# CRITICAL POINTS OF THE MOSER-TRUDINGER FUNCTIONAL ON CLOSED SURFACES

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Abstract. Given a closed Riemann surface $(\Sigma, g)$, we use a minmax scheme together with compactness, quantization results and with sharp energy estimates to prove the existence of positive critical points of the functional

$$J_{p,\beta}(u) = \frac{2-p}{2}\left(\frac{p\|u\|_{H^1}^2}{2\beta}\right)^{\frac{p}{2-p}} - \ln \int_\Sigma \left(e^{u_+^p} - 1\right) dv_g\,,$$

for every $p \in (1,2)$ and $\beta > 0$, or for $p = 1$ and $\beta \in (0,\infty) \setminus 4\pi\mathbb{N}$. Letting $p \uparrow 2$ we obtain positive critical points of the Moser-Trudinger functional

$$F(u) := \int_\Sigma \left(e^{u^2} - 1\right) dv_g$$

constrained to $\mathcal{E}_\beta := \left\{ v \text{ s.t. } \|v\|_{H^1}^2 = \beta \right\}$ for any $\beta > 0$.

## Introduction

We consider a smooth, closed Riemann surface $(\Sigma, g)$ (2-dimensional, connected and without boundary) and we endow the usual Sobolev space $H^1 = H^1(\Sigma)$ with the norm $\|\cdot\|_{H^1}$ given by

$$\|u\|_{H^1}^2 = \int_\Sigma \left(|\nabla u|_g^2 + u^2\right) dv_g\,.$$

Building up on previous works, see e.g. [2, 22, 21, 36, 38, 43], Yuxiang Li [29] proved that the following Moser-Trudinger inequality holds

$$\sup_{u\in H^1\,,\ \|u\|_{H^1}^2=\beta} \int_\Sigma e^{u^2} dv_g < +\infty \quad \Leftrightarrow \quad \beta \le 4\pi\,, \tag{MT}$$

(see also Remark 0.2) and that there is an extremal function for $(MT)$ even in the critical case $\beta = 4\pi$ (see also Remark 5.1). Such an extremal is (up to a sign change) a positive critical point of

$$F(u) := \int_\Sigma \left(e^{u^2} - 1\right) dv_g\,, \tag{0.1}$$

constrained to

$$u \in \mathcal{E}_\beta := \left\{ v \in H^1(\Sigma) \text{ s.t. } \|v\|_{H^1}^2 = \beta \right\} \tag{0.2}$$

when $\beta \in (0, 4\pi]$. A positive function $u$ is a critical point of $F$ constrained to $\mathcal{E}_\beta$ if and only if it satisfies the Euler-Lagrange equation

$$\Delta_g u + u = 2\lambda u e^{u^2}\,, \quad u > 0 \text{ in } \Sigma\,, \tag{0.3}$$

where our convention for the Laplacian is with the sign that makes it positive-definite and where $\lambda > 0$ is given by

$$2\lambda \int_\Sigma u^2 e^{u^2} dv_g = \|u\|_{H^1}^2 = \beta\,. \tag{0.4}$$

For $\beta < 4\pi$, finding critical points of $F$ constrained to $\mathcal{E}_\beta$ reduces to a standard maximization argument. Finding such critical points for larger $\beta$'s is a more challenging problem, since upper bounds on the functional fail, and this will be the main achievement of this paper. Some results in this direction, for planar domains and in *slightly supercritical regimes* $0 < \beta - 4\pi \ll 1$ were obtained in [38] and [26].

In order to do that, we would like to use a variational method, more precisely a minmax method, to produce a converging Palais-Smale sequence. The two main analytic difficulties are that the functional $F$ does not satisfy the Palais-Smale condition and that its criticality is of borderline type, which prevents us from using the methods of [26, 38] for $\beta$ large. To overcome these problems we will introduce a family of subcritical functional $J_{p,\beta}$, $p \in [1, 2)$, that, in some sense, interpolate between a Liouville-type problem and our critical Moser-Trudinger problem, apply the minmax method to obtain critical points of $J_{p,\beta}$, and then prove new compactness and quantization results to such critical points.

More precisely, given $p \in [1, 2)$ and $\beta > 0$, we let $J_{p,\beta}$ be given in $H^1(\Sigma)$ by

$$J_{p,\beta}(u) = \frac{2-p}{2}\left(\frac{p\|u\|_{H^1}^2}{2\beta}\right)^{\frac{p}{2-p}} - \ln \int_\Sigma \left(e^{u_+^p} - 1\right) dv_g\,, \tag{0.5}$$

where $u_+ = \max\{u, 0\}$ and we set $J_{p,\beta}(u) = +\infty$ if $u \le 0$. By Trudinger's result [43], for $p \in (1, 2)$, $J_{p,\beta}$ is finite and of class $C^1$ on the subset of $H^1(\Sigma)$ of functions with non-trivial positive part, and its critical points are the solutions of

$$\Delta_g u + u = p\lambda u^{p-1} e^{u^p}\,, \quad u > 0 \text{ in } \Sigma\,, \tag{0.6}$$

where the positivity follows from the maximum principle, see Lemma 1.1, and $\lambda > 0$ is given by the relation

$$\frac{\lambda p^2}{2}\left(\frac{p\|u\|_{H^1}^2}{2\beta}\right)^{\frac{2(p-1)}{2-p}} \int_\Sigma \left(e^{u^p} - 1\right) dv_g = \beta\,. \tag{0.7}$$

While $J_{1,\beta}$ is not differentiable at functions $u$ vanishing on sets of positive measure, it is differentiable at any $u > 0$ a.e., and $u > 0$ is a critical point if and only if it solves (0.6) with $p = 1$.

Smoothness follows by standard elliptic theory and [43], see Lemma 1.1. Moreover, multiplying (0.6) by $u$ and integrating by parts in $\Sigma$, gives

$$\beta = \frac{\lambda p^2}{2}\left(\int_\Sigma \left(e^{u^p} - 1\right)\, dv_g\right)^{\frac{2-p}{p}} \left(\int_\Sigma u^p e^{u^p}\, dv_g\right)^{\frac{2(p-1)}{p}}\,. \tag{0.8}$$

By $(MT)$ and Young's inequality, $J_{p,\beta}$ is bounded from below for all $\beta \le 4\pi$, and for $\beta < 4\pi$ finding critical points of $J_{p,\beta}$ reduces to a standard minimization argument, as it happens for the constrained extremization of $F$: similarly, finding such critical points for larger $\beta$'s is much more difficult. As we shall discuss, compactness and quantization (see Corollary 4.1) give that, as $p$ approaches the borderline case $p_0 = 2$, the critical points of $J_{p,\beta}$ converge to critical points of the functional $F$ in (0.1) constrained to $\mathcal{E}_\beta$, at least when $\beta > 0$ is given out of $4\pi\mathbb{N}^\star$, where $\mathbb{N}^\star$ denotes the set of the positive integers.

Our main results read as follows:

**Theorem 0.1.** *Let $(\Sigma, g)$ be a closed surface. Let $p \in (1,2)$ and $\beta > 0$ be given. Then the set $\mathcal{C}_{p,\beta}$ of the positive critical points of $J_{p,\beta}$ is not empty and compact. The same is true for $p = 1$ and every $\beta \in (0,\infty) \setminus 4\pi\mathbb{N}^\star$.*

Letting $p \uparrow 2$ suitably, we will obtain the following result, which according to us is the most relevant achievement of this paper.

**Theorem 0.2.** *Let $(\Sigma, g)$ be a closed surface and let $\beta > 0$ be given. Then the set $\mathcal{C}_{2,\beta}$ of the positive critical points of the functional $F$ constrained to $\mathcal{E}_\beta$ is not empty and compact in $C^2$.*

A notable fact in Theorems 0.1 and 0.2 is that, except for $p = 1$, *the full range* $\beta > 0$ *is covered* and in particular also the case $\beta \in 4\pi\mathbb{N}^\star$. If fact we will also prove that the sets

$$\bigcup_{\substack{\beta \in [4\pi(k-1)+\delta, 4\pi k] \\ p \in [1+\delta, 2]}} \mathcal{C}_{p,\beta}, \qquad \bigcup_{\substack{\beta \in [4\pi(k-1)+\delta, 4\pi k - \delta] \\ p \in [1,2]}} \mathcal{C}_{p,\beta}$$

are compact for any $\delta > 0$, i.e. blow-up can occur only for $\beta \downarrow 4\pi\mathbb{N}^\star$ or for $p \to 1$ and $\beta \to 4\pi\mathbb{N}^\star$, as we shall see.

Let us explain the strategy of the proofs. We shall start with the existence part of Theorem 0.1. Here with a minmax scheme based on so called *baricenters*, as originally used in [15], we show that given $p \in (1,2)$ and $\beta \in (4\pi, +\infty) \setminus 4\pi\mathbb{N}^\star$, the very low sublevels of $J_{p,\beta}$ are topologically non-trivial, see Proposition 1.1. This would allow to construct a Palais-Smale sequence at some minmax level, but it is only with a *monotonicity trick* introduced by Struwe, see [39], that we are able to construct Palais-Smale sequences that are bounded for *almost every* $\beta > 0$ and for $p \in (1,2)$. Then, again using the *subcriticality* of $e^{u^p}$ with respect to $(MT)$, a $H^1$-bounded subsequence strongly converges to a positive critical point of $J_{p,\beta}$, see Proposition 1.3 (see also [7, Thm. 5.1] for counterexamples to the strong convergence of bounded Palais-Smale sequences when $p = 2$).

The next step is to extend this result from the existence for a.e. $\beta$ to the existence for *every* $\beta \in (0,\infty) \setminus 4\pi\mathbb{N}^\star$. This is done via the crucial compactness Theorem 4.1, showing that a sequence $(u_\varepsilon)_\varepsilon$ of positive critical points of $J_{p_\varepsilon,\beta_\varepsilon}$ with $p_\varepsilon \in [1,2)$ and $\beta_\varepsilon \to \beta \in [0,\infty)$ can fail to be precompact only if $\beta \in 4\pi\mathbb{N}$. If fact, as $p_\varepsilon \uparrow 2$, this also allows to show that the positive critical points of $J_{p_\varepsilon,\beta_\varepsilon}$ converge to positive critical points of $F|_{\mathcal{E}_\beta}$ if $\beta \notin 4\pi\mathbb{N}$, (see Corollary 4.1), hence proving Theorem 0.2, except for $\beta \in 4\pi\mathbb{N}^\star$. This quantization property ($\beta \in 4\pi\mathbb{N}^\star$ in case of blow up) can be seen as a no-neck energy result, but not only. Indeed, in the specific case where $p = 2$, extending the quantization of [17] to the surface setting, Yang [44] already proved a no-neck energy result for such sequences, but without excluding that some nonzero weak limit $u_0 \not\equiv 0$ appears. We know now that ruling this situation out, or in other words getting the sharp quantization (4.5) instead of (4.6), is a very sensitive property, which depends also on the lower-order terms appearing in the RHS of (0.3) (see for instance [34] for counterexamples with a perturbed version of the nonlinearity $e^{u^2}$) and which requires to be more careful in the way we approach the border case $p = 2$. In this sense, our Theorem 4.1 cannot be seen as a perturbation of previous results, but it is a novelty in itself. We also mention that the proof of Theorem 4.1 never uses the Pohozaev identity, which is however quite classical in proving such quantization results. Instead, we first compare in small disks our blow-up solutions with some radially symmetric functions solving the same PDE, sometimes called "bubbles", and we directly show that the difference must satisfy some balance condition (see (3.1)). From this balance condition, we get that the union of these separate disks is large in the sense that the complementary region cannot contribute in the quantization (4.5). In this last part of the proof, we also show that our specific family of nonlinearities forces the Lagrange multipliers to converge appropriately to 0 (see Step 4.2) as blow-up occurs. One delicate consequence is that each disk only brings the minimal energy $4\pi$ in (4.5) (see also Remark 4.1).

Finally, covering the case $\beta \in 4\pi\mathbb{N}^\star$ relies on delicate energy expansions of the blowing-up sequences carried out in Theorem 5.1 below. When $\beta = 4\pi$ and $p = 2$, it was already observed in a slightly different setting (see [32, 33]) that such expansions do not clearly depend on the geometric quantities of the problem and that the energy always converges to $4\pi$ from above. In the present paper, we observe that this is still true at any level $\beta \in 4\pi\mathbb{N}^\star$ and for all $p \in (1, 2]$, so that if we let $\beta_\varepsilon \uparrow \beta \in 4\pi\mathbb{N}^\star$ no blow-up occurs, while it could occur for $\beta_\varepsilon \downarrow \beta \in 4\pi\mathbb{N}^\star$. In striking contrast, the analogous expansions in [3], dealing with an equation qualitatively similar to the case $p = 1$ (see Remark 0.2 below), are different in nature: for instance, the Gauss curvature of the surface appears and compactness is not always true at critical levels $\beta \in 4\pi\mathbb{N}^\star$ (see the discussion below [3, Corollary 1.2]).

**Remark 0.1.** *When $\Sigma$ is a non-simply connected bounded domain in $\mathbb{R}^2$, in* [18] *the authors compute the Leray-Schauder degree of the Euler-Lagrange equation of the functional $F|_{\mathcal{E}_\beta}$, showing that it is non-zero if $\Sigma$ is not simply connected. Even if we were able to adapt the strategy to the case of a closed manifold $\Sigma$, when the genus of $\Sigma$ is* 0 *(i.e. if $\Sigma$ is topologically a sphere), the Leray-Schauder degree of the Euler-Lagrange equation is expected to be* 1 *for $\beta \in (0, 4\pi]$, $-1$ for $\beta \in (4\pi, 8\pi]$ and* 0 *for $\beta > 8\pi$. Hence this topological method fails to completely answer the question of the existence of critical points of $F|_{\mathcal{E}_\beta}$ on a closed surface.*

*In any case,* ***the Leray-Schauder degree does not depend on*** $p \in [1, 2]$ *by compactness (except for $p = 1$ and $\beta \in 4\pi\mathbb{N}^\star$), and coincides with that of the* mean field equation *(with the full $H^1$-norm, slightly different from* [4] *or* [30]*), namely* (0.6) $p = 1$*. For the case $p \in (1, 2]$ and $\beta = 4\pi k$ the L-S degree is equal to the degree for $\beta \in (4\pi(k-1), 4\pi k)$ by Theorem 5.1.*

**Remark 0.2.** *It is worth mentioning that, on a surface, there is a Moser-Trudinger inequality with a zero average constraint, namely*

$$\sup_{u \in \mathcal{Z}_\beta} \int_\Sigma e^{u^2} dv_g < +\infty \quad \Leftrightarrow \quad \beta \leq 4\pi\,, \qquad (MT_{\mathcal{Z}})$$

*where $\mathcal{Z}_\beta = \left\{u \in H^1 \text{ s.t. } \int_\Sigma |\nabla u|_g^2 dv_g = \beta \text{ and } \int_\Sigma u\, dv_g = 0\right\}$. This inequality was already proven in the original paper of Moser* [36]*, if $(\Sigma, g)$ is the standard* 2*-sphere, and in the general case by Fontana* [22]*, dealing also with the higher dimensional case. The functional $I_\beta$, qualitatively related to $J_{1,\beta}$ in* (0.5) *for $p = 1$,*

$$I_\beta(u) = \frac{1}{4\beta} \int_\Sigma |\nabla u|^2 dv_g + \int_\Sigma u\, dv_g - \ln \int_\Sigma e^u dv_g \qquad (0.9)$$

*attracted a huge attention in the literature (see* [28, 4, 14] *and references therein) and its critical points give rise to the very much studied mean-field equation. As a* *remark, for all $\beta \leq 4\pi$, as $(MT)$ implies that $J_{1,\beta}$ is bounded below, we get from $(MT_{\mathcal{Z}})$ that $I_\beta$ is bounded below.*

**Remark 0.3.** *Different kinds of bubbling solutions for the Moser-Trudinger inequalities on domains and surfaces were built in* [9, 11, 20].

## 1. Variational part

The main goal of the section is to prove the following theorem, with $J_{p,\beta}$ as in (0.5).

**Theorem 1.1.** *Let $(\Sigma, g)$ be a closed surface, and let $p \in (1,2)$ be given. Then, for almost every $\beta > 0$, $J_{p,\beta}$ possesses a smooth and positive critical point $u$, where $J_{p,\beta}$ is as in* (0.5).

As discussed in introduction, $u$ given by Theorem 1.1 is smooth, positive and solves (0.6)-(0.7) for some $\lambda > 0$, as we shall now prove.

**Lemma 1.1.** *Every non-trivial critical point of $J_{p,\beta}$, $p \in (1,2)$, is a smooth and positive solution to* (0.6). *Moreover, for every $p \in [1,2]$ every solution to* (0.6) *is smooth.*

*Proof.* Assume $p \in (1,2)$. One easily verifies that the Euler-Lagrange equation of $J_{p,\beta}$ is

$$\Delta u + u = \lambda u_+^{p-1} e^{u^p}, \tag{1.1}$$

where $\lambda > 0$. Since $e^{u^p} \in L^q(\Sigma)$ for every $q \in [1, \infty)$ thanks to [43], elliptic estimates imply that $u \in C^2(\Sigma)$.

We first claim that $u \geq 0$. Indeed, assume that $\Sigma_- := \{x \in \Sigma : u(x) < 0\} \neq \emptyset$. Then $\Delta u = -u > 0$ in $\Sigma_-$, violating the weak maximum principle at a point of minimum.

Now consider $\Sigma_+ := \{x \in \Sigma : u(x) > 0\}$. We claim that $\Sigma_+ = \Sigma$, i.e. $u > 0$ everywhere. Otherwise $\partial\Sigma_+ \neq \emptyset$. Let then $x_0 \in \partial\Sigma_+$ be a point satisfying the interior sphere condition, and let $D \subset \Sigma_+$ be a disk with $x_0 \in \partial D$ and such that

$$\Delta u = \lambda u^{p-1} e^{u^p} - u > 0 \quad \text{in } D.$$

It is possible to find such $D$ because $u(x_0) = 0$, $\lambda > 0$, and $p < 2$. Then, by the Hopf lemma, see e.g. [23, Lemma 3.4],

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where $\nu$ is the outer normal to $\partial\Sigma_+$ at $x_0$. This violates the non-negativity of $u$, leading to a contradition. Hence $u > 0$. Going back to (1.1), we can now bootstrap regularity, hence $u \in C^\infty(\Sigma)$.

Also for $p = 1, 2$ the regularity of solutions to (0.6) follows from elliptic estimates and [43], which implies that the right-hand side of (0.6) belongs to $L^q(\Sigma)$ for $q \in [1, \infty)$. □

In the rest of the section we consider $p \in (1,2)$ fixed. The first tools we shall need in the proof of Theorem 1.1 are an improven Moser-Trudinger inequalities.

Let us first observe that from Young's inequality $ab \le \frac{a^q}{q} + \frac{b^r}{r}$ applied with $q = \frac{2}{p}$ and $r = q' = \frac{2}{2-p}$ we obtain

$$\begin{aligned} |u|^p &= \left( \frac{|u|}{\|u\|_{H^1}} \sqrt{\frac{8\pi}{p}} \right)^p \left( \|u\|_{H^1} \sqrt{\frac{p}{8\pi}} \right)^p \\ &\le 4\pi \frac{u^2}{\|u\|_{H^1}^2} + \|u\|_{H^1}^{\frac{2p}{2-p}} \frac{2-p}{2} \left( \frac{p}{8\pi} \right)^{\frac{p}{2-p}}, \end{aligned}$$

hence with $(MT)$ we get

$$\ln \int_\Sigma \left( e^{|u|^p} - 1 \right) dv_g \le \frac{2-p}{2} \left( \frac{p\|u\|_{H^1}^2}{8\pi} \right)^{\frac{p}{2-p}} + C. \tag{1.2}$$

It follows that

$$J_{p,\beta}(u) \ge \frac{2-p}{2} \left( p\|u\|_{H^1}^2 \right)^{\frac{p}{2-p}} \left( \left( \frac{1}{2\beta} \right)^{\frac{p}{2-p}} - \left( \frac{1}{8\pi} \right)^{\frac{p}{2-p}} \right) - C,$$

so that $J_{p,\beta}$ is coercive for $\beta < 4\pi$.

On the other hand, if the density $e^{|u|^p} - 1$ is *spread* into $k+1 \ge 1$ disjoint regions we have the following improven Moser-Trudinger inequality which gives a uniform lower bound on $J_{p,\beta}(u)$ for each $\beta < 4\pi(k+1)$, see [6] for a related argument.

**Lemma 1.2.** *For any fixed $k \in \mathbb{N}$, let $\Omega_1, \dots, \Omega_{k+1}$ be subsets of $\Sigma$ satisfying $\mathrm{dist}(\Omega_i, \Omega_j) \ge \delta_0$ for $i \ne j$ and some $\delta_0 > 0$. Let also $\gamma_0 \in \left(0, \frac{1}{k+1}\right)$, $\delta_1 \in (0, 8\pi(k+1))$. Then there exists a constant $C = C(k, \delta_0, \delta_1, \gamma_0, \Sigma)$ such that*

$$\ln \int_\Sigma \left( e^{|u|^p} - 1 \right) dv_g \le \frac{2-p}{2} \left( \frac{p\|u\|_{H^1}^2}{8\pi(k+1) - \delta_1} \right)^{\frac{p}{2-p}} + C \tag{1.3}$$

*for all the functions $u \in H^1(\Sigma)$ satisfying*

$$\frac{\int_{\Omega_i} \left( e^{|u|^p} - 1 \right) dv_g}{\int_\Sigma \left( e^{|u|^p} - 1 \right) dv_g} \ge \gamma_0, \quad \forall\, i \in \{1, \dots, k+1\}. \tag{1.4}$$

*Proof.* Fix $u$ satisfying (1.4). We can find $k+1$ functions $g_1, \dots, g_{k+1}$ such that

$$\begin{cases} g_i(x) \in [0,1] & \text{for every } x \in \Sigma; \\ g_i(x) = 1, & \text{for every } x \in \Omega_i; \\ g_i(x) = 0, & \text{if } \mathrm{dist}(x, \Omega_i) \ge \frac{\delta_0}{2}; \\ \|g_i\|_{C^1} \le C_{\delta_0, \Sigma}. \end{cases} \tag{1.5}$$

For $\varepsilon > 0$ small (to be fixed depending on $k$ and $\delta_1$) using the inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$ we can find a constant $C_{\varepsilon,\delta_0}$ (the dependence of the constants on $\Sigma$ will be omitted) such that, for any $i \in \{1, \dots, k+1\}$ and $v \in H^1(\Sigma)$ there holds

$$\|g_i v\|_{H^1}^2 \le \int_\Omega g_i^2 |\nabla v|^2 dv_g + \varepsilon \int_\Omega |\nabla v|^2 dv_g + C_{\varepsilon,\delta_0} \int_\Omega v^2 dv_g. \tag{1.6}$$

Now let $\lambda_{\varepsilon,\delta_0}$ be an eigenvalue of $\Delta_g + 1$ such that $\frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} < \varepsilon$, where $C_{\varepsilon,\delta_0}$ is as in (1.6), and write

$$u = P_{V_{\varepsilon,\delta_0}} u + P_{V_{\varepsilon,\delta_0}^\perp} u =: u_1 + u_2,$$

where $V_{\varepsilon,\delta_0} \subset H^1(\Sigma)$ is the direct sum of the eigenspaces of $\Delta_g + 1$ with eigenvalues less than or equal to $\lambda_{\varepsilon,\delta_0}$, and $P_{V_{\varepsilon,\delta_0}}, P_{V^\perp_{\varepsilon,\delta_0}}$ denote the projections onto $V_{\varepsilon,\delta_0}$ and $V^\perp_{\varepsilon,\delta_0}$ respectively.

We now choose $i$ such that

$$\int_\Omega g_i^2 |\nabla u_2|^2 dv_g \le \int_\Omega g_j^2 |\nabla u_2|^2 dv_g \quad \text{for every } j \in \{1, \dots, k+1\}.$$

Since the functions $g_1, \dots, g_{k+1}$ have disjoint supports, (1.6) applied with $v = u_2$ gives

$$\|g_i u_2\|^2_{H^1} \le \frac{1}{k+1} \int_\Omega |\nabla u_2|^2 dv_g + \varepsilon \int_\Omega |\nabla u_2|^2 dv_g + C_{\varepsilon,\delta_0} \int_\Omega u_2^2 dv_g.$$

This, together with the inequalities

$$C_{\varepsilon,\delta_0} \int_\Omega u_2^2 dx \le \frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} \|u_2\|^2_{H^1} \le \varepsilon \|u_2\|^2_{H^1},$$

implies

$$\|g_i u_2\|^2_{H^1} \le \left(\frac{1}{k+1} + 2\varepsilon\right) \|u_2\|^2_{H^1} \le \left(\frac{1}{k+1} + 2\varepsilon\right) \|u\|^2_{H^1}. \tag{1.7}$$

In particular from the Moser-Trudinger inequality (1.2) and (1.7), we have for $\varepsilon$ small enough, which we now fix depending on $\delta_1$ and $k$,

$$\begin{aligned} \ln \int_\Sigma e^{(1+\varepsilon)|g_i u_2|^p} dv_g &\le \frac{2-p}{2} \left( \frac{p(1+\varepsilon)^{\frac{2}{p}} \|g_i u_2\|^2_{H^1}}{8\pi} \right)^{\frac{p}{2-p}} + C \\ &\le \frac{2-p}{2} \left( \frac{p\|u_2\|^2_{H^1}}{8\pi(k+1) - \delta_1} \right)^{\frac{p}{2-p}} + C. \end{aligned} \tag{1.8}$$

Notice also that since $V_{\varepsilon,\delta_0}$ is finite dimensional, we have

$$\|v\|_{L^\infty} \le \tilde{C}_{\varepsilon,\delta_0} \|v\|_{L^2} \le \hat{C}_{\varepsilon,\delta_0} \|v\|_{H^1}, \quad \text{for } v \in V_{\varepsilon,\delta_0},$$

hence

$$\|u_1\|_{L^\infty(\Omega)} \le \hat{C}_{\varepsilon,\delta_0} \|u_1\|_{H^1}.$$

Now, using the inequality

$$(a+b)^p \le C_{\varepsilon,p} a^p + (1+\varepsilon) b^p,$$

we get

$$\int_\Sigma e^{|g_i u|^p} dv_g \le e^{C_{\varepsilon,p}\|u_1\|^p_\infty} \int_\Sigma e^{(1+\varepsilon)|g_i u_2|^p} dv_g,$$

hence, from (1.4) and (1.8) we deduce

$$\begin{aligned} \ln \int_\Sigma \left(e^{|u|^p} - 1\right) dv_g &\le \ln \frac{1}{\gamma_0} + \ln \int_{\Omega_i} \left(e^{|u|^p} - 1\right) dv_g \\ &\le \ln \frac{1}{\gamma_0} + \ln \int_\Sigma e^{|g_i u|^p} dv_g \\ &\le \ln \frac{1}{\gamma_0} + C_{\varepsilon,p} \|u_1\|^p_{L^\infty} + \ln \int_\Sigma e^{(1+\varepsilon)|g_i u_2|^p} dv_g \\ &\le \frac{2-p}{2} \left( \frac{p\|u_2\|^2_{H^1}}{8\pi(k+1) - \delta_1} \right)^{\frac{p}{2-p}} + \tilde{C}_{\varepsilon,p} \|u_1\|^p_{H^1} + C', \end{aligned} \tag{1.9}$$

with $C' = C'(k, \delta_0, \delta_1, \gamma_0, \Sigma)$. A further application of Young's inequality to the term $\tilde{C}_{\varepsilon,p}\|u_1\|^p_{H^1}$ and the inequality $a^q + b^q \leq (a+b)^q$ for $q > 1$ then gives

$$\ln \int_\Sigma \left(e^{|u|^p} - 1\right) dv_g \leq \frac{2-p}{2} \left( \frac{p(\|u_2\|^2_{H^1} + \|u_1\|^2_{H^1})}{8\pi(k+1) - \delta_1} \right)^{\frac{p}{2-p}} + C$$

with $C = C(k, \delta_0, \delta_1, \gamma_0, \Sigma)$, and since $\|u_2\|^2_{H^1} + \|u_1\|^2_{H^1} = \|u\|^2_{H^1}$ we conclude. □

The next lemma, proven in [15, Lemma 2.3], is a criterion which implies the situation described by condition (1.4).

**Lemma 1.3.** *Let $k$ be a given positive integer, and consider $\varepsilon, r > 0$. Suppose that for a non-negative function $f \in L^1(\Sigma)$ with $\|f\|_{L^1} = 1$ there holds*

$$\int_{\bigcup_{i=1}^k B_r(x_i)} f_i dx < 1 - \varepsilon \qquad \textit{for every } k\textit{-tuple } x_1, \dots, x_k \in \Sigma. \tag{1.10}$$

*Then there exist $\bar{\varepsilon} > 0$ and $\bar{r} > 0$, depending only on $\varepsilon$, $r$, $k$ and $\Omega$ (but not on $f$), and $k+1$ points $\bar{x}_{1,f}, \dots, \bar{x}_{k+1,f} \in \Sigma$ such that*

$$\int_{B_{\bar{r}}(\bar{x}_{j,f})} f dx \geq \bar{\varepsilon}, \quad \textit{for } j = 1, \dots, k+1,$$

*and $B_{2\bar{r}}(\bar{x}_{i,f}) \cap B_{2\bar{r}}(\bar{x}_{j,f}) = \emptyset$ for $i \neq j$.*

Lemma 1.2 and Lemma 1.3 then imply the following other result.

