ON RIEMANNIAN FOUR-MANIFOLDS AND THEIR TWISTOR SPACES: A MOVING FRAME APPROACH

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Abstract. In this paper we study the twistor space Z of an oriented Riemannian four-manifold M using the moving frame approach, focusing, in particular, on the Einstein, non-self-dual setting. We prove that any general first-order linear condition on the almost complex structures of Z forces the underlying manifold M to be self-dual, also recovering most of the known related rigidity results. Thus, we are naturally lead to consider first-order quadratic conditions, showing that the Atiyah-Hitchin-Singer almost Hermitian twistor space of an Einstein four-manifold bears a resemblance, in a suitable sense, to a nearly Kähler manifold.

Contents

2010 *Mathematics Subject Classification.* 53C28, 53C25, 53B21.
Key words and phrases. Twistor space, four manifold, Einstein manifold, moving frames.
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1. Introduction and main results

Let (M, g) be a Riemannian manifold of dimension $2m$, with metric g. The *twistor space* Z associated to M is defined as the set of all the couples (p, J_p) such that $p \in M$ and J_p is a linear endomorphism of the tangent space T_pM which satisfies the following conditions:

(1) for every $X, Y \in T_pM$, $g_p(J_p(X), J_p(Y)) = g_p(X, Y)$;
(2) for every $X \in T_pM$, $J_p(J_p(X)) = -X$.

Such an endomorphism is called a *g-orthogonal complex structure* on T_pM [4].

The twistor space Z defines a fiber bundle over M *via* the map that assigns to every couple (p, J_p) the point $p \in M$; hence, we can define Z in an equivalent way as

$$Z = O(M)\,/U(m)\,,$$

where $O(M)$ denotes the orthonormal frame bundle over M and the unitary group $U(m)$ is identified with the subgroup of $SO(2m)$ given by

$$\{A \in SO(2m) : AJ_m = J_mA\},$$

where J_m is the representative matrix of the endomorphism of T_pM such that, for every orthonormal basis $\{e_1, \dots, e_{2m}\}$ of T_pM, $J_m(e_{2j-1}) = e_{2j}$, $j = 1, \dots, m$, and $J_m^2 = -I_{2m}$.

Throughout this paper, we consider only the case in which M is oriented, in order to exploit the existence of two connected components $O(M)_+$ and $O(M)_-$ of $O(M)$ and, therefore, define the two connected components of Z as

$$Z_\pm = O(M)_\pm\,/U(m) = SO(M)\,/U(m)\,,$$

where $SO(M)$ is the orthonormal oriented frame bundle over M. Once we have chosen a connected component of Z, it is possible to define a natural family of Riemannian metrics g_t on it, where t is a positive parameter, by taking the pullback of a specific bilinear form defined on $SO(M)$, as explained in [16] and in [18].

These structures, introduced by Penrose ([21]) as an attempt to define an innovative framework for Physics, have been the subject of many investigations by the mathematical community, also in virtue of the numerous geometrical and algebraic tools involved in the definition of their properties. In 1978, Atiyah, Hitchin and Singer ([1]) adapted Penrose's twistor theory to the Riemannian context, introducing the concept of twistor space associated to a Riemannian four-manifold and paving the way for many researches about this topic. Indeed, one of the most attractive feature of the twistor spaces is the strict bond that exists between their geometry and the one of the underlying Riemannian manifolds: many characterizations of certain classes of Riemannian four-manifolds can be obtained by examining the geometrical properties of their twistor spaces.

[4] In this paper, we call *complex structure* an endomorphism J_V of a vector space V such that $J_V^2 = -\,\mathrm{Id}_V$, while we call *almost complex structure* a $(1,1)$-tensor field J on a differentiable manifold M such that J smoothly assigns, to every point p, a complex structure J_p on T_pM.

The particular interest for the four-dimensional geometry, beside the intrinsic importance due to the obvious relation with Relativity, arises from the unique structure of the Riemann curvature operator, which cannot be realized in any other dimension. Indeed, if (M, g) is a Riemannian manifold of dimension m, the Riemann curvature tensor Riem on M admits the well known decomposition

$$\mathrm{Riem} = \mathrm{W} + \frac{1}{m-2}\,\mathrm{Ric} \owedge g - \frac{S}{2(m-1)(m-2)} g \owedge g,$$

where W, Ric and S denote the *Weyl tensor*, the *Ricci tensor* and the *scalar curvature* of M, respectively, and $\owedge$ is the *Kulkarni-Nomizu product*. Moreover, the Riemann curvature tensor defines a symmetric linear operator from the bundle of two-forms Λ^2 to itself

$$\begin{aligned} \mathcal{R}\colon \Lambda^2 &\longrightarrow \Lambda^2 \\ \gamma &\longmapsto \mathcal{R}(\gamma) = \frac{1}{4} R_{ijkt}\gamma_{kt}\theta^i \wedge \theta^j, \end{aligned}$$

where $\{\theta^i\}_{i=1,\dots,m}$ is a local orthonormal coframe on an open set $U \subset M$, with dual frame $\{e_i\}_{i=1,\dots,2m}$, $\gamma_{kt} = \gamma(e_k, e_t)$ and R_{ijkt} are the components of the Riemann tensor with respect to the coframe $\{\theta^i\}$ (note that, here and in the rest of the paper, we adopt Einstein summation convention over repeated indices, unless it is specified otherwise). In general, any $(0, 4)$-tensor P which satisfies the same symmetries as the Riemann curvature tensor induces a symmetric linear operator $\mathcal{P}$ defined as above.

If $m = 4$ and M is oriented, Λ^2 splits, *via* the Hodge $\star$ operator, into the direct sum of two subbundles Λ_+ and Λ_-, which are called the bundles of *self-dual* and *anti-self-dual* forms, respectively. This implies that the Riemann curvature operator gives rise to three linear maps A, B and C, such that A (resp, C) is a symmetric endomorphism of Λ_+ (resp., Λ_-) and B is a linear map from Λ_+ to Λ_- (see [1] and [2]); therefore, $\mathcal{R}$ is represented by a block matrix

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix},$$

where $A^T = A$, $C^T = C$. Moreover, $\mathrm{tr}\, A = \mathrm{tr}\, C = \frac{S}{4}$. This corresponds to a decomposition of the Weyl tensor into a sum

$$W = W^+ + W^-,$$

where W^+ (resp., W^-) is called the *self-dual* (resp., *anti-self-dual*) *part* of W. If $W^+ = 0$ (resp., $W^- = 0$), we say that M is an *anti-self-dual* (resp., *self-dual*) *manifold*. If we consider the symmetric linear operators induced by W^+ and W^-, we have that their representative matrices are $A - \frac{S}{12}I_3$ and $C - \frac{S}{12}I_3$, respectively, with respect to any positively oriented local orthonormal coframe; thus, (M, g) is self-dual (resp., anti-self-dual) if and only if $C = \frac{S}{12}I_3$ (resp., $A = \frac{S}{12}I_3$). Note that, if the coframe is negatively oriented, A and C need to be exchanged in the previous statements.

In the literature, many results about the relation between a Riemannian four-manifold M and its twistor space Z were achieved starting from the hypothesis that M is self-dual: for instance, Atiyah, Hitchin and Singer, in [1], introduced an almost complex structure J on Z_- and showed that M is a self-dual manifold if and only if J is integrable, i.e. the associated Nijenhuis tensor N_J vanishes identically, while Eells and Salamon, in [7], defined an almost complex structure $\mathbf{J}$ on Z_- which is never integrable. In 1985, Friedrich and Grunewald ([9]) showed that (Z_-, g_t) is Einstein if and only if M is Einstein, self-dual and with positive scalar curvature. Some important characterization theorems for Einstein self-dual manifolds were proved by Jensen and Rigoli ([16]) and by Davidov and Muškarov (see [19],[4], [5], [6]), starting from the classification of the almost Hermitian manifolds due to Gray and Hervella ([13]). In the recent paper [10], the authors exploit the moving frame formalism to study, among other things, the so-called balanced and first Gauduchon metric conditions on the twistor spaces of a Riemannian four-manifold.

In this paper we start from the following questions:

(1) Is it possible to introduce a framework that could simplify the study of the Riemannian and Hermitian features of the twistor space associated to a Riemannian four-dimensional manifold?
(2) Given an *Einstein four-manifold* M, is it possible to find new and interesting properties of its associated twistor space?

Our approach to the aforementioned questions is inspired by the works of Jensen and Rigoli ([16]) and of Fu and Zhou ([10]): all our computations of the main Riemannian and Hermitian features of the twistor spaces are based on the method of moving frames *à la* Cartan, which provides an effective answer to question (1). As a consequence of our analysis we are able to easily recover and generalize some classical results. In particular, our first main result is the following

Theorem 1.1. *Let (M, g) be an oriented Riemannian four-manifold and let $(Z_-, g_t, \bar{J})$ be its twistor space, with $\bar{J} = J$ or $\bar{J} = \mathbf{J}$. Suppose that, for every X, Y smooth vector fields on Z_-,*

$$a_1(\nabla_X \bar{J})Y + a_2(\nabla_Y \bar{J})X + a_3(\nabla_{\bar{J}X} \bar{J})Y + a_4(\nabla_{\bar{J}Y} \bar{J})X + a_5(\nabla_{\bar{J}X} \bar{J})\bar{J}Y+$$
$$+ a_6(\nabla_{\bar{J}Y} \bar{J})\bar{J}X + a_7(\nabla_X \bar{J})\bar{J}Y + a_8(\nabla_Y \bar{J})\bar{J}X = 0$$

for some $a_i \in \mathbb{R}$, $i = 1, \dots, 8$, such that $a_j \neq 0$ for some j. Then, M is self-dual.

This theorem allows us to prove in an alternative way the integrability result on the Atiyah-Hitchin-Singer almost complex structure in [1] and the characterization results for Einstein, self-dual manifolds with positive scalar curvature in [19] and [4]. Concerning question (2), since one of our main goals is to study Einstein four-manifolds whose metrics are not necessarily self-dual, the previous theorem naturally lead us to consider first-order quadratic conditions: more precisely, we are able to show a local (i.e., holding only for *orthonormal* frames/coframes), quadratic characterization of Einstein four-manifolds:

Theorem 1.2. *An oriented Riemannian four-manifold (M, g) is Einstein if and only if, for every orthonormal frame in $O(M)_-$ (equivalently, for every negatively oriented orthonormal coframe),*

$$\sum_{t=1}^{6}(J^t_{p,q} + J^t_{q,p})(J^t_{p,p} - J^t_{q,q}) = 0, \quad \forall p, q = 1, \ldots, 6,$$

where $J^t_{p,q}$ are the components of the covariant derivative of J with respect to a local orthonormal coframe on (Z_-, g_t, J).

Moreover, we can prove a quadratic necessary and sufficient condition for Einstein, non-self-dual manifolds:

Theorem 1.3. *Let (M, g) be an oriented Riemannian Einstein four-manifold. Then, for every orthonormal frame in $O(M)_-$,*

$$(J^t_{q,p} + J^t_{p,q})N^t_{pq} = 0 \qquad \textit{(no sum over p,q,t)},$$

where $J^t_{p,q}$ and N^t_{pq} are the components of the covariant derivative of J and of the Nijenhuis tensor, respectively, with respect to a local orthonormal coframe on (Z_-, g_t, J). Conversely, if (M, g) is not self-dual and the equations above hold on $O(M)_-$, then (M, g) is Einstein.

Finally, we compute the components of the Ricci* tensor $\overline{\mathrm{Ric}}^*$ on Z_- in order to prove the following estimates on the holomorphic scalar curvature $\overline{S}_J = \overline{S} - \overline{S}^*$, where $\overline{S}$ and $\overline{S}^*$ are the scalar curvatures of $\overline{\mathrm{Ric}}$ (the Ricci curvature of g_t) and $\overline{\mathrm{Ric}}^*$, respectively (see Section 6 for the precise definitions)

Theorem 1.4. *Let (M, g) be an Einstein four-manifold with positive scalar curvature. Then, on (Z_-, g_t, J) the following inequality holds:*

$$-\frac{1}{2}|\nabla J|^2 \leq \overline{S} - \overline{S}^* \leq |\nabla J|^2,$$

Moreover, one of the equalities holds if and only if M is self-dual.

Note that the Ricci* tensor measures, in some sense, an almost Hermitian manifold is far from being Kähler.

Remark 1.5. It is a well-known fact (see Theorem 5.4) that the twistor space (Z_-, g_t, J) of an Einstein, self-dual manifold with positive scalar curvature (M, g) is nearly-Kähler (indeed, Kähler: see Section 3 for the precise definitions): this means that, for every orthonormal frame in $O(M)_-$, $J^t_{q,p} + J^t_{p,q} = 0$, and also that $\overline{S} - \overline{S}^* = |\nabla J|^2$; moreover, by Hitchin's classification result ([15]), this is the case if and only if (M, g) is isometric (up to quotients) to $\mathbb{S}^4$ or $\mathbb{CP}_2$. If (M, g) is Einstein, but not necessarily self-dual, the properties of its twistor space are not so very well understood: for some interesting results in this direction, see e.g. [22] and [8]. Theorems 1.2, 1.3 and 1.4 show that the Atiyah-Hitchin-Singer almost Hermitian twistor space of an Einstein four-manifold bears a resemblance to a nearly Kähler manifold. Note that, in this work, we do not focus our attention on almost Hermitian manifolds (in particular, twistor space associated to a Riemannian manifold) satisfying these "weak"-nearly Kähler conditions. A natural question would be the following: is it possible to characterize the round metric on $\mathbb{S}^4$ as the unique Einstein metric, by showing that the twistor space (Z_-, g_t, J) of a four-sphere $\mathbb{S}^4$ equipped with an Einstein metric g_{Ein} cannot satisfy the conditions in Theorems 1.2 and 1.3, unless it is Kähler (or nearly Kähler)?

This will be the subject of future investigations, together with the analysis of higher order conditions on the almost complex structures and of curvature properties of Z.

