REGULARITY OF THE OPTIMAL SETS FOR THE SECOND DIRICHLET EIGENVALUE

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Abstract. This paper is dedicated to the regularity of the optimal sets for the second eigenvalue of the Dirichlet Laplacian. Precisely, we prove that if the set Ω minimizes the functional

$$\mathcal{F}_\Lambda(\Omega) = \lambda_2(\Omega) + \Lambda|\Omega|,$$

among all subsets of a smooth bounded open set $D \subset \mathbb{R}^d$, where $\lambda_2(\Omega)$ is the second eigenvalue of the Dirichlet Laplacian on Ω and $\Lambda > 0$ is a fixed constant, then Ω is equivalent to the union of two disjoint open sets Ω_+ and Ω_-, which are $C^{1,\alpha}$-regular up to a (possibly empty) closed set of Hausdorff dimension at most $d-5$, contained in the one-phase free boundaries $D \cap \partial\Omega_+ \setminus \partial\Omega_-$ and $D \cap \partial\Omega_- \setminus \partial\Omega_+$.

1. Introduction

Given a real constant $\Lambda > 0$ and an open set $\Omega \subset \mathbb{R}^d$, we define

$$\mathcal{F}_\Lambda(\Omega) = \lambda_2(\Omega) + \Lambda|\Omega| \ , \tag{1}$$

where $|\Omega|$ is the Lebesgue measure of the set Ω and $\lambda_2(\Omega)$ is the second eigenvalue (counted with the due multiplicity) of the Laplace operator in Ω, with Dirichlet boundary conditions on $\partial\Omega$. Precisely, we recall the following variational characterization of the second eigenvalue:

$$\lambda_2(\Omega) = \min_{E_2 \subset H^1_0(\Omega)} \max\Big\{ \int_\Omega |\nabla u|^2\,dx \ : \ u \in E_2,\ \int_\Omega u^2\,dx = 1\Big\}, \tag{2}$$

where the minimum is taken among all two-dimensional subspaces E_2 of the Sobolev space $H^1_0(\Omega)$, which is the closure, with respect to the H^1 norm, of the space $C^\infty_c(\Omega)$ of smooth functions compactly supported in Ω.

This paper is dedicated to the regularity of the sets that minimize the functional $\mathcal{F}_\Lambda = \lambda_2 + \Lambda|\cdot|$ in a smooth bounded open set D. Shape optimization problems for functionals involving eigenvalues of the Dirichlet Laplacian have received a lot of attention lately (see Section 1.1). Since the classical result of Buttazzo and Dal Maso [9], it is known that, for any $k \geq 1$, optimal (quasi-open) sets for the functional $\lambda_k + \Lambda|\cdot|$ exist in any bounded open set $D \subset \mathbb{R}^d$. Little is known about the regularity of these optimal sets. For $k = 1$, the regularity of the free boundary (the part contained in the interior of D) was obtained by Briançon and Lamboley [4]. Recently, Kriventsov and Lin [24] proved a result which applies to this problem when $k > 1$; they showed that if the free boundary (the part inside D) is sufficiently "flat", then it must be regular. The question of what happens at general "non-flat" points is still open for $k > 1$. Our main result (Theorem 1.1) gives an answer to this question in the case $k = 2$.

Theorem 1.1. *Let $D \subset \mathbb{R}^d$ be an open bounded set of class $C^{1,\beta}$, for some $\beta > 0$, and let $\Lambda > 0$ be a given constant. Let $\Omega \subset D$ be an open set that minimizes $\mathcal{F}_\Lambda$ in D, that is,*

$$\mathcal{F}_\Lambda(\Omega) \leq \mathcal{F}_\Lambda(\widetilde{\Omega}) \quad \text{for every open set} \quad \widetilde{\Omega} \subset D. \tag{3}$$

Then, there are two disjoint open sets Ω_+ and Ω_-, both contained in Ω, such that

$$\lambda_2(\Omega_+ \cup \Omega_-) = \lambda_2(\Omega) \qquad \text{and} \qquad |\Omega \setminus (\Omega_+ \cup \Omega_-)| = 0.$$

Each of the boundaries $\partial\Omega_+$ and $\partial\Omega_-$ can be decomposed as the disjoint union of a regular and of a (possibly empty) singular part, namely

$$\partial\Omega_\pm = \mathrm{Reg}\,(\partial\Omega_\pm) \cup \mathrm{Sing}\,(\partial\Omega_\pm),$$

with the following properties:

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(i) *The regular set* $\mathrm{Reg}(\partial\Omega_\pm)$ *is an open subset of* $\partial\Omega_\pm$, *which is locally the graph of a* $C^{1,\alpha}$ *function, for some* $\alpha > 0$. *Moreover,* $\mathrm{Reg}(\partial\Omega_\pm)$ *contains both the two-phase free boundary* $\partial\Omega_+ \cap \partial\Omega_-$ *and the contact sets with the boundary of the box:* $\partial\Omega_+ \cap \partial D$ *and* $\partial\Omega_- \cap \partial D$.

(ii) *The singular set* $\mathrm{Sing}(\partial\Omega_\pm)$ *is a closed subset of* $\partial\Omega_\pm$ *and contains only one-phase points. Moreover, there exists a critical dimension* $d^* \in \{5, 6, 7\}$ *(see Remark 1.2) such that*
- *if* $d < d^*$, *then the singular set is empty,*
- *if* $d = d^*$, *then the singular set consists of a finite number of points,*
- *if* $d > d^*$, *then the singular set has Hausdorff dimension at most* $d - d^*$.

Remark 1.2. The critical dimension d^* is the lowest dimension in which there exist minimizing one-phase free boundaries with singularities. It is known that d^* is 5, 6 or 7 and conjectured that $d^* = 7$ (see [21] and the references therein).

Remark 1.3. If $\Lambda > 0$ is sufficiently big, then the disjoint union of two balls of the same radius $R_{\Lambda,d}$ is the optimal set for problem (3), thanks to the well-known Krahn-Szegö inequality for the second Dirichlet eigenvalue, see [19, Theorem 4.1.1]. On the other hand, when $\Lambda > 0$ is small, an explicit solution to (3) is not known. In this case, the existence of an open set Ω that minimizes $\mathcal{F}_\Lambda$ in D was proved in [7, Corollary 5.11], while their regularity is given by Theorem 1.1.

1.1. **Optimal sets for the eigenvalues of the Dirichlet Laplacian: an overview.** The optimization problems for functionals involving the eigenvalues of an elliptic operator and the volume (the Lebesgue measure) allow to achieve a better understanding on the interaction between the geometry (the shape) of the domains in $\mathbb{R}^d$ and their spectrum. In the particular case when the elliptic operator is the Laplacian with Dirichlet boundary conditions, these variational problems have a rich, century-long history. We will briefly recall the main results concerning the existence and the regularity of optimal sets and we will refer to the survey papers [8] and [18] for a more detailed introduction to the topic.

1.1.1. *Functionals involving only the first eigenvalue.* For the principal eigenvalue λ_1, the analogous of (1) is the variational problem

$$\min\big\{\lambda_1(A) + \Lambda|A| : A \subset D\big\}. \tag{4}$$

We first recall that the classical Faber-Krahn inequality implies that if Λ is big enough, then balls are the only (up to translation in D) solutions. On the other hand, if Λ is small, then the existence of a minimizer of (4) in the class of quasi-open sets can be easily proved (see Section 1.2.4), but the optimal shapes $\Omega \subset D$ are in general not explicit. In this case, the regularity of the free boundary $\partial\Omega \cap D$ (the part contained in the box D) was obtained by Briançon and Lamboley in [4]. In fact, by the variational characterization of $\lambda_1(\Omega)$, the problem (4) is equivalent to the following variational problem involving functions and not sets

$$\min\left\{\int_D |\nabla u|^2\,dx + \Lambda|\{u > 0\}| \;:\; u \in H^1_0(D),\ \int_D u^2\,dx = 1\right\}.$$

If u is a minimizer and $\xi \in C^\infty_c(D;\mathbb{R}^d)$ is a smooth vector field, then the function

$$t \mapsto \int_D |\nabla u_t|^2\,dx + \Lambda|\{u_t > 0\}| \qquad \text{where} \qquad u_t(x) := u(x + t\xi(x)),$$

is differentiable and has minimum at $t = 0$. The associated first order optimality condition gives that, in some suitable sense, u is a solution of the following (one-phase) free boundary problem

$$-\Delta u = \lambda_1(\Omega)u \quad \text{in} \quad \Omega, \qquad |\nabla u| = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega \cap D, \qquad \Omega = \{u > 0\} \subset D, \tag{5}$$

for which one can apply the techniques developed by Alt and Caffarelli in [1] for the one-phase Bernoulli problem

$$-\Delta u = 0 \quad \text{in} \quad \{u > 0\}, \qquad |\nabla u| = \sqrt{\Lambda} \quad \text{on} \quad \partial\{u > 0\} \cap D,$$

obtained from the minimization (with suitable Dirichlet boundary conditions on ∂D) of the functional

$$\int_D |\nabla u|^2\,dx + \Lambda|\{u > 0\} \cap D|.$$

Finally, we notice that, for solutions Ω of (4), the regularity of the full boundary $\partial\Omega$, including the part touching ∂D, was obtained recently in [28, Theorem 1.2] (by an argument relying on [12]) and in [29] (by the epiperimetric inequality from [30]) .

1.1.2. *Functionals involving higher eigenvalues.* For what concerns functionals $\mathcal{F}$ depending on the higher eigenvalues of the Dirichlet Laplacian, the existence of minimizers is known only in the class of quasi-open sets (the definition of a quasi-open set is recalled in Section 1.2). In this class of domains, Buttazzo and Dal Maso [9] proved the existence of optimal sets for general shape optimization problems

$$\min\big\{\mathcal{F}(A)+\Lambda|A| : A\subset D,\ A \text{ quasi-open}\big\},\tag{6}$$

involving functionals $\mathcal{F}$ of the form

$$\mathcal{F}(\Omega)=F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big),$$

for which the function $F:\mathbb{R}^k\to\mathbb{R}$ satisfies only some mild semicontinuity and monotonicity assumptions.

The regularity of the optimal sets in this more general situation is still to be completely understood even for the simplest model case

$$F\big(\lambda_1,\dots,\lambda_k\big)=\lambda_k.$$

The main difficulty is in the fact that the higher eigenvalues $\lambda_k(\Omega)$ of the Dirichlet Laplacian are variationally characterized by the following min-max principle

$$\lambda_k(A):=\min_{E_k\subset H^1_0(A)}\ \max_{u\in E_k\setminus\{0\}}\frac{\displaystyle\int_A|\nabla u|^2\,dx}{\displaystyle\int_A u^2\,dx}\ ,\qquad k\in\mathbb{N},\tag{7}$$

where the minimum is taken over all k-dimensional linear subspaces E_k of $H^1_0(A)$. One consequence of this min-max formulation is that, for $k\geq 2$, the functional $\Omega\mapsto\lambda_k(\Omega)$ is not differentiable with respect to variations of the domain Ω along smooth vector fields (see for instance [19]), which in particular means that one can not write an overdetermined boundary value problem as (5) for just one of the associated eigenfunctions $u_k\in E_k$.

Several results were obtained recently for functionals involving not only higher eigenvalues but also the principal one $\lambda_1(\Omega)$. In fact, for this type of *non-degenerate* functionals the regularity of the free boundary $\partial\Omega\cap D$ of an optimal set Ω was recently proved in [23] and [25] (see also [31] for the case of more general operators). The main model example of such a functional is

$$F\big(\lambda_1,\dots,\lambda_k\big)=\sum_{j=1}^k\lambda_j,$$

and the crucial observation is that the vector-valued function $U=(u_1,\dots,u_k):D\to\mathbb{R}^k$, whose components are the first k eigenfunctions on Ω and whose norm is

$$|U|=\sqrt{u_1^2+\dots+u_k^2}\ ,$$

is a solution of a free boundary problem

$$-\Delta u_j=\lambda_j(\Omega)u_j\quad\text{in}\quad\Omega\quad\text{for}\quad j=1,\dots,k\,;\qquad\big|\nabla|U|\big|=\sqrt{\Lambda}\quad\text{on}\quad\partial\Omega\cap D,\qquad\Omega=\{|U|>0\}\subset D,\tag{8}$$

which is closely related to the vectorial Bernoulli problem

$$-\Delta U=0\quad\text{in}\quad\{|U|>0\},\qquad\big|\nabla|U|\big|=\sqrt{\Lambda}\quad\text{on}\quad\partial\{|U|>0\}\cap D,\tag{9}$$

obtained from the minimization of the functional

$$\int_D|\nabla U|^2\,dx+\Lambda\big|\{|U|>0\}\cap D\big|,\tag{10}$$

and which was studied in [11], [25], [30], [15], and [26].

For what concerns the optimal sets for *degenerate* functionals of the form $\mathcal{F}(\Omega)=\lambda_k(\Omega)$, the only available regularity result for $k\geq 2$ was obtained by Kriventsov and Lin in [24], where they prove both the existence of an open optimal set Ω and the $C^{1,\alpha}$-regularity of the flat part of the free boundary. The full regularity of the optimal sets is still not completely understood, as $\partial\Omega$ might contain cusp-like singularities (branching points), which a priori might be a large set of the same dimension as the free boundary.

1.1.3. *Optimal sets for* λ_2. Let now $k = 2$ and D be a bounded open subset of $\mathbb{R}^d$. We consider the problem

$$\min\big\{\lambda_2(A) + \Lambda|A| : A \subset D\big\}. \tag{11}$$

Without the constraint $A \subset D$, an optimal set for the functional $\lambda_2 + \Lambda|\cdot|$ is any union of two disjoint balls with the same radius $R_{\Lambda,d}$, which is an explicit constant depending only on Λ and the dimension d (this result is classical and is known as Krahn-Szegö inequality, see [19, Theorem 4.1.1]). In particular, if two disjoint balls of the same radius $R_{\Lambda,d}$ fit into D (this happens for instance when Λ is big or D is large), then the union of these two is a solution to (11). Conversely, if two balls of radius $R_{\Lambda,d}$ do not fit into D, then the optimal domains are not explicit; in this case, one can argue that the free boundary of the solutions to (11) is generated by the presence of the domain D which acts as an obstacle.

The aim of the present paper is to give a complete description of the boundary of the optimal sets for (11), including the branching (cuspidal) points and the contact points with ∂D. Our approach is based on the analysis of the functional λ_2 which, as λ_k, is a singular min-max functional (see (7)). Thus, many of the main obstructions to the regularity of the solutions of

$$\min\big\{\lambda_k(A) + \Lambda|A| : A \subset D\big\}, \tag{12}$$

with D bounded or $D = \mathbb{R}^d$, are already present in (11). The major difference between the two cases $k = 2$ and $k > 2$ is not related to the obstacle D, but to the fact that the first one can be reduced to a two-phase free boundary problem (see Theorem 7.2 and Section 1.1.4), while the latter is expected to be related to a vectorial free boundary problem, for which the analysis of the branching points is not available yet, even for minimizers of (10).

1.1.4. *Multiphase shape optimization problems.* The variational minimization problem

$$\min\Big\{\lambda_2(\Omega) + \Lambda|\Omega| \;:\; \Omega \text{ quasi-open, } \Omega \subset D\Big\} \tag{13}$$

is related to a class of spectral optimization problems involving multiple disjoint sets, the so-called multiphase shape optimization problems. Indeed, (13) is equivalent to the variational problem

$$\min\Big\{\max\big\{\lambda_1(\Omega_1);\lambda_1(\Omega_2)\big\} + \Lambda|\Omega_1\cup\Omega_2| \;:\; \Omega_1 \text{ and } \Omega_2 \text{ are disjoint quasi-open subsets of } D\Big\}. \tag{14}$$

We notice that this multiphase version of (13) was already exploited in [7] in order to prove the existence of open optimal sets for the functional $\mathcal{F}_\Lambda = \lambda_2 + \Lambda|\cdot|$ in D. In the present paper, we will use an equivalent free boundary version (see Section 3).