**Lemma 1.4.** *If $\beta \in (4\pi k, 4\pi(k+1))$ with $k \geq 1$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive constant $L = L(\varepsilon, r, p, \beta)$ such that, for every non-negative $u \in H^1(\Sigma)$ with $J_{p,\beta}(u) \leq -L$ there exist $k$ points $x_1, \dots, x_k \in \Sigma$ such that*

$$\frac{\int_{\Sigma \setminus \cup_{i=1}^k B_r(x_i)} \left(e^{u^p} - 1\right) dv_g}{\int_\Sigma \left(e^{u^p} - 1\right) dv_g} < \varepsilon. \tag{1.11}$$

*Proof.* Fix $\varepsilon$, $r$, $p$, and $\beta$ as in the statement of the lemma and let $u \in H^1(\Sigma)$ be such that $J_{p,\beta}(u) \leq -L$ for some constant $L \geq 0$, and assume by that (1.11) fails for every $k$-tuple of points $x_1, \dots, x_k$. Then setting

$$f := \frac{e^{|u|^p} - 1}{\|e^{|u|^p} - 1\|_{L^1}}$$

we have that (1.10) holds. Therefore, by Lemma 1.3 we can find $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon, r, k, \Sigma)$, $\bar{r} = \bar{r}(\varepsilon, r, k, \Sigma)$ and points $\bar{x}_1, \dots, \bar{x}_{k+1} \in \Sigma$ such that the assumptions of Lemma 1.2 hold with $\Omega_i = B_{\bar{r}}(\bar{x}_i)$, $\gamma_0 = \bar{\varepsilon}$ and $\delta_0 = 2\bar{r}$. Fix also $\delta_1 = 8\pi(k+1) - 2\beta$. Then by Lemma 1.2 there exists a constant $\bar{C}$ depending on $k$, $\delta_0$, $\delta_1$, $\gamma_0$, $p$ and $\Sigma$, hence depending on $\varepsilon$, $r$, $p$, $\beta$, $k$ and $\Sigma$ such that

$$\ln \int_\Sigma \left(e^{|u|^p} - 1\right) dv_g \leq \frac{2-p}{2} \left( \frac{p\|u\|^2_{H^1}}{\beta} \right)^{\frac{p}{2-p}} + \bar{C},$$

hence $J_{p,\beta}(u) \geq -\bar{C}$, and up to choosing $L > \bar{C}$ we obtain a contradiction, unless (1.11) holds for a suitable $k$-tuple $x_1, \dots, x_k \in \Sigma$. □

Given $k \in \mathbb{N}$ we introduce the set of formal *barycenters of* $\Sigma$ *of order* $k$, namely

$$\Sigma_k = \left\{ \sigma = \sum_{i=1}^{k} t_i \delta_{x_i} : x_i \in \Sigma,\, t_i \geq 0,\, \sum_{i=1}^{k} t_i = 1 \right\},$$

where $\delta_{x_i}$ is the Dirac mass at $x_i$, see [15], [31].
We will see $\Sigma_k$ as a subset of $\mathcal{M}(\Sigma)$, the set of all probability Radon measures on $\Sigma$, endowed with the distance defined using duality versus Lipschitz functions:

$$\mathrm{dist}(\mu, \nu) := \sup_{\|h\|_{Lip(\Sigma)} \leq 1} \left| \int_\Sigma h\, d\mu - \int_\Sigma h\, d\nu \right|, \qquad \mu, \nu \in \mathcal{M}(\Sigma), \tag{1.12}$$

which receives the name of *Kantorovich-Rubinstein distance.*

**Lemma 1.5.** *For any* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* $r_\varepsilon > 0$ *such that, for any* $r \in (0, r_\varepsilon]$, *if* $f \in L^1(\Sigma)$ *is a non-negative function such that*

$$\frac{\int_{\Sigma \setminus \cup_{i=1}^k B_r(x_i)} f dv_g}{\int_\Sigma f dv_g} < \delta \tag{1.13}$$

*for some* $x_1, \dots, x_k \in \Sigma$, *then*

$$\mathrm{dist}\left( \frac{f dv_g}{\int_\Sigma f dv_g}, \sigma \right) < \varepsilon,$$

*where*

$$\sigma = \sum_{i=1}^{k} t_i \delta_{x_i}, \qquad t_i = \frac{\int_{B_r(x_i)} f dv_g}{\int_{\cup_{j=1}^k B_r(x_j)} f dv_g}.$$

*Proof.* Consider a function $h$ on $\Sigma$ with $\|h\|_{Lip(\Sigma)} \leq 1$, which we can assume to have zero average, and let us estimate for $\int_\Sigma f dv_g = 1$ (otherwise, we can rescale $f$ by a constant)

$$\left| \int_\Sigma f\, h\, dv_g - \int_\Sigma h\, d\sigma \right| \leq \sum_{i=1}^{k} \left| \int_{B_r(x_i)} f\, h\, dv_g - \int_{B_r(x_i)} h\, d\sigma \right| + \left| \int_{\Sigma \setminus \cup_{i=1}^k B_r(x_i)} f\, h\, dv_g \right|.$$

Since $f$ satisfies (1.10) and since $h$ is uniformly bounded by the diameter of $\Sigma$ (due to the fact that it is 1-Lipschitz and has zero average), by (1.13) we clearly have that

$$\left| \int_{\Sigma \setminus \cup_{i=1}^k B_r(x_i)} f\, h\, dv_g \right| \leq \delta\, diam_g(\Sigma).$$

On the other hand, for the same reason we have that $\int_{\cup_{j=1}^k B_r(x_j)} f dv_g = 1 + O(\delta)$, which implies that $t_i = (1 + O(\delta)) \int_{B_r(x_i)} f dv_g$ and in turn that

$$\begin{aligned} \int_{B_r(x_i)} h d\sigma &= h(x_i)(1 + O(\delta)) \int_{B_r(x_i)} f\, dv_g \\ &= h(x_i) \int_{B_r(x_i)} f dv_g + O(\delta). \end{aligned}$$

Again from the fact that $h$ is 1-Lipschitz, we get that

$$\int_{B_r(x_i)} f\, h\, dv_g = h(x_i) \int_{B_r(x_i)} f dv_g + O(r).$$

The conclusion then follows from the last four formulas and the arbitrarity of $h$. $\square$

An immediate consequence of Lemma 1.4 and Lemma 1.5 is that the low sublevels of $J_{p,\beta}$ can be mapped close to $\Sigma_k$, in the sense of the following lemma.

**Lemma 1.6.** *Given $\beta \in (4\pi k, 4\pi(k+1))$ with $k \geq 1$, $\varepsilon > 0$ there exists $L = L(\varepsilon, p, \beta)$ such that for every non-negative $u \in H^1(\Sigma)$ with $J_{p,\beta} \leq -L$ we have*

$$\mathrm{dist}\left(\frac{\left(e^{u^p}-1\right)dv_g}{\int_\Sigma \left(e^{u^p}-1\right)dv_g}, \Sigma_k\right) < \varepsilon.$$

Let us first recall a well known result about $\Sigma_k$, endowed with the topology induced by $\mathrm{dist}(\cdot,\cdot)$.

**Lemma 1.7** ([31])**.** *For any $k \geq 1$ the set $\Sigma_k$ is non-contractible.*

Our goal is to show that, if $\beta \in (4\pi k, 4\pi(k+1))$, $\Sigma_k$ can be mapped into very negative sublevels of $J_{p,\beta}$ and that this map is non trivial in the sense that it carries some homology. Then, as a consequence of the previous Lemma we will get the non contractibility of low sublevels of $J_{p,\beta}$.

Let us first define the *standard bubble* $\varphi_\gamma : \mathbb{R}^2 \to \mathbb{R}$ for $\gamma > 0$,

$$\varphi_\gamma(x) := \left(\frac{2}{p}\right)^{\frac{1}{p}} \gamma \left(1 - \frac{1}{\gamma^p} \ln\left(1 + \frac{|x|^2}{r_\gamma^2}\right)\right)_+,$$

where $r_\gamma$ is chosen so that

$$r_\gamma = o\left(e^{-\gamma^p}\right), \quad \ln\left(r_\gamma e^{\gamma^p}\right) = o(\gamma^p), \tag{1.14}$$

for instance, $r_\gamma = \gamma^{-1} e^{-\gamma^p}$.

Now, given $x \in \Sigma$ we define the function $\varphi_{\gamma,x} : \Sigma \to \mathbb{R}$ as

$$\varphi_{\gamma,x} = \left(\frac{2}{p}\right)^{\frac{1}{p}} \gamma \left(1 - \frac{1}{\gamma^p} \ln\left(1 + \frac{d^2(y,x)}{r_\gamma^2}\right)\right)_+.$$

Notice that $\varphi_{\gamma,x}(y) > 0$ if and only if $y \in B_{\delta_\gamma}(x)$, where

$$\delta_\gamma^2 := r_\gamma^2 (e^{\gamma^p} - 1) \to 0 \quad \text{as } \gamma \to \infty. \tag{1.15}$$

For a barycenter $\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$ we now want to construct test functions $\varphi_{\gamma,\sigma}$ continuous with respect to $\sigma$ (from $\mathcal{M}(\Sigma)$ into $H^1(\Sigma)$) concentrating mass near the points $x_i$, in the sense that

$$\frac{(e^{\varphi_{\gamma,\sigma}^p} - 1)dv_g}{\int_\Sigma \left(e^{\varphi_{\gamma,\sigma}^p} - 1\right)dv_g} \to \sigma, \quad \text{as } \gamma \to \infty. \tag{1.16}$$

In order to do so, to each $t \in [0,1]$ and $\gamma > 0$ we associate $\tau = \tau(t,\gamma)$ such that

$$\frac{\int_{\mathbb{R}^2} \left(e^{(\varphi_\gamma - \tau)_+^p} - 1\right)dx}{\int_{\mathbb{R}^2} \left(e^{\varphi_\gamma^p} - 1\right)dx} = t. \tag{1.17}$$

Notice that $\tau$ is decreasing with respect to $t$ and that $\tau(0,\gamma) = \gamma$, $\tau(1,\gamma) = 0$ for every $\gamma > 0$. We will need the following elementary estimate.

**Lemma 1.8.** *For any fixed $\bar{t} \in (0,1]$ with have $\tau(t,\gamma) \to 0$ as $\gamma \to \infty$ uniformly for $t \in [\bar{t}, 1]$.*

*Proof.* Given $\gamma, \tau > 0$, consider $L \geq \frac{2}{\gamma}$ to be fixed later. We easily see that

$$\begin{aligned}\int_{\{\varphi_\gamma < L\}} \left(e^{(\varphi_\gamma - \tau)_+^p} - 1\right) dx &\leq \int_{\{\varphi_\gamma < L\}} \left(e^{\varphi_\gamma^p} - 1\right) dx \\ &\leq \left(e^{L^p} - 1\right) \delta_\gamma^2 = o_\gamma(1).\end{aligned} \tag{1.18}$$

Moreover, for $\gamma$ such that

$$\frac{2}{p}\left(1 - \frac{\ln 2}{\gamma^p}\right)^p \geq 1 + \varepsilon > 1,$$

also using that $\varphi_\gamma \geq L$ on $B_{r_\gamma}$ for $\gamma$ large, we get

$$\begin{aligned}\int_{\{\varphi_\gamma \geq L\}} \left(e^{\varphi_\gamma^p} - 1\right) dx &\geq \int_{B_{r_\gamma}} \left(e^{\frac{2}{p}\gamma^p (1 - \frac{\ln 2}{\gamma^p})^p} - 1\right) dx \\ &\geq \int_{B_{r_\gamma}} \left(e^{(1+\varepsilon)\gamma^p} - 1\right) dx \\ &\geq \pi r_\gamma^2 \left(e^{(1+\varepsilon)\gamma^p} - 1\right) \to \infty.\end{aligned} \tag{1.19}$$

By the Taylor expansion

$$(1-x)^p = 1 - px + \frac{p(p-1)}{2(1-\xi)^{2-p}}\xi^2 \leq 1 - px + C_p x^2, \quad 0 \leq \xi \leq x \leq \frac{1}{2},$$

we get for $\varphi_\gamma \geq L \geq 2\tau$

$$(\varphi_\gamma - \tau)_+^p \leq \varphi_\gamma^p - p\tau\varphi_\gamma^{p-1} + C_p \tau^2 \varphi_\gamma^{p-2} \leq \varphi_\gamma^p - \frac{p}{2}\tau\varphi_\gamma^{p-1}$$

up to choosing $L \geq L_0(p)$ sufficiently large. We then infer

$$\begin{aligned}\int_{\{\varphi_\gamma \geq L\}} \left(e^{(\varphi_\gamma - \tau)_+^p} - 1\right) dx &\leq e^{-\frac{p}{2}\tau L^{p-1}} \int_{\{\varphi_\gamma \geq L\}} e^{\varphi_\gamma^p} dx \\ &= o_L\left(\int_{\mathbb{R}^2} \left(e^{\varphi_\gamma^p} - 1\right) dx\right), \quad \text{as } L \to \infty.\end{aligned} \tag{1.20}$$

Putting (1.18)-(1.20) together it follows that

$$t(\tau, \gamma) := \frac{\int_{\mathbb{R}^2} \left(e^{(\varphi_\gamma - \tau)_+^p} - 1\right) dx}{\int_{\mathbb{R}^2} \left(e^{\varphi_\gamma^p} - 1\right) dx} = o(1) \quad \text{as } \gamma \to \infty$$

for any $\tau > 0$. This implies that $\tau(\bar{t}, \gamma) = o(1)$ as $\gamma \to \infty$ for any $\bar{t} \in (0, 1]$ since otherwise there would be sequences $\gamma_\varepsilon \to 0$ and $\tau_\varepsilon \in (0, \gamma_\varepsilon]$ such that $\tau_\varepsilon(\bar{t}, \gamma_\varepsilon) \geq \tau_* > 0$, and by monotonicity

$$0 < \bar{t} = t(\tau_\varepsilon, \gamma_\varepsilon) \leq t(\tau_*, \gamma_\varepsilon) = o(1) \quad \text{as } \varepsilon \to 0,$$

a contradiction. Using the monotonicity of $\tau$ with respect to $t$ the conclusion follows at once. □

Now call $\tau_i = \tau(t_i, \gamma)$, $1 \leq i \leq k$ and define $\varphi_{\gamma,\sigma}$ by the formula

$$e^{\varphi_{\gamma,\sigma}^p} - 1 = \sum_{i=1}^k \left(e^{(\varphi_{\gamma, x_i} - \tau_i)_+^p} - 1\right),$$

or, explicitly

$$\varphi_{\gamma,\sigma} = \ln^{\frac{1}{p}}\left(1+\sum_{i=1}^{k}\left(e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}-1\right)\right). \tag{1.21}$$

*Notation.* Until the end of the section $o(1)$ (resp. $O(1)$) will denote a quantity tending to 0 (resp. a bounded quantity) as $\gamma \to \infty$, uniformly with respect to $x \in \Sigma$ and $\sigma \in \Sigma_k$

**Lemma 1.9.** *For every $x \in \Sigma$, we have*

$$\int_\Sigma |\nabla \varphi_{\gamma,x}|^2\, dv_g = \left(\frac{2}{p}\right)^{\frac{2}{p}} 4\pi\gamma^{2-p}(1+o(1)), \quad \text{as } \gamma \to \infty.$$

*Proof.* By a straightforward computation, for any $y \in B_{\delta_\gamma}(x)$, we get

$$\nabla\varphi_{\gamma,x}(y) = -\left(\frac{2}{p}\right)^{\frac{1}{p}} \gamma^{1-p}\,\frac{r_\gamma^{-2}\nabla_y(d^2(y,x))}{1+r_\gamma^{-2}d^2(y,x)},$$

while $\nabla\varphi_{\gamma,x}(y) = 0$ in $\Sigma \setminus B_{\delta_\gamma}(x)$.
Using geodesic coordinates centered at $x$, with an abuse of notation, we identify the points in $\Sigma$ with their pre-image under the exponential map. Using these coordinates, and recalling that $\delta_\gamma \to 0$ we have that

$$d(y,x) = |y-x|(1+o(1)), \quad |\nabla_y(d^2(y,x))| = 2|y-x|(1+o(1)), \quad y \in B_{\delta_\gamma}(x)$$

hence

$$|\nabla\varphi_{\gamma,x}(y)| = \left(\frac{2}{p}\right)^{\frac{1}{p}}(1+o(1))\frac{2|y-x|}{r_\gamma^2+|y-x|^2}, \quad y \in B_{\delta_\gamma}(x). \tag{1.22}$$

Thanks to the change of variable $s = r_\gamma^2+\rho^2$, we are able to conclude that

$$\begin{aligned}
\int_\Sigma |\nabla\varphi_{\gamma,x}|^2\, dv_g &= \int_{B_{\delta_\gamma}(x)} |\nabla\varphi_{\gamma,x}|^2\, dv_g \\
&= \left(\frac{2}{p}\right)^{\frac{2}{p}}(1+o(1))\int_{B^{\mathbb{R}^2}_{\delta_\gamma}(x)} \frac{4|y-x|^2}{(r_\gamma^2+|y-x|^2)^2}dy \\
&= \left(\frac{2}{p}\right)^{\frac{2}{p}} 4\pi(1+o(1))\int_0^{\delta_\gamma}\frac{2\rho^3}{(r_\gamma^2+\rho^2)^2}d\rho \\
&= \left(\frac{2}{p}\right)^{\frac{2}{p}} 4\pi\gamma^{2-2p}(1+o(1))\int_{r_\gamma^2}^{r_\gamma^2 e^{\gamma^p}}\left(\frac{1}{s}-\frac{r_\gamma^2}{s^2}\right)ds \\
&= \left(\frac{2}{p}\right)^{\frac{2}{p}} 4\pi\gamma^{2-p}(1+o(1)),
\end{aligned}$$

yielding the result. ∎

**Lemma 1.10.** *In the above notation we have, uniformly for $\sigma \in \Sigma_k$*

$$\int_\Sigma |\nabla\varphi_{\gamma,\sigma}|^2\, dv_g \le \left(\frac{2}{p}\right)^{\frac{2}{p}} 4\pi k\gamma^{2-p}(1+o(1)).$$

*Proof.* We compute

$$\nabla\varphi_{\gamma,\sigma} = \frac{\sum_{i=1}^k (\varphi_{\gamma,x_i}-\tau_i)_+^{p-1} e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}\nabla\varphi_{\gamma,x_i}}{\ln^{\frac{p-1}{p}}\left(1+\sum_{i=1}^k\left(e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}-1\right)\right)\left(1+\sum_{i=1}^k\left(e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}-1\right)\right)}.$$

Notice that

$$\begin{aligned}0 &\le \frac{(\varphi_{\gamma,x_i}-\tau_i)_+^{p-1} e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}}{\ln^{\frac{p-1}{p}}\left(1+\sum_{i=1}^k\left(e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}-1\right)\right)\left(1+\sum_{i=1}^k\left(e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}-1\right)\right)}\\ &\le a_i\chi_{\{\varphi_{\gamma,x_i}>\tau_i\}},\end{aligned}$$

where

$$a_i := \frac{e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}}{1+\sum_{i=1}^k\left(e^{(\varphi_{\gamma,x_i}-\tau_i)_+^p}-1\right)},$$

hence

$$|\nabla\varphi_{\gamma,\sigma}(x)| \le \sum_{i=1}^k a_i(x)|\nabla\varphi_{\gamma,x_i}(x)|\chi_{\{\varphi_{\gamma,x_i}>\tau_i\}}(x).$$

Split now $\Sigma$ as a disjoint (up to sets of measure zero) union $\Omega_1\cup\cdots\cup\Omega_k$, such that

$$|\nabla\varphi_{\gamma,x_i}(x)| = \max_{1\le j\le k}|\nabla\varphi_{\gamma,x_j}(x)| \quad \text{for } x\in\Omega_j,$$

and further split $\Sigma$ as $\Sigma=\Sigma_+\cup\Sigma_-$, where

$$\Sigma_+ := \left\{x\in\Sigma : \sum_{j=1}^k e^{(\varphi_j(x)-\tau_j)_+^p}\ge\gamma\right\}, \quad \Sigma_- := \Sigma\setminus\Sigma_+.$$

Notice that

$$\sum_{i=1}^k a_i(x) \le 1+o_\gamma(1) \quad \text{for } x\in\Sigma_+.$$

Then, with the help of Lemma 1.9 we obtain

$$\begin{aligned}\int_{\Sigma_+}|\nabla\varphi_{\gamma,\sigma}|^2dx &\le \sum_{j=1}^k\int_{\Sigma_+\cap\Omega_j}\left(\sum_{i=1}^k a_i|\nabla\varphi_{\gamma,x_j}|\right)^2dx\\ &\le (1+o(1))\sum_{j=1}^k\int_{\Sigma_+}|\nabla\varphi_{\gamma,x_j}|^2dx \qquad (1.23)\\ &\le (1+o(1))\left(\frac{2}{p}\right)^{\frac{2}{p}}4\pi k\gamma^{2-p}.\end{aligned}$$

We now want to prove that the integral over $\Sigma_-$ is negligible. Indeed we have

$$\sum_{i=1}^k a_i(x)\le k \quad \text{for } x\in\Sigma_-,$$

since $\frac{s}{s-k+1}\le k$ for $s\ge k$, and similarly to (1.23) we get

$$\int_{\Sigma_-}|\nabla\varphi_{\gamma,\sigma}|^2dx \le k^2\sum_{j=1}^k\int_{\Sigma_-\cap\Omega_j}|\nabla\varphi_{\gamma,x_j}|^2dx.$$

In order to estimate the right-hand side, observe that

$$1 \le e^{(\varphi_j(x)-\tau_j)_+^p} \le \gamma \quad \text{for } x \in \Sigma_-.$$

This implies that

$$\Sigma_- \cap \Omega_j \subset B_{R_1}(x_j) \setminus B_{r_1}(x_j) \quad \text{for every } j,$$

where $R_1$ and $r_1$ are given by the relations

$$1 \le e^{\left(C_p\gamma\left(1-\frac{1}{\gamma^p}\ln\left(1+\frac{d^2(x,x_j)}{r_\gamma^2}\right)\right)-\tau_j\right)^p} \le \gamma, \quad C_p := \left(\frac{2}{p}\right)^{\frac{1}{p}}.$$

This yields

$$\gamma^p - C_p^{-1}\tau_j\gamma^{p-1} \ge \ln\left(1+\frac{d^2(x,x_j)}{r_\gamma^2}\right) \ge \gamma^p - C_p^{-1}\tau_j\gamma^{p-1} - \gamma^{p-1}\ln^{\frac{1}{p}}\gamma,$$

and

$$R_1^2 = \left(e^{\gamma^p - C_p^{-1}\tau_j\gamma^{p-1}} - 1\right) r_\gamma^2, \quad r_1^2 = \left(e^{\gamma^p - C_p^{-1}\tau_j\gamma^{p-1} - \gamma^{p-1}\ln^{\frac{1}{p}}\gamma} - 1\right) r_\gamma^2.$$

We now integrate as in Lemma 1.9, and with the same change of variables $s = r_\gamma^2 + \rho^2$ we obtain

$$\begin{aligned}\int_{B_{R_1}(x_j)\setminus B_{r_1}(x_j)} |\nabla\varphi_{\gamma,x_j}|^2 dv_g &= O\left(\gamma^{2-2p}\right)\int_{r_\gamma^2+r_1^2}^{r_\gamma^2+R_1^2} \frac{s-r_\gamma^2}{s^2}ds \\ &\le O\left(\gamma^{2-2p}\right)\int_{r_\gamma^2 e^{\gamma^p - C_p^{-1}\tau_j\gamma^{p-1}-\gamma^{p-1}\ln^{\frac{1}{p}}\gamma}}^{r_\gamma^2 e^{\gamma^p - C_p^{-1}\tau_j\gamma^{p-1}}} \frac{ds}{s} \\ &= O\left(\gamma^{1-p}\ln^{\frac{1}{p}}\gamma\right) \\ &= o(\gamma^{2-p}).\end{aligned}$$

Together with (1.23), we conclude. □

**Lemma 1.11.** *We have the following estimates, uniformly for $\sigma \in \Sigma_k$*

$$\int_\Sigma \varphi_{\gamma,\sigma}^2\, dv_g = o(\gamma^{2-p}).$$

*Proof.* Let us first evaluate, for $x \in \Sigma$, $\int_\Sigma \varphi_{\gamma,x}^2\, dv_g$. Being

$$\int_{B_{r_\gamma}(x)} \varphi_{\gamma,x}^2\, dv_g = o(1), \quad \varphi_{\gamma,x} = 0 \text{ in } \Sigma \setminus B_{\delta_\gamma}(x),$$

it is enough to estimate $\int_{B_{\delta_\gamma}(x)\setminus B_{r_\gamma}(x)} \varphi_{\gamma,x}^2\, dv_g$.
Using normal coordinates at $x$ and the change of variables $s = 1+\frac{r^2}{r_\gamma^2}$, we get

$$\begin{aligned}\int_{\Sigma\setminus B_{r_\gamma}(x)} \varphi_{\gamma,x}^2 dv_g &= O(\gamma^2)\int_{r_\gamma}^{\delta_\gamma} r\left(1-\frac{2}{\gamma^p}\ln(1+\frac{r^2}{r_\gamma^2})+\frac{1}{\gamma^{2p}}\ln^2(1+\frac{r^2}{r_\gamma^2})\right)dr\\ &= O(\gamma^2)\int_2^{e^{\gamma^p}} r_\gamma^2\left(1-\frac{2}{\gamma^p}\ln(s)+\frac{1}{\gamma^{2p}}\ln^2(s)\right)ds\\ &= O(\gamma^2 r_\gamma^2)\left[s-\frac{2}{\gamma^p}(-s+s\ln s)+\frac{1}{\gamma^{2p}}(2s-2s\ln s+s\ln^2 s)\right]_2^{e^{\gamma^p}}\\ &= O(\gamma^{4-4p})\\ &= o(\gamma^{2-p}).\end{aligned} \tag{1.24}$$

Splitting $\Sigma$ as a disjoint (up to sets of measure zero) union $\tilde\Omega_1\cup\cdots\cup\tilde\Omega_k$, so that

$$\varphi_{\gamma,x_i}(x) = \max_{1\le j\le k}\varphi_{\gamma,x_j}(x)\quad \text{for } x\in\tilde\Omega_j,$$

we have

$$\begin{aligned}\varphi_{\gamma,\sigma}^2(x) &\le \ln^{\frac{2}{p}}\left(\sum_{i=1}^k e^{\varphi_{\gamma,x_i}^p(x)}\right)\le \sum_{j=1}^k \chi_{\tilde\Omega_j}(x)\ln^{\frac{2}{p}}\left(e^{\varphi_{\gamma,x_j}^p(x)}\right)\\ &\le \sum_{j=1}^k\left(\ln k+\varphi_{\gamma,x_j}^p(x)\right)^{\frac{2}{p}}\\ &\le O(1)+O(1)\sum_{j=1}^k\varphi_{\gamma,x_j}^2(x),\end{aligned}$$

where in the last inequality we used the convexity of the map $t\mapsto t^{\frac{2}{p}}$.

As a consequence

$$\int_\Sigma \varphi_{\gamma,\sigma}^2\, dv_g = O(1)+O(1)\sum_{j=1}^k\int_\Sigma\varphi_{\gamma,x_j}^2(x)\, dv_g \overset{(1.24)}{=} o(\gamma^{2-p}),$$

as desired. □

**Lemma 1.12.** *We have, uniformly for $\sigma\in\Sigma_k$*

$$\ln\int_\Sigma\left(e^{\varphi_{\gamma,\sigma}^p}-1\right)dv_g\ge\frac{2-p}{p}\gamma^p(1+o(1)),\quad \text{as } \gamma\to\infty.$$

*Proof.* Given $\sigma=\sum_{i=1}^k t_i\delta_{x_i}\in\Sigma_k$, fix $i$ such that $t_i\ge\frac{1}{k}$. Then, according to Lemma 1.8 we have $\tau_i=o(1)$ as $\gamma\to\infty$, hence

$$\varphi_{\gamma,\sigma}^p\ge(\varphi_{\gamma,x_i}-\tau_i)_+^p\ge\frac{2}{p}\gamma^p\left(1-\frac{\ln 2}{\gamma^p}-o_\gamma(1)\right)_+^p\ge\frac{2}{p}\gamma^p(1+o(1))\quad \text{on } B_{r_\gamma}(x_i)$$

for $\gamma$ sufficiently large. Then, also using (1.14), it follows

$$\begin{aligned}\ln\int_{\Sigma}\left(e^{\varphi^p_{\gamma,\sigma}}-1\right)dv_g &\geq \ln\int_{B_{r_\gamma}}\left(e^{\varphi^p_{\gamma,\sigma}}-1\right)dv_g\\ &\geq \ln\left((1+o(1))\pi r_\gamma^2 e^{\frac{2}{p}\gamma^p(1+o(1))}\right)\\ &= \left(\frac{2}{p}-1+o(1)\right)\gamma^p\\ &= \frac{2-p}{p}\gamma^p(1+o(1)),\end{aligned}$$

as claimed. □

**Lemma 1.13.** *Given $\beta\in(4\pi k,4\pi(k+1))$, with $k\geq 1$, then as $\gamma\to+\infty$ we have:*

*i.* $J_{p,\beta}(\varphi_{\gamma,\sigma})\to-\infty$ *uniformly for* $\sigma\in\Sigma_k$,

*ii.* $\operatorname{dist}\left(\frac{\left(e^{\varphi^p_{\gamma,\sigma}}-1\right)dv_g}{\int_\Sigma\left(e^{\varphi^p_{\gamma,\sigma}}-1\right)dv_g},\sigma\right)\to 0$ *uniformly for* $\sigma\in\Sigma_k$, *see* (1.12).

*Proof. i.* By definition of $\varphi_{\gamma,\sigma}$ and Lemma 1.12 we have

$$\begin{aligned}J_{p,\beta}(\varphi_{\gamma,\sigma}) &= \frac{2-p}{2}\left(\frac{p\|\varphi_{\gamma,\sigma}\|^2_{H^1}}{2\beta}\right)^{\frac{p}{2-p}}-\ln\int_\Sigma\left(e^{\varphi^p_{\gamma,\sigma}}-1\right)dv_g\\ &\leq \frac{2-p}{2}\left(\frac{p(\frac{2}{p})^{\frac{2}{p}}4\pi k\gamma^{2-p}(1+o(1))}{2\beta}\right)^{\frac{p}{2-p}}-\frac{2-p}{p}\gamma^p(1+o(1))\\ &= \frac{2-p}{2}\left[\left(\frac{4\pi k}{\beta}\right)^{\frac{p}{2-p}}\frac{2}{p}\gamma^p(1+o(1))\right]-\frac{2-p}{p}\gamma^p(1+o(1))\\ &= \frac{2-p}{p}\gamma^p\left[\left(\frac{4\pi k}{\beta}\right)^{\frac{p}{2-p}}-1\right](1+o(1))\to-\infty,\end{aligned}$$

uniformly for $\sigma\in\Sigma_k$.

*ii.* Let us first collect some simple calculations.
Let $\sigma=\sum_{i=1}^k t_i\delta_{x_i}\in\Sigma_k$: then, since $\delta_\gamma\to 0$ when $\gamma\to+\infty$,

$$\begin{aligned}\int_{B_{\delta_\gamma}(x_i)}\left(e^{(\varphi_{\gamma,x_i}-\tau_i)^p_+}-1\right)dv_g &= (1+o(1))\int_{B^{\mathbb{R}^2}_{\delta_\gamma}(0)}\left(e^{(\varphi_\gamma-\tau_i)^p_+}-1\right)dx\\ &\overset{(1.17)}{=}(1+o(1))\,t_i\int_{\mathbb{R}^2}\left(e^{\varphi^p_\gamma}-1\right)dx,\end{aligned}\tag{1.25}$$

as a consequence

$$\begin{aligned}\int_{\cup_{j=1}^k B_{\delta_\gamma}(x_j)}(e^{\varphi^p_{\gamma,\sigma}}-1)\,dv_g &\overset{(1.21)}{=} \int_{\cup_{j=1}^k B_{\delta_\gamma}(x_j)}\left(\sum_{i=1}^k\left(e^{(\varphi_{\gamma,x_i}-\tau_i)^p_+}-1\right)\right)dv_g\\ &= \sum_{i=1}^k\int_{B_{\delta_\gamma}(x_i)}\left(e^{(\varphi_{\gamma,x_i}-\tau_i)^p_+}-1\right)dv_g\\ &\overset{(1.25)}{=}(1+o(1))\int_{\mathbb{R}^2}\left(e^{\varphi^p_\gamma}-1\right)dx,\end{aligned}\tag{1.26}$$

where in the second identity we used that $\varphi_{\gamma,x_i} \equiv 0$ on $\Sigma \setminus B_{\delta_\gamma}(x_i)$. Given $\varepsilon > 0$, we need to show that

$$\operatorname{dist}(f_{\gamma,\sigma} dv_g, \sigma) < 2\varepsilon \qquad \text{for } \gamma \text{ sufficiently large,}$$

uniformly for $\sigma \in \Sigma_k$, where

$$f_{\gamma,\sigma} = \frac{\left(e^{\varphi^p_{\gamma,\sigma}} - 1\right)}{\int_\Sigma \left(e^{\varphi^p_{\gamma,\sigma}} - 1\right) dv_g}.$$

Let $\delta > 0$ and $r_\varepsilon > 0$ be the positive constants of the statement of Lemma 1.5.
It is immediate to see that $f_{\gamma,\sigma}$ satisfies (1.13), being $\varphi_{\gamma,\sigma} \equiv 0$ in $\Sigma \setminus \cup_{i=1}^k B_{\delta_\gamma}$, then by Lemma 1.5 (which holds with $r = \delta_\gamma$, if $\gamma$ is sufficiently large)

$$\operatorname{dist}(f_{\gamma,\sigma}, \sigma_\gamma) < \varepsilon \quad \text{where} \quad \sigma_\gamma := \sum_{i=1}^k \frac{\int_{B_{\delta_\gamma}(x_i)} (e^{\varphi^p_{\gamma,\sigma}} - 1) dv_g}{\int_{\cup_{j=1}^k B_{\delta_\gamma}(x_j)} (e^{\varphi^p_{\gamma,\sigma}} - 1) dv_g} \delta_{x_i}. \tag{1.27}$$

In virtue of (1.25) and (1.26) $\sigma_\gamma = \sum_{i=1}^k t_i(1 + o(1))\delta_{x_i}$, and so

$$\operatorname{dist}(\sigma_\gamma, \sigma) < \varepsilon \quad \text{for } \gamma \text{ sufficiently large.} \tag{1.28}$$

The thesis follows from (1.27) and (1.28). □

Let us set for $L > 0$

$$J^{-L}_{p,\beta} := \{u \in H^1(\Sigma) : J_{p,\beta}(u) \le -L\}.$$

**Proposition 1.1.** *Let $\beta \in (4\pi k + \delta, 4\pi(k+1) - \delta)$, with $k \ge 1$ and $\delta \in (0, \frac{1}{2})$. Then, there exist $L > 0$ and $\gamma > 0$ sufficiently large depending on $p$, $k$ and $\delta$, and a continuous function*

$$\Psi : J^{-L}_{p,\beta} \longrightarrow \Sigma_k$$

*such that i) $\Phi(\sigma) := \varphi_{\gamma,\sigma} \in J^{-2L}_{p,\beta}$ for every $\sigma \in \Sigma_k$ and ii) the map $\Psi \circ \Phi : \Sigma_k \to \Sigma_k$, is homotopically equivalent to the identity on $\Sigma_k$.*

*Proof.* By [1, Proposition 2.2] there exist $\varepsilon > 0$ and a continuous retraction

$$\hat{\Psi} : \{\sigma \in \mathcal{M}(\Sigma) \,:\, \operatorname{dist}(\sigma, \Sigma_k) < \varepsilon\} \to \Sigma_k.$$

By Lemma 1.6 there exists $L = L(\varepsilon, p, \beta)$ such that for every $u \in J^{-L}_{p,\beta}$

$$\operatorname{dist}\left( \frac{\left(e^{u^p} - 1\right) dv_g}{\int_\Sigma \left(e^{u^p} - 1\right) dv_g}, \Sigma_k \right) < \varepsilon.$$

Since the map $u \mapsto \frac{\left(e^{u^p} - 1\right) dv_g}{\int_\Sigma \left(e^{u^p} - 1\right) dv_g}$ is continuous from $J^{-L}_{p,\beta} \subset H^1(\Sigma)$ into $\mathcal{M}(\Sigma)$, for such $L$ the map $\Psi : J^{-L}_{p,\beta} \to \Sigma_k$ defined as

$$\Psi(u) := \hat{\Psi}\left( \frac{\left(e^{u^p} - 1\right) dv_g}{\int_\Sigma \left(e^{u^p} - 1\right) dv_g} \right)$$

is well posed and continuous with respect to the $H^1(\Sigma)$ topology.