The paper is organized as follows: for the sake of completeness, and to fix the notation and conventions of the moving frame formalism, in Section 2 we recall some very well-known facts about the geometry of Riemannian 4-manifolds, while in Section 3 we review some useful definitions from complex and Kähler geometry. The short Section 4 is devoted to the formal definition of the twistor space Z of a Riemannian manifold, with special attention to the case of dimension four. In Section 5 we show that, given a Riemannian four-manifold M and its twistor space Z, any linear condition on the covariant derivative of the almost complex structures on Z implies that M is self-dual; we also show how to recover some of the classical result (due to Atiyah, Hitchin and Singer and to Muškarov). In Section 6 we focus on quadratic conditions, in order to study the case of Einstein four-manifolds whose metric is not necessarily self-dual. We conclude the paper with four brief appendices, where we collect the components (in a local orthonormal coframe) of all the relevant geometric quantities involved in our analysis.

2. Geometry of Riemannian four-manifolds

In this section, for the sake of completeness, we recall some useful and well known features of Riemannian four-manifolds (see e.g. [2] and [23]).

The Hodge $\star$ operator in four dimensions. Let (M, g) a Riemannian oriented manifold of dimension n and let Λ^k be the space (bundle) of the k-differential forms, $1 \leq k \leq n$. Given any local chart (U, φ) that contains $p \in M$, let $\{e_1, \dots, e_n\}$ be a local, positively oriented, orthonormal frame for g on U and let $\{\theta^1, \dots, \theta^n\}$ be its dual orthonormal coframe, with $\theta^i \in \Lambda^1$, $\forall i = 1, \dots, n$. Since M is oriented, we can define a *volume form* locally expressed by

$$\omega = \theta^1 \wedge \dots \wedge \theta^n \in \Lambda^n.$$

Now it is possible to define the *Hodge $\star$ operator*, $\forall\, 1 \leq k \leq n$, locally as

$$\begin{aligned} \star : \Lambda^k &\longrightarrow \Lambda^{n-k} \\ \theta^{i_1} \wedge \dots \wedge \theta^{i_k} &\longmapsto \star(\theta^{i_1} \wedge \dots \wedge \theta^{i_k}) =: \eta \end{aligned} \tag{2.1}$$

where $\eta = \theta^{j_1} \wedge \ldots \wedge \theta^{j_{n-k}} \in \Lambda^{n-k}$ is the unique $(n-k)$-form such that $(\theta^{i_1} \wedge \ldots \wedge \theta^{i_k}) \wedge \eta = \omega$. By construction, $\star$ satisfies the equation

$$\star^2 = (-1)^{k(n-k)} I, \tag{2.2}$$

where I is the identity map from Λ^k to itself. Now, let $n = 4$. We have that, if $k = 2$, the $\star$ operator is an involution: indeed, by definition and (2.2),

$$\star : \Lambda^2 \longrightarrow \Lambda^2 \quad \text{and} \quad \star^2 = (-1)^{2 \cdot 2} I = I.$$

If $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ is an orthonormal coframe for M in a given chart, the set $\{\theta^i \wedge \theta^j\}_{1 \leq i < j \leq 4}$ is an orthonormal basis for Λ^2 with respect to the inner product of differential forms induced on Λ^2 by the metric g. Moreover, since $\star$ is an involution, its only two eigenvalues are ± 1 and it can be easily seen that

$$\begin{aligned} \star(\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4) &= \theta^3 \wedge \theta^4 \pm \theta^1 \wedge \theta^2, \\ \star(\theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2) &= \theta^4 \wedge \theta^2 \pm \theta^1 \wedge \theta^3, \\ \star(\theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3) &= \theta^2 \wedge \theta^3 \pm \theta^1 \wedge \theta^4. \end{aligned} \tag{2.3}$$

This means that

$$\Lambda_{\pm} := span\{\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4, \theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2, \theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3\} \tag{2.4}$$

are the eigenspaces of $\star$ relative to the eigenvalue ± 1, respectively. Thus, by (2.4), we have that Λ^2 decomposes in a direct sum of two three-dimensional subspaces (subbundles)

$$\Lambda^2 = \Lambda_+ \oplus \Lambda_-. \tag{2.5}$$

Note that, if $\{\theta^i\}_{i=1,\ldots,4}$ is a negatively oriented orthonormal coframe, the signs $+$ and $-$ must be exchanged in the right-hand side of (2.4). Moreover it is sufficient to define

$$\begin{aligned} \alpha^1_{\pm} &:= \frac{1}{\sqrt{2}}(\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4) \\ \alpha^2_{\pm} &:= \frac{1}{\sqrt{2}}(\theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2) \\ \alpha^3_{\pm} &:= \frac{1}{\sqrt{2}}(\theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3) \end{aligned} \tag{2.6}$$

to have orthonormal bases for Λ_+ and Λ_-, which are called, respectively, the bundle of *self-dual* and *anti-self-dual* 2-forms of M. Clearly, any 2-form η can be written in a unique way as

$$\eta = \underbrace{\frac{1}{2}(\eta + \star\eta)}_{\in \Lambda_+} + \underbrace{\frac{1}{2}(\eta - \star\eta)}_{\in \Lambda_-} =: \eta_+ + \eta_-, \tag{2.7}$$

where η_+ is the *self-dual* part of η and η_- is the *anti-self-dual* part.

There is an action of $SO(4)$ on Λ^1, defined as

$$\begin{aligned} SO(4)\times\Lambda^1 &\longrightarrow \Lambda^1 \\ (a,\theta^i) &\longmapsto a(\theta^i) := (a^{-1})^i_j\theta^j, \end{aligned} \tag{2.8}$$

which induces an action of $SO(4)$ on Λ^2 given by

$$a(\theta^i\wedge\theta^j) := a(\theta^i)\wedge a(\theta^j) \tag{2.9}$$

(see e.g. [16]). Moreover, it is known that $\mathfrak{so}(4)$ and Λ^2 are isomorphic *via* the map

$$\begin{aligned} f\colon \mathfrak{so}(4) &\longrightarrow \Lambda^2 \\ X=(X_{ij}) &\longmapsto \frac{1}{2}X_{ij}\theta^i\wedge\theta^j \end{aligned} \tag{2.10}$$

(here, $\mathfrak{so}(4)$ denotes the Lie algebra of $SO(4)$). The isomorphism f satisfies, for every $a\in SO(4)$, $X\in\mathfrak{so}(4)$,

$$f(\mathrm{Ad}_a)(X) = a(f(X))$$

where Ad means the adjoint representation of $SO(4)$, and, since $\mathfrak{so}(4)\cong\mathfrak{so}(3)\oplus\mathfrak{so}(3)$, f induces isomorphisms f_+, f_-

$$f_\pm : \mathfrak{so}(3) \longrightarrow \Lambda_\pm.$$

The restriction of the adjoint action of $SO(4)$ to each copy of $\mathfrak{so}(3)$ induces actions of $SO(3)$ on Λ_+ and Λ_-: namely, there exist smooth actions

$$\begin{aligned} SO(3)\times\Lambda_\pm &\longrightarrow \Lambda_\pm \\ (a,\eta_\pm) &\longmapsto a(\eta_\pm) \end{aligned}$$

such that, for every $a\in SO(3)$ and $Y\in\mathfrak{so}(3)$, $a(f_\pm(Y)) = f_\pm^{-1}(\mathrm{Ad}_a(Y))$. Moreover, there exists a surjective Lie group homomorphism

$$\mu : SO(4) \longrightarrow SO(3)\times SO(3) \tag{2.11}$$

such that, for every $a\in SO(4)$, $\mu(a) = (a_+, a_-)$, where, for every $\eta=\eta_++\eta_-\in\Lambda^2=\Lambda_+\oplus\Lambda_-$,

$$a(\eta) = a_+(\eta_+) + a_-(\eta_-).$$

Decomposition of the Riemann curvature tensor. Let (M,g) be again a Riemannian, oriented, manifold of dimension n. We denote by Riem its Riemann curvature tensor and by R_{ijkt} its components with respect to an orthonormal coframe $\{\theta^1,\dots,\theta^n\}$, with $i,j,k,t=1,\dots,n$, which have the following symmetries:

$$\begin{aligned} R_{ijkt} &= -R_{jikt} = -R_{ijtk}, \\ R_{ijkt} &= R_{ktij}, \\ R_{ijkt} + R_{iktj} &+ R_{itjk} = 0. \end{aligned} \tag{2.12}$$

We also define the *curvature forms* Ω^i_j associated to the orthonormal coframe $\{\theta^i\}$ as the 2-forms satisfying the second structure equation

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Omega^i_j, \tag{2.13}$$

where θ^i_j are the Levi-Civita connection 1-forms which satisfy the first structure equation

$$d\theta^i = -\theta^i_j \wedge \theta^j \tag{2.14}$$

(see e.g. [3]). Since, for every $i, j = 1, \dots, n$, $\theta^i_j + \theta^j_i = 0$, we have that $\Theta^i_j + \Theta^j_i = 0$; thus, the matrix of the curvature forms Ω takes values in $\mathfrak{so}(n)$. Moreover, the curvature forms satisfy

$$\Omega^i_j = \frac{1}{2} R_{ijkt}\theta^k \wedge \theta^t, \tag{2.15}$$

where R_{ijkt} are exactly the Riemann curvature tensor components with respect to $\{\theta^i\}$. Let $e, \widetilde{e}$ be orthonormal frames such that there exists a smooth change $A : U \cap \widetilde{U} \longrightarrow O(m)$ for which $\widetilde{e} = eA$ (i.e. $\widetilde{e}_i = A^j_i e_j$). Recall that, if $\widetilde{\Omega}$ is the matrix of the curvature forms associated to the frame $\widetilde{e}$ (equivalently, to the coframe $\widetilde{\theta}$ dual to $\widetilde{e}$), then the following transformation law holds

$$\widetilde{\Omega} = A^{-1}\Omega A. \tag{2.16}$$

Let us define the *Kulkarni-Nomizu product* $\owedge$: if η and κ are two $(0, 2)$-symmetric tensors, we have that $\eta \owedge \kappa$ is the $(0, 4)$-tensor with components

$$(\eta \owedge \kappa)_{ijkt} := \eta_{ik}\kappa_{jt} - \eta_{it}\kappa_{jk} + \eta_{jt}\kappa_{ik} - \eta_{jk}\kappa_{it}.$$

It is well known that, $\forall n \geq 3$, the Riemann curvature tensor admits the decomposition

$$\mathrm{Riem} = \mathrm{W} + \frac{1}{n-2}\,\mathrm{Ric} \owedge g - \frac{S}{2(n-1)(n-2)} g \owedge g, \tag{2.17}$$

where $\mathrm{Ric} = R_{ij}\theta^i \otimes \theta^j$ is the Ricci curvature tensor, S is the scalar curvature and W is the *Weyl tensor*. Equation (2.17) can be written in local form as

(2.18)
$$R_{ijkt} = W_{ijkt} + \frac{1}{n-2}(R_{ik}\delta_{jt} - R_{it}\delta_{jk} + R_{jt}\delta_{ik} - R_{jk}\delta_{it}) - \frac{S}{(n-1)(n-2)}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk})$$

Now, let $n = 4$. It is possible to rewrite the equations (2.17) and (2.18) thanks to (2.5). We know that $\{\theta^i \wedge \theta^j\}_{1\leq i<j\leq 4}$ is an orthonormal basis for Λ^2 and that $\{\alpha^1_\pm, \alpha^2_\pm, \alpha^3_\pm\}$, defined in (2.6), is an orthonormal basis for $\Lambda_\pm$, respectively. The Riemann curvature tensor corresponds to a symmetric operator, called the *Riemann curvature operator*, defined as

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2 \qquad \mathcal{R}(\gamma) = \frac{1}{4} R_{ijkt}\gamma_{kt}\theta^i \wedge \theta^j = \frac{1}{2}\gamma_{kt}\Omega^k_t, \tag{2.19}$$

where $\gamma_{kt} = \gamma(e_k, e_t)$ ($\{e_i\}$ is the orthonormal frame dual to $\{\theta^i\}$). Since (2.5) holds, every 2-form γ can be written as in (2.7) and, since $\mathcal{R}(\gamma) \in \Lambda^2$, it also can be expressed in a unique sum

$$\mathcal{R}(\gamma) = \mathcal{R}(\gamma)_+ + \mathcal{R}(\gamma)_-.$$

Evaluating the Riemann curvature operator on the bases (2.6) in order to find the self-dual and the anti-self-dual parts of the images, we obtain that there exist three 3×3 matrices, $A = (A_{ij})$, $B = (B_{ij})$, $C = (C_{ij})$, $i, j = 1, 2, 3$, such that, again with respect to the basis $\{\alpha^1_\pm, \alpha^2_\pm, \alpha^3_\pm\}$ of $\Lambda_\pm$, the Riemann curvature operator representative matrix takes the form

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \tag{2.20}$$

where $A = A^T$, $C = C^T$ and $\operatorname{tr} A = \operatorname{tr} C = S/4$ (here, tr denotes the matrix trace). More explicitly, if $\gamma = c^+_j \alpha^j_+ + c^-_k \alpha^k_-$,

$$\mathcal{R}(\gamma) = (c^+_j A_{kj} + c^-_t B_{tk})\alpha^k_- + (c^+_j B_{kj} + c^-_t C_{kt})\alpha^k_-.$$

