CALDERÓN-ZYGMUND TYPE ESTIMATES FOR A CLASS OF OBSTACLE PROBLEMS WITH $p(x)$ GROWTH

MICHELA ELEUTERI AND JENS HABERMANN

ABSTRACT. For minimizers $u \in W^{1,p(x)}(\Omega)$ of quasiconvex integral functionals of the type

$$\mathcal{F}[u] := \int_\Omega f(x, Du(x))\, dx$$

with $p(x)$ growth in the class $\mathcal{K} := \{u \in W^{1,p(x)}(\Omega) : u \geq \psi\}$, where $\psi \in W^{1,p(x)}(\Omega)$ is a given obstacle function, we show estimates of Calderón-Zygmund type, i.e.

$$|D\psi|^{p(\cdot)} \in L^q \Longrightarrow |Du|^{p(\cdot)} \in L^q,$$

for any $q > 1$, provided that the modulus of continuity ω of the exponent function p satisfies the condition

$$\omega(\rho) \log \frac{1}{\rho} \to 0 \quad \text{as} \quad \rho \to 0.$$

1. Introduction

The manuscript at hand is concerned with integrability results for solutions of one-sided obstacle problems of $p(x)$-growth type. More precisely, on an open, bounded set $\Omega \subset \mathbb{R}^n$ we consider minimizers u of an integral functional of the type

$$\mathcal{F}(u, \Omega) := \int_\Omega f(x, Du(x))\, dx \tag{1.1}$$

whose integrand function satisfies a $p(x)$-growth condition (see (H1)), and whose class of admissible functions is restricted in the sense that we claim $\{u \geq \psi\}$, for a given obstacle function $\psi : \Omega \to \mathbb{R}$. Additionally we assume the integrand function f to be quasiconvex (see (H2)), continuous with respect to the first variable in the sense of (H3), C^2 with respect to the second variable, and the exponent function $p : \Omega \to (1, \infty)$ to be uniformly continuous with modulus of continuity ω satisfying a so-called strong logarithmic Hölder continuity condition of the form

$$\limsup_{\rho \downarrow 0} \omega(\rho) \log \frac{1}{\rho} = 0. \tag{1.2}$$

In the present paper, the natural space for existence of minimizers in the $p(x)$-growth setting is the space variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, which we specify in Definition 2.2. Generalized spaces, like the Lebesgue spaces $L^{p(x)}$ and the Sobolev spaces $W^{k,p(x)}$ turn out to be interesting by themselves; quite a lot of investigations on their properties have been made in the past years. We refer the reader to [32], [36] and [17], [18], [37], [14] for further discussions, and for example to [25] and [27], together with the references therein, for more recent results.

Functionals with $p(x)$ growth attained the interest of an increasing number of mathematicians in the past 15 years for a variety of reasons: on one hand they represent a borderline case between standard p growth conditions (with constant exponent) and so–called $p - q$ growth condition introduced by Marcellini [33], on the other hand they appear in a natural way also in physical and technical applications, for example in the modeling of anisotropic materials, see [40], electrorheological fluids, see for example [37] or image processing models, as proposed by [9].

The study of nonlinear Calderón-Zygmund type estimates goes back to the fundamental paper of Iwaniec [29] in the case of elliptic equations with constant p growth, and to the paper of DiBenedetto & Manfredi [13] in the case of elliptic systems. Recently, Acerbi & Mingione proved estimates of this kind for parabolic systems [3]. Furthermore, Mingione [34, 35] developed a natural extention of the Calderón Zygmund theory for problems with measure data, showing appropriate fractional differentiability of the solution. Concerning equations with $p(x)$ growth structure, the first result of Calderón-Zygmund type is due to Acerbi & Mingione [2], who proved gradient estimates for nonlinear elliptic equations and the $p(x)$ Laplacean system. Subsequently one of the authors of this manuscript [24] generalized the results (under some natural restriction on the higher integrability exponent) to higher order systems. The linear counterpart to these results, namely the generalization of the classical Calderón-Zygmund Theorem [8] to variable Lebesgue spaces has been done by L.Diening and M. Růžička in [15].

On the other hand, regularity for obstacle problems were studied by Choe [10], who proved Morrey type regularity for minimizers of obstacle problems in the situation of special types of functionals with constant p growth conditions, by Fuchs & Mingione [22], proving $C^{1,\alpha}$ regularity for functionals with non differentiable integrands with nearly linear growth, and by one of the authors of this paper [19], showing Hölder continuity of minimizers of general functionals with constant p growth. Generalizations of Hölder type regularity results for obstacle problems with $p(x)$ growth were done by the authors in [20, 21]. We also would like to quote the paper [26] concerning obstacle problems and superharmonic functions related to partial differential equations with non standard growth (being the Euler Lagrange equations of variational integral of kind (1.1) where the dependence of f on x is omitted). Finally, we mention the paper of Bildhauer, Fuchs & Mingione [5], which is concerned with double obstacle problems in the setting of constant p growth, and very recently the manuscript of Bögelein, Duzaar & Mingione [6], discussing gradient estimates for parabolic obstacle problems.

The aim of this paper is to show Calderón-Zygmund type estimates for obstacle problems with $p(x)$ growth in the following sense: Provided that the obstacle function ψ belongs to $W^{1,p(x)q}_{\mathrm{loc}}$ with arbitrary given $q > 1$, then also the minimizer u belongs to $W^{1,p(x)q}_{\mathrm{loc}}$. We note at this point that provided that the obstacle function itself belongs to $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$, general functional analytic arguments guarantee the existence of a minimizer of the functional (1.1) in the obstacle class $\{u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega) : u \geq \psi\}$.

Some remarks on the proof

The key to the proof of a quantified higher integrability of the gradient of the minimizer u of the functional (1.1) is a decay estimate of the level sets of the maximal funktion of $|Du|^{p(\cdot)}$ to increasing levels, as we can see it in (4.53) (recall therefore also the definitions of μ_1 and μ_2 in (4.49)). Iteration of (4.53) in combination with the well known L^p estimates for the maximal function then directly provides the desired integrability result. To prove (4.53), we take use of Lemma 3.1 which is a direct consequence of a Calderón-Zygmund type covering argument, as it can for example be found (together with the proof of Lemma 3.1) in [7]. To apply this lemma on super level sets of the maximal function (see the definition of X and Y in (4.54) and (4.55)), it turns out to be crutial to show that assumption (ii) in Lemma 3.1 is fulfilled. This is the statement of Lemma 4.2.

To prove Lemma 4.2, the strategy consists in a comparison of the minimizer u of the original problem to the unique solution z of the Dirichlet problem

$$\begin{cases} \displaystyle\int_S a(x_M, Dz)\, D\varphi\, dx = 0 & \forall \varphi \in C_0^\infty(S), \\ z = u & \text{on } \partial S, \end{cases} \tag{1.3}$$

where S denotes a suitable small cube. As we immediately see, on one hand problem (1.3) is frozen in a point x_M and therefore shows standard p growth behaviour, on the other hand the problem is completely independent of the obstacle ψ. To reach these two goals, it turns out to be necessary to include a second comparison process, namely to the unique solution w of a Dirichlet problem of the form

$$\begin{cases} \int_S a(x_M, Dw)\, D\varphi\, dx = \int_S a(x_M, D\psi)\, D\varphi\, dx, & \forall \varphi \in C_0^\infty(S) \\ w = u & \text{on } \partial S. \end{cases} \tag{1.4}$$

The structure conditions of problem (1.3) – a nonlinear degenerate elliptic equation with constant growth exponent – guarantee an L^∞ estimate for the gradient of z. Comparison estimates finally have to be carried out in order to pass the *sup* estimate on the minimizer u. Of course the freezing procedure calls for some quantified continuity of the integrand function with respect to the first variable, i.e.

$$\omega(\rho) \log \frac{1}{\rho} \to 0, \qquad \text{as } \rho \to 0$$

(see hypothesis (H3) and (2.3)). Since the exponent function p is assumed to possess the same quantitative continuity behaviour (see (2.2)), a priori higher integrability (with some higher integrability exponent $\sigma > 0$), which is shown in Lemma 4.1 allows us to localize the problem and therefore to establish suitable comparison estimates. We note that at this point a precise control on the dependence of the constants is essential.

Acknowledgement

The authors wish to acknowledge F. Duzaar and G. Mingione for useful discussions.

2. Results

General notation. In the sequel $\Omega \subset \mathbb{R}^n$ will be a bounded domain; by "cube" we will always mean an open cube with edges parallel to the coordinate axes; when relevant, we will mention the side length, denoting e.g. by Q_R a cube with side length equal to $2R$; with a slight abuse, we will call R the *radius* of such a cube. Moreover, for $\gamma > 0$, we will adopt the convention that γQ or $Q_{\gamma R}$ denote cubes *with the same center* as Q or Q_R, and radius multiplied by γ. Adopting a usual convention, c will denote a constant whose value may change in any two occurrences, and only the relevant dependences will be specified. For the Lebesgue measure of a measurable set A we shall employ the notations

$$|A| = \text{meas}\,(A);$$

then we define the mean value on a cube $Q_R \subset \Omega$ of a locally integrable function $v \in L^1_{\text{loc}}(\Omega)$ by

$$(v)_{Q_R} \equiv (v)_R \equiv \fint_{Q_R} v\, dx := \frac{1}{|Q_R|} \int_{Q_R} v\, dx.$$

Structure conditions. If $\mathcal{F}$ is the functional introduced in (1.1), we set

$$\mathcal{F}(u, \mathcal{A}) := \int_{\mathcal{A}} f(x, Du(x))dx$$

for all $u \in W^{1,1}_{\text{loc}}(\Omega)$ and for all $\mathcal{A} \subset \Omega$. We adopt the following notion of local minimizer.

Definition 2.1. *We say that a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a local minimizer of the functional* (1.1) *if $|Du(x)|^{p(x)} \in L^1_{\text{loc}}(\Omega)$ and*

$$\int_{\text{spt}\,\varphi} f(x, Du(x))dx \le \int_{\text{spt}\,\varphi} f(x, Du(x) + D\varphi(x))dx,$$

for all $\varphi \in W^{1,1}_0(\Omega)$ with compact support in Ω.