In turn, by Lemma 1.13 there exist $\gamma > 0$ such that

$$\varphi_{\gamma,\sigma} \in J^{-2L}_{p,\beta}, \quad \operatorname{dist}\left( \frac{\left(e^{\varphi^p_{\gamma,\sigma}} - 1\right) dv_g}{\int_\Sigma \left(e^{\varphi^p_{\gamma,\sigma}} d - 1\right) v_g}, \sigma \right) < \varepsilon, \quad \text{for any } \sigma \in \Sigma_k. \tag{1.29}$$

Hence $\Psi \circ \Phi(\sigma) = \Psi(\varphi_{\gamma,\sigma})$ is well defined and we only need to show that $\Psi \circ \Phi \simeq \mathrm{Id}_{\Sigma_k}$. Consider the homotopy $H : [0,1] \times \Sigma_k \to \mathcal{M}(\Sigma)$ given by

$$H(s,\sigma) = s\sigma + (1-s)\frac{\left(e^{\varphi^p_{\gamma,\sigma}-1}\right) dv_g}{\int_\Sigma \left(e^{\varphi^p_{\gamma,\sigma}}-1\right) dv_g}.$$

From (1.29) we infer that

$$\mathrm{dist}(H(s,\sigma),\Sigma_k) \le \mathrm{dist}(H(s,\sigma),\sigma) < \varepsilon \quad \text{for } s \in [0,1],\ \sigma \in \Sigma_k,$$

so $\hat{\Psi}$ is well defined on the image of $H$ and we can then define the homotopy $\mathcal{H} : [0,1] \times \Sigma_k \to \Sigma_k$

$$\mathcal{H}(s,\sigma) = \hat{\Psi} \circ H(s,\sigma).$$

Clearly $\mathcal{H}(0,\cdot) = \Psi \circ \Phi$ and $\mathcal{H}(1,\cdot) = \mathrm{Id}_{\Sigma_k}$. □

We are now ready to construct a minmax scheme in the spirit of [15]. Given $p$, $k$ and $\delta > 0$, fix $L > 0$, $\gamma > 0$ and $\Phi : \Sigma_k \to H^1(\Sigma)$ as in Proposition 1.1.

Consider the topological cone $\mathcal{C}_k$ over $\Sigma_k$ defined as

$$\mathcal{C}_k = (\Sigma_k \times [0,1])/\sim$$

where $(\sigma_1, r_1) \sim (\sigma_2, r_2)$ if and only if $r_1 = r_2 = 1$. We shall also identify $\Sigma_k \times \{0\}$ with $\Sigma_k$. Set

$$\mathcal{A}_k := \{\bar{\Phi} \in C^0(\mathcal{C}_k, H^1(\Sigma)) \text{ s.t. } \bar{\Phi}|_{\Sigma_k} = \Phi\},$$

and call

$$\alpha_\beta := \inf_{\bar{\Phi}\in\mathcal{A}_k} \max_{\xi\in\mathcal{C}_k} J_{p,\beta}(\bar{\Phi}(\xi)) \tag{1.30}$$

the minmax value.

**Lemma 1.14.** *With the above choice of $L$ and $\gamma$, depending on $p$, $k$ and $\delta$, we have*

$$\alpha_\beta \ge -L, \qquad \sup_{\bar{\Phi}\in\mathcal{A}_k} \sup_{\xi\in\Sigma_k} J_{p,\beta}(\bar{\Phi}(\xi)) \le -2L. \tag{1.31}$$

*Proof.* The second inequality follows immediately from Proposition 1.1. Assume by contradiction that $\alpha_\beta < -L$: then we can find $\bar{\Phi} \in \mathcal{A}_k$ such that

$$\bar{\Phi}(\mathcal{C}_k) \subset J^{-L}_{p,\beta}.$$

By Proposition 1.1, the map

$$\Psi \circ \bar{\Phi} : \mathcal{C}_k \to \Sigma_k$$

is well-defined and continuous. Moreover, on the one hand

$$\Psi \circ \bar{\Phi}|_{\Sigma_k} = \Psi \circ \Phi \simeq Id_{\Sigma_k}, \tag{1.32}$$

and on the other hand $\Psi \circ \bar{\Phi}$ gives a homotopy between $\Psi \circ \bar{\Phi}(\cdot,0) = \Psi \circ \bar{\Phi}|_{\Sigma_k}$ and the constant map $\Psi \circ \bar{\Phi}(\cdot,1)$. This and (1.32) imply that $\Sigma_k$ is homotopic to a point, which contradicts Lemma 1.7. □

We will now use a well-known monotonocity trick by Struwe to construct bounded Palais-Smale sequences for $J_{p,\beta}$ at level $\alpha_\beta$, as defined in (1.30):

**Proposition 1.2.** *For almost every $\beta > 4\pi$ the functional $J_{p,\beta}$ admits a bounded Palais-Smale sequence at level $\alpha_\beta$, i.e. a sequence $(u_\varepsilon)$ bounded in $H^1(\Sigma)$ such that*

$$J_{p,\beta}(u_\varepsilon) \to \alpha_\beta, \quad J'_{p,\beta}(u_\varepsilon) \to 0 \quad \text{as } k \to \infty. \tag{1.33}$$

*Proof.* Since for all $u \in H^1$ $\beta \mapsto J_{p,\beta}(u)$ is monotone decreasing, the function $\beta \mapsto \alpha_\beta$ is *non-increasing*, hence it is differentiable almost everywhere. Set

$$D_p := \{\beta \in (4\pi, \infty) \setminus 4\pi\mathbb{N} : \alpha_\beta \text{ is differentiable}\}.$$

Take $\beta \in D_p$, fix $\delta \in (0, \frac{1}{2})$ and $k \in \mathbb{N}^\star$ such that $\beta \in (4\pi k + \delta, 4\pi(k+1) - \delta)$, and choose a sequence $\beta_\varepsilon \uparrow \beta$ with $\beta_\varepsilon \in (4\pi k + \delta, 4\pi(k+1) - \delta)$. For every $\varepsilon > 0$ let a function $\bar{\Phi}_\varepsilon \in \mathcal{A}_k$ be given such that

$$\max_{\xi \in \mathcal{C}_k} J_{p,\beta_\varepsilon}(\bar{\Phi}_\varepsilon(\xi)) \leq \alpha_{\beta_\varepsilon} + (\beta - \beta_\varepsilon), \tag{1.34}$$

and let also $\xi_\varepsilon \in \mathcal{C}_k$ be given such that

$$J_{p,\beta}(\bar{\Phi}_\varepsilon(\xi_\varepsilon)) \geq \alpha_\beta. \tag{1.35}$$

Notice that the set of $(\bar{\Phi}_\varepsilon, \xi_\varepsilon)$'s satisfying (1.34)-(1.35) is non-empty thanks to (1.30) (used with $\beta$ and $\beta_\varepsilon$).

Set $v_\varepsilon := \bar{\Phi}_\varepsilon(\xi_\varepsilon)$. Then, posing $C_p := \frac{2-p}{2}\left(\frac{p}{2}\right)^{\frac{p}{2-p}}$, we have that

$$J_{p,\beta_\varepsilon}(v_\varepsilon) - J_{p,\beta}(v_\varepsilon) = C_p \|v_\varepsilon\|_{H^1}^{\frac{2p}{2-p}} \left( \frac{1}{\beta_\varepsilon^{\frac{p}{2-p}}} - \frac{1}{\beta^{\frac{p}{2-p}}} \right),$$

hence, setting $q = \frac{p}{2-p}$, $\alpha'_\beta = \frac{d\alpha_\beta}{d\beta}$, and writing

$$\beta^q - \beta_\varepsilon^q = -q\beta^{q-1}(\beta_\varepsilon - \beta) + o(\beta_\varepsilon - \beta),$$

we bound

$$\begin{aligned}
\|v_\varepsilon\|_{H^1}^{2q} &= \frac{(\beta_\varepsilon\beta)^q}{C_p} \frac{J_{p,\beta_\varepsilon}(v_\varepsilon) - J_{p,\beta}(v_\varepsilon)}{\beta^q - \beta_\varepsilon^q} \\
&\leq \frac{(\beta_\varepsilon\beta)^q}{C_p} \frac{\alpha_{\beta_\varepsilon} - \alpha_\beta + \beta - \beta_\varepsilon}{\beta^q - \beta_\varepsilon^q} \\
&= \frac{\beta^{2q} + o(1)}{C_p} \cdot \frac{-\alpha'_\beta + 1 + o(1)}{q\beta^{q-1}} \\
&\leq \bar{C}_{p,\beta}.
\end{aligned} \tag{1.36}$$

In particular $\|v_\varepsilon\|_{H^1}^{\frac{2p}{p-2}} = O(1)$ as $\varepsilon \to 0$ for any sequence $v_\varepsilon = \Phi_\varepsilon(\xi_\varepsilon)$, where $\Phi_\varepsilon$ and $\xi_\varepsilon$ satisfy (1.34) and (1.35).

We now proceed similarly to [13]. For every $\delta > 0$ (not the same as in Lemma 1.14) consider the set

$$N_{\delta,M} := \left\{ u \in H^1(\Sigma) : \|u\|_{H^1} \leq M, \, |J_{p,\beta}(u) - \alpha_\beta| < \delta \right\}$$

for $M \geq \bar{C}_{p,\beta}^{\frac{p-2}{2p}} + 1$, where $\bar{C}_{p,\beta}$ is as in (1.36). Notice that $N_{\delta,M}$ is non-empty by the previous discussion.

Assume that the claim of the proposition is false, so that there exists $\delta > 0$ small such that

$$\|J'_{p,\beta}(u)\|_{H^{-1}} = \sup_{\|v\|_{H^1} \leq 1} \left\langle J'_{p,\beta}(u), v \right\rangle \geq 2\delta \quad \text{for } u \in N_{\delta,M}.$$

Since $J_{p,\beta}$ is of class $C^1$ (on the open set of $H^1(\Sigma)$ where it is finite), we can construct a locally Lipschitz pseudo-gradient vector field (see e.g. [40, Lemma 3.2])

$$X : H^1(\Sigma) \to H^1(\Sigma)$$

such that

$$\sup_{u\in N_{\delta,M}}\|X(u)\|_{H^1}\le 1,\qquad \sup_{u\in N_{\delta,M}}\left\langle J'_{p,\beta}(u),X(u)\right\rangle\le-\delta.$$

We have

$$\left\langle J'_{p,\beta}(u),v\right\rangle=C_{p,\beta}\|u\|_{H^1}^{\frac{4p-4}{2-p}}\left\langle u,v\right\rangle_{H^1}-\frac{\int_\Sigma pu_+^{p-1}e^{u_+^p}v\,dv_g}{\int_\Sigma\left(e^{u_+^p}-1\right)dv_g},\tag{1.37}$$

where $C_{p,\beta}=p\left(\frac{p}{2\beta}\right)^{\frac{p}{2-p}}$, hence, for any sequence $\beta_\varepsilon\uparrow\beta$

$$\|J'_{p,\beta}(u)-J'_{p,\beta_\varepsilon}(u)\|_{H^{-1}}\le\left(C_{p,\beta}-C_{p,\beta_\varepsilon}\right)\|u\|_{H^1}^{\frac{3p-2}{2-p}}=o(1)\quad\text{as }\varepsilon\to 0,$$

uniformly for $u\in N_{\delta,M}$. Then for $\varepsilon$ small we have

$$\sup_{u\in N_{\delta,M}}\left\langle J'_{p,\beta_\varepsilon}(u),X(u)\right\rangle\le 0.$$

We now choose a Lipschitz cut-off function $\eta:H^1(\Sigma)\to[0,1]$ such that

$$\eta(u)=0\text{ if }u\in H^1(\Sigma)\setminus N_{\delta,M}$$

and

$$\eta(u)=1\text{ if }u\in N_{\frac{\delta}{2},M-1},$$

and consider the flow $\phi_t:H^1(\Sigma)\to H^1(\Sigma)$ generated by the vector field $\eta X$. Assuming with no loss of generality that $-2L<\alpha_\beta-\delta$, since $\Phi(\Sigma_k)\subset J_{p,\beta}^{-2L}$, it follows that

$$\phi_t\circ\bar\Phi|_{\Sigma_k}=\bar\Phi|_{\Sigma_k}=\Phi,$$

hence

$$\phi_t\circ\bar\Phi\in\mathcal{A}_k\quad\text{for every }\bar\Phi\in\mathcal{A}_k,\quad t\ge 0.$$

Moreover

$$\left.\frac{dJ_{p,\beta_\varepsilon}(\phi_t(u))}{dt}\right|_{t=0}\le 0,\quad\text{for }u\in H^1(\Sigma),\tag{1.38}$$

hence if $\bar\Phi_\varepsilon$ satisfies (1.34), so does $\phi_t\circ\bar\Phi_\varepsilon$ for $t\ge 0$. Moreover given any $\bar\Phi_\varepsilon\in\mathcal{A}_k$ satisfying (1.34)

$$\alpha_\beta\le\max_{\xi\in\mathcal{C}_k}J_{p,\beta}(\phi_t(\bar\Phi_\varepsilon(\xi)))=\max_{\xi\in\mathcal{C}_k:\bar\Phi_\varepsilon(\xi)\in N_{\frac{\delta}{2},M-1}}J_{p,\beta}(\phi_t(\bar\Phi_\varepsilon(\xi))),\tag{1.39}$$

since every $\xi_\varepsilon\in\mathcal{C}_k$ attaining the maximum of $J_{p,\beta}(\phi_t(\bar\Phi_\varepsilon(\cdot)))$ satisfies (1.35), so (1.39) follows from (1.36) and our choice of $M$. Therefore, since

$$\left.\frac{dJ_{p,\beta}(\phi_t(u))}{dt}\right|_{t=0}\le-\delta,\quad\text{for }u\in N_{\frac{\delta}{2},M-1},$$

we infer

$$\frac{d}{dt}\sup_{\xi\in\mathcal{C}_k}J_{p,\beta}(\phi_t(\bar\Phi_\varepsilon(\xi)))\le-\delta\quad\text{for }t\ge 0,$$

which contradicts (1.39). □

**Proposition 1.3.** *Given $p\in(1,2)$ and $\beta>0$, let $(u_\varepsilon)\subset H^1(\Sigma)$ be a bounded Palais-Smale sequence for $J_{p,\beta}$. Then up to a subsequence we have $u_\varepsilon\to u_0$ strongly in $H^1(\Sigma)$, where $u_0$ is a positive critical point of $J_{p,\beta}$.*

*Proof.* Up to a subsequence we have $u_\varepsilon \to u_0$ in $L^q(\Sigma)$ for every $q < \infty$, almost everywhere and weakly in $H^1(\Sigma)$. Moreover, by Young's inequality and the Moser-Trudinger inequality we infer

$$\|e^{u_{\varepsilon+}^p}\|_{L^q} \le C(p,q,\|u_\varepsilon\|_{H^1}) \quad \text{for every } q < \infty, \tag{1.40}$$

hence from Vitali's theorem

$$\int_\Sigma e^{u_{\varepsilon+}^p} dv_g \to \int_\Sigma e^{u_{0+}^p} dv_g \quad \text{as } \varepsilon \to 0. \tag{1.41}$$

From (1.37) we deduce that

$$\left\langle J'_{p,\beta}(u_\varepsilon), u_\varepsilon - u_0 \right\rangle = o(1) \quad \text{as } \varepsilon \to 0.$$

Using $u_\varepsilon - u_0$ as test function in $J'_{p,\beta}(u_\varepsilon) \to 0$, we obtain

$$\begin{aligned} o(1) &= \left\langle J'_{p,\beta}(u_\varepsilon) - J'_{p,\beta}(u_0), u_\varepsilon - u_0 \right\rangle \\ &= C_{p,\beta} \left\langle \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} u_\varepsilon - \|u_0\|_{H^1}^{\frac{4p-4}{2-p}} u_0, u_\varepsilon - u_0 \right\rangle_{H^1} \\ &\quad - \frac{\int_\Sigma p u_{\varepsilon+}^{p-1} e^{u_{\varepsilon+}^p}(u_\varepsilon - u_0) dv_g}{\int_\Sigma e^{u_{\varepsilon+}^p} dv_g} + \frac{\int_\Sigma p u_{0+}^{p-1} e^{u_{0+}^p}(u_\varepsilon - u_0) dv_g}{\int_\Sigma e^{u_{0+}^p} dv_g}. \end{aligned}$$

Taking (1.40), (1.41) and the Sobolev embedding into account we notice that the last two terms sum up to $o(1)$, so that

$$\begin{aligned} o(1) &= \left\langle \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} u_\varepsilon - \|u_0\|_{H^1}^{\frac{4p-4}{2-p}} u_0, u_\varepsilon - u_0 \right\rangle_{H^1} \\ &= \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} \|u_\varepsilon - u_0\|_{H^1}^2 + \left( \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} - \|u_0\|_{H^1}^{\frac{4p-4}{2-p}} \right) \langle u_0, u_\varepsilon - u_0 \rangle_{H^1} \\ &= \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} \|u_\varepsilon - u_0\|_{H^1}^2 + o(1), \end{aligned}$$

hence $u_\varepsilon \to u_0$ strongly in $H^1(\Sigma)$.

In order to prove that $u_0$ is a critical point of $J_{p,\beta}$, for $v \in H^1(\Sigma)$ we write

$$\begin{aligned} J'_{p,\beta}(u_0)(v) &= J'_{p,\beta}(u_0)(v) - J'_{p,\beta}(u_\varepsilon)(v) + o(1) \\ &= C_{p,\beta} \left\langle \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} u_\varepsilon - \|u_0\|_{H^1}^{\frac{4p-4}{2-p}} u_0, v \right\rangle_{H^1} \\ &\quad - \frac{\int_\Sigma p u_{\varepsilon+}^{p-1} e^{u_{\varepsilon+}^p} v dv_g}{\int_\Sigma e^{u_{\varepsilon+}^p} dv_g} + \frac{\int_\Sigma p u_{0+}^{p-1} e^{u_{0+}^p} v dv_g}{\int_\Sigma e^{u_{0+}^p} dv_g} + o(1) \\ &= C_{p,\beta} \|u_0\|_{H^1}^{\frac{4p-4}{2-p}} \langle u_\varepsilon - u_0, v \rangle_{H^1} \\ &\quad + \left( \|u_\varepsilon\|_{H^1}^{\frac{4p-4}{2-p}} - \|u_0\|_{H^1}^{\frac{4p-4}{2-p}} \right) \langle u_\varepsilon, v \rangle_{H^1} + o(1) \\ &= o(1), \end{aligned}$$

hence $J'_{p,\beta}(u_0) = 0$.

Were $u_0 \equiv 0$, with (1.41) we would infer that $J_{p,\beta}(u_\varepsilon) \to \infty$, which is impossible since $(u_\varepsilon)$ is a Palais-Smale sequence. Then Lemma 1.1 implies that $u_0 > 0$. □

**Remark 1.1.** *The analogue of proposition 1.3 does not hold in the case $p = 2$ as proven by Costa-Tintarev (Theorem 5.1 in [7]).*

*Proof of Theorem 1.1 (completed).* For every $\beta \in (0, 4\pi)$ the functional $J_{p,\beta}$ has a minimizer, hence a critical point, which can be obtained via direct methods, using (1.2), (1.40) and (1.41). The existence of critical points for a.e. $\beta > 4\pi$ follows at once from Propositions 1.2 and 1.3. □

## 2. A first analysis in the radially symmetric case

Let $(p_\gamma)_\gamma$ be any family of numbers in $[1,2]$, and let $(\mu_\gamma)_\gamma$ be a given family of positive real numbers. Let $\lambda_\gamma > 0$ be given by

$$\lambda_\gamma p_\gamma^2 \gamma^{2(p_\gamma - 1)} \mu_\gamma^2 e^{\gamma^{p_\gamma}} = 8\,, \tag{2.1}$$

and let $t_\gamma, \bar{t}_\gamma$ be defined in $\mathbb{R}^2$ by

$$t_\gamma(x) = \ln\left(1 + \frac{|x|^2}{\mu_\gamma^2}\right); \qquad \bar{t}_\gamma = t_\gamma + 1\,, \tag{2.2}$$

for all $\gamma > 0$ large. In the sequel, for any radially symmetric function $f$ around $0 \in \mathbb{R}^2$, since no confusion is then possible, we often make an abuse of notation and write $f(r)$ instead of $f(x)$ for $|x| = r$. Let $\eta \in (0,1)$ be fixed. Let also $(\bar{r}_\gamma)_\gamma$ be any family of positive real numbers such that

$$\lim_{\gamma \to +\infty} \frac{\mu_\gamma}{\bar{r}_\gamma} = 0\,, \tag{2.3}$$

$$t_\gamma(\bar{r}_\gamma) \le \eta \frac{p_\gamma \gamma^{p_\gamma}}{2}\,, \tag{2.4}$$

$$\gamma^{2p_\gamma} \bar{r}_\gamma^2 = O(1) \tag{2.5}$$

for all $\gamma \gg 1$ large. We study in this section the behavior as $\gamma \to +\infty$ of a family $(B_\gamma)_\gamma$ of functions solving

$$\begin{cases} \Delta B_\gamma + B_\gamma = \lambda_\gamma p_\gamma B_\gamma^{p_\gamma - 1} e^{B_\gamma^{p_\gamma}}\,, \\ B_\gamma(0) = \gamma > 0\,, \\ B_\gamma \text{ is radially symmetric and positive in } B_0(\bar{r}_\gamma)\,, \end{cases} \tag{2.6}$$

where $\Delta = -\partial_{xx} - \partial_{yy}$ denotes the Euclidean Laplace operator in $\mathbb{R}^2$. For $\gamma$ fixed, (2.6) reduces to an ODE with respect to the radial variable $r = |x|$: then we may assume that $B_\gamma$, defined in $[0, s_\gamma)$, is the *maximal* positive solution of (2.6) and it may be checked that it does not blow-up before it vanishes, namely $s_\gamma < +\infty$ implies $\lim_{r \to s_\gamma^-} B_\gamma(r) = 0$. Actually, the proof of Proposition 2.1 below shows that our assumptions (2.4)-(2.5) ensure that $B_\gamma$ is well defined and positive in $B_0(\bar{r}_\gamma)$ for all $\gamma \gg 1$. Let $w_\gamma$ be given by

$$B_\gamma = \gamma\left(1 - \frac{2t_\gamma}{p_\gamma \gamma^{p_\gamma}} + \frac{w_\gamma}{\gamma^{p_\gamma}}\right)\,. \tag{2.7}$$

Then we have the following result:

**Proposition 2.1.** *We have* $B_\gamma \le \gamma$*,*

$$w_\gamma = O(\gamma^{-p_\gamma} t_\gamma)\,, \quad w_\gamma' = O(\gamma^{-p_\gamma} t_\gamma')\,,$$

*and*

$$\lambda_\gamma p_\gamma B_\gamma^{p_\gamma - 1} e^{B_\gamma^{p_\gamma}} = \frac{8 e^{-2t_\gamma}}{\mu_\gamma^2 \gamma^{p_\gamma - 1} p_\gamma}\left(1 + O\left(\frac{e^{\tilde{\eta} t_\gamma}}{\gamma^{p_\gamma}}\right)\right)\,,$$

*uniformly in* $[0, \bar{r}_\gamma]$ *and for all* $\gamma \gg 1$ *large, where* $\tilde{\eta}$ *is any fixed constant in* $(\eta, 1)$ *and* $w_\gamma$ *is as in* (2.7).

Once Proposition 2.1 is proven, we obtain first

$$B_\gamma(r) = \gamma - \frac{2}{p_\gamma \gamma^{p_\gamma - 1}} \left( \ln \frac{1}{\mu_\gamma^2} + \ln(\mu_\gamma^2 + r^2) \right) + O\left(\gamma^{1-p_\gamma}\right)$$

using (2.4) to handle the remainder term, so that we get from (2.1)

$$B_\gamma(r) = -\left(\frac{2}{p_\gamma} - 1\right)\gamma + \frac{2}{p_\gamma \gamma^{p_\gamma - 1}} \ln \frac{1}{\lambda_\gamma \gamma^{2(p_\gamma - 1)}(\mu_\gamma^2 + r^2)} + O\left(\gamma^{1-p_\gamma}\right) \tag{2.8}$$

uniformly in $r \in [0, \bar{r}_\gamma]$ and for all $\gamma \gg 1$ large.

*Proof of Proposition 2.1.* Let $r_\gamma$ be given by

$$r_\gamma = \sup\left\{ r \in [0, \bar{r}_\gamma] \text{ s.t. } |w_\gamma| \le \frac{t_\gamma}{\gamma^{\frac{p_\gamma}{2}}} \text{ in } [0, r] \right\} \tag{2.9}$$

for all $\gamma$. We aim to show that

$$r_\gamma = \bar{r}_\gamma \tag{2.10}$$

for all $\gamma \gg 1$. We start by expanding the RHS in the first equation of (2.6) uniformly in $[0, r_\gamma]$ as $\gamma \to +\infty$, using in a crucial way the control on $w_\gamma$ that we have by (2.9). Fix $\eta_1 < \eta_2 < \eta_3$ such that $\eta_k \in (\eta, 1)$ for all $k$. When not specified, the expansions of this proof are uniform in $[0, r_\gamma]$ as $\gamma \to +\infty$. First, since $|w_\gamma| = o(t_\gamma)$, we get from (2.7) that $B_\gamma / \gamma \ge (1 - \eta_1)$ in $[0, r_\gamma]$ for all $\gamma \gg 1$ large. First, for all $p \in [1, 2]$ and all $x \le 1$, we notice that

$$0 \le (1 - x)^p - (1 - px) \le \frac{p^2}{4} x^2 \,.$$

Then, we have

$$0 \le \frac{B_\gamma^{p_\gamma}}{\gamma^{p_\gamma}} - \left(1 - \frac{2t_\gamma - p_\gamma w_\gamma}{\gamma^{p_\gamma}}\right) \le \frac{t_\gamma^2}{\gamma^{2p_\gamma}} (1 + o(1)) \,,$$

so we get from (2.4) that

$$\exp\left(B_\gamma^{p_\gamma}\right) = e^{\gamma^{p_\gamma}} e^{-2t_\gamma} e^{p_\gamma w_\gamma} \left(1 + O\left(\frac{t_\gamma^2}{\gamma^{p_\gamma}} e^{\eta_1 t_\gamma}\right)\right) .$$

Here and several times in the sequel, we use the elementary inequality

$$\left| e^x - \sum_{j=0}^{n-1} \frac{x^j}{j!} \right| \le \frac{|x|^n}{n!} e^{|x|}$$

for all $x \in \mathbb{R}$ and all integers $n \geq 1$. Using also (2.1) and (2.9) again, we get that

$$\begin{aligned} &\lambda_\gamma p_\gamma B_\gamma^{p_\gamma - 1} e^{B_\gamma^{p_\gamma}} \\ &= \frac{8\ e^{-2t_\gamma}}{\mu_\gamma^2 \gamma^{p_\gamma - 1} p_\gamma} \left(1 + O\left(\frac{t_\gamma}{\gamma^{p_\gamma}}\right)\right) \times \\ &\quad \left(1 + p_\gamma w_\gamma + O\left(\frac{t_\gamma^2}{\gamma^{p_\gamma}} \exp\left(\frac{p_\gamma t_\gamma}{\gamma^{p_\gamma/2}}\right)\right)\right) \left(1 + O\left(\frac{t_\gamma^2}{\gamma^{p_\gamma}} e^{\eta_1 t_\gamma}\right)\right), \\ &= \frac{8\ e^{-2t_\gamma}}{\mu_\gamma^2 \gamma^{p_\gamma - 1} p_\gamma} \left(1 + p_\gamma w_\gamma + O\left(\frac{\bar{t}_\gamma^3}{\gamma^{p_\gamma}} e^{\eta_2 t_\gamma}\right)\right). \end{aligned} \tag{2.11}$$