The study of variational problems for functionals of the form

$$\mathcal{F}(\Omega_1,\Omega_2,\dots,\Omega_N) = F\big(\lambda_1(\Omega_1),\dots,\lambda_1(\Omega_N)\big)$$

was initiated in [7] and was then continued in [3] and [29], where it was proved that if $d = 2$ and if the N-uple $\Omega_1,\dots,\Omega_N$ is a solution of

$$\min\left\{\sum_{j=1}^N\Big(\lambda_1(\Omega_j) + \Lambda|\Omega_j|\Big) \;:\; \Omega_1,\dots,\Omega_N \text{ are disjoint quasi-open subsets of } D\right\}, \tag{15}$$

then each of the sets Ω_j has a $C^{1,\alpha}$ regular boundary. This result was recently extended to any dimension $d \geq 2$ in [14]. As in the one-phase $\mathcal{F}(\Omega) = \lambda_1(\Omega)$ and the vectorial $\mathcal{F}(\Omega) = \sum_{j=1}^k \lambda_j(\Omega)$ problems, the crucial observation is that (15) can be written (at least locally) as a minimization problem involving a single function that changes sign. Precisely, in [7] it was shown that one can reduce the analysis to the case of only two domains ($N = 2$). Then, in [29] it was proved that if u_1 and u_2 are the first eigenfunctions of Ω_1 and Ω_2, then the function $u := u_1 - u_2$ is an almost-minimizer of the functional

$$\int_D |\nabla u|^2\,dx + \Lambda\big|\{u \neq 0\}\cap D\big|.$$

This allowed to prove the regularity of the free boundary for almost-minimizers in dimension two (see [29]) by means of the epiperimetric inequality from [30]. In higher dimension, the analysis was concluded in [14], where it was proved the regularity of both free boundaries

$$\partial\{u > 0\}\cap D \quad\text{and}\quad \partial\{u < 0\}\cap D$$

in a neighborhood of $\partial\{u>0\}\cap\partial\{u<0\}$, for functions u that solve a PDE in $\{u\neq 0\}$ and satisfy the following conditions on the boundary $\partial\{u\neq 0\}\cap D$ in viscosity sense

$$\begin{cases}|\nabla u_+|=\alpha_+>0 \quad \text{on} \quad \partial\{u>0\}\setminus\partial\{u<0\}\cap D,\\ |\nabla u_-|=\alpha_->0 \quad \text{on} \quad \partial\{u<0\}\setminus\partial\{u>0\}\cap D,\\ |\nabla u_\pm|\geq\alpha_\pm \quad \text{and} \quad |\nabla u_+|^2-|\nabla u_-|^2=\alpha_+^2-\alpha_-^2 \quad \text{on} \quad \partial\{u>0\}\cap\partial\{u<0\}\cap D.\end{cases} \tag{16}$$

As it can be easily seen from the analysis in [29], this result applies directly to the multiphase problem (15) by taking the constants α_+ and α_- to be equal to $\sqrt{\Lambda}$. Unfortunately, the regularity theorem from [14] can not be directly applied to (14) and (1). In fact, in the present work, a key point in the proof of Theorem 1.1 is to show that if Ω is an optimal set for (1), then there is a Lipschitz continuous (on the whole $\mathbb{R}^d$) second eigenfunction $u\in H_0^1(\Omega)$ that satisfies (16) in viscosity sense for some strictly positive constants α_+ and α_-. We will discuss the strategy of the proof in Section 1.3.

1.1.5. *Shape optimization problems with measure constraint.* A shape optimization problem closely related to (12) is the following

$$\min\Big\{\lambda_k(\Omega):\Omega\subset D,\ |\Omega|=m\Big\}, \tag{17}$$

where $m\in(0,|D|]$ is a given constant. The equivalence of (12) and (17) is trivial when $D=\mathbb{R}^d$, while for a general open set $D\subset\mathbb{R}^d$ it is only known that any minimizer of (12) is a minimizer of (17) for some $m(\Lambda,D)>0$. In particular, this means that a regularity result for solutions to (17) with $k=2$ will be more general than Theorem 1.1. In the case $k=1$, the regularity of the minimizers of (17) was proved in [4] and [28]. The key point of the argument is in showing that the solutions of (17) with $k=1$ are critical points for the functional

$$\Omega\mapsto\lambda_1(\Omega)+\Lambda|\Omega|,$$

with respect to internal perturbations. This result is then used to prove a monotonicity formula, classify the blow-up limits and write an optimality condition for the first eigenfunction in viscosity sense. For what concerns the case $k=2$ (or $k\geq 2$), we believe that this approach, combined with the ideas from the proof of Theorem 1.1, should still work for minimizers of (17), but would add several technical complications to our proof, so in this paper, we prefer not to follow this direction, but to concentrate on the key issue in the case $k=2$, which is the singular character of the functional.

1.2. **Optimal quasi-open sets.** The variational minimization problem (1) is usually stated in the wider class of the so-called *quasi-open* sets, as

$$\min\Big\{\mathcal{F}_\Lambda(\Omega)\ :\ \Omega\subset D\ ,\ \Omega\ \text{ quasi-open}\Big\}. \tag{18}$$

As explained in the previous section, the main reason is that a general theorem by Buttazzo and Dal Maso [9] provides the existence of optimal sets in this class for a large variety of functionals, including $\mathcal{F}_\Lambda$. Our regularity result holds also for minimizers in this class of sets. Before we state the result in this setting (Theorem 1.4), we briefly recall the main definitions in this context (for more details, we refer to the books [16, 20, 5]).

1.2.1. *Capacity.* The capacity of a set $E\subset\mathbb{R}^d$ is defined as

$$\mathrm{cap}(E)=\inf\Big\{\int_{\mathbb{R}^d}\big(|\nabla u|^2+u^2\big)\,dx\ :\ u\in H^1(\mathbb{R}^d),\ u\geq 1\ \text{ in a neighborhood of }\ E\Big\}. \tag{19}$$

It is well-known (see for instance [16]) that any function $u\in H^1(\mathbb{R}^d)$, which by definition is defined almost-eveywhere in the sense of the Lebesgue measure, is also defined *quasi-everywhere* on $\mathbb{R}^d$ in the following sense: there is a set $E_u\subset\mathbb{R}^d$ such that $\mathrm{cap}(E_u)=0$ and the limit

$$\lim_{r\to 0}\fint_{B_r(x_0)}u(x)\,dx\quad\text{exists for every}\quad x_0\in\mathbb{R}^d\setminus E_u.$$

In particular, this allows to define u pointwise everywhere on $\mathbb{R}^d\setminus E_u$ as

$$u(x_0):=\lim_{r\to 0}\fint_{B_r(x_0)}u(x)\,dx. \tag{20}$$

Notice that the definition does not depend on the choice of rrespresentative of u in $H^1(\mathbb{R}^d)$.

1.2.2. *Quasi-open sets and Sobolev spaces.* For every measurable set $\Omega \subset \mathbb{R}^d$ we define the space $H^1_0(\Omega)$ as

$$H^1_0(\Omega) = \Big\{ u \in H^1(\mathbb{R}^d) \ : \ \text{cap}(\{u \neq 0\} \setminus \Omega) = 0 \Big\}.$$

When Ω is open, $H^1_0(\Omega)$ is precisely the closure of $C^\infty_c(\Omega)$ with respect to the H^1 norm (see for instance [20]). When Ω is bounded, the embedding of $H^1_0(\Omega)$ in $L^2(\Omega)$ is compact.

We say that Ω is a quasi-open set if there is a function $u \in H^1(\mathbb{R}^d)$ satisfying (20) outside a set of zero capacity and such that $\Omega = \{u > 0\}$ up to a set of zero capacity; in particular, for every $u \in H^1(\mathbb{R}^d)$, the set $\Omega = \{u \neq 0\}$ is quasi-open and $u \in H^1_0(\Omega)$.

Notice that a quasi-open set Ω is defined up to a set of zero capacity and that every open set is also quasi-open. Moreover, if E is any subset of $\mathbb{R}^d$, then there is a unique (up to a set of zero capacity) quasi-open set Ω such that $\text{cap}(\Omega \setminus E) = 0$ and $H^1_0(E) = H^1_0(\Omega)$. In other words, when we write $H^1_0(\Omega)$, we can always assume that Ω is quasi-open.

1.2.3. *Spectrum of the Dirichlet Laplacian on quasi-open sets.* Let Ω be a bounded quasi-open set in $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. We say that $u \in H^1_0(\Omega)$ is a solution to

$$-\Delta u = f \quad \text{in} \quad \Omega,$$

if for every $\varphi \in H^1_0(\Omega)$, we have

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi f \, dx.$$

The operator $\mathcal{R}_\Omega : L^2(\Omega) \to L^2(\Omega)$, that associates to each $f \in L^2(\Omega)$ the unique solution u of the above equation, is linear, positive definite, compact and self-adjoint. Thus, its spectrum is discrete and made by eigenvalues that can be ordered in an infinitesimal and monotone decreasing sequence of positive real numbers. By definition, their inverse are the eigenvalues of the Dirichlet Laplacian on Ω and are denoted by $\lambda_k(\Omega)$, $k \in \mathbb{N}$,

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \dots$$

Moreover, there is a sequence of orthonormal (in $L^2(\Omega)$) eigenfunctions $u_k \in H^1_0(\Omega)$, $k \in \mathbb{N}$, satisfying

$$-\Delta u_k = \lambda_k(\Omega)\, u_k \quad \text{in} \quad \Omega\,, \qquad \int_\Omega u_k^2 \, dx = 1.$$

Finally, we recall that for every $k \geq 1$, the eigenvalue λ_k of the Dirichlet Laplacian can be obtained through the following min-max principle,

$$\lambda_k(\Omega) := \min_{E_k \subset H^1_0(\Omega)} \max_{u \in E_k \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}, \tag{21}$$

where the minimum is taken over all k-dimensional linear subspaces E_k of $H^1_0(\Omega)$. For more details, see [20, Section 4.5].

1.2.4. *Existence of optimal quasi-open sets for* (18). This result follows from the Buttazzo-Dal Maso Theorem [9, Theorem 2.5], but we provide a simpler direct proof in our case.
Let Ω_n be a minimizing sequence of quasi-open sets for $\mathcal{F}_\Lambda$ in D, that is

$$\inf \Big\{ \mathcal{F}_\Lambda(\Omega) \ : \ \Omega \subset D \text{ quasi-open} \Big\} = \lim_{n \to \infty} \mathcal{F}_\Lambda(\Omega_n).$$

By the definition of $\lambda_2(\Omega_n)$, there are functions u_n and v_n in $H^1_0(\Omega_n)$ such that

$$\int_D |\nabla u_n|^2 \, dx = \lambda_2(\Omega_n)\,, \quad \int_D |\nabla v_n|^2 \, dx \leq \lambda_2(\Omega_n)\,, \quad \int_D u_n^2 \, dx = \int_D v_n^2 \, dx = 1 \quad \text{and} \quad \int_D u_n v_n \, dx = 0.$$

Moreover, we can assume that $\Omega_n = \{u_n^2 + v_n^2 > 0\}$. Since the sequences u_n and v_n are uniformly bounded in $H^1_0(D)$, up to a subsequence, we can assume that u_n (resp. v_n) converges to a function u_∞ (resp. v_∞) weakly in $H^1_0(D)$, strongly in $L^2(D)$ and pointwise almost everywhere. By the semicontinuity of the H^1 norm, we have

$$\int_D |\nabla u_\infty|^2 \, dx \leq \liminf_{n \to +\infty} \lambda_2(\Omega_n)\,, \quad \int_D |\nabla v_\infty|^2 \, dx \leq \liminf_{n \to +\infty} \lambda_2(\Omega_n)\,,$$

$$\int_D u_\infty^2 \, dx = \int_D v_\infty^2 \, dx = 1 \quad \text{and} \quad \int_D u_\infty v_\infty \, dx = 0,$$

where Ω_∞ is the set $\{u_\infty^2 + v_\infty^2 > 0\}$. Thus,

$$\lambda_2(\Omega_\infty) \leq \liminf_{n \to \infty} \lambda_2(\Omega_n).$$

On the other hand, the pointwise convergence of u_n and v_n gives that

$$\mathbb{1}_{\Omega_\infty} \le \liminf_{n\to\infty} \mathbb{1}_{\Omega_n},$$

and by the Fatou's Lemma, we get

$$|\Omega_\infty| \le \liminf_{n\to\infty} |\Omega_n|.$$

Thus, we obtain

$$\mathcal{F}_\Lambda(\Omega_\infty) \le \liminf_{n\to\infty} \mathcal{F}_\Lambda(\Omega_n) \le \inf\Big\{\mathcal{F}_\Lambda(\Omega)\ :\ \Omega\subset D \text{ quasi-open}\Big\},$$

which proves that Ω_∞ is an optimal quasi-open set.

1.2.5. *Regularity of the optimal quasi-open set.* A regularity result, analogous to Theorem 1.1, holds also for the minimizers of $\mathcal{F}_\Lambda$ among quasi-open sets. In fact, the two results are equivalent (see Section 1.2.6).

Theorem 1.4. *Let $D\subset\mathbb{R}^d$ be an open bounded set of class $C^{1,\beta}$ for some $\beta>0$, and let $\Lambda>0$ be a given constant. Let $\Omega\subset D$ be a quasi-open set that minimizes $\mathcal{F}_\Lambda$ in D, that is,*

$$\mathcal{F}_\Lambda(\Omega) \le \mathcal{F}_\Lambda(\widetilde\Omega) \quad \text{for every quasi-open set} \quad \widetilde\Omega\subset D. \tag{22}$$

Then, there are two disjoint open sets Ω_+ and Ω_- such that:

$$\operatorname{cap}\big((\Omega_+\cup\Omega_-)\setminus\Omega\big)=0\ ,\qquad \lambda_2(\Omega_+\cup\Omega_-)=\lambda_2(\Omega)\qquad\text{and}\qquad |\Omega\setminus(\Omega_+\cup\Omega_-)|=0.$$

The boundaries $\partial\Omega_+$ and $\partial\Omega_-$ can be decomposed as the disjoint union of a regular and a singular part

$$\partial\Omega_\pm = \operatorname{Reg}(\partial\Omega_\pm)\cup\operatorname{Sing}(\partial\Omega_\pm),$$

for which the claims (i) *and* (ii) *of Theorem 1.1 hold.*

1.2.6. *Equivalence of Theorem 1.1 and Theorem 1.4.* We will first show that Theorem 1.1 implies Theorem 1.4. Let $\Omega\subset D$ be an open set satisfying (3). We will prove that it satisfies (22). We will use the fact that if $\widetilde\Omega\subset D$ is any quasi-open set, then there is a sequence of open sets ω_n such that

$$\lim_{n\to\infty}\operatorname{cap}(\omega_n)=0 \qquad\text{and}\qquad \widetilde\Omega\cup\omega_n \ \text{ is open for every } \ n\in\mathbb{N}.$$

In particular, the sets $\widetilde\Omega_n := \widetilde\Omega\cup\big(\omega_n\cap D\big)$ are open and satisfy

$$\lambda_2(\widetilde\Omega_n)\le\lambda_2(\widetilde\Omega) \qquad\text{and}\qquad \lim_{n\to\infty}|\widetilde\Omega_n| = |\widetilde\Omega|.$$

The first inequality follows directly from (2), while the second claim follows from the fact that $|\omega_n|\le\operatorname{cap}(\omega_n)$, which is a consequence of (19). Now, since Ω satisfies (3), we have that $\mathcal{F}_\Lambda(\Omega)\le\mathcal{F}_\Lambda(\widetilde\Omega_n)$, which gives

$$\mathcal{F}_\Lambda(\Omega)\le\liminf_{n\to\infty}\mathcal{F}_\Lambda(\widetilde\Omega_n)\le\mathcal{F}_\Lambda(\widetilde\Omega),$$

and proves that Ω is also a solution to (22).

Conversely, we also have that Theorem 1.4 implies Theorem 1.1. This is a consequence of [7, Corollary 5.11], which states that if Ω_{qo} is a quasi-open set that satisfies (22), then there exists an open set $\Omega_o\subset D$ such that $\Omega_o\subset\Omega_{qo}$ (in the sense that $\operatorname{cap}(\Omega_{qo}\setminus\Omega_o)=0$) and is a solution to (22) (and thus, also to (3)).

We now outline the proof of Theorem 1.4.

1.2.7. *Proof of Theorem 1.4.* Given a minimizer Ω of (22), we use Lemma 5.1 to obtain a Lipschitz continuous function $u : D\to\mathbb{R}$, which is a second eigenfunction for the Laplacian in Ω and a solution to a suitable variational free boundary problem, that is (23). We define

$$\Omega_+ := \{u>0\} \qquad\text{and}\qquad \Omega_- := \{u<0\}.$$

By Proposition 3.4, we know that the sets Ω_+ and Ω_- are inwards minimizing for the functional $\lambda_1+\Lambda|\cdot|$. By the results from Section 4, we get that $\partial\Omega_+\cap\partial\Omega_-$ is contained in D and by Corollary 7.3, there is an open set $A\subset D$ containing $\partial\Omega_+\cap\partial\Omega_-$ such that both

$$A\cap\partial\Omega_+ \qquad\text{and}\qquad A\cap\partial\Omega_-$$

are $C^{1,\alpha}$ regular manifolds. Next, using Corollary 4.5 we get that that there is an open set A_+ such that:

$$\partial\Omega_+\cap D\subset A_+\ ,\qquad A_+\cap\Omega_-=\emptyset \qquad\text{and}\qquad \partial\Omega_+ \text{ is } C^{1,\alpha} \text{ regular in } A_+\ .$$

We next notice that the set $\partial\Omega_+\setminus(A\cup A_+)$ can be covered with a finite number of balls $B_{r_i}(x_i)$ such that

$$B_{r_i}(x_i)\subset D \qquad\text{and}\qquad B_{r_i}(x_i)\cap\Omega_-=\emptyset.$$

Using again Proposition 3.4, we know that Ω_+ solves the one-phase minimum problem (33) in each of the balls $B_{r_i}(x_i)$. Using the results from [4] and [28], we obtain that each of the one-phase free boundaries

$$\Gamma_+^i := \partial\Omega_+ \cap B_{r_i}(x_i)$$

can be decomposed as the disjoint union of a regular part $\mathrm{Reg}\,(\Gamma_+^i)$ a (possibly empty) singular part $\mathrm{Sing}\,(\Gamma_+^i)$, with the following properties:

(i) The regular part $\mathrm{Reg}(\Gamma_+^i)$ is an open subset of Γ_+^i, which is locally the graph of an analytic function.

(ii) The singular set $\mathrm{Sing}(\Gamma_+^i)$ is a closed subset of Γ_+^i and contains only one-phase points. Moreover, there exists a critical dimension $d^* \in \{5, 6, 7\}$ such that
- if $d < d^*$, then the singular set is empty,
- if $d = d^*$, then the singular set consists of a finite number of points,
- if $d > d^*$, then the singular set has Hausdorff dimension at most $d - d^*$.