We shall consider the following growth and ellipticity conditions:

$$L^{-1}(\mu^2+|\zeta|^2)^{p(x)/2} \leq f(x,\zeta) \leq L(\mu^2+|\zeta|^2)^{p(x)/2}, \tag{H1}$$

$$\begin{aligned}&\int_{Q_1}[f(x_0,\zeta_0+D\varphi(x))-f(x_0,\zeta_0)]dx\\ &\qquad\geq L^{-1}\int_{Q_1}(\mu^2+|\zeta_0|^2+|D\varphi(x)|^2)^{\frac{p(x_0)-2}{2}}|D\varphi(x)|^2dx,\end{aligned} \tag{H2}$$

for all $\zeta,\zeta_0\in\mathbb{R}^n$, $x_0\in\Omega$, $\varphi\in\mathcal{C}_0^\infty(Q_1)$, where $Q_1=(0,1)^n$ and the parameter $\mu\in[0,1]$ appears to deal simultaneously with the degenerate and non-degenerate cases. We also consider the continuity condition

$$\begin{aligned}&|f(x,\zeta)-f(x_0,\zeta)|\\ &\qquad\leq L\omega(|x-x_0|)\left[\left(\mu^2+|\zeta|^2\right)^{p(x)/2}+\left(\mu^2+|\zeta|^2\right)^{p(x_0)/2}\right]\left[1+|\log(\mu^2+|\zeta|^2)|\right],\end{aligned} \tag{H3}$$

for all $\zeta\in\mathbb{R}^n$, x and $x_0\in\Omega$, where $L\geq 1$ and $\mu\in[0,1]$. Here the function $p:\Omega\to(1,\infty)$ is supposed to be continuous and to satisfy (which is not restrictive for local results)

$$1<\gamma_1\leq p(x)\leq\gamma_2<\infty, \tag{2.1}$$

while $\omega:\mathbb{R}^+\to\mathbb{R}^+$ denotes the modulus of continuity of the function $p(x)$,

$$|p(x)-p(y)|\leq\omega(|x-y|). \tag{2.2}$$

The main assumption on the function $p(x)$ will be

$$\lim_{\rho\to 0}\omega(\rho)\log\left(\frac{1}{\rho}\right)=0. \tag{2.3}$$

Without loss of generality we can assume $\omega(\cdot)$ to be non-decreasing.
Since all our results are local in nature, without loss of generality we shall suppose that

$$\int_\Omega|Du(x)|^{p(x)}dx<+\infty\ . \tag{2.4}$$

Let $\psi\in W^{1,p(x)}(\Omega;\mathbb{R})$ be a fixed obstacle function and let us set

$$\mathcal{K}:=\{u\in W^{1,p(x)}(\Omega;\mathbb{R}):\ u\geq\psi\}. \tag{2.5}$$

We assume that the Lagrangian f is of class $\mathcal{C}^2$ with respect to the variable ζ in $\Omega\times(\mathbb{R}^n\setminus\{0\})$, with D^2f satisfying

$$L^{-1}\,(\mu^2+|\zeta|^2)^{(p(x)-2)/2}\,|\lambda|^2\leq\ D^2f(x,\zeta)\lambda\otimes\lambda\leq\ L\,(\mu^2+|\zeta|^2)^{(p(x)-2)/2}\,|\lambda|^2, \tag{2.6}$$

for all $\lambda\in\mathbb{R}^n$.
Setting $a(x,\zeta):=Df(x,\zeta)$, we have that (H1), (H2), (H3) and (2.6) entail the following properties for the vector field $a:\Omega\times\mathbb{R}^n\to\mathbb{R}$

$$|a(x,\zeta)-a(x_0,\zeta)|\leq\ \omega(|x-x_0|)\,|\log(\mu^2+|\zeta|^2)|\ \left[(\mu^2+|\zeta|^2)^{(p(x)-1)/2}+(\mu^2+|\zeta|^2)^{(p(x_0)-1)/2}\right], \tag{2.7}$$

for every $x,y\in\Omega$, $\zeta,\lambda\in\mathbb{R}^n$,

$$|a(x,\zeta)|\leq\ L\,(1+|\zeta|^2)^{(p(x)-1)/2}, \tag{2.8}$$

and

$$\nu\,(\mu^2+|\zeta|^2)^{p(x)/2}-L\leq\ \langle a(x,\zeta),\zeta\rangle\qquad\forall\,x\in\Omega,z\in\mathbb{R}^n, \tag{2.9}$$

where $\nu,L\in[1,\infty)$.
Local minimizers of (1.1) **in** $\mathcal{K}$**.** Let u be a local minimizer of the functional (1.1) in the class (2.5). Then it is not difficult to show that u satisfies the following inequality

$$\int_\Omega a(x,Du)\,(Du-D\varphi)\,dx\leq 0, \tag{2.10}$$

for every $\varphi \in K$ such that $\varphi - u$ has compact support in Ω.

Generalized Lebesgue and Sobolev spaces.

Definition 2.2. *For a bounded domain $\Omega \subset \mathbb{R}^n$, a measurable function $p : \Omega \to (1, \infty)$ and $N \geq 1$ we define the generalized Lebesgue space*

$$L^{p(\cdot)}(\Omega) \equiv \left\{ f \in L^1(\Omega) : \int_\Omega |\lambda f(x)|^{p(x)}\, dx < \infty \text{ for some } \lambda > 0 \right\},$$

which, endowed with the Luxemburg norm

$$||f||_{L^{p(\cdot)}(\Omega)} \equiv \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

becomes a Banach space. Furthermore the generalized Sobolev space is defined as

$$W^{1,p(\cdot)}(\Omega) \equiv \left\{ f \in L^{p(\cdot)}(\Omega) : \, Df \in L^{p(\cdot)}(\Omega) \right\},$$

and also becomes a Banach space if endowed with the norm

$$||f||_{W^{1,p(\cdot)}(\Omega)} \equiv ||f||_{L^{p(\cdot)}(\Omega)} + ||Df||_{L^{p(\cdot)}(\Omega)}.$$

We refer for example to [37], [28], [11] *and* [17] *for more details and further references on these spaces.*

The main result of the paper is the following.

Theorem 2.3. *Let $u \in W^{1,p(x)}(\Omega; \mathbb{R})$ be a local minimizer of the functional* (1.1) *in the class* (2.5)*, where the lagrangian f satisfies the assumptions* (H1), (H2) *and* (H3)*, the modulus of continuity for p fulfills* (2.3)*, and where ψ is a given obstacle function which satisfies*

$$|D\psi|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega), \tag{2.11}$$

for some $q > 1$. Then $|Du|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega)$.

In particular there holds: If $\Omega' \Subset \Omega$ and $|D\psi|^{p(\cdot)} \in L^q(\Omega')$, then for any given $\varepsilon \in (0, q-1)$ there exists a positive radius $R_0 > 0$, depending on $n, \gamma_1, \gamma_2, \nu, L, \varepsilon, q, \omega(\cdot), \left\| |Du|^{p(\cdot)} \right\|_{L^1(\Omega')}, \left\| |D\psi|^{p(\cdot)} \right\|_{L^q(\Omega')}$, such that for any cube $Q_{4R} \Subset \Omega'$ and $R \leq R_0$ there holds

$$\left(\fint_{Q_R} |Du|^{p(x)q}\, dx \right)^{1/q} \leq cK^\varepsilon \fint_{Q_{4R}} |Du|^{p(x)}\, dx + cK^\varepsilon \left(\fint_{Q_{4R}} |D\psi|^{p(x)q}\, dx + 1 \right)^{1/q},$$

where $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, q)$ and

$$K := \int_{Q_{4R}} \left(|Du|^{p(x)} + |D\psi|^{p(x)(1+\varepsilon)} \right) dx + 1.$$

3. Preliminary material

In this section we are going to collect a list of preliminary results for later use. Let us start from a restatement of the classical Calderón-Zygmund covering argument; at the same time we will add more notation about dyadic cubes.

Calderón-Zygmund coverings. We consider a cube $Q_0 \subset \mathbb{R}^n$ and define by $\mathcal{D}(Q_0)$ the set of all dyadic subcubes Q of Q_0, i.e. those cubes with sides parallel to the sides of Q_0 that can be obtained from Q_0 by a positive finite number of dyadic subdivisions. We call Q_p *a predecessor* of Q, if Q is obtained from Q_p by a finite number of dyadic subdivisions. In particular we call $\tilde{Q} \in \mathcal{D}(Q_0)$ *the predecessor* of Q, if Q is obtained from $\tilde{Q}$ by exactly one dyadic subdivision from $\tilde{Q}$.

The following lemma will play an essential role in the proof of our results. The proof is a consequence of a Calderón-Zygmund type covering argument and its proof can be found, for instance in [7].

Lemma 3.1. *Let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $\mathcal{X} \subset \mathcal{Y} \subset Q_0$ are measurable sets satisfying the following conditions:*
(i) there exists $\delta > 0$ such that
$$|\mathcal{X}| < \delta\, |Q_0|;$$
(ii) if $Q \in \mathcal{D}(Q_0)$ then
$$|\mathcal{X} \cap Q| > \delta\, |Q| \ \Rightarrow\ \tilde{Q} \subset Y,$$
where $\tilde{Q}$ denotes the predecessor of Q. Then
$$|\mathcal{X}| < \delta\, |\mathcal{Y}|.$$

Maximal Operators. Let $Q_0 \subset \mathbb{R}^n$ be a cube. We will consider the *Restricted Maximal Function Operator relative to* Q_0, which is defined as
$$M^*_{Q_0}(f)(x) := \sup_{Q \subset Q_0,\, x \in Q} \fint_Q |f(y)|\, dy,$$
whenever $f \in L^1(Q_0)$, where Q denotes any cube contained in Q_0, not necessarily with the same center, as long as it contains the point x. In the same way, for $s > 1$ we define
$$M^*_{s,Q_0}(f)(x) := \sup_{Q \subset Q_0,\, x \in Q} \left(\fint_Q |f(y)|^s\, dy \right)^{1/s}$$
whenever $f \in L^s(Q_0)$. We recall the following estimate for $M^*_{Q_0}$:
$$|\{x \in Q_0 : |M^*_{Q_0}(f)(x)| \geq \lambda\}| \leq \frac{c_W}{\lambda} \int_{Q_0} |f(y)|\, dy \qquad \forall\, \lambda > 0, \tag{3.1}$$
which is valid for any $f \in L^1(Q_0)$; the constant c_W depends only on n; for this and related issues we refer to [38]. A standard consequence of the previous inequality is then
$$\int_{Q_0} |M^*_{Q_0}(f)(y)|^q\, dy \leq \frac{c(n,q)}{q-1} \int_{Q_0} |f(y)|^q\, dy, \qquad q > 1. \tag{3.2}$$
A similar estimate for the M^*_{s,Q_0} operator is
$$\int_{Q_0} |M^*_{s,Q_0}(f)(y)|^q\, dy \leq \frac{c(n)q^2}{s(q-s)} \int_{Q_0} |f(y)|^q\, dy, \qquad q > s, \tag{3.3}$$
which can be deduced from (3.2), compare [30], Section 7.

Estimates for the $L \log L$ norm. We recall at this point some well established estimates in $L \log L$ spaces, which we will need later in the comparison estimates. Note that the following statements can be found in [2].

Lemma 3.2. *For any $f \in L^p(\Omega), p > 1$ and $\beta > 1$ there holds*
$$\fint_\Omega |f| \log^\beta \left(e + \frac{|f|}{||f||_{L^1(\Omega)}} \right) dx \leq c(\beta, p) \left(\fint_\Omega |f|^p\, dx \right)^{1/p}, \tag{3.4}$$
where the constant c does not depend on $|\Omega|$ and shows the following asymptotic behaviour:
$$c(\beta, p) \approx \left(\frac{1}{p-1} \right)^\beta \qquad as \quad p \searrow 1. \tag{3.5}$$

Lemma 3.3. *For any $t > 0$, $\beta \in \left[\frac{\gamma_2}{\gamma_2 - 1}, \frac{\gamma_1}{\gamma_1 - 1} \right]$ with $1 < \gamma_1 \leq \gamma_2 < +\infty$ and any $\sigma \in (0,1)$ there holds*
$$(e+t) \log^\beta (e+t) \leq c(\gamma_1, \gamma_2) \sigma^{-\beta} (e+t)^{1+\sigma/4}. \tag{3.6}$$

We conclude the section with the following elementary lemma, whose proof can be immediately adapted from Lemma 2.2 in [12].

Lemma 3.4. *Let* $p \in [\gamma_1, \gamma_2]$ *and* $\mu \in (0, 1]$; *there exists a constant* $c \equiv c(k, \gamma_1, \gamma_2)$ *such that if* $v, w \in \mathbb{R}^k$ *then:*

$$(\mu^2 + |v|^2)^{\frac{p}{2}} \le c\,(\mu^2 + |w|^2)^{\frac{p}{2}} + c\,(\mu^2 + |v|^2 + |w|^2)^{\frac{p-2}{2}}\,|v - w|^2.$$

4. Proof of the results

General setting, I. Here we begin the proof by fixing some objects and notations that will apply to the end of the paper. We consider a "large cube" $Q_{4R_0} \Subset \Omega$; during the development of the section we shall make several restrictions on the size of R_0. Using (2.3) for the second inequality, we shall initially take R_0 small enough in order to have

$$\begin{cases} \omega(8nR_0) \le \sqrt{\dfrac{n+1}{n}} - 1, \\ 0 < \omega(R)\,\log\left(\dfrac{1}{R}\right) \le L \qquad \forall\, R \le 8\,n\,R_0. \end{cases} \tag{4.1}$$

We start with a preliminary version of Theorem 2.3 which rests on an application of Gehring's lemma in the spirit of [1], [39]; we need the following exact statement, emphasizing the precise dependence of the constants.

