,j}(1+|\nabla\alpha_{opt}(\bar{y})|^2) - \partial_{x_i}\alpha_{opt}\,\partial_{x_j}\alpha_{opt}}{(1+|\nabla\alpha_{opt}(\bar{y})|^2)^{\frac{3}{2}}}.$$

In fact, under some assumptions on $a_{i,j}$ (see, for instance, [6]), it is possible to prove a higher regularity on α_{opt}. But, it is not clear if the coefficients $a_{i,j}$ here satisfy the required assumptions or not and even, it is not sure that in higher dimension $d > 2$ we can arrive to prove that the optimal free export/import region $E^{\pm}_{opt}$ is smooth, as $E^{\pm}_{opt}$ may not be, for instance, $\Omega-$convex when $d > 2$.

The following example shows that, in general, the optimal free export set E^+_{opt} is not C^2 on the boundary of $\mathrm{spt}(f)$ if $f \notin C(\Omega)$.

Example 3.12.1. *Let $\Omega = \bar{B}(0,2)$ and let f be a nonnegative density such that $f = 1$ on $[-1,1]\times[-1,1]$. Assume $g = 1$ on $\partial\Omega$. Fix $\lambda > 0$, so the problem (1.4) is equivalent to minimize*

$$\min\left\{\lambda\, Per(E) - |E \cap ([-1,1]\times[-1,1])|\ :\ E \subset \Omega\right\}.$$

Let κ be the curvature of $\partial\Omega$, then we know that $\kappa = \frac{1}{\lambda}$ on $]-1,1[\times]-1,1[$. If $\lambda < \frac{1}{2}$, then it is clear that the optimal set $E_{opt} \neq \emptyset$. Yet, we know that E_{opt} is C^1. This implies that there is part of E_{opt} inside $]-1,1[\times]-1,1[$ and the curvature of this part is $\frac{1}{\lambda}$. However, we can see easily that $E_{opt} \cap \partial(]-1,1[\times]-1,1[) \neq \emptyset$. Consequently, ∂E_{opt} is not C^2.

References

[1] G. Buttazzo, G. Carlier and S. Guarino Lo Bianco, Optimal Regions for Congested Transport, *Math. Model. Numer. Anal.*, 2014.

[2] S. Dweik and F. Santambrogio, Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques, *Control, Optimisation and Calculus of Variations, 2016.*

[3] S. Dweik, Optimal transportation with boundary costs and summability estimates on the transport density, *Journal of Convex Analysis*, 2017.

[4] S. Dweik, Weighted Beckmann problem with boundary costs, *Quarterly of applied mathematics*, 2018.

[5] S. Dweik, N. Ghoussoub and A. Z. Palmer, Optimal controlled transports with free end times subject to import/export tariffs, *Journal of Dynamical and Control Systems*, 2019.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, New York/Berlin, 1983.

[7] A. Henrot and M. Pierre, *Variation et Optimisation de Formes, Une Analyse Géométrique*, Mathématiques & Applications 48, Springer-Verlag, Berlin (2005).

[8] J.M.Mazon, J.Rossi and J.Toledo, An optimal transportation problem with a cost given by the euclidean distance plus import/export taxes on the boundary, *Rev. Mat. Iberoam.* 30 (2014), no. 1, 1-33.

[9] F. Santambrogio, *Optimal Transport for Applied Mathematicians,* in *Progress in Nonlinear Differential Equations and Their Applications,* 87, Birkhäuser Basel (2015).

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