In view of (2.9), to conclude the proof of (2.10), it is sufficient to obtain

$$|w_\gamma| = O\left(\frac{t_\gamma}{\gamma^{p_\gamma}}\right), \tag{2.12}$$

which we prove next. By (2.9), we have that $B_\gamma \leq \gamma$ in $[0, r_\gamma]$ for all $\gamma \gg 1$. Set $\tilde{w}_\gamma = w_\gamma(\cdot/\mu_\gamma)$. Then, since $T_0 := \ln(1 + |\cdot|^2)$ solves

$$\Delta T_0 = -4 e^{-2T_0} \text{ in } \mathbb{R}^2, \tag{2.13}$$

we get from (2.6) and (2.11) that

$$\Delta \tilde{w}_\gamma = 8 e^{-2T_0} \tilde{w}_\gamma + O\left(\mu_\gamma^2 \gamma^{p_\gamma}\right) + O\left(\frac{e^{(-2+\eta_3)T_0}}{\gamma^{p_\gamma}}\right), \tag{2.14}$$

uniformly in $[0, r_\gamma/\mu_\gamma]$ as $\gamma \to +\infty$, applying $\Delta$ to (2.7). By integrating (2.14) in $B_0(r)$ and also by parts, writing merely $|\tilde{w}_\gamma| \leq r\|\tilde{w}_\gamma'\|_\infty$, we get that

$$-2\pi r\ \tilde{w}_\gamma'(r) = O\left(r^2 \mu_\gamma^2 \gamma^{p_\gamma}\right) + O\left(\frac{r^2}{\gamma^{p_\gamma}(1+r^2)}\right) + O\left(\frac{\|\tilde{w}_\gamma'\|_\infty r^3}{1+r^3}\right),$$

where $\|\tilde{w}_\gamma'\|_\infty$ stands for $\|\tilde{w}_\gamma'\|_{L^\infty([0, r_\gamma/\mu_\gamma])}$ and where $\tilde{w}_\gamma' = \frac{d}{dr}\tilde{w}_\gamma$, so that we get

$$|\tilde{w}_\gamma'(r)| = O\left(\frac{r\mu_\gamma^2}{r_\gamma^2 \gamma^{p_\gamma}}\right) + O\left(\frac{r}{1+r^2}\left(\|\tilde{w}_\gamma'\|_\infty + \frac{1}{\gamma^{p_\gamma}}\right)\right), \tag{2.15}$$

uniformly in $r \in [0, r_\gamma/\mu_\gamma]$ as $\gamma \to +\infty$, using (2.5) and $r_\gamma \leq \bar{r}_\gamma$. If $\|\tilde{w}_\gamma'\|_\infty = O(\gamma^{-p_\gamma})$ for all $\gamma$, (2.12) follows from (2.3), (2.15) and from the fundamental theorem of calculus, using again $\tilde{w}_\gamma(0) = 0$. Then, assume by contradiction that the complementary case occurs, namely that

$$\lim_{\gamma \to +\infty} \gamma^{p_\gamma} \|\tilde{w}_\gamma'\|_\infty = +\infty, \tag{2.16}$$

maybe after passing to a subsequence. Let $\rho_\gamma \in [0, r_\gamma/\mu_\gamma]$ be such that $|\tilde{w}_\gamma'(\rho_\gamma)| = \|\tilde{w}_\gamma'\|_\infty$. By (2.3), (2.15) and (2.16), up to a subsequence, $\rho_\gamma \to l$ and $r_\gamma/\mu_\gamma \to L$ as $\gamma \to +\infty$, for some $l \in (0, +\infty)$, $L \in (0, +\infty]$, $l \leq L$. Setting now $\check{w}_\gamma := \tilde{w}_\gamma / \|\tilde{w}_\gamma'\|_\infty$, we then get from (radial) elliptic theory and from (2.14) with (2.3) and (2.5) that, up to a subsequence, $\check{w}_\gamma \to \check{w}_\infty$ in $C^1_{loc}([0, L))$ as $\gamma \to +\infty$ , where $\check{w}_\infty$ solves

$$\begin{cases} \Delta \check{w}_\infty = 8 e^{-2T_0} \check{w}_\infty \text{ in } B_0(L), \\ \check{w}_\infty(0) = 0, \\ \check{w}_\infty \text{ is radially symmetric}, \\ |\check{w}_\infty'(l)| = 1; \end{cases} \tag{2.17}$$

but by ODE theory, the only function satisfying the first three conditions in (2.17) is the null function, which gives the expected contradiction. Observe that we get also a contradiction in the most delicate case where $l = L$. Indeed, since we then have $L \in (0, +\infty)$, writing (2.14) in radial coordinates gives in this case that $(\|\check{w}_\gamma\|_{C^2([0,r_\gamma/\mu_\gamma])})_\gamma$ is bounded, so that $\check{w}'_\infty \in C^1([0,l])$ is well defined at $l$, so that the fourth line in (2.17) makes sense and holds true. As explained above, this concludes the proof of (2.10). Proposition 2.1 clearly follows. □

## 3. Nonradial blow-up analysis: the case of a single bubble

Let $(p_\varepsilon)_\varepsilon$ be a sequence of numbers in $[1,2]$, let $(\mu_\varepsilon)_\varepsilon$ and $(\bar{r}_\varepsilon)_\varepsilon$ be sequences of positive real numbers. Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions in $B_0(\bar{r}_\varepsilon)$, where $B_0(\bar{r}_\varepsilon)$ is the ball of center 0 and radius $\bar{r}_\varepsilon$ in the standard Euclidean space $\mathbb{R}^2$. We assume that

$$\nabla u_\varepsilon(0) = 0 \tag{3.1}$$

for all $\varepsilon$ and that

$$\gamma_\varepsilon := u_\varepsilon(0) \to +\infty \tag{3.2}$$

as $\varepsilon \to 0$. As for (2.1), let $(\lambda_\varepsilon)_\varepsilon$ be given by

$$\lambda_\varepsilon p_\varepsilon^2 \gamma_\varepsilon^{2(p_\varepsilon - 1)} \mu_\varepsilon^2 e^{\gamma_\varepsilon^{p_\varepsilon}} = 8 \tag{3.3}$$

and let $t_\varepsilon, \bar{t}_\varepsilon$ be given by

$$t_\varepsilon = \ln\left(1 + \frac{|\cdot|^2}{\mu_\varepsilon^2}\right); \qquad \bar{t}_\varepsilon = t_\varepsilon + 1$$

for all $\varepsilon$. Let $\eta \in (0,1)$ be fixed; assume also that

$$\frac{\mu_\varepsilon}{\bar{r}_\varepsilon} = o(1)\,, \tag{3.4}$$

$$t_\varepsilon(\bar{r}_\varepsilon) \le \eta \frac{p_\varepsilon \gamma_\varepsilon^{p_\varepsilon}}{2}\,, \tag{3.5}$$

$$\int_{B_0(\bar{r}_\varepsilon)} u_\varepsilon^4 dx \le \bar{C}\,, \tag{3.6}$$

for all $\varepsilon \ll 1$ small and for some given $\bar{C} > 1$, and that

$$\lim_{\varepsilon \to 0} \frac{p_\varepsilon}{2} \gamma_\varepsilon^{p_\varepsilon - 1} \left(\gamma_\varepsilon - u_\varepsilon(\mu_\varepsilon \cdot)\right) = \ln\left(1 + |\cdot|^2\right) \text{ in } C^1_{loc}(\mathbb{R}^2)\,, \tag{3.7}$$

up to a subsequence. As we will see in the subsequent blow-up analysis and in Lemma 4.1, the last two assumptions are indeed natural ones.

Let $(v_\varepsilon)_\varepsilon$ be a sequence of smooth functions solving

$$\begin{cases} \Delta v_\varepsilon + v_\varepsilon = \lambda_\varepsilon p_\varepsilon v_\varepsilon^{p_\varepsilon - 1} e^{v_\varepsilon^{p_\varepsilon}} \text{ in } B_0(\bar{r}_\varepsilon)\,, \\ v_\varepsilon(0) = \gamma_\varepsilon \\ v_\varepsilon \text{ is radially symmetric around } 0 \in \mathbb{R}^2\,, \end{cases} \tag{3.8}$$

for all $\varepsilon$. Let $(\varphi_\varepsilon)_\varepsilon$ be a sequence of smooth functions such that

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(\bar{r}_\varepsilon \cdot) = \varphi_0 \text{ in } C^2\left(\overline{B_0(1)}\right) \text{ and } \varphi_\varepsilon(0) = 0 \tag{3.9}$$

for all $\varepsilon$ small. We assume that $u_\varepsilon$ solves

$$\Delta u_\varepsilon = e^{2\varphi_\varepsilon}\left(-u_\varepsilon + \lambda_\varepsilon p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} e^{u_\varepsilon^{p_\varepsilon}}\right)\,, \quad u_\varepsilon > 0 \text{ in } B_0(\bar{r}_\varepsilon)\,, \tag{3.10}$$

for all $\varepsilon$. At last, we assume that the following key gradient estimate holds true: there exists $C_G > 0$ such that

$$u_\varepsilon^{p_\varepsilon - 1} |\nabla u_\varepsilon| |\cdot| \le C_G \text{ in } B_0(\bar r_\varepsilon) \tag{3.11}$$

for all $\varepsilon$. Letting $w_\varepsilon$ be given by

$$u_\varepsilon = v_\varepsilon + w_\varepsilon\,, \tag{3.12}$$

the following proposition holds true:

**Proposition 3.1.** *We have that*

$$|w_\varepsilon| \le \frac{C_0|\cdot|}{\gamma_\varepsilon^{p_\varepsilon - 1}\bar r_\varepsilon} \text{ in } B_0(\bar r_\varepsilon)\,, \tag{3.13}$$

*and that*

$$\|\nabla w_\varepsilon\|_{L^\infty(B_0(\bar r_\varepsilon))} \le \frac{C_0}{\gamma_\varepsilon^{p_\varepsilon - 1}\bar r_\varepsilon} \tag{3.14}$$

*for all $\varepsilon \ll 1$ small, where $C_0$ is any fixed constant greater than $(C_G/(1-\eta)) + 4$, for $C_G$ as in* (3.11) *and $\eta$ as in* (3.5). *Up to a subsequence, there exists a function $\psi_0$, harmonic in $B_0(1)$, such that we have*

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon^{p_\varepsilon - 1} w_\varepsilon(\bar r_\varepsilon \cdot) = \psi_0 \text{ in } C^1_{loc}(B_0(1)\backslash\{0\})\,, \tag{3.15}$$

$$\nabla \psi_0(0) = 0\,. \tag{3.16}$$

In order to make sure that the estimates of Section 2 can be used to control the $v_\varepsilon$'s, it will be checked in the proof below that our assumptions of this section actually imply

$$\gamma_\varepsilon^{2p_\varepsilon} \bar r_\varepsilon^2 = O(1)\,, \tag{3.17}$$

for all $\varepsilon$ (see (2.5)). Besides, if we strengthen assumption (3.6) and we assume

$$\int_{B_0(\bar r_\varepsilon)} e^{u_\varepsilon^{1/3}} dx = O(1)\,, \tag{3.18}$$

for all $\varepsilon$, again guaranteed by Lemma 4.1, we will also show that (3.17) may be improven to

$$\ln \gamma_\varepsilon = o\left(\ln \frac{1}{\bar r_\varepsilon}\right) \tag{3.19}$$

as $\varepsilon \to 0$.

*Proof of Proposition 3.1.* We first prove (3.13). By (3.8), we have that $v_\varepsilon'(0) = 0$; by (3.2) and (3.8), we have that $u_\varepsilon(0) = v_\varepsilon(0)$ and we then find

$$w_\varepsilon(0) = 0 \text{ and } \nabla w_\varepsilon(0) = 0 \tag{3.20}$$

for all $\varepsilon$, using (3.1) and (3.12). Then, in order to get (3.13), it is sufficient to prove (3.14). Let $r_\varepsilon$ be given by

$$r_\varepsilon = \sup\left\{ r \in [0, \bar r_\varepsilon] \text{ s.t. } \left\{ \begin{array}{l} \gamma_\varepsilon^{p_\varepsilon - 1} r \|\nabla w_\varepsilon\|_{L^\infty(B_0(r))} \le C_0\,, \\ \gamma_\varepsilon^4 r^2 \le \frac{2\bar C}{\pi(1-\eta)^4} \end{array} \right. \right\} \tag{3.21}$$

for all $\varepsilon$, with $C_0 > (C_G/(1-\eta)) + 4$ fixed as in Proposition 3.1 and $\bar C$ as in (3.6). Then proving (3.14) reduces to show that

$$r_\varepsilon = \bar r_\varepsilon \tag{3.22}$$

for all $\varepsilon \ll 1$. By (3.4) and (3.7), there exist numbers $\tilde{r}_\varepsilon$ such that $\mu_\varepsilon = o(\tilde{r}_\varepsilon)$, $\tilde{r}_\varepsilon \le \bar{r}_\varepsilon$ and such that $u_\varepsilon = \gamma_\varepsilon(1+o(1))$ uniformly in $B_0(\tilde{r}_\varepsilon)$: then, we get from (3.6) that

$$\int_{B_0(\tilde{r}_\varepsilon)} u_\varepsilon^4 dx = \pi\gamma_\varepsilon^4\tilde{r}_\varepsilon^2(1+o(1)) \le \bar{C}$$

and that $\gamma_\varepsilon^{2p_\varepsilon}\tilde{r}_\varepsilon^2 \le 2\bar{C}/\pi$ for all $\varepsilon \ll 1$. Then, we may use Proposition 2.1 in $B_0(\tilde{r}_\varepsilon)$, with assumption (2.5), to get that

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon}{2}\gamma_\varepsilon^{p_\varepsilon-1}(\gamma_\varepsilon - v_\varepsilon(\mu_\varepsilon\cdot)) = \ln(1+|\cdot|^2) \text{ in } C^1_{loc}(\mathbb{R}^2)\,,$$

which implies with (3.7) that $\gamma_\varepsilon^{p_\varepsilon-1}\mu_\varepsilon\|\nabla w_\varepsilon\|_{L^\infty(B_0(R\mu_\varepsilon))} = o(1)$ as $\varepsilon \to 0$, for all given $R \gg 1$. Summarizing, both conditions in (3.21) give that $\mu_\varepsilon = o(r_\varepsilon)$ as $\varepsilon \to 0$ and we may now apply Proposition 2.1 in $B_0(r_\varepsilon)$: we have that

$$\sup_{s\in[0,r_\varepsilon]} \frac{p_\varepsilon}{2}\gamma_\varepsilon^{p_\varepsilon-1} s|v_\varepsilon'(s)| \le 2+o(1) \tag{3.23}$$

for all $\varepsilon \ll 1$. Using $w_\varepsilon(0) = 0$, we get from the first condition in (3.21) that $|w_\varepsilon| \le C_0\gamma_\varepsilon^{1-p_\varepsilon}$ so that $u_\varepsilon = v_\varepsilon + O\left(\gamma_\varepsilon^{1-p_\varepsilon}\right)$ in $B_0(r_\varepsilon)$ for all $\varepsilon \ll 1$. Independently, we get from Proposition 2.1 and from (3.5) that

$$v_\varepsilon \ge \gamma_\varepsilon(1-\eta+o(1)) \text{ in } [0,r_\varepsilon]\,, \tag{3.24}$$

for all $\varepsilon \ll 1$. Then, writing $|\nabla w_\varepsilon| \le |\nabla u_\varepsilon| + |\nabla v_\varepsilon|$, using first (3.11) and (3.23), and then (3.24) together with $p_\varepsilon \in [1,2]$, we get that

$$\|\nabla w_\varepsilon\|_{L^\infty(\partial B_0(r_\varepsilon))} \le \frac{1+o(1)}{\gamma_\varepsilon^{p_\varepsilon-1}r_\varepsilon}\left(\frac{C_G}{(1-\eta)^{p_\varepsilon-1}}+4\right) < \frac{C_0}{\gamma_\varepsilon^{p_\varepsilon-1}r_\varepsilon} \tag{3.25}$$

for all $\varepsilon \ll 1$, using our assumption on $C_0$. Independently, Proposition 2.1 gives that $v_\varepsilon(r)' = O\left(r^{-1}\gamma_\varepsilon^{1-p_\varepsilon}\right)$, so we first get that

$$u_\varepsilon = v_\varepsilon(r_\varepsilon) + O\left(\gamma_\varepsilon^{1-p_\varepsilon}\ln\frac{2r_\varepsilon}{|\cdot|}\right)\,, \tag{3.26}$$

then, with (3.24), that also

$$u_\varepsilon(r)^4 = v_\varepsilon(r_\varepsilon)^4\left[1+O\left(\left(\gamma_\varepsilon^{-p_\varepsilon}\ln\frac{2r_\varepsilon}{r}\right)+\left(\gamma_\varepsilon^{-p_\varepsilon}\ln\frac{2r_\varepsilon}{r}\right)^4\right)\right]$$

uniformly in $r \in (0,r_\varepsilon]$, and at last, with (3.6), that

$$\pi v_\varepsilon(r_\varepsilon)^4 r_\varepsilon^2(1+o(1)) = \int_{B_0(r_\varepsilon)} u_\varepsilon^4 dx \le \bar{C}\ :$$

summarizing, the second inequality in (3.21) is strict at $r = r_\varepsilon$ for all $\varepsilon \ll 1$, using (3.24) again. However by (3.25), the first inequality in (3.21) is strict as well at $r = r_\varepsilon$, which concludes the proof of (3.22) by continuity and then, as discussed above, those of (3.13) and (3.14). Since $p_\varepsilon \le 2$, we get at the same time from (3.21) that (3.17) holds true, so that we may apply Proposition 2.1 from now on to estimate the $v_\varepsilon$'s in $B_0(\bar{r}_\varepsilon)$. We turn now to the proofs of (3.15) and (3.16). First, using (3.24) and that $v_\varepsilon \le \gamma_\varepsilon$ by Proposition 2.1, since $|w_\varepsilon| = O\left(\gamma_\varepsilon^{1-p_\varepsilon}\right)$ by (3.13), we may first write $u_\varepsilon^{p_\varepsilon} = v_\varepsilon^{p_\varepsilon} + p_\varepsilon v_\varepsilon^{p_\varepsilon-1} w_\varepsilon(1+o(1))$ and $u_\varepsilon^{p_\varepsilon-1} = v_\varepsilon^{p_\varepsilon-1}\left(1+O\left(|w_\varepsilon|/\gamma_\varepsilon\right)\right)$, then

$$\begin{aligned} & u_\varepsilon^{p_\varepsilon-1} e^{u_\varepsilon^{p_\varepsilon}} \\ & = v_\varepsilon^{p_\varepsilon-1} e^{v_\varepsilon^{p_\varepsilon}} \left(1 + p_\varepsilon v_\varepsilon^{p_\varepsilon-1} w_\varepsilon \left[1 + O\left(\frac{|w_\varepsilon|}{\gamma_\varepsilon} + v_\varepsilon^{p_\varepsilon-1}|w_\varepsilon|\right)\right] + O\left(\frac{|w_\varepsilon|}{\gamma_\varepsilon}\right)\right) \\ & = v_\varepsilon^{p_\varepsilon-1} e^{v_\varepsilon^{p_\varepsilon}} \left(1 + p_\varepsilon v_\varepsilon^{p_\varepsilon-1} w_\varepsilon \left[1 + O\left(\gamma_\varepsilon^{p_\varepsilon-1}|w_\varepsilon|\right) + O\left(\gamma_\varepsilon^{-p_\varepsilon}\right)\right]\right) , \end{aligned}$$

and, observing also $e^{2\varphi_\varepsilon} = 1 + O(|\cdot|)$ by (3.9), $|w_\varepsilon| = O\left(\gamma_\varepsilon^{1-p_\varepsilon}|\cdot|/\bar{r}_\varepsilon\right)$ by (3.13), and using (3.8) and (3.10), we may write at last

$$\begin{aligned} \Delta w_\varepsilon = & -e^{2\varphi_\varepsilon} w_\varepsilon + O\left(|\cdot| v_\varepsilon\right) \\ & + \lambda_\varepsilon p_\varepsilon v_\varepsilon^{p_\varepsilon-1} e^{v_\varepsilon^{p_\varepsilon}} \left(p_\varepsilon v_\varepsilon^{p_\varepsilon-1} w_\varepsilon \left[1 + O\left(\frac{|\cdot|}{\bar{r}_\varepsilon} + \frac{1}{\gamma_\varepsilon^{p_\varepsilon}}\right)\right] + O(|\cdot|)\right) \end{aligned} \tag{3.27}$$

uniformly in $B_0(\bar{r}_\varepsilon)$ and for all $\varepsilon \ll 1$. Setting now $\tilde{w}_\varepsilon = \gamma_\varepsilon^{p_\varepsilon-1} \frac{\bar{r}_\varepsilon}{\mu_\varepsilon} w_\varepsilon(\mu_\varepsilon \cdot)$ and given any $R \gg 1$, we get from Proposition 2.1 and (3.27) that

$$\begin{aligned} \Delta \tilde{w}_\varepsilon & = O\left(\mu_\varepsilon^2 \tilde{w}_\varepsilon\right) + O\left(\mu_\varepsilon^2 \gamma_\varepsilon^{p_\varepsilon} \bar{r}_\varepsilon\right) + \left[\frac{8e^{-2T_0}}{p_\varepsilon \gamma_\varepsilon^{p_\varepsilon-1}}\left(1 + O(\gamma_\varepsilon^{-p_\varepsilon})\right)\right] \times \\ & \left(p_\varepsilon \gamma_\varepsilon^{p_\varepsilon-1} \tilde{w}_\varepsilon \left[1 + O\left(\frac{\mu_\varepsilon}{\bar{r}_\varepsilon} + \gamma_\varepsilon^{-p_\varepsilon}\right)\right] + O\left(\gamma_\varepsilon^{p_\varepsilon-1} \bar{r}_\varepsilon\right)\right) \end{aligned}$$

uniformly in $B_0(R\mu_\varepsilon)$, for all $\varepsilon$. Then, by (3.4), (3.14), (3.17), the first assertion in (3.20) and elliptic theory, we get that, up to a subsequence,

$$\lim_{\varepsilon \to 0} \tilde{w}_\varepsilon = w_0 \text{ in } C^1_{loc}(\mathbb{R}^2)\,, \tag{3.28}$$

where $w_0$ satisfies

$$\begin{cases} \Delta w_0 = 8\exp(-2T_0) w_0 \text{ in } \mathbb{R}^2\,, \\ |w_0| \le C_0 |\cdot| \text{ in } \mathbb{R}^2\,. \end{cases} \tag{3.29}$$

By the second assertion in (3.20) and (3.28), we have $\nabla w_0(0) = 0$. According to the classification result stated by Chen-Lin [3, Lemma 2.3] and also in the generality on the growth assumption that we need here by Laurain [27, Lemma C.1], this last property and (3.29) imply

$$w_0 \equiv 0\,. \tag{3.30}$$

In order to conclude the proofs of (3.15) and (3.16), we establish now the following key estimate:

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon^{p_\varepsilon-1} \bar{r}_\varepsilon \|\nabla(w_\varepsilon - (\psi_\varepsilon - \psi_\varepsilon(0)))\|_{\infty,\varepsilon} = 0\,, \tag{3.31}$$

where $\|\cdot\|_{\infty,\varepsilon}$ denotes $\|\cdot\|_{L^\infty(B_0(\bar{r}_\varepsilon))}$ and where $\psi_\varepsilon$ is given by

$$\begin{cases} \Delta \psi_\varepsilon = 0 \text{ in } B_0(\bar{r}_\varepsilon)\,, \\ \psi_\varepsilon = w_\varepsilon \text{ on } \partial B_0(\bar{r}_\varepsilon)\,, \end{cases} \tag{3.32}$$

for all $\varepsilon$. Let $G^{(\varepsilon)}$ be the Green's function of $\Delta$ in $B_0(\bar{r}_\varepsilon)$ with zero Dirichlet boundary conditions (for an explicit formula for $G^{(\varepsilon)}$, see for instance Han-Lin [24, Proposition 1.22]). Then (see also for instance [19, Appendix B]), there exists $C > 0$ such that

$$|\nabla G_y^{(\varepsilon)}(x)| \le \frac{C}{|x-y|}\,,$$

for all $x, y \in B_0(\bar r_\varepsilon)$, $x \neq y$ and all $\varepsilon$. Let $(y_\varepsilon)_\varepsilon$ be any sequence such that $y_\varepsilon \in B_0(\bar r_\varepsilon)$ for all $\varepsilon$. By the Green's representation formula, we may write

$$\nabla(w_\varepsilon - \psi_\varepsilon)(y_\varepsilon) = \int_{B_0(\bar r_\varepsilon)} \nabla G^{(\varepsilon)}_{y_\varepsilon}(x)(\Delta w_\varepsilon)(x)dx$$

for all $\varepsilon$. Then, using also (3.9), (3.27), Proposition 2.1 and the first assertion in (3.20), we get that

$$\begin{aligned} &|\nabla(w_\varepsilon - \psi_\varepsilon)(y_\varepsilon)| \\ &= O\left(\int_{B_0(\bar r_\varepsilon)} \frac{(\|\nabla w_\varepsilon\|_{\infty,\varepsilon} + \gamma_\varepsilon)|x|dx}{|y_\varepsilon - x|}\right) \\ &\quad + O\left(\int_{B_0(\bar r_\varepsilon)} \frac{|x|e^{(-2+\tilde\eta)t_\varepsilon(x)}\left(\|\nabla w_\varepsilon\|_{\infty,\varepsilon} + \gamma_\varepsilon^{1-p_\varepsilon}\right)dx}{\mu_\varepsilon^2|y_\varepsilon - x|}\right), \end{aligned} \tag{3.33}$$

for all $\varepsilon$, where $\tilde\eta$ is some given constant in $(\eta, 1)$. By the change of variable $x = \bar r_\varepsilon y$, we first deduce

$$\int_{B_0(\bar r_\varepsilon)} \frac{|x|dx}{|y_\varepsilon - x|} = O\left(\bar r_\varepsilon^2\right).$$

If we have $|y_\varepsilon| = O(\mu_\varepsilon)$, we get that

$$\int_{B_0(\bar r_\varepsilon)} \frac{|x|e^{(-2+\tilde\eta)t_\varepsilon(x)}dx}{\mu_\varepsilon^2|y_\varepsilon - x|} = O(1)$$

for all $\varepsilon$, by the change of variable $x = \mu_\varepsilon y$; otherwise, up to a subsequence, we have $\mu_\varepsilon = o(|y_\varepsilon|)$ and

$$\int_{B_0(\bar r_\varepsilon)} \frac{|x|e^{(-2+\tilde\eta)t_\varepsilon(x)}dx}{\mu_\varepsilon^2|y_\varepsilon - x|} = \int_{B_0(\bar r_\varepsilon/|y_\varepsilon|)} \frac{1}{\tilde\mu_\varepsilon^2} \frac{1}{\left(1 + \frac{|y|^2}{\tilde\mu_\varepsilon^2}\right)^{2-\tilde\eta}} \frac{|y|dy}{|\tilde y_\varepsilon - y|} = O\left(\tilde\mu_\varepsilon\right)$$

for all $\varepsilon \ll 1$, by the change of variable $x = |y_\varepsilon|y$, where $\tilde y_\varepsilon = y_\varepsilon/|y_\varepsilon|$ has norm 1 and $\tilde\mu_\varepsilon = \mu_\varepsilon/|y_\varepsilon|$. Plugging these estimates in (3.33), we get in any case

$$\begin{aligned} &|\nabla(w_\varepsilon - (\psi_\varepsilon - \psi_\varepsilon(0)))(y_\varepsilon)| \\ &= O\left((\|\nabla w_\varepsilon\|_{\infty,\varepsilon} + \gamma_\varepsilon)\bar r_\varepsilon^2\right) + O\left(\frac{1}{1 + \frac{|y_\varepsilon|}{\mu_\varepsilon}}\left(\|\nabla w_\varepsilon\|_{\infty,\varepsilon} + \gamma_\varepsilon^{1-p_\varepsilon}\right)\right), \\ &= \frac{1}{\gamma_\varepsilon^{p_\varepsilon - 1}\bar r_\varepsilon}\left(O\left(\frac{1}{1 + \frac{|y_\varepsilon|}{\mu_\varepsilon}}\right) + o(1)\right) \end{aligned} \tag{3.34}$$

for all $\varepsilon$. The last line in (3.34) uses (3.14) and (3.17). We claim now that $(\psi_\varepsilon)_\varepsilon$ from (3.32) satisfies

$$\|\nabla\psi_\varepsilon\|_{\infty,\varepsilon} = O\left(\frac{1}{\gamma_\varepsilon^{p_\varepsilon - 1}\bar r_\varepsilon}\right). \tag{3.35}$$

Writing $\nabla\psi_\varepsilon = \nabla w_\varepsilon + \nabla(\psi_\varepsilon - w_\varepsilon)$, using (3.34) which gives

$$\|\nabla(\psi_\varepsilon - w_\varepsilon)\|_{\infty,\varepsilon} = O\left(\frac{1}{\gamma_\varepsilon^{p_\varepsilon - 1}\bar r_\varepsilon}\right),$$

we indeed get (3.35) from (3.14). Thus, we find from (3.35) and elliptic theory that

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^{p_\varepsilon - 1}(\psi_\varepsilon(r_\varepsilon\cdot) - \psi_\varepsilon(0)) = \psi_0 \text{ in } C^1_{loc}(B_0(1)), \tag{3.36}$$

up to a subsequence, where $\psi_0$ is harmonic in $B_0(1)$, and we obtain at last

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{p_\varepsilon-1}\frac{\bar r_\varepsilon}{\mu_\varepsilon}(\psi_\varepsilon(\mu_\varepsilon\cdot)-\psi_\varepsilon(0))=\langle\nabla\psi_0(0),\cdot\rangle \text{ in } C^1_{loc}(\mathbb{R}^2)\,, \tag{3.37}$$

by (3.4), where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^2$.

Assume now by contradiction that (3.31) does not hold true, in other words that, up to a subsequence,

$$\frac{1}{\gamma_\varepsilon^{p_\varepsilon-1}\bar r_\varepsilon}=O\left(\|\nabla(w_\varepsilon-\psi_\varepsilon)\|_{\infty,\varepsilon}\right) \tag{3.38}$$

for all $\varepsilon$. First, we claim that (3.16) holds true, for $\psi_0$ as in (3.36)-(3.37). Indeed, let $R\gg 1$ be given and let $(y_\varepsilon)_\varepsilon$ be such that $y_\varepsilon\in\partial B_0(R\mu_\varepsilon)$ for all $\varepsilon\ll 1$. We get from (3.28), (3.30) and (3.37) that

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{p_\varepsilon-1}\bar r_\varepsilon\nabla(w_\varepsilon-\psi_\varepsilon)(y_\varepsilon)=\nabla\psi_0(0)\,.$$

This estimate, combined with (3.34), proves (3.16) since $R\gg 1$ may be chosen arbitrarily large. Secondly, we may pick $(y_\varepsilon)_\varepsilon$, such that $y_\varepsilon\in\overline{B_0(\bar r_\varepsilon)}$ and

$$\|\nabla(w_\varepsilon-\psi_\varepsilon)\|_{\infty,\varepsilon}=|\nabla(w_\varepsilon-(\psi_\varepsilon-\psi_\varepsilon(0)))(y_\varepsilon)| \tag{3.39}$$

for all $\varepsilon$, and we get from (3.34) and (3.38) that $|y_\varepsilon|=O\left(\mu_\varepsilon\right)$ for all $\varepsilon$. However, (3.30) and (3.37) with (3.16) contradict (3.38) with (3.39), which concludes the proof of (3.31). Then (3.15) and (3.16) follow from both assertions in (3.20), from (3.31) and from (3.36), which concludes the proof of Proposition 3.1. To end this section, we assume (3.18) and prove (3.19). We have (3.22) and (3.26). Then, using (3.24),

$$(1+t)^{1/3}=1+O(|t|^{1/3}) \text{ for all } t>-1\,,$$

$p_\varepsilon\ge 1$ and $v_\varepsilon(\bar r_\varepsilon)\le\gamma_\varepsilon$, we get first

$$u_\varepsilon^{1/3}=v_\varepsilon(\bar r_\varepsilon)^{1/3}\left(1+O\left(\gamma_\varepsilon^{-p_\varepsilon}\ln\frac{2\bar r_\varepsilon}{|\cdot|}\right)\right)^{1/3}=v_\varepsilon(\bar r_\varepsilon)^{1/3}+O\left(\left(\ln\frac{2\bar r_\varepsilon}{|\cdot|}\right)^{1/3}\right)$$

uniformly in $B_0(\bar r_\varepsilon)\backslash\{0\}$, so that we eventually get

$$\int_{B_0(\bar r_\varepsilon)}e^{u_\varepsilon^{1/3}}dx=e^{v_\varepsilon(\bar r_\varepsilon)^{1/3}}\int_{B_0(\bar r_\varepsilon)}\exp\left(O\left(\left(\ln\frac{2\bar r_\varepsilon}{|\cdot|}\right)^{1/3}\right)\right)dx\gtrsim e^{\left(\frac{(1-\eta)\gamma_\varepsilon}{2}\right)^{1/3}}\bar r_\varepsilon^2\,,$$

for all $\varepsilon\ll 1$, which concludes the proof of (3.19) by (3.18). □

## 4. Nonradial blow-up analysis: the case of several bubbles

The following theorem is the main result of this section. It is a quantization result determining in a precise way the possible blow-up energy levels. Notice that assumption (4.3) will follow from variational reasons.