Finally, we define

$$\mathrm{Sing}(\partial\Omega_+) := \bigcup_i \mathrm{Sing}(\Gamma_+^i) \qquad \text{and} \qquad \mathrm{Reg}(\partial\Omega_+) := \partial\Omega_+ \setminus \mathrm{Sing}(\partial\Omega_+).$$

Since the union is finite, the dimension estimates from point (ii) above remain valid. On the other hand, by construction the regular part $\mathrm{Reg}(\partial\Omega_+)$ is $C^{1,\alpha}$ regular. The same argument can be repeated also for $\partial\Omega_-$. This concludes the proof of Theorem 1.4. □

1.2.8. *Selection of the minima.* In both optimization problems (3) and (22), the regularity results Theorem 1.1 and Theorem 1.4 provide an optimal set of the form $\Omega_+ \cup \Omega_-$ composed of two disjoint opens sets, each one of which is regular in the sense explained in Theorem 1.1 and Theorem 1.4. Still, this set might not be (and in general it is not) the only optimal set. In fact, $\Omega_+ \cup \Omega_-$ is only the *smallest* optimal set in the sense that we briefly explain in this section. We focus on the problem in the class of open sets (3), but analogous remarks hold also for (22).

We start with an open set Ω, which is a solution of (3). By Theorem 1.1, there is a set $\Omega_+ \cup \Omega_-$, which is contained in Ω, has the same Lebesgue measure as Ω, is still a minimizer of (3) and is regular in the sense of Theorem 1.1. Moreover, we notice that since $\Omega_+ \cup \Omega_-$ is optimal, then

$$\Omega_+ \text{ and } \Omega_- \text{ are both connected open sets.}$$

It is immediate to check that any open set $\widetilde{\Omega}$ such that

$$\Omega_+ \cup \Omega_- \subset \widetilde{\Omega} \qquad \text{and} \qquad |\widetilde{\Omega} \setminus (\Omega_+ \cup \Omega_-)| = 0,$$

is also optimal for (3). We denote the family of all open sets $\widetilde{\Omega}$ with this property by $\mathcal{X}(\Omega_+, \Omega_-)$. In particular, $\Omega \in \mathcal{X}(\Omega_+, \Omega_-)$. Moreover, each family $\mathcal{X}(\Omega_+, \Omega_-)$ is a (non totally) ordered set of minimizers, with respect to the natural order relation

$$\Omega_1 \prec \Omega_2 \quad \Leftrightarrow \quad \Omega_1 \subset \Omega_2.$$

As a consequence of Theorem 1.1 we can identify explicitly the *smallest* and the *biggest* elements of $\mathcal{X}(\Omega_+, \Omega_-)$.

Claim A. The set $\Omega_{small} := \Omega_+ \cup \Omega_-$ is the minimal element of $\mathcal{X}(\Omega_+, \Omega_-)$. Precisely, if $\widetilde{\Omega}$ is any solution to (3) contained in Ω_{small}, then $\widetilde{\Omega} = \Omega_{small}$.

Indeed, if $\widetilde{\Omega}$ is optimal, then by Theorem 1.1 there are two connected open sets $\widetilde{\Omega}_+$ and $\widetilde{\Omega}_-$ which are regular in the sense of Theorem 1.1 and are such that

$$\widetilde{\Omega}_+ \cup \widetilde{\Omega}_- \subset \widetilde{\Omega} \qquad \text{and} \qquad |\widetilde{\Omega} \setminus (\widetilde{\Omega}_+ \cup \widetilde{\Omega}_-)| = 0.$$

But then, the connectedness of $\widetilde{\Omega}_\pm$ implies that each of these sets is contained in Ω_+ or in Ω_-, that is, without loss of generality

$$\widetilde{\Omega}_+ \subset \Omega_+ \qquad \text{and} \qquad \widetilde{\Omega}_- \subset \Omega_-.$$

Now, the regularity of $\widetilde{\Omega}_+$ and the fact that it must have the same measure as Ω_+ imply that $\widetilde{\Omega}_+ = \Omega_+$. Analogously, $\widetilde{\Omega}_- = \Omega_-$. This concludes the proof of Claim A.

Claim B. Let $\Omega_{big} := \mathrm{int}\left(\overline{\Omega_+ \cup \Omega_-}\right)$ that is, the interior of the closure of $\Omega_+ \cup \Omega_-$. Then, Ω_{big} is the maximal element of $\mathcal{X}(\Omega_+, \Omega_-)$ in the following sense: $\Omega_{big} \in \mathcal{X}(\Omega_+, \Omega_-)$ and if $\widetilde{\Omega} \in \mathcal{X}(\Omega_+, \Omega_-)$, then $\widetilde{\Omega} \subset \Omega_{big}$.

Notice that the inclusion $\widetilde{\Omega} \subset \Omega_{big}$ follows immediately from the fact that $\widetilde{\Omega}$ contains $\Omega_+ \cup \Omega_-$ and that $\widetilde{\Omega} \setminus (\Omega_+ \cup \Omega_-)$ has zero Lebesgue measure. In order to show that $\Omega_{big} \in \mathcal{X}(\Omega_+, \Omega_-)$, we use the regularity of

$\Omega_\pm$. From Theorem 1.1, it follows that $\partial\Omega_+$ and $\partial\Omega_-$ have zero Lebesgue measure. Thus, $\overline{\Omega_+\cup\Omega_-}\setminus\Omega_+\cup\Omega_-$ has zero Lebesgue measure. This concludes the proof of Claim B.

Remark 1.5. Notice that the set Ω_{big} might not be regular (see FIGURE 1). For instance, even in dimension two, cusp-like singularities may appear on its boundary. From Theorem 1.1 we know that these cusps are generated simply by the contact of two $C^{1,\alpha}$ regular curves (or in $\mathbb{R}^d$, $d-1$ dimensional surfaces) parametrizing $\partial\Omega_+$ and $\partial\Omega_-$. At the moment, this is all that is known about the set

$$\partial\Omega_{big}\cap\partial\Omega_+\cap\partial\Omega_-,$$

which is a set of singular points for the boundary $\partial\Omega_{big}$. It is natural to expect that its Hausdorff dimension is at most $d-2$, but this is currently an open question even for the classical two-phase Bernoulli problem.

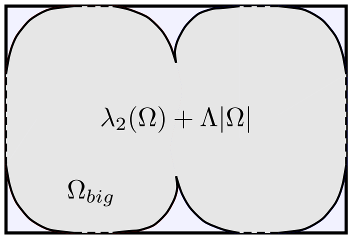

FIGURE 1. A possible solution of the problem (11).

Let now u be the Lipschitz continuous solution to (29) selected in Lemma 5.1. In particular, u is a second eigenfunction on $\Omega_+\cup\Omega_-$ and $\Omega_\pm := \{\pm u>0\}$. By classical elliptic regularity, in a neighborhood of $\partial\Omega_+\cap\partial\Omega_-$, u is $C^{1,\alpha}$ regular (up to the boundary) on the closed sets $\overline{\Omega}_\pm$. Moreover, by Theorem 7.2,

$$|\nabla u|\ge\sqrt{\Lambda}\quad\text{on}\quad\partial\Omega_+\cap\partial\Omega_-\qquad\text{and}\qquad|\nabla u|=\sqrt{\Lambda}\quad\text{on}\quad\big(\partial\Omega_+\cup\partial\Omega_-\big)\setminus\big(\partial\Omega_+\cap\partial\Omega_-\big).$$

Next, we define the following subset of the two-phase free boundary $\partial\Omega_+\cap\partial\Omega_-$

$$\Sigma=\big\{x\in\partial\Omega_+\cap\partial\Omega_-\cap D\ :\ |\nabla u(x)|>\sqrt{\Lambda}\ \big\}.$$

By the continuity of the gradient, Σ is a relatively open set in $\partial\Omega_\pm$ and we know that

$$\overline{\Sigma}\subset\partial\Omega_+\cap\partial\Omega_-.$$

We notice that if $x_0\in(\partial\Omega_+\cup\partial\Omega_-)\setminus\overline{\Sigma}$, then by definition there is some radius $r>0$ such that:

- $\partial\Omega_+$ and $\partial\Omega_-$ are $C^{1,\alpha}$ regular surfaces in $B_r(x_0)$;
- $|\nabla u|=\sqrt{\Lambda}$ on $\partial\Omega_+\cap B_r(x_0)$ and $\partial\Omega_-\cap B_r(x_0)$.

By the classical result of Kinderlehrer and Nirenberg [22], we get that $\partial\Omega_+$ and $\partial\Omega_-$ are $d-1$ dimensional analytic hypersurfaces in $B_r(x_0)$. But then, the contact set $\partial\Omega_+\cap\partial\Omega_-$ must have Hausdorff dimension at most $d-2$ in $B_r(x_0)$. In particular, there should be a sequence of one-phase points in $\partial\Omega_+\setminus\partial\Omega_-$ or $\partial\Omega_-\setminus\partial\Omega_+$ converging to x_0, hence $x_0\notin\Omega_{big}$. Thus,

$$\Omega_{big}\ \subset\ \Omega_{small}\cup\overline{\Sigma}.$$

On the other hand, the continuity of the gradient on the boundary implies that

$$\Omega_{small}\cup\Sigma\ \subset\ \Omega_{big}.$$

In particular, this provides the following dichotomy.

(1) If $\Sigma\neq\emptyset$, then:
   - Ω_{big} is connected and $\lambda_2(\Omega_{big})>\lambda_1(\Omega_{big})$;
   - the set $\partial\Omega_{big}\cap\partial\Omega_+\cap\partial\Omega_-$ is non-empty.

(2) If $\Sigma=\emptyset$, then:
   - $\Omega_{small}=\Omega_{big}$; in particular, Ω_{big} is disconnected and $\lambda_2(\Omega_{big})=\lambda_1(\Omega_{big})$;
   - the set $\partial\Omega_{big}\cap\partial\Omega_+\cap\partial\Omega_-$ might be non-empty, but its Hausdorff dimension is at most $d-2$.

1.3. **Plan of the paper.** In Section 3 we show that (18) is equivalent (in some suitable sense) to the variational free boundary problem

$$\min\Big\{J_\infty(v_+,v_-)+\Lambda\big|\{v\neq0\}\big|\ :\ v\in H^1_0(D),\ \int_D v_+^2\,dx=\int_D v_-^2\,dx=1\Big\},\tag{23}$$

where

$$J_\infty(v_+,v_-)=\max\Big\{\int_D|\nabla v_+|^2\,dx\ ;\ \int_D|\nabla v_-|^2\,dx\Big\}.$$

In Section 4 we prove the non-degeneracy result (Lemma 4.1) for minimizers of (23). The nondegeneracy, together with the three-phase monotonicity formula from [32] implies that if u is any minimizer of (23), then the two-phase free boundary $\partial\Omega_+\cap\partial\Omega_- := \partial\{u>0\}\cap\partial\{u<0\}$ does not touch ∂D. Thus, in Corollary 4.5, using Proposition 3.4 (v) and the one-phase regularity result from [28], we obtain that the one-phase free boundaries $\partial\Omega_\pm\setminus\partial\Omega_\mp$ are $C^{1,\alpha}$ regular in a neighborhood of the contact set $\partial\Omega_\pm\cap\partial D$.

In Section 5 we select the sign changing minimizer u of (23), whose level sets $\{u>0\}$ and $\{u<0\}$ will give us the sets Ω_+ and Ω_- in Theorem 1.4. Precisely, in Lemma 5.1, we prove that if Ω is a solution to (18), then there is a Lipschitz continuous function $u : D \to \mathbb{R}$, which is a sign-changing eigenfunction on Ω and, after a multiplication of the positive and negative parts with appropriate constants, it becomes also a minimizer of (23). In Section 6, we show that the function u, selected in Lemma 5.1, satisfies a first order optimality condition (see Lemma 6.4) with respect to internal variations.

In Section 7 we show that the function u from Lemma 5.1 and Lemma 6.4 satisfies an optimality condition in viscosity sense on the two-phase free boundary $\partial\Omega_+ \cap \partial\Omega_-$, which allows to apply the regularity result from [14] in a neighborhood of the two-phase free boundary (see Corollary 7.3). In order to do this, we study the blow-up limits of u at points of the two-phase free boundary $\partial\Omega_+ \cap \partial\Omega_-$. First, in Section 7 we prove the strong H^1 convergence of the blow-up sequences, which allows us to prove the homogeneity of the blow-up limits in Section 7.2 by the means of a Weiss-type monotonicity formula. Finally, in Theorem 7.2, we use this information to classify the blow-up limits at two-phase points.

Notation. For the whole paper $d \geq 2$ is an integer and denotes the dimension of the space. We use the notation for the positive and negative part of a function:

$$v_+ = \max\{v, 0\} \qquad \text{and} \qquad v_- := \max\{-v, 0\},$$

and if the function has already a subscript, such as v_i, then we use the notation

$$v_i^+ = \max\{v_i, 0\} \qquad \text{and} \qquad v_i^- := \max\{-v_i, 0\}.$$

Given a function $u\colon \mathbb{R}^N \to \mathbb{R}$, we define

$$\Omega_u := \{u \neq 0\}, \qquad \Omega_u^+ := \{u > 0\}, \qquad \Omega_u^- := \{u < 0\},$$

and if a function $u \in H^1_0(D)$ for some domain $D \subset \mathbb{R}^N$, we implicitly extend u to zero outside D.

2. Preliminary facts about the principal eigenfunctions on quasi-open sets

In this section we recall some basic properties of the principal eigenfunctions on quasi-open sets, which we will use several times in the paper. Throughout this section, we consider a quasi-open set $\Omega \subset \mathbb{R}^d$ of finite measure and a first eigenfunction u of the Dirichlet Laplacian on Ω, that is, $u \in H^1_0(\Omega)$ is a non-negative minimizer of

$$\lambda_1(\Omega) = \Big\{ \int_\Omega |\nabla u|^2\,dx \ : \ u \in H^1_0(\Omega),\ \int_\Omega u^2\,dx = 1 \Big\}.$$

We suppose that u is extended as 0 outside Ω and that $u \geq 0$ almost everywhere in $\mathbb{R}^d$.

2.1. Subharmonicity and global L^∞ bound.

We first notice that u is a (weak) solution of the PDE

$$\Delta u + \lambda_1(\Omega)\, u = 0 \quad \text{in} \quad \Omega.$$

Moreover, since u is nonnegative, a standard argument (see for instance [34, Lemma 2.7]) proves that

$$\Delta u + \lambda_1(\Omega)\, u \geq 0 \quad \text{in sense of distributions in } \mathbb{R}^d.$$

Precisely, for every non-negative function $\varphi \in C^\infty_c(\mathbb{R}^d)$ (notice that one can take also $\varphi \in H^1(\mathbb{R}^d)$),

$$\int_{\mathbb{R}^d} \Big(-\nabla u \cdot \nabla \varphi + \lambda_1(\Omega)\, u\, \varphi \Big)\,dx \geq 0.$$

Now, we recall that the supremum of the eigenfunction can be estimated only in terms of the associated eigenvalue. Indeed, there is a dimensional constant $C_d > 0$ (see for instance [13, Example 2.1.8] or [33, Proposition 3.4.37]) such that

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C_d \big(\lambda_1(\Omega)\big)^{d/4}.$$

As a consequence, we get that

$$\Delta u + C_d \big(\lambda_1(\Omega)\big)^{(d+4)/4} \geq 0 \quad \text{in sense of distributions in } \mathbb{R}^d. \tag{24}$$

2.2. **Pointwise definition and local L^∞ bound.** Let now $x_0 \in \mathbb{R}^d$ be any point. By (24), the function

$$u_{x_0}(x) := u(x) + C_d\big(\lambda_1(\Omega)\big)^{(d+4)/4}\,\frac{|x-x_0|^2}{2d}.$$

is subharmonic in $\mathbb{R}^d$ and so, the limit

$$\lim_{r\to 0} \fint_{B_r(x_0)} u_{x_0}(x)\,dx,$$

exists. Now, since by construction $\|u - u_{x_0}\|_{L^\infty(B_r(x_0))} \le Cr^2$, we also get

$$\lim_{r\to 0} \fint_{B_r(x_0)} u(x)\,dx = \lim_{r\to 0} \fint_{B_r(x_0)} u_{x_0}(x)\,dx.$$

Thus, we can choose a representative of u which is defined everywhere in $\mathbb{R}^d$ (recall that $u \in H^1(\mathbb{R}^d)$ is an equivalence class in $L^2(\mathbb{R}^d)$). Precisely, from now on, we will always assume that

$$u(x_0) = \lim_{r\to 0} \fint_{B_r(x_0)} u(x)\,dx \quad \text{for every} \quad x_0 \in \mathbb{R}^d.$$

Finally, as another consequence of the subharmonicity of u_{x_0}, we obtain that, for every $0 < \sigma < 1$ and every $r > 0$, the following estimate holds

$$\|u\|_{L^\infty(B_{\sigma r}(x_0))} \le \frac{1}{(1-\sigma)^d} \fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1} + C_d\big(\lambda_1(\Omega)\big)^{(d+4)/4} r^2. \tag{25}$$

3. Equivalent formulations of the shape optimization problem

3.1. **A variational free boundary problem.** Let Ω be a quasi-open set in $\mathbb{R}^d$. Then, we can give an equivalent formulation of $\lambda_2(\Omega)$ in terms of a two-phase free boundary problem in Ω. Precisely, we have the following lemma.