Theorem 4.1. *Let* $u \in W^{1,p(x)}(\Omega; \mathbb{R})$ *be a minimizer of the functional* (1.1) *in the class* (2.5), *under the assumptions* (H1), (H2) *and* (H3) *and let assume that* (2.11) *holds. Then there exist constants* $c \equiv c(n, \gamma_1, \gamma_2, \nu, L)$ *and* $c_g(n, \gamma_1, \gamma_2, \nu, L)$ *such that the following is true:*
assume R_0 *satisfies* (4.1), *let* $Q_{4R_0} \Subset \Omega$, *set*

$$K_0 := \int_{Q_{4R_0}} |Du|^{p(x)}\,dx + 1, \tag{4.2}$$

and let $\sigma > 0$ *be any number such that*

$$\sigma \le \min\left\{ \frac{c_g}{K_0^{\frac{2q\omega(8nR_0)}{\gamma_1}}}, q - 1, 1 \right\} =: \sigma_0. \tag{4.3}$$

Then for every $Q_R \Subset Q_{4R_0}$ *it holds*

$$\left(⨍_{Q_{R/2}} |Du|^{p(x)(1+\sigma)}\,dx \right)^{\frac{1}{1+\sigma}} \le c\, ⨍_{Q_R} |Du|^{p(x)}\,dx + c\left(⨍_{Q_R} |D\psi|^{p(x)(1+\sigma)}\,dx + 1 \right)^{\frac{1}{1+\sigma}}. \tag{4.4}$$

Proof. The proof of the theorem can be carried out following the proof of Theorem 5 in [2]; in our case we focus on the differences due to the presence of the obstacle.

Let $Q_R \subset Q_{4R_0}$ be cube and

$$p_1 := \inf\{p(x) : x \in Q_R\} \quad p_2 := \sup\{p(x) : x \in Q_R\}.$$

Then, $p_2 - p_1 \le \omega(2nR)$ and by the first inequality in (4.1), we have

$$\frac{p_2}{p_1} = \frac{p_2 - p_1}{p_1} + 1 \le \sqrt{\frac{n+1}{n}} =: \tilde{s}. \tag{4.5}$$

Now let $0 < R/2 \le s < t \le R$. We take a cut-off function $\eta \in \mathcal{C}_0^\infty(Q_R)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on Q_s and $D\eta \le 4/(t-s)$. We would like to test (2.10) with $\varphi := \max\{\tilde{g}, \psi\}$, where

$$\tilde{g} := u - \eta^{p_2}(u - (u)_R).$$

Clearly $\varphi \in K$; let us set $\Sigma := \{x \in Q_R : \tilde{g}(x) \ge \psi(x)\}$. Denoting $a(x, \zeta) := Df(x, \zeta)$, taking into account (2.7), (2.8), (2.9), we have with (2.10)

$$\nu \int_{Q_s} \eta^{p_2} |Du|^{p(x)}\,dx$$

$$
\begin{aligned}
&\leq \int_{Q_R} \eta^{p_2}\langle a(x,Du),Du\rangle + 1\,dx\\
&\leq \int_{Q_R} \eta^{p_2}\langle a(x,Du),D\varphi\rangle + 1\,dx\\
&= \int_{Q_R\cap\Sigma} \eta^{p_2}\langle a(x,Du),D\tilde g\rangle + 1\,dx + \int_{Q_R\setminus\Sigma} \eta^{p_2}\langle a(x,Du),D\psi\rangle + 1\,dx\\
&\leq \int_{Q_R} \eta^{p_2}(1-\eta^{p_2})\langle a(x,Du),Du\rangle\,dx + p_2\int_{Q_R} \eta^{p_2}\eta^{p_2-1}\langle a(x,Du),D\eta\otimes(u-(u)_R)\rangle\,dx\\
&\qquad\qquad + \int_{Q_R} L\eta^{p_2}(1+|Du|^2)^{\frac{p(x)-1}{2}}\,|D\psi| + 1\,dx\\
&\leq \int_{Q_t\setminus Q_s} \langle a(x,Du),Du\rangle\,dx + \frac{\nu}{2}\int_{Q_R}\eta^{p_2}\,|Du|^{p(x)}\,dx + c(\nu)\int_{Q_R}\frac{|u-(u)_R|^{p_2}}{(t-s)^{p_2}} + 1\,dx\\
&\qquad\qquad + \zeta\int_{Q_R}\eta^{p_2}|Du|^{p(x)}\,dx + c_\zeta\int_{Q_R}|D\psi|^{p(x)} + 1\,dx\\
&\leq \bar c\int_{Q_t\setminus Q_s}|Du|^{p(x)}\,dx + \frac{\nu}{2}\int_{Q_s}\eta^{p_2}|Du|^{p(x)}\,dx + c\int_{Q_R}\frac{|u-(u)_R|^{p_2}}{(t-s)^{p_2}} + 1\,dx\\
&\qquad\qquad + \zeta\int_{Q_s}|Du|^{p(x)}\,dx + c_\zeta\int_{Q_R}|D\psi|^{p(x)} + 1\,dx.
\end{aligned}
$$

Now proceeding in a standard way, i.e. “filling the hole” and choosing for example $\zeta = \frac{\nu}{4}$, we have

$$
\nu\int_{Q_s}|Du|^{p(x)}\,dx \leq \theta\int_{Q_t}|Du|^{p(x)}\,dx + c\int_{Q_R}\frac{|u-(u)_R|^{p_2}}{(t-s)^{p_2}} + 1\,dx + c\int_{Q_R}|D\psi|^{p(x)}\,dx,
$$

where $\theta < 1$. Now applying [23], Lemma 6.1 we deduce finally the following Caccioppoli inequality

$$
\frac{\nu}{2}\int_{Q_{R/2}}|Du|^{p(x)}\,dx \leq c\int_{Q_R}\frac{|u-(u)_R|^{p_2}}{R^{p_2}} + 1\,dx + c\int_{Q_R}|D\psi|^{p(x)}\,dx.
$$

The conclusion now comes as in [2]. □

General setting, II. First we observe that, since $K_0 \geq 1$ (see the definition of K_0 in (4.2)), we have for any $K \geq K_0$

$$
\sigma_0 \geq \min\{1, q-1, c_g\}\,K^{-\frac{2q\omega(8nR_0)}{\gamma_1}}, \tag{4.6}
$$

where σ_0 has been introduced in (4.3). We set

$$
K_M := \int_\Omega (|Du|^{p(x)} + |D\psi|^{p(x)q} + 2)\,dx + 1 \tag{4.7}
$$

and

$$
\sigma_m := \min\left\{\frac{c_g}{K_M^{\frac{2q(\gamma_2-\gamma_1)}{\gamma_1}}}, \frac{q-1}{2}, 1\right\} > 0 \qquad \sigma_M := c_g + q. \tag{4.8}
$$

Therefore $K_M \geq K_0$. Furthermore, for any $1 \leq K \leq K_M$ we have

$$
\sigma_m \leq \sigma_0 \leq \sigma_M.
$$

We now choose the higher integrability exponent σ in Theorem 4.1 such that

$$
\sigma := \tilde\sigma\,\sigma_0 \quad \text{with} \quad 0 < \tilde\sigma < \min\{\gamma_1 - 1, 1/2\}. \tag{4.9}
$$

Then by (4.6) we have for any $\beta \in \left[\frac{\gamma_2}{\gamma_2-1}, \frac{\gamma_1}{\gamma_1-1}\right]$ and $K \geq K_0$:

$$
\sigma^{-\beta} \leq c\,\tilde\sigma^{-\beta}\,K^{\beta\frac{2q\omega(8nR_0)}{\gamma_1}} \leq c(n,\gamma_1,\gamma_2,\nu,L,q)\,\tilde\sigma^{-\beta}\,K^{\frac{2q\omega(8nR_0)}{\gamma_1-1}}. \tag{4.10}
$$

By the choice of σ in (4.9) and the structure of the constant σ_0 in Theorem 4.1, we have that

$$\sigma < \frac{q-1}{2}.$$

Now we impose for a fixed choice of $\tilde{\sigma}$ a further restriction on the size of R_0 by claiming

$$\max\left\{2q\omega(8nR_0), \frac{2q\omega(8nR_0)}{\gamma_1 - 1}\right\} \leq \frac{\tilde{\sigma}\,\sigma_m}{4}. \tag{4.11}$$

Therefore R_0 depends on $n, \gamma_1, \gamma_2, L/\nu, q, |||Du(\cdot)|^{p(\cdot)}||_{L^1(\Omega)}, |||D\psi(\cdot)|^{p(\cdot)}||_{L^1(\Omega)}$ and $\tilde{\sigma}$. Now (4.11) immediately implies

$$\omega(8nR_0) \leq \max\left\{2q\omega(8nR_0), \frac{2q\omega(8nR_0)}{\gamma_1 - 1}\right\} \leq \frac{\tilde{\sigma}\sigma_m}{4} \leq \frac{\tilde{\sigma}\,\sigma_0}{4} = \frac{\sigma}{4}. \tag{4.12}$$

Calderón-Zygmund type estimates. The following Lemma will be the crucial point for the proof of our main theorem. The statement is very similar to Lemma 5 in [2]. Nevertheless the proof has to be modified at many points, since we need several steps of comparison in order to be able to exploit the reference estimate having at hand for the solution of a suitable free problem with frozen exponents.

Lemma 4.2. *Let $u \in W^{1,p(x)}(\Omega, \mathbb{R})$ be a minimizer of the functional* (1.1) *in the class* (2.5) *under the assumptions* (H1), (H2) *and* (H3) *and let $\lambda \geq 1$ and $0 < \tilde{\sigma} < 1$ as in* (4.9)*. Then there exists a constant $A \equiv A(n, \gamma_1, \gamma_2, \nu, L)$ independent of $\lambda, \tilde{\sigma}, u, \mathcal{F}, \psi$ such that for every $\delta_1 > 0$ there exists $R_1 \equiv R_1(n, \gamma_1, \gamma_2, \nu, L, q, \tilde{\sigma}, \delta_1) > 0$ such that:*

If $R_0 \leq R_1$ satisfies (4.1), (4.12) *and K_0, σ_0 are as in* (4.2) *and* (4.3)*, setting $\sigma = \tilde{\sigma}\sigma_0$ and*

$$K := \int_{Q_{4R_0}} \left(|Du|^{p(x)} + |D\psi|^{p(x)(1+\sigma)}\right) dx + 1, \tag{4.13}$$

then for every $\delta \geq \delta_1$ there exists $\tilde{\varepsilon} > 0$, independent of λ, such that the following holds:

if $Q \in \mathcal{D}(Q_{R_0})$ satisfies

$$\begin{aligned} \Big| Q \cap \{x \in Q_{R_0} \quad &: \quad M^*_{Q_{4R_0}}(|Du(\cdot)|^{p(\cdot)})(x) > AK^{\sigma}\lambda, \\ &\qquad M^*_{1+\sigma, Q_{4R_0}}(|D\psi(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda\}\Big| > \delta|Q|, \end{aligned} \tag{4.14}$$

then its predecessor $\tilde{Q}$ satisfies

$$\tilde{Q} \subseteq \{x \in Q_{R_0} : M^*_{Q_{4R_0}}(|Du(\cdot)|^{p(\cdot)})(x) > \lambda\}. \tag{4.15}$$

Proof. STEP 1: BEGINNING. As in [2] we prove the statement by contradiction. The constants $A, \tilde{\varepsilon}$ as well as the radius R_1 will be chosen at the end of the proof.