By direct computation, the entries of the matrices A, B and C are

$$\begin{aligned}
A_{11} &= \frac{1}{2}(R_{1212} + 2R_{1234} + R_{3434}),\\
A_{12} = A_{21} &= \frac{1}{2}(R_{1213} + R_{1334} + R_{1242} + R_{3442}),\\
A_{13} = A_{31} &= \frac{1}{2}(R_{1214} + R_{1434} + R_{1223} + R_{2334}),\\
A_{22} &= \frac{1}{2}(R_{1313} + 2R_{1342} + R_{4242}),\\
A_{23} = A_{32} &= \frac{1}{2}(R_{1314} + R_{1442} + R_{1323} + R_{2342}),\\
A_{33} &= \frac{1}{2}(R_{1414} + 2R_{1423} + R_{2323});
\end{aligned}$$

$$\begin{aligned}
B_{11} &= \frac{1}{2}(R_{1212} - R_{3434}) = \frac{1}{4}(R_{11} + R_{22} - R_{33} - R_{44}),\\
B_{12} &= \frac{1}{2}(R_{1213} - R_{1334} + R_{1242} - R_{3442}) = \frac{1}{2}(R_{23} + R_{14}),\\
B_{13} &= \frac{1}{2}(R_{1214} - R_{1434} + R_{1223} - R_{2334}) = \frac{1}{2}(R_{24} - R_{13}),\\
B_{21} &= \frac{1}{2}(R_{1213} + R_{1334} - R_{1242} - R_{3442}) = \frac{1}{2}(R_{23} - R_{14}),\\
B_{22} &= \frac{1}{2}(R_{1313} - R_{4242}) = \frac{1}{4}(R_{11} - R_{22} + R_{33} - R_{44}),\\
B_{23} &= \frac{1}{2}(R_{1314} + R_{1323} - R_{1442} - R_{2342}) = \frac{1}{2}(R_{34} + R_{12}),\\
B_{31} &= \frac{1}{2}(R_{1214} + R_{1434} - R_{1223} - R_{2334}) = \frac{1}{2}(R_{24} + R_{13}),\\
B_{32} &= \frac{1}{2}(R_{1314} - R_{1323} + R_{1442} - R_{2342}) = \frac{1}{2}(R_{34} - R_{12}),\\
B_{33} &= \frac{1}{2}(R_{1414} - R_{2323}) = \frac{1}{4}(R_{11} - R_{22} - R_{33} + R_{44});
\end{aligned}$$

$$
\begin{aligned}
C_{11} &= \frac{1}{2}(R_{1212} - 2R_{1234} + R_{3434}),\\
C_{12} = C_{21} &= \frac{1}{2}(R_{1213} - R_{1334} - R_{1242} + R_{3442}),\\
C_{13} = C_{31} &= \frac{1}{2}(R_{1214} - R_{1434} - R_{1223} + R_{2334}),\\
C_{22} &= \frac{1}{2}(R_{1313} - 2R_{1342} + R_{4242}),\\
C_{23} = C_{32} &= \frac{1}{2}(R_{1314} - R_{1442} - R_{1323} + R_{2342}),\\
C_{33} &= \frac{1}{2}(R_{1414} - 2R_{1423} + R_{2323}).
\end{aligned}
$$

Thus, we can think of A (respectively, C) as a symmetric linear map from Λ_+ (respectively, Λ_-) to itself, that is $A \in \mathrm{End}(\Lambda_+)$, $C \in \mathrm{End}(\Lambda_-)$, and we can think of B as a linear map from Λ_+ to Λ_-, i.e. $B \in \mathrm{Hom}(\Lambda_+, \Lambda_-)$.

Since $\mathcal{R}(\theta^i \wedge \theta^j) = \Omega^i_j$, by the previous considerations we can write the matrix Ω, thanks to the decomposition (2.20), in the form

$$
\Omega = A_{ij}\alpha^i_+ \otimes \alpha^j_+ + B_{ij}\alpha^i_- \otimes \alpha^j_+ + B_{ji}\alpha^i_+ \otimes \alpha^j_- + C_{ij}\alpha^i_- \otimes \alpha^j_-, \tag{2.21}
$$

or, in matrix notation,

$$
\Omega = \alpha_+ \otimes A\alpha_+ + \alpha_- \otimes B\alpha_+ + \alpha_+ \otimes B^T\alpha_- + \alpha_- \otimes C\alpha_-.
$$

It is also explicitly possible to write the transformation laws for A, B and C. Recall that, if $e, \widetilde{e}$ are two orthonormal frames defined on U and $\widetilde{U}$ and $a : U \cap \widetilde{U} \longrightarrow SO(4)$ is a smooth change of frame, the equation (2.16) holds for Ω. Since, for every $a \in SO(4)$, $\mu(a) = (a_+, a_-)$ defines the restriction of the action of a on Λ_+ and Λ_-, we obtain the following transformation laws

$$
\widetilde{A} = a_+^{-1}Aa_+, \qquad \widetilde{B} = a_-^{-1}Ba_+, \qquad \widetilde{C} = a_-^{-1}Ca_-. \tag{2.22}
$$

Since it will simplify all our next computations, we introduce the (purely local) quantities

$$
\overset{2}{\mathcal{Q}}_{ab} := R_{12ab} + R_{34ab}; \qquad \overset{3}{\mathcal{Q}}_{ab} := R_{13ab} + R_{42ab}; \qquad \overset{4}{\mathcal{Q}}_{ab} := R_{14ab} + R_{23ab}. \tag{2.23}
$$

Note that, by the first Bianchi identity, we deduce

$$
\overset{3}{\mathcal{Q}}_{12} + \overset{3}{\mathcal{Q}}_{34} = \overset{2}{\mathcal{Q}}_{13} + \overset{2}{\mathcal{Q}}_{42}; \quad \overset{4}{\mathcal{Q}}_{12} + \overset{4}{\mathcal{Q}}_{34} = \overset{2}{\mathcal{Q}}_{14} + \overset{2}{\mathcal{Q}}_{23}; \quad \overset{3}{\mathcal{Q}}_{14} + \overset{3}{\mathcal{Q}}_{23} = \overset{4}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{42}. \tag{2.24}
$$

We have for A and B the following expressions:

$$
\begin{aligned}
A_{11} &= \frac{1}{2}(R_{1212} + 2R_{1234} + R_{3434}) = \frac{1}{2}(Q^2_{12} + Q^2_{34});\\
A_{22} &= \frac{1}{2}(R_{1313} + 2R_{1342} + R_{4242}) = \frac{1}{2}(Q^3_{13} + Q^3_{42});\\
A_{33} &= \frac{1}{2}(R_{1414} + 2R_{1423} + R_{2323}) = \frac{1}{2}(Q^4_{14} + Q^4_{23});\\
A_{12} = A_{21} &= \frac{1}{2}(R_{1213} + R_{1334} + R_{1242} + R_{3442}) = \frac{1}{2}(Q^3_{12} + Q^3_{34}) = \frac{1}{2}(Q^2_{13} + Q^2_{42})\\
A_{13} = A_{31} &= \frac{1}{2}(R_{1214} + R_{1434} + R_{1223} + R_{2334}) = \frac{1}{2}(Q^4_{12} + Q^4_{34}) = \frac{1}{2}(Q^2_{14} + Q^2_{23});\\
A_{23} = A_{32} &= \frac{1}{2}(R_{1314} + R_{1442} + R_{1323} + R_{2342}) = \frac{1}{2}(Q^3_{14} + Q^3_{23}) = \frac{1}{2}(Q^4_{13} + Q^4_{42});
\end{aligned}
$$

$$
\begin{aligned}
B_{11} &= \frac{1}{2}(Q^2_{12} - Q^2_{34});\\
B_{22} &= \frac{1}{2}(Q^3_{13} - Q^3_{42});\\
B_{33} &= \frac{1}{2}(Q^4_{14} - Q^4_{23});\\
B_{12} &= \frac{1}{2}(Q^3_{12} - Q^3_{34});\\
B_{21} &= \frac{1}{2}(Q^2_{13} - Q^2_{42});\\
B_{13} &= \frac{1}{2}(Q^4_{12} - Q^4_{34});\\
B_{31} &= \frac{1}{2}(Q^2_{14} - Q^2_{23});\\
B_{23} &= \frac{1}{2}(Q^4_{13} - Q^4_{42});\\
B_{32} &= \frac{1}{2}(Q^3_{14} - Q^3_{23}).
\end{aligned}
$$

As far as the Weyl tensor is concerned, by (2.18) and (2.20), we have

$$
W_{1212} = \frac{1}{2}\left(A_{11} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{11} - \frac{S}{12}\right)
$$

$$
W_{1213} = \frac{1}{2}A_{12} + \frac{1}{2}C_{12}, \qquad W_{1214} = \frac{1}{2}A_{13} + \frac{1}{2}C_{13},
$$

$$
W_{1223} = \frac{1}{2}A_{13} - \frac{1}{2}C_{13}, \qquad W_{1242} = \frac{1}{2}A_{12} - \frac{1}{2}C_{12},
$$

$$
W_{1234} = \frac{1}{2}\left(A_{11} - \frac{S}{12}\right) - \frac{1}{2}\left(C_{11} - \frac{S}{12}\right), \qquad W_{1313} = \frac{1}{2}\left(A_{22} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{22} - \frac{S}{12}\right),
$$

$$W_{1314} = \frac{1}{2}A_{23} + \frac{1}{2}C_{23}, \qquad W_{1323} = \frac{1}{2}A_{23} - \frac{1}{2}C_{23},$$

$$W_{1342} = \frac{1}{2}\left(A_{22} - \frac{S}{12}\right) - \frac{1}{2}\left(C_{22} - \frac{S}{12}\right), \qquad W_{1334} = \frac{1}{2}A_{12} - \frac{1}{2}C_{12},$$

$$W_{1414} = \frac{1}{2}\left(A_{33} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{33} - \frac{S}{12}\right), \qquad W_{1423} = \frac{1}{2}\left(A_{33} - \frac{S}{12}\right) - \frac{1}{2}\left(C_{33} - \frac{S}{12}\right),$$

$$W_{1442} = \frac{1}{2}A_{23} - \frac{1}{2}C_{23}, \qquad W_{1434} = \frac{1}{2}A_{13} - \frac{1}{2}C_{13},$$

$$W_{2323} = \frac{1}{2}\left(A_{33} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{33} - \frac{S}{12}\right), \qquad W_{2342} = \frac{1}{2}A_{23} + \frac{1}{2}C_{23},$$

$$W_{2334} = \frac{1}{2}A_{13} + \frac{1}{2}C_{13}, \qquad W_{4242} = \frac{1}{2}\left(A_{22} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{22} - \frac{S}{12}\right),$$

$$W_{3442} = \frac{1}{2}A_{12} + \frac{1}{2}C_{12}, \qquad W_{3434} = \frac{1}{2}\left(A_{11} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{11} - \frac{S}{12}\right).$$

It is apparent that all the components can be written as a sum of two addends

$$W_{ijkt} = W^+_{ijkt} + W^-_{ijkt}$$

This means that the Weyl tensor W splits into a sum of two $(0,4)$-tensors

$$W = W^+ + W^-$$

called, respectively, the *self-dual* and the *anti-self-dual* components of W. A four-dimensional Riemannian manifold is called *self-dual* (respectively *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$). By a direct check of the entries of A, B and C and the coefficients of W^+ and W^-, we easily obtain the following

Theorem 2.1. *Let (M, g) be a Riemannian manifold of dimension 4. Then,*

- *M is self-dual if and only if $C - \frac{S}{12}I_3 = 0$ for every orthonormal positively oriented coframe (respectively, if and only if $A - \frac{S}{12}I_3 = 0$ for every orthonormal negatively oriented coframe);*
- *M is anti-self-dual if and only if $A - \frac{S}{12}I_3 = 0$ for every orthonormal positively oriented coframe (respectively, if and only if $C - \frac{S}{12}I_3 = 0$ for every orthonormal negatively oriented coframe);*
- *M is Einstein if and only if $B = 0$ for every orthonormal positively or negatively oriented coframe.*

3. Remarks on almost complex structures

Let M be a differentiable manifold of dimension n. An *almost complex structure* J on M is a smooth $(1,1)$-tensor field which assigns to every $p \in M$ an endomorphism J_p of T_pM such that

$$J_p^2 = J_p \circ J_p = -\operatorname{Id}_{T_pM},$$

or, equivalently, a linear endomorphism

$$J : TM \longrightarrow TM$$

such that $J^2 = -\operatorname{Id}_{TM}$, where TM is the tangent bundle of M. The couple (M, J) is called an *almost complex manifold*. Note that, by definition, the existence of an almost complex structure J on M implies that the dimension of M is $n = 2m$. If g is a Riemannian metric on M, we say that J is *g-orthogonal* if

$$g(JX, JY) = g(X, Y),$$

for every $X, Y \in \mathfrak{X}(M)$. In this case, we say that (M, g, J) is an almost Hermitian manifold. An almost complex structure define an endomorphism J_p on T_pM for every $p \in M$ such that $J_p^2 = -\operatorname{Id}_{T_pM}$; such an endomorphism is called *complex structure* on T_pM.

If $\{\theta^q\}_{q=1,\dots,2m}$ is a local orthonormal coframe defined on an open set $U \subset M$, with $\{\omega_q^p\}$ the associated Levi-Civita connection 1-forms, and $\{e_q\}$ is its dual frame, we can write

$$J = J_q^p \theta^q \otimes e_p, \tag{3.1}$$

where $J_q^p \in C^\infty(U)$ such that $J_p^q = -J_q^p$. If we denote as ∇ the Levi-Civita connection of (M, g), we can define the covariant derivative ∇J of J as

$$(\nabla J)(Y; X) = (\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y), \tag{3.2}$$

for every $X, Y \in \mathfrak{X}(M)$, or, with respect to an orthonormal coframe,

$$\nabla J = J_{q,t}^p \theta^t \otimes \theta^q \otimes e_p, \tag{3.3}$$

where $J_{q,t}^p \in C^\infty(U)$, $J_{p,t}^q = -J_{q,t}^p$ and

$$J_{q,t}^p \theta^t = dJ_q^p - J_r^p \omega_q^r + J_q^r \omega_r^p.$$

It is easy to show that

$$(\nabla_X J)(JY) = -J((\nabla_X J)(Y)), \tag{3.4}$$

and

$$g((\nabla_X J)(Y), Y) = 0, \tag{3.5}$$

for every $X, Y \in \mathfrak{X}(M)$. We also recall the definition of the *Nijenhuis tensor*:

$$N_J(X, Y) = [X, Y] + J([X, JY]) + J([JX, Y]) - [JX, JY] \quad \forall\, X, Y \in \mathfrak{X}(M); \tag{3.6}$$

there is an equivalent formulation in terms of ∇J, that is

$$N_J(X, Y) = (\nabla_Y J)(JX) - (\nabla_X J)(JY) + (\nabla_{JY} J)(X) - (\nabla_{JX} J)(Y). \tag{3.7}$$

Equation (3.7) follows from (3.6) using the fact that, by definition of covariant derivative,

$$(\nabla J)(Y;X) = (\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y),$$

and $\nabla_X Y - \nabla_Y X = [X,Y]$. The local description of N_J on an open set $U \subset M$ is given by

$$N_J = N^r_{pq}\theta^p \otimes \theta^q \otimes e_r, \tag{3.8}$$

where $N^r_{pq} \in C^\infty(U)$, $N^r_{qp} = -N^r_{pq}$ and

$$N^r_{pq} = J^s_p J^r_{s,q} - J^s_q J^r_{s,p} + J^s_q J^r_{p,s} - J^s_p J^r_{q,s} \tag{3.9}$$

by (3.7). An almost complex structure is said to be *integrable* if it is induced by a complex atlas on (M,J); by the Newlander-Nirenberg Theorem (see [20]), J is integrable if and only if $N_J = 0$ on M. If (M,g,J) is an almost Hermitian manifold, we say that (M,g,J) is Hermitian (and we write $(M,g,J) \in \mathcal{H}$) if J is integrable.