Lemma 3.1 (Second eigenvalue and optimal partitions of a fixed domain)**.** *Let Ω be a bounded open (or quasi-open) set in $\mathbb{R}^d$. Then,*

$$\lambda_2(\Omega) := \min\left\{ J_\infty(v_+, v_-) \ : \ v \in H^1_0(\Omega),\ \int_\Omega v_+^2\,dx = \int_\Omega v_-^2\,dx = 1 \right\}, \tag{26}$$

where the functional $J_\infty : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ is defined by

$$J_\infty(v_+, v_-) := \max\left\{ \int_\Omega |\nabla v_+|^2\,dx \ ;\ \int_\Omega |\nabla v_-|^2\,dx \right\}. \tag{27}$$

Proof. We first notice that by the compactness of the embedding $H^1_0(\Omega)$ into $L^2(\Omega)$, there is a function

$$u = u_+ - u_- \in H^1_0(\Omega)$$

that realizes the minimum in (26), that is $\int_\Omega u_+^2\,dx = \int_\Omega u_-^2\,dx = 1$, and

$$J_\infty(u_+, u_-) = \min\left\{ J_\infty(v_+, v_-) \ : \ v \in H^1_0(\Omega),\ \int_\Omega v_+^2\,dx = \int_\Omega v_-^2\,dx = 1 \right\}.$$

Now, since the space generated by u_+ and u_- is a two-dimensional subspace of $H^1_0(\Omega)$, we get that

$$\lambda_2(\Omega) \le J_\infty(u_+, u_-).$$

On the other hand, let u_1 and u_2 be one first and one second eigenfunction of the Dirichlet Laplacian on Ω. Then, we have that

$$\begin{gathered}\int_\Omega u_1^2\,dx = \int_\Omega u_2^2\,dx = 1 \qquad \text{and} \qquad \int_\Omega u_1u_2\,dx = 0,\\ \int_\Omega |\nabla u_1|^2\,dx = \lambda_1(\Omega) \le \lambda_2(\Omega) = \int_\Omega |\nabla u_2|^2\,dx,\\ -\Delta u_1 = \lambda_1(\Omega)u_1 \quad \text{in} \quad \Omega,\\ -\Delta u_2 = \lambda_2(\Omega)u_2 \quad \text{in} \quad \Omega.\end{gathered} \tag{28}$$

In particular, the space $V \subset H^1_0(\Omega)$ generated by u_1 and u_2 realizes the minimum in (2). We now consider two cases. First, if u_2 changes sign, then we define the functions

$$\varphi_+ := \left(\int_\Omega (u_2^+)^2\,dx\right)^{-1/2} u_2^+ \qquad \text{and} \qquad \varphi_- := \left(\int_\Omega (u_2^-)^2\,dx\right)^{-1/2} u_2^-.$$

By testing the equation (28) with φ_+ and φ_-, we get that

$$\int_\Omega |\nabla\varphi_+|^2\,dx = \lambda_2(\Omega) = \int_\Omega |\nabla\varphi_-|^2\,dx.$$

Thus,

$$J_\infty(u_+,u_-) \le J_\infty(\varphi_+,\varphi_-) = \lambda_2(\Omega),$$

which concludes the proof of (26) in the case when u_2 changes sign. Moreover, by the same argument, we get that if u_1 changes sign, then $\lambda_1(\Omega)=\lambda_2(\Omega)$ and (26) holds. Suppose now that $u_2\ge 0$ and $u_1\ge 0$. Then, the orthogonality in $L^2(\Omega)$ implies that they have disjoint supports and that, by taking $\psi = u_2-u_1$, we have that

$$J_\infty(u_+,u_-) \le J_\infty(\psi_+,\psi_-) = \max\{\lambda_1(\Omega),\lambda_2(\Omega)\} = \lambda_2(\Omega),$$

which concludes the proof. □

As a consequence, we can reformulate (18) as a variational free boundary problem for the functional J_∞

$$\min\left\{ J_\infty(v_+,v_-) + \Lambda\big|\{v\neq 0\}\big| \;:\; v\in H^1_0(D),\ \int_D v_+^2\,dx = \int_D v_-^2\,dx = 1\right\}. \tag{29}$$

We will prove that these two problems are equivalent in Proposition 3.3 and Proposition 3.4. In the proofs we will use several times the following simple fact.

Lemma 3.2. *Suppose that Ω is a bounded quasi-open set in $\mathbb{R}^d$, $d\ge 2$, and let $x_0\in\mathbb{R}^d$. Then*

$$\lim_{r\to 0^+} \lambda_1\big(\Omega\setminus \overline{B}_r(x_0)\big) = \lambda_1(\Omega).$$

Proof. Assume that $d\ge 3$, the case $d=2$ being analogous. Let u be the first (normalized) eigenfunction on Ω and let $\phi_r:\mathbb{R}^d\to[0,1]$ be the function

$$\phi_r = 1 \quad\text{in}\quad \mathbb{R}^d\setminus B_{2r}(x_0),\qquad \phi_r = 0\quad\text{in}\quad B_r(x_0),\qquad \phi_r = \frac{1}{r}(|x|-r)\quad\text{in}\quad B_{2r}(x_0)\setminus B_r(x_0).$$

Since $\lambda_1(\Omega)\le\lambda_1(\Omega\setminus\overline{B}_r(x_0))$, we only have to bound $\lambda_1(\Omega\setminus\overline{B}_r(x_0))$ from above:

$$\lambda_1(\Omega\setminus\overline{B}_r(x_0)) \le \frac{\int|\nabla(u\phi_r)|^2\,dx}{\int(u\phi_r)^2\,dx} \le \Big(1-\int_{B_{2r}} u^2\,dx\Big)^{-1}\Big(\lambda_1(\Omega) + 2\sqrt{\lambda_1(\Omega)}\|\nabla\phi_r\|_{L^2} + \|u\|^2_{L^\infty}\|\nabla\phi_r\|^2_{L^2}\Big).$$

Passing to the limit as $r\to 0$, we get the claim. □

Proposition 3.3. *Let D be a bounded open set in $\mathbb{R}^d$ and let $\Lambda>0$ be a given constant.*

(i) *If $\Omega\subset D$ is a quasi-open set that satisfies (22) and if $u_2\in H^1_0(\Omega)$ is a sign-changing second eigenfunction of the Dirichlet Laplacian on Ω, then the function $u := u_+ - u_-$ defined by*

$$u_+ := \left(\int_\Omega (u_2^+)^2\,dx\right)^{-1} u_2^+ \qquad\textit{and}\qquad u_- := \left(\int_\Omega (u_2^-)^2\,dx\right)^{-1} u_2^-.$$

is a solution to (29).

(ii) *If $\Omega\subset D$ is a quasi-open set that satisfies (22) and if $u_2\in H^1_0(\Omega)$ is a nonnegative and normalized second eigenfunction of the Dirichlet Laplacian on Ω, then $\lambda_1(\Omega)=\lambda_2(\Omega)$ and there exists another nonnegative and normalized eigenfunction u_1 (corresponding to the eigenvalue $\lambda_1(\Omega)=\lambda_2(\Omega)$) orthogonal to u_2 in $L^2(D)$, such that $u := u_2-u_1$ is a solution to (29).*

Proof. We first notice that, by the definition of λ_2, if the function $v\in H^1_0(D)$ is such that

$$\int_D v_+^2\,dx = \int_D v_-^2\,dx = 1,$$

then $\lambda_2(\{v\neq 0\})\le J_\infty(v_+,v_-)$. Now, if u is the function from (i), then

$$J_\infty(u_+,u_-) + \Lambda|\Omega_u| = \mathcal{F}_\Lambda(\Omega_u) \le \mathcal{F}_\Lambda(\Omega_v) \le J_\infty(v_+,v_-) + \Lambda|\Omega_v|,$$

where $\Omega_u = \{u\neq 0\}$ and $\Omega_v=\{v\neq 0\}$. This proves (i).

Let now u_1 and u_2 be as in (ii). Then,

$$\int_D |\nabla u_2|^2\,dx = \lambda_2(\Omega)\,,\qquad \int_D|\nabla u_1|^2\,dx = \lambda_1(\Omega)\,,\qquad \int_D u_1^2\,dx = \int_D u_2^2\,dx = 1\,.$$

Now, suppose that $\lambda_1(\Omega)<\lambda_2(\Omega)$. We pick a point x_0 of Lebesgue density 1 for the set $\{u_1>0\}$ and consider the set

$$\Omega_r := \{u_2>0\}\cup\Big(\{u_1>0\}\setminus\overline{B}_r(x_0)\Big).$$

By Lemma 3.2, we get that for r small enough

$$\lambda_1\big(\{u_2>0\}\big)=\lambda_2(\Omega)>\lambda_1\big(\{u_1>0\}\setminus\overline{B}_r(x_0)\big)>\lambda_1\big(\{u_1>0\}\big)=\lambda_1(\Omega).$$

In particular, this implies that

$$\lambda_2(\Omega_r)=\lambda_2(\Omega),$$

while on the other hand $|\Omega_r|<|\Omega|$, which contradicts the minimality of Ω. This implies that $\lambda_1(\Omega)=\lambda_2(\Omega)$ and the claim now follows as in the proof of (i). ☐

Proposition 3.4. *Let D be a bounded open set in $\mathbb{R}^d$ and let $\Lambda>0$. Suppose that the function $u\in H^1_0(D)$ is a solution to (29). Then,*

$$\int_D|\nabla u_+|^2\,dx=\int_D|\nabla u_-|^2\,dx\,. \tag{30}$$

Moreover, setting

$$\Omega_+=\{u>0\}\,,\quad \Omega_-=\{u<0\}\quad \textit{and}\quad \Omega=\Omega_+\cup\Omega_-\ ,$$

we have that:

(i) *u_+ is the first eigenfunction on Ω_+ and u_- is the first eigenfunction on Ω_-, that is,*

$$\int_D|\nabla u_+|^2\,dx=\lambda_1(\Omega_+)\qquad\textit{and}\qquad\int_D|\nabla u_-|^2\,dx=\lambda_1(\Omega_-)\,. \tag{31}$$

(ii) *The set Ω minimizes (22) and*

$$\lambda_1(\Omega_+)=\lambda_1(\Omega_-)=\lambda_2(\Omega).$$

(iii) *There are constants $a>0$ and $b>0$ such that the function $u_2=au_+-bu_-$ is a second eigenfunction on Ω, that is,*

$$-\Delta u_2=\lambda_2(\Omega)u_2\quad\textit{in}\quad\Omega.$$

(iv) *The sets Ω_+ and Ω_- are inward minimizing for the functional $\lambda_1+\Lambda|\cdot|$, that is,*

$$\lambda_1(\Omega_\pm)+\Lambda|\Omega_\pm|\le\lambda_1(\widetilde\Omega)+\Lambda|\widetilde\Omega|\quad\textit{for every quasi-open set}\quad\widetilde\Omega\subset\Omega_\pm\,. \tag{32}$$

(v) *Setting $c_+=|\Omega_+|$ and $c_-=|\Omega_-|$, we have*

$$\begin{aligned}\lambda_1(\Omega_+)&=\min\Big\{\lambda_1(A):A\subset D\ \textit{quasi-open}\ ,\ |A\cap\Omega_-|=0,\ |A|=c_+\Big\},\\ \lambda_1(\Omega_-)&=\min\Big\{\lambda_1(A):A\subset D\ \textit{quasi-open}\ ,\ |A\cap\Omega_+|=0,\ |A|=c_-\Big\}.\end{aligned} \tag{33}$$

Proof. The first claim (30) follows as in the proof of Lemma 3.2; in fact, if the Dirichlet energy of u_- is smaller than the one of u_+, then we can construct a competitor of the form $u_+-\phi_r u_-$ with the same energy

$$J_\infty(u_+,\phi_r u_-)=J_\infty(u_+,u_-),$$

but with smaller support. The claim (i) now follows directly from the definition of J_∞.

In order to prove (ii) suppose that Ω^* is a solution to (22). Then, by Proposition 3.3, there is a second eigenfunction $u^*\in H^1_0(\Omega^*)$, corresponding to $\lambda_2(\Omega^*)=J_\infty(u_+^*,u_-^*)$ with

$$\int_D(u_+^*)^2=\int_D(u_-^*)^2=1.$$

Thus, the minimality of u gives that

$$J_\infty(u_+^*,u_-^*)+\Lambda|\Omega^*|\ge J_\infty(u_+,u_-)+\Lambda|\Omega|.$$

On the other hand, the minimality of Ω^* implies that

$$\lambda_2(\Omega^*)+\Lambda|\Omega^*|\le\lambda_2(\Omega)+\Lambda|\Omega|,$$

and we can combine these inequalities to get (ii).

In order to prove (iii), we consider two cases. First, if $\lambda_1(\Omega)=\lambda_2(\Omega)$, then both the functions u_+ and u_- are first eigenfunctions on Ω and so the equations

$$-\Delta u_+=\lambda_2(\Omega)u_+\qquad\text{and}\qquad-\Delta u_-=\lambda_2(\Omega)u_-$$

hold in the entire domain Ω, that is, the two equations hold weakly in $H^1_0(\Omega)$: in particular, this proves the claim. Second, we consider the case $\lambda_1(\Omega)<\lambda_2(\Omega)$ and we choose a non-negative eigenfunction u_1 corresponding to

$\lambda_1(\Omega)$. Since $\lambda_1(\Omega) < \lambda_1(\Omega_\pm)$, we have that $\{u_1 > 0\}$ intersects both Ω_+ and Ω_-. In particular, we can find constants a and b such that the function $u_2 := au_+ - bu_-$ is such that:

$$\int_\Omega u_2^2\,dx = 1\,, \qquad \int_\Omega u_2 u_1\,dx = 0 \qquad \text{and} \qquad \int_\Omega |\nabla u_2|^2\,dx = \lambda_2(\Omega).$$

As a consequence, using the variational formulation (2) and comparing the space generated by the couple (u_1, u_2) with the spaces generated by $(u_1, u_2 + \varepsilon\phi)$ for $\phi \in H_0^1(\Omega)$ and ε small, we obtain that u_2 is in fact a second eigenfunction corresponding to the eigenvalue $\lambda_2(\Omega)$:

$$-\Delta u_2 = \lambda_2(\Omega) u_2 \quad \text{in} \quad \Omega.$$

The claim (iv) is an immediate consequence of testing in (29) the optimality of the function u with the functions $u_+ - \widetilde{u}_-$ and $\widetilde{u}_+ - u_-$, where $\widetilde{u}_+$ and $\widetilde{u}_-$ are normalized first eigenfunctions on $\widetilde{\Omega}_+$ and $\widetilde{\Omega}_-$.

We finally deal with (v). Suppose by contradiction that there is a set $\widetilde{\Omega}$ such that

$$\widetilde{\Omega} \subset D\,, \quad |\widetilde{\Omega} \cap \Omega_-| = 0\,, \quad |\widetilde{\Omega}| = c_+ \quad \text{and} \quad \lambda_1(\widetilde{\Omega}) < \lambda_1(\Omega_+).$$

Then, pick a point x_0 of density one for $\widetilde{\Omega}$ and a sufficiently small radius $r > 0$ such that

$$\lambda_1(\Omega_+) > \lambda_1(\widetilde{\Omega} \setminus \overline{B}_r(x_0)) \geq \lambda_1(\widetilde{\Omega}),$$

and let $\widetilde{u}_+$ be the first eigenfunction on $\widetilde{\Omega} \setminus \overline{B}_r(x_0)$. Then,

$$J_\infty(\widetilde{u}_+, u_-) + |\widetilde{\Omega} \setminus \overline{B}_r(x_0)| + |\Omega_-| = \lambda_1(\widetilde{\Omega} \setminus \overline{B}_r(x_0)) + |\widetilde{\Omega} \setminus \overline{B}_r(x_0)| + |\Omega_-| < \lambda_1(\Omega_+) + |\Omega_+| + |\Omega_-|\ ,$$

which contradicts the minimality of u. □

4. Inwards minimizing property, non-degeneracy and two-phase points

Lemma 4.1 (Nondegeneracy)**.** *For every pair of constants $C > 0$ and $\Lambda > 0$, there are constants $r_0 > 0$ and $\eta > 0$, depending on C, Λ and the dimension d, such that the following holds. Suppose that the bounded quasi-open set $\Omega \subset \mathbb{R}^d$ is such that:*

- $\lambda_1(\Omega) \leq C$ *;*
- *Ω satisfies the inwards minimizing property*

$$\lambda_1(\Omega) + \Lambda|\Omega| \leq \lambda_1(\widetilde{\Omega}) + \Lambda|\widetilde{\Omega}| \quad \textit{for every quasi-open set} \quad \widetilde{\Omega} \subset \Omega\,, \tag{34}$$

- $\displaystyle\fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1} \leq \eta r$*, for $r \leq r_0$ and where u is the first eigenfunction on Ω.*

Then $u = 0$ in $B_{r/2}(x_0)$.

This is a well-known result, for a proof see for example [7].

As a consequence of Lemma 4.1, we have the following result. We use the notation $\mathcal{C}_\delta$ for the cone

$$\mathcal{C}_\delta := \big\{x \in \mathbb{R}^d \ :\ x_d > \delta|x|\big\}.$$

Proposition 4.2 (Triple points)**.** *Suppose that Ω_+ and Ω_- are disjoint bounded quasi-open sets in $\mathbb{R}^d$ each one satisfying the inwards minimizing property*

$$\lambda_1(\Omega_\pm) + \Lambda|\Omega_\pm| \leq \lambda_1(\widetilde{\Omega}) + \Lambda|\widetilde{\Omega}| \quad \textit{for every quasi-open set} \quad \widetilde{\Omega} \subset \Omega_\pm\,, \tag{35}$$

for some $\Lambda > 0$. Then, there is a constant $\delta > 0$ such that if

$$\Omega_+ \cap \Omega_- \cap B_R = \emptyset\,, \qquad \mathcal{C}_\delta \cap \Omega_+ \cap B_R = \emptyset \qquad \textit{and} \qquad \mathcal{C}_\delta \cap \Omega_- \cap B_R = \emptyset\,,$$

for some $R > 0$, then there exists $\varepsilon > 0$ such that

$$\Omega_+ \cap B_\varepsilon = \emptyset \qquad \textit{or} \qquad \Omega_- \cap B_\varepsilon = \emptyset\,.$$

In the proof of Proposition 4.2, we will use the following two lemmas.