Let us assume that (4.14) holds, but (4.15) is false. Then there exists a point $x_0 \in \tilde{Q}$ such that

$$M^*_{Q_{4R_0}}\left(|Du(\cdot)|^{p(\cdot)}\right)(x_0) \leq \lambda,$$

i.e. we have

$$\fint_C |Du(x)|^{p(x)}\, dx \leq \lambda, \tag{4.16}$$

for all cubes $C \subseteq Q_{4R_0}$ with $x_0 \in C$. We define $S := 2\tilde{Q}$. Since the cube $\tilde{Q}$ is obtained from Q_{R_0} by at least one dyadic subdivision, we have $\tilde{Q} \subseteq Q_{R_0}$ and therefore $S \subseteq Q_{2R_0}$. With (4.1) there holds

$$s := \operatorname{diam}(2S) \leq 8nR_0, \qquad \omega(s) \leq \sigma/4. \tag{4.17}$$

In particular, since by $x_0 \in 2S$ the cube $2S \subseteq Q_{4R_0}$ is an admissible cube in the maximal function $M^*_{Q_{4R_0}}$, by (4.16) we have

$$\fint_{2S} |Du(x)|^{p(x)}\,dx \le \lambda. \tag{4.18}$$

Additionally (4.14) implies

$$\left|\left\{x \in Q:\ M^*_{1+\sigma,Q_{4R_0}}\left(|D\psi(\cdot)|^{p(\cdot)}+1\right)(x) < \tilde\varepsilon\lambda\right\}\right| > 0, \tag{4.19}$$

so that there exists at least one point $x \in Q$, in which the maximal function $M^*_{1+\sigma,Q_{4R_0}}$ of $|D\psi(\cdot)|^{p(\cdot)}+1$ is small. Since $Q \subset 2S \subset Q_{4R_0}$, this implies

$$\left(\fint_S \left(|D\psi|^{p(x)}+1\right)^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}} < \tilde\varepsilon\lambda, \quad \left(\fint_{2S} \left(|D\psi|^{p(x)}+1\right)^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}} < \tilde\varepsilon\lambda. \tag{4.20}$$

Let us derive some useful preparatory estimates; let

$$p_1 := \min_{\overline{2S}} p(x), \qquad p_2 := p(x_M) = \max_{\overline{2S}} p(x), \qquad x_M \in \overline{2S}; \tag{4.21}$$

observe that the numbers p_1 and p_2 depend on the selected cube Q and vary when Q varies in $\mathcal{D}(Q_{R_0})$. Since $2S \equiv 4\tilde Q \subset Q_{4R_0}$, we get

$$\begin{aligned} p_2 &= (p_2-p_1)+p_1 \\ &\le \omega(s)+p_1 \\ &\le p_1(1+\omega(s)) \\ &\le p(x)(1+\omega(s)) \\ &\le p(x)(1+\omega(s)+\sigma/4) \\ &\le p(x)(1+\sigma) \qquad \forall\, x \in 2S, \end{aligned} \tag{4.22}$$

where we used (4.12) in the last estimate. Also, since (4.9) implies $\sigma \le p_1 - 1$, we have

$$\begin{aligned} p_2(1+\sigma/4) &\le (p_1+\omega(s))(1+\sigma/4) \\ &\le p_1(1+\sigma/4+\omega(s)) \\ &\le p(x)(1+\sigma/4+\omega(s)) \qquad (4.23)\\ &\le p(x)(1+\sigma). \qquad (4.24) \end{aligned}$$

Now, since $\omega(s) \le \sigma/4$ by (4.12), we can use Theorem 4.1 and formula (4.4) as follows:

$$\begin{aligned} \fint_S |Du|^{p_2}\,dx &\le \fint_S (|Du|^{p_2}+1)\,dx \\ &\overset{(4.22)}{\le} 2\fint_S \left(|Du|^{p(x)(1+\omega(s))}+1\right) dx \\ &\overset{(4.4),(4.12)}{\le} c\left(\fint_{2S}\left(|Du|^{p(x)}+1\right) dx\right)^{1+\omega(s)} + c\fint_{2S}(|D\psi|^{p(x)}+1)^{1+\omega(s)}\,dx \\ &\le c\left(\int_{2S}\left(|Du|^{p(x)}+1\right) dx\right)^{\omega(s)} s^{-n\omega(s)} \times \fint_{2S}\left(|Du|^{p(x)}+1\right) dx \\ &\quad + c\left(\int_{2S}\left(|D\psi|^{p(x)(1+\omega(s))}+1\right) dx\right)^{\frac{\omega(s)}{1+\omega(s)}} s^{\frac{-n\omega(s)}{1+\omega(s)}} \\ &\qquad \times \left(\fint_{2S}\left(|D\psi|^{p(x)(1+\omega(s))}+1\right) dx\right)^{\frac{1}{1+\omega(s)}} \\ &\overset{(4.12)}{\le} c\,K^{\frac{\sigma}{4}}\fint_{2S}\left(|Du|^{p(x)}+1\right) dx + c\,K^{\frac{\sigma}{4}}\left(\fint_{2S}|D\psi|^{p(x)(1+\sigma)}+1\,dx\right)^{\frac{1}{1+\sigma}} \end{aligned} \tag{4.25}$$

$$\overset{(4.18),(4.20)}{\leq} \quad c(n,\gamma_1,\gamma_2,\nu,L)\, K^{\frac{\sigma}{4}}\, \lambda.$$

Here we crucially used the fact that $s^{-n\,\omega(s)}$ stays bounded as $0 < s < 8nR_0$ by (4.1).

Moreover we also obtain

$$\int_S |Du|^{p_2}\, dx \leq c \left(\int_{2S} \left(|Du|^{p(x)} + 1\right) dx \right)^{\omega(s)} s^{-n\,\omega(s)} \int_{2S} \left(|Du|^{p(x)} + 1\right) dx \tag{4.26}$$

$$+c \int_{2S} \left(|D\psi|^{p(x)(1+\omega(s))} + 1\right) dx$$

$$\leq c\, K^{\frac{\sigma}{4}} \int_{2S} \left(|Du|^{p(x)} + |D\psi|^{p(x)(1+\sigma)} + 1\right) dx$$

$$\leq c(n,\gamma_1,\gamma_2,\nu,L)\, K^{1+\frac{\sigma}{4}}. \tag{4.27}$$

STEP 2: COMPARISON TO A REFERENCE PROBLEM.

By (4.27) it follows that $u \in W^{1,p_2}(S)$, therefore we are able to define $v \in (u + W_0^{1,p_2}(S)) \cap W^{1,p_2}(S)$ as the unique minimizer of the functional

$$\mathcal{G}(v) := \int_S f(x_M, Dv(x))\, dx =: \int_S g(Dv(x))\, dx$$

in the class $\tilde{\mathcal{K}} := \{v \in u + W_0^{1,p_2}(S) :\ v \geq \psi\}$, where $x_M \in \overline{2S}$ denotes the point according to (4.21). Then it is not difficult to see that v satisfies the following inequality

$$\int_S a(x_M, Dv)\,(Dv - D\varphi)\, dx \leq 0 \tag{4.28}$$

for all $\varphi \in K$ such that $\varphi - v$ has compact support in S.

Moreover we define $w \in (u + W_0^{1,p_2}(S)) \cap W^{1,p_2}(S)$ as the unique solution of the Dirichlet problem

$$\begin{cases} \displaystyle\int_S a(x_M, Dw)\, D\varphi\, dx = \int_S a(x_M, D\psi)\, D\varphi\, dx, & \text{for all } \varphi \in W_0^{1,p_2}(S), \\ w = u & \text{on } \partial S. \end{cases} \tag{4.29}$$

Let us notice that by the maximum principle we have $w \geq \psi$ on S since $w \geq \psi$ on ∂S.

Finally we define $z \in (u + W_0^{1,p_2}(S)) \cap W^{1,p_2}(S)$ as the unique solution of the Dirichlet problem

$$\begin{cases} \displaystyle\int_S a(x_M, Dz)\, D\varphi\, dx = 0, & \text{for all } \varphi \in W_0^{1,p_2}(S), \\ z = u & \text{on } \partial S. \end{cases} \tag{4.30}$$

The vector field $\zeta \mapsto a(x_M, \zeta)$ satisfies the following growth and coercivity conditions (with respect to the z variable)

$$c^*(\nu)\,(\mu^2 + |\zeta_1|^2 + |\zeta_2|^2)^{\frac{p_2-1}{2}}\, |\zeta_2 - \zeta_1|^2 \leq \langle a(x_M,\zeta_2) - a(x_M,\zeta_1), \zeta_2 - \zeta_1 \rangle, \tag{4.31}$$

$$|a(x_M,\zeta)| \leq L\,(1 + |\zeta|^2)^{\frac{p_2-1}{2}} \tag{4.32}$$

and

$$\nu\,|\zeta|^{p_2} \leq \langle a(x_M,\zeta), \zeta \rangle + c(L), \tag{4.33}$$

for every $\zeta, \zeta_1, \zeta_2 \in \mathbb{R}^n$ and $c^* \equiv c^*(n,\gamma_1,\gamma_2,\nu) > 0$.

By the theory for degenerate elliptic equations, for z the following estimate holds true (for more details we refer to [2], estimate (64), together with the reference therein)

$$\sup_{\frac{3}{2}\tilde{Q}} (\mu^2 + |Dz|^2)^{\frac{p_2}{2}} \leq c(n,\gamma_1,\gamma_2,\nu,L) \fint_S (\mu^2 + |Dz|^2)^{\frac{p_2}{2}}\, dx. \tag{4.34}$$

Let us test (4.30) with $\varphi = z - u$. Using (4.32) and (4.33) we get

$$\begin{aligned} \nu \int_S |Dz|^{p_2}\, dx &\leq c \int_S \left(\langle a(x_M, Dz), Dz\rangle + 1\right)\, dx \\ &= c \int_S \langle (a(x_M, Dz), Du\rangle + 1)\, dx \\ &\leq c \int_S \big((1+|Dz|)^{p_2-1}\,|Du| + 1\big)\, dx. \end{aligned}$$

Now, averaging, observing that $\gamma_1 \leq p_2 \leq \gamma_2$ and applying Young's inequality we conclude that

$$\fint_S |Dz|^{p_2}\, dx \leq c \fint_S (|Du|^{p_2} + 1)\, dx, \tag{4.35}$$

with a constant $c \equiv c(n, \gamma_1, \gamma_2, \nu, L)$. Together with (4.34) and (4.25) this gives

$$\sup_{\frac{3}{2}\tilde{Q}} (\mu^2 + |Dz|^2)^{\frac{p_2}{2}} \leq c_1 K^{\frac{\sigma}{4}} \lambda. \tag{4.36}$$

On the other hand, testing (4.29) with $\varphi = w - u$ and using again (4.32), (4.33) and Young's inequality, we deduce

$$\begin{aligned} \nu \int_S |Dw|^{p_2}\, dx &\leq c \int_S \left(\langle a(x_M, Dw), Dw\rangle + 1\right)\, dx \\ &= \int_S c\left(\langle a(x_M, Dw), Du\rangle + 1\right)\, dx + c\int_S \langle (a(x_M, D\psi), Dw - Du\rangle + 1)\, dx \\ &\leq \frac{\nu}{2} \int_S |Dw|^{p_2}\, dx + c\int_S |Du|^{p_2}\, dx + c\int_S (|D\psi|^{p_2} + 1)\, dx \end{aligned}$$

which gives by averaging

$$\fint_S |Dw|^{p_2}\, dx \leq c \fint_S |Du|^{p_2}\, dx + c \fint_S (|D\psi|^{p_2} + 1)\, dx, \tag{4.37}$$

with constants $c \equiv c(n, \gamma_1, \gamma_2, \nu, L)$. Finally, exploiting (4.28), we deduce

$$\begin{aligned} \nu \int_S |Dv|^{p_2}\, dx &\leq c \int_S \left(\langle a(x_M, Dv), Dv\rangle + 1\right)\, dx \\ &= c \int_S \left(\langle a(x_M, Dv), Dv - Du\rangle + 1\right)\, dx + c\int_S \left(\langle a(x_M, Dv), Du\rangle + 1\right)\, dx \\ &\leq \frac{\nu}{2} \int_S |Dv|^{p_2}\, dx + c\int_S (|Du|^{p_2} + 1)\, dx, \end{aligned}$$

which gives

$$\fint_S |Dv|^{p_2}\, dx \leq c \fint_S (|Du|^{p_2} + 1)\, dx, \tag{4.38}$$

with a constant $c \equiv c(n, \gamma_1, \gamma_2, \nu, L)$.