It is possible to define the *Kähler form* ω of (M,g,J), which is a two-form, as

$$\omega(X,Y) := g(JX,Y) \tag{3.10}$$

for every $X,Y \in \mathfrak{X}(M)$. In components, we have

$$\omega = \frac{1}{2}J^q_p\theta^p \wedge \theta^q = \sum_{p<q} J^q_p\theta^p \wedge \theta^q =: \sum_{p<q}\omega(e_p,e_q)\theta^p \wedge \theta^q; \tag{3.11}$$

moreover, computing the exterior derivative $d\omega$, we obtain

$$\begin{aligned} d\omega &= \frac{1}{2}J^q_{p,t}\theta^t \wedge \theta^p \wedge \theta^q = \\ &= \sum_{t<p<q}(J^q_{p,t} + J^t_{q,p} + J^p_{t,q})\theta^t \wedge \theta^p \wedge \theta^q =: \sum_{t<p<q} d\omega(e_t,e_p,e_q)\theta^t \wedge \theta^p \wedge \theta^q. \end{aligned} \tag{3.12}$$

We define the following classes of almost Hermitian manifolds, according to Gray-Hervella classification (see [13]):

- (M,g,J) is a **Kähler manifold** ($(M,g,J) \in \mathcal{K}$) if

$$\nabla J \equiv 0, \tag{3.13}$$

  or, equivalently, if

$$J^q_{p,t} = 0, \quad \forall t,p,q = 1,\dots,2m; \tag{3.14}$$

- (M,g,J) is an **almost Kähler manifold** ($(M,g,J) \in \mathcal{AK}$) if

$$g((\nabla_X J)Y,W) + g((\nabla_Y J)W,X) + g((\nabla_W J)X,Y) = 0 \tag{3.15}$$

  for every $X,Y,W \in \mathfrak{X}(M)$, or, equivalently, if

$$J^q_{p,t} + J^p_{t,q} + J^t_{q,p} = 0, \quad \forall t,p,q = 1,\dots,2m; \tag{3.16}$$

- (M, g, J) is a **nearly Kähler manifold** ($(M, g, J) \in \mathcal{NK}$) if

$$(\nabla_X J)X = 0, \tag{3.17}$$

for every $X \in \mathfrak{X}(M)$, or, equivalently, if

$$J^q_{p,t} + J^q_{t,p} = 0, \quad \forall t, p, q = 1, \ldots, 2m; \tag{3.18}$$

- (M, g, J) is a **quasi-Kähler manifold** ($(M, g, J) \in \mathcal{QK}$) if

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0, \tag{3.19}$$

for every $X, Y \in \mathfrak{X}(M)$, or, equivalently, if

$$J^q_{p,t} + J^r_t J^s_p J^q_{s,r} = 0, \quad \forall t, p, q = 1, \ldots, 2m; \tag{3.20}$$

- (M, g, J) is a $\mathbf{q^2}$**-Kähler manifold** ($(M, g, J) \in \mathcal{QQK}$) if

$$(\nabla_X J)X + (\nabla_{JX} J)JX = 0, \tag{3.21}$$

for every $X \in \mathfrak{X}(M)$, or, equivalently, if

$$J^q_{p,t} + J^q_{t,p} + J^r_t J^s_p J^q_{s,r} + J^r_p J^s_t J^q_{s,r} = 0, \quad \forall t, p, q = 1, \ldots, 2m; \tag{3.22}$$

- (M, g, J) is a **semi-Kähler manifold** ($(M, g, J) \in \mathcal{SK}$) if

$$g((\nabla_{e_p} J)e_p, X) + g((\nabla_{J(e_p)} J)J(e_p), X) = 0, \quad \forall X \in \mathfrak{X}(M), \tag{3.23}$$

or, equivalently, if

$$\sum_{p=1}^{2m} J^q_{p,p} = 0, \quad \forall q = 1, \ldots, 2m. \tag{3.24}$$

Using (3.12) and some identities involving N_J, $d\omega$ and ∇J (see e.g. Gray [11]), we have that

- $(M, g, J) \in \mathcal{K}$ if and only if $d\omega = 0$ and $N_J = 0$;
- $(M, g, J) \in \mathcal{AK}$ if and only if $d\omega = 0$;
- $(M, g, J) \in \mathcal{NK}$ if and only if $3J^q_{p,t} = d\omega(e_t, e_p, e_q)$, $\forall t, p, q$;
- $(M, g, J) \in \mathcal{QK}$ if and only if, $\forall t, p, q$,

$$d\omega(e_t, e_p, e_q) = d\omega(e_t, J(e_p), J(e_q)) - d\omega(e_p, J(e_q), J(e_t)) - d\omega(e_q, J(e_t), J(e_p));$$

- $(M, g, J) \in \mathcal{QQK}$ if and only if, $\forall t, p, q$,

$$d\omega(e_p, J(e_q), J(e_t)) + d\omega(e_t, J(e_q), J(e_p)) = 0;$$

- $(M, g, J) \in \mathcal{SK}$ if and only if $\delta\omega = 0$.

Here $\delta\omega$ is the codifferential of ω, defined by

$$\delta\eta := (-1)^{m(k-1)+1}(\star \circ d \circ \star)(\eta),$$

for every $\eta \in \Lambda^k$. Moreover, it is not difficult to see that the following chain of inclusions holds:

$$
\begin{array}{ccccccccc}
 & & & \mathcal{AK} & & & & & \\
 & & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\subset$} & & & & \\
\mathcal{H} & \supset & \mathcal{K} & & & \mathcal{QK} & \subset & \mathcal{QQK} & \subset\ \mathcal{SK} \\
 & & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\subset$} & & & & \\
 & & & \mathcal{NK} & & & & &
\end{array}
$$

and $\mathcal{K} = \mathcal{AK} \cap \mathcal{NK}$.

4. The twistor space of a four-manifold

Let (M, g_M) be a connected Riemannian manifold of dimension $2m$. We define its *twistor space* Z associated to M as the set of the pairs (p, J_p), where $p \in M$ and J_p is a g-orthogonal complex structure on T_pM. It is not hard to show that the set of all g-orthogonal complex structures is diffeomorphic to $O(2m)/U(m)$, where

$$U(m) := \{A \in O(2m) : A^T J_m = J_m A\}$$

and J_m is a matrix in $O(2m) \cap \mathfrak{so}(2m)$ with entries $(J_m)_{kl} = \delta_k^{l+1} - \delta_l^{k+1}$; therefore, it can be shown that, if we denote as $O(M)$ as the orthonormal frame bundle of M, Z is the associated bundle

$$Z = O(M) \times_{O(2m)} (O(2m)/U(m)) = O(M)/U(m).$$

This means that there exists a surjective smooth map $\sigma : O(M) \longrightarrow Z$ such that σ defines a principal bundle $(O(M), Z, U(m))$ with structure group $U(m)$. Moreover, the map

$$\pi_Z : Z \longrightarrow M$$

$$(p, J_p) \longmapsto p$$

determines a fiber bundle $(Z, M, O(2m)/U(m), O(2m))$ with structure group $O(2m)$ and standard fiber $O(2m)/U(m)$ (see [17]).

It is known that, in general, there exists a one-parameter family of Riemannian metrics g_t on Z, with $t > 0$ (see [18] and [16]). Let $m = 2$; from now on, we adopt the index conventions $1 \leq a, b, c, \ldots \leq 4$ and $1 \leq p, q, \ldots, \leq 6$. Given a local orthonormal coframe $\{\omega^a\}_{a=1,\ldots,4}$ on an open set $U \subset M$, with Levi-Civita connection forms $\{\omega_b^a\}$, we define

$$\omega^5 := \frac{1}{2}\big(\omega_3^1 + \omega_2^4\big), \quad \omega^6 := \frac{1}{2}\big(\omega_4^1 + \omega_3^2\big);$$

a local orthonormal coframe on (Z, g_t) is obtained by considering the pullbacks of $\omega^1, \ldots, \omega^6$ *via* a smooth section $u : U \longrightarrow Z$ of the twistor bundle determined by π_Z. This means that

$$g_t = \sum_{p=1}^{6} (\theta^p)^2, \tag{4.1}$$

where

$$\theta^a := u^*(\omega^a), \quad \theta^5 := 2tu^*(\omega^5), \quad \theta^6 := 2tu^*(\omega^6); \tag{4.2}$$

for the sake of simplicity, we write ω^a for $u^*(\omega^a)$ and similarly for $2t\omega^5$ and $2t\omega^6$. By (4.1) and (4.2), we can write

$$g_t = g_M + 4t^2\Big[\big(\omega^5\big)^2 + \big(\omega^6\big)^2\Big] = g_M + (\theta^5)^2 + (\theta^6)^2 \tag{4.3}$$

(again the pullback notation is omitted). In order to compute the Levi-Civita connection forms θ^p_q and the curvature forms Θ^p_q for the orthonormal coframe defined in (4.2), recall the structure equations (2.14) and (2.13). By direct computation, we obtain (see [16])

$$\theta^a_b = \omega^a_b + \frac{1}{2}t\big(\overset{3}{\mathcal{Q}}_{ba}\,\theta^5 + \overset{4}{\mathcal{Q}}_{ba}\,\theta^6\big), \tag{4.4}$$

$$\theta^5_b = \frac{1}{2}t\,\overset{3}{\mathcal{Q}}_{ba}\;\theta^a, \quad \theta^6_b = \frac{1}{2}t\,\overset{4}{\mathcal{Q}}_{ba}\;\theta^a; \tag{4.5}$$

$$\theta^5_6 = \omega^1_2 + \omega^3_4. \tag{4.6}$$

Now, let us denote as $\overline{\mathrm{Riem}}$, $\overline{\mathrm{Ric}}$ and $\overline{S}$ the Riemann curvature tensor, the Ricci tensor and the scalar curvature of (Z, g_t), respectively; by (2.13) and (2.15), it is easy to obtain the coefficients $\overline{R}_{pqrs}$, $\overline{R}_{pq}$ and the scalar curvature $\overline{S}$ in terms of $\overset{2}{\mathcal{Q}}_{ab}$, $\overset{3}{\mathcal{Q}}_{ab}$ and $\overset{4}{\mathcal{Q}}_{ab}$ (see (A.1), (A.2) and (A.3)).

It is possible to introduce two almost complex structures on (Z, g_t). Indeed, let $\{\theta^p\}$ be the orthonormal coframe defined in (4.2) and $\{e_p\}$ be its dual frame; then, we define

$$J^{\pm} = \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3 \pm \theta^5 \otimes e_6 \mp \theta^6 \otimes e_5. \tag{4.7}$$

It is clear that J^+ and J^-, which were introduced by Atiyah, Hitchin and Singer ([1]) and by Eells and Salamon ([7]), respectively, are g_t-orthogonal almost complex structures on (Z, g_t), that is, $(Z, g_t, J_{\pm})$ is an almost Hermitian manifold. From now on, we denote $J^+ = J$ and $J^- = \mathbf{J}$. By (3.1), we can write

$$J = J^q_p \theta^p \otimes e_q$$
$$\mathbf{J} = \mathbf{J}^q_p \theta^p \otimes e_q,$$

where

$$J^2_1 = J^4_3 = J^6_5 = \mathbf{J}^2_1 = \mathbf{J}^4_3 = -\mathbf{J}^6_5 = 1,$$

and all other components are zero. By (3.11), the Kähler forms of (Z, g_t, J) and $(Z, g_t, \mathbf{J})$ are, respectively,

$$\omega_{\pm} = \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4 \pm \theta^5 \wedge \theta^6 \tag{4.8}$$

(see the appendix B for the expressions of $d\omega_+$, $d\omega_-$, $\delta\omega_+$ and $\delta\omega_-$, that is equations (B.5), (B.6) and (B.7)). By (3.3), we can compute the coefficients of ∇J and $\nabla \mathbf{J}$ (see (B.2) and (B.4)); moreover, by (3.8), we obtain the components of the Nijenhuis tensors N_J

$$N_J = N^p_{tq} \theta^t \otimes \theta^q \otimes e_p, \quad N^p_{tq} = -N^p_{qt}, \tag{4.9}$$

and $N_{\mathbf{J}}$

$$N_{\mathbf{J}} = \mathbf{N}^p_{tq}\theta^t \otimes \theta^q \otimes e_p, \quad \mathbf{N}^p_{tq} = -\mathbf{N}^p_{qt}, \tag{4.10}$$

associated to J and $\mathbf{J}$, respectively (see (C.1) and (C.2)).

5. Linear conditions on J and $\mathbf{J}$: proof of Theorem 1.1

In this section we show that, given a Riemannian four-manifold M and its twistor space Z, any linear condition on the covariant derivative of the almost complex structures J and $\mathbf{J}$ on Z implies that M is self-dual. From now on, we will constantly make use of the components of ∇J, $\nabla \mathbf{J}$, N_J and $N_{\mathbf{J}}$ listed in the appendixes.