Lemma 4.3 (Three-phase monotonicity formula [32, 7])**.** *Let $u_i \in H^1(B_1)$, $i = 1, 2, 3$, be three non-negative functions such that:*

- *$\Delta u_i + 1 \geq 0$ in B_1 in sense of distributions, for every $i = 1, 2, 3$;*
- $\displaystyle\int_{\mathbb{R}^d} u_i u_j\,dx = 0$*, for every pair $i \neq j \in \{1, 2, 3\}$.*

Then there are dimensional constants $\varepsilon > 0$ *and* $C_d > 0$ *such that, for every* $r \in (0, \frac{1}{2})$*, we have*

$$\prod_{i=1}^{3}\left(\frac{1}{r^{2+\varepsilon}}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right) \le C_d\left(1+\sum_{i=1}^{3}\int_{B_1}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right)^3. \tag{36}$$

Lemma 4.4 (Alt-Caffarelli potential estimate [1])**.** *For every* $u \in H^1(B_r)$ *we have the following estimate:*

$$\frac{1}{r^2}\big|\{u=0\}\cap B_r\big|\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2 \le C_d\int_{B_r}|\nabla(u-h)|^2\,dx \le C_d\int_{B_r}|\nabla u|^2\,dx, \tag{37}$$

where:

- C_d *is a constant that depends only on the dimension* d*;*
- h *is the harmonic extension of* u *in* B_r*, that is,*

$$\Delta h = 0 \quad \text{in} \quad B_r\,, \qquad u = h \quad \text{on} \quad \partial B_r\,.$$

Proof of Proposition 4.2. Let u_+ and u_- be the first eigenfunctions on Ω_+ and Ω_-, normalized in $L^2(\Omega)$. Let $v \in H^1(\mathbb{R}^d)$ be the $(1+\gamma)$-homogeneous, non-negative harmonic function on $\mathcal{C}_\delta$, which vanishes on $\partial\mathcal{C}_\delta$. In polar coordinates

$$v = r^{1+\gamma}\phi(\theta),$$

where ϕ is the first eigenfunction of the spherical Laplacian on $\mathcal{C}_\delta \cap \mathbb{S}^{d-1}$, that is,

$$-\Delta_{\mathbb{S}^{d-1}}\phi = (1+\gamma)(d-1+\gamma)\phi \quad \text{in} \quad \mathcal{C}_\delta\cap\mathbb{S}^{d-1}, \qquad \phi = 0 \quad \text{on} \quad \partial\mathcal{C}_\delta\cap\mathbb{S}^{d-1}, \qquad \int_{\mathbb{S}^{d-1}}\phi^2(\theta)\,d\theta = 1,$$

where we notice that γ is uniquely determined by δ (and the dimension d) and

$$\lim_{\delta\to 0}\gamma(\delta) = 0.$$

Moreover, we have that

$$\Delta v \ge 0 \quad \text{in sense of distributions in } \mathbb{R}^d.$$

By the three-phase monotonicity formula (Lemma 4.3), that we can apply thanks to (24), there are constants $C > 0$ and $\varepsilon > 0$ such that

$$Cr^\varepsilon \ge \left(\frac{1}{|B_r|}\int_{B_r}|\nabla u_+|^2\,dx\right)\left(\frac{1}{|B_r|}\int_{B_r}|\nabla u_-|^2\,dx\right)\left(\frac{1}{|B_r|}\int_{B_r}|\nabla v|^2\,dx\right).$$

Now, using (37) and the fact that $\big|\{u_\pm = 0\}\cap B_r\big| \ge |\mathcal{C}_\delta\cap B_r| \ge \frac{1}{2}|B_r|$, we get

$$Cr^\varepsilon \ge \left(\fint_{\partial B_r} u_+\,d\mathcal{H}^{d-1}\right)^2\left(\fint_{\partial B_r} u_-\,d\mathcal{H}^{d-1}\right)^2\left(\frac{1}{|B_r|}\int_{B_r}|\nabla v|^2\,dx\right),$$

for some different constant C. Now, using the non-degeneracy (Lemma 4.1), we obtain

$$\begin{aligned} Cr^\varepsilon &\ge \frac{1}{|B_r|}\int_{B_r}|\nabla v|^2\,dx \\ &= \frac{1}{|B_r|}\int_0^r\int_{\mathbb{S}^{d-1}}\Big((1+\gamma)^2\phi^2(\theta) + |\nabla_\theta\phi(\theta)|^2\Big)\rho^{d-1+2\gamma}\,d\theta\,d\rho = (1+\gamma)r^{2\gamma}, \end{aligned}$$

which is impossible when δ (and thus γ) is small enough (ε being a fixed constant, depending on d, $\lambda_1(\Omega_+)$ and $\lambda_1(\Omega_-)$, but not on δ). □

As a corollary of Proposition 4.2, we obtain the following regularity result for the solutions of (29).

Corollary 4.5 (Regularity of the one-phase free boundaries)**.** *Suppose that* D *is a bounded open set in* $\mathbb{R}^d$ *with* $C^{1,\beta}$ *regular boundary, for some* $\beta > 0$*. Suppose that* $u \in H^1_0(D)$ *is a solution to the problem* (29) *and that* Ω_u^+ *and* Ω_u^- *are the sets* $\Omega_u^+ = \{u > 0\}$ *and* $\Omega_u^- = \{u < 0\}$*. Then:*

(i) *there are no two-phase points on the boundary of* D*, that is, for every* $x_0 \in \partial D$*, there is* $\varepsilon > 0$ *such that*

$$\Omega_u^+\cap B_\varepsilon(x_0) = \emptyset \quad \text{or} \quad \Omega_u^-\cap B_\varepsilon(x_0) = \emptyset\,;$$

(ii) *the one-phase free boundaries* $\partial\Omega_u^\pm$ *are* $C^{1,\alpha}$*-regular in a neighborhood of* ∂D*. Precisely, if* $x_0 \in \partial D$ *is such that* $B_r(x_0)\cap\Omega_u^- = \emptyset$ *for some* $r > 0$*, then* $\partial\Omega_u^+\cap B_r(x_0)$ *is a* $C^{1,\alpha}$ *manifold, for some* $\alpha > 0$*.*

Proof. From Proposition 3.4 point (iv), we know that the sets Ω_u^+ and Ω_u^- are inwards minimizing. Now, since ∂D is $C^{1,\beta}$ regular, at every point $x_0 \in \partial D$ there is, up to a rotation of the coordinate system, a cone $\mathcal{C}_\delta$ contained in $\mathbb{R}^d \setminus D$. Thus, Proposition 4.2 implies that if $x_0 \in \partial\Omega_u^+ \cap \partial D$, then in a small ball $B_\varepsilon(x_0)$, the set Ω_u^- is empty. This proves (i). In order to prove (ii), we use again Proposition 3.4 point (v), and we deduce that Ω_u^+ is a solution to the problem

$$\min\Big\{\lambda_1(\Omega)\ :\ \Omega \text{ quasi-open},\ \Omega \subset \Omega_u^+ \cup B_\varepsilon(x_0) \cap D,\ |\Omega| = |\Omega_u^+|\Big\}.$$

Thus, by [28, Proposition 5.35], $\partial\Omega_u^+$ is $C^{1,\alpha}$ regular in $B_\varepsilon(x_0)$. □

5. Lipschitz continuous solutions

In this section, we show that to every solution Ω of the shape optimization problem (18), we can associate a Lipschitz continuous solution $u \in H_0^1(\Omega)$ for the free boundary problem (29). Our main result of the section is Lemma 5.1 here below.

Lemma 5.1. *Let D be a bounded open set in $\mathbb{R}^d$ with $C^{1,\beta}$ regular boundary. Let $\Lambda > 0$ be fixed and let Ω be a solution to (18). Then, there exists a function $u : \mathbb{R}^d \to \mathbb{R}$, $u \in H_0^1(\Omega)$, such that:*

- *u is a sign-changing second eigenfunction on Ω;*
- *u is a solution to (29);*
- *u is Lipschitz continuous on $\mathbb{R}^d$.*

Before dealing with the proof of Lemma 5.1, we need a technical result. It is well-known that if Ω minimizes the first eigenvalue among all quasi-open sets with a fixed measure, then the first eigenfunction on Ω is Lipschitz (when extended as zero outside Ω). This was already proved by Briançon and Lamboley in [4] through an Alt-Caffarelli argument [1]. In the proof of Lemma 5.1, we need to know the Lipschitz constant explicitely, so we briefly give a quantitative local version of this result in the next lemma by a method already used in several other works (see for instance [28, 6, 1]).

Lemma 5.2. *Suppose that Ω is a bounded quasi-open set in $\mathbb{R}^d$ and that the function $u \in H_0^1(\Omega)$ is the first eigenfunction on Ω, that is, $u \geq 0$ in $\mathbb{R}^d$, $\int_\Omega u^2\,dx = 1$ and*

$$\lambda_1(\Omega) = \int_\Omega |\nabla u|^2\,dx = \min\left\{\int_\Omega |\nabla\phi|^2\,dx\ :\ \phi \in H_0^1(\Omega),\ \int_\Omega \phi^2\,dx = 1\right\}.$$

Suppose that B_R is a ball of radius $R \leq 1$ and that there are constants $r > 0$ and $K > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \leq \frac{\int_{\mathbb{R}^d} |\nabla(u+\varphi)|^2\,dx}{\int_{\mathbb{R}^d} (u+\varphi)^2\,dx} + K\rho^d, \tag{38}$$

for every $\varphi \in H_0^1\big(B_\rho(x_0)\big)$ and every ball $B_\rho(x_0) \subset B_R$. Then, there is a constant C, depending only on $\lambda_1(\Omega)$, K and d, such that, if $u(0) = 0$, then

$$u(x_0) = 0 \qquad\Longrightarrow\qquad \|\nabla u\|_{L^\infty(B_{R/8})} \leq C.$$

Proof. Let $\varphi \in C_c^\infty(B_\rho(x_0))$ be such that $\varphi = 1$ in $B_{\rho/2}(x_0)$ and $|\nabla\varphi| \lesssim \frac{1}{\rho}$. We can compute

$$\begin{aligned}\lambda_1(\Omega) &\leq \frac{\int_{\mathbb{R}^d} |\nabla(u+t\varphi)|^2\,dx}{\int_{\mathbb{R}^d} (u+t\varphi)^2\,dx} + K\rho^d\\ &= \frac{\lambda_1(\Omega) + 2t\int \nabla u\cdot\nabla\varphi\,dx + t^2\int |\nabla\varphi|^2\,dx}{1 + 2t\int u\varphi\,dx + t^2\int \varphi^2\,dx} + K\rho^d,\end{aligned}$$

which implies that

$$2t\left(-\int \nabla u\cdot\nabla\varphi\,dx + \lambda_1(\Omega)\int u\varphi\,dx\right) \leq t^2\int |\nabla\varphi|^2\,dx + 2\left(1 + t^2\int \varphi^2\,dx\right)K\rho^d,$$

Choosing $t = \rho \le 1$ and using that $\Delta u + \lambda_1(\Omega)u$ is a positive Radon measure on $\mathbb{R}^d$ (see Section 2.1), we get

$$\big(\Delta u + \lambda_1(\Omega)u\big)\big(B_{\rho/2}(x_0)\big) \le \left(-\int \nabla u\cdot\nabla\varphi\,dx + \lambda_1(\Omega)\int u\varphi\,dx\right) \le C_d(1+K)\rho^{d-1}.$$

As a consequence, if $u(x_0) = 0$, using [6, formula (2.26)] and an integration by parts we obtain

$$\fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1} = \int_0^r \frac{\Delta u(B_\rho(x_0))}{d\,\omega_d\,\rho^{d-1}}\,d\rho \le C_d(1+K)r. \tag{39}$$

Now, let $y_0 \in B_{R/8}$ and let x_0 be the projection of y_0 on the set $\{u = 0\}$, which is closed as a consequence of (39). Since $u(0) = 0$, we have that

$$r_0 := |x_0 - y_0| \le {}^R/_8.$$

Notice that we have

$$B_{r_0}(y_0) \subset B_{2r_0}(x_0) \subset B_{R/2}.$$

Thus, applying (39), we get

$$\fint_{\partial B_{2r_0}(x_0)} u\,d\mathcal{H}^{d-1} \le C_d(1+K)r_0.$$

Now, since there is a constant $C(d,\lambda_1)$, depending on d and $\lambda_1(\Omega)$ such that

$$u(x) + C(d,\lambda_1)|x - x_0|^2$$

is subharmonic (see Section 2.1), we have that

$$\begin{aligned}\|u\|_{L^\infty(B_{r_0/2}(y_0))} &\le \fint_{B_{r_0}(y_0)} u\,dx + C(d,\lambda_1)r_0^2 \le 2^d \fint_{B_{2r_0}(x_0)} u\,dx + C(d,\lambda_1)r_0^2\\ &\le 2^d\left(\fint_{\partial B_{2r_0}(x_0)} u\,d\mathcal{H}^{d-1} + C(d,\lambda_1)r_0^2\right) + C(d,\lambda_1)r_0^2 \le C_d(1+K)r_0 + C(d,\lambda_1)r_0^2.\end{aligned}$$

Now, the gradient estimate (41) gives the claim. □

Proof of Lemma 5.1. Let Ω be as in the assumptions. By [6, Theorem 5.3] there exists a second eigenfunction $w \in H_0^1(\Omega)$, which is Lipschitz continuous on $\mathbb{R}^d$. We consider two cases.

Case 1. If w changes sign in Ω, then w is a solution of (29) (by Proposition 3.3), so we can take $u = w$.

Case 1a. Suppose that w does not change sign and that the open set $\{w \neq 0\}$ is disconnected. Let Ω_1 and Ω_2 be two connected components of $\{w \neq 0\}$ and let

$$\widetilde{w} = a_1 w \mathbb{1}_{\Omega_1} - a_2 w \mathbb{1}_{\Omega_2} \quad \text{where} \quad a_i = \left(\int_{\Omega_i} w^2\right)^{-1/2}.$$

It is immediate to check that $\widetilde{w}$ is a Lipschitz continuous sign-changing second eigenfunction on Ω and a solution to (27).

Case 2. Suppose that w does not change sign and that the open set $\Omega_w := \{w \neq 0\}$ is connected. Without loss of generality, we can assume that w is non-negative.

We will show that there is a non-negative first eigenfunction $v \in H_0^1(\Omega)$ such that :

- v and w have disjoint supports : $vw = 0$ on $\mathbb{R}^d$;
- $w - v$ is a solution to (29) ;
- there are positive constants α and β such that the function $u := \alpha w - \beta v$ is a (sign-changing and normalized) second eigenfunction on Ω.

It is enough to prove that there is a non-negative first eigenfunctions for which the first point holds; the other two claims follow by Proposition 3.3 and Proposition 3.4. Suppose that there is a non-negative eigenfunction u_1 on Ω such that

$$\int_\Omega u_1^2\,dx = \int_\Omega w^2\,dx = 1 \qquad \text{and} \qquad 0 < \int_\Omega u_1 w\,dx < 1\,. \tag{40}$$

In particular, since both u_1 and w are eigenfunctions on Ω and since they are not orthogonal in $L^2(\Omega)$, we get that

$$\lambda_1(\Omega) = \lambda_2(\Omega).$$

Moreover, w is a solution also of

$$-\Delta w = \lambda_2(\Omega)w \quad \text{in} \quad \Omega_w := \{w > 0\}\ , \qquad w \in H_0^1(\Omega).$$

Thus, it is also an eigenfunction on Ω_w. Since w is positive on Ω_w and Ω_w is connected, we get that w is the first eigenfunction on Ω_w and

$$\lambda_1(\Omega) = \lambda_2(\Omega) = \lambda_1(\Omega_w).$$

We also notice that another consequence of the hypothesis that Ω_w is connected is that any eigenfunction on Ω_w corresponding to the first eigenvalue $\lambda_1(\Omega_w)$ is necessarily proportional to w. In particular, this implies that $u_1 \notin H^1_0(\Omega_w)$, otherwise we would get $w = u_1$, which is not possible by (40). Moreover, the strong maximum principle on the open set Ω_w implies that $u_1 > 0$ on Ω_w. Now, consider the function

$$(w - u_1)_+ := \sup\{(w - u_1), 0\}.$$

We first notice that $(w-u_1)_+$ is in $H^1_0(\Omega_w)$. This is true since $0 \le (w-u_1)_+ \le w$ and $w \in H^1_0(\Omega_w)$. Moreover, since both w and u_1 are normalized in $L^2(\Omega)$, the function $(w - u_1)_+$ is not identically zero. Now, testing $(w - u_1)_+$ with itself, we get that

$$\begin{aligned}\int_\Omega |\nabla(w-u_1)_+|^2\,dx &= \int_\Omega \nabla w\cdot\nabla(w-u_1)_+\,dx - \int_\Omega \nabla u_1\cdot\nabla(w-u_1)_+\,dx\\ &= \lambda_2(\Omega)\int_\Omega w(w-u_1)_+\,dx - \lambda_1(\Omega)\int_\Omega u_1(w-u_1)_+\,dx\\ &= \lambda_1(\Omega_w)\int_\Omega (w-u_1)(w-u_1)_+\,dx = \lambda_1(\Omega_w)\int_\Omega (w-u_1)_+^2\,dx.\end{aligned}$$

By the variational characterization of $\lambda_1(\Omega_w)$, $(w - u_1)_+$ is also a first eigenfunction on Ω_w. Thus,

$$(w - u_1)_+ = cw,$$

for some $c > 0$. In particular, this means that $w > u_1$ on Ω_w and that

$$w = (1 + c)u_1 \quad \text{on} \quad \Omega_w.$$

Finally, choosing

$$v := u_1 - \frac{1}{1+c} w\,,$$

we get that, by construction, v is a first eigenfunction on Ω and $v = 0$ on Ω_w.