COMPARISON ESTIMATES.

We now establish the following comparison estimates

$$
\begin{aligned}
I &:= \int_S (\mu^2 + |Dw|^2 + |Dz|^2)^{\frac{p_2-2}{2}} |Dw - Dz|^2 \, dx \le c\, K^{\frac{\sigma}{4}}\, \tilde{\varepsilon}^{\frac{p_2-1}{p_2}}\, s^n\, \lambda, \\
II &:= \int_S (\mu^2 + |Dv|^2 + |Dw|^2)^{\frac{p_2-2}{2}} |Dv - Dw|^2 \, dx \le c\, K^{\frac{\sigma}{4}}\, \tilde{\varepsilon}^{\frac{p_2-1}{p_2}}\, s^n\, \lambda, \\
III &:= \int_S (\mu^2 + |Du|^2 + |Dv|^2)^{\frac{p_2-2}{2}} |Du - Dv|^2 \, dx \\
&\qquad \le c\, \omega(s)\, \log\left(\frac{1}{s}\right) K^{\sigma}\, s^n\, \lambda + c\, \omega(s)\, \tilde{\sigma}^{-1}\, K^{\sigma}\, s^n\, \lambda,
\end{aligned}
\tag{4.39}
$$

with constants $c \equiv c(n, \gamma_1, \gamma_2, \nu, L)$.

To prove these estimates we first use the continuity of p together with the localization in terms of (4.23) and (4.24), to control the p_2 energy of the obstacle funtion ψ on the set S. Applying Hölder's inequality, exploiting (4.23) and (4.24) and finally inserting (4.13) and the smallness of the radius R_0 in terms of (4.12), we deduce

$$
\begin{aligned}
\fint_S |D\psi|^{p_2} dx &\le c \left(\fint_S |D\psi|^{p_2(1+\sigma/4)} dx \right)^{\frac{1}{1+\sigma/4}} \\
&\le c \left(\fint_S |D\psi|^{p(x)(1+\sigma/4+\omega(s))} dx \right)^{\frac{1}{1+\sigma/4}} \\
&\le c \left(\int_S |D\psi|^{p(x)(1+\sigma/4+\omega(s))} dx \right)^{\frac{\omega(s)}{(1+\sigma/4)(1+\sigma/4+\omega(s))}} \\
&\qquad \times s^{\frac{-n\omega(s)}{(1+\sigma/4)(1+\sigma/4+\omega(s))}} \left(\fint_S |D\psi|^{p(x)(1+\sigma/4+\omega(s))} dx \right)^{\frac{1}{1+\sigma/4+\omega(s)}} \\
&\le c K^{\frac{\sigma}{4}} \left(\fint_S |D\psi|^{p(x)(1+\sigma)} dx \right)^{\frac{1}{1+\sigma}},
\end{aligned}
\tag{4.40}
$$

where $c \equiv c(n, L)$. Again we used the fact that $s^{-n\,\omega(s)}$ stays bounded as $0 < s < 8nR_0$ by (4.1) and we used (4.12) to apply Hölder's inequality in the last estimate.

At this point, exploiting (4.31), (4.30) and (4.29), the growth (4.32) and Hölder's inequality, we deduce

$$
\begin{aligned}
c^* I &\le \int_S \langle a(x_M, Dw) - a(x_M, Dz), Dw - Dz \rangle \, dx \\
&= \int_S \langle a(x_M, Dw), Dw - Dz \rangle \, dx \\
&= \int_S \langle a(x_M, D\psi), Dw - Dz \rangle \, dx \\
&\le c \int_S (|D\psi|^{p_2-1} + 1)\, |Dw - Dz| \, dx \\
&\le c\, s^n \left(\fint_S (|D\psi|^{p_2-1} + 1)^{\frac{p_2}{p_2-1}} \, dx \right)^{\frac{p_2-1}{p_2}} \left(\fint_S |Dw - Dz|^{p_2} \right)^{\frac{1}{p_2}} \\
&\le c\, s^n \left(\fint_S (|D\psi|^{p_2} + 1) \, dx \right)^{\frac{p_2-1}{p_2}} \left(\fint_S (|Dw|^{p_2} + |Dz|^{p_2}) \, dx \right)^{\frac{1}{p_2}}.
\end{aligned}
$$

Considering (4.35) and (4.37), we estimate the second integral according to

$$
\fint_S (|Dw|^{p_2} + |Dz|^{p_2}) \, dx \le \fint_S (|Du|^{p_2} + 1) \, dx + \fint_S (|D\psi|^{p_2} + 1) \, dx,
$$

Taking now use of the energy estimate (4.40) for ψ and finally exploiting extimate (4.25) for the p_2 energy of u and (4.20), we conclude

$$\begin{aligned}
c^* I &\le c\, s^n \left(\fint_S (|D\psi|^{p_2}+1)\; dx \right) + c\, s^n \left(\fint_S (|D\psi|^{p_2}+1)\; dx \right)^{\frac{p_2-1}{p_2}} \left(\fint_S |Du|^{p_2}+1\, dx \right)^{\frac{1}{p_2}} \\
&\le c\, s^n\, K^{\frac{\sigma}{4}} \left(\fint_S (|D\psi|^{p(x)}+1)^{(1+\sigma)}\, dx \right)^{\frac{1}{1+\sigma}} \\
&\qquad + c\, s^n\, K^{\frac{\sigma}{4}\,\frac{p_2-1}{p_2}} \left(\fint_S (|D\psi|^{p(x)}+1)^{(1+\sigma)}\, dx \right)^{\frac{p_2-1}{p_2(1+\sigma)}} \left(\fint_S (|Du|^{p_2}+1)\; dx \right)^{\frac{1}{p_2}} \\
&\le c\, s^n\, K^{\frac{\sigma}{4}}\, \tilde{\varepsilon}\, \lambda + c\, s^n\, K^{\frac{\sigma}{4}}\, \lambda\, \tilde{\varepsilon}^{\frac{p_2-1}{p_2}} \\
&\le c(n,\gamma_1,\gamma_2,\nu,L)\, K^{\frac{\sigma}{4}}\, \tilde{\varepsilon}^{\frac{p_2-1}{p_2}}\; s^n\, \lambda.
\end{aligned}$$

On the other hand, working as we did to estimate I, first exploiting (4.28),(4.29) and (4.32), then (4.37) and (4.38) and finally as before (4.40), (4.20) and (4.25) we deduce

$$\begin{aligned}
c^* II &\le \int_S \langle a(x_M, Dv) - a(x_M, Dw), Dv - Dw \rangle\, dx \\
&\le \int_S \langle a(x_M, Dw), Dw - Dv \rangle\, dx \\
&= \int_S \langle a(x_M, D\psi), Dw - Dv \rangle\, dx \\
&\le c \int_S (|D\psi|^{p_2-1}+1)\, |Dw - Dv|\, dx \\
&\le c\, s^n \left(\fint_S (|D\psi|^{p_2}+1)\; dx \right) + c\, s^n \left(\fint_S (|D\psi|^{p_2}+1)\; dx \right)^{\frac{p_2-1}{p_2}} \left(\fint_S (|Du|^{p_2}+1)\; dx \right)^{\frac{1}{p_2}} \\
&\le c\, s^n\, K^{\frac{\sigma}{4}} \left(\fint_S (|D\psi|^{p(x)}+1)^{(1+\sigma)}\, dx \right)^{\frac{1}{1+\sigma}} \\
&\le c(n,\gamma_1,\gamma_2,\nu,L)\, K^{\frac{\sigma}{4}}\, \tilde{\varepsilon}^{\frac{p_2-1}{p_2}}\; s^n\, \lambda.
\end{aligned}$$

We start estimating III by (4.28), (2.10), obtaining by Hölder's inequality and (4.38)

$$\begin{aligned}
c^* III &\le \int_S \langle a(x_M, Du) - a(x_M, Dv), Du - Dv \rangle\, dx \\
&\le \int_S \langle a(x_M, Du) - a(x, Du), Du - Dv \rangle\, dx + \int_S \langle a(x, Du), Du - Dv \rangle\, dx \\
&\le \int_S \langle a(x_M, Du) - a(x, Du), Du - Dv \rangle\, dx \\
&\le c\, \omega(s) \int_S (\mu + |Du|)^{p_2-1}\; |\log(\mu + |Du|)|\; |Du - Dv|\, dx \\
&\le c\, \omega(s) \left(\int_S (\mu + |Du|)^{p_2}\; |\log(\mu + |Du|)|^{\frac{p_2}{p_2-1}}\; dx \right)^{\frac{p_2-1}{p_2}} \left(\int_S |Du - Dv|^{p_2}\, dx \right)^{\frac{1}{p_2}} \\
&\le c\, \omega(s) \left(\int_S (\mu + |Du|)^{p_2}\; |\log(\mu + |Du|)|^{\frac{p_2}{p_2-1}}\; dx \right)^{\frac{p_2-1}{p_2}} \left(\int_S |Du|^{p_2} + 1\, dx \right)^{\frac{1}{p_2}}.
\end{aligned}$$

To estimate the first integral on the right hand side we proceed as in [2]. For the convienience of the reader we restate the main arguments, using the well established tools from the theory of the spaces $L\log^{\beta} L$ (for details see [30], [31], [4]). Setting

$$\beta := \frac{p_2}{p_2-1} \in \left[\frac{\gamma_2}{\gamma_2-1}, \frac{\gamma_1}{\gamma_1-1}\right],$$

and noting that for such β and for $\gamma_1 \le p_2 \le \gamma_2$ we have

$$t^{p_2}|\log t|^{\beta} \le c(\gamma_1,\gamma_2) \quad \text{for any} \quad 0<t<e+1,$$

we obtain by decomposing S into $S^- := \{x \in S : |Du| < e\}$ and $S^+ := \{x \in S : |Du| \ge e\}$ and proceeding exactly as in [2], page 136

$$\int_{S^-} (\mu+|Du|)^{p_2} \,|\log|^{\beta}\,(\mu+|Du|)\, dx \le c(\gamma_1,\gamma_2)|S| \overset{(\lambda\ge 1)}{\le} cs^n\lambda. \tag{4.41}$$

On the set S^+ the estimates are a little more involved: exploiting $\mu \le e \le |Du|$, and using the elementary estimate (being a direct consequence of the concavity of the logarithm)

$$\log^{\beta}(e+ab) \le 2^{\frac{\gamma_1}{\gamma_1-1}-1}\left(\log^{\beta}(e+a)+\log^{\beta}(e+b)\right),$$

for all $a,b>0$ and any $\beta \le \frac{\gamma_1}{\gamma_1-1}$, by we split as follows:

$$\begin{aligned}\int_{S^+} (\mu+|Du|)^{p_2}\,|\log(\mu+|Du|)|^{\beta}\, dx &\le 2^{p_2}\int_{S^+} |Du|^{p_2}\log^{\beta}(e+|Du|^{p_2})\, dx \\ &\le cs^n \fint_S |Du|^{p_2}\log^{\beta}\left(e+|||Du|^{p_2}||_{L^1(S)}\right) dx \\ &\quad + cs^n \fint_S |Du|^{p_2}\log^{\beta}\left(e+\frac{|Du|^{p_2}}{|||Du|^{p_2}||_{L^1(S)}}\right) dx.\end{aligned} \tag{4.42}$$