**Theorem 4.1.** *Let $(\lambda_\varepsilon)_\varepsilon$ be any sequence of positive real numbers and $(p_\varepsilon)_\varepsilon$ be any sequence of numbers in $[1,2]$. Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions solving*

$$\Delta_g u_\varepsilon+u_\varepsilon=\lambda_\varepsilon p_\varepsilon u_\varepsilon^{p_\varepsilon-1}e^{u_\varepsilon^{p_\varepsilon}}\,,\quad u_\varepsilon>0 \text{ in } \Sigma\,, \tag{4.1}$$

*for all $\varepsilon$. Let $(\beta_\varepsilon)_\varepsilon$ be given by*

$$\beta_\varepsilon=\frac{\lambda_\varepsilon p_\varepsilon^2}{2}\left(\int_\Sigma\left(e^{u_\varepsilon^{p_\varepsilon}}-1\right)dv_g\right)^{\frac{2-p_\varepsilon}{p_\varepsilon}}\left(\int_\Sigma u_\varepsilon^{p_\varepsilon}e^{u_\varepsilon^{p_\varepsilon}}\,dv_g\right)^{\frac{2(p_\varepsilon-1)}{p_\varepsilon}} \tag{4.2}$$

*for all $\varepsilon$. If we assume the energy bound*

$$\lim_{\varepsilon\to 0}\beta_\varepsilon = \beta \in [0,+\infty)\,, \tag{4.3}$$

*but the pointwise blow-up of the $u_\varepsilon$'s, namely*

$$\lim_{\varepsilon\to 0}\max_{\Sigma} u_\varepsilon = +\infty\,, \tag{4.4}$$

*then, there exists an integer $k \geq 1$ such that*

$$\beta = 4\pi k\,. \tag{4.5}$$

A quantization result on a surface and in the specific case $p_\varepsilon = 2$ was partially obtained by Yang [44], following basically the scheme of proof developed in [17] to get an analoguous result on a bounded domain. However, even in this specific case, Theorem 4.1 is stronger (see also Remark 4.1). Indeed the analysis in [44] does not exclude that a nonzero $H^1$-weak limit $u_0$ of the $u_\varepsilon$'s contributes and breaks (4.5), that would become

$$\beta = 4\pi k + \|u_0\|^2_{H^1}\,. \tag{4.6}$$

On a bounded domain and still in this specific case $p_\varepsilon = 2$, starting from the so-called weak pointwise estimates and *using the first quantization* in [17], a more precise blow-up analysis was carried out and in particular the precise quantization (4.5) was obtained recently in [19]. Here on a surface and for general $p_\varepsilon$'s in $[1,2]$, our proof starts also from the weak pointwise estimates, but *gives at once the precise quantization, without using any intermediate one,* by pushing techniques in the spirit of [19]. As mentioned in introduction, perturbing the standard critical nonlinearity in the RHS of (0.3), as we do here, requires to be very careful, if one wants to keep the precise quantization (4.5), which is crucial for the overall strategy of the present paper to work. Indeed, it was recently proven in [34] that (4.5) may actually break down for some perturbations of the nonlinearity in (0.3) which are surprisingly weaker in some sense than the ones that we consider here.

As a byproduct of Theorem 4.1, we easily get the following corollary, allowing to get critical points of $F$ in (0.1) constrained to $\mathcal{E}_\beta$ in (0.2), as the limit of critical points of $J_{p,\beta}$ as $p \to 2$, for any fixed $\beta \notin 4\pi\mathbb{N}^\star$.

**Corollary 4.1.** *Let $\beta \in (0,+\infty)\backslash 4\pi\mathbb{N}^\star$ be given. Let $(p_\varepsilon)_\varepsilon$ be any sequence of numbers in $[1,2)$ such that $p_\varepsilon \to 2$ as $\varepsilon \to 0$. Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions such that* (4.1) *holds true for $\lambda_\varepsilon > 0$ given by* (4.2) *and for $\beta_\varepsilon := \beta$ for all $\varepsilon$. Then, up to a subsequence, we have that $u_\varepsilon \to u$ in $C^2$, where $u > 0$ is smooth and solves* (0.3)-(0.4)

For any $\lambda > 0$, $p \in [1,2]$ and $u$ satisfying (0.6), observe first that we necessarily have

$$\lambda \leq \frac{1}{2}\,, \tag{4.7}$$

by integrating (0.6) in $\Sigma$, by using $qt^{q-1}e^{t^q} \geq 2t$, for all $t > 0$ and all $q \in [1,2]$, and the assumption in (0.6) that $u$ is positive on $\Sigma$.

*Proof of Corollary 4.1.* Let $\beta$, $(p_\varepsilon)_\varepsilon$, $(u_\varepsilon)_\varepsilon$ and $(\lambda_\varepsilon)_\varepsilon$ be given as in Corollary 4.1. Since $\beta \notin 4\pi\mathbb{N}^\star$, we get from Theorem 4.1 that (4.4) cannot hold true. Then, by (4.7) and by standard elliptic theory as developed in [23], up to a subsequence, $\lambda_\varepsilon \to \lambda$ and $u_\varepsilon \to u$ in $C^2$ as $\varepsilon \to 0$, for some $C^2$-function $u \geq 0$ and some $\lambda \geq 0$ satisfying the equation in (0.3) and (0.4). If $u \equiv 0$, we clearly get a contradiction to

(0.4), since $\beta > 0$. Then, $u \not\equiv 0$ and $u > 0$ in $\Sigma$ by the maximum principle, which concludes the proof of Corollary 4.1. ☐

We now turn to the proof of Theorem 4.1 itself. From now on, we let $(\lambda_\varepsilon)_\varepsilon$ be a sequence of positive real numbers, we let $(p_\varepsilon)_\varepsilon$ be a sequence of numbers in $[1,2]$ and we let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions solving (4.1). Let $(\beta_\varepsilon)_\varepsilon$ be given by (4.2). *We also assume* (4.3). Then, since

$$\frac{2-p_\varepsilon}{p_\varepsilon} + \frac{2(p_\varepsilon - 1)}{p_\varepsilon} = 1\,, \tag{4.8}$$

Hölder's inequality gives that

$$\lambda_\varepsilon \int_\Sigma u_\varepsilon^{2(p_\varepsilon-1)} \left(e^{u_\varepsilon^{p_\varepsilon}} - 1\right) dv_g = O(1)\,. \tag{4.9}$$

By (4.7), $p_\varepsilon \in [1,2]$ and the fact that $\Sigma$ has finite volume

$$\begin{aligned} \lambda_\varepsilon \int_\Sigma u_\varepsilon^{2(p_\varepsilon-1)} dv_g &= \lambda_\varepsilon \int_{\{u_\varepsilon \le 2\}} u_\varepsilon^{2(p_\varepsilon-1)} dv_g + \lambda_\varepsilon \int_{\{u_\varepsilon > 2\}} u_\varepsilon^{2(p_\varepsilon-1)} dv_g \\ &\le O(1) + \lambda_\varepsilon e^{-2} \int_\Sigma e^{u_\varepsilon^{p_\varepsilon}} u_\varepsilon^{2(p_\varepsilon-1)} dv_g, \end{aligned} \tag{4.10}$$

then as a consequence

$$\lambda_\varepsilon \int_\Sigma u_\varepsilon^p e^{u_\varepsilon^{p_\varepsilon}} dv_g = O(1) \tag{4.11}$$

for all $p \in [0, 2(p_\varepsilon - 1)]$ and all $\varepsilon$. We get (4.11), for $p = 2(p_\varepsilon - 1)$, combining (4.9) and (4.10), and then also for $p \in [0, 2(p_\varepsilon - 1))$, using that $\Sigma$ has finite volume and (4.7).

As a first step, observe that we may directly get the following rough, subcritical but global bounds on the $u_\varepsilon$'s.

**Lemma 4.1.** *There exists $C > 0$ such that*

$$\int_\Sigma e^{u_\varepsilon^{1/3}} dv_g \le C$$

*for all $\varepsilon$. In particular, for all given $p < +\infty$, $(u_\varepsilon)_\varepsilon$ is bounded in $L^p$.*

Lemma 4.1 strongly relies on (4.3) and is actually the very first step to get Proposition 4.1 below, already obtained in [44] for $p_\varepsilon = 2$. This lemma is relevant to handle the term $u_\varepsilon$ in the LHS of (4.1), appearing in the present surface setting.

*Proof of Lemma 4.1.* Integrating (4.1) in $\Sigma$, we get from the consequence (4.11) of (4.3) that $(u_\varepsilon)_\varepsilon$ is bounded in $L^1$. Set now $\check{u}_\varepsilon = \max\{u_\varepsilon, 1\}$. Multiplying (4.1) by $\check{u}_\varepsilon^{-1/3}$ and integrating by parts in $\Sigma$ (see for instance [25, Proposition 2.5]), we get

$$3\int_\Sigma |\nabla(\check{u}_\varepsilon^{1/3})|^2 dv_g = + \int_\Sigma \check{u}_\varepsilon^{-1/3} u_\varepsilon dv_g - \lambda_\varepsilon p_\varepsilon \int_\Sigma \check{u}_\varepsilon^{-1/3} u_\varepsilon^{p_\varepsilon - 1} e^{u_\varepsilon^{p_\varepsilon}} dv_g\,.$$

Since $\check{u}_\varepsilon \ge 1$ and $(u_\varepsilon)_\varepsilon$ is bounded in $L^1$, it is clear that $\int_\Sigma \check{u}_\varepsilon^{-1/3} u_\varepsilon dv_g = O(1)$. Concerning the last integral, writing $\Sigma = \{x \text{ s.t. } u_\varepsilon > 1\} \cup \{x \text{ s.t. } u_\varepsilon \le 1\}$, we find that the integral on the latter set is of order $O(1)$ since $\Sigma$ has finite volume and by (4.7), while the integral on the complement is of order $O(1)$ by (4.11) for $p = p_\varepsilon - 1$, using $\check{u}_\varepsilon \ge 1$. Similarly, since $(u_\varepsilon)_\varepsilon$ is bounded in $L^1$, $(\check{u}_\varepsilon^{1/3})_\varepsilon$ is bounded in $L^2$. Then, by the Moser-Trudinger inequality, $(\exp(\check{u}_\varepsilon^{1/3}))_\varepsilon$ is bounded in $L^1$. Obviously, the same property also holds for $(\exp(u_\varepsilon^{1/3}))_\varepsilon$, which concludes the proof. ☐

From now on, we also assume that the $u_\varepsilon$'s blow-up, namely *we assume that* (4.4) *holds*. In order to prove Theorem 4.1, we need to introduce some notation and a first set of pointwise estimates on the $u_\varepsilon$'s gathered in Proposition 4.1 below. As aforementioned, these estimates have already been proven by Yang [44] in the case where $p_\varepsilon$ equals 2 for all $\varepsilon$. Yet, if this last specific condition is not satisfied, note that, even in the case $p_\varepsilon \to 2^-$, we are not here in the suitable framework to use the results from [44], since the nonlinearity appearing in the RHS of (4.1) is not of *uniform* Moser-Trudinger critical growth (see [17, Definition 1]). However, as it was already observed in the literature (see for instance [8]), the technique of the *pointwise* exhaustion of concentration points introduced in [17] is rather robust and may be successfully adapted to a much broader class of problems. Once Lemma 4.1 is obtained, the proof of Proposition 4.1 for general $p_\varepsilon$'s is very similar to the corresponding proof for $p_\varepsilon = 2$ in [44].

Concerning the notation, for all $i \in \{1, ..., N\}$ and $\varepsilon \ll 1$, we may choose isothermal coordinates $(B_{x_{i,\varepsilon}}(\kappa_1), \phi_{i,\varepsilon}, U_{i,\varepsilon})$ around $x_{i,\varepsilon}$, such that $\phi_{i,\varepsilon}$ is a diffeomorphism from $B_{x_{i,\varepsilon}}(\kappa_1) \subset \Sigma$ to $U_{i,\varepsilon} \subset \mathbb{R}^2$, where $\kappa_1 > 0$ is some appropriate given positive constant and $B_{x_{i,\varepsilon}}(\kappa_1)$ is the ball of radius $\kappa_1$ and center $x_{i,\varepsilon}$ for the metric $g$, such that $\phi_{i,\varepsilon}(x_{i,\varepsilon}) = 0$, such that $B_0(2\kappa) \subset U_{i,\varepsilon}$, for some $\kappa > 0$, and such that $(\phi_{i,\varepsilon})_\star g = e^{2\varphi_{i,\varepsilon}}\xi$, where $\mathbb{R}^2$ is endowed with its standard metric $\xi$ (see for instance [16, 41]). We may also assume that $(\varphi_{i,\varepsilon})_\varepsilon$ satisfies

$$\forall \varepsilon\,, \quad \varphi_{i,\varepsilon}(0) = 0 \text{ and } \lim_{\varepsilon \to 0} \varphi_{i,\varepsilon} = \varphi_i \text{ in } C^2_{loc}(B_0(2\kappa))\,. \tag{4.12}$$

At last, we set

$$u_{i,\varepsilon} = u_\varepsilon \circ \phi_{i,\varepsilon}^{-1}$$

in $B_0(2\kappa)$. We denote also by $d_g(\cdot,\cdot)$ the Riemannian distance on $(\Sigma, g)$.

**Proposition 4.1.** *Up to a subsequence, there exist an integer $N \geq 1$ and sequences $(x_{i,\varepsilon})_\varepsilon$ of points in $\Sigma$ such that $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$, such that, setting $\gamma_{i,\varepsilon} := u_\varepsilon(x_{i,\varepsilon})$,*

$$\mu_{i,\varepsilon} := \left(\frac{8}{\lambda_\varepsilon p_\varepsilon^2 \gamma_{i,\varepsilon}^{2(p_\varepsilon - 1)} e^{\gamma_{i,\varepsilon}^{p_\varepsilon}}}\right)^{\frac{1}{2}} \to 0\,, \tag{4.13}$$

*such that*

$$\forall j \in \{1, ..., N\} \backslash \{i\}\,, \frac{d_g(x_{j,\varepsilon}, x_{i,\varepsilon})}{\mu_{i,\varepsilon}} \to +\infty\,, \tag{4.14}$$

*and such that*

$$\frac{p_\varepsilon}{2}\gamma_{i,\varepsilon}^{p_\varepsilon - 1}(\gamma_{i,\varepsilon} - u_{i,\varepsilon}(\mu_{i,\varepsilon}\cdot)) \to T_0 := \ln\left(1 + |\cdot|^2\right) \text{ in } C^1_{loc}(\mathbb{R}^2)\,, \tag{4.15}$$

*as $\varepsilon \to 0$, for all $i \in \{1, ..., N\}$. Moreover, there exist $C_1, C_2 > 0$ such that we have*

$$\min_{i \in \{1,...,N\}} u_\varepsilon^{p_\varepsilon - 1} d_g(x_{i,\varepsilon}, \cdot)^2 |\Delta_g u_\varepsilon| \leq C_1 \text{ in } \Sigma \tag{4.16}$$

*and*

$$\min_{i \in \{1,...,N\}} u_\varepsilon^{p_\varepsilon - 1} d_g(x_{i,\varepsilon}, \cdot) |\nabla u_\varepsilon|_g \leq C_2 \text{ in } \Sigma \tag{4.17}$$

*for all $\varepsilon$. We also have that $\lim_{\varepsilon \to 0} x_{i,\varepsilon} = x_i$ for all $i$, and that there exists $u_0 \in C^2(\Sigma \backslash \mathcal{S})$ such that*

$$\lim_{\varepsilon \to 0} u_\varepsilon = u_0 \text{ in } C^2_{loc}(\Sigma \backslash \mathcal{S})\,, \tag{4.18}$$

*where $\mathcal{S} := \{x_1, ..., x_N\}$.*

Observe first that $\gamma_{i,\varepsilon} \to +\infty$ as $\varepsilon \to 0$ by (4.7) and (4.13). As an other remark, by (4.14) and (4.15), we have that

$$4\pi N = N \int_{\mathbb{R}^2} 4e^{-2T_0} dx \leq \liminf_{\varepsilon \to 0} \frac{\lambda_\varepsilon p_\varepsilon^2}{2} \int_\Sigma u_\varepsilon^{2(p_\varepsilon - 1)} e^{u_\varepsilon^{p_\varepsilon}} dv_g \,,$$

so that (4.3) and its consequence (4.11) for $p = 2(p_\varepsilon - 1)$ are not only used to get (4.15) from the classification in [5], but also to get that the extraction procedure of the blow-up points $(x_{i,\varepsilon})_\varepsilon$ has to stop after a finite number $N$ of steps, which eventually gives (4.16) (see [17, Section 3]).

**Remark 4.1.** *At this stage, we have only extracted the "highest bubbles" in* (4.15) *and it is not yet clear at all whether N in Proposition 4.1 is a good candidate to be k in* (4.5) *(see also the discussion in* [19, Section 2]*)). Indeed, for p* = 2*, it is now known (see* [35]*) that a tower of k-bubbles may exist for nonlinearities which are lower order perturbations of the one in* (0.3) *and we may then have only one "highest bubble" (i.e.* $N = 1$*) with any* $k \in \mathbb{N}^\star$ *in* (4.5)*.*

We get from (4.1) that

$$\Delta u_{i,\varepsilon} = e^{2\varphi_{i,\varepsilon}} \left( -u_{i,\varepsilon} + \lambda_\varepsilon p_\varepsilon u_{i,\varepsilon}^{p_\varepsilon - 1} e^{u_{i,\varepsilon}^{p_\varepsilon}} \right) \,, \quad u_{i,\varepsilon} > 0 \text{ in } B_0(2\kappa) \,, \tag{4.19}$$

for all $i$ and $\varepsilon$, where $\Delta = \Delta_\xi$ throughout the paper. For all $i \in \{1, ..., N\}$, we set

$$r_{i,\varepsilon} = \begin{cases} \kappa \text{ if } N = 1 \,, \\ \min\left( \frac{1}{3} \min_{j \in \{1,\dots,N\} \setminus \{i\}} d_g(x_{i,\varepsilon}, x_{j,\varepsilon}), \kappa \right) \text{ otherwise} \,, \end{cases} \tag{4.20}$$

for all $\varepsilon$, so that we get from (4.13) and (4.14) that

$$\lim_{\varepsilon \to 0} \frac{\mu_{i,\varepsilon}}{r_{i,\varepsilon}} = 0 \,. \tag{4.21}$$

We set $t_{i,\varepsilon} := \ln \left( 1 + \frac{|\cdot|^2}{\mu_{i,\varepsilon}^2} \right)$ in $\mathbb{R}^2$. We set also

$$v_{i,\varepsilon} = B_{\gamma_{i,\varepsilon}} \,, \tag{4.22}$$

where $B_\gamma$ is as in (2.6) for $(p_\gamma)_\gamma$ and $(\mu_\gamma)_\gamma$ satisfying $p_{\gamma_{i,\varepsilon}} = p_\varepsilon$ and $\mu_{\gamma_{i,\varepsilon}} = \mu_{i,\varepsilon}$, for all $\varepsilon$ and all $i \in \{1, ..., N\}$. Up to renumbering, we may also assume that

$$r_{1,\varepsilon} \leq r_{2,\varepsilon} \leq ... \leq r_{N,\varepsilon} \tag{4.23}$$

for all $\varepsilon$.

In order to link the present situation to the results of Sections 2 and 3, we need some preliminary observations. Let $l \in \{1, ..., N\}$ be given. Given a parameter $\eta \in (0, 1)$ that is going to take several values in the proof below, we let $r_{l,\varepsilon}^{(\eta)}$ be given by

$$t_{l,\varepsilon} \left( r_{l,\varepsilon}^{(\eta)} \right) = \eta \frac{p_\varepsilon \gamma_{l,\varepsilon}^{p_\varepsilon}}{2} \,, \tag{4.24}$$

and, for $r_{l,\varepsilon}$ as in (4.20), we set

$$\bar{r}_{l,\varepsilon}^{(\eta)} = \min \left( r_{l,\varepsilon}, r_{l,\varepsilon}^{(\eta)} \right) \tag{4.25}$$

for all $\varepsilon$. By collecting the above preliminary information, we can check that Proposition 3.1 applies with $\bar{r}_\varepsilon = \bar{r}_{l,\varepsilon}^{(\eta)}$, $\varphi_\varepsilon = \varphi_{l,\varepsilon}$, $u_\varepsilon = u_{l,\varepsilon}$, $\gamma_\varepsilon = \gamma_{l,\varepsilon}$ and $v_\varepsilon = v_{l,\varepsilon}$. In particular, the definition (4.20) of $r_{l,\varepsilon}$ is used to get (3.11) from (4.17), while Lemma

4.1 is used to get (3.6) and (3.18). As a remark, the metrics $(\phi_{l,\varepsilon})_\star g$ and $\xi$ are equivalent in $B_0(\kappa)$ by (4.12): we use this fact here and currently in the sequel. We get in particular (see (3.19)) that $\bar{r}_{l,\varepsilon}^{(\eta)} = o(1)$ and even that

$$\ln \gamma_{l,\varepsilon} = o\left(\ln \frac{1}{\bar{r}_{l,\varepsilon}^{(\eta)}}\right), \tag{4.26}$$

so that Proposition 3.1 also applies (see the remark involving (3.17)), and so that we get

$$\gamma_{l,\varepsilon} \geq v_{l,\varepsilon} = \gamma_{l,\varepsilon}\left(1 - \frac{2t_{l,\varepsilon}\left(1 + O\left(\gamma_{l,\varepsilon}^{-p_\varepsilon}\right)\right)}{p_\varepsilon \gamma_{l,\varepsilon}^{p_\varepsilon}}\right) \geq (1-\eta)\gamma_{l,\varepsilon} + O\left(\gamma_{l,\varepsilon}^{1-p_\varepsilon}\right), \tag{4.27}$$

uniformly in $\left[0, \bar{r}_{l,\varepsilon}^{(\eta)}\right]$ and for all $\varepsilon \ll 1$, using Proposition 2.1 and (4.24). We also get from Section 3 (see (3.13)) that

$$|u_{l,\varepsilon} - v_{l,\varepsilon}| = O\left(\frac{|\cdot|}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}\bar{r}_{l,\varepsilon}^{(\eta)}}\right) \tag{4.28}$$

and (see (3.14))

$$|\nabla(u_{l,\varepsilon} - v_{l,\varepsilon})| = O\left(\frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}\bar{r}_{l,\varepsilon}^{(\eta)}}\right) \tag{4.29}$$

uniformly in $B_0\left(\bar{r}_{l,\varepsilon}^{(\eta)}\right)$ and for all $\varepsilon \ll 1$. We get now the following result:

**Step 4.1.** *For all $i \in \{1, ..., N\}$, we have that*

$$\liminf_{\varepsilon \to 0} \frac{2t_{i,\varepsilon}(r_{i,\varepsilon})}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon}} \geq 1, \tag{4.30}$$

*and that there exists $C \gg 1$ such that*

$$0 < \bar{u}_{i,\varepsilon}(r) \leq -\left(\frac{2}{p_\varepsilon} - 1\right)\gamma_{i,\varepsilon} + \frac{2}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{C}{\lambda_\varepsilon \gamma_{i,\varepsilon}^{2(p_\varepsilon - 1)} r^2} + O\left(r^{3/2}\right) \tag{4.31}$$

*uniformly in $r \in (0, \kappa]$ and for all $\varepsilon \ll 1$, where $\bar{u}_{i,\varepsilon}$ is continuous in $[0, 2\kappa)$ and given by*

$$\bar{u}_{i,\varepsilon}(r) = \frac{1}{2\pi r}\int_{\partial B_0(r)} u_{i,\varepsilon}\, d\sigma_\xi, \tag{4.32}$$

*for all $r \in (0, 2\kappa)$, where $d\sigma_\xi$ is the volume element for the metric induced in $\partial B_0(r)$ by the standard metric $\xi$ in $\mathbb{R}^2$.*

*Proof of Step 4.1.* We divide the proof of Step 4.1 into two parts.

*Proof of* (4.31). Here we show (4.31), assuming that (4.30) is already obtained for some $i$. Let $\eta_1 < \eta_2$ be two given numbers in $(0, 1)$. Then by (4.24), (4.25) and (4.30), we get

$$\bar{r}_{i,\varepsilon}^{(\eta_1)} = r_{i,\varepsilon}^{(\eta_1)} \text{ and } \bar{r}_{i,\varepsilon}^{(\eta_2)} = r_{i,\varepsilon}^{(\eta_2)}$$

for all $\varepsilon \ll 1$. Then (4.31) holds true uniformly in $\left(0, r_{i,\varepsilon}^{(\eta_2)}\right]$ using (2.8) and (4.28) for $l = i$ and parameters $\eta_1$ or $\eta_2$. We get also from (4.27) and (4.28) that

$$\bar{u}_{i,\varepsilon}\left(r_{i,\varepsilon}^{(\eta_1)}\right) = v_{i,\varepsilon}\left(r_{i,\varepsilon}^{(\eta_1)}\right) + O\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\right) \leq \gamma_{i,\varepsilon} + O\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\right), \tag{4.33}$$

and from (4.29) that

$$\|\nabla(u_{i,\varepsilon}-v_{i,\varepsilon})\|_{L^\infty\left(\partial B_0\left(r_{i,\varepsilon}^{(\eta_1)}\right)\right)} = O\left(\frac{1}{\gamma_{i,\varepsilon}^{p_\varepsilon-1} r_{i,\varepsilon}^{(\eta_2)}}\right) \tag{4.34}$$

for all $\varepsilon \ll 1$. For $f$ a $C^2$ function around $0 \in \mathbb{R}^2$ and $r \geq 0$, we let $\bar{f}(r)$ (see (4.32)) be the average of $f$ on $\partial B_0(r)$; integrating by parts, we get

$$-2\pi r \bar{f}'(r) = \int_{B_0(r)} (\Delta f)(x)dx \tag{4.35}$$

with the usual radial (abuse of) notation. We write with (4.12) and (4.19) that

$$\int_{B_0(r)} (\Delta u_{i,\varepsilon})dx \geq \int_{B_0\left(r_{i,\varepsilon}^{(\eta_1)}\right)} (\Delta u_{i,\varepsilon})dx + O\left(\int_{B_0(r)\setminus B_0\left(r_{i,\varepsilon}^{(\eta_1)}\right)} u_{i,\varepsilon}dx\right),$$

that $\int_{B_0(r)} u_{i,\varepsilon}dx = O\left(r^{3/2}\right)$ by Hölder's inequality with Lemma 4.1 for $p=4$, and then, with (4.35), that

$$\bar{u}_{i,\varepsilon}'(r) \leq -\frac{1}{2\pi r}\left(-2\pi r_{i,\varepsilon}^{(\eta_1)} \bar{u}_{i,\varepsilon}'\left(r_{i,\varepsilon}^{(\eta_1)}\right)\right) + O\left(r^{1/2}\right) \tag{4.36}$$

uniformly in $r \in \left[r_{i,\varepsilon}^{(\eta_1)}, \kappa\right]$ and for all $\varepsilon \ll 1$. We get from the definition (4.24) of $r_{l,\varepsilon}^{(\eta_j)}$ for $l=i$ and $j \in \{1,2\}$ that

$$\ln \frac{r_{i,\varepsilon}^{(\eta_1)}}{r_{i,\varepsilon}^{(\eta_2)}} = -\frac{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon}}{4}(\eta_2-\eta_1) + o(1) \tag{4.37}$$

as $\varepsilon \to 0$. We now write

$$\bar{u}_{i,\varepsilon}' = v_{i,\varepsilon}' + \left(\bar{u}_{i,\varepsilon}' - v_{i,\varepsilon}'\right) = -\frac{2t_{i,\varepsilon}'}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon-1}}\left[1 + O\left(\gamma_{i,\varepsilon}^{-p_\varepsilon}\right) + O\left(\frac{r_{i,\varepsilon}^{(\eta_1)}}{r_{i,\varepsilon}^{(\eta_2)}}\right)\right],$$

at $r_{i,\varepsilon}^{(\eta_1)}$ for all $\varepsilon \ll 1$, using Proposition 2.1 and (4.34). This implies with (4.37) that

$$-2\pi r_{i,\varepsilon}^{(\eta_1)} \bar{u}_{i,\varepsilon}'\left(r_{i,\varepsilon}^{(\eta_1)}\right) = \frac{8\pi}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon-1}} + O\left(\gamma_{i,\varepsilon}^{1-2p_\varepsilon}\right), \tag{4.38}$$

using also that

$$r_{i,\varepsilon}^{(\eta_1)} t_{i,\varepsilon}'\left(r_{i,\varepsilon}^{(\eta_1)}\right) = 2 + O\left(\mu_{i,\varepsilon}^2 / \left(r_{i,\varepsilon}^{(\eta_1)}\right)^2\right) = 2 + O\left(\gamma_{i,\varepsilon}^{-p_\varepsilon}\right)$$

for all $\varepsilon \ll 1$, by the definition (4.24) of $r_{i,\varepsilon}^{(\eta_1)}$. Then, integrating (4.36) in $[r_{i,\varepsilon}^{(\eta_1)}, s]$ and using the fundamental theorem of calculus and (4.38), we get that

$$\bar{u}_{i,\varepsilon}(s) - \bar{u}_{i,\varepsilon}\left(r_{i,\varepsilon}^{(\eta_1)}\right) \leq -\frac{4}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon-1}} \ln \frac{s}{r_{i,\varepsilon}^{(\eta_1)}}\left(1 + O\left(\gamma_{i,\varepsilon}^{-p_\varepsilon}\right)\right) + O\left(s^{3/2}\right) \tag{4.39}$$

uniformly in $s \in [r_{i,\varepsilon}^{(\eta_1)}, \kappa]$, for all $\varepsilon \ll 1$, and conclude the proof of (4.31) by evaluating $\bar{u}_{i,\varepsilon}\left(r_{i,\varepsilon}^{(\eta_1)}\right)$ with (2.8) and (4.33). To get the existence of $C>0$ in (4.31) from the remainder in (4.39), we use that (4.39) and $\bar{u}_{i,\varepsilon}(\kappa) > 0$ imply

$$0 \leq \ln \frac{s}{r_{i,\varepsilon}^{(\eta_1)}} = O\left(\gamma_{i,\varepsilon}^{p_\varepsilon-1} \bar{u}_{i,\varepsilon}\left(r_{i,\varepsilon}^{(\eta_1)}\right)\right) + O\left(\gamma_{i,\varepsilon}^{p_\varepsilon-1}\right) = O(\gamma_{i,\varepsilon}^{p_\varepsilon})$$

uniformly in $s \in [r_{i,\varepsilon}^{(\eta_1)}, \kappa]$ and for all $\varepsilon \ll 1$, thanks to (4.33). $\square$

*Proof of* (4.30). We now turn to the proof of (4.30). We prove it by induction on $i \in \{1, ..., N\}$. In particular, we assume that (4.30) holds true at steps $1, ..., i-1$ if $i \geq 2$. By contradiction, assume in addition that (4.30) does not hold true at step $i$. Thus, by (4.24)-(4.25), up to a subsequence, we may choose and fix $\eta \in (0,1)$ sufficiently close to 1 such that

$$\bar{r}_{i,\varepsilon}^{(\eta)} = r_{i,\varepsilon} \tag{4.40}$$

for all $\varepsilon \ll 1$. Set $J_i = \{j \in \{1, ..., N\} \text{ s.t. } d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) = O\left(r_{i,\varepsilon}\right)\}$. Obviously, we get from (4.20) that

$$r_{l,\varepsilon} = O\left(r_{i,\varepsilon}\right) \tag{4.41}$$

for all $\varepsilon \ll 1$ and all $l \in J_i$. We also find from (4.26) for $l = i$ and from (4.40) that $r_{i,\varepsilon} \to 0$, so we get from (4.12) that

$$g_{l,\varepsilon} := \left[\left(\phi_{l,\varepsilon}\right)_\star g\right] \left(r_{i,\varepsilon} \cdot\right) \to \xi \text{ in } C^2_{loc}(\mathbb{R}^2)\,, \tag{4.42}$$

as $\varepsilon \to 0$ for all $l \in J_i$. Up to a subsequence, we may assume that

$$\lim_{\varepsilon \to 0} \frac{\phi_{i,\varepsilon}(x_{l,\varepsilon})}{r_{i,\varepsilon}} = \check{x}_l \in \mathbb{R}^2$$

for all $l \in J_i$, and we have that $\mathcal{S}_i := \{\check{x}_l, l \in J_i\}$ contains at least two distinct points, by (4.20), since $r_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$. We may now choose and fix $\tau \in (0,1)$ small enough such that

$$3\tau < \min_{\left\{(x,y) \in \mathcal{S}_i^2\,, x \neq y\right\}} |x - y|$$

and such that $\mathcal{S}_i \subset B_0(1/(3\tau))$. We can check that there exists $C > 0$ such that any point in

$$\Omega_{i,\varepsilon} := B_0(r_{i,\varepsilon}/\tau) \backslash \cup_{j \in J_i} B_{\phi_{i,\varepsilon}(x_{j,\varepsilon})}(\tau r_{i,\varepsilon})$$

may be joined to $\partial B_0(\tau r_{i,\varepsilon})$ by a $C^1$ path in $\Omega_{i,\varepsilon}$ of $\xi$-length at most $C r_{i,\varepsilon}$, for all $\varepsilon \ll 1$. Therefore, by (4.40) with (4.27) and (4.28) for $l = i$, we may estimate first $u_{i,\varepsilon}$ on $\partial B_0(\tau r_{i,\varepsilon})$ and then get from (4.17) and (4.42) that

$$u_{i,\varepsilon} = \bar{u}_{i,\varepsilon}(\tau r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\right) \geq (1-\eta)\gamma_{i,\varepsilon} + O(1) \tag{4.43}$$

uniformly in $\Omega_{i,\varepsilon}$ and for all $\varepsilon \ll 1$, with $\eta \in (0,1)$ still as initially fixed in (4.40). Independently, we get from (2.8), (4.28) for $l = i$ and (4.40) that

$$\bar{u}_{i,\varepsilon}(\tau r_{i,\varepsilon}) = -\left(\frac{2}{p_\varepsilon} - 1\right)\gamma_{i,\varepsilon} + \frac{2}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon - 1}}\left(\ln \frac{1}{\lambda_\varepsilon \gamma_{i,\varepsilon}^{2(p_\varepsilon - 1)} r_{i,\varepsilon}^2} + O(1)\right) \tag{4.44}$$

for all $\varepsilon \ll 1$.