Let us start with the following proposition, which should be compared with Theorem 2.1:

Proposition 5.1. *Let (M, g) be an oriented Riemannian four-manifold. Let*

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

be the decomposition of the Riemann curvature operator on M, with $A = (A_{ij})_{i,j=1,2,3}$, $B = (B_{ij})_{i,j=1,2,3}$ and $C = (C_{ij})_{i,j=1,2,3}$. Then,

1. *M is self-dual if and only if, for every negatively oriented local orthonormal coframe, $A_{ij} = 0$ for some $i \neq j$ or $A_{kk} = A_{ll}$ for some k, l;*
2. *M is Einstein if and only if, for every negatively oriented local orthonormal coframe, $B_{ij} = 0$ for some i, j.*

Proof. Recall the transformation laws for A, B and C defined in (2.22) and the surjective Lie group homomorphism

$$\mu : SO(4) \longrightarrow SO(3) \times SO(3).$$

(1) If M is self-dual, then A is a scalar matrix with $A_{ij} = \frac{S}{12}\delta_{ij}$. Conversely, let us prove the claim for $A_{12} = 0$ and $A_{11} = A_{22}$, since the other cases can be shown in an analogous way. If, for every negatively oriented orthonormal coframe, $A_{12} = 0$, then the matrix A has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

on $O(M)_-$. Equivalently, for every smooth change of frames $a : U \longrightarrow SO(4)$, the transformed matrix $\widetilde{A}$ is such that $\widetilde{A}_{12} = 0$. Thus, let us choose $a \in SO(4)$ such that $\mu(a) = (a_+, a_-)$, where

$$a_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have that

$$\widetilde{A} = a_+^{-1} A a_+ = \begin{pmatrix} A_{33} & A_{23} & -A_{13} \\ A_{23} & A_{22} & 0 \\ -A_{13} & 0 & A_{11} \end{pmatrix};$$

thus, $\widetilde{A}_{12} = A_{23} = 0$ on $O(M)_-$, that is, A has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{13} & 0 & A_{33} \end{pmatrix}.$$

Repeating the argument on A with

$$a_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

we obtain that $A_{13} = 0$; hence, A is a diagonal matrix. Choosing other suitable changes of frames, it is not hard to show that $A_{11} = A_{22} = A_{33}$, i.e. A is a scalar matrix; by Theorem (2.1), M is self-dual.

Similar computations show that, if $A_{11} = A_{22}$ on $O(M)_-$, then A is a scalar matrix, i.e. M is self-dual.

(2) If M is Einstein, then $B = 0$ by Theorem (2.1). Conversely, suppose, for instance, that $B_{11} = 0$ on $O(M)_-$ (the other cases can be proved analogously). Again, this means that, for every change of frames a, the transformed matrix $\widetilde{B}$ is such that $\widetilde{B}_{11} = 0$. Let us choose $a \in SO(4)$ such that $\mu(a) = (a_+, a_-)$, where

$$a_- = I_3, \qquad a_+ = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, we have that

$$\widetilde{B} = a_-^{-1} B a_+ = \begin{pmatrix} B_{12} & 0 & B_{13} \\ B_{22} & -B_{21} & B_{23} \\ B_{32} & -B_{31} & B_{33} \end{pmatrix};$$

this implies that $\widetilde{B}_{11} = B_{12} = 0$. By the same argument, if we choose

$$a_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we conclude that $B_{13} = 0$. Now, repeating the argument with suitable choices of a_+ and a_-, we obtain

$$B_{21} = B_{22} = B_{23} = B_{31} = B_{32} = B_{33} = 0,$$

that is, $B = 0$. Therefore, M is Einstein by Theorem (2.1). □

From now on, given an oriented Riemannian four-manifold M, we denote its twistor space Z_- as Z. The first main result of this section is the following (see Theorem 1.1 in the introduction)

Theorem 5.2. *Let (M, g) be a Riemannian four-manifold and let (Z, g_t, J) be its twistor space. If, for every $X, Y \in \mathfrak{X}(Z)$,*

$$\begin{aligned} a_1(\nabla_X J)Y + a_2(\nabla_Y J)X + a_3(\nabla_{JX} J)Y + a_4(\nabla_{JY} J)X + a_5(\nabla_{JX} J)JY+ \\ + a_6(\nabla_{JY} J)JX + a_7(\nabla_X J)JY + a_8(\nabla_Y J)JX = 0 \end{aligned} \tag{5.1}$$

for some $a_i \in \mathbb{R}$, $i = 1, \ldots, 8$, such that $a_j \neq 0$ for some j, then, M is self-dual.

Proof. First, we rewrite the equality in (5.1) with respect to a local orthonormal frame $\{e_t\}_{t=1,\ldots,6}$ by putting $X = e_p, Y = e_q$. This implies that

(5.2)
$$a_1 J^t_{q,p} + a_2 J^t_{p,q} + a_3 J^r_p J^t_{q,r} + a_4 J^r_q J^t_{p,r} + a_5 J^r_p J^s_q J^t_{s,r} + a_6 J^r_q J^s_p J^t_{s,r} + a_7 J^s_q J^t_{s,p} + a_8 J^s_p J^t_{s,q} = 0$$

for every $p, q, t = 1, \ldots, 6$. We now proceed by steps.

(1) We start by considering (5.2) with $p = 5, q = 2, t = 1$, i.e.

$$(a_8 - a_4)\, \mathcal{Q}^3_{12} - (a_2 + a_6)\mathcal{Q}^4_{12} = 0.$$

Putting $p = 5, q = 4, t = 3$, we easily obtain

$$(a_8 - a_4)\, \mathcal{Q}^3_{34} - (a_2 + a_6)\mathcal{Q}^4_{34} = 0;$$

Summing these two equalities, we can write

$$(a_8 - a_4)A_{12} - (a_2 + a_6)A_{13} = 0.$$

Repeating the argument with $p = 6, q = 2, t = 1$ and $p = 6, q = 4, t = 3$ we have that

$$(a_2 + a_6)A_{12} + (a_8 - a_4)A_{13} = 0.$$

Thus, we deduce the following system of equations:

$$\begin{cases} (a_8 - a_4)A_{12} - (a_2 + a_6)A_{13} = 0, \\ (a_2 + a_6)A_{12} + (a_8 - a_4)A_{13} = 0. \end{cases}$$

If at least one of the coefficients $(a_8 - a_4)$ and $(a_2 + a_6)$ is different from 0, we must have that $A_{12} = A_{13} = 0$ on $O(M)_-$, since (5.1) is a global condition. By Proposition 5.1, M is self-dual.

Note that, if we exchange the values of p and q in all the previous calculations, the following system holds:

$$\begin{cases} (a_7 - a_3)A_{12} - (a_1 + a_5)A_{13} = 0, \\ (a_1 + a_5)A_{12} + (a_7 - a_3)A_{13} = 0. \end{cases}$$

As before, if at least one of the coefficients $(a_1 + a_5)$ and $(a_7 - a_3)$ is different from zero, then M is self-dual.

(2) Now, we have to study the case

$$a_1 = -a_5, \quad a_2 = -a_6, \quad a_3 = a_7, \quad a_4 = a_8,$$

that is

$$a_1(J^t_{q,p} - J^r_p J^s_q J^t_{s,r}) + a_2(J^t_{p,q} - J^r_q J^s_p J^t_{s,r}) + a_3(J^r_p J^t_{q,r} + J^s_q J^t_{s,p}) + a_4(J^r_q J^t_{p,r} + J^s_p J^t_{s,q})a = 0.$$

By choosing $p = 5, q = 3, t = 1$ and $p = 6, q = 3, t = 1$, we obtain

$$\begin{cases} (a_1 + a_2)(A_{22} - A_{33}) + 2(a_3 + a_4)A_{23} = 0, \\ (a_3 + a_4)(A_{22} - A_{33}) - 2(a_1 + a_2)A_{23} = 0. \end{cases}$$

Again, if not all the coefficients vanish, the system holds if and only if $A_{22} = A_{33}$ and $A_{23} = 0$. By Proposition 5.1, M is self-dual.
Finally, we have to show the claim when

$$a_1 = -a_2, \quad a_3 = -a_4.$$

Choosing $p = 3, q = 1, t = 5$ and $p = 4, q = 1, t = 5$, we get the system

$$\begin{cases} a_1(A_{22} - A_{33}) + 2a_3A_{23} = 0, \\ -2a_3(A_{22} - A_{33}) + a_1A_{23} = 0. \end{cases}$$

Since, by hypothesis, at least one of the coefficients does not vanish, we conclude that $A_{22} = A_{33}$ and $A_{23} = 0$, i.e. M is self-dual. □

The previous theorem allows us to easily prove a well-known result, due to Atiyah, Hitchin and Singer ([1]):

Theorem 5.3. *Let (M, g) be a Riemannian four-manifold and let (Z, g_t, J) be its twistor space. Then, the almost complex structure J is integrable if and only if M is self-dual.*

Proof. Recall that an almost complex structure is integrable if and only if the associated Nijenhuis tensor identically vanishes. Thus, by direct inspection of the components, if M is self-dual, the Nijenhuis tensor N_J is identically null. Conversely, note that the condition of integrability for J corresponds to the equation (5.1) with

$$a_1 = a_2 = a_5 = a_6 = 0, \qquad a_4 = a_8 = -a_3 = -a_7 = 1.$$

Thus, if N_J vanishes identically, then M is self-dual. □

The following characterization result is due to Muškarov (see [19]):

Theorem 5.4. (1) *$(Z, g_t, J) \in \mathcal{K} \cup \mathcal{AK} \cup \mathcal{NK} \cup \mathcal{QK} \cup \mathcal{QQK}$ if and only if M is Einstein, self-dual, with positive scalar curvature equal to $12/t^2$.*
(2) *$(Z, g_t, J) \in \mathcal{SK}$ if and only if M is self-dual.*

Proof. (1) Let us suppose that M is Einstein, self-dual with $S = 12/t^2$. A direct check of the components of ∇J shows immediately that

$$J^t_{q,p} = 0$$

for every p, q, t, i.e. $Z \in \mathcal{K}$. This obviously implies that $Z \in \mathcal{AK} \cup \mathcal{NK} \cup \mathcal{QK} \cup \mathcal{QQK}$, since $\mathcal{K} \subset \mathcal{AK} \cup \mathcal{NK} \subset \mathcal{QK} \subset \mathcal{QQK}$. Conversely, let us suppose that $Z \in \mathcal{K}$. This means that the covariant derivative ∇J satisfies the equation in (5.1) with

$$a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0.$$

Thus, M is self-dual. Since the components $J^1_{3,6}$ and $J^1_{4,5}$ vanish, we obtain

$$A_{22} = A_{33} = \frac{1}{t^2},$$

which implies that $S = 12/t^2$, since M is self-dual. Furthermore, the condition $J^1_{5,1} = J^3_{5,3} = 0$ leads to

$$\mathcal{Q}_{12} = \mathcal{Q}_{34} = 0,$$

that is, $B_{12} = 0$ on $O(M)_-$. By Proposition 5.1, M is Einstein. Now, let us suppose that $Z \in \mathcal{NK}$. Then, M is self-dual, since ∇J satisfies (5.1) with

$$a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0, \qquad a_1 = a_2.$$

Rewriting the components of ∇J and using the nearly Kähler condition on Z, we obtain the equations

$$J^1_{5,1} = J^3_{5,3} = 0,$$

which imply that $B_{12} = 0$. By Proposition 5.1, M is Einstein. By using $J^1_{3,6} + J^1_{6,3} = 0$ and the Einstein condition, we easily obtain $S = 12/t^2$.

Let us now consider the case $Z \in \mathcal{QK}$, which is equivalent to say that ∇J satisfies (5.1) with

$$a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = 0, \qquad a_1 = a_5.$$

Thus, M is self-dual. Obviously, the equation

$$J^1_{3,6} - J^1_{4,5} = 0$$

implies that $S = 12/t^2$, since $A_{22} = A_{33}$. Writing the quasi-Kähler condition for $J^1_{5,1}$ and $J^3_{5,3}$ we easily obtain

$$B_{12} = 0,$$

which implies that M is Einstein.

Now, let us consider the q^2-Kähler case ($Z \in \mathcal{QQK}$). This is just the $\mathcal{QK}$ condition with $X = Y$. Note that we can rewrite the equation as

$$(\nabla_X J)Y + (\nabla_Y J)X + (\nabla_{JX} J)JY + (\nabla_{JY} J)JX = 0$$

for every $X, Y \in \mathfrak{X}(Z)$, which is (5.1) with

$$a_3 = a_4 = a_7 = a_8 = 0, \qquad a_1 = a_2 = a_5 = a_6.$$

This immediately implies that M is self-dual. With some straightforward calculation, we obtain the system

$$\begin{cases} B_{22} + B_{33} = \dfrac{S}{6} - \dfrac{2}{t^2}, \\ B_{33} - B_{22} = -\dfrac{S}{6} + \dfrac{2}{t^2}. \end{cases}$$

Summing the two equations, we obtain $B_{33} = 0$, i.e. M is Einstein by Proposition 5.1. Therefore, by the same system, we obtain that $S = 12/t^2$.

Finally, suppose that $Z \in \mathcal{AK}$. By our previous considerations, this means that $d\omega_+ = 0$; therefore, the vanishing of all the components in (B.5) implies the following equations:

$$\begin{cases} B_{12} = B_{13} = B_{22} = B_{23} = B_{32} = B_{33} = 0, \\ A_{12} = A_{13} = A_{23} = 0, \\ A_{22} = A_{33} = \dfrac{1}{t^2}. \end{cases}$$

Thus, M is Einstein, self-dual with $S = 12/t^2$.

(2) Let us consider the semi-Kähler equation

$$\sum_{p=1}^{6} J^t_{p,p} + J^r_p J^s_p J^t_{s,r} = 2\sum_{p=1}^{6} J^t_{p,p} = 0$$

for every $t = 1, \ldots, 6$. It is easy to see that the left-hand side of this equation is identically zero if $t = 1, \ldots, 4$. Thus, by putting $t = 5, 6$, we obtain that the equation holds, i.e., $Z \in \mathcal{SK}$ if and only if $A_{12} = A_{13} = 0$, i.e. if and only if M is self-dual (equivalently, by (B.7), it is clear that $\delta\omega = 0$ if and only if $A_{12} = A_{13} = 0$). □

Now, we prove the analogous of Theorem 5.2 for $\mathbf{J}$:

Theorem 5.5. *Let (M, g) be a Riemannian four-manifold and let $(Z, g_t, \mathbf{J})$ be its twistor space. If $\nabla\mathbf{J}$ satisfies the equation in (5.1) for every $X, Y \in \mathfrak{X}(Z)$, then M is self-dual. Moreover, at least one of the coefficients $a_1 + a_5$, $a_2 + a_6$, $a_3 - a_7$ or $a_4 - a_8$ is different from zero.*

Proof. The first step of the proof is identical to the one in Theorem 5.2; thus, if at least one of the coefficients $a_2 + a_6$, $a_8 - a_4$, $a_1 + a_5$ or $a_7 - a_3$ is different from 0, we conclude that M is self-dual. Now, suppose that

$$a_1 = -a_5, \quad a_2 = -a_6, \quad a_3 = a_7, \quad a_4 = a_8.$$

If we choose $p = 5, q = 3, t = 1$ and $p = 1, q = 5, t = 3$ in (5.2), we obtain the system

$$\begin{cases} a_3\left(A_{22} + A_{33} - \dfrac{2}{t^2}\right) + a_4(A_{22} + A_{33}) = 0, \\ a_3(A_{22} + A_{33}) - a_4\left(A_{22} + A_{33} - \dfrac{2}{t^2}\right) = 0. \end{cases}$$

If $A_{22} + A_{33} = \frac{2}{t^2}$ or $A_{22} = -A_{33}$, we obtain that $a_3 = a_4 = 0$. If $A_{22} + A_{33} \neq \frac{2}{t^2}, 0$, we have again that $a_3 = a_4 = 0$. Note that, exchanging the values of p and q in the previous argument, we obtain the system

$$\begin{cases} a_1\left(A_{22} + A_{33} - \dfrac{2}{t^2}\right) + a_2(A_{22} + A_{33}) = 0, \\ a_1(A_{22} + A_{33}) - a_2\left(A_{22} + A_{33} - \dfrac{2}{t^2}\right) = 0, \end{cases}$$

which implies that $a_1 = a_2 = 0$. Since not all the coefficients a_i can be equal to zero, there is a contradiction and the claim is proved. □

Theorem 5.5 can be exploited to show another characterization result by Muškarov (see [19]):

Theorem 5.6.