The first integral on the right hand side of (4.42) can be treated by Lemma 3.3, (4.10), (4.12) and finally (4.25) and (4.27):

$$\begin{aligned}&cs^n \fint_S |Du|^{p_2}\log^{\beta}\left(e+|||Du|^{p_2}||_{L^1(S)}\right) dx \\ &\quad\le c\log^{\beta}\left(s^{-n}e+s^{-n}\int_S |Du|^{p_2}\,dx\right)\int_S |Du|^{p_2}\,dx \\ &\quad\le c\int_S |Du|^{p_2}\,dx\log^{\beta}\left(e+\int_S |Du|^{p_2}\,dx\right)+c\log^{\beta}\left(\frac{1}{s}\right)\int_S |Du|^{p_2}\,dx \\ &\quad\le c(\gamma_1,\gamma_2)\sigma^{-\beta}\left(1+\int_S |Du|^{p_2}\,dx\right)^{\sigma/4}\int_S |Du|^{p_2}\,dx + c\log^{\beta}\left(\frac{1}{s}\right)\int_S |Du|^{p_2}\,dx \\ &\quad\le c\tilde{\sigma}^{-\beta}K^{\frac{2q\omega(2nR_0)}{\gamma_1-1}}s^n\left(1+\int_S |Du|^{p_2}\,dx\right)^{\sigma/4}\fint_S |Du|^{p_2}\,dx + c\log^{\beta}\left(\frac{1}{s}\right)s^n\fint_S |Du|^{p_2}\,dx \\ &\quad\le c\tilde{\sigma}^{-\beta}K^{\sigma/4}\left(1+K^{1+\sigma/4}\right)^{\sigma/4}K^{\sigma/4}\lambda s^n + c\log^{\beta}\left(\frac{1}{2}\right)K^{\sigma/4}\lambda s^n \\ &\quad\le c\tilde{\sigma}^{-\beta}K^{\sigma}\lambda s^n + c\log^{\beta}\left(\frac{1}{s}\right)K^{\sigma/4}\lambda s^n.\end{aligned}$$

The second integral on the right hand side of (4.42) is handled via Lemma 3.2, (4.23), Theorem 4.1, (4.40), again (4.12), (4.10) and finally (4.20):

$$
\begin{aligned}
cs^n \fint_S |Du|^{p_2} \log^\beta \left(e + \frac{|Du|^{p_2}}{|||Du|^{p_2}||_{L^1(S)}} \right) dx
&\le c\sigma^{-\beta} s^n \left(\fint_S |Du|^{p_2(1+\sigma/4)}\, dx \right)^{\frac{1}{1+\sigma/4}} \\
&\le c\sigma^{-\beta} s^n + c\sigma^{-\beta} s^n \left(\fint_S |Du|^{p(x)(1+\sigma/4+\omega(s))}\, dx \right)^{\frac{1}{1+\sigma/4}} \\
&\le c(q)\, \tilde\sigma^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1-1}}\, s^n \left(\fint_{2S} |Du|^{p(x)}\, dx \right)^{\frac{1+\sigma/4+\omega(s)}{1+\sigma/4}} \\
&\quad + c(q)\, \tilde\sigma^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1-1}}\, s^n \left(\fint_{2S} |D\psi|^{p(x)(1+\sigma/4+\omega(s))}\, dx \right)^{\frac{1}{1+\sigma/4}} \\
&\quad + c(q)\, \tilde\sigma^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1-1}}\, s^n \\
&\le c\, \tilde\sigma^{-\beta} K^{\frac{\sigma}{4}}\, s^{-n\omega(s)} \left(\int_{2S} |Du|^{p(x)}\, dx \right)^{\frac{\omega(s)}{1+\sigma/4}} \int_{2S} |Du|^{p(x)}\, dx \\
&\quad + c\, \tilde\sigma^{-\beta} K^{\frac{\sigma}{4}}\, s^n \left(\fint_{2S} |D\psi|^{p(x)(1+\sigma)}\, dx \right)^{\frac{1}{1+\sigma}} \\
&\quad + c\, \tilde\sigma^{-\beta} K^{\frac{\sigma}{4}}\, s^n \\
&\le c(n, \gamma_1, \gamma_2, \nu, L, q)\, \tilde\sigma^{-\beta} K^{\sigma}\, s^n\, \lambda.
\end{aligned}
$$

Taking the estimates on S^- and S^+ together we arrive at

$$
\int_S (\mu + |Du|)^{p_2} \left|\log(\mu + |Du|)\right|^\beta\, dx \le cs^n\lambda + c\log^\beta\left(\frac{1}{s}\right) K^{\sigma/4} s^n \lambda + c\tilde\sigma^{-\beta} K^\sigma\, s^n\, \lambda
$$

Thus, again exploiting (4.25), we deduce

$$
\begin{aligned}
c^* III &\le c\,\omega(s) \left(\int_S (\mu + |Du|)^{p_2} |\log|^\beta (\mu + |Du|)\, dx \right)^{\frac{1}{\beta}} \left(\int_S |Du|^{p_2} + 1\, dx \right)^{\frac{1}{p_2}} \\
&\le c\,\omega(s) \left[\log^\beta\left(\frac{1}{s}\right) K^\sigma\, s^n\, \lambda + \tilde\sigma^{-\beta} K^\sigma\, s^n\, \lambda \right]^{\frac{1}{\beta}} \left[s^n (K^{\frac{\sigma}{4}}\, \lambda + 1) \right]^{\frac{1}{p_2}} \\
&\le c\,\omega(s) \left[\log\left(\frac{1}{s}\right) K^{\frac{\sigma}{\beta}}\, s^{\frac{n}{\beta}}\, \lambda^{\frac{1}{\beta}} + \tilde\sigma^{-1} K^{\frac{\sigma}{\beta}}\, s^{\frac{n}{\beta}}\, \lambda^{\frac{1}{\beta}} \right] \left[s^{\frac{n}{p_2}}\, K^{\frac{\sigma}{p_2}}\, \lambda^{\frac{1}{p_2}} \right] \\
&\le c\,\omega(s) \log\left(\frac{1}{s}\right) K^\sigma\, s^n\, \lambda + c\,\omega(s)\, \tilde\sigma^{-1} K^\sigma\, s^n\, \lambda,
\end{aligned}
$$

where $c \equiv c(n, \gamma_1, \gamma_2, \nu, L)$.

These are the desired comparison estimates (4.39).

STEP 4: ESTIMATES OF THE MAXIMAL FUNCTION ON LEVEL SETS. At this point of the proof we combine the a priori estimate for the solution of the frozen problem with the comparison estimates in order to estimate the super level sets of the maximal function of $|Du|^{p_2}$ on increasing levels.

We define the restricted maximal function to the cube $\frac{3}{2}\tilde Q$ by

$$
M^{**} := M^*_{\frac{3}{2}\tilde Q},
$$

whereas

$$
M^* := M^*_{Q_{4R_0}}
$$

denotes the maximal function on Q_{4R_0} (see the statement of Lemma 4.2).

We would now like to estimate the measure of the set

$$\Big\{x \in Q \,:\, M^{**}(|Du|^{p_2})(x) > CK^\sigma\lambda,\, M^*_{1+\sigma}(|D\psi(\cdot)|^{p(\cdot)}+1)(x) < \tilde\varepsilon\lambda\Big\},$$

where C will be chosen later.

At this point, using repeatedly Lemma 3.4, we deduce

$$\begin{aligned}(\mu^2+|Du|^2)^{\frac{p_2}{2}} &\leq c_3(\mu^2+|Dz|^2)^{\frac{p_2}{2}} + c_3\,(\mu^2+|Dw|^2+|Dz|^2)^{\frac{p_2-2}{2}}\,|Dw-Dz|^2\\ &\qquad + c_3\,(\mu^2+|Dv|^2+|Dw|^2)^{\frac{p_2-2}{2}}\,|Dv-Dw|^2\\ &\qquad + c_3\,(\mu^2+|Du|^2+|Dv|^2)^{\frac{p_2-2}{2}}\,|Du-Dv|^2\\ &=: c_3(\mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3+\mathcal{G}_4),\end{aligned}$$

with $c_3 \equiv c_3(n,\gamma_1,\gamma_2)$ and the obvious labelling of $\mathcal{G}_1$ to $\mathcal{G}_4$. Therefore we immediately have

$$\begin{aligned}&\Big|\Big\{x\in Q : M^{**}(|Du|^{p_2})(x) > AK^\sigma\lambda,\ M^*_{1+\sigma}(|D\psi(\cdot)|^{p(\cdot)}+1)(x)<\tilde\varepsilon\lambda\Big\}\Big|\\ &\leq \Big|\Big\{x\in Q : M^{**}(\mathcal{G}_1)(x) > \tfrac{AK^\sigma\lambda}{4c_3}\Big\}\Big| + \Big|\Big\{x\in Q : M^{**}(\mathcal{G}_2)(x) > \tfrac{AK^\sigma\lambda}{4c_3}\Big\}\Big|\\ &\quad + \Big|\Big\{x\in Q : M^{**}(\mathcal{G}_3)(x) > \tfrac{AK^\sigma\lambda}{4c_3}\Big\}\Big| + \Big|\Big\{x\in Q : M^{**}(\mathcal{G}_4)(x) > \tfrac{AK^\sigma\lambda}{4c_3}\Big\}\Big|\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

Estimate for I_1: By (4.36) we deduce

$$M^{**}(\mathcal{G}_1)(x) \leq c_1K^\sigma\lambda, \qquad \forall x\in \tfrac{3}{2}\tilde Q,$$

and therefore

$$I_1 = 0.$$

Estimate for I_2: We use estimate (3.1) for the maximal function, the comparison estimate (4.39a) and the inclusion $\frac{3}{2}\tilde Q \subset S$ to conclude

$$\begin{aligned}I_2 &\leq \frac{c(n)c_3}{CK^\sigma\lambda}\int_S (\mu^2+|Dw|^2+|Dz|^2)^{\frac{p_2-2}{2}}|Dw-Dz|^2\,dx\\ &\leq \frac{c}{CK^\sigma\lambda}c_3K^{\sigma/4}\tilde\varepsilon^{\frac{\gamma_2-1}{\gamma_2}}s^n\lambda\\ &\leq c_4\tilde\varepsilon^{\frac{\gamma_2-1}{\gamma_2}}|Q|.\end{aligned}$$

Estimate for I_3: Again by (3.1) and by (4.39b) we deduce

$$I_3 \leq c_4\tilde\varepsilon^{\frac{\gamma_2-1}{\gamma_2}}|Q|.$$

Estimate for I_4: We use (4.39c) together with (3.1) to obtain

$$\begin{aligned}I_4 &\leq \frac{c(n)c_3}{CK^\sigma\lambda}\int_S (\mu^2+|Du|^2+|Dv|^2)^{\frac{p_2-2}{2}}|Du-Dv|^2\,dx\\ &\leq \frac{c(n)c_3}{CK^\sigma\lambda}c\,\omega(s)\,\log\left(\frac{1}{s}\right)K^\sigma s^n\lambda + c\,\omega(s)\,\tilde\sigma^{-1}K^\sigma s^n\lambda\\ &\leq c_5\omega(s)\log\left(\tfrac{1}{s}\right)|Q| + c_6\omega(s)\tilde\sigma^{-1}|Q|.\end{aligned}$$

So alltogether we conclude

$$\begin{aligned}&\Big|\Big\{x\in Q : M^{**}(|Du|^{p_2})(x) > CK^\sigma\lambda,\, M^*_{1+\sigma}(|D\psi(\cdot)|^{p(\cdot)}+1)(x)<\tilde\varepsilon\lambda\Big\}\Big|\\ &\qquad\leq \Big[c_4\tilde\varepsilon^{\frac{\gamma_2-1}{\gamma_2}} + c_6\omega(s)\log\left(\tfrac{1}{s}\right) + c_6\omega(s)\tilde\sigma^{-1}\Big]\,|Q|.\end{aligned}\tag{4.43}$$