• We prove now that, for all $j \in J_i$

$$j < i \implies \lim_{\varepsilon \to 0} \frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}} = 0\,, \tag{4.45}$$

up to a subsequence. Then, let $j \in J_i$ such that $j < i$. By (4.23), we have that $r_{j,\varepsilon} \leq r_{i,\varepsilon}$ and, by our induction assumption, we know from (4.30) at step $j$ and from (4.24)-(4.25) that, given any $\eta_2 \in (0,1)$,

$$\bar{r}_{j,\varepsilon}^{(\eta_2)} = r_{j,\varepsilon}^{(\eta_2)} \tag{4.46}$$

for all $\varepsilon \ll 1$. Then, by (4.27), by (4.28) for $l = j$ with parameter $\eta = \eta_2$, and by the definition (4.24) of $r_{j,\varepsilon}^{(\eta_2)}$ we have that

$$\bar{u}_{j,\varepsilon}\left(\bar{r}_{j,\varepsilon}^{(\eta_2)}\right) \le (1-\eta_2)\gamma_{j,\varepsilon}(1+o(1))$$

as $\varepsilon \to 0$. For all $l \in J_i$, let $w_{l,\varepsilon}$ be given by

$$\begin{cases} \Delta w_{l,\varepsilon} = -e^{2\varphi_{l,\varepsilon}} u_{l,\varepsilon} \text{ in } B_0(r_{i,\varepsilon}/(2\tau))\,, \\ w_{l,\varepsilon} = 0 \text{ on } \partial B_0(r_{i,\varepsilon}/(2\tau))\,. \end{cases} \tag{4.47}$$

By observing that $\Delta(u_{l,\varepsilon} - w_{l,\varepsilon}) \ge 0$ in $B_0(r_{i,\varepsilon}/(2\tau))$ by (4.19), the maximum principle yields that $u_{l,\varepsilon} - w_{l,\varepsilon}$ *attains its infimum on* $B_0(r_{i,\varepsilon}/(2\tau))$ *at some point in* $\partial B_0(r_{i,\varepsilon}/(2\tau))$. Moreover, for all given $p \in (1,+\infty)$, we get from Lemma 4.1 and (4.12) that

$$\|\Delta(w_{l,\varepsilon}(r_{i,\varepsilon}\cdot))\|_{L^p(B_0(1/(2\tau)))} = O\left(r_{i,\varepsilon}^{\frac{2(p-1)}{p}}\right)\,,$$

so, by elliptic theory, (4.24)-(4.26) and (4.40), we get

$$w_{l,\varepsilon}(r_{i,\varepsilon}\cdot) = O\left(r_{i,\varepsilon}^{\frac{2(p-1)}{p}}\right) = o\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\right) \tag{4.48}$$

uniformly in $B_0(1/(2\tau))$ as $\varepsilon \to 0$. Summarizing this argument for $l = j$, we get

$$(1-\eta)\gamma_{i,\varepsilon} \le (1-\eta_2)\gamma_{j,\varepsilon}(1+o(1)) \tag{4.49}$$

as $\varepsilon \to 0$, using also (4.42)-(4.43). Indeed, by (4.42), observe that we may choose $\tau > 0$ sufficiently small from the beginning to have

$$\partial B_0(r_{i,\varepsilon}/(2\tau)) \subset \phi_{l,\varepsilon}^{-1} \circ \phi_{i,\varepsilon}(\Omega_{i,\varepsilon})\,, \tag{4.50}$$

so that we may estimate $u_{l,\varepsilon}$ on $\partial B_0(r_{i,\varepsilon}/(2\tau))$ with (4.43), for all $l \in J_i$ and all $\varepsilon \ll 1$. Since $\eta_2 < 1$ may be chosen arbitrarily close to 1, (4.49) gives (4.45).

• We prove now that, for all $j \in J_i$,

$$\gamma_{j,\varepsilon} = O(\gamma_{i,\varepsilon})\,. \tag{4.51}$$

By contradiction, if (4.51) does not hold true, we choose $j \in J_i$ such that

$$\lim_{\varepsilon \to 0} \frac{\gamma_{i,\varepsilon}}{\gamma_{j,\varepsilon}} = 0\,, \tag{4.52}$$

up to a subsequence. In particular, we have $j \neq i$. If $j > i$, we may write that

$$\begin{aligned} t_{j,\varepsilon}(r_{j,\varepsilon}) &= \ln \frac{r_{j,\varepsilon}^2}{\mu_{j,\varepsilon}^2} + o(1)\,, \\ &= \ln \frac{r_{j,\varepsilon}^2}{r_{i,\varepsilon}^2} + t_{i,\varepsilon}(r_{i,\varepsilon}) + \ln \frac{\mu_{i,\varepsilon}^2}{\mu_{j,\varepsilon}^2} + o(1)\,, \\ &= O(1) + \eta \frac{p_\varepsilon}{2}\gamma_{i,\varepsilon}^{p_\varepsilon} + \gamma_{j,\varepsilon}^{p_\varepsilon} - \gamma_{i,\varepsilon}^{p_\varepsilon} + O(\ln \gamma_{i,\varepsilon} + \ln \gamma_{j,\varepsilon})\,, \\ &= \gamma_{j,\varepsilon}^{p_\varepsilon}(1+o(1)) \end{aligned} \tag{4.53}$$

as $\varepsilon \to 0$. The first two equalities use (4.21); the third one uses first our assumption $j > i$ with (4.23) and (4.41), then the definition (4.24) for $\eta$ as in (4.40), and at last (4.13); the last equality uses (4.52). Thus, given any $\eta_2 \in (0,1)$, we get in complement of (4.45) and the paragraph below that (4.46) holds true also if $j > i$. As a first consequence, for all given $0 < \eta_2' < \eta_2 < 1$, we get that

$$\lim_{\varepsilon\to 0} \frac{r_{j,\varepsilon}^{(\eta_2')}}{r_{i,\varepsilon}} = 0\,, \tag{4.54}$$

using (4.41). We get from (2.8) and (4.28), for $l = j$ and parameter $\eta_2'$, that

$$\bar{u}_{j,\varepsilon}\left(r_{j,\varepsilon}^{(\eta_2')}\right) = -\left(\frac{2}{p_\varepsilon} - 1\right)\gamma_{j,\varepsilon} + \frac{2}{p_\varepsilon \gamma_{j,\varepsilon}^{p_\varepsilon - 1}}\left(\ln \frac{1}{\lambda_\varepsilon \gamma_{j,\varepsilon}^{2(p_\varepsilon-1)}\left(r_{j,\varepsilon}^{(\eta_2')}\right)^2} + O(1)\right) \tag{4.55}$$

for all $\varepsilon \ll 1$.

In order to have the desired contradiction to (4.52), fixing $\eta_2 \in (0,1)$, we prove now the following estimate

$$\bar{u}_{j,\varepsilon}\left(r_{j,\varepsilon}^{(\eta_2)}\right) \geq \bar{u}_{i,\varepsilon}(\tau r_{i,\varepsilon}) + \frac{2}{p_\varepsilon \gamma_{j,\varepsilon}^{p_\varepsilon-1}} \ln \frac{r_{i,\varepsilon}^2}{\left(r_{j,\varepsilon}^{(\eta_2)}\right)^2} + O\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\right) \tag{4.56}$$

for all $\varepsilon \ll 1$. Let $\psi_\varepsilon$ be given by

$$\begin{cases} \Delta\psi_\varepsilon = 0 \text{ in } B_0(r_{i,\varepsilon}/(2\tau))\,, \\ \psi_\varepsilon = u_{j,\varepsilon} \text{ on } \partial B_0(r_{i,\varepsilon}/(2\tau))\,, \end{cases}$$

for all $\varepsilon$. We get first

$$\psi_\varepsilon = \bar{u}_{i,\varepsilon}(\tau r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\right) \tag{4.57}$$

for all $\varepsilon \ll 1$, by (4.43), (4.50) and the maximum principle for the harmonic function $\psi_\varepsilon$. Let $(z_\varepsilon)_\varepsilon$ be any sequence of points such that $|z_\varepsilon| = r_{j,\varepsilon}^{(\eta_2)}$ for all $\varepsilon$. Let $G_\varepsilon$ be the Green's function of $\Delta$ in $B_0(r_{i,\varepsilon}/(2\tau))$ with zero Dirichlet boundary conditions. We know that $G_\varepsilon(x,y) > 0$ by the maximum principle for all $x, y \in B_0(r_{i,\varepsilon}/(2\tau))$, $x \neq y$ and for all $\varepsilon$. Let $\eta_1 \in (0, \eta_2)$ be fixed. By Green's representation formula and (4.19), using the positivity of $\Delta u_{j,\varepsilon} + e^{2\varphi_{j,\varepsilon}} u_{j,\varepsilon}$ and that of $G_\varepsilon(z_\varepsilon, \cdot)$, we have that

$$(u_{j,\varepsilon} - \psi_\varepsilon - w_{j,\varepsilon})(z_\varepsilon) \geq \lambda_\varepsilon p_\varepsilon \int_{B_0\left(r_{j,\varepsilon}^{(\eta_1)}\right)} G_\varepsilon(z_\varepsilon, y) e^{2\varphi_{j,\varepsilon}} u_{j,\varepsilon}(y)^{p_\varepsilon-1} e^{u_{j,\varepsilon}^{p_\varepsilon}(y)} dy \tag{4.58}$$

for all $\varepsilon \ll 1$, with $w_{j,\varepsilon}$ given by (4.47). There exists $C > 0$ (see [19, Appendix B]) such that

$$\left| G_\varepsilon(z,y) - \frac{1}{2\pi} \ln \frac{r_{i,\varepsilon}}{|z-y|} \right| \leq C$$

for all $y \in B_0(r_{i,\varepsilon}/(2\tau))$, for all $z \in B_0(5r_{i,\varepsilon}/(12\tau))$, $y \neq z$ and for all $\varepsilon \ll 1$. Observe also that (4.37) holds true for $l = j$. Then, since $|z_\varepsilon| = r_{j,\varepsilon}^{(\eta_2)}$, we first get that

$$G_\varepsilon(z_\varepsilon, \cdot) = \frac{1}{2\pi} \ln \frac{r_{i,\varepsilon}}{|z_\varepsilon|} + O(1) + O\left(\frac{|\cdot|}{|z_\varepsilon|}\right) = \frac{1}{2\pi} \ln \frac{r_{i,\varepsilon}}{r_{j,\varepsilon}^{(\eta_2)}} + O(1) \tag{4.59}$$

uniformly in $B_0(r_{j,\varepsilon}^{(\eta_1)})$ and for all $\varepsilon \ll 1$. Now, by (4.27) and (4.28) for $l = j$ with parameter $\eta = \eta_2$, computing as in Proposition 2.1 or in the argument involving

(3.27), we get that, for some given $\tilde\eta \in (0,1)$,

$$\lambda_\varepsilon p_\varepsilon e^{2\varphi_{j,\varepsilon}} u_{j,\varepsilon}^{p_\varepsilon-1} e^{u_{j,\varepsilon}^{p_\varepsilon}} = \frac{8e^{-2t_{j,\varepsilon}}}{\mu_{j,\varepsilon}^2 \gamma_{j,\varepsilon}^{p_\varepsilon-1} p_\varepsilon}\left(1+O\left(e^{\tilde\eta t_{j,\varepsilon}}\left[\frac{|\cdot|}{r_{j,\varepsilon}^{(\eta_2)}}+|\cdot|+\frac{1}{\gamma_{j,\varepsilon}^{p_\varepsilon}}\right]\right)\right) \tag{4.60}$$

in $B_0(r_{j,\varepsilon}^{(\eta_1)})$ and for all $\varepsilon \ll 1$. Resuming arguments in (4.53) and using (4.54), we have that

$$0 < \ln \frac{r_{i,\varepsilon}}{r_{j,\varepsilon}^{(\eta_2)}} \le \ln \frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}} + \ln \frac{\mu_{i,\varepsilon}}{\mu_{j,\varepsilon}} = \gamma_{j,\varepsilon}^{p_\varepsilon}(1+o(1)) \tag{4.61}$$

as $\varepsilon \to 0$, since (4.52) is assumed to be true. By (4.59), (4.60) and (4.61), we get that

$$\begin{aligned}
&\lambda_\varepsilon p_\varepsilon \int_{B_0\left(r_{j,\varepsilon}^{(\eta_1)}\right)} G_\varepsilon(z_\varepsilon,y) u_{j,\varepsilon}(y)^{p_\varepsilon-1} e^{u_{j,\varepsilon}^{p_\varepsilon}(y)} dy \\
&= \left(\frac{1}{2\pi}\ln\frac{r_{i,\varepsilon}}{r_{j,\varepsilon}^{(\eta_2)}} + O(1)\right) \frac{8\pi}{\gamma_{j,\varepsilon}^{p_\varepsilon-1} p_\varepsilon}\left(1+O\left(\left[\mu_{j,\varepsilon}/r_{j,\varepsilon}^{(\eta_1)}\right]^2\right)+O\left(\frac{r_{j,\varepsilon}^{(\eta_1)}}{r_{j,\varepsilon}^{(\eta_2)}}+\frac{1}{\gamma_{j,\varepsilon}^{p_\varepsilon}}\right)\right) \\
&= \frac{2}{p_\varepsilon \gamma_{j,\varepsilon}^{p_\varepsilon-1}} \ln \frac{r_{i,\varepsilon}^2}{\left(r_{j,\varepsilon}^{(\eta_2)}\right)^2} + O\left(\gamma_{j,\varepsilon}^{1-p_\varepsilon}\right)
\end{aligned}$$

for all $\varepsilon \ll 1$, using the definition of $r_{j,\varepsilon}^{(\eta_1)}$, (4.37) and (4.46) with (4.26) for $l = j$. By plugging this last estimate with (4.48), (4.52) and (4.57) in (4.58), since $(z_\varepsilon)_\varepsilon$ is arbitrary, this concludes the proof of (4.56).

We now plug (4.44) and (4.55) in (4.56) and we get

$$\left(\frac{2}{p_\varepsilon}-1\right)\gamma_{j,\varepsilon}(1+o(1)) + \frac{2+o(1)}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon-1}}\left(\ln\frac{1}{r_{i,\varepsilon}^2}+\ln\frac{1}{\lambda_\varepsilon}\right) \le O\left(\gamma_{i,\varepsilon}^{1-p_\varepsilon}\ln\gamma_{i,\varepsilon}\right)$$

still using (4.52). However this estimate gives a contradiction for $\varepsilon \ll 1$, by (4.7) and (4.26) for $l = i$ and (4.40): (4.51) is proven.

• Then, using (4.23) and (4.45), (4.51) implies that for all $l \in J_i$

$$r_{i,\varepsilon} \le r_{l,\varepsilon} \tag{4.62}$$

for all $\varepsilon \ll 1$. We now claim that there exists $\eta_3 \in (\eta, 1)$ such that

$$\bar r_{j,\varepsilon}^{(\eta_3)} = r_{j,\varepsilon} \tag{4.63}$$

for all $j \in J_i$ and all $\varepsilon \ll 1$. Coming back otherwise to (4.24)-(4.25), up to a subsequence, we may assume by contradiction that there exists $j \in J_i$ such that

$$\frac{2t_{j,\varepsilon}(r_{j,\varepsilon})}{p_\varepsilon \gamma_{j,\varepsilon}^{p_\varepsilon}} \ge 1 + o(1)$$

as $\varepsilon \to 0$. As a remark, we must have $j \ne i$ by (4.40). Then, for all given $\eta_2 \in (0,1)$, (4.46) holds true and the argument between (4.46) and (4.51) gives (4.52), which does not occur by (4.51) and proves (4.63). For $j \in J_i$, since

$$\phi_{i,\varepsilon} \circ \phi_{j,\varepsilon}^{-1}\left(\partial B_0(r_{j,\varepsilon}/2)\right) \subset \Omega_{i,\varepsilon}$$

by (4.20), (4.42), (4.62) and the definition of $\tau$, we get from the equality in (4.43)

$$\bar u_{j,\varepsilon}(r_{j,\varepsilon}/2) = \bar u_{i,\varepsilon}(\tau r_{i,\varepsilon}) + O\left(\gamma_{i,\varepsilon}^{p_\varepsilon-1}\right),$$

so that we eventually have

$$\gamma_{i,\varepsilon} = O\left(\gamma_{j,\varepsilon}\right)\,, \tag{4.64}$$

using the inequality in (4.43) and since $\bar{u}_{j,\varepsilon}(\bar{r}^{(\eta_3)}_{j,\varepsilon}/2) \le 2\gamma_{j,\varepsilon}$ by (4.27), (4.28) and (4.63), for all $\varepsilon \ll 1$.

• We are now in position to conclude the proof of (4.30). Setting

$$\tilde{u}_\varepsilon := \gamma_{i,\varepsilon}^{p_\varepsilon-1}\left(u_{i,\varepsilon}(r_{i,\varepsilon}\cdot) - \bar{u}_{i,\varepsilon}(r_{i,\varepsilon})\right)\,,$$

with an argument similar to the proof of (4.43) one deduces from (4.17) and (4.42) that $(\tilde{u}_\varepsilon)_\varepsilon$ is uniformly locally bounded in $\mathbb{R}^2\backslash\mathcal{S}_i$ for all $\varepsilon \ll 1$, where $\mathcal{S}_i$ is given below (4.42). Then, using (4.27) and (4.28) for $l = i$ with (4.40), we get from (4.12) and (4.19) that

$$\Delta\tilde{u}_\varepsilon = O\left(\gamma_{i,\varepsilon}^{p_\varepsilon} r_{i,\varepsilon}^2\right) + O\left(r_{i,\varepsilon}^2\lambda_\varepsilon\left(\gamma_{i,\varepsilon}^{p_\varepsilon-1} v_{i,\varepsilon}^{p_\varepsilon-1} e^{v_{i,\varepsilon}^{p_\varepsilon}}\right)(r_{i,\varepsilon}\cdot)\right) = o(1)$$

uniformly locally in $\mathbb{R}^2\backslash\mathcal{S}_i$ for all $\varepsilon \ll 1$. To get the last estimate, we use (4.26) for $l = i$ to control the first term, while we estimate the second one first by $O((\mu_{i,\varepsilon}/r_{i,\varepsilon})^{2(1-\tilde{\eta})})$ (see Proposition 2.1) and then we conclude with (4.21). Hence, there exists a harmonic function $\tilde{u}_0$ such that $\tilde{u}_\varepsilon \to \tilde{u}_0$ in $C^1_{loc}(\mathbb{R}^2\backslash\mathcal{S}_i)$ as $\varepsilon \to 0$. Now observe that (4.17) also gives the existence of $C > 0$ such that

$$|\nabla\tilde{u}_0| \le C\sum_{x\in\mathcal{S}_i}\frac{1}{|x-\cdot|} \text{ in } \mathbb{R}^2\backslash\mathcal{S}_i\,,$$

using the local convergence of the $\tilde{u}_\varepsilon$'s in $\mathbb{R}^2\backslash\mathcal{S}_i$ and the lower estimate in (4.27) for $l = i$. Then, by harmonic function's theory, there exist real numbers $\alpha_x$ and $\Lambda$ such that

$$\tilde{u}_0 = \Lambda + \sum_{x\in\mathcal{S}_i}\alpha_x \ln\frac{1}{|x-\cdot|} \text{ in } \mathbb{R}^2\backslash\mathcal{S}_i\,. \tag{4.65}$$

However, by (4.41) and (4.62), by (4.51) and (4.64), Proposition 3.1 gives that the $\alpha_x$ are positive and in particular (3.16) gives that

$$\nabla\left(\tilde{u}_0 - \alpha_x\ln\frac{1}{|x-\cdot|}\right)(x) = 0$$

for all $x \in \mathcal{S}_i$. Picking now $y$ an extreme point of the convex hull of $\mathcal{S}_i$, we get from (4.65) that this last property fails for $x = y$, since $\mathcal{S}_i$ possesses at least two points. This gives the expected contradiction to (4.40) and concludes the proof of (4.30). □

Step 4.1 is proven. □

Up to a subsequence, we assume from now on that

$$\lim_{\varepsilon\to 0} p_\varepsilon = p_0\,, \tag{4.66}$$

for some $p_0 \in [1, 2]$. As a first consequence of Step 4.1, we improve (4.7) and conclude the proof of (4.5) and thus that of Theorem 4.1 in the subcritical case. A key ingredient to get the sharp quantization (4.5) (and not (4.6) for $u_0 \not\equiv 0$, for instance) is given by (4.31) in Step 4.1: roughly speaking, the only way for the RHS of (4.31) to be positive at some $r$ not too small is that $\lambda_\varepsilon$ *is quite small* (see (4.68) and (4.107) below).

**Step 4.2.** *In any case, we have that*

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon = 0\,. \tag{4.67}$$

*Moreover, assuming that $p_0 \in [1,2)$, (4.5) holds true for $k = N$ and $N$ given by Proposition 4.1.*

*Proof of Step 4.2.* By evaluating (4.31) at $r = \kappa\gamma_{i,\varepsilon}^{2(1-p_\varepsilon)/3}$, we get that

$$\left(1-\frac{p_\varepsilon}{2}\right)\gamma_{i,\varepsilon}^{p_\varepsilon} + \frac{2}{3}(p_\varepsilon - 1)\ln\gamma_{i,\varepsilon} \le \ln\frac{1}{\lambda_\varepsilon} + O(1) \tag{4.68}$$

for all $\varepsilon \ll 1$ and all $i \in \{1,...,N\}$, which clearly proves (4.67). Now assume that $p_0 < 2$ in (4.66). Up to renumbering, fix $i$ such that $\gamma_{i,\varepsilon}$ is the largest of the $\gamma_{j,\varepsilon}$'s for all $\varepsilon \ll 1$ and all $j$. Given any $\eta \in (0,1)$ to be chosen later, setting $r_{l,\varepsilon}^{(\eta)}$ as in (4.24), we know from (4.30) that $\bar{r}_{l,\varepsilon}^{(\eta)} = r_{l,\varepsilon}^{(\eta)}$ for all $\varepsilon \ll 1$ and all $l$. Then, we get from (4.27) and (4.28) (see also Proposition 2.1) that

$$\begin{aligned} &\int_{B_0\left(r_{l,\varepsilon}^{(\eta)}\right)} \frac{\lambda_\varepsilon p_\varepsilon^2}{2} u_{l,\varepsilon}^{p_\varepsilon} e^{u_{l,\varepsilon}^{p_\varepsilon}}\, e^{2\varphi_{l,\varepsilon}} dx = (4\pi + o(1))\,\gamma_{l,\varepsilon}^{2-p_\varepsilon}\,, \text{ that}\\ &\int_{B_0\left(r_{l,\varepsilon}^{(\eta)}\right)} \frac{\lambda_\varepsilon p_\varepsilon^2}{2} e^{u_{l,\varepsilon}^{p_\varepsilon}}\, e^{2\varphi_{l,\varepsilon}} dx = \frac{4\pi + o(1)}{\gamma_{l,\varepsilon}^{2(p_\varepsilon-1)}} \end{aligned} \tag{4.69}$$

and that

$$\begin{aligned} u_{l,\varepsilon} &= (1-\eta)\gamma_{l,\varepsilon} + O\left(\gamma_{l,\varepsilon}^{1-p_\varepsilon}\right)\,,\\ &= -\left(\frac{2}{p_\varepsilon} - 1\right)\gamma_{l,\varepsilon} + \frac{2}{p_\varepsilon\gamma_{l,\varepsilon}^{p_\varepsilon-1}}\ln\frac{1}{\lambda_\varepsilon (r_{l,\varepsilon}^{(\eta)})^2} + O(1) \end{aligned} \tag{4.70}$$

uniformly in $\partial B_0(r_{l,\varepsilon}^{(\eta)})$ for all $\varepsilon \ll 1$ and for all $l$. The second equality uses also (2.8) with $\gamma = \gamma_{l,\varepsilon}$ and $p_\gamma = p_\varepsilon$. Up to a subsequence, by comparing the two RHS of (4.70), by using $p_0 < 2$, $r_{l,\varepsilon}^{(\eta)} \le \kappa$ and that $\eta$ (moving only here) may be arbitrarily close to 1, we may complement (4.68) here and get that

$$\left(1-\frac{p_0}{2}\right)\gamma_{l,\varepsilon}^{p_\varepsilon}(1+o(1)) = \ln\frac{1}{\lambda_\varepsilon} \tag{4.71}$$

for all $l$ as $\varepsilon \to 0$, so that we have in particular

$$\gamma_{l,\varepsilon} = (1+o(1))\gamma_{i,\varepsilon} \tag{4.72}$$

for all $l$. Given any $\check{\eta} \in (0,\eta)$, we claim that the first equality in (4.70) implies that

$$u_\varepsilon \le (1-\check{\eta})\,\gamma_{i,\varepsilon} \text{ in } \Omega_\varepsilon := \Sigma\backslash \cup_{l=1}^N \phi_{l,\varepsilon}^{-1}\left(B_0\left(r_{l,\varepsilon}^{(\eta)}\right)\right) \tag{4.73}$$

for all $\varepsilon \ll 1$. Otherwise, as when proving Proposition 4.1, if $x_\varepsilon \in \Omega_\varepsilon$ satisfies $u_\varepsilon(x_\varepsilon) = \max_{\Omega_\varepsilon} u_\varepsilon$, then $x_\varepsilon$ is a good candidate to be another concentration point for $u_\varepsilon$: we get that $\mu_{l,\varepsilon} = o\left(d_g(x_{l,\varepsilon}, x_\varepsilon)\right)$ for all $l$ by (4.15) and that $\min_{l\in\{1,...,N\}} u_\varepsilon^{p_\varepsilon-1}(x_\varepsilon) d_g(x_{l,\varepsilon}, x_\varepsilon)^2 |(\Delta_g u_\varepsilon)(x_\varepsilon)| \to +\infty$ as $\varepsilon \to 0$, which contradicts (4.16) and establishes (4.73). Independently, (4.71) gives

$$\lambda_\varepsilon \int_{\Omega_\varepsilon} (1+u_\varepsilon^{p_\varepsilon})\, e^{u_\varepsilon^{p_\varepsilon}}\, dv_g = O\left(\exp\left(\left((1-\check{\eta})^{p_0} - \left(1-\frac{p_0}{2}\right)\right)\gamma_{i,\varepsilon}^{p_\varepsilon} + o(\gamma_{i,\varepsilon}^{p_\varepsilon})\right)\right)$$

for all $\varepsilon \ll 1$. Choosing $0 < \check{\eta} < \eta < 1$ sufficiently close to 1 from the beginning (depending on the smallness of $2 - p_0 > 0$ here), we may plug this estimate and (4.69) in (4.2) to conclude the proof of (4.5), using also (4.8) and (4.72). □

In contrast to the case $p_0 = 2$ handled below (see also [17]), it is interesting to note that, due to the global nature of both integrals in (4.2), we need also (4.72) to get the quantization (4.5), at least for $k > 1$ and $1 < p_0 < 2$ in (4.66). At that stage, we are left with the proof of (4.5) in the more delicate borderline case $p_0 = 2$. *We assume from now on that* $p_0 = 2$ *in* (4.66).