(1) $(Z, g_t, \mathbf{J}) \in \mathcal{AK}$ *if and only if* M *is Einstein, self-dual with, scalar curvature equal to* $-12/t^2$*;*
(2) $(Z, g_t, \mathbf{J}) \in \mathcal{NK}$ *if and only if* M *is Einstein, self-dual, with constant scalar curvature equal to* $6/t^2$*;*
(3) $(Z, g_t, \mathbf{J}) \in \mathcal{QK}$ *if and only if* M *is Einstein and self-dual;*
(4) $(Z, g_t, \mathbf{J}) \in \mathcal{SK}$ *if and only if* M *is self-dual.*

Proof. If M is Einstein and self-dual, a direct check of the components of $\nabla \mathbf{J}$ (see (B.4)) shows that $Z \in \mathcal{QK}$. Moreover, if $S = 6/t^2$, $Z \in \mathcal{NK}$. Conversely, let us suppose $Z \in \mathcal{QK}$. By the same argument used in the proof of Theorem 5.4, we obtain that M is Einstein and self-dual (note that we do not have any condition on the scalar curvature, since the term $2/t^2$ does not appear for any p, q, t in (5.2)). If, in addition, $Z \in \mathcal{NK}$, by

$$J^1_{3,6} + J^1_{6,3} = 0$$

and the fact that M is Einstein and self-dual, we easily obtain $S = 6/t^2$. Then, as in Theorem 5.4, $\nabla \mathbf{J}$ satisfies the semi-Kähler equation, i.e. $Z \in \mathcal{SK}$, if and only if $A_{12} = A_{13} = 0$, i.e., if and only if M is self-dual (again, this can be seen by using (B.7)). Finally, we have that $Z \in \mathcal{AK}$ if and only if $d\omega_- = 0$, that is, if and only if all the terms in (B.6) vanish. This holds if and only if

$$\begin{cases} B_{12} = B_{13} = B_{22} = B_{23} = B_{32} = B_{33} = 0; \\ A_{12} = A_{13} = A_{23} = 0; \\ A_{22} = A_{33} = -\dfrac{1}{t^2}. \end{cases}$$

Thus, $Z \in \mathcal{AK}$ if and only if M is Einstein, self-dual with scalar curvature equal to $-12/t^2$. □

6. Quadratic conditions: proofs of Theorems 1.2, 1.3 and 1.4

In this section we present some new results about the twistor space associated to an Einstein four-dimensional manifold whose metric is not necessarily self-dual; in particular, we partially generalize the characterization Theorem 5.4 and we introduce a new necessary condition for a manifold to be Einstein, which leads to a characterization of Einstein, non-self-dual manifolds.

As before, let (M, g) be a connected, oriented Riemannian manifold of dimension 4 and (Z, g_t, J) be its twistor space, with J the Atiyah-Hitchin-Singer almost complex structure defined in (4.7). We have the following (see Theorem 1.2 in the introduction):

Theorem 6.1. *(M, g) is Einstein if and only if, for every orthonormal frame in $O(M)_-$ (equivalently, for every negatively oriented orthonormal coframe),*

$$\sum_{t=1}^{6} (J^t_{p,q} + J^t_{q,p})(J^t_{p,p} - J^t_{q,q}) = 0, \quad \forall p, q = 1, \dots, 6. \tag{6.1}$$

Proof. First, suppose that M is Einstein. Then, a direct computation over the components in (B.2) shows that (6.1) holds; indeed, for instance, by Theorem 2.1

$$\sum_{t=1}^{6} (J^t_{1,3} + J^t_{3,1})(J^t_{1,1} - J^t_{3,3}) = B_{32}B_{12} + B_{13}B_{33} = 0.$$

Conversely, suppose that (6.1) holds. By choosing $p = 1, q = 3$ and $p = 1, q = 4$, we obtain the following system

$$\begin{cases} B_{12}B_{22} + B_{13}B_{23} = 0; \\ B_{12}B_{32} + B_{13}B_{33} = 0. \end{cases}$$

By hypothesis, the two equations must hold on all $O(M)_-$. Let us choose $e \in O(M)_-$ and suppose that $B_{ij} = 0$ for some i, j; we want to prove that $B = 0$. For instance, let $B_{11} = 0$ (the other cases can be shown analogously). Let us choose a smooth change of frames $a \in SO(4)$ such that

$$a_+ = a_- = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

by (2.22), we obtain that the matrix $\widetilde{B}$ associated to the frame $\widetilde{e} = ea$ has the form

$$\widetilde{B} = \begin{pmatrix} \widetilde{B_{11}} & \widetilde{B_{12}} & \widetilde{B_{13}} \\ \widetilde{B_{21}} & \widetilde{B_{22}} & \widetilde{B_{23}} \\ \widetilde{B_{31}} & \widetilde{B_{32}} & \widetilde{B_{33}} \end{pmatrix} = \begin{pmatrix} B_{22} & -B_{21} & -B_{23} \\ -B_{12} & 0 & B_{13} \\ -B_{32} & B_{31} & B_{33}. \end{pmatrix}$$

By hypothesis, we have that

$$0 = \widetilde{B}_{12}\widetilde{B}_{22} + \widetilde{B}_{13}\widetilde{B}_{23} = -B_{23}B_{13},$$

that is, $B_{23} = 0$ or $B_{13} = 0$. In both cases, with similar computations, it can be shown that all the other B_{ij} are zero, i.e. $B = 0$, which means that M is Einstein by (2.22). In particular, if one of the B_{ij} in the system is 0, then M is Einstein.

Let us now suppose $B_{12}, B_{13}, B_{22}, B_{23}, B_{32}, B_{33} \neq 0$. By the previous system of equations, we obtain that

$$B_{12} = -\frac{B_{13}B_{23}}{B_{22}} = -\frac{B_{13}B_{33}}{B_{32}},$$

which implies that

$$B_{23}B_{32} = B_{22}B_{33}.$$

Choosing the matrices

$$a_+ = I_3, \qquad a_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

in (2.22), we deduce

$$B_{13}B_{22} = B_{12}B_{23} = -\frac{B_{13}B_{23}}{B_{22}}B_{23},$$

that is,

$$B_{22} = -\frac{B_{23}^2}{B_{22}}.$$

But this implies that $B_{22}^2 = -B_{23}^2$, i.e. $B_{22} = B_{23} = 0$, which is a contradiction. This means that at least one of the terms in the system above has to be equal to 0; therefore, by the previous considerations, M is Einstein. □

We highlight that the fact that Theorem 6.1 is only true *locally*: this means that, if M is Einstein, the condition (6.1) holds only for orthonormal frames. Indeed, if $X, Y \in \mathfrak{X}(Z)$, the global version of the equation (6.1), namely

$$\langle (\nabla_X J)(Y) + (\nabla_Y J)(X), (\nabla_X J)(X) - (\nabla_Y J)(Y) \rangle = 0,$$

is not satisfied, in general, if the norm of X and Y are different from 1 (for instance, it is sufficient to consider $Y = 2X$). However, it is important to underline that, in order to find characterizations of the Einstein manifolds *via* polynomial conditions on ∇J, we have to investigate equations of order higher than 1, since, by Theorem 5.2, every linear condition on ∇J implies that M is self-dual.

Now we show the following (see Theorem 1.3 in the introduction):

Theorem 6.2. *Let (M,g) be an oriented Riemannian Einstein four-manifold. Then, for every orthonormal frame in $O(M)_-$,*

$$(J^t_{q,p}+J^t_{p,q})N^t_{pq}=0 \qquad (\textit{no sum over } p,q,t), \tag{6.2}$$

where $J^t_{p,q}$ and N^t_{pq} are the components of the covariant derivative of J and of the Nijenhuis tensor, respectively, with respect to a local orthonormal coframe on (Z_-,g_t,J). Conversely, if (M,g) is not self-dual and the equation (6.2) holds on $O(M)_-$, then (M,g) is Einstein.

Proof. If (M,g) is Einstein, by (B.2) and (C.1) we obtain that

$$\begin{aligned}
(J^5_{3,1}+J^5_{1,3})N^5_{13} &= -(J^5_{4,2}+J^5_{2,4})N^5_{24}=t^2B_{32}(A_{22}-A_{33});\\
(J^5_{4,1}+J^5_{1,4})N^5_{14} &= -(J^5_{3,2}+J^5_{2,3})N^5_{23}=-2t^2B_{22}A_{23};\\
(J^6_{3,1}+J^6_{1,3})N^6_{13} &= -(J^6_{4,2}+J^6_{2,4})N^6_{24}=2t^2B_{33}A_{23};\\
(J^6_{4,1}+J^6_{1,4})N^6_{14} &= -(J^6_{3,2}+J^6_{2,3})N^6_{23}=t^2B_{23}(A_{22}-A_{33});
\end{aligned} \tag{6.3}$$

since $B=0$ by hypothesis, (6.2) holds. Conversely, let us suppose that (M,g) is not self-dual. Since the matrix A is symmetric, there exists $e\in O(M)_-$ such that A is diagonal, i.e.

$$A=\begin{pmatrix} x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix};$$

since A is not self-dual, we can assume that $y\neq z$. Let

$$a_+=\begin{pmatrix} 1 & 0 & 0\\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}\in SO(3);$$

by (2.22), we have that, with respect to the transformed frame $\widetilde{e}\in O(M)_-$,

$$A=\begin{pmatrix} x & 0 & 0\\ 0 & \frac12(y+z) & \frac12(z-y)\\ 0 & \frac12(z-y) & \frac12(y+z)\end{pmatrix}.$$

By hypothesis, (6.2) holds on $O(M)_-$, which implies that $B_{22}=B_{33}=0$ with respect to $\widetilde{e}$, by (6.3) and the fact that $y\neq z$. Putting

$$a_+=\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},$$

by the transformation laws (2.22), we obtain

$$\widetilde{A}=\begin{pmatrix} x & 0 & 0\\ 0 & \frac12(y+z) & \frac12(y-z)\\ 0 & \frac12(y-z) & \frac12(y+z)\end{pmatrix}, \qquad \widetilde{B}=\begin{pmatrix} B_{11} & B_{13} & -B_{12}\\ B_{21} & B_{23} & 0\\ B_{31} & 0 & -B_{32}\end{pmatrix};$$

by (6.2), $B_{23} = B_{32} = 0$ (note that $\widetilde{A}_{23} \neq 0$). By (2.22), if $a \in SO(4)$ is a change of frames such that $\mu(a) = (a_+, a_-)$, A is invariant under the action of a_-: therefore, by similar computations on B with $a_- = I_3$, it is easy to show that $B_{12} = B_{13} = 0$. Now, let us consider two cases:

(1) **Case** $\mathbf{x} \neq \frac{1}{2}(\mathbf{y}+\mathbf{z})$. Putting

$$a_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

we have

$$\widetilde{A} = \begin{pmatrix} \frac{1}{2}(y+z) & \frac{1}{2}(y-z) & 0 \\ \frac{1}{2}(y-z) & \frac{1}{2}(y+z) & 0 \\ 0 & 0 & x \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} 0 & 0 & B_{11} \\ 0 & 0 & B_{21} \\ 0 & 0 & B_{31} \end{pmatrix};$$

since $\widetilde{A}_{22} \neq \widetilde{A}_{33}$, by (6.2) we obtain $B_{21} = 0$. Again, since A is invariant under the action of a_-, by analogous computations $B_{11} = 0$. Finally, putting

$$a_+ = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we conclude that $B_{31} = 0$, that is, M is Einstein.

(2) **Case** $\mathbf{x} = \frac{1}{2}(\mathbf{y}+\mathbf{z})$. In this case, A and B have the form

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & x & x-z \\ 0 & x-z & x \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & 0 & 0 \\ B_{31} & 0 & 0 \end{pmatrix}$$

(note that $x \neq z$, otherwise M would be self-dual). Choosing

$$a_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$\widetilde{A} = \begin{pmatrix} x & 0 & \frac{1}{\sqrt{2}}(x-z) \\ 0 & x & \frac{1}{\sqrt{2}}(x-z) \\ \frac{1}{\sqrt{2}}(x-z) & \frac{1}{\sqrt{2}}(x-z) & x \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} \frac{1}{\sqrt{2}}B_{11} & -\frac{1}{\sqrt{2}}B_{11} & 0 \\ \frac{1}{\sqrt{2}}B_{21} & -\frac{1}{\sqrt{2}}B_{21} & 0 \\ \frac{1}{\sqrt{2}}B_{31} & -\frac{1}{\sqrt{2}}B_{31} & 0 \end{pmatrix};$$

by $x \neq z$ and (6.2), $B_{21} = 0$. As we did earlier, the invariance of A under the action of a_- implies that $B_{11} = 0$. Finally, choosing

$$a_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

we conclude that $B_{31} = 0$, i.e. M is Einstein.