Now we come to the appropriate choice of constants and radii: let δ_1 be given as in the statement. We choose the radius $R_1 \equiv R_1(n, \gamma_1, \gamma_2, L/\nu, q, \omega(\cdot), \tilde{\sigma}, \delta_1)$ small enough to have

$$c_6\omega(s)\log\left(\frac{1}{s}\right) \le \frac{\delta_1}{8}, \qquad c_6\omega(s) \le \frac{\delta_1\tilde{\sigma}}{8}, \qquad \text{for any } x \le 8nR_1. \tag{4.44}$$

Then if $R_0 \le R_1$ satisfies (4.1) and (4.12), then

$$R_0 \equiv R_0(n, \gamma_1, \gamma_2, L/\nu, q, \omega(\cdot), \tilde{\sigma}, \delta_1, |||Du(\cdot)|^{p(\cdot)}||_{L^1(\Omega)}, |||D\psi(\cdot)|^{p(\cdot)}||_{L^q(\Omega)}). \tag{4.45}$$

Then for any $\delta \ge \delta_1$ we have

$$c_6\omega(s)\log\left(\frac{1}{s}\right) \le \frac{\delta}{8}, \qquad c_6\omega(s) \le \frac{\delta\tilde{\sigma}}{8}, \qquad \text{for any } s \le 8nR_0.$$

Next we choose $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(n, \gamma_1, \gamma_2, L, \nu, \delta)$ in such a way that

$$c_4\tilde{\varepsilon}^{\frac{\gamma_2-1}{\gamma_2}} \le \frac{\delta}{8}. \tag{4.46}$$

Thus with the above choices we obtain

$$\left|\left\{x \in Q : M^{**}(|Du(\cdot)|^{p_2})(x) > CK^\sigma\lambda,\ M^*(|D\psi(\cdot)|^{p(\cdot)}+1)(x) < \tilde{\varepsilon}\lambda\right\}\right| \le \frac{\delta}{2}|Q|. \tag{4.47}$$

We now turn this estimate for the maximal function of $|Du|^{p_2}$ into an estimate for $|Du|^{p(\cdot)}$. Therefore we find that, since $p_2 \ge p(x)$ for any $x \in 2\tilde{Q}$, we have for any cube $Q \subset \frac{3}{2}\tilde{Q}$ the estimate

$$⨍_Q |Du(x)|^{p(x)}\,dx ⨍_Q |Du(x)|^{p_2}\,dx + 1$$

holds. Hence for $x \in Q$ we have

$$M^{**}(|Du(\cdot)|^{p(\cdot)})(x) \le M^{**}(|Du(\cdot)|^{p_2}+1)(x).$$

Since $C, K^\sigma, \lambda > 1$, we have in particular that $CK^\sigma\lambda > 1$ and therefore

$$M^{**}(|Du(\cdot)|^{p(\cdot)})(x) > AK^\sigma\lambda$$

implies

$$M^{**}(|Du(\cdot)|^{p_2})(x) + CK^\sigma\lambda \ge M^{**}(|Du(\cdot)|^{p_2}+1)(x) > 2CK^\sigma\lambda = AK^\sigma\lambda.$$

By (4.47) we therefore obtain

$$\left|\left\{x \in Q : M^{**}(|Du(\cdot)|^{p(\cdot)})(x) > AK^\sigma\lambda,\ M^*(|D\psi(\cdot)|^{p(\cdot)}+1)(x) < \tilde{\varepsilon}\lambda\right\}\right| \le \frac{\delta}{2}|Q|. \tag{4.48}$$

In order to pass from the maximal function M^{**} to the restricted maximal function M^*, we argue exactly as in [2]: let ℓ be the sidelength of the cube Q. For an arbitrary point $x \in Q$ both x itself and the point x_0 chosen in (4.16) are contained in the cube $\tilde{Q}$ which has sidelength 2ℓ.
Now if $C' \subseteq Q_{4R_0}$ is a cube, containing x and having side length ℓ' larger than $\ell/2$, there holds $C' \cap \tilde{Q} \neq \emptyset$. Thus there exists a cube $C'' \subseteq Q_{4R_0}$, containing C' and $\tilde{Q}$, and whose side length ℓ'' is bounded by

$$\ell'' \le 2\ell + \ell' \le 5\ell'.$$

Therefore, by (4.16) there holds

$$⨍_{C'} |Du(x)|^{p(x)}\,dx \le \frac{1}{|C'|} ⨍_{C''} |Du(x)|^{p(x)}\,dx \le \frac{|C''|}{|C'|}\lambda \le 5^n\lambda,$$

while in the case $\ell' \le \frac{\ell}{2}$, we have $C' \subset \frac{3}{2}\tilde{Q}$ and

$$⨍_{C'} |Du(x)|^{p(x)}\,dx \le M^{**}(|Du(\cdot)|^{p(\cdot)})(x).$$

This implies that

$$M^*(|Du(\cdot)|^{p(\cdot)})(x) \le \max\left\{M^{**}(|Du(\cdot)|^{p(\cdot)})(x), 5^n\lambda\right\} \qquad \text{for all } \ x \in Q.$$

¿From the choice of C we infer that $CK^\sigma \geq 5^{n+1}$.

$$\Big\{x \in Q : M^*(|Du(\cdot)|^{p(\cdot)})(x) > AK^\sigma\lambda\Big\} \subseteq \Big\{x \in Q : M^{**}(|Du(\cdot)|^{p(\cdot)})(x) > AK^\sigma\lambda\Big\},$$

and therefore

$$|\{x \in Q : M^*(|Du(\cdot)|^{p(\cdot)})(x) > AK^\sigma\lambda, M^*_{1+\sigma}(|D\psi(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda\}| \leq \tfrac{\delta}{2}|Q|.$$

This contradicts (4.14) and completes the proof of Lemma 4.2. □

Proof of Theorem 2.3. We now take use of Lemma 4.2 to prove the main theorem of this paper. The result of Lemma 4.2 provides the hypothesis for Lemma 3.1, applied on the sets which appear in the definition of μ_1 and μ_2. This is the key to the proof of the desired higher integrability result. Although the following procedure is more or less standard in Calderón-Zagmund theory, for the conveinience of the reader we sketch the main steps. See [2, pp 141-146] for a more detailed argumentation.

We start by defining the quantities

$$\begin{aligned} \mu_1(t) &:= \left|\Big\{x \in Q_{R_0} : M^*(|Du(\cdot)|^{p(\cdot)})(x) > t\Big\}\right|, \\ \mu_2(t) &:= \left|\Big\{x \in Q_{R_0} : M^*_{1+\sigma}(|D\psi(\cdot)|^{p(\cdot)} + 1)(x) > t\Big\}\right|, \end{aligned} \tag{4.49}$$

with $M^* \equiv M^*_{Q_{4R_0}}$ and $M^*_{1+\sigma} = M^*_{1+\sigma,Q_{4R_0}}$. We apply Lemma 4.2 with the choice

$$\delta_1 := \frac{1}{2A^q K_M^{\sigma_M q}},$$

where K_M, σ_M are the constants defined in (4.7) and (4.8). We fix R_1 as in (4.44). Taking, as done in (4.45), the greatest number $R_0 \leq R_1$, satisfying (4.1) and (4.12), also R_0 is fixed with the dependencies of (4.45). With the definition of K_0 in (4.2), this fixes σ_0 in (4.3). We set $\sigma := \tilde{\sigma}\sigma_0$ and K as in (4.13).

Now we define

$$\delta := \frac{1}{2A^q K^{\sigma q}}. \tag{4.50}$$

We define

$$\lambda_0 := \frac{5^{n+2}c_W}{\delta} \fint_{Q_{4R_0}} |Du(x)|^{p(x)}\, dx + 1. \tag{4.51}$$

By (3.1) we have

$$\mu_1(\lambda_0) \leq \frac{c_W}{\lambda_0}|Q_{4R_0}| \fint_{Q_{4R_0}} |Du(x)|^{p(x)}\, dx \leq \frac{4^n|Q_{R_0}|\delta}{5^{n+2}} \leq \frac{\delta}{2}|Q_{R_0}|. \tag{4.52}$$

In a second step, we eploit the results of Lemma 4.2 to deduce the following estimate

$$\mu_1\left((AK^\sigma)^{h+1}\lambda_0\right) \leq \frac{1}{2(AK^\sigma)^q}\mu_1\left((AK^\sigma)^h\lambda_0\right) + \mu_2\left(\tilde{\varepsilon}(AK^\sigma)^h\lambda_0\right), \tag{4.53}$$

where the quantity $\tilde{\varepsilon}$ is the quantity appearing in Lemma 4.2. Let us note that $\tilde{\varepsilon}$ does not depend on h.

To prove the preceding estimate, we define the sets

$$\begin{aligned} \mathcal{X} := \Big\{x \in Q_{R_0} : M^*\left(|Du(\cdot)|^{p(\cdot)}\right)(x) > (AK^\sigma)^{h+1}\lambda_0, \\ M^*_{1+\sigma}\left(|D\psi(\cdot)|^{p(\cdot)} + 1\right)(x) < \tilde{\varepsilon}A^h\lambda_0\Big\}, \end{aligned} \tag{4.54}$$

and

$$\mathcal{Y} := \left\{ x \in Q_{R_0} : M^* \left(|Du(\cdot)|^{p(\cdot)} \right)(x) > (AK^\sigma)^h \lambda_0 \right\}. \tag{4.55}$$

Taking into account (4.53) and the fact that $AK^\sigma > 1$, we see that

$$|\mathcal{X}| \leq \frac{\delta}{2} |Q_{R_0}|.$$

Let $Q \subset Q_{R_0}$ be a dyadic subcube with

$$|\mathcal{X} \cap Q| > \delta |Q|,$$

we apply Lemma 4.2 with the choice $\lambda := (AK^\sigma)^h \lambda_0 \geq 1$ to conclude that the predecessor $\tilde{Q}$ of Q satisfies

$$\tilde{Q} \subset \left\{ x \in Q_{R_0} :\ M^* \left(|Du(\cdot)|^{p(\cdot)} \right)(x) > (AK^\sigma)^h \lambda_0 \right\}.$$

At this stage Lemma 3.1 shows

$$|\mathcal{X}| < \delta |\mathcal{Y}|,$$

which translates into the desired estimate (4.53).