It remains to prove that v is Lipschitz continuous. Let $x_0 \in \Omega_v^+ := \{v > 0\}$ and let r be the largest radius for which the ball $B_r(x_0)$ is contained in $\{v > 0\}$. We fix a constant $r_0 > 0$ (that we will later choose small enough) and we consider four cases:

Case 2a. $r \ge r_0$;

Case 2b. $r < r_0$ and in $B_{10r}(x_0)$ there is a point lying outside D;

Case 2c. $r < r_0$, $B_{10r}(x_0)$ is contained in D and in $B_{4r}(x_0)$ there is a point lying in Ω_w^+;

Case 2d. $r < r_0$, $B_{10r}(x_0)$ is contained in D and $B_{4r}(x_0) \cap \{w > 0\} = \emptyset$.

We start with the case $2a$. Since v solves

$$-\Delta v = \lambda_2(\Omega) v \quad \text{in} \quad B_r(x_0),$$

the classical gradient estimate (see [17]) gives

$$\|\nabla v\|_{L^\infty(B_{r/2}(x_0))} \le C_d \|\lambda_2(\Omega) v\|_{L^\infty(B_r(x_0))} + \frac{2d}{r}\|v\|_{L^\infty(B_r(x_0))}. \tag{41}$$

Since v satisfies the global L^∞ bound $\|v\|_{L^\infty(\mathbb{R}^d)} \le C_d\big(\lambda_2(\Omega)\big)^{d/4}$ and since $r \ge r_0$, we get that there is a constant $C(d, \lambda_2, r_0)$, depending on d, $\lambda_2(\Omega)$ and r_0, such that

$$|\nabla v|(x_0) \le C(d, \lambda_2, r_0).$$

We now consider the case $2b$. Let w_D be the solution of

$$-\Delta w_D = 1 \quad \text{in} \quad D, \qquad w_D = 0 \quad \text{on} \quad \mathbb{R}^d \setminus D.$$

Since D is $C^{1,\beta}$ regular, the function w_D is Lipschitz continuous on $\mathbb{R}^d$. We denote by L its Lipschitz constant. Setting

$$C := C_d\big(\lambda_1(\Omega)\big)^{(d+4)/4}$$

to be the constant from (24), we know that $Cw_D \ge v$ everywhere in $\mathbb{R}^d$. Then, we have

$$v \le 11CLr \quad \text{in} \quad B_r(x_0).$$

Using again the gradient estimate (41), we get that there is a constant $C(D,d,\lambda_2)$ depending only on D, d and $\lambda_2(\Omega)$ such that

$$|\nabla v|(x_0)\le C(D,d,\lambda_2).$$

We next consider the case 2c. Let y_0 be a point in $\{w>0\}$. By the two-phase monotonicity formula of Caffarelli-Jerison-Kënig (see [10] and [32]), we know that there is a constant C, depending on $\lambda_2(\Omega)$ and the dimension such that

$$C\ge\left(\fint_{B_R(y_0)}|\nabla w|^2\,dx\right)\left(\fint_{B_R(y_0)}|\nabla v|^2\,dx\right).$$

Applying Lemma 4.4, we get that (up to multiplying C by a factor depending only on the dimension)

$$C\ge\frac{|\{w=0\}\cap B_R(y_0)|}{|B_R|}\left(\frac{1}{R}\fint_{\partial B_R(y_0)}w\,d\mathcal{H}^{d-1}\right)^2\frac{|\{v=0\}\cap B_R(y_0)|}{|B_R|}\left(\frac{1}{R}\fint_{\partial B_R(y_0)}v\,d\mathcal{H}^{d-1}\right)^2.$$

and, since w and v have disjoint supports,

$$C\ge\frac{|\{v>0\}\cap B_R(y_0)|}{|B_R|}\left(\frac{1}{R}\fint_{\partial B_R(y_0)}w\,d\mathcal{H}^{d-1}\right)^2\frac{|\{w>0\}\cap B_R(y_0)|}{|B_R|}\left(\frac{1}{R}\fint_{\partial B_R(y_0)}v\,d\mathcal{H}^{d-1}\right)^2.$$

We next choose $R=4r$. Thus, the non-degeneracy of w (in order to use the non-degeneracy Lemma 4.1, we choose r_0 small enough from the beginning) gives that

$$\frac{1}{R}\fint_{\partial B_R(y_0)}w\,d\mathcal{H}^{d-1}\ge\eta.$$

In particular, there is a point $z_0\in\partial B_R(y_0)$ such that $w(z_0)\ge\eta$. But now, the Lipschitz continuity of w (say $|\nabla w|\le L_w$) gives that $w>0$ in $B_{\eta/L_w}(z_0)$. Thus, we get also that

$$\frac{|\{w>0\}\cap B_R(y_0)|}{|B_R|}\ge\frac{|B_{\eta/L_w}(z_0)\cap B_R(y_0)|}{|B_R|}\ge C(L_w,\eta,d).$$

Similarly, since v is positive in $B_r(x_0)\cap B_{4r}(y_0)$, we have that there is a dimensional constant c_d such that

$$\frac{|\{v>0\}\cap B_R(y_0)|}{|B_R|}\ge c_d.$$

This finally gives that there is a constant $C(w,d)$, depending on w and the dimension, such that

$$C\ge\frac{1}{R}\fint_{\partial B_R(y_0)}v\,d\mathcal{H}^{d-1}.$$

Applying (25) the gradient estimate as in the case *2a*, we get that

$$|\nabla v|(x_0)\le C(w,d).$$

Finally, we consider the case *2d*. First of all, we suppose that there is at least one point $x_1\in\partial\Omega_v\cap D$ and a radius $r_1>0$ such that $B_{r_1}(x_1)\subset D$ and $B_{r_1}(x_1)\cap\{w>0\}=\emptyset$ (in fact if there were not such x_1 and r_1, the proof of the lemma would be concluded with the case *2c*). Now, by [4], we know that v is Lipschitz in $B_{r_1}(x_1)$ and that the free boundary $\partial\Omega_v\cap B_{r_1}(x_1)$ is C^∞ up to a small closed set. In particular, we may assume that in D there are two distinct points x_1 and x_2, and a radius $0<R_{12}<\frac{1}{3}|x_1-x_2|$ such that:

- $B_{R_{12}}(x_1)\subset D$ and $B_{R_{12}}(x_2)\subset D$;
- $B_{R_{12}}(x_1)\cap\{w>0\}=\emptyset$ and $B_{R_{12}}(x_2)\cap\{w>0\}=\emptyset$;
- $\partial\Omega_v$ is C^∞ in $B_{R_{12}}(x_1)$ and $B_{R_{12}}(x_2)$;
- there are constants $m>0$ and $C>0$ such that, for every $i=1,2$ and every $t\in(-m,m)$, there is a function $v_{i,t}\in H^1(B_{R_{12}}(x_i))$ such that:

$$v_{i,t}=v\quad\text{on}\quad\partial B_{R_{12}}(x_i).$$

$$\big|B_{R_{12}}(x_1)\cap\{v_{i,t}>0\}\big|-\big|B_{R_{12}}(x_1)\cap\{v>0\}\big|=t\tag{42}$$

$$\int_{B_{R_{12}}(x_1)}\big(|\nabla v_{i,t}|^2+v_{i,t}^2\big)\,dx\le Kt.\tag{43}$$

We notice that for the construction of $v_{i,t}$ it is sufficient to take smooth vector fields $\xi_i \in C_c^\infty\big(B_{R_{12}}(x_i);\mathbb{R}^d\big)$, $i = 1, 2$, orthogonal to $\partial\Omega_v$ (parallel to the outgoing normal ν) and pointing outwards and to define the functions

$$v_{i,t}(x) := v(x + \xi_{i,t}(x)).$$

the claims (42) and (43) now follow from the well-known (see [4]) first variation formulas

$$\frac{d}{dt}\Big|_{t=0} \int |\nabla v_{i,t}|^2\,dx = -\int_{\partial\Omega_v} (\xi\cdot\nu)|\nabla v|^2\,d\mathcal{H}^{d-1} \qquad\text{and}\qquad \frac{d}{dt}\Big|_{t=0} |\{v_{i,t}>0\}| = \int_{\partial\Omega_v} \xi\cdot\nu\,d\mathcal{H}^{d-1},$$

and the inverse function theorem. Now, with this family of functions in hand, we get back to the case *2d*. Notice that, by choosing $r_0 > 0$ small enough, we can assume that the ball $B_{4r}(x_0)$ intersects at most one of the balls $B_{R_{12}}(x_1)$ and $B_{R_{12}}(x_2)$ (say, the first one). Thus, if φ is a function compactly supported in $B_{4r}(x_0)$, we can consider the competitor

$$\widetilde{v} = \begin{cases} v & \text{in} \quad \mathbb{R}^d \setminus (B_{4r}(x_0) \cup B_{R_{12}}(x_1)),\\ v+\varphi & \text{in} \quad B_{4r}(x_0),\\ v_{i,t} & \text{in} \quad B_{R_{12}}(x_1),\end{cases}$$

where we choose t such that $\{\widetilde{v} > 0\} = \{v > 0\}$. Thus, from (42) and (43), we get that v satisfies the almost-minimality condition (38). Thus, we can use the universal estimate from Lemma 5.2 and this concludes the proof. □

6. First variation formula

Let Ω be a solution to (18). From now on, we will take $\Lambda = 1$, without loss of generality. We know that there is a sign-changing function $u \in H_0^1(\Omega)$, which is Lipschitz continuous on $\mathbb{R}^d$ and a solution to (29). Our next objective is to prove that the function u is a solution, in the viscosity sense, of a free boundary problem. In order to do so, we will first try to deduce a first order optimality condition coming from internal perturbations with vector fields. Since the function $\mathbb{R}^2 \ni (a, b) \mapsto \max\{a, b\}$ is not differentiable, we will approximate J_∞ with smooth functionals, inspired by [27] and [23].

In what follows we will use the notation

$$\mathcal{R}(v) := \frac{\displaystyle\int_{\{v>0\}} |\nabla v|^2\,dx}{\displaystyle\int_{\{v>0\}} v^2\,dx} \quad \text{for every nonnegative function} \quad v \in H^1(\mathbb{R}^d),\ v \neq 0,$$

while, when $v = 0$, we simply set $\mathcal{R}(0) = +\infty$. For every $p \in (1, +\infty)$, we consider the problem

$$\min\left\{ J_p\big(\mathcal{R}(v_+)\,;\,\mathcal{R}(v_-)\big) + \int_D |u-v|^2 + |\Omega_v| : v \in H_0^1(D)\right\}. \tag{44}$$

where as usual $v_+ = \max\{v, 0\}$, $v_- = \max\{-v, 0\}$ and $\Omega_v = \{v \neq 0\}$, and where J_p is the function

$$J_p(X, Y) := \big(X^p + Y^p\big)^{1/p}.$$

Remark 6.1. For all $p \in (1, +\infty)$, there exists a solution to the problem (44): the proof is standard and follows by the same argument as the one in Section 1.2.4.

Lemma 6.2 (Convergence of the minima)**.** *For every $p \geq 2$, let $v_p \in H_0^1(D)$ be a solution to (44) such that*

$$\int_D (v_p^+)^2\,dx = \int_D (v_p^-)^2\,dx = 1.$$

Then, as $p \to \infty$, v_p converges strongly in $H_0^1(D)$ to the function u, solution to (29). Moreover, the characteristic functions $\mathbb{1}_{\Omega_{v_p}^+}$ and $\mathbb{1}_{\Omega_{v_p}^-}$ converge strongly in L^1 and pointwise almost-everywhere to $\mathbb{1}_{\Omega_u^+}$ and $\mathbb{1}_{\Omega_u^-}$, respectively.

Proof. We first notice that, by testing the minimality of v_p with u, we get

$$\begin{aligned} J_p\left(\int_D |\nabla v_p^+|^2\,;\,\int_D |\nabla v_p^-|^2\right) + \int_D |u-v_p|^2 + |\Omega_{v_p}| &\leq J_p\left(\int_D |\nabla u_+|^2\,;\,\int_D |\nabla u_-|^2\right) + |\Omega_u|\\ &\leq 2\,J_\infty\left(\int_D |\nabla u_+|^2\,;\,\int_D |\nabla u_-|^2\right) + |\Omega_u|. \end{aligned}$$

Thus, v_p is bounded in H^1 and so, up to a subsequence, v_p^+ and v_p^- converge weakly in H^1, strongly in L^2 and pointwise almost-everywhere to a function $v_\infty \in H_0^1(D)$. The convergence and the minimality of v_p now give

$$\begin{aligned}
J_\infty\Big(\int_D|\nabla u_+|^2\,;\,\int_D|\nabla u_-|^2\Big)&+\int_D|u-v_\infty|^2+|\Omega_u|\\
&\le J_\infty\Big(\int_D|\nabla v_\infty^+|^2\,;\,\int_D|\nabla v_\infty^-|^2\Big)+\int_D|u-v_\infty|^2+|\Omega_{v_\infty}|\\
&\le J_\infty\Big(\liminf_{p\to\infty}\int_D|\nabla v_p^+|^2\,;\,\liminf_{p\to\infty}\int_D|\nabla v_p^-|^2\Big)+\int_D|u-v_\infty|^2+|\Omega_{v_\infty}|\\
&\le \liminf_{p\to\infty}\Big\{J_p\Big(\int_D|\nabla v_p^+|^2\,;\,\int_D|\nabla v_p^-|^2\Big)+\int_D|u-v_p|^2+|\Omega_{v_p}|\Big\}\\
&\le \lim_{p\to\infty}\Big\{J_p\Big(\int_D|\nabla u_+|^2\,;\,\int_D|\nabla u_-|^2\Big)+|\Omega_u|\Big\}\\
&= J_\infty\Big(\int_D|\nabla u_+|^2\,;\,\int_D|\nabla u_-|^2\Big)+|\Omega_u|,
\end{aligned}$$

which proves that $v_\infty = u$ and that all the inequalities above are equalities. In particular,

$$\begin{aligned}
\int_D|\nabla u_+|^2\le\liminf_{p\to\infty}\int_D|\nabla v_p^+|^2&=J_\infty\Big(\liminf_{p\to\infty}\int_D|\nabla v_p^+|^2\,;\,\liminf_{p\to\infty}\int_D|\nabla v_p^-|^2\Big)\\
&=J_\infty\Big(\int_D|\nabla u_+|^2\,;\,\int_D|\nabla u_-|^2\Big)=\int_D|\nabla u_+|^2,
\end{aligned}$$

which means that the convergence is strong in H^1. Finally, the strong convergence of the characteristic functions follows from the equalities

$$|\Omega_u^\pm|=\liminf_{p\to\infty}|\Omega_{v_p}^\pm|. \qquad \square$$

In what follows we will use the notation $\delta J(u)[\xi]$ to indicate the first variation of a functional J at a function u in the direction of a smooth vector field ξ. Precisely, for every $u\in H_0^1(D)$, $\xi\in C_c^\infty(D;\mathbb{R}^d)$, we define the diffeomorphism Φ_t as

$$\Phi_t=\Psi_t^{-1},\quad\text{where}\quad \Psi_t(x):=x+t\xi(x). \tag{45}$$

Then, if the derivative $\dfrac{\partial}{\partial t}\Big|_{t=0}J(u\circ\Phi_t)$ exists, we set

$$\delta J(u)[\xi]:=\frac{\partial}{\partial t}\Big|_{t=0}J(u\circ\Phi_t).$$

It is well-known that $\delta\mathcal{R}(u)[\xi]$ exists for any $u\in H_0^1(D)$ and $\xi\in C_c^\infty(D;\mathbb{R}^d)$ and that

$$\delta\mathcal{R}(u)[\xi]=\int_D\Big(|\nabla u|^2-\lambda u^2\Big)\operatorname{div}\xi-2\,\nabla u\,D\xi\,(\nabla u)^t\,dx, \tag{46}$$

where $\lambda:=\mathcal{R}(u)$. Moreover, setting $\operatorname{Vol}(u)=|\Omega_u|$, we have that $\delta\operatorname{Vol}(u)[\xi]$ exists for all $u\in H_0^1(D)$ and $\xi\in C_c^\infty(D;\mathbb{R}^d)$, and

$$\delta\operatorname{Vol}(u)[\xi]=\int_{\Omega_u}\operatorname{div}\xi\,dx. \tag{47}$$

Now, using the formulas (46) and (47), we can compute the optimality condition for the minimizers of (44). Precisely, we have the following lemma.