□

Now, let us introduce the *Ricci* tensor* $\overline{\mathrm{Ric}}^*$ associated to J on Z, which is defined as (see also [12])

$$\overline{R}^*_{pq} := \overline{\mathrm{Riem}}(e_p, e_t, J(e_q), J(e_t)) = J^s_q J^u_t \overline{R}_{ptsu} \tag{6.4}$$

with respect to a local orthonormal coframe on Z. By (A.1), we easily obtain the components $\overline{R}^*_{pq}$ (see (D.1)). We also define the *scalar* curvature* $\overline{S}^*$ as

$$\overline{S}^* = \sum_{a=1}^{4} \overline{R}^*_{aa} + \overline{R}^*_{55} + \overline{R}^*_{66}; \tag{6.5}$$

thus, we can introduce the *holomorphic scalar curvature* $\overline{S}_J$ (see also [12], [13] and [14]):

$$\overline{S}_J := \overline{S} - \overline{S}^*. \tag{6.6}$$

It is well-known that, for every almost Hermitian manifold Z, (see [13])

$$\begin{cases} \overline{S}_J = |\nabla J|^2 & \text{if } Z \in \mathcal{NK}; \\ \overline{S}_J = -\dfrac{1}{2}|\nabla J|^2 & \text{if } Z \in \mathcal{AK}, \end{cases}$$

where, by definition,

$$|\nabla J|^2 = \sum_{p,q,t=1}^{6} J^t_{p,q} J^t_{p,q}; \tag{6.7}$$

moreover, if (Z, g_t, J) is the twistor space associated to a Riemannian manifold (M, g), by Theorem (5.4), in both cases $\overline{S}_J = 0$, since $(Z, g_t, J) \in \mathcal{NK} \cup \mathcal{AK}$ implies $(Z, g_t, J) \in \mathcal{K}$.

We now show the following upper and lower bounds for the holomorphic scalar curvature $\overline{S}_J$ of the twistor space (Z, J) (see Theorem 1.4):

Theorem 6.3. *Let (M, g) be a four-dimensional Einstein manifold with positive scalar curvature. Then, on the twistor space (Z, J) the following estimates hold*

$$-\frac{1}{2}|\nabla J|^2 \leq \overline{S}_J \leq |\nabla J|^2.$$

Moreover, one of the equality holds if and only if (M, g) is Einstein, self-dual with positive scalar curvature.

Proof. We deal first with the lower bound. Since (M, g) is Einstein, using (B.3) and (D.2), we obtain

$$\begin{aligned}\overline{S} - \overline{S}^* + \frac{1}{2}|\nabla J|^2 &= 6\big({\overset{3}{\mathcal{Q}}_{12}}^2 + {\overset{4}{\mathcal{Q}}_{12}}^2\big) + 2\big({\overset{3}{\mathcal{Q}}_{14}}^2 + {\overset{4}{\mathcal{Q}}_{13}}^2\big) + \\ &\qquad + (\overset{3}{\mathcal{Q}}_{13} - \overset{4}{\mathcal{Q}}_{14})^2 + (\overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14})^2 + \frac{4}{t^4} - \frac{8}{t^2}\overset{2}{\mathcal{Q}}_{12} \\ &\geq \frac{1}{t^2}\left[t^2(\overset{3}{\mathcal{Q}}_{12} + \overset{4}{\mathcal{Q}}_{14})^2 + \frac{4}{t^2} - 8\overset{2}{\mathcal{Q}}_{12}\right].\end{aligned}$$

Since g is Einstein with positive scalar curvature, we set

$$S = \frac{12}{t^2} = 4(\overset{2}{\mathcal{Q}}_{12} + \overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14})$$

obtaining

$$\begin{aligned}\overline{S} - \overline{S}^* + \frac{1}{2}|\nabla J|^2 &\geq \frac{1}{t^2}\left[\frac{12}{S}\left(\frac{S}{4} - \overset{2}{\mathcal{Q}}_{12}\right)^2 + \frac{S}{3} - 8\overset{2}{\mathcal{Q}}_{12}\right] \\ &= \frac{1}{12}\left(\frac{13}{12}S^2 + 12\,{\overset{2}{\mathcal{Q}}_{12}}^2 - 14S\,\overset{2}{\mathcal{Q}}_{12}\right) \\ &= \frac{1}{144}(S - 12\overset{2}{\mathcal{Q}}_{12})(13S - 12\overset{2}{\mathcal{Q}}_{12}) \\ &= \frac{1}{144}(S - 12A_{11})(13S - 12A_{11}) \geq 0,\end{aligned}$$

since we can assume

$$S = 4(A_{11} + A_{22} + A_{33}) \geq 12A_{11}.$$

Moreover, we have equality if and only if A is diagonal, i.e. (M, g) is self-dual.

We show now the upper bound. Since (M, g) is Einstein, using again (B.3) and (D.2), we obtain

$$\begin{aligned}\overline{S} - \overline{S}^* - |\nabla J|^2 &= -2\big(5\,{\overset{3}{\mathcal{Q}}_{13}}^2 + 5\,{\overset{4}{\mathcal{Q}}_{14}}^2 - 6\,\overset{3}{\mathcal{Q}}_{13}\,\overset{4}{\mathcal{Q}}_{14}\big) - 32{\overset{3}{\mathcal{Q}}_{14}}^2 \\ &\qquad - \frac{8}{t^2}\,\overset{2}{\mathcal{Q}}_{12} + \frac{12}{t^2}(\overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14}) - \frac{8}{t^4} \\ &= -2(\overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14})^2 - 8(\overset{3}{\mathcal{Q}}_{13} - \overset{4}{\mathcal{Q}}_{14})^2 - 32{\overset{3}{\mathcal{Q}}_{14}}^2 \\ &\qquad - \frac{8}{t^2}\,\overset{2}{\mathcal{Q}}_{12} + \frac{12}{t^2}(\overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14}) - \frac{8}{t^4} \\ &\leq -\frac{2}{t^2}\left[t^2(\overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14})^2 + 4\,\overset{2}{\mathcal{Q}}_{12} - 6(\overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14}) + \frac{4}{t^2}\right].\end{aligned}$$

Since g is Einstein with positive scalar curvature, we set

$$S = \frac{12}{t^2} = 4(\overset{2}{\mathcal{Q}}_{12} + \overset{3}{\mathcal{Q}}_{13} + \overset{4}{\mathcal{Q}}_{14})$$

obtaining

$$\begin{aligned}
\overline{S} - \overline{S}^* - |\nabla J|^2 &\leq -\frac{2}{t^2}\left[\frac{12}{S}\left(\frac{S}{4} - \overset{2}{Q}_{12}\right)^2 + 4\,\overset{2}{Q}_{12} - 6\left(\frac{S}{4} - \overset{2}{Q}_{12}\right) + \frac{S}{3}\right] \\
&= -\frac{1}{6}\left(12\,\overset{2}{Q}_{12}{}^2 - \frac{5}{12}S^2 + 4S\,\overset{2}{Q}_{12}\right) \\
&= -\frac{1}{72}(12\,\overset{2}{Q}_{12} - S)(12\,\overset{2}{Q}_{12} + 5S) \\
&= -\frac{1}{72}(12A_{11} - S)(12A_{11} + 5S) \leq 0,
\end{aligned}$$

since we can assume

$$S = 4(A_{11} + A_{22} + A_{33}) \leq 12A_{11}.$$

Moreover, we have equality if and only if A is diagonal, i.e. (M, g) is self-dual. □

Appendix A. Riemann curvature of a twistor space

In this appendix we recall all the components of the Riemann tensor, the Ricci tensor and the scalar curvature of the twistor space for a Riemannian four-manifold (see also [16]).

Components of the Riemann curvature tensor $\overline{\mathrm{Riem}}$ on (Z, g_t):

$$\begin{aligned}
\overline{R}_{abcd} &= R_{abcd} - \frac{1}{4}t^2[(\overset{3}{Q}_{ac}\,\overset{3}{Q}_{bd} - \overset{3}{Q}_{ad}\,\overset{3}{Q}_{bc}) + (\overset{4}{Q}_{ac}\,\overset{4}{Q}_{bd} - \overset{4}{Q}_{ad}\,\overset{4}{Q}_{bc})] \\
&\quad - \frac{1}{2}t^2(\overset{3}{Q}_{ab}\,\overset{3}{Q}_{cd} + \overset{4}{Q}_{ab}\,\overset{4}{Q}_{cd}); \\
\overline{R}_{ab56} &= \overset{2}{Q}_{ab} - \frac{1}{4}t^2\sum_{c=1}^{4}(\overset{3}{Q}_{ac}\,\overset{4}{Q}_{bc} - \overset{3}{Q}_{bc}\,\overset{4}{Q}_{ac}); \\
\overline{R}_{abc5} &= -\frac{1}{2}t(\overset{3}{Q}_{ab})_c; \qquad \overline{R}_{abc6} = -\frac{1}{2}t(\overset{4}{Q}_{ab})_c; \\
\overline{R}_{5656} &= \frac{1}{t^2}; \quad \overline{R}_{565b} = \overline{R}_{56a6} = 0; \\
\overline{R}_{5ab5} &= -\frac{1}{4}t^2\sum_{c=1}^{4}\overset{3}{Q}_{ac}\overset{3}{Q}_{bc}; \quad \overline{R}_{5ab6} = -\frac{1}{2}\,\overset{2}{Q}_{ab} - \frac{1}{4}t^2\sum_{c=1}^{4}\overset{3}{Q}_{bc}\overset{4}{Q}_{ac}; \\
\overline{R}_{6ab6} &= -\frac{1}{4}t^2\sum_{c=1}^{4}\overset{4}{Q}_{ac}\overset{4}{Q}_{bc}; \quad \overline{R}_{6ab5} = \frac{1}{2}\,\overset{2}{Q}_{ab} - \frac{1}{4}t^2\sum_{c=1}^{4}\overset{4}{Q}_{bc}\,\overset{3}{Q}_{ac}\,.
\end{aligned} \tag{A.1}$$

Components of the Ricci tensor $\overline{\mathrm{Ric}}$:

$$\overline{R}_{ab} = R_{ab} - \frac{1}{2}t^2 \sum_{c=1}^{4} (Q^3_{ac}\, Q^3_{bc} + Q^4_{ac}\, Q^4_{bc}); \tag{A.2}$$

$$\overline{R}_{a5} = \frac{1}{2}t \sum_{c=1}^{4} (Q^3_{ac})_c;$$

$$\overline{R}_{a6} = \frac{1}{2}t \sum_{c=1}^{4} (Q^4_{ac})_c;$$

$$\overline{R}_{55} = \frac{1}{t^2} + \frac{1}{4}|Q^3_{ab}|^2;$$

$$\overline{R}_{56} = \frac{1}{4}t^2 \sum_{c=1}^{4} Q^3_{ac}\, Q^4_{bc};$$

$$\overline{R}_{66} = \frac{1}{t^2} + \frac{1}{4}|Q^4_{ab}|^2,$$

(here: $|Q^3_{ab}|^2 = \sum_{a,b=1}^{4} (Q^3_{ab})^2$, and similarly for Q^4_{ab}).

Scalar curvature $\overline{S}$:

$$\overline{S} = S + \frac{2}{t^2} - \frac{1}{4}t^2\Big(|Q^3_{ab}|^2 + |Q^4_{ab}|^2\Big). \tag{A.3}$$

Appendix B. Covariant derivative of the almost complex structures and differential of the Kähler forms

In this appendix we list all the components of the covariant derivative of the almost complex structures $J^{\pm}$ on Z. Recall that

$$\begin{aligned} J^{\pm} &= \sum_{k=1}^{3} \Big(\theta^{2k-1} \otimes e_{2k} - \theta^{2k} \otimes e_{2k-1}\Big) \\ &= \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3 \pm \theta^5 \otimes e_6 \mp \theta^6 \otimes e_5; \end{aligned} \tag{B.1}$$

using the same notation of the article, we write $J^+ = J$, $J^- = \mathbf{J}$.

Computation of ∇J on (Z, g_t, J):
Using (3.3), a long but straightforward computation shows that:

(B.2)
$$\begin{aligned}
&J^1_{2,t} = J^3_{4,t} = J^5_{6,t} = 0;\\
&J^1_{3,a} = 0;\\
&J^1_{3,5} = -\frac{1}{2}t(\mathcal{Q}^3_{14} + \mathcal{Q}^3_{23});\\
&J^1_{3,6} = \frac{1}{2}t\left[\frac{2}{t^2} - (\mathcal{Q}^4_{14} + \mathcal{Q}^4_{23})\right];\\
&J^1_{4,a} = 0;\\
&J^1_{4,5} = -\frac{1}{2}t\left[\frac{2}{t^2} - (\mathcal{Q}^3_{13} + \mathcal{Q}^3_{42})\right];\\
&J^1_{4,6} = \frac{1}{2}t(\mathcal{Q}^4_{13} + \mathcal{Q}^4_{42});\\
&J^1_{5,a} = -\frac{1}{2}t(\mathcal{Q}^3_{2a} + \mathcal{Q}^4_{1a});\\
&J^1_{5,5} = J^1_{5,6} = 0;\\
&J^1_{6,a} = \frac{1}{2}t(\mathcal{Q}^3_{1a} - \mathcal{Q}^4_{2a});\\
&J^1_{6,5} = J^1_{6,6} = 0;\\
&J^2_{3,t} = J^1_{4,t};\\
&J^2_{4,t} = -J^1_{3,t}\\
&J^2_{5,t} = J^1_{6,t}\\
&J^2_{6,t} = -J^1_{5,t}\\
&J^3_{5,a} = -\frac{1}{2}t(\mathcal{Q}^3_{4a} + \mathcal{Q}^4_{3a});\\
&J^3_{5,5} = J^3_{5,6} = 0;\\
&J^3_{6,a} = \frac{1}{2}t(\mathcal{Q}^3_{3a} - \mathcal{Q}^4_{4a});\\
&J^3_{6,5} = J^3_{6,6} = 0;\\
&J^4_{5,t} = J^3_{6,t}\\
&J^4_{6,t} = -J^3_{5,t}.
\end{aligned}$$

The square norm $|\nabla J|^2 = \sum_{p,q,t=1}^6 J^t_{p,q} J^t_{p,q}$ is given by