We note that inequality (4.53) holds for any $h \in \mathbb{N} \cup \{0\}$ Iterating the estimate directly gives

$$\mu_1 \left((AK^\sigma)^{h+1} \lambda_0 \right) \leq \left(\frac{1}{2(AK^\sigma)^q} \right)^{h+1} \mu_1(\lambda_0) + \sum_{i=0}^{h} \left(2(AK^\sigma)^q \right)^{-(h-i)} \mu_2 \left(\tilde{\varepsilon} (AK^\sigma)^i \lambda_0 \right).$$

Therefore for $J \in \mathbb{N}$ arbitrary we have

$$\begin{aligned}
&\sum_{h=0}^{J} (AK^\sigma)^{q(h+1)} \mu_1 \left((AK^\sigma)^{h+1} \lambda_0 \right) \\
&\quad \leq \sum_{h=0}^{J} 2^{-(h+1)} \mu_1(\lambda_0) + \sum_{h=0}^{J} \sum_{i=0}^{h} (AK^\sigma)^{q(i+1)} 2^{-(h-i)} \mu_2 \left(\tilde{\varepsilon} (AK^\sigma)^i \lambda_0 \right) \\
&\quad \leq \mu_1(\lambda_0) + (A),
\end{aligned}$$

with the obvious labelling of (A). Interchanging the order or summation in (A) and, exploiting the geometric series, we deduce

$$(A) \leq 2(AK^\sigma)^q \sum_{i=0}^{J} (AK^\sigma)^{qi} \mu_2 \left(\tilde{\varepsilon} (AK^\sigma)^i \lambda_0 \right).$$

Passing to the limit provides

$$\sum_{h=1}^{\infty} (AK^\sigma)^{qh} \mu_1 \left((AK^\sigma)^h \lambda_0 \right) \leq \mu_1(\lambda_0) + 2(AK^\sigma)^q \sum_{h=0}^{\infty} (AK^\sigma)^{qh} \mu_2 \left(\tilde{\varepsilon} (AK^\sigma)^h \lambda_0 \right). \tag{4.56}$$

Applying the elementary identity

$$\int_Q g^q \, dx = \int_0^\infty q \lambda^{q-1} \left| \{ x \in Q : g(x) > \lambda \} \right| d\lambda,$$

which holds for $g \in L^q(Q), g \geq 0, q \geq 1$ to the function $g \equiv M^* \left(|Du(\cdot)|^{p(\cdot)} \right)$, the preceeding estimate can be turned into an estimate for the maximal function. Decomposing the interval $[0, \infty)$ into intervals $[0, \lambda_0]$ and $[(AK^\sigma)^n \lambda_0, (AK^\sigma)^{n+1} \lambda_0]$ and exploiting (4.56) in combination with the monotonicity of the functions $\mu_1(t)$ and $\mu_2(t)$, finally using the L^p estimate for the maximal function, we calculate

$$\int_{Q_{R_0}} |Du|^{p(x)q} \, dx \leq |Q_{R_0}| \lambda_0^q + 2(AK^\sigma \lambda_0)^q \mu_1(\lambda_0) + \frac{c(n) q^2}{q-1} \left(\frac{(AK^\sigma)^2}{\tilde{\varepsilon}} \right)^q \int_{Q_{4R_0}} \left(|D\psi|^{p(x)q} + 1 \right) dx.$$

By (4.52) and the choice of δ in (4.50) the second term on the right hand side is estimated from above by $1/2|Q_{R_0}|\lambda_0^q$. Recalling the definition of λ_0 in (4.51) and the dependencies of the constant A in Lemma 4.2, we conclude that

$$\lambda_0 \le c(n,\gamma_1,\gamma_2,\nu,l)K^{\sigma q} \fint_{Q_{4R_0}} \left(|Du|^{p(x)}+1\right)\,dx.$$

On the other hand, recalling the choice of $\tilde\varepsilon$ in (4.46) and the dependencies of the constants A and c_4, we see that

$$\frac{(AK^\sigma)^2}{\tilde\varepsilon} = c(n,\gamma_1,\gamma_2,\nu,L)K^{2\sigma+\sigma q\frac{\gamma_2}{\gamma_2-1}}.$$

Taking together these estimates we deduce

$$\begin{aligned}&\left(\fint_{Q_{R_0}}|Du|^{p(x)q}\,dx\right)^{1/q}\\ &\qquad\le cK^{q\sigma}\fint_{Q_{4R_0}}\left(|Du|^{p(x)}+1\right)dx + cK^{2\sigma+\sigma q\frac{\gamma_2}{\gamma_2-1}}\left(\fint_{Q_{4R_0}}|D\psi|^{p(x)q}\,dx+1\right)^{1/q}.\end{aligned}$$

Now, for given $\varepsilon>0$ we want to have satisfied that

$$q\sigma<\varepsilon,\qquad 2\sigma<\frac{\varepsilon}{2},\quad \sigma q\frac{\gamma_2}{\gamma_2-1}<\frac{\varepsilon}{2}.$$

This can be reached by claiming that

$$\sigma<\varepsilon\min\left\{\frac14,\frac{1}{2q}\frac{\gamma_2-1}{\gamma_2}\right\}.$$

Setting

$$\bar\sigma := \frac{\varepsilon}{\sigma_M}\min\left\{\frac14,\frac{1}{2q}\frac{\gamma_2-1}{\gamma_2}\right\}\qquad\text{and}\qquad \tilde\sigma:=\min\left\{\bar\sigma,\gamma_1-1,\frac12\right\},$$

we have fixed $\tilde\sigma\equiv\tilde\sigma(n,q,\gamma_1,\gamma_2,c_g)$ and (4.9) and the above smallness condition for σ are satisfied (recall at this point also the definition of σ_M in (4.8)). In particular we have that $\sigma<\varepsilon$, which implies that (taking into account the fact that $|Q_{4R_0}|\le 1$)

$$K=\int_{Q_{4R_0}}\left(|Du|^{p(x)}+|D\psi|^{p(x)(1+\sigma)}\right)dx+1\le\int_{Q_{4R_0}}\left(|Du|^{p(x)}+|D\psi|^{p(x)(1+\varepsilon)}\right)dx+2.$$

Claiming here that $\varepsilon\le q-1$ guarantees that the right hand side is finite.

Note that by the choice of $\tilde\sigma\equiv\tilde\sigma(\varepsilon)$, also $R_1\equiv R_1(\tilde\sigma)\equiv R_1(\varepsilon)$ is fixed via Lemma 4.2, and finally also $R_0\equiv R_0(\varepsilon)$ via (4.11). Therefore for any cube Q_R with $R\le R_0, Q_{4R}\Subset\Omega$ there holds

$$\left(\fint_{Q_R}|Du|^{p(x)q}\,dx\right)^{1/q}\le cK^\varepsilon\fint_{Q_{4R}}\left(|Du|^{p(x)}+1\right)dx+cK^\varepsilon\left(\fint_{Q_{4R}}\left(|D\psi|^{p(x)q}+1\right)dx\right)^{1/q},$$

where the constant depends on $n,\gamma_1,\gamma_2,L/\nu$ and q, and where

$$K=\int_{Q_{4R}}\left(|Du|^{p(x)}+|D\psi|^{p(x)(1+\varepsilon)}\right)dx+1.$$

Therefore the statement $|Du|^{p(\cdot)}\in L^q_{loc}(\Omega)$ follows by a standard covering argument. □

References

[1] E. Acerbi, G. Mingione: *Regulairy results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal., **164** (2002), 213-259.

[2] E. Acerbi, G. Mingione: *Gradient estimates for the $p(x)$-laplacean system*, J. Reine Angew. Math., **584** (2005), 117-148.

[3] E. Acerbi, G. Mingione: *Gradient estimates for a class of parabolic systems*, Duke Math. J., **136** (2007) 285-320.

[4] K. Astala, T. Iwaniec, P. Koskela, G. Martin: *Mappings of BMO-bounded distorsion*, Math. Ann., **317** (2000), 703-726.

[5] M. Bildhauer, M. Fuchs, G. Mingione: *A priori gradient bounds and local $C^{1,\alpha}$-estimates for (double) obstacle problems under non-standard growth conditions*, Z. Anal. Anwend., **20** (4) (2001), 959-985.

[6] V. Bögelein, F. Duzaar, G. Mingione: *Degenerate problems with irregular obstacles*, Preprint (2009).

[7] L. Caffarelli, I. Peral: *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math., **51** (1998), 1-21.

[8] A.P. Calderón, A. Zygmund: *On singular integrals*, Am. J. Math., **78** (1956), 289-309.

[9] Y. Chen, S. Levine, M. Rao: *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., **66** (2006), no. 4, 1383-1406.

[10] H. J. Choe: *A regularity theory for a general class of quasilinear elliptic partial differential equations and obstacle problems*, Arch. Ration. Mech. Anal., **114** (1991), 383-394.

[11] D. Cruz-Uribe, A. Fiorenza, C.J. Neugebauer: *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. (A I Math.), **28** (2003), 223-238.

[12] G. Cupini, N. Fusco, R. Petti: *Hölder continuity of local minimizers*, J. Math. Anal. Appl., **235** (1999), 578-597.

[13] E. DiBenedetto, J. Manfredi: *On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems*, Am. J. Math., **115** (5) (1993), 1107-1134.

[14] L. Diening: *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr., **268** (2004), 31-43.

[15] L. Diening, M. Růžička: *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math., **563** (2003), 197-220.

[16] F. Duzaar, G. Mingione, K. Steffen: *Parabolic systems with polynomial growth and regularity*, Preprint (2009)

[17] D. Edmunds, J. Rákosník: *Sobolev embeddings with variable exponent*, Stud. Math., **143** No. 3 (2000), 267-293.

[18] D. Edmunds, J. Rákosník: *Sobolev embeddings with variable exponent II*, Math. Nachr., **246-27** (2002), 53-67.

[19] M. Eleuteri: *Regularity results for a class of obstacle problems*, Appl. Math., **52**, No. 2 (2007), 137-169.

[20] M. Eleuteri, J. Habermann: *Regularity results for a class of obstacle problems under non standard growth conditions*, J. Math. Anal. Appl., **344** (2), (2008), 1120 - 1142.

[21] M. Eleuteri, J. Habermann: *A Hölder continuity result for a class of obstacle problems under non standard growth conditions*, to appear in Math. Nachr.

[22] M. Fuchs, G. Mingione: *Full $C^{1,a}$-regularity for free and constrained local minimizers of integral functionals with nearly linear growth*, manus. math., **102** (2000), 227-250

[23] E. Giusti: *Direct methods in the calculus of variations*, Singapore: World Scientific, vii, (2003).

[24] J. Habermann: *Calderón-Zygmund estimates for higher order systems with $p(x)$ growth*, Math. Z., **258** (2008), no. 2, 427-462.

[25] P. Hästö: *On the variable exponent Dirichlet energy integral*, Comm. Pure Appl. Anal., **5** (2006), no. 3, 413-420.

[26] P. Harjulehto, P. Hästö, M. Koskenoja, T. Lukkari and N. Marola: *An obstacle problems and superharmonic functions with nonstandard growth*, Nonlinear Anal., **67** (2007), 3424-3440.

[27] P. Harjulehto, P. Hästö and M. Koskenoja: *Properties of capacities in variable exponent Sobolev spaces*, J. Anal. Appl., **5** (2007), no. 2, 71-92.

[28] P. Harjulehto, P. Hästö, O. Martio: *Fuglede's theorem in variable exponent Sobolev spaces*, Collet. Math., **55** No. 3 (2004), 315-324.

[29] T. Iwaniec: *Projections onto gradient fields and L^p-estimates for degenerated elliptic operators*, Stud. Math., **75** (1983), 293312.

[30] T. Iwaniec: *$p-$harmonic tensors and quasiregular mappings*, Annals of Mathematics (2) **136** (1992), 589-624.

[31] T. Iwaniec, A. Verde: *On the operator $\mathcal{L}(f) = f \log|f|$*, J. Functional Analysis, **169** (1999), 391-420.

[32] O. Kováčik, Ž. & J. Rákosník: *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 116.

[33] P. Marcellini: *Regularity and existence of solutions of elliptic equations with p, q–growth conditions*, J. Diff. Equ., **90** (1991), 1-30.

[34] G. Mingione: *The Calderón-Zygmund theory for elliptic problems with measure data*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (Ser. V) **6** (2007) 195-261.

[35] G. Mingione: *Calderón-Zygmund estimates for measure data problems*, C. R., Math., Acad. Sci. Paris, **344** (7) (2007), 437-442.

[36] J. Musielak: *O*rlicz spaces and modular spaces, Springer, Berlin, 1983.
[37] K. R. Rajagopal, M. Růžička: *Mathematical modelling of electro-rheological fluids*, Cont. Mech. Therm., **13** (2001), 59-78.
[38] E.M. Stein: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, N.J., 1993.
[39] V.V. Zhikov: *On Lavrentiev's phenomenon*, Russian J. Math. Physics, **3** (1995), 249-269.
[40] V.V. Zhikov: *On some variational problems*, Russian J. Math. Physics, **5** (1997), 105-116.

Michela Eleuteri, Dipartimento di Matematica di Trento, via Sommarive 14, 38100 Povo (Trento), Italy;

E-mail address: eleuteri@science.unitn.it

Jens Habermann, Department Mathematik, Friedrich-Alexander Universität, Bismarckstr. 1 1/2, 91054 Erlangen, Germany;

E-mail address: habermann@mi.uni-erlangen.de