*Conclusion of the proof of Theorem 4.1.* We still resume the notation and observations of (4.24)-(4.25) and below. On the other hand, by (4.30) in Step 4.1, for all given $\eta \in (0,1)$, we have that

$$r_{l,\varepsilon}^{(\eta)} = o(r_{l,\varepsilon}) \quad \Longrightarrow \quad \bar{r}_{l,\varepsilon}^{(\eta)} = r_{l,\varepsilon}^{(\eta)} \tag{4.74}$$

for all $\varepsilon \ll 1$ and all $l \in \{1, ..., N\}$. Then, as a consequence of Propositions 2.1 and 3.1, we get that (4.26)-(4.29) hold true. In particular, for all given $\eta' < \eta$ in $(0,1)$, we get from (4.29) that

$$|\nabla(u_{l,\varepsilon} - v_{l,\varepsilon})| = o\left(\frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1} r_{l,\varepsilon}^{(\eta')}}\right) \tag{4.75}$$

uniformly in $B_0(r_{l,\varepsilon}^{(\eta')})$ for all $\varepsilon \ll 1$ and all $l$. Then, for all given $\eta' \in (0,1)$, since we also have

$$0 \le v_{l,\varepsilon} - v_{l,\varepsilon}\left(r_{l,\varepsilon}^{(\eta')}\right) \le \frac{2 + o(1)}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{r_{l,\varepsilon}^{(\eta')}}{|\cdot|},$$

using the estimate in $w'_\gamma$ in Proposition 2.1, we eventually get that

$$\left|u_{l,\varepsilon} - \overline{u}_{l,\varepsilon}\left(r_{l,\varepsilon}^{(\eta')}\right)\right| \le \frac{2 + o(1)}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{2 r_{l,\varepsilon}^{(\eta')}}{|\cdot|} \tag{4.76}$$

uniformly in $B_0(r_{l,\varepsilon}^{(\eta')}) \backslash \{0\}$ for all $\varepsilon \ll 1$ and all $l$. During the whole proof below, we choose and fix $\eta_0 \in (0,1)$ and set

$$\nu_{j,\varepsilon} = \sup \left\{ r \in \left(r_{j,\varepsilon}^{(\eta_0)}, \kappa\right] \text{ s.t. } \left\{ \begin{array}{l} |u_{j,\varepsilon} - \bar{u}_{j,\varepsilon}(r)| \\ < 5\Big(\pi C_2 \bar{u}_{j,\varepsilon}(r)^{1-p_\varepsilon} \\ \qquad + 2 \sum_{l \in I_{j,\varepsilon}(r)} \gamma_{l,\varepsilon}^{1-p_\varepsilon} \ln \frac{6r}{|\cdot - \phi_{j,\varepsilon}(x_{l,\varepsilon})|}\Big) \\ \text{in } B_0(r) \backslash \cup_{l \in I_{j,\varepsilon}(r)} B_{\phi_{j,\varepsilon}(x_{l,\varepsilon})}\left(r_{l,\varepsilon}^{(\eta_0)}\right) \end{array} \right\} \right\} \tag{4.77}$$

for all $j \in \{1, ..., N\}$ and all $\varepsilon \ll 1$, where $C_2 > 0$ is as in (4.17) and where $I_{j,\varepsilon}(r)$ is given by

$$I_{j,\varepsilon}(r) = \left\{ l \in \{1, ..., N\} \text{ s.t. } \phi_{j,\varepsilon}(x_{l,\varepsilon}) \in B_0\left(\frac{3r}{2}\right) \right\}.$$

As a first remark, it follows from the very definition (4.77) of $\nu_{j,\varepsilon}$ and from (4.76) that, for all given $\eta_2 \in [\eta_0, 1)$, we have

$$\nu_{l,\varepsilon} \ge r_{l,\varepsilon}^{(\eta_2)} \tag{4.78}$$

for all $\varepsilon \ll 1$ and all $l \in \{1, ..., N\}$. Our main goal now is to show that

$$\bar{u}_{j,\varepsilon}(\nu_{j,\varepsilon}) = O(1) \tag{4.79}$$

for all $\varepsilon \ll 1$ and all $j \in \{1, ..., N\}$. For all $j$, we may assume up to a subsequence that either (4.79) or

$$\lim_{\varepsilon \to 0} \bar{u}_{j,\varepsilon}(\nu_{j,\varepsilon}) = +\infty \tag{4.80}$$

hold true. Assume from now on by contradiction that (4.79) does not hold true for all $j$ so that we may choose and fix $i \in \{1, ..., N\}$ such that

$$\nu_{i,\varepsilon} = \min \{\nu_{j,\varepsilon} \text{ s.t. (4.80) holds true}\} . \tag{4.81}$$

Clearly, we then have

$$\lim_{\varepsilon \to 0} \bar{u}_{i,\varepsilon}(\nu_{i,\varepsilon}) = +\infty . \tag{4.82}$$

By (4.18), we also have that

$$\lim_{\varepsilon \to 0} \nu_{i,\varepsilon} = 0 , \tag{4.83}$$

so that, using (4.12), the following property currently used in the sequel holds true:

$$\tilde{g}_{l,\varepsilon} := \left( (\phi_{l,\varepsilon})_\star g \right) (\nu_{i,\varepsilon} \cdot) \to \xi \text{ in } C^2_{loc}(\mathbb{R}^2) \tag{4.84}$$

as $\varepsilon \to 0$, for all $l \in I$, where

$$I := \{l \in \{1, ..., N\} \text{ s.t. } d_g(x_{i,\varepsilon}, x_{l,\varepsilon}) = O\left(\nu_{i,\varepsilon}\right) \text{ for all } \varepsilon \ll 1\} .$$

Up to a further subsequence, we may also assume that

$$\lim_{\varepsilon \to 0} \frac{\phi_{i,\varepsilon}(x_{l,\varepsilon})}{\nu_{i,\varepsilon}} = \tilde{x}_l \in \mathbb{R}^2$$

for all $l \in I$. Set also $\mathcal{S} = \{\tilde{x}_l \mid l \in I\}$ so that clearly $0 \in \mathcal{S}$. Fix $\tau \in (0, 1)$ and $R \geq 1$ to be chosen properly later on such that

$$3\tau < \begin{cases} 1 \text{ if } \mathcal{S} = \{0\} , \\ \min_{\{(x,y) \in \mathcal{S}^2 | x \neq y\}} |x - y| \text{ otherwise} , \end{cases}$$

and such that $\mathcal{S} \subset B_0(3R)$. Set $D_\varepsilon = B_0(R\nu_{i,\varepsilon}) \setminus \cup_{l \in I} B_{\phi_{i,\varepsilon}(x_{l,\varepsilon})}(\tau\nu_{i,\varepsilon}/3)$ for all $\varepsilon \ll 1$. Let now $\tilde{w}_\varepsilon$ be given by

$$\begin{cases} \Delta \tilde{w}_\varepsilon = -e^{2\varphi_{i,\varepsilon}} u_{i,\varepsilon} \text{ in } B_0(R\nu_{i,\varepsilon}) , \\ \tilde{w}_\varepsilon = 0 \text{ on } \partial B_0(R\nu_{i,\varepsilon}) , \end{cases} \tag{4.85}$$

for all $\varepsilon$. Observe first by (4.19) that $\Delta(u_{i,\varepsilon} - \tilde{w}_\varepsilon) \geq 0$ in $B_0(R\nu_{i,\varepsilon})$ so that $\overline{u_{i,\varepsilon} - \tilde{w}_\varepsilon}$ is radially nonincreasing in $[0, R\nu_{i,\varepsilon}]$. Moreover, the maximum principle gives that $u_{i,\varepsilon} - \tilde{w}_\varepsilon$ attains its infimum in $B_0(R\nu_{i,\varepsilon})$ at some point on $\partial B_0(R\nu_{i,\varepsilon})$. Independently, for all given $p > 2$, by elliptic theory, we get from Lemma 4.1 and (4.12) that

$$\|\tilde{w}_\varepsilon(\nu_{i,\varepsilon} \cdot)\|_{L^\infty(B_0(R))} = O\left(\|\Delta\left(\tilde{w}_\varepsilon(\nu_{i,\varepsilon} \cdot)\right)\|_{L^p(B_0(R))}\right) = O\left(\nu_{i,\varepsilon}^{\frac{2(p-1)}{p}}\right) \tag{4.86}$$

for all $\varepsilon \ll 1$. Summarizing, by (4.83) and since $\tau < 1$, this argument for $R = 1$ (only there) gives that $\bar{u}_{i,\varepsilon}(\tau\nu_{i,\varepsilon}) \geq \bar{u}_{i,\varepsilon}(\nu_{i,\varepsilon}) + o(1)$, so that (4.82) leads to

$$\Gamma_\varepsilon := \bar{u}_{i,\varepsilon}(\tau\nu_{i,\varepsilon}) \to +\infty \tag{4.87}$$

as $\varepsilon \to 0$. Then, as a consequence of (4.17) (and (4.84) again), we get that

$$u_{i,\varepsilon} = \Gamma_\varepsilon + O\left(\Gamma_\varepsilon^{1-p_\varepsilon}\right) \tag{4.88}$$

uniformly in $D_\varepsilon$ and for all $\varepsilon \ll 1$, using once more the mean value property on $\partial B_0(\tau\nu_{i,\varepsilon})$ and the definition of $\tau$. Then, by the maximum principle-based argument below (4.85), with (4.83) and (4.86), we get that

$$\inf_{B_0(R\nu_{i,\varepsilon})} u_{i,\varepsilon} \geq \min_{\partial B_0(R\nu_{i,\varepsilon})} u_{i,\varepsilon} + o(1) = \Gamma_\varepsilon + o(1) \tag{4.89}$$

as $\varepsilon \to 0$.

We prove now that

$$\Gamma_\varepsilon = o(\gamma_{j,\varepsilon}) \tag{4.90}$$

as $\varepsilon \to 0$, for all $j \in I$, up to a subsequence. Consider first the case $j = i$ in (4.90). For all given $\eta_2 \in [\eta_0, 1)$, we have that

$$\bar{u}_{j,\varepsilon}\left(r_{j,\varepsilon}^{(\eta_2)}\right) = (1-\eta_2)\gamma_{j,\varepsilon}(1+o(1)) \geq \Gamma_\varepsilon(1+o(1)) \tag{4.91}$$

for all $\varepsilon \ll 1$. The first equality comes from the definition (4.24) of $r_{i,\varepsilon}^{(\eta_2)}$, from (4.74), from the equality in (4.27) and from (4.28) for $l = i$, while the inequality comes from (4.78), (4.89) and the above largeness assumption $\mathcal{S} \subset B_0(3R)$ on $R \gg 1$. Observe that (4.78) implies that (4.92) below holds true for $t = i$. Since $\eta_2$ may be arbitrarily close to 1, (4.91) concludes the proof of (4.90) for $j = i$. If now $I \neq \{i\}$, we may pick $j \in I \backslash \{i\}$ and we get from the very definition of $I$ with (4.20) and (4.84) again that $r_{j,\varepsilon} = O(\nu_{i,\varepsilon})$ for all $\varepsilon$. Then, also in the last present case $j \neq i$, using now (4.74) for $l = j$, we get

$$\lim_{\varepsilon \to 0} \frac{r_{t,\varepsilon}^{(\eta_0)}}{\nu_{i,\varepsilon}} = 0\,, \tag{4.92}$$

for all $t \in I$, and then similarly (4.91), to conclude the proof of (4.90).

At that stage, we may improve the estimate in (4.86). As a consequence of (4.87), (4.88) and Lemma 4.1, writing merely that $\|u_{i,\varepsilon}\|_{L^p(D_\varepsilon)} = O(1)$, we get that $\nu_{i,\varepsilon}^2 \Gamma_\varepsilon^p = O(1)$ for all $\varepsilon$, so that (4.86) gives

$$|\tilde{w}_\varepsilon| = O\left(\Gamma_\varepsilon^{1-p}\right) = o(\Gamma_\varepsilon^{1-p_\varepsilon}) \tag{4.93}$$

uniformly in $D_\varepsilon$, for all $\varepsilon \ll 1$, since $p$ is fixed greater than 2 just above (4.86). Let $\zeta_\varepsilon$ be given by

$$\begin{cases} \Delta\zeta_\varepsilon = 0 \text{ in } B_0(R\nu_{i,\varepsilon})\,, \\ \zeta_\varepsilon = u_{i,\varepsilon} \text{ on } \partial B_0(R\nu_{i,\varepsilon}) \end{cases}$$

for all $\varepsilon$. By keeping track of the constant $C_2$ of (4.17) and choosing $R \gg 1$ large enough (depending only on $\mathcal{S}$) from the beginning, using a mean value theorem on $\partial B_0(R\nu_{i,\varepsilon})$, (4.84) and (4.87), we may get a slightly more precise version of (4.88) on $\partial B_0(R\nu_{i,\varepsilon})$, namely we have that

$$\sup_{B_0(R\nu_{i,\varepsilon})} |\zeta_\varepsilon - \bar{u}_{i,\varepsilon}(R\nu_{i,\varepsilon})| \leq \sup_{\partial B_0(R\nu_{i,\varepsilon})} |u_{i,\varepsilon} - \bar{u}_{i,\varepsilon}(R\nu_{i,\varepsilon})| \leq \frac{2\pi C_2}{\Gamma_\varepsilon^{p_\varepsilon - 1}} \tag{4.94}$$

for all $\varepsilon \ll 1$, using also the maximum principle. Observe in particular that $u_{i,\varepsilon} = (1+o(1))\Gamma_\varepsilon$ uniformly in $D_\varepsilon$. Let $\tilde{G}_\varepsilon$ be the Green's function of $\Delta$ in $B_0(R\nu_{i,\varepsilon})$ with zero Dirichlet boundary condition. Let $(z_\varepsilon)_\varepsilon$ be any sequence of points such that

$$z_\varepsilon \in \overline{B_0(R\nu_{i,\varepsilon}) \backslash \cup_{l \in I} B_{\phi_{i,\varepsilon}(x_{l,\varepsilon})}\left(r_{l,\varepsilon}^{(\eta_0)}\right)} \tag{4.95}$$

for all $\varepsilon$. We have that

$$0 < \tilde{G}_\varepsilon(z_\varepsilon, \cdot) \le \frac{1}{2\pi} \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - \cdot|}$$

in $B_0(R\nu_{i,\varepsilon})\backslash\{z_\varepsilon\}$ for all $\varepsilon \ll 1$. Thus, the Green's reprentation formula gives that

$$\begin{aligned} 0 &\le (u_{i,\varepsilon} - \tilde{w}_\varepsilon - \zeta_\varepsilon)(z_\varepsilon) \\ &\le \frac{\lambda_\varepsilon p_\varepsilon}{2\pi} \int_{B_0(R\nu_{i,\varepsilon})} \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - y|} \left(e^{2\varphi_{i,\varepsilon}} u_{i,\varepsilon}^{p_\varepsilon - 1} e^{u_{i,\varepsilon}^{p_\varepsilon}}\right)(y)\, dy \end{aligned} \tag{4.96}$$

for all $\varepsilon$, using (4.19). Using Step 4.1 as above to use Proposition 2.1 and (4.29), we have that for all $l \in \{1, ..., N\}$

$$|\nabla u_{l,\varepsilon}| = O\left(\frac{1}{r_{l,\varepsilon}^{(\eta_0)} \gamma_{l,\varepsilon}^{p_\varepsilon - 1}}\right) \text{ uniformly in } B_0\left(3r_{l,\varepsilon}^{(\eta_0)}\right) \backslash B_0\left(\frac{r_{l,\varepsilon}^{(\eta_0)}}{3}\right)$$

so that, for all $j \in I$, we get as a byproduct of (4.77) and (4.81) with $\tau < 1$ that

$$|\bar{u}_{j,\varepsilon}(\tau\nu_{i,\varepsilon}) - u_{j,\varepsilon}| = O\left(\bar{u}_{j,\varepsilon}(\tau\nu_{i,\varepsilon})^{1-p_\varepsilon}\right) + O\left(\sum_{l \in I_{j,\varepsilon}(\tau\nu_{i,\varepsilon})} \frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{j,\varepsilon}(x_{l,\varepsilon})|}\right)$$

uniformly in $B_0(\tau\nu_{i,\varepsilon})\backslash \cup_{l \in I_{j,\varepsilon}(\tau\nu_{i,\varepsilon})} B_{\phi_{j,\varepsilon}(x_{l,\varepsilon})}\left(2r_{l,\varepsilon}^{(\eta_0)}/5\right)$, and then we eventually obtain with (4.88) and our definition of $\tau$ that

$$|\Gamma_\varepsilon - u_{i,\varepsilon}| = O\left(\Gamma_\varepsilon^{1-p_\varepsilon}\right) + O\left(\sum_{l \in I_{j,\varepsilon}(\tau\nu_{i,\varepsilon})} \frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|}\right) \tag{4.97}$$

uniformly in $D_{j,\varepsilon} := B_{\phi_{i,\varepsilon}(x_{j,\varepsilon})}(\tau\nu_{i,\varepsilon}/2)\backslash \cup_{l \in I_{j,\varepsilon}(\tau\nu_{i,\varepsilon})} B_{\phi_{i,\varepsilon}(x_{l,\varepsilon})}\left(r_{l,\varepsilon}^{(\eta_0)}/2\right)$ for all $\varepsilon$, still using (4.84). Independently, using that $|z_\varepsilon - \phi_{i,\varepsilon}(x_{l,\varepsilon})| \ge r_{l,\varepsilon}^{(\eta_0)}$, we have

$$\ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - \cdot|} = \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - \phi_{i,\varepsilon}(x_{l,\varepsilon})|} + O\left(\frac{r_{l,\varepsilon}^{(\eta_0)}}{r_{l,\varepsilon}^{(\eta_0)} + |z_\varepsilon - \phi_{i,\varepsilon}(x_{l,\varepsilon})|}\right) \tag{4.98}$$

uniformly in $B_{\phi_{i,\varepsilon}(x_{l,\varepsilon})}\left(r_{l,\varepsilon}^{(\eta_0)}/2\right)$ and for all $\varepsilon \ll 1$. By (4.75) for some given $\eta' \in (\eta_0, 1)$ and since $u_{l,\varepsilon}(0) = v_{l,\varepsilon}(0)$, we get $u_{l,\varepsilon} - v_{l,\varepsilon} = o(\gamma_{l,\varepsilon}^{1-p_\varepsilon})$, so we eventually get for all given $\tilde{\eta} \in (\eta_0, 1)$ that

$$\lambda_\varepsilon p_\varepsilon u_{l,\varepsilon}^{p_\varepsilon - 1} e^{u_{l,\varepsilon}^{p_\varepsilon}} = \frac{8e^{-2t_{l,\varepsilon}}(1 + o(e^{\tilde{\eta} t_{l,\varepsilon}}))}{\mu_{l,\varepsilon}^2 \gamma_{l,\varepsilon}^{p_\varepsilon - 1} p_\varepsilon} \tag{4.99}$$

uniformly in $B_0\left(r_{l,\varepsilon}^{(\eta_0)}\right)$ and for all $\varepsilon \ll 1$, still applying Proposition 2.1. Then, using also (4.12) and (4.84), we get from (4.98) and (4.99) that

$$\begin{aligned} &\frac{\lambda_\varepsilon p_\varepsilon}{2\pi} \int_{B_{\phi_{i,\varepsilon}(x_{l,\varepsilon})}\left(r_{l,\varepsilon}^{(\eta_0)}/2\right)} \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - y|} \left(e^{2\varphi_{i,\varepsilon}} u_{i,\varepsilon}^{p_\varepsilon - 1} e^{u_{i,\varepsilon}^{p_\varepsilon}}\right)(y)\, dy \\ &= \frac{(4 + o(1))}{p_\varepsilon \gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{\nu_{i,\varepsilon}}{|z_\varepsilon - \phi_{i,\varepsilon}(x_{l,\varepsilon})|} + O\left(\frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}}\right), \end{aligned} \tag{4.100}$$

as $\varepsilon \to 0$ and for all $l \in I$. Using the basic inequalities

$$|(1+t)^p - 1| \le C\left(|t| + |t|^p\right)$$

for all $t > -1$, and

$$\left(\sum_{t=1}^{N} a_t\right)^p \le C \sum_{t=1}^{N} a_t^p$$

for all $a_t \ge 0$ and all $p \in [1,2]$, we get first from (4.97) that

$$\begin{aligned} u_{i,\varepsilon}^{p_\varepsilon} &= \Gamma_\varepsilon^{p_\varepsilon} + O(1) + O\left(\sum_{l \in I_{j,\varepsilon}(\tau\nu_{i,\varepsilon})} \left(\frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|}\right)^{p_\varepsilon}\right) \\ &\quad + O\left(\sum_{l \in I_{j,\varepsilon}(\tau\nu_{i,\varepsilon})} \left(\frac{\Gamma_\varepsilon}{\gamma_{l,\varepsilon}}\right)^{p_\varepsilon - 1} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|}\right) \end{aligned} \tag{4.101}$$

uniformly in $D_{j,\varepsilon}$ and for all $\varepsilon$. Independently, we get from (4.7), (4.13), (4.24) and (4.74) that

$$\begin{aligned} \ln \frac{1}{\left(r_{l,\varepsilon}^{(\eta_0)}\right)^2} &= -t_{l,\varepsilon}(r_{l,\varepsilon}^{(\eta_0)}) + o(1) + \ln \frac{1}{\mu_{l,\varepsilon}^2}\,, \\ &\le \left(-\frac{p_0\eta_0}{2} + 1 + o(1)\right)\gamma_{l,\varepsilon}^{p_\varepsilon}\,, \end{aligned} \tag{4.102}$$

as $\varepsilon \to 0$ and for all $l$. Recall that we are now assuming that $p_0 = 2$ in (4.66). Then, we may get from (4.83), (4.92) and (4.102) that

$$\begin{aligned} &\left(\frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|^2}\right)^{p_\varepsilon} \\ &= \left(\frac{1}{\gamma_{l,\varepsilon}^{p_\varepsilon}} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|^2}\right)^{p_\varepsilon - 1} \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|^2}\,, \\ &\le C(1 - \eta_0 + o(1)) \ln \frac{4\tau\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|^2} \end{aligned} \tag{4.103}$$

uniformly in $D_{j,\varepsilon}$ as $\varepsilon \to 0$ and for all $l \in I$. Choose now $j_1, ..., j_{|\mathcal{S}|}$ in $I$ such that $\{\tilde{x}_{j_1}, ..., \tilde{x}_{j_{|\mathcal{S}|}}\} = \mathcal{S}$. We compute and then get from (4.101)-(4.103) and from (4.90) that

$$\begin{aligned} &\frac{\lambda_\varepsilon p_\varepsilon}{2\pi} \int_{D_{j_t,\varepsilon}} \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - y|} \left(e^{2\varphi_{i,\varepsilon}} u_{i,\varepsilon}^{p_\varepsilon - 1} e^{u_{i,\varepsilon}^{p_\varepsilon}}\right)(y)\, dy \\ &= O\Bigg(\lambda_\varepsilon \Gamma_\varepsilon^{p_\varepsilon - 1} \exp\left(\Gamma_\varepsilon^{p_\varepsilon}\right) \times \\ &\quad \sum_{l \in I_{j_t,\varepsilon}(\tau\nu_{i,\varepsilon})} \int_{B_{\phi_{i,\varepsilon}(x_{j_t,\varepsilon})}\left(\frac{\tau\nu_{i,\varepsilon}}{2}\right)} \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - y|} \left(\frac{4\tau\nu_{i,\varepsilon}}{|y - \phi_{i,\varepsilon}(x_{l,\varepsilon})|^2}\right)^{1 - \frac{\eta_0}{2}} dy\Bigg)\,, \\ &= O\left(\lambda_\varepsilon \Gamma_\varepsilon^{p_\varepsilon - 1} \exp\left(\Gamma_\varepsilon^{p_\varepsilon}\right) \nu_{i,\varepsilon}^2\right)\,, \end{aligned} \tag{4.104}$$

for all $t \in \{1, ..., |\mathcal{S}|\}$ and all $\varepsilon \ll 1$, using that $\eta_0 > 0$ to get the last estimate. At last, it readily follows from (4.88) that

$$\begin{aligned} &\frac{\lambda_\varepsilon p_\varepsilon}{2\pi} \int_{D_{0,\varepsilon}} \ln \frac{2R\nu_{i,\varepsilon}}{|z_\varepsilon - y|} \left(e^{2\varphi_{i,\varepsilon}} u_{i,\varepsilon}^{p_\varepsilon - 1} e^{u_{i,\varepsilon}^{p_\varepsilon}}\right)(y)\, dy \\ &\qquad = O\left(\lambda_\varepsilon \Gamma_\varepsilon^{p_\varepsilon - 1} \exp\left(\Gamma_\varepsilon^{p_\varepsilon}\right) \nu_{i,\varepsilon}^2\right) \end{aligned} \tag{4.105}$$

for all $\varepsilon \ll 1$, where

$$D_{0,\varepsilon} = B_0(R\nu_{i,\varepsilon}) \backslash \cup_{t=1}^{|\mathcal{S}|} B_{\phi_{i,\varepsilon}(x_{j_t,\varepsilon})} \left(\frac{\tau \nu_{i,\varepsilon}}{2}\right) .$$

Summarizing, by plugging (4.93), (4.94), (4.100), (4.104) and (4.105) in (4.96), we get that

$$\begin{aligned} & |u_{i,\varepsilon}(z_\varepsilon) - \bar{u}_{i,\varepsilon}(R\nu_{i,\varepsilon})| \\ & \le 2\pi C_2 \Gamma_\varepsilon^{1-p_\varepsilon} + \sum_{l \in I} \frac{2+o(1)}{p_\varepsilon \gamma_{l,\varepsilon}^{p_\varepsilon - 1}} \left(2 \ln \frac{4\tau\nu_{i,\varepsilon}}{|z_\varepsilon - \phi_{i,\varepsilon}(x_{l,\varepsilon})|} + O(1)\right) \\ & \quad + O\left(\lambda_\varepsilon \Gamma_\varepsilon^{p_\varepsilon - 1} \exp\left(\Gamma_\varepsilon^{p_\varepsilon}\right) \nu_{i,\varepsilon}^2\right) \end{aligned} \tag{4.106}$$

for all $\varepsilon$, given $(z_\varepsilon)_\varepsilon$ as in (4.95). By the estimate $\nu_{i,\varepsilon}^2 \Gamma_\varepsilon^p = O(1)$ just above (4.93) for $p > 4/3$, we get that $\nu_{i,\varepsilon}^{3/2} = o(\Gamma_\varepsilon^{1-p_\varepsilon})$. Then, evaluating (4.31) at $\tau\nu_{i,\varepsilon}$ and by (4.87), we get that

$$\Gamma_\varepsilon \le \frac{2}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon - 1}} \left(\ln \frac{1}{\lambda_\varepsilon \gamma_{i,\varepsilon}^{2(p_\varepsilon-1)} \nu_{i,\varepsilon}^2} + O(1)\right) + o\left(\Gamma_\varepsilon^{1-p_\varepsilon}\right) , \tag{4.107}$$

then with (4.90) that

$$\exp\left(\Gamma_\varepsilon^{p_\varepsilon}\right) \le \exp\left(\frac{2\Gamma_\varepsilon^{p_\varepsilon-1}}{p_\varepsilon \gamma_{i,\varepsilon}^{p_\varepsilon-1}} \ln \frac{1}{\lambda_\varepsilon \gamma_{i,\varepsilon}^{2(p_\varepsilon-1)} \nu_{i,\varepsilon}^2} + o(1)\right) ,$$

that

$$\lambda_\varepsilon \gamma_{i,\varepsilon}^{2(p_\varepsilon-1)} \nu_{i,\varepsilon}^2 \le \exp\left(-\frac{p_\varepsilon}{2} \Gamma_\varepsilon (1+o(1)) \gamma_{i,\varepsilon}^{p_\varepsilon-1}\right)$$

and eventually that

$$\lambda_\varepsilon \Gamma_\varepsilon^{p_\varepsilon-1} \nu_{i,\varepsilon}^2 \exp\left(\Gamma_\varepsilon^{p_\varepsilon}\right) = o(\Gamma_\varepsilon^{1-p_\varepsilon}) \tag{4.108}$$

for all $\varepsilon \ll 1$. By (4.76) and (4.106) with (4.108), we get that

$$|u_{i,\varepsilon} - \bar{u}_{i,\varepsilon}(R\nu_{i,\varepsilon})| \le (2\pi C_2 + o(1)) \Gamma_\varepsilon^{1-p_\varepsilon} + O\left(\sum_{l \in I} \frac{1}{\gamma_{l,\varepsilon}} \ln \frac{3R\nu_{i,\varepsilon}}{|\cdot - \phi_{i,\varepsilon}(x_{l,\varepsilon})|}\right)$$

uniformly in $B_0(R\nu_{i,\varepsilon}) \backslash \{\phi_{i,\varepsilon}(x_{j_1,\varepsilon}), ..., \phi_{i,\varepsilon}(x_{j_{|\mathcal{S}|},\varepsilon})\}$. In particular, using (4.90) again, we get

$$|\bar{u}_{i,\varepsilon}(\nu_{i,\varepsilon}) - \bar{u}_{i,\varepsilon}(R\nu_{i,\varepsilon})| \le (2\pi C_2 + o(1)) \Gamma_\varepsilon^{1-p_\varepsilon} \tag{4.109}$$

as $\varepsilon \to 0$. Then, $p_0 = 2$, (4.90), (4.106), (4.108) and (4.109) give that

$$\begin{aligned} & |u_{i,\varepsilon} - \bar{u}_{i,\varepsilon}(\nu_{i,\varepsilon})| \\ & \le \frac{9}{2} \pi C_2 \Gamma_\varepsilon^{1-p_\varepsilon} + \sum_{l \in I_{i,\varepsilon}(\nu_{i,\varepsilon})} \frac{2+o(1)}{\gamma_{l,\varepsilon}^{p_\varepsilon-1}} \ln \frac{4\tau\nu_{i,\varepsilon}}{|z_\varepsilon - \phi_{i,\varepsilon}(x_{l,\varepsilon})|} \end{aligned} \tag{4.110}$$

uniformly in $B_0(\nu_{i,\varepsilon}) \backslash \cup_{l \in I_{i,\varepsilon}(\nu_{i,\varepsilon})} B_{\phi_{i,\varepsilon}}(x_{l,\varepsilon}) \left(r_{l,\varepsilon}^{(\eta_0)}\right)$ and for all $\varepsilon$. But by (4.78) for $l = i$, our assumption (4.80) and by (4.18), the inequality in (4.77) for $j = i$ and $r = \nu_{i,\varepsilon}$ should be an equality somewhere on $\partial B_0(\nu_{i,\varepsilon})$ of this set for all $\varepsilon \ll 1$, which gives a contradiction to (4.110) and concludes the proof of (4.79).

Then, picking now a sequence $(\tilde{\Gamma}_\varepsilon)_\varepsilon$ such that $\lim_{\varepsilon \to 0} \tilde{\Gamma}_\varepsilon = +\infty$ and $\tilde{\Gamma}_\varepsilon = o(\gamma_{j,\varepsilon})$, and setting

$$\tilde{\nu}_{j,\varepsilon} = \inf\left\{r > 0 \text{ s.t. } \bar{u}_{j,\varepsilon} \ge \tilde{\Gamma}_\varepsilon \text{ in } [0, r]\right\} ,$$

we get from (4.79) that

$$\tilde{\nu}_{j,\varepsilon} \le \nu_{j,\varepsilon} \tag{4.111}$$

for all $j \in \{1, ..., N\}$ and all $\varepsilon \ll 1$. By (4.15), $\tilde{\nu}_{j,\varepsilon} = o(1)$. As in (4.88), we get from (4.17) and (4.18) that we can fix $0 < R < 1$ such that $u_\varepsilon = \tilde{\Gamma}_\varepsilon(1 + o(1))$ uniformly in $\partial\phi_{j,\varepsilon}^{-1}(B_0(R\tilde{\nu}_{j,\varepsilon}))$ for all $\varepsilon \ll 1$ and all $j$. Arguing now as below (4.73), we get from (4.16) that

$$\sup_{\Sigma\backslash \cup_j \phi_{j,\varepsilon}^{-1}(B_0(R\tilde{\nu}_{j,\varepsilon}))} u_\varepsilon \le 2\tilde{\Gamma}_\varepsilon \tag{4.112}$$

for all $\varepsilon \ll 1$. Then choose and fix? $(\tilde{\Gamma}_\varepsilon)_\varepsilon$ growing slowly to $+\infty$ and more precisely such that

$$\begin{aligned} &\lambda_\varepsilon \tilde{\Gamma}_\varepsilon^{p_\varepsilon} \exp\left((2\tilde{\Gamma}_\varepsilon)^{p_\varepsilon}\right) = o\left(\gamma_{j,\varepsilon}^{2-p_\varepsilon}\right) \text{ and} \\ &\qquad (2-p_\varepsilon)\ln\left(1 + \lambda_\varepsilon \gamma_{j,\varepsilon}^{2(p_\varepsilon-1)} \exp((2\tilde{\Gamma}_\varepsilon)^{p_\varepsilon})\right) = o(1) \end{aligned} \tag{4.113}$$

for all $j$ as $\varepsilon \to 0$. The first condition is clearly possible by (4.67). The second one is also possible since $\lambda_\varepsilon \gamma_{j,\varepsilon}^{2(p_\varepsilon-1)} = O(1)$ by (4.68) and since now $p_0 = 2$ in (4.66). We may now compute and use either (4.112) in $\Sigma\backslash \cup_j \phi_{j,\varepsilon}^{-1}(B_0(R\tilde{\nu}_{j,\varepsilon}))$, or the controls given by the inequality in (4.77) for $r = \tilde{\nu}_{j,\varepsilon}$ thanks to (4.111), allowing to estimate the nonlinearity as in (4.101)-(4.104). This leads to the following integral estimates:

$$\begin{aligned} &\frac{\lambda_\varepsilon p_\varepsilon^2}{2} \int_{\Sigma\backslash \cup_j \phi_{j,\varepsilon}^{-1}\left(B_0\left(r_{j,\varepsilon}^{(\eta_0)}\right)\right)} u_\varepsilon^{p_\varepsilon} e^{u_\varepsilon^{p_\varepsilon}}\,dx = o\left(\gamma_{j,\varepsilon}^{2-p_\varepsilon}\right), \\ &\frac{\lambda_\varepsilon p_\varepsilon^2}{2} \int_{\Sigma\backslash \cup_j \phi_{j,\varepsilon}^{-1}\left(B_0\left(r_{j,\varepsilon}^{(\eta_0)}\right)\right)} \left(e^{u_\varepsilon^{p_\varepsilon}} - 1\right) dx = O\left(\lambda_\varepsilon \exp\left((2\tilde{\Gamma}_\varepsilon)^{p_\varepsilon}\right)\right), \end{aligned} \tag{4.114}$$

while, computing as in (4.100), we get that

$$\begin{aligned} &\frac{\lambda_\varepsilon p_\varepsilon^2}{2} \int_{\phi_{j,\varepsilon}^{-1}\left(B_0\left(r_{j,\varepsilon}^{(\eta_0)}\right)\right)} u_\varepsilon^{p_\varepsilon} e^{u_\varepsilon^{p_\varepsilon}}\,dx = (4\pi + o(1))\,\gamma_{j,\varepsilon}^{2-p_\varepsilon}, \\ &\frac{\lambda_\varepsilon p_\varepsilon^2}{2} \int_{\phi_{j,\varepsilon}^{-1}\left(B_0\left(r_{j,\varepsilon}^{(\eta_0)}\right)\right)} \left(e^{u_\varepsilon^{p_\varepsilon}} - 1\right) dx = \frac{4\pi + o(1)}{\gamma_{j,\varepsilon}^{2(p_\varepsilon-1)}} \end{aligned} \tag{4.115}$$

as $\varepsilon \to 0$. Thus, by plugging (4.114)-(4.115) in (4.2) and by using our conditions (4.113) on $(\tilde{\Gamma}_\varepsilon)_\varepsilon$, we get that

$$\begin{aligned} \beta_\varepsilon &= \left(\sum_{j=1}^N \frac{4\pi + o(1)}{\gamma_{j,\varepsilon}^{2(p_\varepsilon-1)}}\right)^{\frac{2-p_\varepsilon}{p_\varepsilon}} \left((4\pi + o(1))\sum_{j=1}^N \gamma_{j,\varepsilon}^{2-p_\varepsilon}\right)^{\frac{2(p_\varepsilon-1)}{p_\varepsilon}}, \\ &= 4\pi(1+o(1)) \underbrace{\left(1 + \sum_{j\neq j_0}\left(\frac{\gamma_{j,\varepsilon}}{\gamma_{j_0,\varepsilon}}\right)^{2-p_\varepsilon}\right)^{\frac{2(p_\varepsilon-1)}{p_\varepsilon}}}_{(\star)}, \end{aligned}$$

using that

$$\left(1 + \sum_{j\neq j_0}\left(\frac{\gamma_{j_0,\varepsilon}}{\gamma_{j,\varepsilon}}\right)^{2(p_\varepsilon-1)}\right)^{\frac{2-p_\varepsilon}{p_\varepsilon}} = 1 + o(1)$$

since $p_\varepsilon \to 2$, where we choose $j_0 \in \{1, ..., N\}$ such that $\gamma_{j_0,\varepsilon} = \min_{j\in\{1,...,N\}} \gamma_{j,\varepsilon}$ for all $\varepsilon$, up to a subsequence. Then, in order to conclude the proof of (4.5) for $k = N$, it is then sufficient to get that the term $(\star)$ converges to $N$, namely that

$$\forall j \in \{1, ..., N\}\,, \quad \lim_{\varepsilon\to 0}(2 - p_\varepsilon) \ln \frac{\gamma_{j,\varepsilon}}{\gamma_{j_0,\varepsilon}} = 0\,.$$

To get this, we use (4.67), (4.68) and argue as below (4.70) for $\eta = 1/2$ to write

$$(2 - p_\varepsilon)\, \gamma_{j,\varepsilon}^{p_\varepsilon} \le (1 + o(1)) \ln \frac{1}{\lambda_\varepsilon^2} \le \frac{1 + o(1)}{2} \gamma_{j,\varepsilon}^{p_\varepsilon}$$

for all $j$, so that $1 \le (\gamma_{j,\varepsilon}/\gamma_{j_0,\varepsilon})^{p_\varepsilon} = O\left(1/(2 - p_\varepsilon)\right) \le +\infty$. Theorem 4.1 is proven. □

## 5. Compactness at the critical levels $\beta \in 4\pi\mathbb{N}^\star$ for $p \in (1, 2]$

Our main goal in this section is to prove the following result:

**Theorem 5.1.** *Let $(\lambda_\varepsilon)_\varepsilon$ be any sequence of positive real numbers. Let $p \in (1, 2]$ be given and set $p_\varepsilon = p$ for all $\varepsilon$. Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions solving (4.1). Let $(\beta_\varepsilon)_\varepsilon$ be given by (4.2). Assume that (4.4) holds true, so that (4.3) holds true for some $\beta \in 4\pi\mathbb{N}^\star$ (see Theorem 4.1). Then we have that*

$$\beta_\varepsilon > \beta \tag{5.1}$$

*for all $\varepsilon \ll 1$.*

More precisely, if $(\gamma_{1,\varepsilon})_\varepsilon\,, ...\,, (\gamma_{k,\varepsilon})_\varepsilon$ are the sequences of positive real numbers diverging to $+\infty$ given by Proposition 4.1, we show in the proof below that

$$\beta_\varepsilon \ge 4\pi \left( k + \frac{4(p-1)(1 + o(1))}{p^2} \sum_{i=1}^{k} \gamma_{i,\varepsilon}^{-2p} \right) \tag{5.2}$$

as $\varepsilon \to 0$. As a remark, according to the proof of Theorem 4.1, $N$ in Proposition 4.1 equals $k$ in (4.5). Interestlingly enough, the cancellation of terms of order $\gamma_{i,\varepsilon}^{-p}$ still occurs here on a surface for all $p \in (1, 2]$ and for arbitrary energies, as pointed out in [32] concerning the unit disk for $p = 2$ and in the minimal energy case $\beta = 4\pi$.