(B.3)
$$\begin{aligned}
\frac{1}{t^2}|\nabla J|^2 = {} & [(\mathcal{Q}^3_{14} + \mathcal{Q}^3_{23})^2 + (\mathcal{Q}^4_{14} + \mathcal{Q}^4_{23})^2 + (\mathcal{Q}^3_{13} + \mathcal{Q}^3_{42})^2 + (\mathcal{Q}^4_{13} + \mathcal{Q}^4_{42})^2 +\\
&+ (\mathcal{Q}^3_{23} - \mathcal{Q}^4_{14})^2 + (\mathcal{Q}^3_{42} - \mathcal{Q}^4_{14})^2 + (\mathcal{Q}^3_{13} - \mathcal{Q}^4_{23})^2 + (\mathcal{Q}^3_{14} + \mathcal{Q}^4_{42})^2 +\\
&+ (\mathcal{Q}^3_{14} + \mathcal{Q}^4_{13})^2 + (\mathcal{Q}^4_{23} - \mathcal{Q}^3_{42})^2 + (\mathcal{Q}^4_{14} - \mathcal{Q}^3_{13})^2 + (\mathcal{Q}^3_{23} + \mathcal{Q}^4_{42})^2] +\\
&+ 2[{\mathcal{Q}^3_{12}}^2 + {\mathcal{Q}^4_{12}}^2 + {\mathcal{Q}^3_{34}}^2 + {\mathcal{Q}^4_{34}}^2] - (\mathcal{Q}^3_{13} + \mathcal{Q}^3_{42}) - (\mathcal{Q}^4_{14} + \mathcal{Q}^4_{23}) + \frac{8}{t^4}.
\end{aligned}$$

Computation of $\nabla\mathbf{J}$ on (Z, g_t, J):
Again, using (3.3), we have:

(B.4)
$$\begin{aligned}
&\mathbf{J}^1_{2,t} = \mathbf{J}^3_{4,t} = \mathbf{J}^5_{6,t} = 0;\\
&\mathbf{J}^1_{3,a} = 0;\\
&\mathbf{J}^1_{3,5} = -\frac{1}{2}t(\mathcal{Q}^{3}_{14} + \mathcal{Q}^{3}_{23});\\
&\mathbf{J}^1_{3,6} = \frac{1}{2}t\left[\frac{2}{t^2} - (\mathcal{Q}^{4}_{14} + \mathcal{Q}^{4}_{23})\right];\\
&\mathbf{J}^1_{4,a} = 0;\\
&\mathbf{J}^1_{4,5} = -\frac{1}{2}t\left[\frac{2}{t^2} - (\mathcal{Q}^{3}_{13} + \mathcal{Q}^{3}_{42})\right];\\
&\mathbf{J}^1_{4,6} = \frac{1}{2}t(\mathcal{Q}^{4}_{13} + \mathcal{Q}^{4}_{42});\\
&\mathbf{J}^1_{5,a} = -\frac{1}{2}t(\mathcal{Q}^{3}_{2a} - \mathcal{Q}^{4}_{1a});\\
&\mathbf{J}^1_{5,5} = \mathbf{J}^1_{5,6} = 0;\\
&\mathbf{J}^1_{6,a} = -\frac{1}{2}t(\mathcal{Q}^{3}_{1a} + \mathcal{Q}^{4}_{2a});\\
&\mathbf{J}^1_{6,5} = \mathbf{J}^1_{6,6} = 0;\\
&\mathbf{J}^2_{3,t} = \mathbf{J}^1_{4,t};\\
&\mathbf{J}^2_{4,t} = -\mathbf{J}^1_{3,t}\\
&\mathbf{J}^2_{5,t} = -\mathbf{J}^1_{6,t}\\
&\mathbf{J}^2_{6,t} = \mathbf{J}^1_{5,t}\\
&\mathbf{J}^3_{5,a} = -\frac{1}{2}t(\mathcal{Q}^{3}_{4a} - \mathcal{Q}^{4}_{3a});\\
&\mathbf{J}^3_{5,5} = \mathbf{J}^3_{5,6} = 0;\\
&\mathbf{J}^3_{6,a} = -\frac{1}{2}t(\mathcal{Q}^{3}_{3a} + \mathcal{Q}^{4}_{4a});\\
&\mathbf{J}^3_{6,5} = \mathbf{J}^3_{6,6} = 0;\\
&\mathbf{J}^4_{5,t} = -\mathbf{J}^3_{6,t}\\
&\mathbf{J}^4_{6,t} = \mathbf{J}^3_{5,t}.
\end{aligned}$$

Kähler forms of J and $\mathbf{J}$:
denoting by ω_+ and ω_- the Kähler forms of J and $\mathbf{J}$, respectively, we have:

$$
\begin{aligned}
d\omega_+ = &-t\,\overset{4}{Q}_{12}\,\theta^1\wedge\theta^2\wedge\theta^5 + t\,\overset{3}{Q}_{12}\,\theta^1\wedge\theta^2\wedge\theta^6 - t\,\overset{4}{Q}_{13}\,\theta^1\wedge\theta^3\wedge\theta^5 + \\
&+\left(t\,\overset{3}{Q}_{13} - \frac{1}{t}\right)\theta^1\wedge\theta^3\wedge\theta^6 + \left(\frac{1}{t} - t\overset{4}{Q}_{14}\right)\theta^1\wedge\theta^4\wedge\theta^5 + \\
&+ t\,\overset{3}{Q}_{14}\,\theta^1\wedge\theta^4\wedge\theta^6 + \left(\frac{1}{t} - t\overset{4}{Q}_{23}\right)\theta^2\wedge\theta^3\wedge\theta^5 + t\,\overset{3}{Q}_{23}\,\theta^2\wedge\theta^3\wedge\theta^6 + \\
&- t\,\overset{4}{Q}_{42}\,\theta^4\wedge\theta^2\wedge\theta^5 + \left(t\,\overset{3}{Q}_{42} - \frac{1}{t}\right)\theta^4\wedge\theta^2\wedge\theta^6 - t\,\overset{4}{Q}_{34}\,\theta^3\wedge\theta^4\wedge\theta^5 + \\
&+ t\,\overset{3}{Q}_{34}\,\theta^3\wedge\theta^4\wedge\theta^6.
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
d\omega_- = &\;t\,\overset{4}{Q}_{12}\,\theta^1\wedge\theta^2\wedge\theta^5 - t\,\overset{3}{Q}_{12}\,\theta^1\wedge\theta^2\wedge\theta^6 + t\,\overset{4}{Q}_{13}\,\theta^1\wedge\theta^3\wedge\theta^5 + \\
&-\left(t\,\overset{3}{Q}_{13} + \frac{1}{t}\right)\theta^1\wedge\theta^3\wedge\theta^6 + \left(\frac{1}{t} + t\overset{4}{Q}_{14}\right)\theta^1\wedge\theta^4\wedge\theta^5 + \\
&- t\,\overset{3}{Q}_{14}\,\theta^1\wedge\theta^4\wedge\theta^6 + \left(\frac{1}{t} + t\overset{4}{Q}_{23}\right)\theta^2\wedge\theta^3\wedge\theta^5 - t\,\overset{3}{Q}_{23}\,\theta^2\wedge\theta^3\wedge\theta^6 + \\
&+ t\,\overset{4}{Q}_{42}\,\theta^4\wedge\theta^2\wedge\theta^5 - \left(t\,\overset{3}{Q}_{42} + \frac{1}{t}\right)\theta^4\wedge\theta^2\wedge\theta^6 + t\,\overset{4}{Q}_{34}\,\theta^3\wedge\theta^4\wedge\theta^5 + \\
&- t\,\overset{3}{Q}_{34}\,\theta^3\wedge\theta^4\wedge\theta^6.
\end{aligned}
\tag{B.6}
$$

As far as the codifferentials of ω_+ and ω_- are concerned, we have:

$$
\delta\omega_+ = \delta\omega_- = t(\overset{3}{Q}_{12} + \overset{3}{Q}_{34})\theta^5 + t(\overset{4}{Q}_{12} + \overset{4}{Q}_{34})\theta^6. \tag{B.7}
$$

Appendix C. Computation of N_J, $N_{\mathbf{J}}$

Using equation (3.9) and the components of ∇J, we deduce that the non-zero components of the Nijenhuis tensors N_J of (Z, g_t, J) and $N_{\mathbf{J}}$ of $(Z, g_t, \mathbf{J})$ are, respectively:

$$
\begin{aligned}
&N^a_{pq} = 0; \\
&N^5_{13} = -t(\overset{3}{Q}_{13} + \overset{3}{Q}_{42} - \overset{4}{Q}_{14} - \overset{4}{Q}_{23}) = 2t(A_{33} - A_{22}); \\
&N^5_{14} = -t(\overset{3}{Q}_{14} + \overset{3}{Q}_{23} + \overset{4}{Q}_{13} + \overset{4}{Q}_{42}) \\
&\qquad = -2t(\overset{3}{Q}_{14} + \overset{3}{Q}_{23}) = -2t(\overset{4}{Q}_{13} + \overset{4}{Q}_{42}) = -4tA_{23}; \\
\\
&N^5_{13} = -N^5_{24} = -N^6_{14} = -N^6_{23}; \\
&N^5_{14} = N^5_{23} = N^6_{13} = -N^6_{24};
\end{aligned}
\tag{C.1}
$$

(C.2)
$$\mathbf{N}^5_{13} = -t(\overset{3}{Q}_{13} + \overset{3}{Q}_{42} + \overset{4}{Q}_{14} + \overset{4}{Q}_{23}) = -2t(A_{22} + A_{33});$$
$$\mathbf{N}^1_{35} = -\frac{2}{t^2};$$

$$\mathbf{N}^5_{13} = -\mathbf{N}^5_{24} = \mathbf{N}^6_{14} = \mathbf{N}^6_{23};$$
$$\mathbf{N}^1_{35} = -\mathbf{N}^3_{15} = -\mathbf{N}^4_{16} = \mathbf{N}^4_{25} = -\mathbf{N}^3_{26} = \mathbf{N}^2_{36} = -\mathbf{N}^2_{45} = \mathbf{N}^1_{46};$$

Appendix D. Ricci ⋆ tensor for J

In this final appendix we collect the components of the Ricci* tensor $\overline{\mathrm{Ric}}^*$ of (Z, g_t, J); from the definition (6.4), a straightforward compuation shows that

(D.1)
$$\begin{aligned}
\overline{R}^*_{ab} &= J^c_b(J^d_e\overline{R}_{aecd} - \overline{R}_{a6c5} + \overline{R}_{a5c6}) = \\
&= J^c_b\Big\{J^d_e\Big(R_{aecd} - \frac{1}{4}t^2[(\overset{3}{Q}_{ac}\,\overset{3}{Q}_{ed} - \overset{3}{Q}_{ad}\,\overset{3}{Q}_{ec}) + (\overset{4}{Q}_{ac}\,\overset{4}{Q}_{ed} - \overset{4}{Q}_{ad}\,\overset{4}{Q}_{ec})]+ \\
&\quad - \frac{1}{2}t^2(\overset{3}{Q}_{ae}\,\overset{3}{Q}_{cd} + \overset{4}{Q}_{ae}\,\overset{4}{Q}_{cd})\Big) - \overset{2}{Q}_{ca} - \frac{1}{4}t^2\sum_{d=1}^{4}(\overset{3}{Q}_{ad}\,\overset{4}{Q}_{cd} - \overset{3}{Q}_{cd}\,\overset{4}{Q}_{ad})\Big\}; \\
\overline{R}^*_{a5} &= J^c_d\overline{R}_{ad6c} = \frac{1}{2}tJ^c_d(\overset{4}{Q}_{ad})_c; \\
\overline{R}^*_{5a} &= J^c_aJ^d_e\overline{R}_{5ecd} = \frac{1}{2}tJ^c_aJ^d_e(\overset{3}{Q}_{cd})_e; \\
\overline{R}^*_{a6} &= -J^c_d\overline{R}_{ad5c} = -\frac{1}{2}tJ^c_d(\overset{3}{Q}_{ad})_c; \\
\overline{R}^*_{6a} &= J^c_aJ^d_e\overline{R}_{6ecd} = \frac{1}{2}tJ^c_aJ^d_e(\overset{4}{Q}_{cd})_e; \\
\overline{R}^*_{55} &= J^c_d\overline{R}_{5d6c} + \overline{R}_{5656} = J^c_d\Big(\frac{1}{2}\,\overset{2}{Q}_{dc} + \frac{1}{4}t^2\sum_{a=1}^{4}\overset{3}{Q}_{ca}\,\overset{4}{Q}_{da}\Big) + \frac{1}{t^2}; \\
\overline{R}^*_{56} &= J^d_c\overline{R}_{5d5c} = 0; \\
\overline{R}^*_{65} &= J^c_d\overline{R}_{6d6c} = 0; \\
\overline{R}^*_{66} &= J^c_d\overline{R}_{5d6c} + \overline{R}_{5656} = \overline{R}^*_{55}.
\end{aligned}$$

The scalar* curvature is

(D.2)
$$\begin{aligned}
\overline{S}^* &= -\frac{3}{2}t^2(\overset{3}{Q}_{12}{}^2 + \overset{4}{Q}_{12}{}^2 + \overset{3}{Q}_{34}{}^2 + \overset{4}{Q}_{34}{}^2)+ \\
&+ t^2(\overset{3}{Q}_{13}\,\overset{3}{Q}_{42} + \overset{3}{Q}_{14}\,\overset{3}{Q}_{23} + \overset{4}{Q}_{13}\,\overset{4}{Q}_{42} + \overset{4}{Q}_{14}\,\overset{4}{Q}_{23}) - 2t^2(\overset{3}{Q}_{12}\,\overset{3}{Q}_{34} + \overset{4}{Q}_{12}\,\overset{4}{Q}_{34})+ \\
&+ 6(\overset{2}{Q}_{12} + \overset{2}{Q}_{34}) - t^2[(\overset{3}{Q}_{13} + \overset{3}{Q}_{42})(\overset{4}{Q}_{14} + \overset{4}{Q}_{23}) - (\overset{4}{Q}_{13} + \overset{4}{Q}_{42})(\overset{3}{Q}_{14} + \overset{3}{Q}_{23})] + \frac{2}{t^2}.
\end{aligned}$$

Acknowledgments . *The first and the third author are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and they are partially supported by GNAMPA project "Equazioni ellittiche e paraboliche in Analisi Geometrica".*

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