Lemma 6.3. *Let $p>1$ and let $u_p\in H_0^1(D)$ be a solution to (44) such that*

$$\int_D(u_p^+)^2\,dx=\int_D(u_p^-)^2\,dx=1.$$

Then, setting

$$a_p^\pm:=\frac{\big(\mathcal{R}(u_p^\pm)\big)^{p-1}}{\Big[\big(\mathcal{R}(u_p^+)\big)^p+\big(\mathcal{R}(u_p^-)\big)^p\Big]^{1-\frac1p}},$$

we have that for any smooth vector field $\xi\in C_c^\infty(D;\mathbb{R}^d)$,

$$a_p^+\delta\mathcal{R}(u_p^+)[\xi]+a_p^-\delta\mathcal{R}(u_p^-)[\xi]+2\int_D(u_p-u)\,\xi\cdot\nabla u_p\,dx+\delta\operatorname{Vol}(u_p^+)[\xi]+\delta\operatorname{Vol}(u_p^-)[\xi]=0. \tag{48}$$

Proof. Let $\xi \in C_c^\infty(D;\mathbb{R}^d)$ and Φ_t be as in (45). Since we already have (46) and (47), it is sufficient to compute the variation of the fidelity term. We have

$$\frac{\partial}{\partial t}\Big|_{t=0}\int_D |u_p\circ\Phi_t - u|^2\,dx = \int_D 2(u_p-u)\,\xi\cdot\nabla u_p\,dx.$$

Then, using the optimality of u_p, we get that

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\Big|_{t=0}\Big[J_p\Big(\mathcal{R}\big((u_p\circ\Phi_t)_+\big),\mathcal{R}\big((u_p\circ\Phi_t)_-\big)\Big) + \int_D |u_p\circ\Phi_t-u|^2\,dx + \big|\{u_p\circ\Phi_t\neq 0\}\big|\Big]\\ &= a_p^+\delta\mathcal{R}(u_p^+)[\xi] + a_p^-\delta\mathcal{R}(u_p^-)[\xi] + \int_D 2(u_p-u)\,\xi\cdot\nabla u_p\,dx + \delta\mathrm{Vol}(u_p^+)[\xi] + \delta\mathrm{Vol}(u_p^-)[\xi],\end{aligned}$$

which gives the claim. □

We now pass to the limit as $p\to+\infty$.

Lemma 6.4. *Let D be a bounded open set and let $u\in H_0^1(D)$ be a Lipschitz continuous solution of (29). Then, there are constants $a_+\geq 0$ and $a_-\geq 0$ such that*

$$a_+ + a_- = 1,$$

and, for every smooth vector field $\xi\in C_c^\infty(D;\mathbb{R}^d)$, we have

$$a_+\delta\mathcal{R}(u_+)[\xi] + a_-\delta\mathcal{R}(u_-)[\xi] + \delta\mathrm{Vol}(u_+)[\xi] + \delta\mathrm{Vol}(u_-)[\xi] = 0.$$

Proof. Using Lemma 6.3 and the convergence of the solutions u_p proved in Lemma 6.2, we have that the variation of the fidelity term vanishes. Indeed,

$$\lim_{p\to\infty}\int_D (u_p-u)\,\xi\cdot\nabla u_p\,dx = 0.$$

Thus, passing to the limit in Lemma 6.3 and using again Lemma 6.2, we get the claim. Finally, the equality $a_+ + a_- = 1$ follows from the fact that

$$\lim_{p\to\infty} a_p^\pm = a_\pm \qquad\text{and}\qquad (a_p^+)^{\frac{p}{p-1}} + (a_p^-)^{\frac{p}{p-1}} = 1 \quad\text{for every}\quad p\geq 1.$$ □

7. Two-phase free boundary: blow-up limits and regularity

Let $u:\mathbb{R}^d\to\mathbb{R}$ be a Lipschitz continuous solution to (29). Let x_0 be a point of the free boundary, that is,

$$x_0\in\partial\Omega_u\cap D,$$

and we define the rescaled function

$$u_{x_0,r}(x) = \frac{u(x_0+rx)}{r}, \qquad \text{for } r>0,$$

on the set $\{x\in\mathbb{R}^d : x_0+rx\in D\}$. For any vanishing sequence (r_n), we say that u_{x_0,r_n} is a blow-up sequence (with fixed center). It is clear that, for all $R>0$, for all n large enough, we have

$$B_R\subset\{x\in\mathbb{R}^d : x_0+r_nx\in D\},$$

and moreover, by Lipschitz continuity of u and the definition of the blow-up sequence with $u(x_0)=0$, we have that there is a locally Lipschitz continuous function $u_0\colon\mathbb{R}^d\to\mathbb{R}$ such that

$$\|u_{x_0,r_n} - u_0\|_{L^\infty(B_R)}\to 0, \qquad \text{for all } R>0, \tag{49}$$

up to pass to a suitable subsequence with a diagonal argument.

Definition 7.1. We will say that $u_0\colon\mathbb{R}^d\to\mathbb{R}$ is a *blow-up* limit of u at x_0.

Our main result is the following

Theorem 7.2. *Let u be a Lipschitz continuous solution of (29) and let $a_+\geq 0$ and $a_-\geq 0$ be the constants from Lemma 6.4. Then*

$$a_+>0 \qquad\text{and}\qquad a_->0. \tag{50}$$

Moreover, if $x_0\in\partial\Omega_u^+\cap\partial\Omega_u^-$ then every blow-up limit u_0 of u at x_0 is of the form

$$u_0(x) := \beta_+(x\cdot\nu)_+ - \beta_-(x\cdot\nu)_-, \tag{51}$$

where $\nu\in\partial B_1$ and the coefficients β_+ and β_- are such that

$$\beta_+\geq\frac{1}{\sqrt{a_+}}, \quad \beta_-\geq\frac{1}{\sqrt{a_-}} \quad\text{and}\quad a_+\beta_+^2 = a_-\beta_-^2. \tag{52}$$

As a corollary, we obtain the regularity of the two-phase free boundary.

Corollary 7.3. *Let D be a bounded open set and let $u : D \to \mathbb{R}$ be a Lipschitz continuous solution to (29). Then, in a neighborhood of the two-phase free boundary $\partial\Omega_u^+ \cap \partial\Omega_u^-$, both $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are $C^{1,\alpha}$ regular.*

Proof. We define the function v as

$$v = \sqrt{a_+}\, u_+ - \sqrt{a_-}\, u_-.$$

Then:

- v is Lipschitz continuous;
- v satisfies the equations
$$-\Delta v = \lambda v \quad \text{in} \quad \Omega_v^+ \cup \Omega_v^-.$$
- on the one-phase free boundaries $D \cap \partial\Omega_v^+ \setminus \partial\Omega_v^-$ and $D \cap \partial\Omega_v^- \setminus \partial\Omega_v^+$, we have that $|\nabla v| = 1$ in viscosity sense (see for example [28, Section 5]);
- for every two-phase point $x_0 \in \partial\Omega_v^+ \cap \partial\Omega_v^-$, v satisfies the equations
$$|\nabla v_+| \geq 1, \quad |\nabla v_-| \geq 1, \quad |\nabla v_+| = |\nabla v_-|,$$
in viscosity sense. This is an immediate consequence of the classification of the blow-up limits of Theorem 7.2, and can be done as in [14, Section 2].

Thus, the claim follows from [14, Theorem 1.1 and 4.3]. □

7.1. **Convergence of the blow-up sequences.** In this section we prove the strong convergence of the blow-up sequences. The main result is the following.

Lemma 7.4. *Let D be an open subset of $\mathbb{R}^d$, u a Lipschitz continuous solution of (29) and $y_0 \in \partial\Omega_u \cap D$. Let $r_n > 0$ be a vanishing sequence and $u_n := u_{y_0,r_n}$ be the corresponding blow-up sequence converging locally uniformly to the blow-up limit $u_0 \colon \mathbb{R}^d \to \mathbb{R}$. Then, for every $R > 0$,*

(i) *the sequence of rescalings u_{y_0,r_n} converges strongly in $H^1(B_R)$ to u_0;*

(ii) *the sequences of characteristic functions $\mathbb{1}_{\Omega_n^+}$ and $\mathbb{1}_{\Omega_n^-}$, where $\Omega_n^\pm := \{\pm u_n > 0\}$, converge in $L^1(B_R)$ and pointwise almost-everywhere to the characteristic functions $\mathbb{1}_{\Omega_0^+}$ and $\mathbb{1}_{\Omega_0^-}$ of the sets $\Omega_0^\pm := \{\pm u_0 > 0\}$.*

Proof. We first prove (i). We will proceed as in [27, Step 5 of the proof of Theorem 3.1]. We notice that u_n is a weak (in $H^1(\mathbb{R}^d)$) solution of the equation

$$\Delta u_n^\pm + r_n \lambda_1(\Omega^\pm) u_n^\pm = \mu_n^\pm \quad \text{in} \quad \mathbb{R}^d, \tag{53}$$

for certain positive Radon measures μ_n^+ and μ_n^-. On the other hand u_0^+ and u_0^- are nonnegative and harmonic on $\{u_0 > 0\}$ and $\{u_0 < 0\}$. Thus, there are positive Radon measures μ^+ and μ^- such that

$$\Delta u_0^\pm = \mu^\pm \quad \text{in} \quad \mathbb{R}^d. \tag{54}$$

Let now $R > 0$ be fixed. Since u_n and u_0 are uniformly Lipschitz continuous in B_R, there is a constant $C_R > 0$, depending only on R such that

$$\mu_n^\pm(B_R) + \mu^\pm(B_R) \leq C_R \qquad \text{for every} \qquad n \geq 0.$$

Let now $\varphi \in C_0^\infty(\mathbb{R}^d)$, be a test function such that

$$0 \leq \varphi \leq 1 \quad \text{in} \quad \mathbb{R}^d\,, \quad \varphi = 1 \quad \text{in} \quad B_R\,, \quad \text{and} \quad \varphi = 0 \quad \text{in} \quad \mathbb{R}^d \setminus B_{2R}.$$

We test the difference of the two equations (53) and (54) with $\varphi(u_n^\pm - u^\pm)$

$$\int_{\mathbb{R}^d} \nabla(u_n^\pm - u_0^\pm) \cdot \nabla[\varphi(u_n^\pm - u_0^\pm)]\, dx = \int_{\mathbb{R}^d} \varphi(u_n^\pm - u_0^\pm)\, d(\mu^\pm - \mu_n^\pm) + r_n \lambda_1(\Omega^\pm) \int_{\mathbb{R}^d} \varphi\, u_n^\pm\, (u_n^\pm - u_0^\pm)\, dx.$$

We now observe that, first of all, by definition of φ

$$\begin{aligned}\int_{B_R} |\nabla(u_n^\pm - u_0^\pm)|^2\, dx &\leq \int_{B_{2R}} \varphi |\nabla(u_n^\pm - u_0^\pm)|^2\, dx \\ &\leq \int_{\mathbb{R}^d} \nabla(u_n^\pm - u_0^\pm) \cdot \nabla[\varphi(u_n^\pm - u_0^\pm)]\, dx - \int_{B_{2R}} (u_n^\pm - u_0^\pm) \nabla(u_n^\pm - u_0^\pm) \cdot \nabla\varphi\, dx.\end{aligned}$$

It is easy to check that, thanks to the weak H^1_{loc} convergence and the uniform convergence

$$\lim_{n\to\infty} \|u_n^\pm - u^\pm\|_{L^\infty(B_{2R})} = 0,$$

therefore we get that the last term in the right-hand side converges to zero as $n \to \infty$. Moreover, we have

$$\lim_{n\to\infty} \lambda_1(\Omega^\pm) \int_{\mathbb{R}^d} \varphi\, u_n^\pm (u_n^\pm - u_0^\pm)\, dx = 0.$$

Finally, using again the local uniform convergence, we get

$$\begin{aligned}\left| \int_{\mathbb{R}^d} \varphi (u_n^\pm - u_0^\pm)\, d(\mu_n^\pm - \mu^\pm) \right| &\le \Big(\mu_n^\pm(B_{2R}) + \mu_0^\pm(B_{2R}) \Big) \|u_n^\pm - u_0^\pm\|_{L^\infty(B_{2R})} \\ &\le C_{2R} \|u_n^\pm - u_0^\pm\|_{L^\infty(B_{2R})} \to 0,\end{aligned}$$

which finally implies that $u_n^\pm$ strongly converges to $u_0^\pm$ in $H^1(B_R)$.

We now prove (ii). We will show that $\mathbb{1}_{\Omega_n^+}$ converges pointwise almost-everywhere to $\mathbb{1}_{\Omega_0^+}$. We first consider the case when $x_0 \in \mathbb{R}^d$ is a point of Lebesgue density one for Ω_0^+. If $x_0 \in \Omega_0^+$, then $u_0(x_0) > 0$ and by the uniform convergence of u_n to u_0, we get that $u_n(x_0) > 0$ for n large enough. This gives that

$$\mathbb{1}_{\Omega_0^+}(x_0) = 1 = \lim_{n\to\infty} \mathbb{1}_{\Omega_n^+}(x_0).$$

We will next show that x_0 can not be on the boundary of Ω_0^+. Let $\rho > 0$ be fixed and small. If there was a sequence of points x_n converging to x_0 such that $u_n(x_n) < 0$, then by the nondegeneracy of u_n^- we have that $\|u_n^-\|_{L^\infty(B_\rho(x_n))} > \rho\eta$, which passing to the limit as $n \to 0$ implies that $\|u_0^-\|_{L^\infty(B_{2\rho}(x_0))} > \rho\eta$. Thus, the L-Lipschitz continuity of u_0^- implies that in $B_{3\rho}(x_0)$ there is a ball of radius $\rho\eta/L$, where u_0^- is strictly positive (and so u_0^+ is zero). Since ρ is arbitrary, we obtain a contradiction with the fact that x_0 is of density 1 for Ω_0^+. This means that there is a ball $B_{r_0}(x_0)$ such that $\Omega_n^- \cap B_{r_0}(x_0) = \emptyset$, for every n large enough. In particular, in $B_{r_0}(x_0)$ the function u_0^+ is a blow-up limit of eigenfunctions on optimal sets for the first eigenvalue λ_1. Thus, by [28], u_0 is a local minimizer of the one-phase Alt-Caffarelli functional and so, it satisfies an exterior density estimate, that is, there are no points of density one on the boundary of Ω_0^+. This concludes the proof in the case when x_0 has density one.

Let now x_0 be a point of Lebesgue density 0 for Ω_0^+. By the continuity of u_0^+ we have that $u_0^+(x_0) = 0$ and $\mathbb{1}_{\Omega_u^+}(x_0) = 0$. Suppose for the sake of contradiction that (for some subsequence that we still denote by u_n^+) $u_n^+(x_0) > 0$ for every $n > 0$. But then the nondegeneracy of u_n at x_0 implies that there is a constant $\eta > 0$ such that

$$\|u_n^+\|_{L^\infty(B_\rho(x_0))} > \eta\rho,$$

for every $\rho > 0$ and every $n \ge 0$. As a consequence, the uniform L-Lipschitz continuity of u_n implies that there are points $x_n \in B_\rho(x_0)$ such that

$$u_n^+ \ge \frac{\eta}{2} \quad \text{in} \quad B_{\rho\eta/2L}(x_n).$$

Notice that, up to extracting a subsequence x_n converges to some point $x_\infty \in \overline{B}_\rho(x_0)$. The uniform convergence of u_n^+ now implies that

$$u_0^+ \ge \frac{\eta}{2} \quad \text{in} \quad B_{\rho\eta/2L}(x_\infty).$$

Since ρ is arbitrary this contradicts the initial assumption that x_0 has Lebesgue density 0. Thus, we get that for n large enough $u_n^+(x_0) = 0$, which implies that

$$\mathbb{1}_{\Omega_0^+}(x_0) = 0 = \lim_{n\to\infty} \mathbb{1}_{\Omega_n^+}(x_0),$$

and this concludes the proof. □

As an immediate corollary of Lemma 7.4 and Lemma 6.4, we obtain the following stationarity condition for the blow-up limits of u.

Lemma 7.5. *Let u be a Lipschitz continuous solution of (29) in the open set $D \subset \mathbb{R}^d$ and let $x_0 \in \partial\Omega_u \cap D$. Then, for every blow-up limit $u_0 : \mathbb{R}^d \to \mathbb{R}$ of u at x_0, we have the first variation formula*

$$\begin{aligned}0 = a_+ &\int_{\mathbb{R}^d} |\nabla u_0^+|^2 \operatorname{div}\xi - 2\nabla u_0^+\, D\xi\, (\nabla u_0^+)^t\, dx \\ &+ a_- \int_{\mathbb{R}^d} |\nabla u_0^-|^2 \operatorname{div}\xi - 2\nabla u_0^-\, D\xi\, (\nabla u_0^-)^t\, dx + \int_{\Omega_{u_0}} \operatorname{div}\xi\, dx\,,\end{aligned} \tag{55}$$

for every smooth vector field $\xi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, where a_+ and a_- are the nonnegative constants from Lemma 6.4.

7.2. **Homogeneity of the blow-up limits.** For every $u \in H^1(B_1)$, we consider the following Weiss-type boundary adjusted energy

$$\begin{aligned} W(u) = &\left[a^+ \int_{B_1} |\nabla u_+|^2\,dx + a^- \int_{B_1} |\nabla u_-|^2\,dx\right] \\ &- \left[a^+ \int_{\partial B_1} u_+^2\,d\mathcal{H}^{d-1} + a^- \int_{\partial B_1} u_-^2\,d\mathcal{H}^{d-1}\right] + |\Omega_u^+ \cup \Omega_u^- \cap B_1|. \end{aligned} \tag{56}$$

We will prove a monotonicity formula for W, which we will use to show that the blow-up limits are 1-homogeneous functions. The argument is standard (see [35]) and is based on the first variation formula (48) and a computation of the derivative of $W(u_{r,x_0})$ in r. We sketch the proof and we refer to [28] for the detailed computations.