### 5.1. Further estimates in the radially symmetric case.

Let $p \in (1, 2]$ be given, let $(\mu_\gamma)_\gamma$ be a family of positive real numbers, and let $(\lambda_\gamma)_\gamma$ be such that (2.1) holds true, where $p_\gamma = p$ for all $\gamma$, let $t_\gamma, \bar{t}_\gamma$ be given by (2.2) and let $(B_\gamma)_\gamma$ be given by (2.6). Let also $(\bar{r}_\gamma)_\gamma$ be a family of positive real numbers such that (2.3) holds true, and such that

$$t_\gamma(\bar{r}_\gamma) \le \sqrt{\gamma}\,, \tag{5.3}$$

$$\gamma^{4p}\bar{r}_\gamma^2 = O(1) \tag{5.4}$$

for all $\gamma \gg 1$. In this section we aim to get more precise estimates on the $B_\gamma$'s than in Section 2, but at smaller scales around 0, in order to be technically as simple as possible: namely, (5.3)-(5.4) imply (2.4)-(2.5). We also restrict here to the specific case where $p$ is fixed. As already mentioned in the introduction, some issues may arise when studying compactness at the critical levels $\beta \in 4\pi\mathbb{N}^\star$ in the case $p = 1$. Following [32, 33] and still abusing the radial notation $r = |x|$, we let $w_0$ be given by

$$w_0(r) = -T_0(r) + \frac{2r^2}{1 + r^2} - \frac{1}{2}T_0(r)^2 + \frac{1 - r^2}{1 + r^2} \int_1^{1+r^2} \frac{\ln t}{1 - t} dt$$

for $T_0$ as in (2.13), so that, by ODE theory, $w_0$ is the unique solution of

$$\begin{cases} \Delta w_0 = 4e^{-2T_0}\left(2w_0 + T_0^2 - T_0\right) \text{ in } \mathbb{R}^2\,, \\ w_0(0) = 0\,, \\ w_0 \text{ is radially symmetric around } 0 \in \mathbb{R}^2\,. \end{cases} \tag{5.5}$$

We further set

$$\begin{aligned} F = \; & 2(p-1)w_0 + (p-2)T_0^2 - 8(p-1)T_0 w_0 - \frac{8p-10}{3}T_0^3 \\ & + 4\,(p-1)\,w_0^2 + 4(p-1)T_0^2 w_0 + (p-1)T_0^4\,, \end{aligned} \tag{5.6}$$

and let $w_1$ be the unique solution of

$$\begin{cases} \Delta w_1 = 4e^{-2T_0}\left(2w_1 + \frac{4(p-1)}{p^3}F\right) \text{ in } \mathbb{R}^2\,, \\ w_1(0) = 0\,, \\ w_1 \text{ is radially symmetric around } 0 \in \mathbb{R}^2\,. \end{cases} \tag{5.7}$$

Resuming the strategy and the explicit computations in [33, Section 3], even if we do not have $w_1$ in closed form, we know that

$$\begin{aligned} \int_{\mathbb{R}^2} \Delta w_1 dx = \frac{16(p-1)}{p^3}\Bigg[&(p-1)\left(\frac{\pi^3}{3} + \frac{33\pi}{2}\right) \\ & + \frac{3\pi}{2}\,(p-2) - \frac{7(4p-5)\pi}{2}\Bigg]\,. \end{aligned} \tag{5.8}$$

We also have that

$$\begin{aligned} w_0(r) &= -T_0(r) + O(1)\,, \\ w_1(r) &= -\frac{T_0(r)}{4\pi}\int_{\mathbb{R}^2} \Delta w_1 dx + O(1) \end{aligned} \tag{5.9}$$

as $r \to +\infty$. Note that the convention on the sign of the Laplace operator here is not the same as that in [33]. In complement of the computations already done in [33], we compute also

$$\int_{\mathbb{R}^2} \frac{|x|^2 - 1}{(1+|x|^2)^3} T_0(x)^2 dx = \frac{3\pi}{2}$$

to get (5.8). Let $w_{0,\gamma}$, $w_{1,\gamma}$ be given by $w_{0,\gamma} = w_0(\cdot/\mu_\gamma)$ and $w_{1,\gamma} = w_1(\cdot/\mu_\gamma)$, and let $w_\gamma$ be given by

$$B_\gamma = \gamma - \frac{2t_\gamma}{p\gamma^{p-1}} + \frac{4(p-1)w_{0,\gamma}}{p^2\gamma^{2p-1}} + \frac{w_{1,\gamma} + w_\gamma}{\gamma^{3p-1}}\,. \tag{5.10}$$

Proposition 2.1 already gives $B_\gamma \le \gamma$ and some estimates on $w_\gamma$ given by (5.10) in $[0, \bar{r}_\gamma]$ for all $\gamma \gg 1$. Much more precisely here, we get that $w_\gamma$ is actually a small remainder term in the following sense:

**Proposition 5.1.** *We have*

$$w_\gamma = O(\gamma^{-p} t_\gamma)\,, \quad w_\gamma' = O(\gamma^{-p} t_\gamma')\,,$$

*and*

$$\lambda_\gamma p B_\gamma^{p-1} e^{B_\gamma^p} = -\frac{2}{p\gamma^{p-1}}\Delta t_\gamma\left(1 + O\left(\frac{e^{t_\gamma/2}}{\gamma^{3p}}\right)\right) + \frac{4(p-1)}{p^2\gamma^{2p-1}}\Delta w_{0,\gamma} + \frac{\Delta w_{1,\gamma}}{\gamma^{3p-1}}\,,$$

*uniformly in* $[0, \bar{r}_\gamma]$ *and for all* $\gamma \gg 1$ *large, where* $w_\gamma$ *is as in* (5.10).

The proof of Proposition 5.1 follows the strategy of the proof of Proposition 2.1, but the stronger assumption (5.3) basically reduces now the computations to Taylor expansions.

*Proof of Proposition 5.1.* Let $r_\gamma$ be given by

$$r_\gamma = \sup \{r \in [0, \bar{r}_\gamma] \text{ s.t. } |w_\gamma| \le t_\gamma\}$$

for all $\gamma$. Taking advantage of the control on $w_\gamma$ in $[0, r_\gamma]$ given by this definition, we may perform the following computations uniformly in $[0, r_\gamma]$ as $\gamma \to +\infty$. We first get

$$\begin{aligned} B_\gamma^p = \; & \gamma^p - 2t_\gamma + \frac{2(p-1)}{p\gamma^p}\left(2w_{0,\gamma} + t_\gamma^2\right) + \frac{p(w_{1,\gamma} + w_\gamma)}{\gamma^{2p}} \\ & - \frac{8(p-1)^2 t_\gamma w_{0,\gamma}}{p^2\gamma^{2p}} - \frac{8(p-1)(p-2)t_\gamma^3}{6p^2\gamma^{2p}} + O\left(\frac{\bar{t}_\gamma^4}{\gamma^{3p}}\right). \end{aligned}$$

and

$$B_\gamma^{p-1} = \gamma^{p-1}\left(1 - \frac{2(p-1)t_\gamma}{p\gamma^p} + \frac{4(p-1)^2 w_{0,\gamma}}{p^2\gamma^{2p}} + \frac{2(p-1)(p-2)t_\gamma^2}{p^2\gamma^{2p}} + O\left(\frac{\bar{t}_\gamma^3}{\gamma^{3p}}\right)\right).$$

We use for this (5.3) and (5.9) and the expansions of $(1+\varepsilon)^q$ as $\varepsilon \to 0$. Then, using (2.1), we similarly compute and get

$$\begin{aligned} & \lambda_\gamma p e^{B_\gamma^p} \\ & = \frac{8e^{-2t_\gamma}}{p\gamma^{2(p-1)}\mu_\gamma^2}\Bigg[1 + \frac{2(p-1)}{p\gamma^p}\left(2w_{0,\gamma} + t_\gamma^2\right) + \\ & \frac{1}{p^2\gamma^{2p}}\Bigg(p^3(w_{1,\gamma} + w_\gamma) - 8(p-1)^2 t_\gamma w_{0,\gamma} - \frac{4}{3}(p-1)(p-2)t_\gamma^3 \\ & + 8(p-1)^2 t_\gamma^2 w_{0,\gamma} + 2(p-1)^2 t_\gamma^4 + 8(p-1)^2 w_{0,\gamma}^2\Bigg) + O\Bigg(\frac{1}{\gamma^{3p}} e^{C\bar{t}_\gamma \times \left(\frac{\bar{t}_\gamma^3}{\gamma^{3p}}\right)}\Bigg)\Bigg], \end{aligned}$$

so that we eventually have

$$\begin{aligned} & \lambda_\gamma p B_\gamma^{p-1} e^{B_\gamma^p} \\ & = \frac{8e^{-2t_\gamma}}{p\gamma^{p-1}\mu_\gamma^2}\left[1 + \frac{2(p-1)}{p\gamma^p}\left(2w_{0,\gamma} + t_\gamma^2 - t_\gamma\right) + O\left(\frac{e^{t_\gamma/2}}{\gamma^{3p}}\right)\right] + \\ & \qquad \frac{4e^{-2t_\gamma}}{\gamma^{3p-1}\mu_\gamma^2}\left[2(w_{1,\gamma} + w_\gamma) + \frac{4(p-1)}{p^3} F\left(\frac{\cdot}{\mu_\gamma}\right)\right], \end{aligned} \tag{5.11}$$

for $F$ as in (5.6), using again (5.3) to write $\bar{t}_\gamma^3/\gamma^{3p} = o(1)$. Then, setting $\tilde{w}_\gamma = w_\gamma(\cdot/\mu_\gamma)$, using now not only (2.13), but also (5.5) and (5.7), we get from (2.6) that

$$\Delta\tilde{w}_\gamma = 8e^{-2T_0}\tilde{w}_\gamma + O\left(\mu_\gamma^2\gamma^{3p}\right) + O\left(\frac{e^{-3T_0/2}}{\gamma^p}\right), \tag{5.12}$$

uniformly in $[0, r_\gamma/\mu_\gamma]$ as $\gamma \to +\infty$, applying $\Delta$ to (5.10). The second-last term in (2.14) is obtained when controlling $B_\gamma$ in the LHS of (2.6), since our definition of $r_\gamma$ implies $B_\gamma \le \gamma$ in $[0, r_\gamma]$ for all $\gamma \gg 1$. Then, (2.15) may be obtained from (2.14) by using also (5.4). At that stage, we may conclude the proof of Proposition 5.1 by following closely the lines below (2.15) and showing mainly that (2.10) holds true for all $\gamma \gg 1$. □

As a direct corollary of Proposition 5.1, we get the following estimates:

**Corollary 5.1.** *Assume that* (5.3) *is an equality, namely that*

$$t_\gamma(\bar r_\gamma) = \sqrt{\gamma} \tag{5.13}$$

*for all* $\gamma \gg 1$*, then we have that*

$$\begin{aligned}
&\frac{\lambda_\gamma p^2}{2}\int_{B_0(\bar r_\gamma)} B_\gamma^p e^{B_\gamma^p} dx \\
&= 4\pi\gamma^{2-p}\Bigg[1+\frac{2(p-2)}{p\gamma^p}+o\left(\frac{1}{\gamma^{2p}}\right) \\
&+\frac{p-1}{p^2\gamma^{2p}}\left(-8-\frac{2\pi^2}{3}+2(p-1)\left(\frac{\pi^2}{3}+\frac{33}{2}\right)+3(p-2)-7\left(4p-5\right)\right)\Bigg],\\
&\frac{\lambda_\gamma p^2}{2}\int_{B_0(\bar r_\gamma)} e^{B_\gamma^p} dx \\
&= \frac{4\pi}{\gamma^{2(p-1)}}\Bigg[1+\frac{4(p-1)}{p\gamma^p}+o\left(\frac{1}{\gamma^{2p}}\right) \\
&\qquad+\frac{1}{\gamma^{2p}}\Bigg(\frac{2(p-1)}{p^2}\left((p-1)\left(\frac{\pi^2}{3}+\frac{33}{2}\right)+\frac{3}{2}(p-2)-\frac{7(4p-5)}{2}\right) \\
&\qquad\qquad+\frac{4(p-1)}{p}+\frac{(p-1)^2}{p^2}\left(8+\frac{2\pi^2}{3}\right)\Bigg)\Bigg],
\end{aligned}$$

*and then that*

$$\begin{aligned}
&\left(\frac{\lambda_\gamma p^2}{2}\int_{B_0(\bar r_\gamma)} e^{B_\gamma^p} dx\right)^{\frac{2-p}{p}}\left(\frac{\lambda_\gamma p^2}{2}\int_{B_0(\bar r_\gamma)} B_\gamma^p e^{B_\gamma^p} dx\right)^{\frac{2(p-1)}{p}} \\
&\qquad = 4\pi\left(1+\frac{4(p-1)}{p^2\gamma^{2p}}+o\left(\frac{1}{\gamma^{2p}}\right)\right)
\end{aligned} \tag{5.14}$$

*as* $\gamma \to +\infty$.

Since the computations to get Corollary 5.1 from Proposition 5.1 basically resume those in [33], we leave them to the reader. In particular, proving the first two estimates in Corollary 5.1 uses (5.8) and the following computations

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Delta w_0 dx = -\int_{\mathbb{R}^2} \Delta T_0 dx = -\int_{\mathbb{R}^2} T_0\Delta T_0 dx = -\frac{1}{2}\int_{\mathbb{R}^2} T_0^2\Delta T_0 dx = 4\pi\,,\\
&\int_{\mathbb{R}^2} \left(w_0(\Delta T_0)+T_0\Delta w_0\right) dx = 8\pi+\frac{2\pi^3}{3}\,.
\end{aligned}$$

Once the first two estimates of Corollary 5.1 are obtained, proving (5.14) is quite elementary: in particular, we observe in (5.14) the aforementioned cancellation of the term $\gamma^{-p}$. Besides, the term $\gamma^{-2p}$ vanishes as well for $p = 1$. That is the technical reason why the approach of this section does not work for $p = 1$ and why we assume $p > 1$ in Theorem 5.1 (see also the paragraph above Remark 0.1).

5.2. **Conclusion of the proof of Theorem 5.1.** Let $(\lambda_\varepsilon)_\varepsilon$ be any sequence of positive real numbers. Let $p \in (1, 2]$ be given and set $p_\varepsilon = p$ for all $\varepsilon$. Let $(u_\varepsilon)_\varepsilon$ be a sequence of smooth functions solving (4.1). Let $(\beta_\varepsilon)_\varepsilon$ be given by (4.2). Assume that (4.4) holds true, so that (4.3) holds true for some $\beta \in 4\pi\mathbb{N}^\star$ by Theorem 4.1. We may also apply Proposition 4.1, getting in particular sequences $(\mu_{i,\varepsilon})_\varepsilon$, $(x_{i,\varepsilon})_\varepsilon$,

$(\gamma_{i,\varepsilon})_\varepsilon$ and $(\varphi_{i,\varepsilon})_\varepsilon$, and we resume the notation $r_{i,\varepsilon}$, $t_{i,\varepsilon}$ and $v_{i,\varepsilon}$ in (4.20)-(4.22); let also $\bar r_{i,\varepsilon}$ be given by

$$t_{i,\varepsilon}(\bar r_{i,\varepsilon}) = \sqrt{\gamma_{i,\varepsilon}} \tag{5.15}$$

for all $i \in \{1,...,k\}$ and all $\varepsilon$. By (4.30) in Step 4.1, we know that $\bar r_{l,\varepsilon}^{(1/2)}$ given by (4.25) equals $r_{l,\varepsilon}^{(1/2)}$ in (4.24) for all $l \in \{1,...,k\}$ and all $\varepsilon \ll 1$. Moreover, since $r_{i,\varepsilon} = O(1)$ according to (4.20), we get that

$$r_{i,\varepsilon}^{(1/2)} = o(r_{i,\varepsilon}) = o(1) \tag{5.16}$$

for all $\varepsilon \ll 1$ and all $i$. By (4.24) and (5.15), we deduce that

$$\ln \frac{\bar r_{i,\varepsilon}^2}{\left(r_{i,\varepsilon}^{(1/2)}\right)^2} = t_{i,\varepsilon}(\bar r_{i,\varepsilon}) - t_{i,\varepsilon}\left(r_{i,\varepsilon}^{(1/2)}\right) + o(1) \le -3\gamma_{i,\varepsilon}$$

for all $i$ and all $\varepsilon \ll 1$. Then, we find from (5.16) that

$$\bar r_{i,\varepsilon} = O\left(e^{-\gamma_{i,\varepsilon}}\right) \tag{5.17}$$

for all $\varepsilon \ll 1$ and all $i$. Proposition 2.1 may be applied as below (4.25). We get that

$$|u_{i,\varepsilon} - v_{i,\varepsilon}| = O\left(\frac{\bar r_{i,\varepsilon}}{r_{i,\varepsilon}^{(1/2)}\gamma_{i,\varepsilon}^{p-1}}\right) = O\left(e^{-\gamma_{i,\varepsilon}}\right)$$

uniformly in $B_0(\bar r_{i,\varepsilon})$ for all $\varepsilon \ll 1$ and all $i$, using also (4.28). Then, using similarly Proposition 2.1 to get that $v_{i,\varepsilon} = \gamma_{i,\varepsilon}(1+o(1))$, we obtain that

$$u_{i,\varepsilon}^p = v_{i,\varepsilon}^p\left(1 + O\left(\frac{e^{-\gamma_{i,\varepsilon}}}{\gamma_{i,\varepsilon}}\right)\right), \tag{5.18}$$

so that we have

$$e^{u_{i,\varepsilon}^p} = e^{v_{i,\varepsilon}^p}\left(1 + o\left(\frac{1}{\gamma_{i,\varepsilon}^{2p}}\right)\right) \tag{5.19}$$

uniformly in $B_0(\bar r_{i,\varepsilon})$, for all $\varepsilon \ll 1$ and all $i$. An easy consequence of (4.12), (4.20) and (5.16) is that the domains $\phi_{i,\varepsilon}^{-1}(B_0(\bar r_{i,\varepsilon}))$ are two by two disjoint for all $\varepsilon \ll 1$. Then we may write that

$$\begin{aligned} \frac{\lambda_\varepsilon p^2}{2}\int_\Sigma u_\varepsilon^p e^{u_\varepsilon^p} dv_g &\ge \sum_{i=1}^k \underbrace{\frac{\lambda_\varepsilon p^2}{2}\int_{B_0(\bar r_{i,\varepsilon})} u_{i,\varepsilon}^p e^{u_{i,\varepsilon}^p} e^{2\varphi_{i,\varepsilon}} dx}_{:=a_{i,\varepsilon}}, \\ \frac{\lambda_\varepsilon p^2}{2}\int_\Sigma \left(e^{u_\varepsilon^p} - 1\right) dv_g &\ge \sum_{i=1}^k \underbrace{\frac{\lambda_\varepsilon p^2}{2}\int_{B_0(\bar r_{i,\varepsilon})} \left(e^{u_{i,\varepsilon}^p} - 1\right) e^{2\varphi_{i,\varepsilon}} dx}_{:=b_{i,\varepsilon}}. \end{aligned} \tag{5.20}$$

Using (4.12), (5.17), (5.18) and (5.19), we write $e^{2\varphi_{i,\varepsilon}} = 1 + O(\bar r_{i,\varepsilon})$ and get

$$\int_{B_0(\bar r_{i,\varepsilon})} u_{i,\varepsilon}^p e^{u_{i,\varepsilon}^p} e^{2\varphi_{i,\varepsilon}} dx = \left(\int_{B_0(\bar r_{i,\varepsilon})} v_{i,\varepsilon}^p e^{v_{i,\varepsilon}^p} dx\right)\left(1 + o\left(\gamma_{i,\varepsilon}^{-2p}\right)\right), \tag{5.21}$$

for all $\varepsilon \ll 1$ and all $i$. Similar arguments give that

$$\int_{B_0(\bar r_{i,\varepsilon})} \left(e^{u_{i,\varepsilon}^p} - 1\right) e^{2\varphi_{i,\varepsilon}} dx = \left(\int_{B_0(\bar r_{i,\varepsilon})} e^{v_{i,\varepsilon}^p} dx\right)\left(1 + o\left(\gamma_{i,\varepsilon}^{-2p}\right)\right), \tag{5.22}$$

for all $\varepsilon \ll 1$ and all $i$. By plugging (5.21)-(5.22) in (5.20) and coming back to the definition (4.2), we obtain

$$\beta_\varepsilon \geq \left(\sum_{i=1}^k b_{i,\varepsilon}\right)^{\frac{2-p}{p}} \left(\sum_{i=1}^k a_{i,\varepsilon}\right)^{\frac{2(p-1)}{p}} \geq \sum_{i=1}^k b_{i,\varepsilon}^{\frac{2-p}{p}} a_{i,\varepsilon}^{\frac{2(p-1)}{p}},$$

by Hölder's inequality for vectors in $\mathbb{R}^k$. In order to compute the RHS, since we have (5.17), so that (see (5.4)) we may apply also Proposition 5.1 to $v_{i,\varepsilon}$ in $B_0(\bar{r}_{i,\varepsilon})$ and thus use (5.14). This proves (5.2) and concludes the proof of Theorem 5.1.

**Remark 5.1.** *The minimization of $I_\beta$ in* (0.9) *for $\beta = 4\pi$ attracted some attention (see for instance* [12, 37]*): in this case we basically have $p = 1$. Then, turn now to the case $p \in (1, 2]$ of this section. First if $p = 2$, we may get by following the strategy in* [33] *that the convergence of $(\beta_\varepsilon)_\varepsilon$ to $4\pi$ from above in* (5.2) *for $k = 1$ gives back the existence of a maximizer for $(MT)$ if $\beta = 4\pi$ (see also* [2, 38, 21]*). Now, if $p \in (1, 2)$, we already pointed out in the introduction that*

$$-\infty < I_{p,\varepsilon} := \inf_{u \in H^1} J_{p,4\pi(1-\varepsilon)}(u)$$

*for all $\varepsilon \in [0, 1)$, where $J_{p,\beta}$ is as in* (0.5)*. Moreover, the existence of a minimizer $u_\varepsilon$ for $I_{p,\varepsilon}$ follows from a standard minimization argument for all given $\varepsilon \in (0, 1)$. Here again, the convergence of $(\beta_\varepsilon)_\varepsilon$ to $4\pi$ from above in* (5.2) *for $k = 1$ gives the existence of a minimizer for $I_{p,0}$, since the present $u_\varepsilon$'s then have to converge strongly in $C^2$ as $\varepsilon \to 0$.*

*We conclude this remark by a curiosity. If $G > 0$ is the Green's function of $\Delta_g + 1$ in $M$, we may write $G(x, y) = \frac{1}{4\pi}\left(\ln \frac{1}{|x-y|^2} + \mathcal{H}(x, y)\right)$ for all $x \neq y$. We know that $\mathcal{H} \in C^0(\Sigma \times \Sigma)$ and we set $M = \max_{x\in\Sigma} \mathcal{H}(x, x)$. As a byproduct of the analysis in the present paper, it can be also checked that*

$$\ln \frac{1}{\lambda_\varepsilon} = \left(1 - \frac{p}{2}\right) \gamma_\varepsilon^p + \ln \frac{p^2 \gamma_\varepsilon^{2(p-1)}}{8} + H_x(x) + (p - 1) + o(1),$$

*as $\varepsilon \to 0$, if the $u_\varepsilon$'s blow-up at some $x \in \Sigma$ for $k = 1$ in* (4.5) *and solve* (4.1)*, with $\lambda_\varepsilon$ given by* (4.2)*, for $\beta_\varepsilon = 4\pi(1 - \varepsilon)$, $p_\varepsilon = p$ and $\gamma_\varepsilon = \max_\Sigma u_\varepsilon$ for all $\varepsilon$. We may also get that*

$$I_{p,0} = \inf_{u\in H^1} J_{p,4\pi}(u) < -\left(\ln \pi + M + (p - 1) + \frac{(2 - p)(p - 1)}{p}\right). \tag{5.23}$$

*The large inequality in* (5.23) *is a byproduct of a by now rather standard test function computations (see for instance* [42, Step 3.1]*). The strict inequality is more subtle and can be seen as a consequence of the convergence of the $\beta_\varepsilon$'s from above, picking the **refined test functions provided by the blow-up analysis**, in the spirit of* [42, Section 4]*. At last, observe that the exponential of the opposite of the RHS of* (5.23) *converges to $\pi \exp(1 + M)$ as $p \to 2$, which turns out to be consistent with the original works* [2, 21].

## Conclusion of the proofs of Theorems 0.2 and 0.1

Let $\beta > 0$ be given. Assume first that $p$ is given in $(1, 2)$. By Theorem 1.1, there exist a sequence $(\beta_\varepsilon)_\varepsilon$ increasing to $\beta^-$ as $\varepsilon \to 0$, and $u_\varepsilon$ such that (4.1) is satisfied for $p_\varepsilon = p$ and $\lambda_\varepsilon$ given by (4.2) for all $\varepsilon$. Now, we claim that the $u_\varepsilon$'s are uniformly bounded: this is a direct consequence of (4.5) in Theorem 4.1

if $\beta \notin 4\pi\mathbb{N}^\star$ and follows from Theorem 5.1 if $\beta \in 4\pi\mathbb{N}^\star$, since the present sequence $(\beta_\varepsilon)_\varepsilon$ is assumed to increase. By elliptic theory in (4.1) and (4.7), we easily then get that, up to a subsequence, the $\lambda_\varepsilon$'s converge to some $\lambda$ and the $u_\varepsilon$'s converge in $C^2$ to some $u$ solving the equation in (0.6) and (0.7). Observe in particular that since $\beta > 0$, (0.7) gives that $u \geq 0$ is not identically zero, so that $u > 0$ in $\Sigma$ by Lemma 1.1. Then $\mathcal{C}_{p,\beta} \ni u$ is not empty in Theorem 0.1. The compactness of $\mathcal{C}_{p,\beta}$ also clearly follows from Theorems 4.1 and 5.1. For $p = 1$, and $\beta \notin 4\pi\mathbb{N}^*$, we take a sequences $(p_\varepsilon)$, $p_\varepsilon \downarrow 1$ and $u_\varepsilon \in \mathcal{C}_{p_\varepsilon,\beta}$. As before, by Theorem 4.1, up to a subsequence $(u_\varepsilon)$ converges to a positive function $u \in \mathcal{C}_{1,\beta}$, and Theorem 0.1 is proven. Assume now that $p = 2$. By Theorem 1.1 again, there exist a sequence $(\beta_\varepsilon)_\varepsilon$ increasing to $\beta^-$, a sequence $(p_\varepsilon)_\varepsilon$ increasing to $2^-$ as $\varepsilon \to 0$, and $u_\varepsilon$ such that (4.1) is satisfied for $\lambda_\varepsilon$ given by (4.2) for all $\varepsilon$. First, if we have in addition $\beta \notin 4\pi\mathbb{N}^\star$, we get similarly from Theorem 4.1 that, up to a subsequence, the $\lambda_\varepsilon$'s converge to some $\lambda$ and the $u_\varepsilon$'s converge in $C^2$ to some $u$ solving the equation in (0.3) and (0.4). Then, we use again that $\beta$ is positive to get from (0.4) that $u$ is actually positive in $\Sigma$ and then that $u \in \mathcal{C}_{2,\beta}$. Thus, if we have now $\beta \in 4\pi\mathbb{N}^\star$, setting $\beta_\varepsilon = \beta - \varepsilon$ and $p_\varepsilon = 2$, there exists $u_\varepsilon$ such that (4.1) is satisfied for $\lambda_\varepsilon$ given by (4.2) for all $0 < \varepsilon \ll 1$. By Theorem 5.1, we similarly get that the $u_\varepsilon$'s converge in $C^2$ to some $u \in \mathcal{C}_{2,\beta}$ solving (0.3)-(0.4), up to a subsequence. The compactness of $\mathcal{C}_{2,\beta}$ follows from Theorems 4.1 and 5.1 again, which concludes the proof of Theorem 0.2 in any case.

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