Lemma 7.6 (Homogeneity of the blow-up limits)**.** *Let u be a Lipschitz continuous solution of (29) in the open set $D \subset \mathbb{R}^d$ and let $x_0 \in \partial\Omega_u \cap D$. Then, there is a constant $C > 0$ such that*

$$\frac{\partial}{\partial r} W(u_{x_0,r}) \ge \frac{2}{r}\left[a^+ \int_{\partial B_1} |x\cdot\nabla u_{x_0,r}^+ - u_{x_0,r}^+|^2\,d\mathcal{H}^{d-1} + a^- \int_{\partial B_1} |x\cdot\nabla u_{x_0,r}^- - u_{x_0,r}^-|^2\,d\mathcal{H}^{d-1}\right] - C, \tag{57}$$

where a_+ and a_- are the nonnegative constants from Lemma 6.4 and

$$u_{x_0,r}(x) := \frac{1}{r}u(x_0 + rx).$$

As a consequence, if $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a blow-up limit of u at x_0, then

(i) *if $a_+ > 0$, then u_0^+ is 1-homogeneous;*
(ii) *if $a_- > 0$, then u_0^- is 1-homogeneous.*

Proof. The estimate (57) follows directly from the first variation formula (48), just as in [28, Lemma 5.37]. Now, this implies that the function $r \mapsto W(u_{r,x_0}) + Cr$ is non-decreasing and so the limit

$$\Theta = \lim_{r\to 0} W(u_{x_0,r})$$

exists. If u_0 is a blow-up limit of u at x_0, u_0 is the locally uniform limit

$$u_0 = \lim_{n\to\infty} u_{r_n,x_0} \quad \text{for some sequence} \quad r_n \to 0,$$

then, setting $(u_0)_\rho(x) := \dfrac{1}{\rho}u(\rho x)$, we have

$$W\big((u_0)_\rho\big) = \lim_{n\to\infty} W(u_{x_0,\rho r_n}) = \Theta \quad \text{for every} \quad \rho > 0.$$

On the other hand, we know that u_0 satisfies the optimality condition (55). Thus, using again the computations from [28, Lemma 5.37], we get that

$$\frac{\partial}{\partial r} W\big((u_0)_r\big) \ge \frac{2}{r}\left[a^+ \int_{\partial B_1} |x\cdot\nabla(u_0)_r^+ - (u_0)_r^+|^2\,d\mathcal{H}^{d-1} + a^- \int_{\partial B_1} |x\cdot\nabla(u_0)_r^- - (u_0)_r^-|^2\,d\mathcal{H}^{d-1}\right]. \tag{58}$$

On the other hand, we know that $W\big((u_0)_r\big)$ is constant: $W\big((u_0)_r\big) = \Theta$ for every $r > 0$. Thus the right-hand side of (58) is zero. This gives the claims (i) and (ii). □

7.3. **Proof of Theorem 7.2.** We are now in position to prove Theorem 7.2, which will imply Corollary 7.3 and conclude the proof of Theorem 1.4 (and also the one of Theorem 1.1). We proceed in several steps.

Step 1. *The nondegeneracy of the coefficients (50) implies the classification of the blow-up limits (51) and (52).* Indeed, if $a_+ > 0$ and $a_- > 0$, then by Lemma 7.6 any blow-up limit u_0 of u at a two-phase point x_0 is one-homogeneous. Moreover, since u_0 is harmonic on $\Omega_0^+ := \{u_0 > 0\}$ and $\Omega_0^- := \{u_0 < 0\}$, we have that it can be written in polar coordinates as

$$u_0(r,\theta) = r\phi(\theta),$$

where the positive and the negative parts of $\phi : \mathbb{S}^{d-1} \to \mathbb{R}$ are non-zero (due to the nondegeneracy of u_0^+ and u_0^-) and are eigenfunctions on their supports, that is

$$-\Delta_{\mathbb{S}^{d-1}}\phi_\pm = (d-1)\phi_\pm \quad \text{on} \quad \partial B_1 \cap \Omega_0^\pm.$$

We now choose α and β such that

$$\int_{\mathbb{S}^{d-1}} \big(\alpha^2\phi_+^2 + \beta^2\phi_-^2\big)\,d\theta = 1 \qquad \text{and} \qquad \int_{\mathbb{S}^{d-1}} \big(\alpha\phi_+ + \beta\phi_-\big)\,d\theta = 0.$$

Moreover, integrating by parts on the sphere, we have that

$$\int_{\mathbb{S}^{d-1}} \left|\nabla\big(\alpha\phi_+ + \beta\phi_-\big)\right|^2 d\theta = d-1.$$

Now, by the variational formula for the eigenfunctions of the spherical Laplacian

$$d-1 = \min\Big\{ \int_{\mathbb{S}^{d-1}} |\nabla\psi|^2\,d\theta \ :\ \psi\in H^1(\mathbb{S}^{d-1}),\ \int_{\mathbb{S}^{d-1}}\psi\,d\theta = 0,\ \int_{\mathbb{S}^{d-1}}\psi^2\,d\theta = 1\Big\},$$

we get that the function

$$\alpha\phi_+ + \beta\phi_- \,:\, \mathbb{S}^{d-1}\to\mathbb{R},$$

is an eigenfunction of the Laplace-Beltrami operator corresponding to the eigenvalue $d-1$. Thus, u_0^+ and u_0^- are linear functions, which gives (51), that is, there are a unit vector $\nu\in\partial B_1$ and constants $\beta_+>0$ and $\beta_->0$ (notice that these constants are not a priori related to the auxiliary constants α and β above) such that

$$u_0(x) := \beta_+(x\cdot\nu)_+ - \beta_-(x\cdot\nu)_-.$$

Now, in order to prove that β_+ and β_- satisfy (52), we use the stationarity of u_0. Indeed, integrating by parts (55) we get that for every smooth vector field $\xi\in C^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$, we have

$$\int_{H_\nu}\Big(a_+|\nabla u_0^+|^2 - a_-|\nabla u_0^-|^2\Big)\,\xi\cdot\nu\,d\mathcal{H}^{d-1} = 0,$$

where H_ν is the hyperplane $\big\{x\in\mathbb{R}^d\ :\ x\cdot\nu = 0\big\}$. Since the vector field ξ is arbitrary, we get that

$$a_+\beta_+^2 - a_-\beta_-^2 = 0.$$

Step 2. *Strict positivity of the coefficients a_+ and a_-.*
Since $a_+\geq 0$, $a_-\geq 0$ and $a_++a_-=1$, we only need to exclude the case when one of the coefficients is zero and the other one is 1. We argue by contradiction and we suppose that $a^-=0$ and $a^+=1$. We consider two cases.

Step 2 - Case 1. *There are no two-phase points in D.* In this case, we have that Ω_u^- and Ω_u^+ lie at a positive distance in D. Now, if $a_-=0$, we have that

$$\int_{\Omega_u^-}\operatorname{div}\xi\,dx = 0 \quad \text{for every} \quad \xi\in C^\infty_c\big(D\setminus\overline{\Omega}_u^+;\mathbb{R}^d\big).$$

Choosing vector fields of the form $(x-x_0)\phi_{\varepsilon,r}(x-x_0)$, where the family of functions $\phi_{\varepsilon,r}\in C^\infty_c(B_r)$ is such that

$$\phi = 1 \quad\text{in}\quad B_{(1-\varepsilon)r},\qquad \phi_{\varepsilon,r}(x) = \frac{r-|x|}{\varepsilon r}\quad\text{in}\quad B_r\setminus B_{(1-\varepsilon)r},$$

and passing to the limit as $\varepsilon\to 0$, we get that (for almost-every $r>0$)

$$|B_r(x_0)\cap\Omega_u^-| = \frac{r}{d}\mathcal{H}^{d-1}\big(\partial B_r(x_0)\cap\Omega_u^-\big).$$

Thus, the function

$$r\mapsto \frac{|B_r(x_0)\cap\Omega_u^-|}{|B_r|}$$

is constant for every $x_0\in D\setminus\overline{\Omega}_u^+$, which is impossible in the neighborhood of any one-phase point $x_0\in\partial\Omega_u^-\cap D$.

Step 2 - Case 2. *There is at least one two-phase point $x_0\in\partial\Omega_u^+\cap\partial\Omega_u^-\cap D$.* Let $r>0$ be small enough such that $B_r(x_0)\subset D$ and let y_0 be any point such that

$$y_0\in B_{r/2}(x_0)\qquad\text{and}\qquad y_0\in\Omega_u^-.$$

Let z_0 be the projection of y_0 at $\partial\Omega_u^+$. Notice that by construction, we have that $z_0\in D$. Let now u_0 be a blow-up limit of u at z_0. Since $B_{r/2}(y_0)\cap\Omega_u^+=\emptyset$, we know that u_0^+ vanishes in the half space

$$H_\nu^+ = \{x\in\mathbb{R}^d\ :\ x\cdot\nu>0\},\qquad\text{where}\qquad \nu = \frac{z_0-y_0}{|z_0-y_0|}.$$

On the other hand, u_0^+ is harmonic in $\{u_0>0\}$ and, by Lemma 7.6, 1-homogeneous. But then u_0^+ should be a linear function:

$$u_0^+(x) = c(x\cdot\nu)_+\quad\text{for every}\quad x\in\mathbb{R}^d,$$

for some positive constant c. Conversely, for the negative part u_0^-, we know that Ω_u^- lies in the oppposite half-space

$$H_\nu^- = \{x\in\mathbb{R}^d\ :\ x\cdot\nu<0\},$$

and that

$$\int_{\Omega_u^-} \operatorname{div}\xi\,dx = 0 \quad \text{for every} \quad \xi \in C_c^\infty\big(H_\nu^-;\mathbb{R}^d\big).$$

Now, reasoning as in Step 2 - Case 1 and knowing that u_0^- is not identically zero in B_1 (due to the nondegeneracy of u_-), we get that $\Omega_u^- = H_\nu^-$. But now the optimality condition (55) gives that

$$0 = \int_{\partial H_\nu^+} a_+|\nabla u_0^+|^2(\xi\cdot\nu)\,d\mathcal{H}^{d-1} = c^2\int_{\partial H_\nu^+} \xi\cdot\nu\,d\mathcal{H}^{d-1},$$

for every smooth vector field $\xi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, which is a contradiction. This concludes the proof of Step 2.

Step 3. *Local inwards minimality of* u_+. We suppose that at least one of the one-phase free boundaries $\partial\Omega_u^+\setminus\partial\Omega_u^-$ and $\partial\Omega_u^-\setminus\partial\Omega_u^+$ is non-empty in D. Without loss of generality, there exists some point

$$y_0 \in D\cap\partial\Omega_u^+\setminus\partial\Omega_u^-.$$

Then, there is some $r>0$ such that $B_r(y_0)\cap\Omega_u^- = \emptyset$ and we can assume that $\partial\Omega_u^+$ is smooth in $B_r(y_0)$. Let now ξ be a smooth vector field in $B_r(y_0)$ and let

$$u_t(x) = u(\Psi_t(x)) \qquad \text{where} \qquad \Phi_t = (Id+t\xi)^{-1}.$$

. We can choose ξ in such a way that

$$|\{u_t>0\}\cap B_r(y_0)| - |\{u>0\}\cap B_r(y_0)| = t+o(t),$$

and

$$\int_{B_r(y_0)}|\nabla u_t|^2\,dx - \int_{B_r(y_0)}|\nabla u|^2\,dx = -\frac{1}{a_+}t+o(t),$$

Now, suppose that ρ is small enough and that $v\in H^1(B_\rho)$ is such that

$$u=v \quad \text{on} \quad \partial B_\rho, \qquad v\le u \quad \text{in} \quad B_\rho,$$

and consider the test function

$$\tilde v = v \quad \text{in} \quad B_\rho, \qquad \tilde v = u \quad \text{in} \quad D\setminus\big(B_\rho\cup B_r(y_0)\big), \qquad \tilde v = u_t \quad \text{in} \quad B_r(y_0),$$

where t is such that

$$\big|\{u_t>0\}\cap B_r(y_0)\big| + \big|\{v>0\}\cap B_\rho\big| = \big|\{u>0\}\cap B_\rho\big| + \big|\{u>0\}\cap B_r(y_0)\big|,$$

and in particular, $t=O(\rho^d)$. Thus, the minimality of u implies that

$$\frac{\lambda_1(\Omega_u^+) - \int_{B_\rho}\big(|\nabla u|^2-|\nabla v|^2\big) - \int_{B_r(y_0)}\big(|\nabla u|^2-|\nabla u_t|^2\big)}{1-\int_{B_\rho}\big(u^2-v^2\big)-\int_{B_r(y_0)}\big(u^2-u_t^2\big)} \ge \lambda_1(\Omega_u^+),$$

and so

$$\begin{aligned}
\int_{B_\rho}|\nabla v|^2\,dx &\ge \int_{B_\rho}|\nabla u|^2\,dx + \int_{B_r(y_0)}\big(|\nabla u|^2-|\nabla u_t|^2\big)\,dx + o(\rho^d)\\
&\ge \int_{B_\rho}|\nabla u|^2\,dx + \frac{1}{a_+}\Big(\big|\{u>0\}\cap B_\rho\big| - \big|\{v>0\}\cap B_\rho\big|\Big) + o(t)+o(\rho^d)\\
&= \int_{B_\rho}|\nabla u|^2\,dx + \frac{1}{a_+}\Big(\big|\{u>0\}\cap B_\rho\big| - \big|\{v>0\}\cap B_\rho\big|\Big) + o(\rho^d).
\end{aligned}$$

Thus, the rescaling $u_\rho(x) = \frac{1}{\rho}u(\rho x)$ satisfies

$$\int_{B_1}|\nabla v|^2\,dx \ge \int_{B_1}|\nabla u_\rho|^2\,dx + \frac{1}{a_+}\Big(\big|\{u_\rho>0\}\cap B_1\big| - \big|\{v>0\}\cap B_1\big|\Big) + o(1), \tag{59}$$

for all test functions v such that

$$v = u_\rho^+ \quad \text{on} \quad \partial B_1\,, \quad v\le u_\rho^+ \quad \text{in} \quad B_1\,.$$

Step 4. *Local inwards minimality of* u_0^+. Let $v : B_1\to\mathbb{R}$ be such that

$$v = u_0^+ \quad \text{on} \quad \partial B_1\,, \quad v\le u_0^+ \quad \text{in} \quad B_1\,.$$

Fix $\varepsilon > 0$ and consider the function

$$v_\varepsilon : B_1 \to \mathbb{R}, \quad v_\varepsilon(x) = v(x) + \varepsilon\big(|x| - (1-\varepsilon)\big)_+$$

Thus, if $u_n := u_{r_n}$ is a blow-up sequence that converges uniformly to u_0^+, then $v_\varepsilon \geq u_n^+$ on ∂B_1, for every n. Thus, we can use $v_\varepsilon \wedge u_n^+$ to test the minimality of u_n^+ in (59), thus obtaining that

$$\int_{B_1} |\nabla(v_\varepsilon \wedge u_n^+)|^2\,dx \geq \int_{B_1} |\nabla u_n^+|^2\,dx + \frac{1}{a_+}\Big(\big|\{u_n^+ > 0\} \cap B_1\big| - \big|\{v_\varepsilon \wedge u_n^+ > 0\} \cap B_1\big|\Big) + o(1). \tag{60}$$

Now, using Lemma 7.4 and passing to the limit as $n \to \infty$ gives

$$\begin{aligned}\int_{B_1} |\nabla(v_\varepsilon \wedge u_0^+)|^2\,dx &\geq \int_{B_1} |\nabla u_0^+|^2\,dx + \frac{1}{a_+}\Big(\big|\{u_0 > 0\} \cap B_1\big| - \big|\{v_\varepsilon \wedge u_0^+ > 0\} \cap B_1\big|\Big)\\ &\geq \int_{B_1} |\nabla u_0^+|^2\,dx + \frac{1}{a_+}\Big(\big|\{u_0 > 0\} \cap B_1\big| - \big|\{v > 0\} \cap B_1\big|\Big) - \frac{1}{a_+}|B_1 \setminus B_{1-\varepsilon}|,\end{aligned}$$

which, since ε was arbitrary gives that

$$\int_{B_1} |\nabla v|^2\,dx + \frac{1}{a_+}\big|\{v > 0\} \cap B_1\big| \geq \int_{B_1} |\nabla u_0^+|^2\,dx + \frac{1}{a_+}\big|\{u_0^+ > 0\} \cap B_1\big|,$$

and concludes the proof of Step 4.

Step 5. *Non-degeneracy of β_+ and β_-.* Let now x_0 be a two-phase point in D and u_0 be a blow-up limit of u at x_0. We know that u_0 is of the form (51), where β_+ and β_- are such that $a_+\beta_+^2 = a_-\beta_-^2$. Let ξ be any smooth vector field entering the half-space H_ν^+. Then, the inwards minimizing property of u_0^+ implies that

$$\int_{\partial H_\nu^+} \big(a_+\beta_+^2 - 1\big)|\xi \cdot \nu|\,d\mathcal{H}^{d-1} \geq 0.$$

Thus, $a_+\beta_+^2 \geq 1$ and, as a consequence, $a_-\beta_-^2 \geq 1$. This concludes the proof of Theorem 7.2. □

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